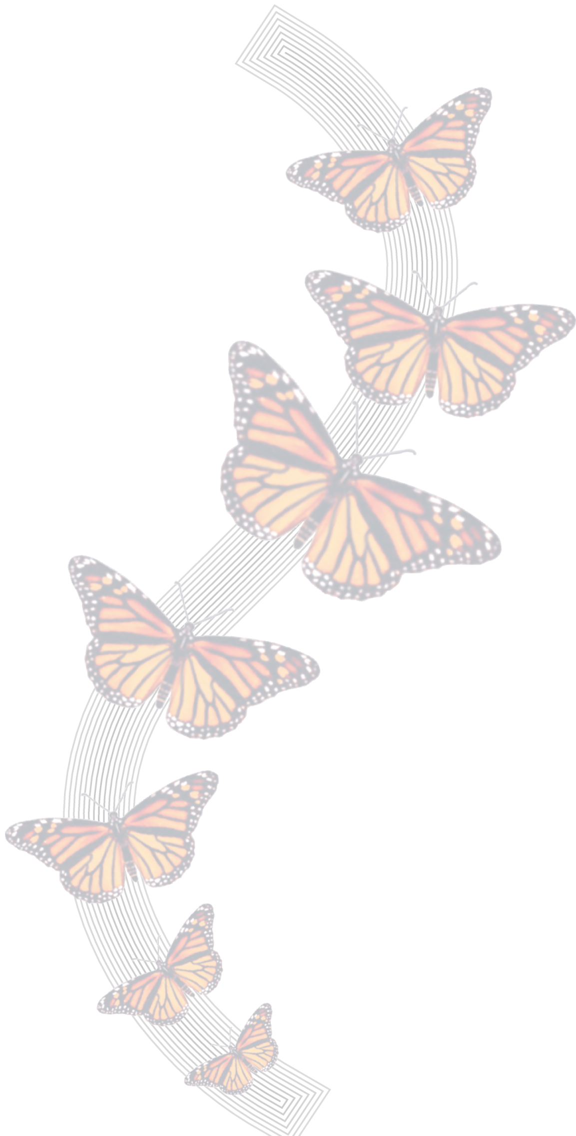
Mathematical Ideas: History, Education, and Cognition



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Morelia, México, 2008

PME 32

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Proceedings of the Joint Meeting of PME 32 and PME-NA XXX

Editors
Olimpia Figueras
José Luis Cortina
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Cinvestav

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Research Reports
Mul-Zac

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Mul- Zac

PROMOTING MATHEMATICAL PATTERN AND STRUCTURE IN THE FIRST YEAR OF SCHOOLING: AN INTERVENTION STUDY

Joanne Mulligan, Michael Mitchelmore, Jennifer Marston, Kate Highfield and Coral Kemp

Macquarie University

Using a design approach, this study monitors the growth in mathematics learning of ten Kindergarten students engaged in a Pattern and Structure Mathematics Awareness Program (PASMAP) over 15 weekly teaching episodes. Pre- and post- interviews using a Pattern and Structure Assessment (PASA) and standardised measures indicated substantial growth across number, measurement and geometry concepts. Qualitative analyses of digital recordings and students' representations provided complementary evidence of their invented symbolisations and generalisations in repetitions and growing patterns. Improvements in mathematical processes such as skip counting, multiplicative thinking, unitising and partitioning, similarity and congruence, and area measurement were observed.

Pattern and structure are the basis of mathematical abstraction and understanding relationships. There is growing research evidence that students of all ages have a poor grasp of mathematical pattern and structure. Rather than attributing this to immutable "low ability", we believe that it gives the clue to preventing difficulties in learning mathematics. In this paper we report the findings of an early intervention program with Kindergarten students that focused on the development of pattern and structure in a range of mathematical areas.

BACKGROUND

The study of pattern and structure is embedded in research on the development of a wide range of topics such as counting, unitising, partitioning, numeration, arithmetic problems and analogical reasoning (Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997; English, 2004; Lamon, 1996; Thomas, Mulligan, & Goldin, 2002). Other researchers have studied structural development through measurement: Unitising is central to this development (Lehrer & Lesh, 2003; Slovin & Dougherty, 2004).

Studies of spatial and geometric concepts with young children have highlighted "spatial structuring" in the learning process (Clements & Sarama, 2007; van Nes & de Lange, 2007). Battista, Clements, Arnoff, Battista and Borrow (1998) and Outhred and Mitchelmore (2000) have studied the development of spatial structuring in rectangular figures and arrays in the elementary grades. Most children learn to construct the row-by-column structure of rectangular arrays and also acquire the equal-groups structure required for counting rows and layers in multiples. However there is little known about how these structures, which underlie multiplicative and algebraic thinking, originally develop in children as young as 4 or 5 years of age.

There is increasing evidence that algebraic thinking develops from the ability to recognise and represent patterns and relationships in early childhood (Mason, Graham, & Johnson-Wilder, 2005). Recent studies of algebraic thinking (Carraher, Schliemann, Brizuela, & Earnest, 2006) have given support to the idea that, given appropriate opportunities, young children can develop abstract mathematical skills that can give them an advantage with basic concept formation. Other studies support the notion that young children can develop understanding of repetition (Papic & Mulligan, 2007) and functional thinking (Blanton & Kaput, 2005; Warren, 2006).

The Pattern and Structure Mathematics Awareness Project (PASMAP)

Our studies on the role of structure in early mathematics developed from earlier work on intuitive models of multiplicative reasoning and structural development of representations (Thomas, Mulligan & Goldin, 2002). A recent study of 103 first graders and 16 case studies found that children's perception and representation of structure generalised across a wide range of mathematical domains (Mulligan, Mitchelmore & Prescott, 2006). We found that early mathematics achievement was highly correlated with individuals' development of mathematical structure, classified as one of four broad stages of structural development: *pre-structural, emergent,* and *partial* and *structural* stages. Multiplicative structure, including unitising and partitioning, and spatial structuring were found central to concept development.

At PME 29 we reported the high consistency of this structural development for eight high-achieving and eight low-achieving individuals, who were tracked over a two-year period. The high achievers drew out and extended structural features, and demonstrated relational understanding. In contrast, low achievers tended to focus on non-mathematical features of special interest to them, and their representations often varied over time without showing progress towards use of structure. These students did not seem to look for, or did not recognise, underlying mathematical structure. The questions that naturally arose from our findings was: *Can we teach young children to become aware of mathematical pattern and structure? If so, will it affect their mathematics learning and achievement?* The Pattern and Structure Mathematics Awareness Program (PASMAP) was developed to address these questions.

A school-based numeracy project suggested that young students can be taught to seek and recognise mathematical structure, and that the effect on their overall mathematics achievement can be substantial (Mulligan, Prescott, Papic, & Mitchelmore, 2006). Further support for our focus on pattern and structure was found in a recent study of preschoolers (Papic & Mulligan, 2007). This study showed that four-year-olds involved in a 6-month intervention promoting the development of repeating and spatial patterns outperformed comparison children at post and follow-up interviews-including on growing patterns, which they had not been exposed to.

THEORETICAL ORIENTATION

Our studies indicate that initial recognition of similarities and differences in mathematical representations plays a critical role in the development of pattern and structure, abstraction and generalisation. The development of multiplicative concepts, (including understanding the base ten system, grouping and partitioning) are integral to building structural relationships in early mathematics. Spatial structuring is necessary in visualising and organising these structures. There is also compelling evidence that young students are capable of developing complex mathematical ideas, rather than being limited to unitary counting, simple arithmetic, shape recognition and informal units of measure (Clements & Sarama, 2007).

However, consistent with Gray and Tall (2001), we propose that some young students fail to perceive structural features and are impeded in mathematics learning because they focus on idiosyncratic, non–mathematical aspects. Thus, we conjecture that if a child does not notice salient features of structure in specific situations, they are unlikely to notice or seek them in other mathematical representations or contexts. Moreover, they may not be attuned to seeking mathematical similarities and differences within or between patterns and structures.

The broad aim of PASMAP is to scaffold experiences where children seek out and represent pattern and structure in a wide range of concepts and transfer this awareness to varied mathematical situations. In other words, we intend to promote generalisation in early mathematical thinking. More detail is provided below.

METHOD AND ANALYSES

We implemented PASMAP with 10 Kindergarten students aged 4 to 6 years. The program involved 15 weekly teaching episodes of one-hour duration conducted during Terms 2 and 3. Students were selected by classroom teachers on the basis that they were representative of the lowest quartile of the cohort in terms of perceived mathematical ability. Students were pre- and post-assessed by the researchers using a Pattern and Structure Assessment (PASA) and two subsets of the Woodcock-Johnson mathematics test (Woodcock, McGrew, & Mather, 2001). A teacher and an assistant trained in PASMAP withdrew the group to a discrete learning area. The children were engaged in carefully scaffolded tasks that differentiated for individual levels of patterning skills.

PASMAP tasks are known as *pattern-eliciting tasks*. The broad instructional approach using such tasks may be summarised as follows:

- The task requires children to produce a model or other representation (*representing*)
- Children explain their initial, perhaps inaccurate representation (*intuitive justification*)
- The teacher scaffolds probing questions, comparing children's patterns with correct patterns produced by others (*modelling*)
- The teacher asks children how they can make their pattern the same as the given pattern, and to explain why this will make it the same (*similarities and differences*)

- Children's attention is drawn to crucial attributes of shape, size, spacing, repetition, etc. (*focus on spatial or numerical structure*)
- Children reproduce entire pattern with parts increasingly obscured (*successive screening*)
- Children justify why their pattern is correct (or incorrect) (*justification*)
- Children reproduce patterns from memory (*visual memory*)
- The task is repeated regularly, extended and linked to prior learning and other tasks *(repetition and transformation).*

Teaching episodes focused on pattern as simple repetition, pattern as spatial structuring, and spatial properties of congruence and similarity. These activities extended the students' already developed notion of simple repetition to include generalising and functional thinking. Table 1 provides an overview of the key components, which were introduced in the stated order but regularly revisited later.

Component	Focus
Counting Rhythmic and perceptual counting	Counting orally by twos and threes, with and without materials Constructing simple patterns using perceptual counting
Repetition Simple and complex	Constructing, drawing, symbolising and justifying linear and cyclic patterns using a variety of materials
Unit of repeat	Chunking, ordering, symbolising and translating
Similarity and congruence (2D shapes)	Comparing and drawing similar triangles and squares, distinguishing congruence
Symmetry and transformations	Identifying symmetry through matching and congruence
Subitising	Identifying number and shape in subitising patterns, three to nine. Spatial structuring of subitising patterns
Grids	Identifying number of units in simple grids, 2 x 2, 3 x 3, 4 x 4, 5 x 5 squares and 2 x 3 rectangle. Deconstructing and reconstructing from memory the properties of grids
Arrays	Identifying number of units in simple arrays. Deconstructing and reconstructing from memory the spatial properties of arrays
Table of data: functional thinking	Constructing tables of data, representing ratio as a pattern

Table 1. Key components of the Intervention teaching episodes

The data analysis involved systematic coding of videotaped interactions with individual students and interpretation of drawn and written representations for structural features. We did not, however, impose our previously classified stages

(Mulligan et al., 2006). Instead, our focus was on describing students' idiosyncratic growth of structure. Detailed individual profiles of learning were constructed, enabling us to track individual's thinking for each component of the program and to look for developments in the student's use of pattern and structure over time.

DISCUSSION OF RESULTS

Every student showed improvement on the PASA interview, not only in gaining correct solutions but in the growth of their representational skills and the way they could justify their responses (see Figure 1).

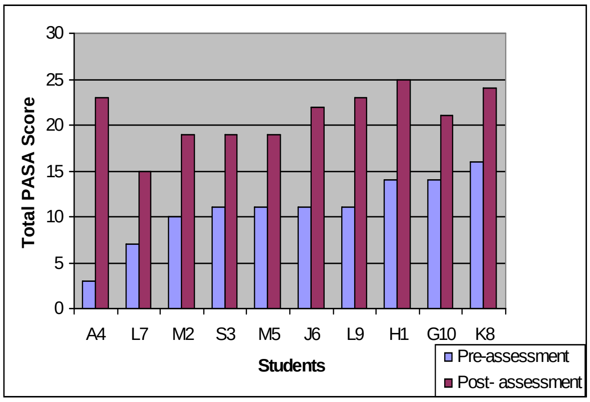

Figure 1. Pre and post PASA assessment.

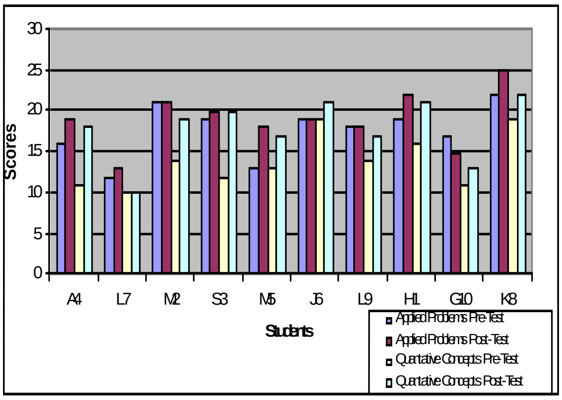

Figure 2. Pre and post Woodcock Johnson.

All students made substantial gains; however these improvements were not necessarily consistent across tasks for individuals. Those PASA tasks where the largest gains were made included skip counting by 2s and 3s, using a numeral track, unitising length, area, mass, time (clock face), subitising patterns and structuring partition and quotition problems. Increased performance on the Woodcock-Johnson post-test was observed for most children.

In the teaching episodes impressive growth was observed in representing, symbolising and translating simple and complex repetitions, structuring arrays and grids and unitising and partitioning in a variety of ways. The students' construction of a simple table of data showing functional thinking was also demonstrated in the final teaching episodes. Improvements in recognising subitising patterns and counting in multiples in 2's, 3's and 4's were also observed as well as some grouping strategies. This improvement could be explained by the varied experiences in grouping and patterning using a unit of repeat, consistent with Papic and Mulligan (2007).

Three significant features of learning captured in our data were translating, connecting and symbolising. Translating is exemplified where the student translates the same structure to a different mode or situations and explains and justifies why the structure is the same. Figure 3 shows the child's translation of an AAB pattern justified as: "It's the same pattern (structure) but different colours". A second feature was students' connection of particular features of pattern and structure between their representations over time. For example, "It's like the one I made with the two triangles last time but it's got squares: big then little" (see Figure 4). The use of invented symbolism was elicited through translation and representation of patterns. An example is where the student describes the pattern as "DB, DB, DB…my pattern is my initials" or where the student uses symbols to represent colours (see Figure 5).

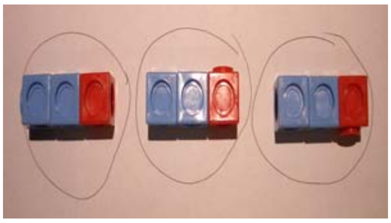

Figure 3. AAB structure re- represented.

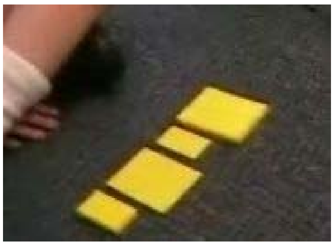

Figure 4. Simple repetition with squares.

Figure 5. Invented symbols for simple repetition.

Another pertinent example of spatial structuring was the development of understanding of the structure of grids. For example students were asked to cover a 2 x 2 grid with squares, discuss similarities and differences, and describe patterns and other mathematical aspects such as the alignment or counting corners. Students progressed to a pattern of squares using counters, and gradually highlighted vertices rather than scattering counters at random. Other tasks were designed to assist students to replicate a unit of repeat (in this case squares).

Students later drew grids from memory. Figures 6, 7, and 8 show an individual's growth of understanding of the structure of unit squares, with increased focus on the size and structure of the unit squares rather than 'crowding' the image (Figure 7).

Figure 6. drawing of 2 x 2 squares, initial attempt.

Figure 7. drawing of 2 x 2 squares, later attempt.

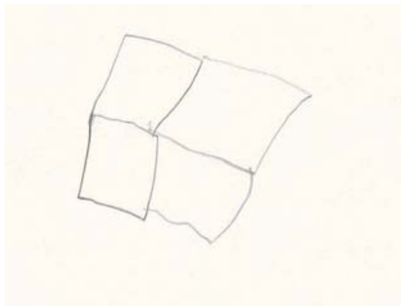

Figure 8. drawing of 2 x 2 squares, final attempt.

CONCLUSIONS AND LIMITATIONS

This project illustrates the rich and diverse learning experiences of ten young students in their development of structural awareness. The intervention was limited to a small group of students withdrawn for individualised instruction, supported by specialist teachers and well-formulated resources. We cannot assume that the success of this program can be generalised to other pedagogical settings. Nevertheless, our data suggest that explicit assessment and teaching of structure has the potential to effectively improve students' abstraction of mathematical processes within a relatively short time frame.

There is some evidence from our study that PASMAP teachers could increased their awareness of the essential role of pattern and structure in mathematical learning and their belief that young students are capable of learning relatively complex mathematical ideas. Both changes are likely to result in more effective teaching.

PASMAP has recently been further developed to explicitly reflect aspects of early algebraic reasoning and data exploration. Currently we are conducting an empirical evaluation of the impact of this extended program on mathematics teaching and learning in the first year of formal schooling in New South Wales and Queensland, supported by a grant from the Australian Research Council.

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THE USE OF THE EMPTY NUMBER LINE IN ENGLAND AND THE NETHERLANDS

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This paper contrasts the development of the Empty Number Line (ENL) in the Netherlands with its use in the teaching of mental calculations in England. In doing so the paper investigates the Dutch adoption of Soviet socio-cultural philosophies, in particular Gal'perin's theory, and relates the notion of internalisation to the analysis of the ENL and its use within the English Primary National Strategy (PNS). By contrasting the English and Dutch approaches a better understanding of the specificity of the development of number sense in relation to mental calculation strategies is attained.

INTRODUCTION

The use of the Dutch Empty Number Line (ENL) would now seem to be established as an effective model to support children's development of mental calculation strategies (Rousham, 2003; Foxman and Beishuizen, 2003). A first glance at the 'Guidance paper' Calculation would suggest that the English Primary National Strategy's (PNS) new framework for teaching mathematics (DfES, 2006) has firmly adopted its use. A closer look suggests that this adoption has been piecemeal and that there are ambiguities. Thompson (2007) critiqued the PNS's use (or misuse) of the ENL and suggested a misunderstanding of its purpose.

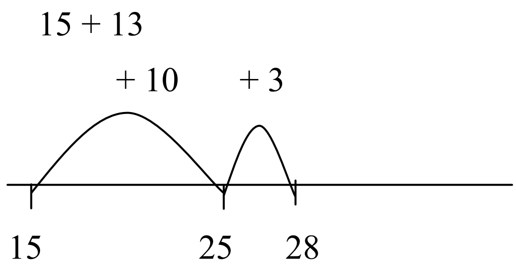
This paper proposes that the English use of the ENL is the result of fundamental differences in pedagogies related to the teaching of mental calculation strategies in England and the Netherlands. Both the Dutch Realistic Mathematics Education (RME) and the English PNS promote the explicit teaching of mental calculation strategies and the development of a sense of number. In order to compare the teaching approaches in the two countries, the socio-cultural notion of internalisation is reviewed. The intention is not to further the internalisation debate, but to relate the review to the teaching of number in order to better understand the teaching-learning process.

MENTAL CALCULATION STRATEGIES AND NUMBER SENSE

Both the Dutch and English mathematics curricula aim to provide children with the ability to use mental calculation strategies flexibly, that is, to select an appropriate strategy to solve a given problem. Thompson (1999) defined mental strategies as the "application of known or quickly calculated number facts in combination with specific properties of the number system to find the solution of a calculation…" (p.2). The facility to apply number facts and number properties in conjunction with number operations requires an understanding or sense of number. McIntosh, Reys and Reys (1992) referred the notion of number sense to "a person's general understanding of number and operations" (p.3). The intention would be to encourage the development of 'a well organised conceptual framework' (Bobis, 1991) that can enable a person to solve calculations using flexible strategies that are not bound to the traditional standard algorithms. What is less clear is how children attain number sense. The development of number sense has been seen as highly personalised, relating to an individual's ideas and how those ideas have been established (McIntosh et al, 1992; Anghileri, 2000). Is number sense developed individually through a child's mathematical experiences or is it desirable to teach it? If this knowledge is individualised, can it be taught?

In England the publication of the Cockroft Report (DES, 1982) coined the term 'at-homeness' with number with an emphasis on mental skills and the application of number in real life problem solving and mathematical investigations. The engagement in problem solving and investigations was seen to encourage mathematical thinking where decision-making encouraged the active use of number and the development of an individual's number sense.

With the introduction of the National Numeracy Strategy (NNS) (DfEE 1998) and the later revision of the mathematics framework by the PNS (DfES, 2006) emphasis was put on the development of mental skills through the modelling of a range of calculation strategies. The aim was for children to acquire a repertoire of mental calculation strategies from which they could select the most effective for a given problem. Rather than encouraging mathematical thinking through problem solving and investigations the encouragement was to work mentally and to recognise patterns and relations. The use of materials and visual images were promoted to model the relationships and mental calculation strategies. Adopted from the Dutch, the ENL provided a visual model to support the teaching of mental calculation strategies (Figure 1).



15 + 10 = 25; 25 + 3 = 28

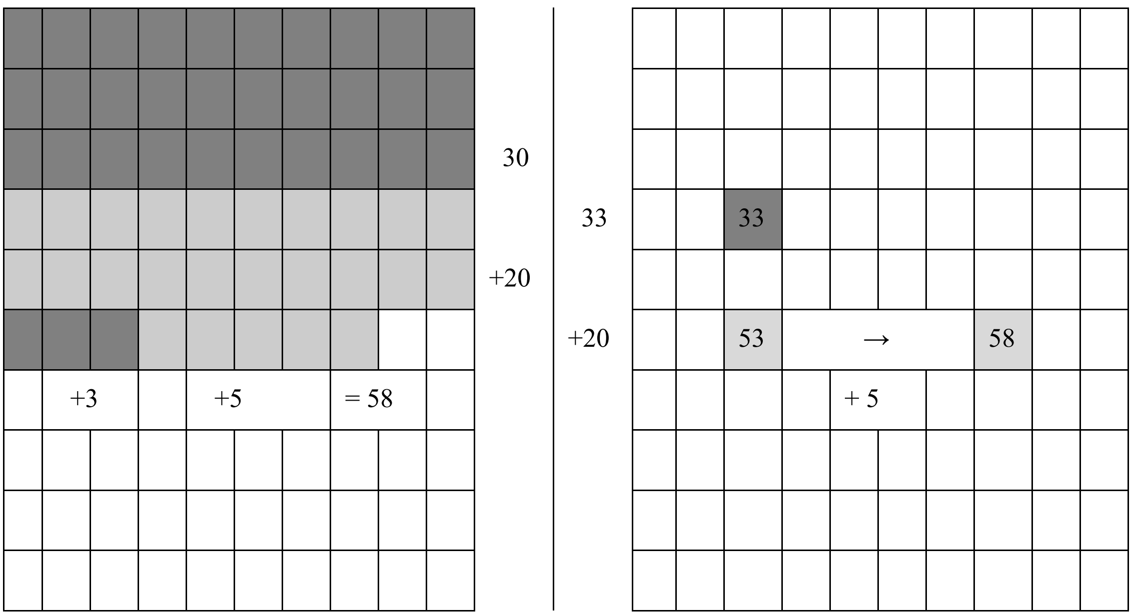
Figure 1. Use of the Empty Number Line to solve 15 + 13.

THEORETICAL ORIGINS OF THE EMPTY NUMBER LINE

The influence of Piaget and Bruner's theories led to the use of materials such as Dienes base ten blocks as a way of concretising the place value structure in whole number in both England and the Netherlands. In the 1970's ideas from Soviet psychology inspired the Dutch to review the role of concrete experiences in the teaching of calculation (Beishuizen, 1985). Traditionally in western psychology "meaning and abstraction tend to be reduced to properties of individual consciousness" (van Geert, 1987, p.357) but with the Dutch adoption of Soviet psychological perspectives the social origin of the human mind, and Vygotsky's notion of internalisation (Arievitch and van der Veer, 1995) were introduced. The Soviet insights that particularly influenced the Dutch reform were those of Gal'perin, a contemporary of Vygotsky. Following the early work of Leont'ev, Gal'perin's focus was on the transformation of external object-related activities to internal mental acts.

Whilst not disputing that an individual has 'something in mind', Gal'perin proposed that the very nature of individual mental acts means they could not be taught. Instead Gal'perin's interest was in the meanings and abstractions that have been formed in the cultural-historical process and can be transmitted from one generation to another (van Geert, 1987). So rather than starting with the individual, Gal'perin's theory starts with the functional, concrete regularities of knowledge and skill acquisition within a discipline. His theory explains the transformation of initially external activities into internal forms through a step-wise formation of mental actions from the first materialised stage to the last mental stage via perception and verbalisation (Beishuizen, 1985; van Geert, 1987, Haenen, 2001).

In contrast to Piaget and Bruner, Gal'perin did not see physical objects as stimulating discovery but as "materialised orienting bases" (Beishuizen, 1985). The material representations orientate the learner to carry out the act by presenting the learner with a model or scheme of the act to be acquired (van Geert, 1987). Images and mental actions constitute the basis for the orienting activity where images and concepts are formed on the basis of actions.



100square representing number 'Empty' 100square representing operation

Figure 2. 100squares for 33 + 25 (adapted from Beishuizen, 1985 p.248).

This shift in perspective is illustrated by Beishuizen's (1985) study that compared the representation of the traditional hundred number square (100square) with the representation of an 'empty' 100square in the teaching of mental calculation strategies. Whilst the former represents the number the latter represents the operation (Figure 2). Where the traditional 100square was used in schools the children were more likely to use the model as a step-by-step animated guide to simulate the process with little evidence of understanding. It seemed that there was little internalisation of the processes involved. With the 'empty' 100square the representation provided a more condensed and abstract structured form that could be said to activate the children's cognitive processes. The children were no longer imitating processes but had internalised them. Further research at the Freudenthal Institute combined the abstractness of the 'empty' 100square to the number line that resulted in the ENL format.

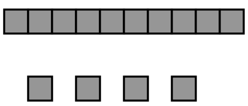
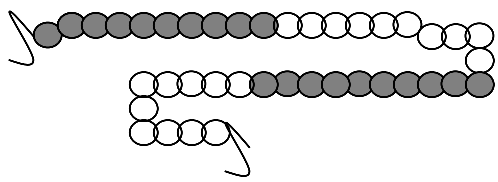
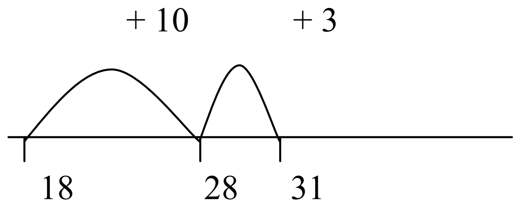
<u>Model concept and link to process</u>	<u>Orienting Activity</u>
Concrete representation of number Concept, Place Value 	Representation of operation Children's activities, Counting 
Manipulating material Modelling, visualisation	Informal strategies Structuring
Abstraction 18 = 10 + 8 13 = 10 +3	
10 + 8 + 10 + 3; 10 + 10 + 8 + 3; 20 + 11; 31	$18 \xrightarrow{+10} 28 \xrightarrow{+3} 31$
Concept used to inform process Process is imitated	Pre-structuring of cognition Action constitutes the mental image

Figure 3. Contrasting approaches to 18 + 13.

Within the Dutch RME material representations are not seen as stimulation for discovering a concept. The aim is not to try to teach the concept by concretising it but to present the organising phenomenology that will compel the learner to constitute a mental object and the ENL is seen to provide a condensed and abstract representation of both the number system and the operation (Freudenthal, 1983). As Freudenthal explained the empty number line visualises magnitude and articulates numbers. By positioning a number x on the number line there is a sense that the number x is 'accomplished' on the line by scanning or pacing it with real or imaginary fingers or feet. Whatever is being counted is 'accomplished' at this point, where it is 'full'. A child can step or jump to this point. The number line can dispense with numbers as it mirrors magnitudes that are visualised and bound to the number line. This provides a starting abstraction that is focused on the key essential features. The representation abstracts the operation as a development from children's informal counting strategies. In this way the ENL is not used as a material representation for imitation. Instead it compels the learner to carry out the action and it is the action that constitutes the mental image (Figure 3).

Rather than concretising a concept and imitating processes, the number line represents the essential position and magnitude of number. The ENL builds on intuitive counting strategies to develop a structure of the number system and the operations themselves. Counting and adding are seen as being closely 'knit'. "Counting is again and again adding one, and additions are performed by counting". (Freudenthal, 1983, p.101). In this way the ENL "gives a more directed pre-structuring of the cognitive addition operation itself" (Beishuizen, 1985, p. 249). The structure of the action compels the learner to use the abstract concept. Or to see it from the other perspective: concept is abstracted through the action of the learner. "Cognition does not start with concepts but the other way around: concepts are the result of cognitive processed" (Freudenthal, 1991 p. 18).

PEDAGOGICAL IMPLICATIONS

By relating the underlying principles of the Dutch RME to Gal'perin's theories of internalisation, the ENL can be seen as an orienting basis that abstracts the concrete regularities of the number system and number operations. In line with the social origin of the human mind it provides an organising phenomenon based on "certain principles of structural similarity between the architecture of the minds of people…" and arrives "…at common ideas and concepts…" (van Geert, 1987, p.376). It represents the meanings and abstractions that can be transmitted as collective knowledge. Even though children's learning can take individual paths, the ENL represents common ideas and concepts as part of a broad outline that can be identified in children's learning in number.

Based on this notion of a broad outline of common ideas, the TAL project at the Freudenthal institute set out a learning-teaching trajectory for whole number calculation (van den Heuvel-Panhuizen, 2001a). The teacher is given a broad 'helicopter' view or 'educational map' in order to grasp the course of development in a few large steps and not get lost in detail. The outline reflects the progression from children's intuitive counting and calculation by counting, to calculation by structuring and, later, to formal calculation based on a grouping view of place value.

In this way the teacher has a framework to support didactical decisions and adaptations. As children put forward their ideas, they are discussed and compared. The teacher can assess the children's progression on the 'educational map' and provide guidance to raise the level of the children's work. Whole class teaching is maintained but it does not mean "every student is following the same track and reaching the same level of development at the same moment" (van den Heuvel-Panhuizen, 2001b, p.54). Individual differences are catered for by providing problems that can be solved at different levels.

It is difficult to identify common progressive steps from the many specific objectives presented in the English PNS mathematics framework. Guidance is given on making connections and supporting children in seeing the relationship in number but lesson objectives focus on small aspects of mathematics with the notion that all children will meet that objective. Yackel (2001) has critiqued this approach in the USA and suggested that the specific objectives are replaced with broader aims that support a common conceptual development so that teaching moves children's understanding forward despite individual differences.

Without the broad picture the tendency is to present visual images, such as the ENL, as models to emulate. Doritou's (2006) study of the use of the ENL in one English school suggested that the there was more concern with establishing the ENL as a valid metaphor. Its introduction to the children was often confused and some children would make procedural errors in an attempt to imitate the methods demonstrated by the teachers. Children often found other, to them simpler, ways of solving problems.

CONCLUSION

If number sense is defined as a personal understanding of number it would seem desirable that activities are provided that encourage a child's individual understanding. The use of investigations and problem solving to develop an individual's own decision-making and mathematical thinking had been seen as a way to develop this in England. However this approach does not allow for the transferral of functional concrete regularities of knowledge within number and calculations.

Our main concern as educators is to support those children who are unable to move to more flexible strategies and the transferral of such knowledge would be seen as useful in supporting this move and so explicit instruction in developing mental calculation strategies would seem desirable. The NNS and the later PNS framework for mathematics emphasise the teaching of mental calculation strategies. Number concepts and relationships are concretised using a range of materials and visual images such as the ENL. There is the possibility that this approach, whilst attempting to engage in the transmission of social knowledge has not fully acknowledged the type of material knowledge that can be transferred. Children may attempt to imitate the processes rather than internalise them.

The modelling and emulating of the English approach has been questioned (Threlfall, 2002) and Thompson (2007) continues to critique the lack of coherence in the guidance given to teachers by the PNS. A review of what are considered to be the common building blocks within the teaching and learning of mathematics in England is worth investigating.

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INTEGRATING THE MEASURE AND QUOTIENT INTERPRETATION OF FRACTIONS

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The study reported here is part of a longer study on developing students' fraction concepts as a preparation for solving ratio proportion problems. Here we investigate the impact of instruction aimed at developing the measure and quotient interpretation of fractions in an integrated manner. Grade 5 students, who participated in the study, had been previously exposed to the part-whole interpretation in school. The inadequacy of an exclusive emphasis on the part-whole interpretation, and the effectiveness of supplementary instruction as described are discussed. Student responses on pre- mid- and post tests, interviews and classroom discussion are analysed. The responses show that students' performance on representation and comparison tasks improved significantly after the instruction.

INTRODUCTION

A number of educators have stated that the learning of fractions is probably one of the most serious obstacles to the mathematical maturation of children (Behr et al., 1992; Kieren, 1988; Streefland, 1994). Many students have little conceptual understanding of fractions and are dependent on procedures that are learnt by rote, which are often incorrect. The fraction concept is also complex since it consists of multiple subconstructs, which are not adequately developed in traditional school curricula (Kieren, 1988. Charalambous and Pitta-Pantazi, 2007). Some researchers have located the source of many of these problems in the almost exclusive focus on the part-whole subconstruct of fraction in traditional instruction (Streefland, 1994).

In this article, we discuss the responses of grade 5 students (11 year olds), who have been exposed to traditional fraction curriculum based on the part-whole subconstruct in school, to supplementary instruction emphasizing the measure and quotient subconstructs. The instruction was a part of a teaching design experiment aimed at developing an understanding of fraction to prepare students to solve ratio and proportion problems. In this report, we restrict discussion to students' understanding of the magnitude of fractions as inferred from the tasks of pictorial representation and comparison of fractions.

BACKGROUND

The sub-construct theory of fraction understanding analyses the fraction concept as comprising the five sub-constructs of part-whole, measure, quotient, ratio and operator (Kieren, 1988; Behr et al. 1992). It has been suggested that instruction must aim to develop an adequately integrated understanding of the multiple subconstructs (Post et al., 1993). Indian textbooks emphasize only the part-whole subconstruct in introductory fraction activity through the use of the area model. This model is used to introduce the fraction symbol and vocabulary, typically by comparing the number of shaded parts to the total number of parts. The approach then rapidly shifts to procedures for operating with fractions, drawing from time to time on the area model to explain the procedures.

There are several problems with this manner of introducing the part-whole interpretation. Although teachers often stress that the parts into which the whole is divided must be equal, students do not absorb this fact very well (as indicated also by student responses in our study, discussed below). An explanation of this may be found in the fact that the concept of unit is not highlighted. In situations where counting is used to arrive at the numerosity of a discrete set, the size (or even the nature) of each unit may be ignored, for example, when counting the number of people standing in a queue to enter a bus in order to find out if there are enough vacant seats. Thus children who operate on the basis of whole number thinking would only count the number of shaded parts, and the total number of parts, ignoring the size of the parts. In contrast, in measurement contexts the size of the unit is critically important. When a part of a whole is taken, we divide the whole into equal parts in order to obtain a subunit. This unit is then iterated to measure out the part that is taken. Traditional teaching typically emphasizes the counting aspect while missing out the measurement aspect. While the partitioning scheme is drawn upon and reinforced, the fact that partitioning involves formation of unit structures is underplayed (Lamon 1996).

In our approach, we brought the measurement aspect centrally into focus by emphasizing the notion of unit fraction. Although, many studies on various constructs of fractions have been reported, few have focused on exploiting the power of unit fractions in the construction of fraction understanding. Some approaches such as Lamon's (2002), which have emphasized the unit concept, and the strengthening of the unitizing schema within the part-whole interpretation, are indeed powerful. In Lamon's approach, the designated part is compared with the whole as a ratio by taking different, arbitrary units in a flexible manner. Thus 3/5 may be symbolized as both 6/10 and (1½)/(2½). A sense of the magnitude of the designated part is based on grasping the ratio of the part to the whole, while the unit recedes to the background. In contrast, in our approach, the role of the unit fraction is more central, and the magnitude of a part or a quantity is grasped in terms of the named unit fraction, which establishes the relation of the new subunit to the base unit. Since unit fractions are given prominence as objects in their own right, their properties become worthy of study. An important property that is easy for students to construct is that the unit fractions form a regular sequence ordered in terms of decreasing magnitude.

The milieu in which students learn in Indian schools is frequently multilingual. For the vast majority of students who study in the English language, English is not a language of comfort. So the noun forms that signify unit fractions such as 'fourths', 'fifths' are easily missed; indeed, they are often avoided and replaced by fraction names such as ‘3 out of 5’ or ‘3 over 5’. Indian languages have fraction names similar to ‘3 over 5’. Hence the language support for calling attention to unit fractions is very weak, justifying the need for special emphasis on them to be built into instruction.

THE STUDY

The study reported here forms part of a larger ongoing study that uses a design experiment methodology (Cobb et al., 2003) and is aimed at developing an approach to teaching fractions as a preparation for understanding ratio and proportion. The study was conducted with grade 5 students from nearby English and vernacular medium schools during the summer vacation period between grade 5 and 6. An earlier cycle focusing on developing the operator interpretation of fractions as a preparation for understanding ratio, revealed the inadequate understanding of basic fraction concepts of students exposed to the traditional school curriculum. Accordingly, a unit emphasizing the measure and quotient interpretations of fractions in an integrated manner was developed and implemented in a vacation programme for students from schools in the summer of 2007. Students from two schools, one studying in the English language (N=41, Avg. age: 10.5 y) and one in the Marathi language (N=30, Avg. age: 11 y) volunteered for the programme. The English and Marathi groups received 16 and 14 days of instruction respectively, of approximately 1.5 hours per day. Each group was taught by a separate teacher (one of the authors). All the lessons were video recorded. Six students from the English group and five students from the Marathi group, whom the teachers judged to have a weak understanding of fractions, were chosen to be interviewed. The purpose of these interviews was to probe the nature of students’ difficulties.

In the curriculum adopted in both the schools, fractions are introduced in grade 3. The whole gamut of fraction concepts and tasks: like and unlike fractions, equivalent fractions, improper fractions, mixed numbers and comparison of fractions is introduced in grade 4. All the four operations with fractions are introduced in grade 5. Thus students had received instruction on fractions over the previous three years before participating in our programme. Nearly all the conceptual work is done with the part-whole interpretation of fractions. Fraction as division is cursorily introduced without any explanation in grade 5.

The teaching unit in the programme consisted of two segments: the first dealing with interpretations of fractions in terms of the measure and quotient subconstructs, and the second dealing with equivalent fractions and the operator subconstruct. Representation and comparison tasks were the primary tasks used in the first segment. We include for analysis in this paper only the first segment of the programme that formed 9 days of instruction for the English language group and 6 days for the Marathi language group. Students were taught to write fractions in measure and share situations. All the other competencies such as representation and comparison of fractions were developed through reasoning about fractions. The main focus of the teaching was to move the students away from a sequential, procedural understanding of fractions to a conceptual and meaningful understanding of fractions.

Representing Fractions

The students were encouraged to represent fractions in two ways. The measure interpretation was developed through the notion of the unit fraction 1/n. We defined unit fraction as *when a whole is partitioned in to equal number of parts and then each such part represents a unit fraction which is 1/(number of equal parts).* This definition was further simplified by the students as *whenever one cake (or unit) is shared among a number of students equally the share of each child is a unit fraction.* The relation of the unit fraction to the basic unit as well as the relative sizes of unit fractions were highlighted, while also preserving the connection with the share interpretation. After practicing the representations for 1/7, 1/5 students represented the non-unit or composite fractions such as 2/5, 3/7, etc. Composite fractions were needed to specify quantities that could not be measured by whole units, and were introduced as built up of unit fractions.

The quotient interpretation was introduced as equal shares (Streefland, 1994). Students found the share of each person when m cakes were shared among n persons, and represented it as m/n of a cake. In the process, they linked this with the use of the traditional symbol for the division operation, as well as with the measure interpretation of *m/n* as *m* '*1/n*' units.

In a variety of situations such as sharing cakes equally, students solved the problem by actually drawing a cake and stick figures for students, followed by the act of equal partitioning. The video records show that students were usually careful about making exactly equal parts in trying to make fair shares. The measure interpretation was helpful in answering the question - *find how much of a cake is obtained by each child.*

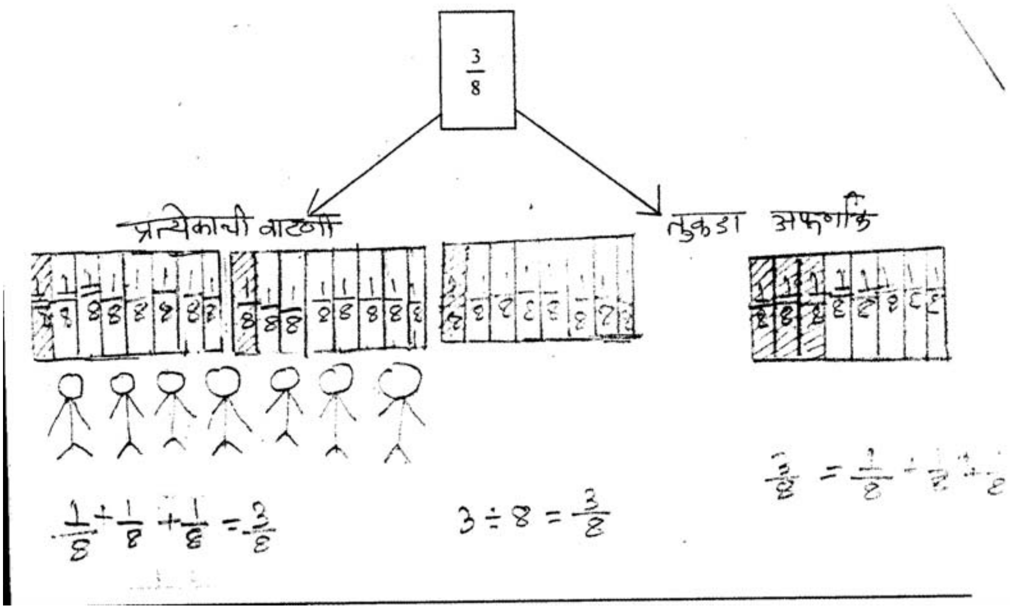

Figure1. Representing fractions in two ways.

The illustration in Figure 1 exemplifies the instructional approach adopted. It shows two representations for the fraction 2/7: '*share of each child*' on the left and as '*built* *up from unit fractions*' on the right. As the act of sharing invokes the idea of division in whole numbers, it is consistently extended in this situation which results in the division fact 2 ÷ 7 = 2/7. (This understanding of students is used in the later part of the study dealing with the operator concept, but is not the focus of this paper.) The presentation on the right side, indicates more clearly the quantity of cake that each child has received: it shows formation of the unit 1/7 and construction of 2/7 as 1/7 + 1/7. This is the portion of cake that each child has received. The discussion in the classroom consistently emphasized an integration of the measure and the quotient sub-constructs.

RESULTS AND DISCUSSION

The table below presents the data on student performance with respect to two kinds of tasks: representation of fractions and comparison of fractions. The data is from three tests: pre-test conducted at the beginning of the camp, the mid-test conducted after the first segment on representation and comparison of fractions using the measure and quotient sub-constructs was completed, and the post-test taken after the entire programme was completed.

Task	***Percentage of correct responses***		
	Pre-test	*Mid-test*	*Post-test*
Writing a fraction for a shaded part (marked parts are unequal, need to be remarked)	8.7	31.5	70.1
Writing a fraction for a shaded part (more than a whole)	9.9	56.3	59.4
Pictorial representation of improper fraction	18	59.8	62
Pictorial representation of mixed number	24.3	73	66.3
Comparison of fractions (all items)	37.3	81.3	83.9
Comparison of unit fractions	21.2	84.5	83.2

The items in the test were items that students typically find difficult. In the area representation, unequal parts were marked and students needed to remark the parts. The representation tasks contained improper and mixed fractions, and the comparison tasks included tricky comparisons. Students showed considerable improvement over their pre-test performance on all these items. It was observed that a few students had used a sharing picture to represent the fraction either together with or without the measure picture. Another minor observation is that the performance marginally improved when students were asked to show the composition of a fraction in terms of unit fractions before drawing a picture representing the fraction.

Classroom discussion on comparing fractions

The comparison of quantities is based on an understanding of the magnitude of a quantity. Hence understanding how to compare quantities and understand how much a quantity is, are not two disjoint cognitive abilities. Students' performance in the pre-test comparison tasks showed evidence of whole number thinking, rather than an understanding of fraction magnitudes. For example, students judged that 1/2 < 1/3 because 2 is smaller than 3. However they soon replaced their representation of fractions with the new interpretations of measuring and sharing that became available to them. Students were strongly encouraged to justify and give reasons for their answers. The classroom video recordings showed the variety of strategies that students used to solve these tasks. As we see below, the sharing and measure interpretations were both drawn upon.

Fractions with the same denominator:

As the number of children to share the cakes are same, children from the group with more number of cakes get bigger share.

As the unit piece is same in both the fractions more number of pieces represents the bigger fraction.

Fractions with the same numerator:

As the number of cakes to share are same, the group where more number of children are there will have a smaller share.

As the number of pieces are same what matters is the size of the unit.

Comparison with half:

The number of cakes are exactly half of the number of children to share.

The number of pieces picked are exactly half of the total number of pieces in the whole

Apart from these kinds of comparison, students reasoned about questions such as - which is the smallest unit fraction? Which is the biggest unit fraction?

In some of the open ended tasks students showed their ability to reason about fractions. Students understood every fraction as constructed from its unit. This constructive nature of unit fractions allowed students to explain how a improper fraction is constructed. They understood 5/4 as : 5/4 = $\underline{1/4 + 1/4 + 1/4 + 1/4}$ + 1/4 = 4/4 + 1/4 = 1 + 1/4 = 1 1/4. As the pre-test data reveal representing improper fractions was especially hard for the students. Some students were also comfortable in the idea of sharing and interpreted improper fraction as follows: '*as number of cakes are more than the number of children to share. Obviously each child will get at least one cake.'*

In one of the discussion tasks in the classroom, several fractions were written on the board and students were asked to compare them and give reasons for their responses. Many of the students first wrote all the fractions in the form of their unit fractions, and then made statements about the comparison of those fractions. While comparing the two fractions 4/5 and 6/7, students found the similarity that both of these need one piece to complete the whole. When asked which one of these is closer to one whole, many students could say that it is 6/7. When asked how, students came up with following reason -

Even though both the fractions 4/5 and 6/7 need one more piece to complete a whole. 4/5 needs one piece of 1/5 and 6/7 needs one piece of 1/7. but 1/5 is more than 1/7 as one cake is shared among 5 children only hence 4/5 is away from the whole.

Towards the end of the programme individual interviews were conducted of six students from the English language group and four students from the Marathi language group. The students interviewed were judged to have a weak understanding of fractions on the basis of their classroom work and the interviews were aimed at understanding the nature of their difficulty. Students were asked to represent the fraction 14/9 by drawing a picture. After they made a drawing, an alternative drawing was shown to them and they were asked if it was correct. For students who drew the correct representation, a representation of 9/14 was shown, and for those who drew a wrong picture, a picture with one whole and a 5/9 shaded was shown. Five of the ten students were confident about the correct response that they made initially. All of them used a measure picture, except one student who began with a sharing picture and changed to a measure picture. Of the remaining students, one could not complete the task. The remaining had difficulty in completing the task and changed their response midway, but eventually managed to do so. Two students who worked with the sharing picture had difficulty representing the share of each child, and both moved eventually towards a measure picture. Two students completed the task by rewriting the improper fraction as a mixed number, but were unsure about the representation of 14/9.

For the second task of representing and comparing the three fractions: 1/5, 5/5 and 5/1, all except one student completed the comparison task successfully. Eight of nine students could also represent the fraction either by a picture or by description. Seven students used the share interpretation to justify their response. Two students, who used the measure interpretation, expressed themselves clearly and confidently. One of the students who reasoned on the basis of both the share and the measure interpretation was sure that 5/1 is more than 5/5, but was hesitant about drawing a picture or describing precisely how much each portion was.

From the interviews, it appears that students readily draw on both the interpretations, especially on the sharing interpretation for comparison. The share meaning in some cases did not lead to a clear picture of how much the fraction exactly was. In the case of the fraction 14/9, it was difficult to draw a picture showing the sharing completely and students either hesitated to do so or withdrew after trying. We interpret this as suggesting that both the interpretations are useful and mutually reinforce each other. The results taken as a whole indicate that instruction emphasizing the measure and share meaning can positively contribute to students' understanding of fractions and can supplement part-whole understanding, which by itself is inadequate.

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PERCEPTIONS OF GOALS IN THE MENTAL MATHEMATICAL PRACTICES OF A BUS CONDUCTOR IN CHENNAI, INDIA

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Sociocultural dimensions of mathematical knowledge have greatly influenced research in the field of mathematics education in the past few decades, resulting in the rise of different areas of research that include ethnomathematics, everyday mathematics, situated cognition, and workplace mathematics. Although over the past 15 years, mathematics education research has begun to explore the nature of the mathematics used in different workplaces, very few studies have investigated the nature of workplace mathematics in India. The general aim of this study is to develop a better understanding of the mathematics used in the workplace of bus conductors in Chennai, India. In particular, in this paper we highlight one conductor's perception of his goals, and the researchers' concomitant perception of his mental mathematics.

INTRODUCTION

Mathematics knowledge has been traditionally considered as absolute and infallible (Ernest, 1991). This absolutist view of mathematics has been challenged since the beginning of the twentieth century and a new wave of 'fallibilist' philosophies of mathematics has evolved. Researchers have recognized that the study of mathematical knowledge has a value dimension and hence it is a social and a cultural phenomenon as well (Bishop, 1988). This evolution has greatly influenced mathematics education research in the past few decades and major shifts have occurred in relation to the teaching and learning of mathematics (Presmeg, 2007).

Mathematical thinking and learning take place in any culture and it is important to study the mathematics of different groups of people from all over the world (Ascher, 2002). Such investigations broaden and modify the history of mathematics to one that has a multicultural, global perspective, and emphasize the interplay between mathematics and culture (Ascher). In the last two decades several researchers have analyzed and documented the mathematics practices of adults as well as children that take place outside the school settings (e.g., Carraher, Carraher, & Schliemann, 1987; Gerdes, 1996; Saxe, 1991). This association has given rise to the recognition of different forms of mathematics such as academic mathematics, everyday mathematics, folk mathematics, and ethnomathematics.

One area of research that emanates from the research field of ethnomathematics is the research area of everyday mathematics (Vithal & Skovsmose, 1997). One subset of the research field of everyday cognition is concerned with research conducted in workplaces. This line of research gives some insight into how people conceptualize the role of mathematics in their work. Although the past two decades have seen a surge in research dealing with the workplace mathematics of adults, many of these studies were conducted in the western hemisphere and developed nations of the world (Presmeg, 2007). Very few studies have investigated the nature of workplace mathematics in developing nations. In India, for example, there are several professions that have not seen the influence of technology. Thus, people holding occupations in these workplaces serve as rich sources for researchers to investigate and shed some insight into the mathematical practices of adults. Despite this fact, there have been very few contributions to the research area of everyday mathematics from India (e.g., Mukhopadhyay, Resnick, & Schauble, 1990). The research reported in this paper is part of a larger project that addresses this lacuna by investigating the workplace mathematics of bus conductors in Chennai, India. In particular, in this paper we focus on the goals of one bus conductor in his practice, as perceived by the researchers and as perceived by the actor himself.

THEORETICAL FIELD

Ethnomathematics, everyday mathematics, and workplace mathematics

Ubiratan D'Ambrosio launched his *ethnomathematical program* in the 1970s "as a methodology to track and analyze the processes of generation, transmission, diffusion and institutionalization of mathematical knowledge in diverse cultural systems" (D'Ambrosio, 1990, p. 78). Growing concern among educators in non-Western nations about the imported Westernized mathematics curriculum resulted in the rise of ethnomathematics as a research field. Vithal and Skovsmose (1997) added a new dimension to the term ethnomathematics by stating, "ethnomathematics can refer to a certain practice as well as the study of this practice" (p. 133). The research field of everyday mathematics could be viewed as a strand in the ethnomathematics literature. Research on workplace mathematics can be viewed as a subset of the research field of everyday mathematics. This line of research has not only explained the mathematics embedded in the everyday cultural practices of people but has also explored and described in detail these cultural practices. However, it must be noted here that people who are engaged in these cultural practices did not consider "themselves to be engaged in doing significant mathematics" (Presmeg, 1998, p. 328). It was the researchers who interpreted their cultural practices and brought out the mathematics embedded in their practices. More recent research in workplace mathematics has attempted to uncover the mathematical practices of specific groups such as nurses (Noss, Hoyles, & Pozzi, 2000), automobile workers (Magajna & Monaghan, 2003), and carpet layers (Masingila, 1994).

Starting with a broad theoretical field of everyday cognition, we narrowed our focus to concentrate on workplace mathematics. In particular, the research purpose associated with this study is to observe, understand, describe, and analyze the mental mathematical practices of the bus conductors in their workplace and examine what this knowledge can add to the study of everyday mathematics. As researchers we are interested in understanding the mathematical practices of bus conductors. However, the bus conductors themselves may not perceive that they are engaged in mathematical practices. While the conductors' goals may be work related, our goal is to describe their everyday practice with a focus on bringing out the mental mathematical activities involved in their practice. Thus the following research question specifically addresses the goals of the researchers and the conductors.

> What is the structure of bus conductors' everyday mathematical practices and how is it related to their goal-directed activities?

In this paper we address the question with regard to perceptions of the goals of one of the five conductors who participated, chosen because of salient elements in his case study.

THEORETICAL FRAMEWORK FOR A CASE STUDY

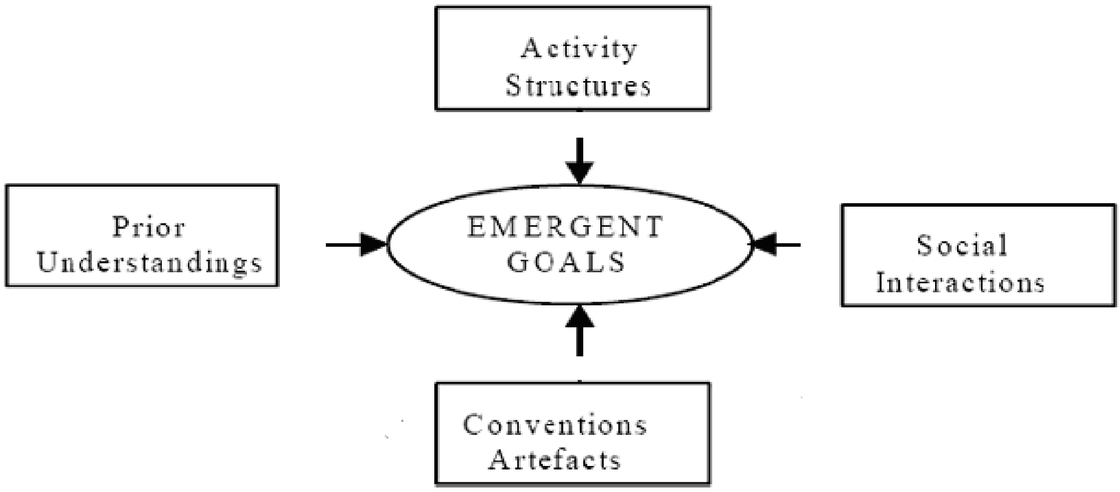

Figure 1. Four-parameter model (Saxe, 1991, p. 17).

We used Saxe's (1991) four-parameter model (figure 1) to explore the research question. Saxe formulated this model in his quest for developing a framework that would account for the developmental cognitive processes of participants in a cultural practice, and that would not ignore the sociocultural context in which the practice took place. Saxe used his model to explain the complex relationships between Oksapmin traders' goals and actions and the trading practice in which their actions were performed.

The assumptions underlying the present study are that the conductors' workplace mathematical practices are influenced by their working conditions, and that their practice-linked goals emerge and change as individual conductors participate in this practice. Hence, we needed a theoretical framework that acknowledged the influence of context on their mathematical practices. Second, we needed a framework that could inform our investigation of the cognitive processes in the social context and one that would take into consideration the complex relationships between the conductors' mathematical tasks and the practices in which they were performed. Saxe's (1991) four-parameter model brings out the context related parameters that influence a bus conductor's work-related goals and the concomitant mathematical goals related to his work activities.

METHODOLOGY

Research methods and data collection

The overall purpose of this research study is to understand a general phenomenon- the nature of the bus conductors' mathematical practices. By investigating individual cases (bus conductors), we seek to show how the phenomenon itself (their practice) can be described, and hence we used an instrumental case study approach (Stake, 2000).

The bus conductors are employees of the government organization, Metropolitan Transport Corporation (MTC). Due to the hierarchical structures of the MTC, the primary researcher first contacted the higher authorities of the MTC to gain permission to carry out her research with an employee (bus conductor) of the MTC. After gaining entry into the organization, a convenience sampling was done to choose a bus depot for investigation. A purposive sampling was employed, and five participants were carefully and appropriately chosen, based on the following criteria:

- conductors' years of service (less than 10 years, 10-15 years, more than 20 years),
- conductors' educational qualifications (high school graduation, college graduation),
- conductors with good service records only,
- conductors' willingness to participate in the study.

In this paper, we discuss one conductor's perception of goals associated with his practice.

Mr. Raju, who has been a bus conductor with the MTC for the past 27 years, was highly recommended by his superiors for this study. During his work shift, he commuted several times on a bus from point A to point B. He picked up and dropped off commuters en route and regulated their entry into and exit from the bus. His duties as a bus conductor included issuing a ticket to a commuter based on the entry and exit point, tendering the exact change back to the commuter when the commuter gave more money than the required amount, keeping a record of the number of tickets sold, calculating the daily allowance based on the day's collection, and submitting the trip earnings to the supervisor at the end of his shift.

The primary researcher (the first author) accompanied and observed Mr. Raju during his work shift four times a week, observing a total of 15 trips. Based on the first few observations, she singled out factors that were helpful in pursuing the research question. She reviewed field notes and personal reflections at the end of each day and identified features of the workplace, participant's actions, and events that needed further scrutiny. Further, she conducted audiorecorded interviews with participants, and she collected and researched several official documents related to the research purposes associated with the study. She created a case study database to organize information obtained from all of the above sources.

ANALYSIS AND RESULTS

Mr. Raju did not have fond memories about his school mathematics learning experience. He said that he memorized formulas and solved problems routinely without understanding underlying mathematical ideas. However, his bitter experiences in school did not seem to have had any negative impact on his views about mathematics. In one of our conversations, he expressed his views about mathematics as follows.

> Addition, subtraction, multiplication, and division alone are not mathematics. In everyday life I do a lot of things that need mathematics. I have to think and make decision every day based on circumstances, which involves time and money. This decision-making process itself involves great deal of logical thinking. All of these involve some aspect of mathematics.

Although Mr. Raju was very sincere and dedicated to his job he did not hold his job in high esteem. He regretted not having a college degree and a desk job at an office. When questioned about this perception of his job he responded as follows.

> Conductor's job is stressful, risky, and highly demanding. In an office job, if you make mistakes, you get the time and chance to correct it later. But in our job, we cannot do today's work tomorrow-or this minute' work the next minute. We work with the public and deal with their money. We are also responsible for their belongings and their safety. If there are shortages at the end of the day, we are held responsible.

Emergent goals

Every day Mr. Raju reported to work with several work–related goals in mind. As he described them, these goals were to have an incident free day, to avoid confrontations with the commuters, and to earn maximum collection. In addition, he set new goals that specifically related to his duties in the bus. These included issuing a ticket to a passenger and updating ticket information into the traffic return document. His goals at the end of his work shift included calculating the daily driver and conductor allowance, filling out official documents to calculate the total earnings for the day, and submitting the day's earnings to his superior. Depending upon what the situation demanded, Mr. Raju engaged in the process of reformulating his goals and the ways in which he could accomplish these goals:

> If there is a technical problem and the bus cannot be operated further then I have to think about changing some of my goals. I know in this situation than I cannot achieve the target of 2000 rupees as I have lost passengers on this trip. At present my goal is to make sure the passengers are not stranded and arrange for alternate travel arrangements. I should also request for another bus so we can continue to work. If we cannot get a substitute bus we will have to sign off for the day.

Based on her observations, the primary researcher probed Mr. Raju about his views related to his work-related goals. In order to understand his views, we present below a snippet from one of Mr. Raju's several work related activities (fieldnote):

> A passenger boarded the bus at point P along with four members of his family. Mr. Raju approached the passenger. The passenger handed out a 50-rupee note and requested the

conductor for five tickets to point Q. Mr. Raju requested the passenger to give him an extra 50 paise coin and the passenger obliged. He then gave the passenger five tickets and handed 33 rupees back to the passenger.

While Mr. Raju described the goal associated with this event as " I issued tickets to a passenger", we described the mathematical goal associated with this event as "Mr. Raju performed a mental mathematics transaction". We now use Saxe's model to explore both Mr. Raju's and our perceptions of these goals using Saxe's four context-related parameters: *activity structures, conventions and artifacts, social interactions, and prior understandin*gs (see figure 1).

Analysis of Mr. Raju's perceptions

Mr. Raju completed a sequence of *activities* in order to issue tickets to this passenger. These included obtaining information from the passenger regarding destination, receiving money, and issuing tickets.

Mr. Raju constantly engaged in *social interactions* with the passenger in completing these activities. Although as a conductor, Mr. Raju was the main actor in this scenario, the passenger played a major role as well. Mr. Raju initiated action in this situation by approaching the passenger. The passenger responded by requesting the conductor for tickets. Mr. Raju did not have a 50 paise coin at hand. Hence, he requested the passenger for extra money and the passenger obliged. By negotiating with the passenger, Mr. Raju was able to overcome a tricky situation and complete the activity.

Mr. Raju followed certain *routines* to accomplish activities associated with the goal. He approached the passenger as soon as he boarded the bus. He did not hesitate to ask the passenger for extra money to avoid giving incorrect money back to the passenger. Further, tickets printed with denominations and currencies served as *artifacts* that helped Mr. Raju to complete his goal.

Mr. Raju's *prior understandings* about his practice included knowledge about the bus routes, the number of bus stops, and ticket fares based on entry and exit points. His thorough knowledge on these topics helped him determine the ticket fares and issue tickets without delays.

Analysis of researchers' perceptions

We propose that Mr. Raju performed a transaction by engaging in the following *mathematical activity* mentally: find the value of $50 – (3.50 * 5)$. This was done in two steps. First Mr. Raju calculated the value of $3.50 * 5$. He then subtracted this number from 50 to get the solution.

Mr. Raju's *interactions* with the passenger were very crucial to this whole process. It was the passenger who posed the problem to the conductor in the first place. In this situation, the passenger helped Mr. Raju in reformulating the problem by giving him extra money.

Currency and printed tickets with denominations served as *artifacts* that helped Mr. Raju to complete this mathematical activity with ease. He used the time to locate and tear out five tickets worth 3.50 to mentally calculate the fare for five passengers. The passenger handed out a 50-*rupee* note, which Mr. Raju knew was equivalent to five ten-*rupee* bills. As explained above Mr. Raju reformulated the original problem and then thought of 50.50 in terms of currency – Five ten-*rupee* bills and one 50 *paise* coin. He then had no trouble locating and giving back 33 *rupees* back to the passenger.

Mr. Raju's *prior understandings* about whole number and decimal operations, and his prior experience working with money greatly helped him solve this problem. According to his reporting, this is how Mr. Raju calculated the value of 3.5 * 5: He first doubled 3.50 to get to 7.00. He further doubled 7 to get to 14. He then added 3.5 to 14 to get 17.50 as the solution. The next step was to calculate 50 – 17.50 At this point, he used his prior experience with money-related transactions to get the answer. He knew that 17.50 was 2.50 short of 20 *rupees.* Adding another 30 *rupees* gave him 50 *rupees*. Thus he added 2.50 to 30 to get 32.50 as the final answer. In this situation, he added another 50 to this and arrived at an answer of 33. The fact that he took another 50 *paise* from the passenger forced us to modify our original problem to the following problem. 50.50 – (3.50 * 5) to which the answer is 33.00. Mr. Raju reformulated the given problem into another problem, used whole number facts, doubling, and compensating strategies effectively to solve the problem and complete the transaction.

DISCUSSION

Mr. Raju's workplace setting demanded that he perform mental mathematical calculations quickly and efficiently. In the absence of technological devices, he had to work carefully to avoid errors in calculations. As researchers we observed and acknowledged that at work Mr. Raju established and achieved goals that involved significant mental mathematical activity. However, according to Mr. Raju, these goals were just the demands of his work place and part of his job. On hearing the primary researcher's interpretation of his work-related goals, this is how he responded:

> You may be right. However, when I work, I am completely focused. I don't think about what mathematics I am using. I see money, I hear words, my hands deal with tickets and money and I calculate in my heart. There is little time to think about anything else. I don't gloat about the mathematics that I do everyday. I cannot afford to make mistakes in my calculations because I may end up losing department's money. So I only concentrate on doing my job quickly and correctly.

These differing perceptions of Mr. Raju's work-related goals are complementary, and completely consonant with Saxe's four-parameter model. The bus conductor's mental mathematics is invisible to him in the press of his practice; in the researchers' perception it is an example of ethnomathematics.

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ETHNIC AND GENDER GAPS IN MATHEMATICAL SELF-CONCEPT: THE CASE OF BEDOUIN AND JEWISH STUDENTS[1]

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This study focuses on the mathematical self-concept of mathematically promising students, participating in "Kidumatica" - a prestigious extra curricular, after-school math club, which integrates Jewish and Bedouin students. Bedouin girls' mathematical self-concept was found to be significantly lower than the Jewish girls'. In all other comparisons, no significant differences were found. Possible explanations for this phenomenon are discussed.

BACKGROUND

Rational

One of the most troubling and longstanding problems in gifted education is the low and disproportionate participation of minority students. This world wide phenomenon is also true for mathematical programs. While discriminating qualification processes can lead to minorities' under-representation, high drop-out rates are evident even after their acceptance to gifted programs, due to cultural, social or affective factors (Donovan & Cross, 2002; Maker, 1996; Stormont, Stebbins & Holliday, 2001). Therefore it is crucial to find and define those aspects that might be leading to the low attendance rates of minority students in programs for the talented and promising.

The Bedouins in Israel

The Israeli society is heterogeneous, comprising a variety of Jewish ethnic groups and Arabs that constitutes approximately 19% of the population, The Bedouin minority, which is a part of the Arab population constitutes 2% of the state population. Most of the Bedouins live in the Negev region, in southern Israel. Nowadays, there are more than 130,000 Bedouins in the Negev (Statistical Yearbook of the Negev Bedouin, 2004), about half of whom live in seven exclusively Bedouin towns established by the Israeli government. These Bedouin towns were set up so that public services could be provided efficiently. Despite governmental attempts to modernize the Bedouin's lives, the Negev Bedouins remain the weakest socio-economic status group in Israel. They have twice as many children compared to the Israeli average (the average Bedouin family size is 9-10 persons), half the income and living space, and the highest unemployment rate in the country (Abu-Saad, 1997; Amit, Fried, & Abu-Naja, 2007; Statistical Yearbook of the Negev Bedouin, 2004).

The radical social changes in the Bedouin community, from traditional nomadic living style to modern-western living patterns, have vastly improved the educational

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system, but a tension between traditional values and Western values is apparent. The lowest success rates for the high school matriculations, which are a prerequisite for higher education, are found in the Bedouin localities in the Negev. Consequently, few Bedouin students pursue higher education in general, and mathematical subjects in particular (Abu-Saad, 1997; Amit et al., 2007; Statistical Yearbook of the Negev Bedouin, 2004).

Kidumatica Math club

Kidumatica Math club was founded in 1988 in Ben-Gurion University of the Negev. The name “Kidumatica” is a play on the words *kidum*, “progress”, and the Hebrew word for mathematics. Every year, around 400 students ranging from ages 10-16, from 60 schools, participate in the clubs' activities. The weekly activities increase their creative thinking and mathematical skills, through subjects such as game theory, logic, combinatorics, and algebra. The students also benefit from competitions, experiments, museum tours, and more. Students are chosen for their high mathematical abilities and their interest to develop these talents. The activities are run by experienced educators, who have been specially trained to instruct gifted students. Kidumatica’s students have won countless national and international contests, participate in university courses at a young age, and, most importantly, acquire the tools that will equip them to be the potential future technological and scientific leaders of Israel.

The club members represent various ethnic and religious groups of the region: Jews and Moslem Arabs participate in the clubs' activities. The Jewish students themselves are not a homogenous group: Among them are immigrants and Israeli-born students, both religious and non-religious. In fact, it was found that the club members’ families are from over 15 different countries, and speak 11 different languages.

Since its establishment, the Kidumatica math club has become a prestigious program that draws a multitude of applicants.

Bedouin students in Kidumatica

Bedouin students have participated in Kidumatica since 2002, starting with 35 students, with a gradual increase - this year (2007-8) about 50 students have joined. The club was the first program to deal specifically with mathematic excellence in the Bedouin sector, and is the only mathematical program that actively integrates Bedouin and Jewish pupils. Statistics collected annually show that Bedouin attendance and participation exceeds that of any other group. Their attendance percentage is over 90%, a contrast to the low attendance in school by the Bedouin pupils in general (Amit et al., 2007).

The teaching in Bedouin schools is conducted in Arabic, while Hebrew is taught as a second language. Higher studies are conducted in Hebrew. While the program's activities are conducted in Hebrew, examinations are given in both Hebrew and Arabic to reduce exam pressure. In order to overcome the language barrier during classes, two Bedouin university students work as tutors and are always present in group activities for translating, explaining terminology, and providing moral support for the Bedouin club members (Amit et al., 2007).

All students are equal members of an elite group. Recognition of common goals of academic excellence removes any ethnic tension, and this is reflected in the social connections created between the students. Jews and Bedouins alike participate in all the social activities, which include extended day-long activities, competitions, field trips and museums visits.

Academic Self-Concept

Self concept is defined as the image and collection of ideas one has about oneself and is a dynamical multidimensional construct. It refers to the attitudes, feelings and knowledge about one self's abilities, skills, appearance, and social acceptability (Hoge & Renzulli, 1993; Marsh, 1992; Plucker & Stocking, 2001).

An individuals' academic self-concept is based on two simultaneous sets of comparisons: External-social comparisons, in which students compare their self-perceived performance in a particular school subject with the perceived performance of other students and internal comparisons in which students compare their self-perceived performance in a particular school subject with their performance in other school subjects (Marsh, 1992; Plucker & Stocking, 2001). Academic self-concept is divided into three categories: mathematics, verbal and general school self-concept (Marsh, 1992).

Numerous studies have shown positive correlations between attainment of a positive academic self-concept and academic achievement, academic effort and persistence, coursework selection and future educational aspirations (e.g. Marsh et al., 2005; O'Brien, Martinez-Pons & Kopala, 1999; Plucker & Stocking, 2001).

Studies concerning minority students conducted in the Unites States have concluded that minority students have lower self-concept than the majority group (O'Brien et al., 1999; Stormont et al., 2001). Regarding the Arab minority in Israel, there have been contradicting results. While Hofman, Beit-Hallahmi and Hertz-Lazarowitz (1982) found that Arab-Israeli students have lower self-esteem[2] than Jewish-Israeli students, Sherer and Enbal (2006) found that the self-esteem of Arab-Israeli students is higher than the self-esteem of Jewish-Israeli students. Abu-Saad (1999) found that among the Arab-Israeli students, Bedouin students have lower self-esteem than their other Arab peers.

The current research differs from previous studies in several key aspects: First, it focuses specifically on mathematical self-concept (and not the more general dimensions of the construct); second, it refers to the Bedouins specifically (and not to Arab-Israeli students in general), and finally, it focuses specifically on mathematically talented

[2] We use the terms "self-esteem" and "self-concept" as in the different cited studies, although the distinction between them is somewhat blurred: "construct validity research ….has been unsuccessful in providing empirical evidence of such discriminability (Byrne, 1996, p. 6) .

Bedouins, within a unique educational program in which various ethnic groups study together (Unlike the school system, which is largely separated for Arabs and for Jews).

In the course of tracking the integration process of the Bedouin mathematically promising students that were accepted to the Kidumatica math club, mathematical academic self-concept was assessed.

The research questions were:

1. Are there differences in mathematical self-concept between Bedouin and Jewish students beginning their studies in Kidumatica?
2. Are there gender differences in mathematical self-concept between Bedouin and Jewish students beginning their studies in Kidumatica?

METHODOLOGY

Population

The Research population comprised 80 eight and nine grade mathematically capable students beginning their studies in “Kidumatica". All first year students in the aforementioned grade levels were included in the study: 34 Bedouin students (18 boys, 16 girls) and 46 Jewish students (16 boys, 30 girls).

Instrument

Academic self-concept was assessed by the items referring to the academic dimension of the Self-Description-Questionnaire II (Marsh, 1992) which is a questionnaire designed to measure the self-concept of young adolescents. The questionnaire consisted of 30 declarative statements, to each of which the respondents answered on a 6 point Likert scale (1=strongly disagree, 6= strongly agree). Note: this report is restricted to the 10 statements relating to the mathematical self-concept. Sample items: "I have always done well in mathematics"; "I have trouble understanding anything with mathematics in it".

Interviews

Semi-structured interviews (in Arabic) were conducted with six of the Bedouin students (three boys and three girls), in order to gain further understanding of students’ attitudes toward mathematics and their participation in the club. The interviews were carried out by one of the researchers in Arabic, and were recorded, transcribed and translated to Hebrew.

RESULTS

Self-Description-Questionnaire

Below (table 1) are Means and standard deviations of self-concept scores. A two-tailed unpaired t-tests was used for Comparisons.

The only significant difference in mathematical self-concept was found between the Bedouin girls' and the Jewish girls' score ($t(44)=2.58, p=0.013$).

No significant differences in mathematical self-concept were found between Bedouin students' score and the Jewish students or between Bedouin boys and Jewish boys.

No significant differences were found in gender comparisons, between Bedouin boys and girls, or between Jewish boys and girls.

	Bedouin Students	Jewish students
All the students	48.35 (8.78) N=34	50.78 (5.67) N=46
Boys	50.56 (9.52) N=18	50.25 (5.14) N=16
Girls	45.88 (7.37) N=16	51.07 (6.00) N=30

Note: standard deviations appear in parenthesis

Table 1. Mean scores of mathematical of self-concept

Interviews

In the interviews, the Bedouin students were asked general questions concerning their academic and social integration in the club, the club's influence on their math studies at school, and the reactions of families and friends to their acceptance. Although direct questions about mathematical self-concept were not asked, from the analysis of the interviews an immense degree of confidence in their mathematical abilities and pride of their acceptance was apparent. For example, when talking about mathematics a 9th grade student said*:"...my friends tell me that what I'm doing is difficult, but I keep telling them that for me difficult is not an option, if I want to (succeed), nothing is difficult……*". They all held the view that participating in the club is a springboard to future academic studies and found an extra value in gaining confidence in Hebrew. Statements such as: "*I like the club, this is a chance to get to know the university, improve my mathematics and my Hebrew. This way I will be better prepared for university…*" repeated themselves in all the interviews.

DISCUSSION

This study focuses on the mathematical self-concept of Bedouin and Jewish new members of the Kidumatica math club. Previous studies regarding affective aspects of mathematical studies of middle-school Arab students in Israel compared to Jewish students, have usually considered the Arab population as a whole. There are substantial differences between the various sectors of the Arab population in Israel. The Bedouin population is considered to be more conservative, less educated and more economically disadvantaged than other Arab sectors (Abu-Saad, 1999).

In comparing the mathematical self-concept of Bedouin students' with the Jewish students (as a whole), no statistically significant differences were found. These results are inconsistent with the claim that minority students have lower self-concept than the majority group (O'Brien et al., 1999; Stormont et al., 2001). The contradiction has several possible explanations. One aspect is the diversity of the

Kidumatica's' members. While the Bedouin students may differ from their peers in the math club by speaking Arabic, many of the students in the club speak other languages at home (Russian, Spanish and more), and therefore the Bedouins are not singled out as a minority. This is reinforced by the metalanguage of mathematics, considered a "cultureless discipline," thus fostering integration and reducing feelings of inferiority.

Another possible factor could be the high calibre of participants in this study. These students were hand-picked by their teachers and passed entrance exams, which "confirmed" their abilities, and led to strengthening the academic self-concept (Hoge & Renzulli, 1993).

However, focusing on gender, we found a considerable gap between the mathematical self-concept of Bedouin girls and Jewish girls,

Studies focusing on the affective aspects of mathematics, conducted in the Israeli context, have reached contradicting results. While Mittelberg and Lev-Ari (1999) found that Arab girls have a slightly lower mathematical self-confidence than Arab boys, Nasser and Birenbaum (2005) found higher mathematical self-efficacy in Arab girls than for Arab boys. However, these studies did not treat the different sectors of the Arab population, but related to them as a whole.

In this study, the Bedouin girls' mathematical self-concept was found to be significantly lower than that of the Jewish girls participating in the club, in contrast to the lack of significant differences between the Bedouin boys' academic self-concept and the Jewish boys'. As girls are more sensitive to the affective aspects of mathematics (Amit, 1988; Freeman, 2004), it is possible that the external comparisons, with their other peers, had resulted in their relatively low mathematical self-concept.

Social influence might be another factor that may contribute to the Bedouin girls' lower mathematical self-concept. High achieving girls in mathematics have been found to be more sensitive and influenced by teachers and parents views about their studies (Amit, 1988; Freeman, 2004; Reis & Park, 2001). Khattab (2003) found that Arab girls in Israel are "aware that an academic career is not a realistic option for them and thus are likely to aspire to a level of education that does not conflict with their traditional future roles" (p. 291). For Bedouin girls, this prospect is more extreme.

In the Bedouin society, traditional gender roles are prominent and women's status is considerably lower, (the Bedouin girls attending the club are dressed in the traditional clothing, including full head covering, hijab, in contrast to Bedouin the boys, which are dressed in modern, western style clothing).

Despite their participation in a prestigious mathematical program that sets high academic aspirations and expectations, the dictated traditional values of the Bedouin society may cause dissonance resulting in the girls’ lower mathematical self-concept, relatively to the Jewish girls in the club.

Study limitations

The study population of this research is relatively small, due to the number of Bedouin students attending the club. As mentioned above, this study was conducted at the beginning of the academic year. The next phase of the study will be conducted at the end of the year, and changes over time will be looked for, as well as expansion to students who are in their second and third year in the club.

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UNDERSTANDING WORD PROBLEMS

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When students' read a word problem they construct a mental representations of the problem text. This mental representation is the basis for their work towards solving the problem. In a study 19 students have given verbal protocols while working through eight multistep word problems. This paper presents and discusses data on their mental representations in connection to measures for reading comprehension and domain knowledge

Over many years much effort has been put into researching student work on word problems (see for instance Reed, 1999 or Verschaffel, Greer & De Corte, 2000). Much of the earlier work concerned itself with researching students' work on different prototypes of word problems. The last years have produced an interest in investigating students' comprehension of problem text (see for instance Cook, 2006; Thevenot, Devidal, Barrouillet & Fayol, 2007). Still, a focus on students' strategies for reading word problems or students' mental representations of problem text are not a novelty. Many of the recent research studies refer to Cummins, Kintsch, Reusser & Wiemer (1988) and Nathan, Kintsch & Young (1992).

What is presented in this paper is a small part of a larger study aiming at researching how reading strategies are intertwined with (other) mathematical strategies. The research question for the overall study is: *What can be categorised as "typical" strategy use for grade (7) 8 students at different levels of competence?*

In this paper I write about students' mental representation in connection to other measures, especially for reading. My aim is to present a part of the study where differences considering students' mental representations are investigated.

WORD PROBLEMS

Word problems are traditionally associated with a school setting wherein the student is asked to solve a problem in connection to a mathematics lesson, a test situation or as homework (Verschaffel et al, 2000). Semadeni (1995) defines word problems as verbal descriptions of problem situations.

> Word problems can be defined as verbal descriptions of problem situations wherein one or more questions are raised the answer to which can be obtained by the application of mathematical operations to numerical data available in the problem statement. In their most typical form, word problems take the form of brief texts describing the essentials of some situation wherein some quantities are explicitly given and some are not, and wherein the solver – […] is required to give a numerical answer to a specific question by making explicit and exclusive use of the quantities given in the text and mathematical relationships between those quantities inferred from the text (Verschaffel et al., 2000 p. ix).

In a word problem the procedure of solving the exercise is not given in the text, and the student has to decide on an appropriate solving procedure or strategy him/ her self. This means that the student must be able to both read and understand the text. While reading the text, the student constructs a mental representation of the text (Bråten, 1994). This mental representation is the basis for solving the word problem (Cummins et al., 1988).

PRIOR RESEARCH ON WORD PROBLEMS

Whether student' errors in solving problems are caused by difficulties in comprehending word problem text or not, are and have been debated. Knifong and Holtan (1977) in their studies found no clear indications for reading difficulties to be the cause for erred word problems. Other studies however establish a link between reading comprehension and success in solving word problems (see for instance Cummins et al., 1988; Reed, 1999). Roe and Taube (2006) found that overall scores in reading and mathematics in PISA 2003 had a correlation coefficient of .57 for Norwegian and Swedish students. The correlation is not dependent on text length, but varies more strongly with mathematical content.

Mental representations

Students' mental representations of problem texts might be a key to understanding how reading is connected to solving word problems. Cummins et al. (1988) suggest that a student's mental representation of a word problem has the form of a schema. Comprehension demands that solvers "map linguistic input onto knowledge about the problem domain" (ibid. p. 406). Nathan et al. (1992) suggest that this mental model in reality consists of two types of representations: a more qualitative situation model containing the social context of the word problem and a quantitative model containing the algebraic structure or schema embedded in the problem. More recent computer laboratory studies allow for manipulation of problem format, like presenting the question before or after revealing the problem text. Thevenot et al. (2007) hypothesized that if the mental model is a schema, placing the question in front of the word problem would help high achieving students recall schemas while the placement would be of lesser importance to low achieving students with fewer schemas. However raises in success rate demonstrate the opposite. Placement of question proves to be more important to low achieving students. Thevenot et al. (ibid. p. 45) suggest that the mental representation rather than a formal schema, is a more qualitative situation model, that is a

> temporary structure stored in working memory that contains in addition to the mathematical information necessary to solve the problem, nonmathematical information that is related to the context in which the situation described by the problem takes place. A situation model is there for more qualitative and less formal than a schema

Nathan et als. (1992) model for mental representations is widely used, also in recent research (see for instance Cook, 2006 or Reed, 1999). I will however in this paper refer to the situation model when I write about the mental representations (or SMs) of the 19 students participating in the study.

Understanding word problems

Reading and solving word problems is not necessarily a linear process. When working on a word problem, students might return to the text and refine or change their mental model (Hegarty, Mayer & Monk, 1995). When children make errors, it is often due to misconceptions of the problem situation grounded in insufficient understanding of the semantic schemes of the word problems (Verschaffel et al., 2000). Cummins et al. (1988) found that when students arrive at a wrong solution on a word problem, they often make a correct solution to the problem as they comprehend it. The use of key words like *altogether*, *sum* or *less* might lead students to think they shall add or subtract key numbers in the text (Cummins et al. 1988, Reed 1999). Scanning for keywords without considering the relationship between numbers can be termed a surface strategy (Cook, 2006). Hegarty, Meyer and Monk found that less successful students use a direct-translation strategy where they search for key numbers and directly translate keywords into mathematical actions. Alexander, Buehl, Sperl, Fives & Chiu (2004) would name such a strategy a text-base strategy as opposed to a deep-level strategy where students elaborate on relationships between text elements. A parallel to Alexander's deep-level strategy would be Hegarty et als. (1995) problem model or Cooks (2006) discrimination strategies.

Cook (2006) found that college and elementary school students use the same strategies to discriminate between relevant and irrelevant information in word problems. These same strategies also apply when students do not succeed in solving a word problem. Multiple strategy use is more frequent when students do not manage to solve problems. Littlefield and Rieser (1993) found that when successful students base their discrimination on a feature analysis of the text, less successful students are more likely to use surface-level aspects like positioning of numbers or number grabbing. Later Cook and Rieser (2005) found that mathematically disabled students attempt to apply the same strategies as successful students, only are not able to implement the strategies effectively. Cook (2006) suggests that when students do not succeed in constructing a fitting mental model, this could be due to lack of mathematical knowledge relevant in the given situation. This view is supported by Alexander (1997) who claims that domain knowledge and strategies together with domain interest are the pillars of students' competence

RESEARCH DESIGN

In my study to understand is connected to comprehension of word problems: To understand means to create a mental representation of the problem situation to an extent that enables you to solve the problem (see for instance Thevenot et al 2007, Cook 2006 or Nathan et al 1992). What is reported in this paper is a small part of a bigger study researching students' comprehension and work on word problems. See Nortvedt (2008) for methodology. Test scores on national tests have been collected from grade 8 students from two schools, giving measures of domain knowledge within numeracy and reading comprehension. A sample of the students also has given individual verbal protocols while solving a collection of word problems. Scaffolding was offered when students were not able to proceed on their own. Scores on the national test in numeracy were used to divide students into four groups. Setting cut scores were based on scores form a national sample, after dividing the sample into four groups of equal size. Reading scores were transformed likewise. The main focus of the analysis of the protocols is to investigate students' strategy use. For this paper analysis is focused on mental representations in connection to solving the word problems assigned for the verbal protocols.

Informants

Students come from two combined primary and lower secondary schools situated in Oslo. Both schools have a combination of majority and minority language students representing a variety of socio-economic backgrounds. Students were chosen to represent a variety of achievement levels within their group, partly based on teacher judgement, partly based on classroom observations and partly based on test scores on national tests. The sample from school B has higher test scores that the sample from school A.

Data collection - verbal protocols

Verbal protocols have been collected from 19 students in grade 7 and 8. Students were asked to think out aloud while working through eight word problems and one practice problem as if they were assigned for homework or lesson work. All problems were multistep problems that could be solved using arithmetic only. Some were consistent while others contained wording that might seem inconsistent to the students. Text complexity and mathematical difficulty varied over the problems. All problems were collected from a school context: textbooks, exams and national tests.

RESULTS

A score for the word problems students worked on while giving their verbal protocols has been computed for every student (Score WP, see table 1). Partial scores were assigned for partially correct solutions. For each student the total number of appropriate mental representations has been counted. When protocol analysis reveal that students formed an understanding of the social context but struggle to form a correct situation model, protocols are labelled Sm (social context understood, partial understanding of mathematical relationships) or S (social context only). In about 10 % of the protocols evidence is unclear and no judgement about mental representations is made.

Students with high scores on national tests in numeracy also solve more word problems correct and more often demonstrate an appropriate mental representation. Students with low test scores in general are more familiar with the social context and struggle more to form a mental representation. This is not surprising. The low achieving students do not have the same domain knowledge as high achieving students, and it is probable that deep-level discrimination are not to the same extent a possibility to them (Alexander et al., 2004). For consistent problems where keywords can be directly translated, this will not be the same problematic.

WP2 – Socks and T-shirts
Aud bought three pairs of socks and four T-shirts. The T-shirts were 79 kroner each. All together she paid 364 kroner. How much did a pair of socks cost?

Example 1. WP2 – Socks and T-shirts (my translation).

WP 2 (see example 1) is from a grade 8 textbook. It contains no irrelevant information and no inconsistent key words. WP2 is among the easiest to comprehend. 11 out of 19 students form appropriate SM. However this proves to be difficult for two of the students of interest.

Results national tests, verbal protocols and mental representations							
Numeracy	School	Student	Gender	Score WP	SM	Sm or S	Reading
1	A	AA13	Boy	1,5	1	3	1
1	B	BA07	Girl	3,5	1	3	2
1	A	AA14	Boy	3,5	2	3	1
2	A	AA04	Girl	1	1	4	1
2	A	AA12	Girl	2,5	1	2	1
2	B	BB11	Boy	2,5	2	5	1
2	A	AA08	Girl	2,5	4	2	4
2*	B	BA04	Girl	3	1	4	4
2	B	BC02	Boy	4	3	4	3
2	A	AB05	Boy	4,5	3	3	3
2	A	AA03	Girl	4,5	4	2	1
3	B	BB02	Boy	4,5	4	4	3
3	B	BC08	Girl	5,5	6	2	4
4 *	B	BA08	Boy	6,5	5	2	2
4	A	AB02	Boy	6,5	7		4
4	B	BA09	Girl	6,5	8		4
4	B	BC06	Boy	7,5	7	1	2
4	A	AA09	Girl	7,5	8		4
4	B	BC10	Girl	8	8		4

Table 1

The first student (BA08) is a boy from school B. His reading score is just above the 25^{th} percentile while he is a high achieving mathematics student. He reads most word problems several times and also rereads during solving to check or to adjust his work. WP2 might appear to be easy to him? This is the only problem he reads only once. As the transcript demonstrates, his mental representation only contains parts of the problem, although the relationship between numbers seems to be comprehended. His error in computing might be explained as a wrongful recall of number facts.

Boy, BA08: (Reads problem text silently). Then we do four times 79. And yes (mumbles) to do it in my head (mumbles) round up to 80. Times four. That is 240 minus four. Is 236. 236. Three - six - four - twelve (mumbles). Yes. Then the answer is 122. (Turns the page and moves to next WP)

The second student (BA04), a girl, has high scores for reading and low scores for mathematics, again just above the 25^{th} percentile. She guesses a lot, and uses the context situation to form expectations to elaborate on.

Girl, BA04: (reads out WP3 loud). Mm. Shall I do three divided by 79 or something like that then?

I: Mm. Why do you want to do that?

Girl, BA04: Or something else. Oh. Yes. T-shirts are 79. Yes.

I: Mm

Girl, BA04: So four times 79. I do not know if I can do that (laughs excusingly)

For WP2 it could be suggested that she guesses at first before adjusting her understanding. On general she often, on the basis of the social context, reduce the problem to something she can work on.

DISCUSSION AND CONCLUDING REMARKS

Cook (2006) found multiple strategy use over a group of word problems. Multiple and flexible strategy use is one of the characteristics of experts (Alexander, 1997). A quick review of table 1 demonstrates that students in general formed mental representations that mapped the social context for as many as or more problems than they solved successfully. However transparent patters do not appear from these simple comparisons. Boy BA08 would not be expected to be so successful solving the word problems based on his reading score and the number of appropriate mental representations, as he is. Girl BA04 might find that her general reading comprehension supports her mathematical activities.

It is difficult to draw conclusions as to whether the two students in the transcript situations apply text-base or deep-level strategies. It could be argued that guessing and reducing problems on the basis of the social context, are surface strategies to the same extent as number grabbing and direct translation and hence text-base strategies. Rereading several times is performed by both novices and experts, but for different reasons (Alexander et al. 2004). When boy BA08 rereads before and during solving for other word problems, it enables him to solve them correctly. Maybe what he manages to do is to elaborate on his domain knowledge in consideration to the text and moves beyond text-base strategies and towards deep-level strategies?

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TYPES OF LINKING IN TEACHING LINEAR EQUATIONS

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In the paper, Shimizu's notion of explicit linking is used as the starting point for an analysis and classification of teachers' repertoire of prerequisites that they employ to connect students' previous knowledge to the didactical situation of linear equations. The method of data collection is based on the Learner's Perspective Study framework. To illustrate the theoretical classification, the video recordings of two Czech classes are referred to. A comparison of the two different approaches of Czech teachers is presented. The findings are connected to Brousseau's Theory of Didactical Situations in Mathematics.

INTRODUCTION

To learn why some mathematics teachers are more successful in their effort to create suitable climate for their students' learning and understanding is still one of the most intriguing themes in research in the field of didactic of mathematics. Our understanding of teaching as relational (in agreement with Franke, Kazemi, Battey, 2007) is the reason why we have paid attention to discourse in mathematics classroom for a long time (Novotná, Hošpesová, 2007). It is through analyses of classroom discourse that we try to find out how teachers support students' involvement and development of their knowledge.

Shimizu (1999) mentions Stigler's study (Stigler et al., 1999, p. 117, in Shimizu 1999) on importance of explicit linking of knowledge across lessons and within a single lesson. In the study, linking is defined as an explicit verbal teacher's reference to ideas or events from another lesson or part of the lesson. The reference is taken as linking to single events (i.e., referring to a particular time, not to some general idea) and the reference must be related to the current activity. Later Shimizu (2007) specifies the idea on the interrelations "in the way that mathematical ideas in the current lesson are connected to students' experience in the previous or forthcoming lessons as well as part of the same lesson" (p. 177). It covers both, "looking back" as well as "looking ahead".

In this paper, we focus on one aspect of Shimizu's point of view, the connections to students' previous experience – "looking back": What are the prerequisites that the teacher refers to when solving new problems, developing new domains of school mathematics? In agreement with Shimizu (2007), we call these teachers' actions "linking". We consider both, linking within a single lesson and across lessons. Compared to Shimizu (2007), we include into linking also referring back to "general" ideas (see the example of multiple linking later). Our considerations are restricted to the domain of linear equations in the eighth grade in the Czech Republic.

In our study we come out from the belief that teachers have a certain repertoire of prerequisites ("pieces" of knowledge) that they see as unavoidable in the new didactical situation and that they have a pattern for recalling them.

We formulated our goals into the following questions:

- How can be linking uncovered in Czech lessons classified?
- What are the positive and negative aspects of frequent linking?
- What is the role of didactical contract (Brousseau, 1987) in developing links to students' previous knowledge?

DATA COLLECTION

In the research reported we analyse linking in video recordings of ten consecutive lessons on the solution of linear equations and their systems in the 8^{th} grade (students aged 14-15) of two lower secondary schools in the Czech Republic – in the following text, they will be labelled CZ1 and CZ2. Both schools are located in a county town with approximately 100 000 inhabitants. The method of data collection is based on the Learner's Perspective Study (LPS) framework (Clarke, Keitel, Shimizu, 2006). We compare our findings with our long-time experience from observations of lessons in different research projects.

Both teachers are experienced and respected by parents, colleagues and educators, although their teaching strategies differ: CZ2 teacher mostly concentrates on the question "How?". She develops problem solving strategies in the perspective of each problem. CZ1 teacher pays more attention to the question "Why?" She tries to build the new knowledge on her students' previous knowledge. On the other hand, CZ2 teacher trusts much more in her students' independent discoveries.

FORMS OF LINKING

During the analysis of the lesson we detected two basic categories of linking characterized by the teacher's prior intention to make linking (we call it a priori linking, shortly APL) or not to make linking (ad hoc linking, AHL). Our differentiation does not claim to be exhaustive, to cover all forms used by mathematics teachers in all educational settings. Nevertheless, our long time experience from our observations of many mathematics lessons in several countries lead us to the conviction that the main forms of linking are covered in the following text.

When presenting our classification of linking, we do not claim to give a hierarchy of their usefulness and influence on the level of students' understanding of mathematics.

a) A priori linking (APL)

The teacher, using his/her experience from teaching the particular topic, plans to recall in advance the necessary knowledge that should be already known to his/her students. Bearing in mind the prerequisites for successful learning of the piece of knowledge, he/she recalls the part of mathematics that the students have already met. The newly introduced content is placed in the network of existing knowledge and skills with the aim to promote understanding.

APL may have the form of a "theoretical discourse" when the teacher mentions the knowledge from one of the previous lessons or can be presented as a solving procedure for a suitable problem:

CZ1-L02, 7:36[1]

T: And we will need one more thing today or in the next lesson. The difference of sets. We have already seen it. Do you remember? How do we draw it? ...

APL can be of two forms, *linking across lessons (APL-Across)* or *linking within a single lesson (APL-Within)*. In both cases, APL might be used by the teacher *before* or *inside* a solving procedure of a problem or a proof (we will shortly speak about an "activity").[2]

When the teacher uses the "before activity" type of APL, the students often do not see the immediate use of the recalled piece(s) of knowledge. It is known to the teacher, not to the students (see also paradoxes of didactical contract in the TDSM, Brousseau, 1997). They are supposed to apply it at the appropriate moment later on.

CZ1-L02, 03:10 (linking to a more remote lesson)

T: Today and also in the following lessons, we will also need this, we have not revised it for quite a long time. How did we say it, how is it with the numbers that mathematics calculates with? …

The "before activity" type of APL is often used by the teacher as a reaction to a students' common mistake. The teacher presents the linked knowledge related to a mistake before correcting it or before solving a similar problem. (For example CZ1-L04 starts with the correction of the common mistake in students' homework.)

The teacher often uses the "inside activity" type of APL in the course of the work on a new piece of mathematics. Usually it is the case when the teacher expects the students not to remember an algorithmic step or to make a common mistake and plans to use it only in case that the necessity occurs.

Examples of the "inside activity" type of APL-Across:

CZ1-L02, 18:37 (linking to the previous lesson when solving the first equation with an unknown in the denominator, with infinitely many roots)

T: And yesterday we said that we could do this step if, if …

Student: If I do not multiply by zero.

CZ1-L02, 03:10 (linking to a more remote lesson when solving a "speed/distance" word problem)

T: So, what type of a problem is it? I am sure you will recognize it. You must surely remember …

The "inside activity" type of APL can be successfully applied as motivation for a more complicated problem. For example in CZ1-L5 (10:40), students' results when solving an equation with a parameter for various values of the parameter are used as motivation for solving the equation with the "general" parameter.

[1] CZ1-L02 means the second lesson in CZ1; 7:36 refers to the moment in the lesson when the dialogue/discussion started; similarly in the other excerpts. The excerpts are translated from authentic classroom discourse including all inaccuracies and clumsy expressions.

[2] If the reference occurs after the activity, we see it as a form of institutionalisation (Brousseau, 1997), (Brousseau, Sarrazy, 2002, p. 6), not as a linking.

A special case of APL is linking a priori to the previous lesson (usually during a series of lessons developing the same topic). The teacher reminds the students what they were supposed to learn from the activities of the last lesson. The most frequent reason is the teacher's belief that the students need more than one lesson to grasp the new piece of knowledge including the desired terminology.

CZ1-L02, 43:10 ("before activity" linked to CZ1-L02, 15:35, see above)

T: Let's return to the equation where there were infinitely many solutions. If I meet a condition that I have to record, don't forget about it in the end. ... In the problem that we went through together it was important to include the condition in the conclusion.

CZ1-L02, 31:00 ("inside activity" before solving the second equation with the unknown in the denominator, the root is 5, the domain of the equation contains all real numbers different from 3)

T: One more thing. Look here, you found that the root of this equation is 5, so what about it? We got around the condition and it's not necessary to take it into account any more. But in this case (she points to the first equation) that we went through together when we got …

CZ1 teacher often starts a new topic/procedure by "multiple linking" – linking to several items at the same time (e.g. *CZ1-L02, 15:30).*

T: We already know that it's not always so straight forward – the solving of an equation. What has always happened so far?

Student: We isolated the unknown.

T: We isolated the unknown. We calculated always the one, the number, the root of the equation. It might also turn out differently ... Vítek, how?

Vítek: That it does not have any soltuion.

T: How can I recognize that the equation does not have any solution, Denisa?

Denisa: The left-hand side does not equal the right-hand side.

T: Yes, there will be a contradiciton. For example what, Vasek?

Vasek: For example 2 equals 12.

b) *Ad hoc* linking (AHL)

By *ad hoc linking* we understand the linking integrated by the teacher as a reaction to what the students do or say when solving a new problem or when presented a new mathematical topic. Mostly it is a reaction to an incorrect answer or a step of the solving procedure requiring the use of knowledge that should already be known. It is the teacher's reaction to the immediate situation in the classroom. The success of AHL depends on the teacher's experience and pedagogical skills.

AHL can also occur as the teacher's reaction to a student's/students' questions e.g. in case of their individual solving of problems. For example in CZ1-L10 (13:30), the AHL-Across to the properties of a tetrahedron is provoked by the context of the word problem chosen by one of the students for his solving.

AHL can again be of two forms, *linking across lessons (AHL-Across)* or *linking within a single lesson (AHL-Within)*. It is used by the teacher *inside* an activity. A special case of AHL-Within is the individual help of the teacher during individual work in the lesson.[3]

a) AHL-Across *CZ2-L03, 27:25 (solving the equation 3(2 – x) – 4 = 1 – 2(x – 2))*

T: Let's start the solving process. Together, together, Lucka. If there are brackets in the equation, let's try to get rid of them. As, apart from brackets, there are also other expressions there, we have to remove them by multiplying out the brackets. And we already know how to multiply a binomial by a number. Attention when multiplying by minus! …

b) AHL-Within *CZ1-L02, 37:20 (creation of the equation for the "speed/distance" word problem, x = time)*

Student: $x + x + 1$

T: How do we calculate the distance, Michal? It was said a few minutes ago.

Michal: Speed times time.

THE TEACHING STYLES AND LINKING

Let us start by comparing incidence of linking in schools CZ1 and CZ2. This will later serve as the basis for comparing the teaching styles of the two teachers. Comments regarding "real life situations" done by the teacher are not included; the table is restricted to linking to school subject content.

Differences between the lessons

Lesson	CZ1	APL/AHL	CZ2	APL/AHL
L1	Presentation of new subject matter	8/2	Presentation of new subject matter through problems	5/3
L2	Presentation of new subject matter	13/4	Presentation of new subject matter through problems	5/2
L3	Practice and application	5/6	Practice and application	1/1
L4	Practice and application, students' individual work	2/8	Practising prerequisites, development of the new subject matter through problem solving	3/7
L5	Correction of individual work, new subject matter	3/3	Development of the new subject matter through problem solving	3/3
L6	Practice and application, new subject matter	1/2	Test, practice	0/0
L7	Practice and application, students' individual work, new subject matter	5/2	Word problems presented through problem types	6/0
L8	Correction of individual work, new subject matter	11/2	Special types of word problems presented through their solving	1/0
L9	Comprehension check, written test	1/2	Special types of word problems presented through their solving	4/1
L10	Correction of the test, summary of the learned subject matter	4/3	Special types of word problems presented through their solving	0/3

Table 1. Lesson orientation and number of APL/AHL

[3] AHL is often introduced similarly to the "inside activity" type of APL. The type can be recognized only when looking at it in the context of the students' reactions and action or during the post-lesson interview with the teacher.

Differences observed in the two classes can be explained as the consequence of the lesson orientation. Tab. 1 indicates that in both classes, the number of APL decreases and AHL increases when the lesson focuses on practice.

Differences between the classes

There is a significant difference between the number and types of linking used in the two classes – see graph in Figure 1.

In order to create a consolidated net of knowledge, CZ1 teacher roots the new subject matter in the theory that her students are supposed to be familiar with. In most cases, she inserts it before solving problems, sometimes even far before the activity where it is really needed (e.g. in L2, 3:20, number sets are recalled but not used until L2, 22:10 and later). It is obvious from the students' reactions that this type of linking became one part of the didactical contract. The number of APL applied inside the solving procedure of a problem is considerably lower. Earlier we (Novotná, Hošpesová, 2007) suggested that the reason for this teacher's behaviour might be her distrust in students' abilities resulting in students' lack of self-confidence.

CZ2 teacher prefers the re-discovery of necessary facts and procedures through problem solving to linking across or within lessons. She believes that her students' success in mathematics can be reached by their successful completion of assigned tasks. On the contrary, AHL occurs much more often. It is not only the reaction to her students' mistake but often the teacher's immediate reaction to her students' performance and her attempt to prevent the occurrence of too many mistakes. The teacher's experience plays an important role in her decision to use a certain linking type.

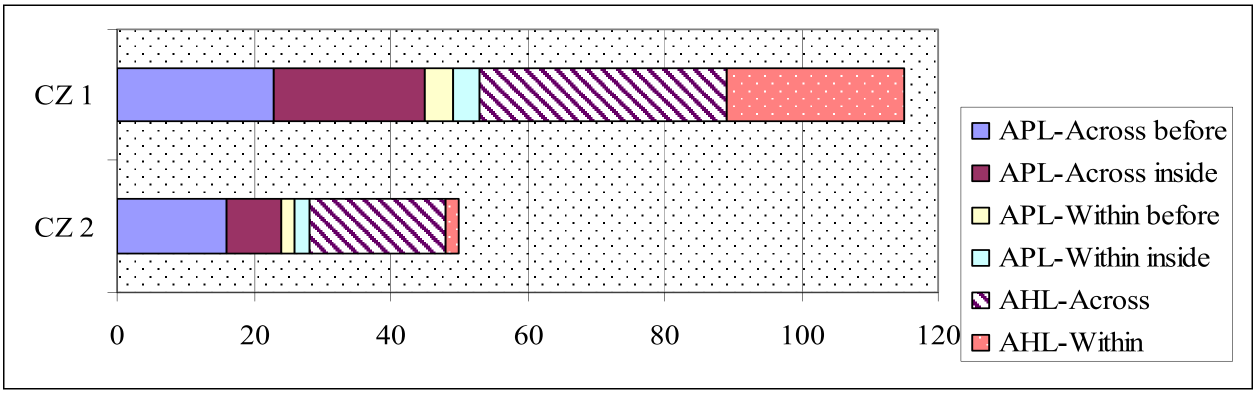

Figure 1. Graph comparing CZ and CZ 2.

Discussion

In some cases it could seem that linking within school mathematics is a special case of Topaze effect. In our previous work (Novotná, Hošpesová, 2007) we have studied the Topaze effect as a means of controlling students' uncertainty. In the situation of Topaze effect, "the teacher begs for a sign that the student is following him, and steadily lowers the conditions under which the student will wind up producing the desired response. In the end the teacher has taken on everything important about the work. The answer that the student is supposed to give is determined at the outset, and the teacher chooses questions to which this answer can be given. Obviously the knowledge required to produce the answer changes its meaning as well." (Brousseau, Sarrazy, 2002, p. 9, item 16). In this paper, we focus on the "useful linking"; the teacher coherently or intuitively (experience-based) recalls previous pupils' knowledge or experience that is useful for their successful work on the piece of mathematics. The aim is to build the nets of knowledge, structures, and links.

The teachers' decision to include APL seems to correspond to their analysis a priori of the didactical situation. Predicting possible students' solving strategies, previous knowledge needed for successful application of them etc. results in preparing the corresponding types and places of linking in the classroom.

Linking, especially AHL is susceptible to becoming a form of the Topaze effect. Used as a hint without checking students' understanding of the relationship between the linked and the "new" knowledge, we see it as Topaze effect. In contrast to Topaze effect, linking is used as an aid to grasp the new knowledge and to place it into the existing grid of knowledge.

The presented analysis of linking indicates that linking can play different roles in mathematical activities. It might be used for example:

- as a means of recalling needed previous knowledge,
- as scaffolding applied when helping the students to overcome the failure in understanding the new knowledge due to lack of previous knowledge,
- as a tool for checking previous understanding,
- as a tool for clarifying something puzzling from previous activities.

Our analysis compared to the Stigler's (in Shimizu, 1999) findings shows that linking could be an important feature of classroom culture. It significantly influences the quality of the grasped knowledge.

Endnote

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EXAMINING GENDER DIFFERENCES IN LANGUAGE USED WHEN BOTH A MOTHER AND FATHER WORK ON MATHEMATICS TASKS WITH THEIR CHILD

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This paper examines differences in the use of cognitively demanding language among four types of child-parent dyads (daughter-mother, son-mother, daughter-father, son-father) working together on mathematical tasks in number, algebra, and geometry. Parents and their children from third and fourth grade classrooms participated in the study. This paper reports on 20 of the 110 child-parent dyads that represent 10 children for which both the mother and father participated. These dyads are balanced by gender of the parent and of the child.

INTRODUCTION

This paper focuses on preliminary analysis of data collected as part of a three-year project, *The role of gender in language used by children and parents working on mathematical tasks,* funded by the National Science Foundation. While there were several research questions related to the project, this paper addresses the question: To what extent are there differences in the use of cognitively demanding language among four types of child-parent dyads (daughter-mother, son-mother, daughter-father and son-father) working together on mathematical tasks that initiate high levels of interactions?

Parents and their children from third and fourth grade classrooms participated in the study by working on mathematical tasks in child-parent dyads. To initiate a high level of interaction, the mathematical tasks had multiple solutions and/or multiple solution methods. Each dyad worked on three tasks, one representing each of three content strands, Number and Operation, Algebra and Geometry (NCTM, 2000). The 20 dyads on which we are reporting are balanced by gender of the parent and of the child.

THEORETICAL FRAMEWORK

Tenenbaum and Leaper (2003) investigated parents' teaching language during science and nonscience tasks among families who were recruited from public schools, summer camps, and after-school activities. Their findings indicated that fathers used more cognitively demanding speech with sons than with daughters when working with their children on a physics task, but not on a biology task. The researchers noted that biology is generally viewed as a more gender-neutral field of study.

Based on prior research that has shown gender differences in mathematics performance within different content areas (Casey, M. B., Nuttall, R. L., & Pezaris, E 2001), (OECD, 2004), it is hypothesized that the types of mathematics tasks will also affect the cognitively demanding language used by children and parents. The importance of the mathematical tasks was also noted by Junge & Dretzke (1995) who found that girls judged their self-efficacy lower than boys for occupations requiring quantitative skills.

Research showing gender differences on spatial skills and geometry along with research by Baenninger & Newcombe (1995) reporting that girls have fewer out-of-school spatial experiences, gave reason to anticipate there would be gender differences for children and parents working on the spatial and geometry task in our research. These findings are complemented by the recent findings (OECD, 2004) that males outperformed females on mathematics/space and shapes scales. Based on the longitudinal study by Fennema, E., Carpenter, T. P., Jacobs, V. R., Franke, M. L., & Levi, L. W. (1998), it is anticipated that some gender differences on the algebra task may also occur. In their longitudinal study, it was found that girls in first and second grades were more likely to use concrete solution strategies like modeling and counting while boys tended to use more abstract solution strategies that reflected conceptual understanding. Third grade boys showed that they were better at applying their knowledge to extension problems.

METHODOLOGY

The larger study included 110 child-parent dyads recruited from third and fourth grade classrooms at five elementary public schools and one charter school in Hawai‘i. These schools were chosen because of their students' low SES, ethnically diverse student populations, and willingness to participate in the study. Personnel at the schools assisted with recruiting parents and their children by distributing and collecting appropriate forms and by providing facilities for after-school or weekend sessions with parents and their children. Parents and children were videotaped at their home schools as they worked on the three tasks, each timed for 10 minutes. They were informed that it was okay if the tasks were not completed in the given time, and that the researchers were not looking for right or wrong answers. After each task was explained and necessary materials provided, researchers remained accessible but did not interfere with the working dyads.

Videotapes were transcribed verbatim and coded for instances of cognitively demanding language between the parent and the child. A coding instrument was developed using six pilot video sessions of dyads from the University Laboratory School at the University of Hawai‘i. The coding instrument consisted of three main categories: Getting started, Discussion Mode, and Vocabulary Usage. The Getting Started section informed how the tasks got started, i.e. who read the directions. The Discussion Mode focused on types of questions asked, who directed the tasks and how they were directed, types of explanations given and who provided encouragement. Questioning, directing, and explaining were coded at three levels to capture the levels of cognitive demand in each. Vocabulary Usage classified the mathematics vocabulary used and was coded at two levels. The levels will be explained in the results section.

The following are used to identify dyads:

M(mother), S(son), F(father), D(daughter)

MS(mother-son, when mother initiates)

MD(mother-daughter, when mother initiates)

FS(father-son, when father initiates)

FD(father-daughter, when father initiates)

SM(mother-son, when son initiates)

DM(mother-daughter, when daughter initiates)

SF(father-son, when son initiates)

DF(father-daughter, when daughter initiates).

Analysis was conducted on 20 dyads consisting of mothers and fathers of five daughters and five sons. Although the sample is small, the process provided a model for the analysis of all dyads. More importantly, it examines the use of language when the same child works with her/his mother and father. In this paper we present the analysis related to mathematics vocabulary, questioning, explaining and encouragement.

Chi-square tests were used to compare the cell frequencies of the codes for MD, MS, FD, and FS dyads. If there were significant differences among the dyads, post hoc follow-up chi-square tests were used to determine which pairs of cells were significantly different. In order to compensate for the alpha inflation problem, we set the alpha value required for significance to the $p < 0.01$ level.

RESULTS

Mathematics Vocabulary

The mathematics vocabulary used by parents and children as they worked on the tasks was coded as either M2, procedural or contextualizing familiar words to describe shapes and patterns, or M3, conceptual or more advanced mathematical terms (see Table 1). Overall among all three tasks, mothers and daughters used mathematics vocabulary significantly more than did fathers and sons. Across tasks, mothers used more procedural, M2, mathematics vocabulary with both daughters and sons than fathers did with daughters and sons. Mothers working with sons on the geometry task used significantly more procedural mathematics vocabulary than fathers did with sons. Mothers working with daughters used more procedural vocabulary on the algebra task that did fathers working with daughters or mothers and fathers working with sons.

Also on the algebra task, daughters working with mothers used more procedural vocabulary than did sons working with mothers or fathers. Daughters working with fathers also used more procedural vocabulary than sons working with fathers or mothers.

However, fathers used more vocabulary that was conceptual or consisted of more advanced mathematical terms, M3, more with their daughters than with their sons. Although this difference was not significant for any task, it was more likely when parents and children worked on the algebra task. The summary of differences in mathematics vocabulary is given in Table 1.

Area of significance at 0.01 level	Differences that were found
M2 - Across Tasks	M > F, MD > FD, MS > FD, MD > FS, MS > FS, D > S, DM > SM, DM > SF, DF > SF
M2 - Geometry	M > F, MS > FS
M2 - Algebra	M > F, MD > MS, MD > FD, MD > FS, D > S, DM > SM, DM > SF, DF > SM, DF > SF
M3 - Across Tasks	FD > FS

Table 1. Differences found in uses of mathematics vocabulary

Questioning

Questions asked by either parents or children were coded in three levels. Questions formulated directly from the task card were coded as CQ1; perceptual questions to elicit concrete or one-word answers were coded as CQ2; and conceptual questions to elicit abstract ideas or relationships were coded as CQ3. Table 2 provides information on the perceptual and conceptual questions asked by parents and children. There were significant differences across tasks and for each of the three tasks for perceptual questioning, CQ2. Across tasks mothers asked sons more perceptual questions than mothers or fathers asked daughters. Fathers asked sons more perceptual questions than mothers asked daughters.

When dyads worked on the number task, fathers asked sons more perceptual questions than fathers asked daughters or mothers asked daughters. While working on the geometry task, mothers asked sons more perceptual questions than mothers or fathers asked daughters while fathers asked sons more perceptual questions more than fathers asked daughters.

The significant differences for questioning were found for the algebra task where mothers asked sons more perceptual questions than mothers asked daughters or fathers asked sons. When fathers and mothers worked with their daughters on the algebra task, fathers asked perceptual questions more of daughters than mothers asked of daughters.

Area of significance at 0.01 level	Differences that were found
CQ2 - Across Tasks	MS > MD, FS > MD, MS > FD
CQ2 - Number	FS > MD, FS > FD
CQ2 - Geometry	MS > MD, MS > FD, FS > FD
CQ2 - Algebra	MS > MD, FD > MD, MS > FS
CQ3 - Across Tasks	FD > MD
CQ3 - Algebra	F > M, FD > MD

Table 2. Differences found in uses of questioning

Across tasks, conceptual questions to elicit abstract ideas or relationships, CQ3, were asked significantly more by fathers working with their daughters than mothers working with their daughters. On the algebra task, fathers overall asked conceptual questions more than mothers and furthermore, fathers asked significantly more conceptual questions with daughters than mothers with daughters.

Explaining

The verbal interactions between parents and children were coded to capture three levels of explanations that were provided by either parents or children. Whenever parents or children explained the task presented on the card, the language was coded as E1. Explanation of the mathematics was coded as E2, while E3 was used when the explanation also included reasoning about the mathematics (see Table 3).

Across the tasks, mothers explained the task, E1, more to sons than to daughters. This was true on the number and algebra tasks, but not the geometry task. Across the tasks and on the algebra task, fathers explained more to daughters than mothers did to daughters. Also, fathers explained the algebra task more to sons than mothers did to daughters. Therefore, in comparison to the other dyad groups, the mothers did not explain the task much when working with daughters. On the algebra task, sons explained the task more than daughters and, in particular, they explained the task more to fathers than did daughters to mothers.

There were significant differences for explaining the mathematics, E2, among the children. Across tasks, sons explained the mathematics more than daughters and sons explained the mathematics more to fathers than either daughters to fathers or mothers. On the geometry task, sons explained the mathematics more to fathers than to mothers.

Significant differences were also found for explaining the mathematics with reasoning, E3. Across tasks and for the algebra task, daughters explained the mathematics with reasoning more to fathers than did sons to fathers. On the algebra task, daughters also explained the mathematics with reasoning more to fathers than daughters to mothers.

Area of significance at 0.01 level	Differences that were found
E1 - Across Tasks	MS > MD, FD > MD
E1 - Number	MS > MD
E1 - Algebra	F > M, MS > MD, FD > MD, FS > MD, S > D, SF > DM
E2 - Across Tasks	S > D, SF > DM, SF > DF
E2 - Geometry	S > D, SF > DM
E3 - Across Tasks	DF > SF
E3 - Algebra	DF > DM, DF > SF

Table 3. Differences found in uses of explaining

Encouragement

The language used by parents and children as they worked on the task was also coded for encouragement (see Table 4). Across the tasks, mothers encouraged sons more than fathers encouraged sons or daughters. Fathers encouraged daughters more than mothers encouraged daughters. However, fathers encouraged sons more than mothers encouraged daughters. On individual tasks, significant differences were found for the number and algebra tasks but not for the geometry task. On the number task, mothers with sons and fathers with daughters in total were more encouraging than mothers with daughters. On the algebra task, mothers with sons and fathers with sons both were more encouraging than mothers with daughters. Also, on the algebra task, mothers encouraged sons than fathers encouraged daughters.

Area of significance at 0.01 level	Differences that were found
Encouragement - Across Tasks	MS > MD, MS > FD, MS > FS, FD > MD, FS > MD
Encouragement - Number	MS > MD, FD > MD
Encouragement - Algebra	MS > MD, MS > FD, FS > MD

Table 4. Differences found in uses of encouragement

SUMMARY

In the analysis of the data presented above, mothers and daughters used mathematics vocabulary more than fathers and sons, respectively. However, mothers used more procedural vocabulary with both daughters and sons than fathers did. On the algebra task, mothers used more procedural vocabulary with daughters than fathers with daughters or mothers and fathers with sons. Across tasks, daughters also used more procedural vocabulary than sons. On the algebra task, daughters used more procedural vocabulary with mothers than sons with mothers or fathers; and daughters used more procedural vocabulary with fathers than sons with fathers or mothers. However, across tasks, fathers used more total conceptual vocabulary with daughters than with sons.

In questioning, mothers asked more perceptual questions with sons than either mothers or fathers with daughters. Parents tended not to ask daughters perceptual questions as often as sons. However, on the algebra task, fathers asked daughters more perceptual questions than did mothers. Across tasks and on the algebra task, fathers asked daughters more conceptual questions mothers did.

In explanations, mothers explained the tasks more to sons than to daughters while fathers explained the tasks more to sons and to daughters than mothers did to daughters. Sons explained the mathematics more to fathers than to mothers and more than daughters explained to fathers. Interestingly, daughters explained the mathematics with reasoning more to fathers than to mothers and more than sons explained the mathematics with reasoning to fathers.

Mothers encouraged sons more than they did daughters, more than fathers encouraged daughters and more than fathers encouraged sons. Fathers encouraged daughters and sons more than mothers encouraged daughters. It is interesting to note there was no difference in the encouragement that fathers provided sons or daughters while mothers encouraged sons about seven times more than they encouraged daughters.

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LEARNING OF DIVISION WITH DECIMALS TOWARDS UNDERSTANDING FUNCTIONAL GRAPHS

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This research explores the development of the concept of rate for the steepness of cliffs through learning division with decimals with a teaching experiment and interview. Our analysis suggests that students' investigation of steepness by division may be connected to their understanding of functional graphs. In more detail, we show that steepness can be conceived through division as a result of students' coordination of change in width with height, that the understanding of the two lines in the number lines enables them to overcome misconceptions of measurement division, and that their investigation of cliff steepness help them determine slopes in a graph and cultivate a number sense of slopes.

INTRODUCTION

Ideally we hope that secondary students can explore and understand functional graphs based on a rate of change. However, it is well known that it is difficult for students to conceptualize this (Kunimune, 1996). Sakitani and Kunioka (2001) mentioned that difficulty in understanding rate of change stems from the concept of rate not being settling in students' minds in primary education. Here we can see a vicious circle in learning to understand functional graphs through rate of change, as even the central idea (rate) in understanding rate of change is problematic. Thus, here, we concentrate on the process of how primary students learn to understand rate with an expectation that they will have a sound understanding of a rate of change in the future secondary stage.

Thompson and Thompson (1992) argued that a ratio is "the result of comparing two quantities multiplicatively" and a rate is "a reflectively abstracted constant ratio" and identified four levels in the development of children's ratio/rate schemes (ratio, internalized ratio, interiorized ratio and rate). They examined the development of images of speed that consist of the rates of quantities of different attributes such as time and distance in terms of children's conception of ratio/rate in co-varying accumulation of quantities. We think that this scheme is useful since it suggests the concept of rate needed to explore functions. However, study of the understanding of the rates of quantities of the same attributes remains as a research task. We think that the process by which students understand how large one thing is in terms of another must be considered in ratio/rate schemes of quantities of the same attribute. We examine this through a design experiment below.

Simon and Blume (1994) studied prospective elementary teachers' ability to identify a ratio as the appropriate measure of a given attribute (ratio-as- measure), and reported that they tended to invoke a way of comparing by difference rather than by ratio. Also, they suggested that "generic knowledge of mathematization is a requisite component of an understanding of ratio-as-measure" (p. 191). We conducted a design experiment in a learning situation in which students mathematize the steepness of the cliffs using "division with decimals". Division implies rate as a generalization of quotitive division, namely rate and division can be connected within multiplicative concepts (Vergnaud, 1988) and can lead to development of functional concepts (Carraher, 1993).

In summary, in this paper we will explore how primary students develop their understanding of rate through learning division with decimals, specifically for the steepness of cliffs, so as to help them interpret functional graphs.

DESIGN EXPERIMENT

Background: Equilibration theory and overcoming misconceptions

We developed a design experiment based on our previous study into division with decimals in terms of Piaget's equilibration theory (Okazaki, 1997).

In Piaget's theory, mathematics is by nature an operation or transformation, and is constructed through a reflective abstraction which has two phases (Piaget, 1970). One is projection which is based on the success of action, and the other is reflection as its conceptual reconstruction. For measurement division, the first phase is measurement and the second giving the process a meaning of a rate.

Equilibration refers to the process of overcoming disequilibrium. For measurement division, some children cannot see "2.3 times as large as something" but just 2 of something and the remainder. In particular, they experience difficulties when the solution is smaller than 1 (Greer, 1992). However cognitive development generally does not occur without overcoming some kind of difficulty (Piaget, 1985).

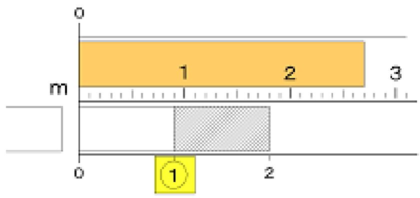

Figure 1. Operative material for learning measurement division.

Given these results we devised operative material (Figure 1) that permits children to understand how many times larger one thing is than another through measurement on double number lines. In the first setting no graduations appear on the lower line since we think it is essential for students to conceptualize the unit "1" for themselves.

Design experiment as a methodology

For our work here we included a new learning situation "the steepness of the cliffs" (Figure 2) in the scenarios used in our previous study. In this situation students compare the steepness of cliffs (right triangles) [cliff A (0.5 meters wide and 2 meters height); cliff B (0.9m, 2.7m); cliff C (1.5m, 4.65m); cliff D (4.8m, 2.4m); cliff E (0.5m, 0.4m)]. Our conjecture is that students' inquiry into the slopes might form an important step towards their understanding of rates of change.

Figure 2. The situation of the cliffs.

We conducted 10 class lessons for 36 children in the 5th grade at the university attached elementary school with a teacher with 12 years experience. After this, a short lesson of 30 minutes was given in which learning division was applied to investigation of a temperature graph. After each lesson, a reflective session was held. One month after the lessons finished, we conducted interviews with 16 children chosen at random to investigate how they conceive division and its application to graphs. These lessons and interviews were recorded with video and audio equipments, field notes were made, and transcripts were also made of the video and audio data.

Two types of data analysis were conducted as per Cobb et al. (2003). The first, of these was ongoing analysis in the reflective session after each lesson. There, we analyzed what happened in the classroom in terms of students' thinking, and redesigned the subsequent lesson by taking into account both the original plan and the ongoing analysis. Second was retrospective analysis which was done after all the class activities had finished. We first divided the classroom episodes into meaningful entities in terms of the situations that appeared to have changed students' conceptions. Next, the overall pattern of learning was considered by reviewing all the analyses in terms of consistency of interpretations of the episodes.

The data below was obtained from the first four lessons and the interview data. As the first presentation of our analysis we discuss the characteristics of learning division and their effects on the students' comprehension of a temperature graph.

RESULTS

The characteristics of students' mathematization of the steepness of cliffs

The teacher put a picture of mountain climbing (Figure 2) and the figures of cliffs A and B (Figure 3) on the blackboard, and explained an image of steepness with gestures. He then asked students which of cliffs A (W0.5, H2) and B (W0.9, H2.7) was steeper.

One student, Mura, noticed the width and stated "When the width is short, it is steep. So, the cliff B is gentle and the cliff A is steeper". Next, another student, Kaya, tried to justify Mura's idea by continual transformation: "the triangle comes to be flat when we increase the width". The students were convinced by this idea and it was agreed in the class.

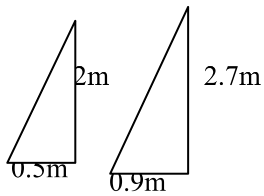

Figure 3.

Some children noticed that height is also related to steepness, but could not regulate width with height. After a while, Naga said, "Route A is 2 meters' height divided by 0.5 meters' width, and the answer is 4. B is 2.7 meters divided by 0.9 meters, and the answer is 3. So, the route A is steeper because of this large number". However, the other children did not understand well. Then, he tried again to justify his idea by presenting an extreme example (10 m wide, 1 m high) and comparing the small number 1 ÷ 10 with the visual gentleness of

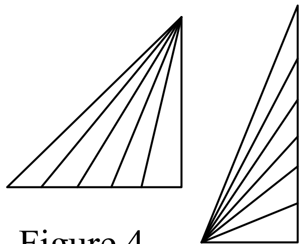

Figure 4.

the right triangle he described. However, as this example can also be applied to the idea of the width only, his idea was not recognized again. Finally, Matsu gave two examples (W6, H2) and (W3, H1); similar triangles to cliff B and explained “all are the same in balance. I used division to check how many widths are in the height.” The children then began to recognize the idea, and the teacher confirmed it on the blackboard. However in the reflective session for the day the collaborating participants reported that several children continued to stick to the idea that the width determined the steepness. It was determined that the teacher should cover steepness as the topic again in the next lesson.

The teacher began the second lesson by examining change in steepness using continual transformations (Figure 4). Soon the students noticed that the height did not determine the steepness, because cliff B is steeper than the cliff A; the opposite of what they believed. Then, the teacher asked them about cases where both the height and width change. Here Kaya said, “Four of 0.5 meters comes to 2 meters. Here, only three 0.9s are included in 2.7” (Figure 5). And, the teacher said “What do you call that?” Then, most students said “Division!” in unison. Through these interactions, the students realized that the steepness was decided by division and that the idea of ‘width only’ and ‘height only’ was incorrect. In the students’ reports for that day, many of them wrote statements like “I was so surprised to know that division determined the steepness”.

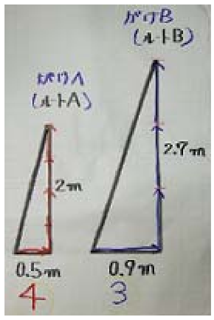

Figure 5.

The characteristics of the students’ understanding of “how many times as large as” by measurement

In the third lesson, the teacher asked the students to measure the slope of cliff C (W1.5, H4.65,) using the operative material. Several students already knew algorithm for division and stated the result was 3.1 using the calculation. However, they experienced difficulties measuring when the teacher asked them to measure from the material. That is, their measurement result was three and a remainder, which was different from their solution 3.1.

The teacher again confirmed the operative nature of the material and made clear what part should be sought (Figure 6). After a while, Koji marked 10 divisions on the strip of the divisor, and wrote 0.1 on the lower line and 0.15 on the upper line (Figure 7).

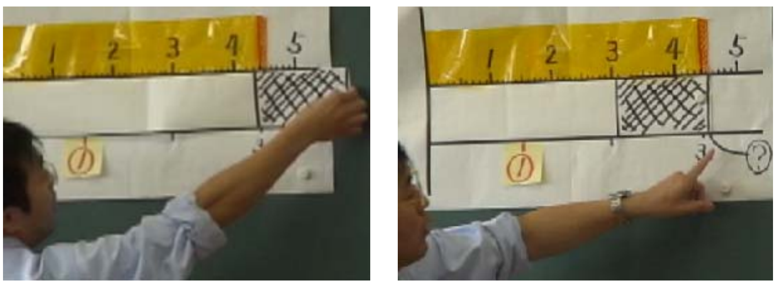

Figure 6.

Figure 7.

Koji: When 10 reminders are included… If we consider these (upper) graduations, each is 0.15. But, using these (lower) graduations, this is 0.1. We have to use these (lower) graduations in calculating the steepness. One, two, three and 0.1. The steepness is 3.1. But we have to say 0.15 for the length. We must distinguish between these graduations.

Koji insisted that the meaning of the upper line was different from that of the lower line, which was recognized by the classmates. We can therefore conclude that this measurement activity using double number lines enabled the students to make sense of the role of each line.

However, the collaborating fellows reported in the reflective session that some students considered the value to be 3.15 where they added the measurement value 3 to the remainder 0.15. Thus, we decided to take 3.15 as the theme of discussion in the next lesson.

In the fourth lesson, the teacher reported that some classmates considered the value to be 3.15 and asked them to explain why they thought this.

Kaya: I think that as it advances a small step of 0.15 from three, it's 3.15.

Iga: If you measure it by the lower graduations but not by 0.15, it comes to be 0.1. You have to consider the lower line as different from the upper line. I think that they did 3 + 0.15 because they mistook 0.1 of the lower line for 0.15 of the upper line.

Koji: They measured three's using the strip, but investigated the remaining little bit using the upper line. So, they made a mistake.

Although Iga and Koji made sense of both lines in the same way in their explanations, the way they restated why the answer was 3.1 suggested that their recognition of the concept of divisor was different:

Iga: When we consider 1.5 as 1… As it is a tenth of 1, it is 1÷10. So, it is 0.1.

Koji: We don't consider 1.5 as 1. If we consider so, it comes to be hard to understand it. I think we should consider it as it is.

Namely, Koji had an idea of measuring using the divisor as a standard, but could not regard a divisor as 1. After this, learning division with decimals was progressed by measuring the slopes of cliff D and E using the operative material in the same way.

The effects of investigation of slopes using division

For just 30 minutes at the last day, a line graph of temperature change was introduced, and the students were taught that their view of slope could be applied to the temperature graph (Figure 8).

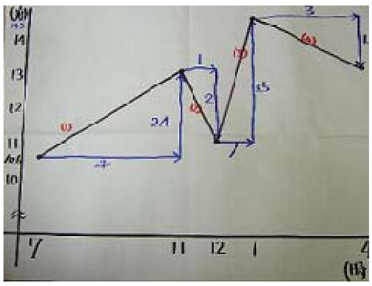

Figure 8.

In the interview conducted one month after, we asked the students to read the slopes and to predict the values in the curved temperature graph. The interview items were (A) Find the slopes of 7 o'clock to 8, 8 to 10, 10 to 11, 7 to 10 and 7 to 11, (B) How do these slopes change with time?, (C) When the temperature increases in slope 2 from 7 o'clock, what is the temperature at 2 o'clock?, (D) When the temperature decreases in slope 5 from 2 o'clock, what is the temperature at 4 o'clock?; and (E) To understand the changes in temperature in detail, what do you pay attention to? Here, we present data for Naga which shows a response common to 10 of the 16 subjects and data for Matsu which shows more advanced recognition.

Naga said "I see if I use a straight line" on the slope from 7 to 8 o'clock, and applied a right triangle. As to the change of slopes between 7 and 11 o'clock, he explained: "it first gets high, next low, and it again gets high". All subjects, including Naga, responded in a similar way to this question. Thus, we suggest that investigation of the cliff slopes was successfully applied to the temperature graph.

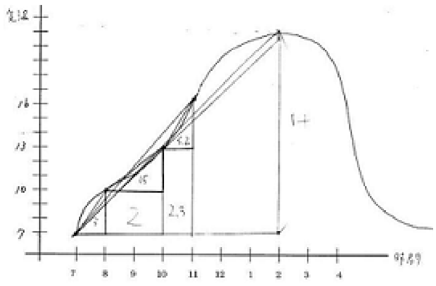

Figure 9. Naga's sheet.

For the task of finding the temperature at 2 o'clock when it increases by slope 2 from 7 o'clock, Naga predicted the temperature by drawing a similar right triangle using slope 2 from 7 to 10 o'clock that was already known. Namely, he used slope as a predictor, although the temperature was not accurate for a straight line drawn as it deviated a little. Also, it seemed that he had acquired the number sense of slopes, as he made statements like "it is bigger than 2", "it is gentler than 1.5" regarding slopes after 2 o'clock.

Matsu showed more advanced recognition of the graph. He first tried to apply a right triangle, but stopped and then stated,

> Matsu: I am not sure how I decide the straight line. If you determine the triangle by one minute unit, you have to consider where one minute passed. If you take it for one hour, you draw a straight line like this.... It changes depending on what we use as 'one'.

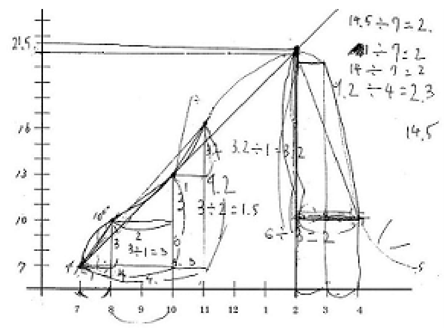

Figure 10. Matsu's sheet.

He explained that slopes change with time based on the idea of a unit that he had acquired when learning division. This suggests that he possessed a concept of rate of change. Furthermore he used words like "same pace" to find the temperature at 4 o'clock when decreasing by slope 5 from 2 o'clock, implying that the rate of change was constant. Last, Matsu stated how the graph is better recognized:

> Matsu: From where to where should we determine the change of slope? The important thing is we cannot know it unless we investigate the slopes in small widths.

We think seeing slopes in small intervals is essential to understanding rates of change in functional graphs, and we can find a beginning of such recognition in his utterance.

CONCLUSIONS

This paper explores the development of the concept of rate in learning division with decimals and the possibility this is connected to understanding of functional graphs through a design experiment and interviews. On the whole, our data analysis indicates affirmatively that exploring the steepness of cliffs by division can lead to understanding slope and rate of change in functional graphs. In more detail, we can state this in terms of how students mathematize a slope in the cliff situation, how they may make sense of rate through measurement, and how their understanding of division leads to understanding functional graphs.

First, we suggest that students tend to conceive of steepness by (change in) width or height only and that change in width needs to be coordinated with that in height so that the slope can be conceived of through division. Also, given that the students felt surprise when they discovered that slopes can be described using division, it may be worth preparing teaching content for students linking these concepts before they learn functions in secondary school.

Second, it is suggested that the meaning of the upper and lower lines in number lines and the meaning of "how many times as large as" can be clarified through measurement activities and by measuring remainders using a new unit of one tenth of the divisor. We also observed the phenomenon that the several students considered the integer part and the remainder as the measurement value, and that this misconception can be overcome if they recognized the meanings of the two lines in the number lines. We think that these findings describe components in understanding ratio/rate schemes of quantities of the same attributes.

Third, it was indicated that students' investigation of the cliff slopes by division not only leads to applying right triangles to graphs and determining the slopes of the hypotenuses, but also cultivates a number sense of slopes. It was also suggested that seeing (1/10 of) a divisor as 1 in measurement can be transferred to investigation of rate of change by small intervals.

We could not, however, fully discuss students' learning since our data was limited here to the four of 10 lessons and interviews. Thus, our future task is to clarify all the components of ratio/rate schemes of quantities of same attributes through continuing our analysis of the remaining data from the students.

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CASES STUDIES OF MATHEMATICAL THINKING ABOUT AREA IN PAPUA NEW GUINEA

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This paper is a distillation of part of the ground-breaking systematic recording of measurement data from many Indigenous communities of Papua New Guinea. It focuses on area thinking and measurement. Implications for teaching measurement and making the implicit knowledge of students explicit are discussed.

CULTURAL CONTEXT AND EDUCATION REFORM

In Papua New Guinea, an educational reform program reorganised the schooling structure. One purpose of this reform was to ensure children began school in their home language. Elementary schools (pre-elementary, grades 1 and 2) were introduced so 5/6 year old children could start at a school near home in their home language. Reform syllabuses emphasise the importance of culture throughout schooling. In elementary school, in line with the syllabus *Culture and Mathematics*, teachers should introduce counting, arithmetic and other mathematical ideas relevant to the school's context in the children's home language. There is a transition to English in the second half of the third year and in early primary school (Grades 3-8).

There are over 800 languages with a diversity of patterns and types of counting (Lean, 1993; Owens, 2001). In elementary schools, counting systems are being recited but generally with English and resulting in an interesting mix of continuities and discontinuities (de Abreu, Bishop, & Presmeg, 2002; Valsiner, 2000) between vernacular counting systems and base 10 English arithmetic. Part of this relates to teachers' limited knowledge of how their own system relates to other systems and how to use this analysis to advantage (Owens, 2000; Matang, 2005; Paraides, 2003). A detailed study on in- and out-of-school mathematics among the Oksapmin illustrates such continuities and discontinuities (Esmonde & Saxe, 2003).

One issue for the elementary and primary school teachers is an understanding of how their cultural knowledge can best be used for schooling in mathematics by making tacit knowledge explicit (Frade & Borges, 2006). This requires communication of the mathematical thinking behind activities. We set out to collect and analyse approaches to measurement for as many language groups as possible to mirror Lean's (1993) work on counting. However, for measurement, unlike analysing patterns for counting words, our analysis involves deeper thinking about comparisons and mathematical thinking in activities like gardening and house building.

RESEARCH DESIGN

Ethnographic study is appropriate to investigate "ways of acting, interacting, talking, valuing, and thinking, with associated objects, settings, and events (that impact on)

… the mental networks" that constitute meaning (Gee, 1992, p. 141). Data is being collected and analysed and the analysis checked with other participants' and communities' data as the grounded theory develops. Our research project requires participants who are familiar with their own communities' activities and preferably investigating their own cultural practices. We are conscious of the various relationships between ourselves, the participants and the village elders whom we interview with a participant researcher (Owens, 2006). Villages referred to in this paper come from a variety of environments (mountains, coastal areas and large valleys) and language types (Austronesian and non-Austronesian languages). For this paper, data from 16 in-depth interviews (demonstrations, discussions, and observations with some semi-structured questions) either in the village (visited by at least two researchers), at the University of Goroka or with linguists have been complemented by questionnaire and focus group data.

RESEARCH CHALLENGES

Language Diversity

The mostly oral languages are rapidly changing and being overtaken by Tok Pisin (the main creole). Linguistic ways of comparing vary (Smith, 1988). For example, there may be a limited number of comparative adjectives or very general concepts like size. Some languages may use a prefix or suffix or reduplication or similar morphology, e.g. in Dobu, pumpkin *kaprika* changes to *kapukapurika* for small pumpkin (Capell, 1943). Some languages have an adjective for a measurable attribute that is used for all objects and others have several adjectives for specific classifications e.g. round objects, flat objects, people, food, and in Korafe (Farr, linguist) *big* for fish is different to *big* for people. Two linguistic features: order of words and way of indicating emphasis can impact on discussions about measurement activities (Tupper, 2007). Metaphorical use of words makes it more complex e.g. a child is a small version, chunk, of father in Korafe. Our main communication language was Tok Pisin which has a limited vocabulary expanded by metaphors. For example, one participant researcher referred to *cm, m* to imply units and composite units (or at least small and large units) but not the actual size of the cultural unit. The word *leg* would be used for various uses of the leg (e.g. walking steps, metre steps, heel-to-toe steps) or part of the leg (e.g. length, or width of foot).

Cultural Values and Practices

In these Indigenous societies where marriage exchanges, compensation and many other practices require the community to participate in a display of wealth, what is valued is complex and size is not the only criterion. For the garden which is so important for a subsistence farmer, size is considered along with other attributes. Elders make decisions about how land will be distributed with varying degrees of discussion. Other considerations might be the kind of crops that best grow in a particular area, the position of the various garden plots or the use to be made of other land areas (e.g. for hunting, for food gathering, for pigs, as a barrier between tribal or clan groups, for relatives and in-laws). In addition, the elder son may receive more land. The land may be lent to a family member or passed on to the sons (examples from Matafao, Markham Valley & Kopnung, Western Highlands) or daughters.

Much of the cultural practices are not spoken about but observed. People measure by "eye" and "think in the head" and those who are competent at these tasks are involved in decisions. In order not to lose cultural context, we must ask who and how do people make decisions, what tacit knowledge is used, and how might culture and decision-making impact on the measurement process? How might this link to school mathematics and school learning?

Subsistence farmers are reliant on the environment which impacts on activities and hence on their mathematics. For example, gardening techniques in the forest differ to those in the kunai grass. Some people own both kinds of land, others do not. Rugged, steep mountains and soggy valleys change practices in other ways. Different kinds of food plants, trees, grasses, birds, and winds impact on different aspects of food and house production and time descriptions. The time of year and time of day may be described in terms of the position of the sun on the local landscape. Seas and rivers provide other mathematical activities such as canoe making, travel and fishing.

DIVERSITY OF FINDINGS FROM THE MEASUREMENT RESEARCH

Measurement Purposes

The communities place differing levels of importance on comparisons, estimations and accuracy. For some communities, marking land is carefully carried out with bright tanget plants that last for several generations. Natural phenomena like creeks are also used to identify a person's land. Some people are less concerned with the measurement of the area to decide the boundary. Some groups are careful to compare heights of doors or walls with parts of the body or compare the slope of the roof with the image of other roofs (e.g. in Kopnung village). Accuracy may be decided by a group of people and/or a temporary standard (stick or rope) used by all helpers to get lengths equal (e.g. in Malalamai village). The person who owns land or builds the house may set the standard of one or more steps, arm spans or hand spans. Convenient objects (arms, legs, rope and sticks) are used. Some lengths are kept for future reference and these may be the total length or a series of lengths marked on a stick or knotted on a rope. Estimates of amounts are made based on knowledge gained from other constructions. Some people are not too concerned about cutting the exact amount of material for frames, walls or fence posts as excess material will be used by other or more can be collected (Panim). By contrast, Kate speakers measure the floor width with rope equal to the circumference of the bamboo and they count the number of pieces that are needed. This practice helps preserve the bamboo which must be cut and carried a long distance.

Measurement of Area

The following provide representative cases from different geographical places in Papua New Guinea. Data from nearby villages generally confirm the summary but also indicate some variance. Some effort is made to explore the significance of these differences. For example, the need to be accurate, the nature of the ground, the availability of land, the family relationships or needs might impact on the decisions.

Around Goroka in the Eastern Highlands Province, people speak Alekano (also referred to as Gahuku-Asaro). The data were obtained from a field trip of three days to Kaveve village with demonstrations, discussions, observations and follow-up checks of our notes. Six men from other villages completed the measurement questionnaires and six schools were visited to observe and discuss elementary school teaching issues with teachers. Land area is compared by looking at it, discussing it and marking the boundary with tanget plants but there is no formal measurement. When making a garden close to the house, people will decide half of it by standing in the middle of one side and deciding where the half way line should go. Half the garden is left fallow. Kaukau (sweet potato) is planted in mounds, generally with two mounds between drains. The fifth pair of mounds might be marked with a sugar cane or other plant. The various garden sizes seem to be well established in the mind suggesting that people have a good visual image of the areas involved with the garden, halves of the garden, rows of mounds, and blocks of mounds between drains.

In this area, houses are round. A pole is placed in the centre with a rope attached, people gather to decide an appropriate area for the house determining the radius. The other end of the rope is tied to the ankle of the house builder who drags his foot around to form the circle. If the house is big, then it will have a larger volume to warm with the fire and the builder needs more help from family and friends to collect materials and build. In tandem, gardens will be planted to provide food for the feast for the helpers confirming relationships between people. The roof provides triangular areas between rafters of the conical shaped roof. Each area is well pictured for the collection of kunai grass to cover it. Areas at ground, bed and ceiling level within the house are imagined in terms of space for their purpose.

In the Whagi Valley (Yu Wooi or Mid-Whagi speakers) in the Western Highlands area, the drains frequently form squares (data from three days in Kopnung village with demonstrations, discussions, and observations; and 16 measurement questionnaires from participants from other villages). The kaukau mounds are generally larger. For the men’s house, people discuss its floor plan in terms of the number of men who might sleep in it and by comparison to another house. Round numbers are used e.g. “it is for 25 men” and for the area of rooms e.g “7 feet by 7 feet”. The square sleeping room is visualised as sleeping around seven men with the length for the man also being 7 feet. The prone position image seems as strong as the vertical. After discussion, the ground is levelled and the expected floor plan traced out on the ground. The rectangle is divided equally into three. In the middle is the area for sitting around the fire, opposite the door. The outer thirds are each divided into two squares for sleeping. Care is taken to ensure the walls are at right-angles and straight by ensuring markers are lined up. Adjustments might be made to the outer walls especially to make the ends curved.

The coastal village of Malalamai in the Madang Province is one of only two large villages speaking this language with strong customs and relationship patterns. A four day visit was made with demonstrations, discussions and observations occurring throughout the time. From other small language groups living further inland, data was subtly different. Floor areas are decided by what space the villager wants for family's expected size and activities and the extent to which he can afford to build such a house given the amount of manpower that it requires. Plans may be modified by what is available to them in the bush. There needs to be space for sleeping, sitting to talk and eat, and cook if there is no separate cook house. People think of the floor space in terms of the number of rows of posts. These are 6, 9 or 12 posts with the base row of 3. From house to house, the space between posts is about the same. People talk of the house as half as big again (i.e., 12 post compared to 9 post) or twice the size (12 post compared to 6). In other language groups (e.g. at Panim), further inland where the winds are not so strong and the posts shorter, posts may be further apart.

Several full lengths of the roof *morata* (made from limbom timber and sewn sago leaves) determines the length of the house. They look for trees that provide sufficient sago leaves for that house, carefully selecting as the size of the leaves may vary and morata must overlap for waterproofing. Some people spread morata further apart and yet people are able to look at a pile of morata and decide if it is enough for a particular roof. In Panim the roof is exceptionally steep to make the house look nice requiring more materials. In places further inland or for smaller structures that use kunai, people estimate the area of kunai needed for the roof of a certain house. People talked of the house as requiring, for example, 40 or 70 bundles of kunai. In Gua (Yupno language), the large round roof houses look like "stones" from a distance requiring large amounts of kunai.

In these areas, agriculture officers have encouraged cash crops like cocoa to be planted out at the vertices of tessellated triangles. Two standard bamboo lengths are used. The baseline is first marked with sticks and is parallel to the edge of the garden and equally spaced using the standard length. Then with two sticks of the same length, a second row is marked as the vertex of the equilateral triangle. These tessellated triangles form diagonal rows that can be kept clean and extended, the plants are spaced more efficiently in the sense of maximum circular space for each tree's roots and the trees can break up the flow of water. Figure 1 illustrates this pattern. Other crops on slopes may be planted in holes placed around the roots of the hewn secondary growth trees. Yams may be carefully grown in rows but not necessarily. People have a practical idea of how big the area needs to be for the number of yams wanted in gardens near their houses for the *taim hungri* period when the rivers flood and they are cut off from their main gardens.

Walls of bamboo are common in many regions. The tessellated shapes, often squares, cover areas. Dried bamboo lengths that equal the length of the wall plus a bit more (a forearm length) are laid out with a small space between them and then bamboo lengths equal to the height plus a bit more are woven into them. Women (Figure 2) have traditionally made squares in their continuous figure-8 knot bilums (string bags). Nowadays women develop new designs using up to 20 needles with several colours (some repeated) and they carefully take account of the area covered by the part of each shape. For example, a hexagon may be made up of two sections knotted separately using two needles. Women count knots to assist the pattern making.

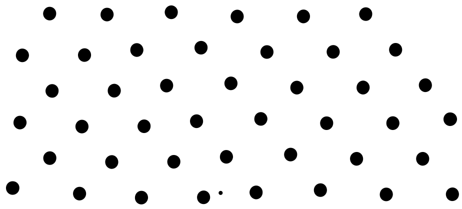

Figure 1. Pattern for cash crop

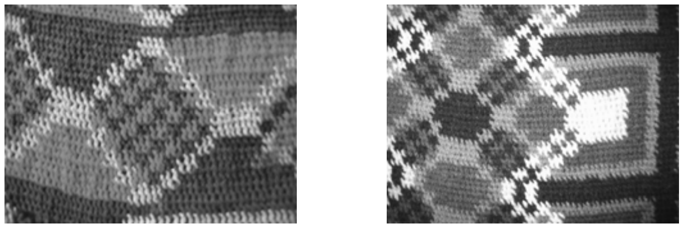

Figure 2. Bilum squares.

Related Measures

Interestingly, whenever people were asked how they measure garden areas, they talked about pacing lengths. Unlike Highlands people, coastal people tend not to pace out their gardens while groups from various places pace out the lengths for fencing. On the coast, gardens tend to be planted around tree roots and in kunai grass so that carefully spaced plants and size of land is less significant. In the highlands in particular, there are times (e.g. distribution of land between children) when the size of the garden might be considered along with other features. Garden lengths are paced out as the width is fairly standard in each place. Widths are measured for the purpose of deciding the number of kaukau and drains.

Composite units are generally decided when a rope or stick equals a length measured by a number of steps, hand or arm spans. The space between tanget plants might also indicate a composite unit like 20 paces but this is rarely remarked upon.

Estimating areas involves an intuitive form of proportional reasoning. The area of kunai needed for a 6, 9, or 12 post house and the increase in floor for additional posts are examples given above. Kate people know how much area will be watered by three, four or five nodes of both a large and small diameter bamboo. Some people use the area of the pig's foot to compare the size of pigs whereas others consider length such as how far the belly is from the ground, the length from nose to tail, or the girth.

IDEAS FOR TEACHING IN ELEMENTARY SCHOOL

Children could view the different gardens and skip count (2, 4, 6, 8, 10) the groups of mounds or holes if plants are placed in regular rectangular rows. The rows of plants or mounds form a composite unit. This can then be modelled in the classroom as squares, each containing a kaukau mound, to give a sense of rectangular area measure. The mapping of the triangular plantings could be illustrated by joining the dots to form tessellated equilateral triangles. Older children could look at the touching circles (representing the tree roots/branches) and compare the number of trees with those planted in rectangular rows. Material, e.g. the common lowland woven coconut frond (these only approximate a rectangular unit), could be used as an informal area unit to represent people lying down to see how big areas are for sleeping people. Similarly, people can sit on these for informal composite area units. The tessellations of woven walls and mats and bags could be used to calculate areas.

Besides exploring the gardens and making maps and models of houses for ways to compare and measure areas, teachers can prepare reading books in their own language telling a story about planting a garden or making or living in a house. Teachers do refer to standard length units that are used for a particular activity such as spaces between posts or morata, and the size of the fishing net holes for catching different types of fish (e.g. in Malalamai). The concepts of an area unit and of composite measurement units also need to be established by these activities.

Communities also need to develop a lexicon of words and grammar related to measurement (see earlier comments on language). For example, how will they select words to refer to *area* (not place), volume and mass, smaller and larger units (not just *big* and *small*), and how will they refer to lengths generally (cf. the English use of length for the longer side of a rectangle and as a generic attribute). Linguistic issues can be illustrated by Korafe– the nominal word e.g. *length* is also the adjective *long,* similarly *depth of the sea* for *deep* but *width* has the meaning of only the one *side*. The word for *big* refers to volume. The word *ai* is used for pile so it is like a volume unit but they do not have a word for *group* unlike the neighbouring language Baraga *nosi.* Korafe provides comparatives for *bigger, biggest* etc by using suffixes.

DEVELOPMENT OF THE CONCEPTS OF AREA AND AREA UNITS

Frade and Borges (2006) discussed the issues of making implicit knowledge explicit. We can generalise to say that people have a sense of area (tacit knowledge) developed through sleeping, gardening and house building in particular. People are able to use this idea of area to make judgements such as the estimated amount of material needed for a house of a particular floor size. Many participant researchers referred to the length of a garden as a measure of a garden. However, people would visualise a garden by knowing its length. Some visualised the kaukau mounds, others visualised a garden with a common width. Similar comments could be made about floor plans and roof areas. The static environment provides some mathematics whereas mathematical thinking occurs during the process or activity. By making these points explicit, teachers can reduce the discontinuities in knowledge and hence build a firm basis for school mathematics.

Endnote

Charly Muke (Kopnung), Rex Matang (Kate & Matafao), Sorongke Sondo (Malalamai), Zuze Hizoke (Kaveve) and their extended families, Annica Andersson (assistant), Cindi Farr (Korafe), Lillian Supa (bilum-maker), staff and students at the University of Goroka and Madang Teachers College, and many villagers who shared their experiences.

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VOICES IN SHAPING THE SUBJECTIVITY OF PEDAGOGICAL CONTENT KNOWLEDGE

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In this paper we consider pedagogical content knowledge as involving subjective decisions on making the content instructional and are concerned with the development of this subjectivity. In our attempts to understand how pre-service teachers decide upon what approach is the best in delivering the mathematical content we find Bakhtinian concept of voice useful. Our examination of the pre-service teachers' microteachings and retrospective interviews suggests that teacher candidates' discourse and practice were greatly shaped by the voices of others who are distant in space and time. These voices are selectively assimilated and this assimilation reflects certain value judgements. We exemplify our arguments with a pre-service teacher's introduction of derivative during a microteaching activity.

INTRODUCTION

The term 'pedagogical content knowledge' (PCK) has a common usage in teacher education discourse. It is first introduced by Shulman and broadly deals with the understanding of how particular issues (topics, problems and so on) are organised, represented and adapted for instruction (Shulman, 1987). Hence it has a connotation with knowledge for successful teaching of the subject at hand. Shulman (1986) described pedagogical content knowledge as:

> the most useful forms of [content] representation… the most powerful analogies, illustrations, examples, explanations, and demonstrations – in a word, the ways of representing and formulating the subject that makes it comprehensible for others (p. 9).

'Useful' and 'powerful' are value-laden words and require subjective decisions on the part of a teacher in finding the ways to represent and formulate the content in a comprehensible manner. Given this subjectivity, we, in this paper, ask: How do teachers come to know what the best representation and/or formulation of the content for a successful instruction is? Our starting point in answering this question was to focus on the term PCK itself. PCK is viewed to represent the blending of content and pedagogy for instruction (Shulman, 1987). Surely depth of content knowledge is important in making content instructional and this is convincingly documented by the relevant research studies (see Manouchehri, 1997). However, we are also convinced of the importance of 'others' voices' (Bakhtin, 1981) in shaping teachers' pedagogical practice and hence their subjectivity.

In what follows we describe the theoretical framework of this study by drawing on the concept of pedagogy and voice. Then the context of study is detailed. We later present data and analyse it through the lens of voice. The paper ends with a discussion of the importance of others' voices in shaping the subjectivity of PCK.

THEORETICAL FRAMEWORK

Central to the theoretical framework of this paper are two main concepts: pedagogy and voice, both of which are crucial in explaining the formation of subjectivity of PCK. One's view on pedagogy, not necessarily held consciously, shapes his/her approaches to teaching and hence subjectivity (Simon, 1992). In Simon's view it is pedagogy "through which we are encouraged to know, to form a particular way of ordering the world, giving and making sense of it" (p.56). Simon recognises the influence of pedagogy on developing particular world-views not only of knowing and thinking but also of (re)production of knowledge by recognising and accepting certain meanings as well as challenging and dismissing others. This partly echoes in Giroux and Simon's (1988, p.12) line of thought, pedagogy, they note:

> organise[s] a view of… and specifies a particular version of what knowledge is of most worth, in what direction we should desire, what it means to know something, and how we might construct representations of ourselves, others, and our physical and social environment.

This view stresses the power of one's understanding of pedagogy in shaping his/her approach to teaching. However pedagogy is also related to how one interprets the educational aims and their realisation. This is eloquently noted by Davies (1994):

> pedagogy involves a vision (theory, set of beliefs) about society, human nature, knowledge and production, in relation to educational ends, with terms and rules inserted as to the practical and mundane means of their realisation (p. 26).

As this brief consideration indicates, pedagogy involves worldviews, ideas, beliefs about and value judgements on, broadly speaking, the knowledge, how its produced and learnt in relation to educational aims and their realisation and these are all equally important to consider in explaining teachers' subjectivity in the preferred approaches to make content knowledge instructional. Formation of worldviews, ideas and value judgements are also important to Bakhtin; and central to his ideas is the notion of voice. The voice is the speaking personality, the speaking consciousness and always has a desire behind it (Bakhtin, 1981). By voice Bakhtin puts the emphasis on broader issues of speaking person's perspective, conceptual horizon, intention and worldview (Wertsch, 1991). Bakhtin (1986) argues that a personal voice in producing particular utterances (whether spoken or written) takes on and reproduces other people's voices either directly through speaking their words as if they were their own, or through the use of reported speech. This process of producing unique utterances by invoking other's voices involves ventriloquation, "the process whereby one voice speaks through another voice or voice type in a social language" (Wertsch, 1991, p.59). However, words and voices of others do not simply enter into one's own voice, they are selectively assimilated by the speaking consciousness. On account of the value-laden nature of language, this selective assimilation of the words and voices of others, Bakhtin suggests, is part of "the ideological becoming of a human being" (1981, p.341). Hence a voice and its particular utterances inescapably convey commitments and enact particular value judgements (Cazden, 1993).

In the light of these considerations, we believe that subjectivity in teachers' PCK is closely related to their understanding of pedagogy which manifests itself through their practice and discourse. How a teacher's understanding of pedagogy relates to their subjectivity of PCK is our concern in this paper. We examine this issue from the lenses of Bakhtin and will argue that teachers' formation of pedagogy takes place with the interaction of the others' voices (some of which ventriloquate through teachers' voices and hence not necessarily physically present within the immediate context), and their practices are saturated with the words and voices of others, reflect certain value judgements and hence determine their subjectivity. Next we detail the context of the study and present data from a mathematics teacher-training program.

SETTING OF THE STUDY AND THE DATA

This paper is a by-product of a research study which aimed to examine the progress of pre-service mathematics teachers in terms of PCK during a teacher education program in Turkey. The program admits graduates of mathematics departments and lasts for one and a half years. During the program, pre-service teachers attend general and subject specific pedagogy courses (which aim to help pre-service teachers develop PCK) and do teaching in actual classrooms. In their attendance, pre-service teachers study learning theories in mathematics education, design and carry out microteaching activities, find opportunities to follow their peers during microteachings, and examine and reflect on their and peers' teaching approaches.

The data on which this paper reports was collected during pre-service teachers' microteachings, each of which took about 45 minutes. Every pre-service teacher was assigned a mathematics topic (including limit, derivative and integral) and asked to prepare a lesson plan and have teaching notes to use during the microteaching. Before the microteaching, the candidates were given a set of questions on the target topic to see if they have a conceptual understanding of the content they would be teaching. The microteachings were video-recorded. Pre-service teachers were interviewed after the microteachings to gain insights into a variety of issues including how they planned the lesson, what sources they used, and also to hear their reflections on how, in their view, the planned lesson went. Hence the data for this study is composed of video-records of microteaching, interviews, written documents (e.g., lesson plans, teaching notes, topic tests given to pre-service teachers prior to microteaching.

While analysing the data, we examined microteaching videos along with lesson plans and teaching notes, and critically evaluate the pre-service teachers' approaches to the delivery of content. Note that every topic was taught by four pre-service teachers, each of whom displayed different approaches to the delivery of the same content. For instance, while introducing the concept of derivative, one pre-service teacher started with a problem regarding the velocity and motion whereas another directly provided the formal definition of derivative. Noting such this difference might not be surprising; of course pre-service teachers make their own decisions while delivering the content in a way that they think appropriate. However, we wished to understand how (pre-service) teachers decide what approach is the best in delivering the topics and why they approached the same topic in different ways despite the fact that they drew on the same curriculum, and same textbooks and that they were members of the same preparation program. To gain insights into this issue, we examined teacher candidates' lesson plans and teaching notes as well as microteaching videos and interview transcripts. During this examination, we identified certain trends in the pre-service teachers' approaches and carried out retrospective interviews on their actions and decisions while delivering the content. In what follows, we provide data from a pre-service teacher's microteaching and retrospective interviews with her, and present our analyses with regard to the notion of voice.

THE DATA

In this section, we detail a pre-service teacher's (called Banu) introduction of derivative during her microteaching and our retrospective interview with her. We briefly describe how she introduced the concept of derivative in her microteaching.

Banu started her teaching by noting that the concept of derivative is important for it is used in such domains as banking, engineering, economy and physics (also gave a few examples). Then she began with, what she called, physical and geometrical interpretation of derivative. For the physical interpretation, she drew on distance equation (velocity multiplied by time) and calculated the instantaneous velocity at t=4 by evaluating average velocity over various intervals in the neighbourhood of 4. For the geometrical interpretation, she calculated the slope of tangent at a point on a curve by using the concept of limit. She then related these two interpretations to the formal definition of derivative, providing the algebraic expression of the concept. By using the formal definition of derivative, she explained how to calculate slope of a tangent line at x=2 on the graph of $y=x^2$. Following this she stated that there are rules to find the derivative of functions in short ways. She ended her teaching by assigning homework for students to search where and how derivative is used.

Following the microteaching we interviewed Banu to understand how she was prepared and to hear her reflections. We also interviewed Banu after our analysis of her microteaching videos. We provide here selective excerpts from our interviews insofar as they are related to our focus in this paper. As mentioned earlier, Banu started her lesson by explaining where derivative is used and gave several examples. We asked her purpose with this:

> Banu: Students always say, "we learn these but where we use them, what use they have?" and generally this is the breaking point from the topic taught. I thought explaining where it is used is important to motivate students. For example engineering students often say "we learnt derivative but only now (when they

apply it in various contexts) we understood why it is important and useful." To stress this I explained where it is used. My father and brother are engineers… my brother often uses derivative during his sketching in Autocad and asks me about it.

Banu's pedagogic decision of exemplifying the fields where derivative is of use is saturated with the others' voices: that of students questioning the use of this notion, students from engineering departments, and her brother. Other's voices shape her practice in that she felt the need to note the use of derivative in other fields. This pedagogic decision is related to Banu's vision on human nature and learning: learners should be motivated or else this might be a breaking point for learning.

While introducing derivative in her lesson, Banu said that she would obtain the definition of derivative from physical and geometrical interpretations. We asked why:

Banu: I tried to obtain the definitions (from physical and geometrical interpretations) because in textbooks I realised that these were treated as if they are completely unrelated (to each other and to the definition of derivative)… their approach is confusing… geometrical and physical interpretations are kind of introduction to derivative and I thought of obtaining the definition from these.

Banu is not only critical of textbook approach but also of the teachers in the schools where she observed as part of her teacher preparation program and noted, "In many schools teachers give rules of derivative before explaining geometrical and physical interpretation… like abstract mathematics… with proofs." She also stated that she looked at her university calculus notes which were the sources that inspired her approach. In approaching derivative through physical and geometrical interpretations, she was in fact interacting with the voices of many including at least those of the textbook writers existent in texts and those of her university lecturers existent in their teaching notes. Banu was recognising both textbooks' (and teachers she observed for that matter) and university lecturers' approaches to derivative, but while accepting the latter she was challenging and in fact dismissing the former. It is through this process (interacting with the voices of others) Banu was structuring her lesson.

An interesting incident occurred during Banu's microteaching following the presentation of the formal definition of derivative that she applied to calculate the derivative of x^2 at x=2. She told the class "later we won't be tussling all over these, there is a short way", and we asked why she mentioned about the short way:

Banu: In a private tutoring (of a student), while defining the derivative, the student asked if there isn't a short way of finding derivative. So (during the microteaching) I wanted to tell that students wouldn't need to tussle with all these (application of formal definition to find derivative of a function at a point), there are short ways to find derivative.

Despite the fact that she was not asked about the short way, she felt the need to mention in the very moment when she completed the application of formal definition. Banu was in a sense interacting (by responding) with a student's voice who was distant in space and time. Such an interaction reflects certain value judgements. For example, Banu saw the application of formal definition as a 'tussle' and hence by implication valued the short way (which refers to the rules of differentiation).

Banu also had her understanding of what elements are important in teaching mathematics and these were expressed in different occasions during our interviews; these are visualisation, concretising, moving from the simple to the complex, constructing definitions, and making connections. To Banu, visualisation is important because, she stated, "verbal expressions are difficult to visualise. It is still firmly in my mind what my lecturers did while getting the graphs sketched" and to realise this she "used graphs and tables because numbers stand still in the air". Concretising is also another element of her teaching mathematics because, as she stated, derivative "shouldn't remain as an abstract thing" and hence she said she gave "students homework to search and find concrete examples of its (derivative's) use in a field". Banu also believed that instruction should provide a transition from the simple to the complex and she reflected her understanding during her teaching by choosing "the simplest examples, which were doable immediately following the definition". Banu considered definitions as indispensable because "only then you can build something on them". Finally, for Banu, making connections is important while teaching:

> Banu: I tried to build up on their old, existing knowledge. Step by step. Wanted them to make connections with their old knowledge and thus construct new knowledge… Teach them to use what is known.

This has become especially evident while Banu explains her attempts to "get the students to sense the applicability of their knowledge of (slope) from analytic geometry to the interpretation of derivative". Certainly these elements reflect Banu's vision of how (mathematical) knowledge is learnt/reproduced and her interpretation of how teaching of mathematics should be done.

DISCUSSION

In our analysis of Banu's microteaching along with interview transcripts, it became evident to us that Banu had a conviction that her approach to teaching derivative was useful and better than at least textbooks' and those teachers' approaches that she observed in schools. Clearly she thinks that the way she structured her lesson and her formulation of derivative through "physical and geometrical interpretations" was appropriate to make this concept comprehensible to the learners. We do not put our value judgements (good or bad, appropriate or not) on her approach or on textbooks' here in this paper. But we suggest that her approach surely reflects her understanding of how to explain and teach derivative, and hence her PCK of derivative. This understanding involves, as evidenced with her practice and accounts, her subjective judgements as to what a successful teaching of mathematics requires: creating opportunities for students to visualise, connect the bits of old-knowledge, concretise as well as organising and giving examples from the simple to the complex and constructing definitions. This in fact points to a vision, Banu's vision, on learning and learner, on development and on reproduction of knowledge and hence closely related to her understanding of pedagogy of teaching mathematics. It is through this vision that content (i.e. introduction of derivative) and pedagogy are blended and hence that Banu's PCK is formed with terms and rules inserted into her discourse and practice (Davies, 1994) both in structuring and delivering the content: using tables and graphs for visualisation, eliciting what students know about average velocity and slope from analytic geometry for making connections with derivative, giving homework for students to exemplify the use of derivative in other fields for concretising, and application of the formal definition of derivative to a simple example as a start.

Banu's discourse and pedagogical decisions are, as our analysis suggests, saturated with the words and voices of others who are distant in space and time but whose voices ventriloquate through Banu's voice: those of earlier students, her brother who is an engineer, past teachers, university lecturers, friends/students from engineering departments, and authors of textbooks. However other voices do not simply enter into one's discourse and practice: they are selectively assimilated, that is, some voices are accepted and accentuated (i.e. appropriated) but some others are challenged and rejected. This is evident in Banu's teaching: while she was, for example, challenging and dismissing the voices of textbook writers and of those teachers who teach derivative in an abstract way privileging the algebraic rules of differentiation, she was appropriating the voices of, for instance, her university lecturer, those students who questions the merit of learning derivative and the one who asks if there is a shorter way of finding derivative other than applying the formal definition. Hence we contend that this selective assimilation of the others' voices constitutes an important part of teacher subjectivity of PCK with regard to the decisions on what are the most useful forms of content representation and on formulating the content in ways comprehensible to the students (as noted earlier content knowledge is also important in explaining this subjectivity but this is not our focus in this paper).

These voices also enact particular value judgements which, though not necessarily consciously, guide the practice of teaching and hence strongly shape the development of PCK (see also Gudmundsdottir, 1990). We see Banu's practice, decisions and discourse as imbued with value judgements: considering the application of formal definition to find derivative as 'tussles' and valuing the 'short-ways', considering the exemplification of the use of derivative as a necessity, attaching importance to the construction of the algebraic definition of derivative, valuing to "teach them [students] to use what is known". Her understanding of what a successful teaching of mathematics, as exemplified in our discussion hitherto, reflects further value judgements (e.g., valuing visualisation, concretisation, elicitation of old-knowledge from the students and so on), effects of which on Banu's formulation and representation of the content was all too apparent. On the basis of this, we claim that PCK cannot be isolated from value judgements which necessarily include personal views on what knowledge is worthwhile to learn, how it should be learnt, in what direction we should desire it and how the representations can be constructed (Giroux and Simon, 1988; Gudmundsdottir, 1990). Therefore we believe that value judgements are the cement through which content and pedagogy are blended and are the significant sources for the subjectivity of PCK.

These observations made us rethink the statements viewing PCK in relation to supplementing (pre-service) teachers with knowledge of students and learning, and knowledge of teaching (Manouchehri, 1997). These issues can all be viewed within the realm of pedagogy and we do not deny the importance of these. However, we should also be aware that even the inexperienced pre-service teachers are already imbued with their own understandings of learning and teaching, and hence of a pedagogy which they apply to the content to make it comprehensible to the others. Hence a critical question here is to what extent do we, as teacher educators, need to work with or against these understandings? We do not have a definitive answer to this question, yet we here suggest that impact of other voices in development of PCK is critical and we may need to create situations in which teacher candidates find opportunities to face those voices and value judgements with which they are instilled.

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DEVELOPING STUDENTS GEOMETRICAL THINKING THROUGH LINKING REPRESENTATIONS IN A DYNAMIC GEOMETRY ENVIRONMENT

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The paper draws on an experiment conducted in a secondary school mathematics classroom in Greece which aimed to explore ways in which students develop problem representations, reasoning and rigorous proof through building visual active representations. The paper introduces the meanings of Linking Visual Active Representations (LVAR) and Reflective Visual Reaction (RVR). Two distinct themes emerge from an analysis of the results: the relationship between: a) the building /design of the activities and the RVR by the students and b) LVAR in the activities and the development of geometric thinking. LVAR, RVR and van Hiele levels of the students are used as descriptors for the analysis.

RESEARCH REGARDING PROOF AND VAN HIELE

Many researchers have investigated the role of teaching and learning of proof in school, tackling it from different perspectives. Specifically they analyse the cognitive processes during the construction of proofs (e.g. Duval, 1991; Hanna, 1998) or the role of the teacher regarding the proof process (e.g. Bartolini Bussi & Mariotti, 1998). The impact of the computer technologies in the class regarding proof is another branch of the research (e.g. Schwartz & Yerushalmy, 1992) and more concretely another branch investigates which is the impact of the dynamic geometry software regarding the proving process (e.g Goldenberg, 1995; Olivero, 1999). Fuys, Geddes & Tischler (1988) support that dynamic manipulations help students to transit from the first to the second van Hiele level. The model of van Hiele in geometry has motivated considerable research. The van Hiele model provides a concise description of students' geometric concept development and consequently their status and progress (i.e Fuys et al. 1988). The van Hiele theory distinguishes five different levels of thought which are level 1 (recognition or visualization), level 2 (analysis), level 3 (ordering), level 4 (deduction) and level 5 (rigor) (Fuys et al., 1988). Fuys et al. (ibid.) specified that to be "on a level" students had to consistently exhibit behaviours indicative of that level. The level 3 (deduction) is identified as the level which is connected with the construction of the "if ...then" statements and consequently with proof (e.g Gawlick, 2005). The introduction of new *representational infrastructures* (Kaput, Noss, and Hoyles 2002, p.2) such as dynamic geometry systems in the teaching and learning process makes it necessary to investigate the way in which students create mathematics and support reasoning. By way of a brief overview, many researchers have conducted studies and concluded that students who used the Sketchpad displayed: achieved significantly higher scores on a test containing concepts [e.g Dixon (1996), who concludes that students who were taught about the concepts of reflection and rotation in a GSP environment significantly outperformed their peers who had received traditional instruction in the content measures of these concepts]; achieved significantly higher scores between the pre- and post-tests [e.g Yousef (1997); Almeqdadi (2000)]. Researchers around the world concur in the view that greater emphasis should be put on activities in software that actively involve students. As Gawlick (2005, p.362) declares "progression through these levels will not occur all by itself, but needs to be triggered by giving the students suitable tasks that really afford the building of new concepts". The role of the designer of the activities in the DGS environment - taking the designer to be the teacher in his/her professional practice - is crucial as s/he can develop the connection of interaction techniques in the computer software by evolving mathematical problem-solving techniques. The rationale underlying the design of activities/ problems for a computer environment is therefore crucial if students' transition to higher level is to be facilitated. Consequently, it becomes an object of discussion in the next section.

THE DESIGN OF THE PROBLEM IN THE DGS ENVIRONMENT

The problem that was explored by the pupils in our paper is a revision of the pirate problem, which has been conceived by Russian George Gamow (1948, reprinted 1988), enriched with historical evidence by the researcher in order to motivate the pupils to be more interested in the ancient history through the math class. Gamow proposes a problem suggested by a treasure map found in a grandfather's attic: "In the Odyssey, Homer (c74-77) mentions that the pirates also raided Greek islands. The pirate in our story has buried his treasure on the Greek island of Thasos and noted its location on an old parchment. "You walk directly from the flag (point F) to the palm tree (point P), counting your paces as you walk. Then turn a quarter of a circle to the right and go to the same number of paces. When you reach the end, put a stick in the ground (point K). Return to the flag and walk directly to the oak tree (point O), again counting your paces and turning a quarter of a circle to the left and going the same number of paces. Put another stick in the ground (point L). The treasure is buried in the middle of the distance of the two sticks (point T)."(Figure 1) After some years the flag was destroyed and the treasure could not be found through the location of the flag. Can you find the treasure now or is it impossible?" The problem has attracted many researchers (for example Scher (2003) who designed the activity in multiple pages making interactive constructions by using the Geometer's Sketchpad. The proof usually involves complex numbers or algebra /geometry. In our case we will use Geometer's Sketchpad v4 and a geometry framework as medium for investigation and productive reasoning. For this reason, the researcher (S.P.) designed the multiple pages of the software using interaction techniques such as 'hide/show action buttons' or 'link buttons'. For example, a pre-designed hide/show action button gives the user the choice/ability to hide the flag and, as a consequence, point F. This action results in the simultaneous disappearance of the lines which could lead to the treasure point T. The researcher took care so that point T remained on screen to assist the students in their investigation of the problem. Sedig, Rowhani and Liang (2005, p.422) pinpoint that the interaction with representations in a computing environment has two aspects: the action upon a representation by the user through the intermediary of a human-computer interface, and the representation communicating back through some form of reaction or response. In the present problem, the interaction technique that affected most crucially the way students thought was the rotation command. The rotation technique "has been referred to as direct concept manipulation, as opposed to direct object manipulation" (Sedig and Sumner 2006, p.35). The researcher also took into account the theories referred to in the next paragraph relating to knowledge, learning and teaching during the design/construction and implementation of the activities.

The complex process of instrumental genesis takes place into a class of students who have the same objective and are processing/manipulating the same tools in order to accomplish a goal. Guin and Trouche (1999) characterize the dual interacting process of instrumental genesis, distinguishing in two processes: the tool affects and shapes the users' thought (instrumentation process); the tool is shaped by the user (instrumentalization process). During the instrumental genesis both the phases (instrumentation and instrumentalization) coexist and interact. Then the user structures that Rabardel calls utilization schemes or using schemes of the tool/artefact. Utilization schemes are the mental schemes that organize the activity though the tool/artefact (Trouche, 2004).

For example, the process of rotating a segment by specifying a mark angle of 90^o and marking point F as the centre results in a segment the same length as the original segment but rotated through 90^o (it is vertical to the initial segment). Any effort to modify the length of the original segment by dragging its endpoints will result in an equivalent modification of the dependant rotated segment due to transformation (in our case, rotation). This transformation has a significant impact: during the instrumental approach, the student structures a utilization scheme of the tool, and consequently a mental image of the functional/operational process of rotation, since any modification/ transformation of the initial segment (input) results in the modification /transformation of the final segment (output). The rotation of the segment in the software leads the students to conceptually grasp the meaning of a) perpendicularity/a right angle; b) the isosceles and right triangle.

The researcher had verbally linked questions to the transformations of the shape. She took care to highlight the equal segments or the equal triangles with the same colour, which would appear in the students' correct answer when pressing a hide-show button. In this way, the sequence of increasingly sophisticated construction steps in the activity could correspond to the numbering of the action buttons, allowing the student to interact with the tool on his/her own volition or on being encouraged to do so by the teacher during the time allotted for the activity. This leads to a cognitive linking of the representations which (Kaput, 1989) "creates a whole that is more than the sum of its parts...It enables us to see complex ideas…" Building on the above-reviewed theoretical background and new views, the meanings of Linking Visual Active Representations, and Reflective Visual Reaction during a dynamic geometry problem solving, is introduced / defined in the present study as follows:

Linking Visual Active Representations are the successive phases of the dynamic representations of the problem which link the constructional representational integrated steps of the problem in order to reveal an ever increasing constructive complexity; since the representations build on what has come before, each one is more complex, sophisticated and integrated than its predecessors, thanks to the student's (or teacher's, in a half-preconstructed activity) choice of interaction techniques during the problem-solving process, aiming to externalize the transformational steps they have visualized mentally (or exist in their mind).

Reflective Visual Reaction is that reaction which is based on a reflective mode of thought, derived from interaction with LVAR in the software, thus facilitating the comprehension and integration of mathematical meanings.

RESEARCH METHODOLOGY

This research sought to investigate the effects of dynamic geometry software on secondary students' representations, reasoning and problem-solving. The qualitative study was conducted in a class at a public high school in Athens, and involved 65 students aged 15-16 during the second term of the academic year. The first step was to examine student's level of geometric thought according to the van Hiele model using the test developed by Usiskin at the University of Chicago (Usiskin 1982). Twenty eight volunteers were randomly divided between the 'experimental' and the 'control' teams, with 14 students in each. The researcher ensured that both teams consisted of equal number of: students at levels 1 to 3 respectively; boys and girls. The students were friends, which fostered group discussion. The experimental team had participated in 3 previous sequential phases consisting of instructional / interactional sessions with the geometry software before exploring the concrete problem. During the inquiring process the researcher acted as a co-actor in/co-ordinator of the activity, assuming the role of the teacher, coordinating the discourse and leading pupils towards the Euclidean proof. The experimental sessions were videotaped. The analysis of the results that follows is based on observations in class and of the video. Later sections present the sessions of the solution process the students undertook. One pair's session from the experimental team interplayed with the linking pages LVAR. An orchestration with most students of the experimental team interplayed with LVAR to solve the problem with rigorous proof. It is also important to examine how students' extant knowledge affected the results of the research. As Kaput (1998, p.7) claims "Just because representations are linked, if there is no connection to other knowledge in the learner's schema / experience the linked representations are just as meaningless". Within this theoretical framework, two significant research questions are posed:

- To what degree can LVAR contribute to the students constructing rigorous Euclidean proofs?

- How important is pre-existing knowledge of the theorems to problem solving?

We will now present the most important instants from the peer session. Later sections present the sessions of the solution process the students undertook. One pair's session from the experimental team interplayed with LVAR and a teaching session with most students of the experimental team interplayed with LVAR to solve the problem with rigourous proof. Below, the points in the dialogues at which the researcher observed the RVR will be noted, along with those points at which the students constructed a utilization schema. In the dialogues, the phrases marked in bold are indicative of the students' levels.

A: an excerpt from dialogue recorded during the pair session

M_1 is a male pupil (van Hiele level: 1 at the pre-test) and M_2 is a female pupil (van Hiele level: 2 at the pre-test) tested about 4 months ago. The students had dragged point F so that FF´ was perpendicular to PQ. (figure 2)

M_1: **if** we prove that these are parallel lines (points to QK, SL) **then** the quadrilateral is a parallelogram **because** these are equal (**RVR**), **so** the diagonals will be intersected, **so** the diagonals will be dichotomized.

M_2: we must prove that these are **right triangles** (points out the triangles PFF', OSL ...) …**but they are (right)** (**utilization schema**) because we have constructed them with the rotation. (**RVR**)

B: an excerpt from dialogue recorded during the teaching session

The teaching session was designed to record how students, who had never come into contact with the problem in question, would react while interacting with the multiple pages and interaction techniques. The researcher's aim was in this way to adapt the experiment to real classroom conditions. Pupils M_7 (van Hiele level 2) and M_8 (van Hiele level 2) had participated in the solution of other problems, but had never been faced with the pirate problem until the researcher presented it in class in the software's multiple linking pages. In fact, it transpired that it was these students who had the highest reaction. (Figure 3)

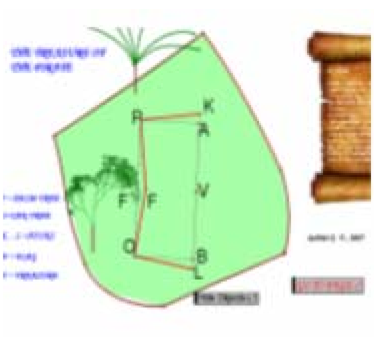

Figure 1

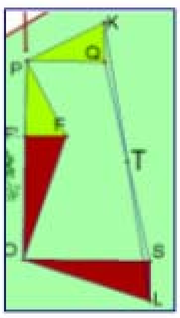

Figure 2

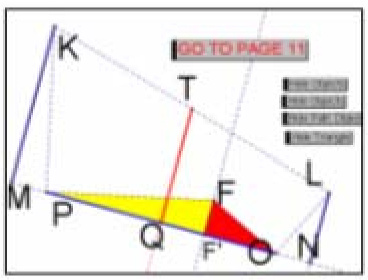

Figure 3

M_7: the segments MK and PF´ are equal, **because** the triangles MKP και FF´P are congruent **as they are right triangles**, they have KP = PF , and angle < MKP is equal to <FPF´ angle – **because** <KPF´ angle is external to the triangle MKP

so it is constituted from an angle of 90^o and the angle <FPF′ **so** it is equal with the opposite angles <(MKP+ 90^o)

M_8: the quadrilateral KMNL is a trapezium and TQ is equal with the sum (KM + LN)/2. **But** KM + LN= PF′+F′O =PO, **so** the segment TQ is the half of the segment PO.

CONCLUSIONS

The LVAR on multiple pages helped students to react instantaneously and connect their thoughts. Consequently, an inductive mental procedure led them to a productive reasoning. The transformations which occurred due to techniques had a significant impact: during the instrumental approach, the student structured utilization schemes of the tools, and consequently mental images of the operational processes, since any modification/transformation of the initial figure (input) resulted in the modification/transformation of the final figure (output). In the dialogues, the phrases marked in bold are indicative of the students' levels. An observation of their answers would indicate that the software has helped the students answer at a "higher level" than that indicated by the van Hiele test. For example: a) M_1's logical sentence includes a hypothesis and concludes in a semperasma (Greek word meaning "thought which is formulated from new evidence": terminology used during a Euclidean proof) which will additionally be a new hypothesis. This is a complex way of thinking that leads us to conjecture that his van Hiele level is in transition to a higher level; b) students M_7 and M_8 would seem to belong on level 4, since they reached conclusions on the problem by correlating the theorems they already know and they were "starting to develop longer sequences of statements and beginning to understand the significance of deduction" (de Villiers, 2004). During the last session, the pupils used theorems of relevance to trapeziums. In the pair-work session, the pupils hadn't used the same approach to the using the related theorems, because they were unfamiliar with the relative concepts (and the activities were not oriented to construct the relative meanings). We can therefore conclude that prior knowledge is very important for this type of teaching, which includes a problem-solving situation. LVAR play a significant role in developing pupil understanding and reasoning is clear from the fact that students demonstrated a shift from visual to formal proof during the experiment. The RVR led students to formulate "if …then" propositions, thanks to the instrumental genesis evoked through the research process.

Consequently: a pupil can develop his/her level of knowledge by proceeding through increasingly complex, sophisticated and integrated figures and visualizations to a more complex linked representation of problem, and thereby moving instantaneously between two successive Linking Visual Active Representations only by means of mental consideration, without returning to previous representations to reorganize his/her thoughts.

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PROBLEM POSING STRATEGIES OF FIRST YEAR MATHEMATICS STUDENTS

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In our paper we report results about first year university students' strategies to pose sequence problems. The analysis shows that on meta-level the students follow a formulate-improve-solve cycle that also fits their generation process in a generate-test view of creativity. On the strategic level, students appeal to a problem-type in order to bootstrap their posing process and then try to modify an initial formulation such to comply with some self or task imposed criteria. The commonly used techniques are domain specific ones. Another finding is that in some cases techniques are employed algorithmically leading to ill-formulated problems. An especially difficult point is the question of the problem: students don't seem to have experience to search for interesting questions in a given situation.

INTRODUCTION

For mathematicians problem posing refers to the process by which they obtain a problem that has not been solved yet by anyone. In most empirical studies, though, problem posing means the formulation of novel problems with the solution unknown at least for its creator (Van den Heuval-Panhuizen et al., 1995). In other contexts it is understood as reformulation of an existing problem (Cohen & Stover, 1981), mostly ill-defined ones. In order to define what classroom problem posing is we need to consider the nature of the mathematical knowledge, but also the way in which this is structured for classroom activities. Ervynck (1991) described the formal theory of mathematics as a "framework consisting of definitions of concepts and relations between defined concepts, the latter being of a very particular kind: the relations emerge from the implementation of very strictly prescribed (deductive) rules". On other hand, Brinkmann (2005) claims that relations between mathematical objects around a topic may be graphically represented by mind and concept maps in a way that corresponds to the structure of mathematics. Therefore, we describe problem posing, in a particular topic, as the formulation of questions about 1) the existence of a mathematical object; 2) the relation between different mathematical objects; 3) new properties of a given object deduced or related to a set of specified properties. Obviously, not any question leads to a problem. We consider only the specification of Pólya (1967) who characterized a problem as having an intrinsic difficulty that differentiate it from an exercise. Therefore, we consider the result of the posing process as a problem if the recall of factual knowledge is not enough to answer it.

Research on problem posing can be described along three lines of studies. First, there is an inquiry on the relation between problem posing and mathematical understanding; second, studies concern the development of student's abilities and the posing processes, and the third line is on classification of problem posing tasks.

As research, our analysis fits in the second category. On this line of research, English (1997a, 1997b, 1998) presented results regarding third, fifth and seventh graders abilities to pose problems. Her major conclusions were that problem posing ability is related, on one hand, to the perception of the problem structure as independent of a particular context and, on other hand, to the ability to focus on semantic structure of the problem. Silver & Cai (1996) conducted a study with sixth and seventh grade students who had to pose problems to a given story problem. They found that, in this situation, students used association in order to obtain new questions based on already formulated ones. Christou *et al.* (2005) studied the cognitive processes involved in problem posing for semi- and structured situations. Semi-structured situations were described by Stoyanova (1998) as ones in which students need to formulate problems based on specific diagrams, meanwhile in structured ones students need to reformulate already solved problems. However, none of the above studies have concentrated strictly on the strategies students use in problem posing.

Problem posing strategies

The term strategy is used here as defined by Campistrous & Rizo (2000, cited in Cruz, 2006) for problem solving and it is a "generalized procedure that is made up by schemes of actions whose content is not a specific one, but a general content, applicable in situations of different contents, which someone uses to orient himself to situations in which he or she does not have an 'ad hoc' procedure and on its bases he or she decides and controls the course of the action of finding the solution" (p. 8). Thus, a strategy is a general way of accomplish the problem posing task. Brown & Walter (1990) propose a five stage procedure for obtaining new problems from given ones: 1) choosing a starting point; 2) listing attributes; 3) what-if-not; 4) question asking and 5) analysing the problem. The "what-if-not" phase refers to techniques used to modify objects or properties by replacing them with more general or restricted ones. Cruz (2006) takes further this idea and proposes a mathematical problem posing strategy consisting of six non-sequentially connected actions (Figure 1).

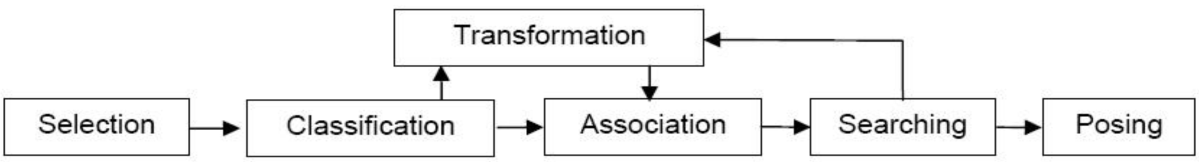

Figure 1. A mathematical problem-posing strategy (Cruz, 2006).

Selection refers to the action of determining what kind of mathematical object is proper for the purpose of the task. Classification refers to an analysis and synthesis of the object's components. During the following action, transformation, an object can be completely or partially changed, meanwhile during the association concepts, similar under some criteria are joined with the current object. The search consists from searching for dependencies between the newly aggregated concepts. The problem posing occurs once a decision is made with regard to the question to be selected from a set of possible questions. The different techniques that help to pose a new problem are included in the transformation and association actions as specific tools.

It is interesting, and it is our aim, to see how freshmen problem posing strategies fit into the above general view. Do they have some more specific strategy or all the six actions can be identified? We are also interested by the criteria students' use to choose their questions about the objects and to fit partial results with their conception of some problem characteristics.

In the following we describe our methodology, the analysis of the student's process and, then, the patterns in students' strategy by analysing some particular papers. We finish by concluding remarks.

METHODOLOGY

Subjects

In the present study, 18 first year mathematics students completed a problem posing task. Students were of 18-20 years old and entered to university after completing an admission exam. None of the students has been subject of training in problem posing, however it is possible that some of them would have competed in Olympiads during their high school studies.

Task

Students had to generate three sequence problems (as home assignment task) such that to have an easy, one of average difficulty and a difficult problem. They had a week at their disposal to finish; at the end, they responded a questionnaire regarding their problem posing process. It was requested to hand in not only the final formulations, but also the scratch work. The questions were about the following aspects of the problem posing process: the existence of an initial idea (for each problem of different difficulty), change of the idea during generation, problem types from which to start the generation process, a theorem or generalization as from where to trigger the problem posing process and difficulty criteria they used.

ANALYSIS OF THE PROCESS

Student's responses concerning the process of generation revealed some interesting things. First, 13 from the 18 students said that they had in mind some type of sequence problem and tried to formulate problems that fit that type. For the used problem types they reported arithmetic and geometric progression and second order linear recurrence. We also observed that the techniques they employ to modify the problems are, in most cases, typical for that problem type. Such techniques can be learned, mostly, from problem solving sessions. This situation suggests that students' notion of problem type is close to what is called in mathematics education literature as problem-type schemata. According to Bernardo (1994), "knowledge about problem categories includes information about the relevant underlying principles, concepts, relations, procedures, rules, operations and so on". Thinking in structures has the advantage that the techniques preserve the problem type, but in the same time it can be a disadvantage, especially when the students tried to build more difficult problems (some examples will be given in the next section). However, student's success in building problems relates to their expertise, manifested in their meta-cognition. This thinking skill is the knowledge used in planning, monitoring, controlling, selecting and evaluating cognitive activities (Bernardo, 1997). Our results suggest that some of the students can successfully employ the problem-type specific techniques, however in their intent to make a more difficult problem they do not have enough control of modifications (meta-cognition) and they fail. This is an interesting result, for two reasons. First, it suggests that problem posing can be quite algorithmic in its nature, even when it leads to new (and potentially difficult) problems. In the same time, the techniques employed by students need not to be very complicated ones, instead it is enough (or maybe necessary) to have a good control of them. Second, it tells us that the relation between problem posing ability and student's understanding is quite complex: a resulting good problem is not yet a proof of a deep understanding of a topic. It has to be remarked that, in the special case of our students, the assimilation of problem schemata in calculus it is facilitated even by the organization of the text- and problem books, that often present this topic in sections of problem types.

The second group of students (3 from 18) started from a concrete problem (or result) and generalized it. We underline the fact that students make a clear distinction between problem type and a concrete problem from which to start to generalize. Interestingly enough, students' conceptions of what it means to generalize seem to be very different: one sees generalization as omission of some required data in the problem (but not as release of some property of an object), meanwhile the other, in fact, applies a rule for composing new convergent sequences. This suggest that students have problems with generalizations starting from an adequate understanding of the notion, but even having a kind of interpretation for it, they have problems in its application as a way for obtaining new problems (with increased difficulty).

The third group, consisting of only two students, claimed that they did not started from any particular element nor did they have any initial idea of what they will produce at the end. These cases are especially interesting. One could ask if it is really possible not to start from anything or, rather, the situation is that these students do not ascribe their thinking to these categories. By analysing their problems, one can see that one started from a type of problem, however not frequent in textbooks. He asked questions about the existence of a particular sequence having some specified properties. More interestingly, he said that he built the second problem by asking questions about the first one. Such a generation process has a lot of common with creative thinking, since needs to explore in detail the object one deals with. The second, although says that had not thought of anything in particular, wrote down that he had in mind to make something related to convex sequences. We claim that this vague initial idea shaped his way of thinking of the problem posing process, such that then he was trying to make a connection between convex function values and sequences. His technique can be interpreted as particularization, a domain-general procedure that in some cases asks for transfer from one domain to another. We consider that, in all, the domain-general techniques are available to students with expert knowledge, meanwhile others prefer to use problem-type schemata that can work (in many cases) as a recipe, guaranteed to succeed, for making new problems.

As conclusion to this part we argue that students' strategy to generate problems can be described by a simpler model than the one proposed by Cruz (2006); students seem to make an initial selection, some transformation and posing. This is not to say that Cruz's model is not valid, we do think it is, rather that students expertise level combined with the specific task they had to perform lead to an overall more simplified strategy use.

ANALYSIS PARTICULAR CASES

In this section we give examples for the above exposed ideas by presenting some of the students' proposed problems.

Starting points in generating problems

Some students start from a particular problem type to generate all three problems, case in which they apply their difficulty criteria to guide the generation process. We present the case of Ana.

1. Let $(a_n)_n$ be the sequence defined by: $a_1 = 6, a_2 = 72$, and $a_{n+1} = 12a_n - 36a_{n-1}$, for $n \geq 3$. Determine a formula for the general term of the sequence.
2. Consider the matrix $D = \begin{bmatrix} 5 & 4 \\ -4 & -3 \end{bmatrix}$. Compute D^n.
3. Consider $(a_n)_n$ defined by $a_1 = 12, a_2 = 288$, and $a_{n+1} = 24a_n - 144a_{n-1}$, for $n \geq 3$. Compute $b_n = \sum_1^n a_k$ and decide its monotony and convergence.

Ana said, she thought of Weierstrass' theorem and, from the beginning, she had in mind second order linear recurrences. The first problem was formulated so "that any student having basic knowledge of the topic can solve it". By "basic knowledge" Ana means the algorithm for determining the general term of a sequence defined by a linear, second order recurrence. For the other two problems, Ana used the same type of problem as starting point, i.e. linear, second order recurrences. Note that, in all three problems, the recurrences lead to a characteristic equation with one double root.

Other students also start from a problem-type, but different for each difficulty level. For example, Ciprian generated the following problem for the difficult one:

3. Let $(a_n)_n$ be the sequence defined by $a_{n+1} = 2a_n - a_{n-1} + n! \cdot n$ with $a_0 = 0, a_1 = 1$. Prove that $a_n - 153 | 720$, for $n \geq 6$, then compute $\lim_{n \to \infty} \frac{1}{n} \sum_1^n \frac{\sqrt[k]{1! \cdot 2! \cdot \ldots k!}}{a_k}$.

Ciprian claim that for his third problem he thought of "problems like in Olympiads" and starts from the particular the sequence: $a_n = 1!+2!+...+n!$. His explanation of its own problem posing process is: "We start with the sequence $a_n = 1!+2!+...+n!$ and we find a recurrence between a_{n-1}, a_n, and a_{n+1}. Then, we use the inequality between arithmetical and geometrical means for the terms of the sum $1!+2!+...+n!$." He added that, from the beginning, he wanted to use the "ham-and-sandwich" and Stoltz-Cesaro's theorem. This case is also illustrative for the fact that one can obtain interesting problems by proper handle of simple techniques.

Cosmin has a similar idea: for the generation of the most difficult problem, he starts from the alternated harmonic series: $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+...+(-1)^{n+1}\frac{1}{n}$ that he rewrites as a recurrence relation between two consecutive terms.

Arguments regarding the difficulty of problems

Although Ana's three problems were proposed by starting from the same problem type, they have, in her opinion, different difficulties. For example, she said about the second problem "this is a problem of medium difficulty, because it uses more complex notions than the first one". Explaining further, she said: "in order to get to sequences in this problem, I would ask the students to compute the matrix $D^{n+1} = D^n \cdot D$". The difference in difficulty for Ana's problems comes from the usage of "the form factor" (Pelczer & Gamboa, 2006). The increase in difficulty is given, in Ana's vision, by the addition of new requests, regardless of the solving techniques needed in the problem.

Cristian had similar view when he proposed the following problem as a difficult one:

> Study the convergence of the sequence (a_n), defined by $a_{n+1} = \sin(a_n) + \cos(a_{n-1})$, $n \geq 3$ and $a_1 = a_2 = 1$.

He argued "I thought of trigonometric functions, in order to make the computation really hard and to determine the solver to use inspiration". As an irony of fate, Cristian did not solve this problem!

Andreea, in turn, has a different perception concerning the difficulty of problems. She stated: "the quality and quantity of the given information generate the difficulty; the fewer given data, the more difficult the problem is".

In order to increase the difficulty of his proposal, Mihai introduced parameters in the recurrence relations, and asked for a discussion over the convergence nature of the sequence, as a function of parameters used.

Difficulty criteria was important in this task, since it acts as a guiding line in choosing proper start points and techniques tog et to the final form of the problem.

Strategies in formulating the conclusion

In Ana's first problem, the first two members of the sequence are $a_1 = 6, a_2 = 72$. These numbers are not randomly chosen, as the general term of the sequence can be expressed by the "nice" formula $a_n = n \cdot 6^n$. A more experimented student maybe would have formulated some of these problems differently. For example, instead of the sequence (b_n) in the third problem (obviously divergent), it would have been more inspired to ask for the convergence of the sequence (c_n), where $c_n = \sum_{1}^{n} \frac{1}{a_k}$.

Let's consider Cosmin's case. For the difficult problem he proposed:

> Let (a_n) be the sequence defined by $a_{n+1} = a_n + a_{n-1} (\mathrm{mod}\ 100)$, where $a_0 = 0$ and $a_1 = 1$. Prove that the given sequence is periodical.

One with experience would have immediately noticed that, using this formulation, the problem's proof, based on the pigeonhole principle, is obvious. It would have been more interesting to ask how many different values we could find among the terms of this sequence, or which its main period would be, and so on.

The above situation is common among students and tells us that asking the right questions is a tough task itself, but also that problem posing has bottlenecks in each part of it: start point, techniques and question posing.

CONCLUSION

We analysed first year university students' problem posing from strategic point of view. Results suggest that most students start the generation from a specific problem type and then try to transform it such to comply with some self or task-imposed criteria. In that sense they appeal to a simplified strategy that consists of *selection-transformation-posing*. In their transformation most of them use domain-specific techniques, abstracted, learned or memorized during problem solving sessions. Just few of them appeal to more general techniques or methods such as generalization, paraphrasing, analogy or combination of elements. In general, their views on problem difficulty are related to problem or question type, knowledge or data necessary for solving the problem so that often the three generated problems are of the same type and only parts of it modified. We also found that even when disposing of techniques, often very effective ones, they do not have strong control of its use: when and how to employ it and how close they got to what they wanted to obtain. It seems that many of them have problems with choosing the question from a set of possible ones, maybe because they don't realize the availability of other ones or because they can't judge beforehand the way in which the question impacts on the overall quality of the problem. As a final remark, we observe that quality of the final problem is not strictly related to the start point: it is possible to obtain an interesting problem by the successive transformations to an initially simple problem, but also it is possible to get an uninteresting problem from a good start point by not asking the right question.

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WANTED - ONE GREAT MATHS TEACHER!

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What makes an effective mathematics teacher? Research has identified a number of attributes but most of these are noted from an adults' point of view. What is it that the students consider essential in an effective mathematics teacher? Here a picture of the ideal mathematics teacher as described by a group of 11-12 year old students is presented. The attributes they identified fell into two categories, one addressing factors associated with personal attributes of the teacher and the second factors relating to the learning environment the students saw as most supportive to their mathematical learning.

INTRODUCTION

The response of the New Zealand government to the Third International Mathematics and Science Study was the implementation of the Numeracy Development Project [NDP]. This initiative addressed the underachievement in mathematics through teacher professional development. The NDP has succeeded in improving the level of mathematics for all children although the gap between the higher and lower achievers is still a concern (Young-Loveridge, 2007).

In politics and education it is invariably the adults voice that is heard, however much of what is decided impacts on children. The extent to which children's voices are heard is limited. This paper seeks to address this imbalance by investigating the views on effective teachers with a group of 11 and 12 year old students. The justification for listening to children's voices can be seen in the United Nations [UN] declaration on Human Rights which states that children should be given a voice in matters that have an impact on them (New Zealand Ministry of Foreign Affairs & Trade, 1997). Similarly Bishop (2003) argues that children need to be involved in the exploration of teaching and learning issues if a long-term solution is to be found.

Drawing on literature and professional expertise the New Zealand Teachers Council [NZTC] (2007) identifies criteria for teachers' registration that must be considered as foundation criteria for effective teachers of mathematics, but to date the adults voice has been privileged. Do teachers match what students' want in a teacher? This study looks at what a group of 11 and 12 year old students consider the ideal mathematics teacher to be like.

THEORETICAL BACKGROUND

Judgements about teachers and aspects of teaching are often heard from groups such as government agencies, school communities, parents and the child in the classroom. The New Zealand Teacher Registration Board identifies criteria that indicate an individual is fit to be a teacher. Four of the criteria identified by the registration board that link to teachers personal behaviour include; recognise and respect others, support and inspire others in their work, generate excitement and satisfaction in learning, show respect for learning and inspire a love of learning (New Zealand Teachers Council, 2007).

As the criteria above are considered important for teacher registration they must also be considered essential in the making of an effective teacher of mathematics. A recent Ministry of Education funded study – Effective Pedagogy in Mathematics / Pangarau: Best Evidence Synthesis Iteration (Anthony & Walshaw, 2007) drew on over 800 national and international research studies with one of the findings being that effective teachers of mathematics were those who enhanced both social and academic outcomes for their students, were committed to teaching and took students' mathematical thinking seriously. These attributes support the NZTC criteria of showing respect for others and for learning and inspiring a love of learning.

If a teacher is to inspire and excite their students in mathematical learning (NZTC, 2007) they themselves need a positive attitude towards mathematics. Recent research relating to the New Zealand Numeracy Project found that students were more successful when taught by teachers with a positive attitude, as this built positive attitudes in their students (Irwin & Irwin, 2005). In Brown and McIntyres' (1993) study involving hundreds of responses from 12 – 14 year olds about what made an effective teacher, students recognised that the positive attitude of the teacher motivated them to learn and encouraged them to achieve.

Another attribute of an effective teacher identified in the NZTC criteria is one involving respect. Anthony and Walshaw (2007) stated that effective teachers of mathematics were those who worked at developing partnerships in learning where students' ideas were received with respect. Pasifika[1] culture fosters a respect for adults, and teachers have a right to be respected, although traditionally this was not necessarily reciprocal. Secondary school aged Pasifika students (13 – 18 years old) in Clark's (2001) and Jones's (1991) studies would not ask a teacher questions if they did not understand what had been taught as they considered it disrespectful to show that you had not understood something the teacher had told you. The Pasifika secondary school students in Hill and Hawks' (2000) study however noted that effective teachers were those where respect was a two way process.

As well as the personal attributes noted above an effective teacher needs to provide a successful learning environment. The mathematics classroom is a community where teacher/student and student/student interactions happen every day. Anthony & Walshaw (2007) outlined a successful learning environment as one where students are encouraged to participate in the discussion and learning, and where provided with an opportunity to work co-operatively with peers. A Kershner and Pointon (2000) study involving 70 children aged between six and nine supported the use of a co-

[1] In New Zealand Pasifika students are those who identify themselves with, or were born in the island nations of the Cook Islands, Fiji, Niue, Samoa, Tokelau and Tonga.

operative learning approach as helpful to learning mathematics. In contrast the Pasifika students in Clark's (2001) and Jones's (1991) studies noted this approach as unpopular yet Pasifika students in Hill and Hawks' (2000) study recognised effective teachers were those who encouraged students to work in pairs or small groups.

Traditionally teachers have been seen as the authority figure, a person whose post in the community demanded respect. The teacher was the source of all knowledge and it was the students' role to acquire this knowledge. Recent research (Anthony & Walshaw, 2007) identifies the effective mathematics teachers as someone who engages students in the learning process, challenging their thinking and encouraging them to take ownership of their own learning. Kershner and Pointon (2000) demonstrated that students as young as six can identify practices that support learning. This paper presents the attributes for the ideal mathematics teachers as described by a group of 11 and 12 year old students.

THIS STUDY

Eighteen Pasifika students from a large co-educational, urban intermediate school (11 – 12 year olds) shared their perceptions about what makes the ideal mathematics teacher.[2] The school was located in a low socio-economic area with the school population representative of twenty-two different ethnic groups. Pasifika students made up 31% of this school roll. The school employed a mathematics specialist teacher who worked with both teachers and students to improve mathematical skills throughout the school. Mathematics specialist teachers are the norm in New Zealand secondary schools although not common in primary or intermediate schools. The mathematics specialist teacher in this school had continued her education to a Masters level with mathematics education and assessment the focus of her study. She was a mathematics teacher more in line with New Zealand secondary school systems although her training and background experience was more in line with the primary school system where one teacher teaches all subjects.

Students were chosen to attend special mathematics classes at the beginning of the school year through the use of The New Zealand Progressive Achievement Test. During the year the students' chosen would attend either an extension class or a class providing extra help for two of the four school terms, both classes taught by the mathematics specialist teacher. The participants involved in the study were the top nine Pasifika students attending the extension class and the bottom nine Pasifika students attending the class requiring help. Of these students fourteen had been born in New Zealand but all eighteen were entirely schooled in New Zealand. The group consisted of ten females and eight males. For the two terms where students did not attend the mathematics specialist class they were included in their homeroom mathematics programme, taught by their homeroom teacher. This provided students with two very different environments for learning mathematics

[2] Pasifika students make up to 21% of those attending Primary and Intermediates schools in the Auckland region (New Zealand).

to draw on when identifying their ideal teacher. One where they had an enthusiastic, constructivist, mathematics specialist teacher, the other a more traditional teacher with a multi subject focus.

Initially students participated in semi-structured individual interviews where they presented their ideas about the ideal mathematics-learning environment. Students were asked to identify practices they considered important for the successful learning of mathematics and also what they considered the ideal mathematics teacher would be like. Once all individual interviews were completed students were observed working in the mathematics specialist class then two group interviews were conducted, one with the higher achievers and one with the lower achievers. The group interviews provided an opportunity for the groups to share information from the individual interviews and for the researcher to clarify the meaning behind some of the students' responses. During the next stage of the group interview students were given the scenario that a school principal was considering employing a mathematics specialist teacher. The principal had approached them to help identify criteria that could be used when interviewing prospective teachers. Here both groups of students had the information from all the individual interviews to base their ideas on.

RESULTS AND DISCUSSION

Students in this study, as did those in Kershner and Pointons' (2000) study held definite ideas about what they thought the ideal mathematics teacher would be like. Both groups (higher and lower achievers) held similar ideas about what made the ideal teacher. The quotes presented here are indicative of a number of statements made by students when discussing attributes that both groups agreed important enough to be included in either an advertisement or interview for a mathematics specialist teacher. These criteria fell into two categories; personal attributes and the learning environment they wished the teacher to provide.

Students saw the ideal teacher as a person who could influence how they felt about mathematics. They wanted someone who not only took their job of teaching mathematics seriously, but someone who also enjoyed mathematics – someone who would inspire them. The Pasifika students in this study stated that if their teacher appeared to enjoy mathematics they would be more inclined to participate in the subject themselves, "They [the teacher] need to inspire you, they need to look like they are enjoying maths cause then you [the student] will like maths too, cause you like your teacher". The students recognised that what the teacher did influenced how they perceived mathematics. They wanted someone who could inspire them, criteria noted by the NZTC (2007).

The 11 and 12 year old students in this study saw the ideal teacher as one they could interact with, from discussing rugby, "Hey Miss did you see the game on Saturday, Hewitt was pretty awesome ay?" to sharing jokes or discussing strategies used and possible solutions for a given mathematics problem "but Miss couldn't you also have done it …". They saw themselves as participants in these discussions and expected their contributions to be valued. Although they identified the teacher as a person they could interact with they recognised that it was the teachers' responsibility to maintain discipline and expected the teacher to do this "be firm but take - like jokes - not too strict". The teacher was still the authority figure and someone to be respected as noted by the secondary school students in Clark's (2001) and Jones's (1991) studies but the students in this study saw that they had valuable contributions to make in interactions with the teacher.

Students wanted a teacher that would challenge them so that they had to think as one student stated, "thinking is fun". While secondary school students noted that listening to the teacher was important to gain the knowledge in the teachers head (Clark, 2001; Jones, 1991) the 11 and 12 year olds in this study stated that the importance was so that you would find out about tasks to be completed; "if you didn't listen to the teacher you won't know what to do". They recognised that to learn they had to take an active part in the learning and that meant more than just listening to the teacher, as "you had to do things as well as listen to the teacher, that's how you learn". In contrast to Clark's (2001) and Jones's (1991) findings the students in this study saw the ideal teacher as someone who would scaffold their learning, not just give them the information to remember. Their ideal teacher was one they could approach to ask for help when needed. They did not see asking for help as a sign of disrespect as the secondary students in Clark's (2001) and Jones's (1991) studies. Instead it was a way to clarify instructions or new learning or gain clues to help in solving a given problem. The students in this study were adamant that teachers should only help with clues and not answers; "ask for clues on how to do a question – clues not answers".

The personal attributes students would like in their ideal mathematics teacher included having a positive attitude towards mathematics so that they could enthuse and inspire others. They wanted someone they could respect but wanted to be respected in return. They wanted someone who would challenge them although support them in developing their own mathematical understanding. The 11 and 12 year old Pasifika students in this study had a different understanding of the role of ideal teacher than the secondary student in earlier studies (Clark, 2001; Jones, 1991).

The learning environment students in this study wanted to work in was one where they felt comfortable sharing their ideas as they stated that "it's important to share your ideas and prove your answer" They saw the ideal teachers' role in this as a facilitator, someone who was 'cheerful and helpful' but made them do the work. They wanted the teacher to provide them with challenging problems, "cause you need it a little bit hard so that you will think and learn. Easy questions are boring," then leave them to work with their peers to solve the problem. Like students in Kershner and Pointons' (2000) study the Pasifika students recognised that working co-operatively with peers was useful as "people have different ideas and you can put them together to get one big idea". Although both higher and lower achievers identified that being allowed to work together was of high importance on further investigation the two groups had a different understanding of the term 'working together'. The higher achievers understood this as a way to learn "cause [they] might have solved it another way - like you might not think of that way – it might be a good way". In contrast the lower achievers saw it as a strategy to use if problems became difficult "sometimes you need to work with others for hard problems". The environment that allowed students to work co-operatively was identified as vital for their mathematical learning even if for different reasons.

Students wanted an environment where they could work in groups of up to five members (the most popular chosen group size was three or four). This meant students also recognised that they wanted an environment where they were able to talk. They did not want to work in a silent classroom. They recognised that the teacher was not the only source of information or help, as the secondary students in Clark's (2001) and Jones's (1991) studies had and therefore wanted to be free to interact with peers. Students recognised the "need to talk to others cause talking helps you work through understanding". They noted that talking through a problem not only helped you solve problems but also allowed for clarification of their own thought processes "you talk through the problem so that you can think clearly". An environment that allowed students to talk when learning mathematics was considered important but this focus on being allowed to talk created a personal dilemma for one of the lower achieving students. This student saw that being able to talk was a useful strategy, and promoted in the mathematics specialist teachers room but generally in the more traditional homeroom talking was not encouraged. This conflict was noticeable in his comment about working together "it's good to talk but only if you have permission cause otherwise you'll get into trouble". Students recognised that the freedom to work co-operatively with others went hand in hand with being able to talk when solving problems but at the same time noted that there were times when talking was not appropriate. Students considered being able to ask peers or the teacher for help as acceptable, not disrespectful as Clark's (2001) and Jones's (1991) students did. Asking for help was apposite as long as you were not asking for the answer and "only when questions were hard but not in a test". It was also recognised that there were times when you needed to think something through on your own "thinking about the question, like the one the teacher gives you, you need to think about it to be able to talk properly about it in your group" as "you need to think about the question or you might get it mixed up if you talk too soon". Students identified that discussing problems with peers was a useful strategy in building understanding although there were times when a person needed time to think things through for themselves.

From observations of the mathematics specialist class the attributes chosen by students as their ideal reflected more the mathematics specialist class than the more traditional environments. Students in this study identified the ideal working environment as one where it was safe to share ideas and where their contributions would be valued. An environment that had them working co-operatively with others and where the teacher was a facilitator scaffolding their mathematical understanding by providing challenging problems and guidance to develop the mathematical skills to solve them.

CONCLUSION

The 11 and 12 year old Pasifika students in this study identified attributes and practices that supported previous research on effective teachers and learning environments (Anthony & Walshaw, 2007; Brown & McIntyre, 1993; Hill and Hawk, 2000; Kershner & Pointon, 2000). Yet at the same time disagreed with some of the previous research involving Pasifika students (Clark, 2001; Jones, 1991). The differences in the findings in Clark's (2001) and Jones's (1991) studies and this one could be due to the ages of the students. Students in this study were still in the primary school system (intermediate schools are part of the primary school system) while students in Clark (2001) and Jones' (1991) studies had experienced the different secondary school system. This raised the issue about what underlying factors might explain the differences. Secondary systems in New Zealand have specialist teachers for all subjects while the primary systems have generally one teacher teaching all subjects. Do students ideas about the ideal learning environment change as they grow older or is it the different school systems that influence this change?

Listening to the teacher no longer meant that the teacher held all the knowledge and it was the students' role to learn this knowledge. Asking the teacher for help is no longer a sign of disrespect but an acceptable learning strategy. Pasifika students in this study showed that they could identify attributes required to make the ideal mathematics teacher. They described a teacher who inspired them, enthused them but at the same time respected them and their contributions to the learning environment. They wanted to work in an environment that challenged them, that allowed them to work co-operatively with others and let them interact with their teacher in developing mathematical skills and understandings for themselves.

Students stressed the importance of working co-operatively yet there were different understandings about what this entailed. Through linking students' responses from the individual interviews with general observations and group interviews, where further clarification to responses was sort, a deeper understanding of the identified criteria can be gained. Higher achievers saw the importance of working with others as a way to learn while the lower achievers recognised the importance as a strategy for when problems were difficult. This indicates that if we are exploring what students identify as important practices in learning mathematics we also need to identify what students believe the practice entails and see how they are implemented into everyday work to really understand what is meant by the responses students have given.

Although students are able to state practices such as working co-operatively are central to their ideal learning environment are they able to implement these practices into their everyday mathematical learning. Do they have the skills that enable them to do this or are there other unknown factors that prevent them using these practices? Is there still an underlying cultural belief inhibiting the full implementation of them? Co-operative learning requires specific skills if students are to get the most out of this approach teachers need to make sure that students have the skills required to make the most of learning in this way.

Finally, if a generation of students where the challenge of mathematics generates excitement is the goal, teachers need to project a positive attitude towards learning mathematics for in a students words they [the student] will be more successful “Cause your teacher likes it and makes it interesting and you like your teacher.”

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CYPRIOT PRESERVICE TEACHERS' CONTENT KNOWLEDGE AND ITS RELATIONSHIP TO THEIR TEACHING

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What is the role of Cypriot preservice primary teachers' content knowledge in teaching mathematics? In Cyprus research has focused on investigating different aspects of preservice teachers' Subject Matter Knowledge of mathematics, while investigation of the relationship between their content knowledge (Subject Matter Knowledge and Pedagogical Content Knowledge) and teaching has been neglected. In this paper, I demonstrate how an empirically-based conceptual framework, developed for the analysis of mathematics lessons taught by preservice teachers, was used with the aim to understand what relationship can be observed between preservice teachers' understanding of mathematics and their teaching.

INTRODUCTION

The object of the study discussed is based on the classic distinction by Shulman (1986) between two aspects of teachers' mathematical content knowledge, Subject Matter Knowledge (SMK) and Pedagogical Content Knowledge (PCK). In his work Shulman provided a broader conception of teachers' knowledge that emphasised content knowledge and initiated a new wave of interest in the conceptualisation of teachers' knowledge. Particularly, he identified a special domain of teachers' knowledge that he called '*pedagogical content knowledge*'. PCK includes the representations, examples and applications that teachers use in order to make the subject matter comprehensible to students.

SMK consists of substantive and syntactic knowledge (Schwab, 1978). Substantive knowledge focuses on the organisation of key facts, theories, and concepts and syntactic knowledge on the processes by which theories and models are generated and established as valid.

From a variety of perspectives, research in the field of preservice teachers' knowledge focuses on their SMK and PCK. Some researchers have investigated preservice teachers' understanding of different topics in mathematics (Ball, 1990; Philippou and Christou, 1994; Rowland, Martyn, Barber and Heal, 2001) and others have focused on investigating the relationship between SMK and PCK and teaching (Rowland, Huckstep and Thwaites, 2004; Hill, Rowan and Ball, 2005) and have suggested that content knowledge might affect the process of teaching. These studies have shown that preservice teachers' substantive knowledge of mathematics was significantly better than their syntactic knowledge, and this was reflected in their teaching. Finally, research has suggested that teachers were unsuccessful in promoting mathematical learning outside the limits of their own understanding, and their knowledge for teaching was significantly related to student achievement gains (Hill et al, 2005).

In Cyprus, policy makers' concern about students' achievement in mathematics has grown recently, and many attempts have been made to improve the instructional practices in public primary schools. The most recent attempt was to develop new mathematics textbooks and curriculum. The attempts to change the curriculum were important: however, it seems that policy makers did not take into consideration that in order to implement the new mathematics curriculum effectively, skilled teachers who understand the subject matter would be needed (McNamara, 1991). Attempts of improving mathematics teaching in Cyprus have focused on learners and the curriculum, but none is focused on teachers. Research on teachers' knowledge has been neglected in the Cypriot literature. The few studies in this field (e.g. Philippou and Christou, 1994) focused on investigating aspects of Cypriot preservice teachers' substantive and syntactic knowledge of mathematics and have shown that the participants were poorly prepared to examine different mathematical concepts and procedures conceptually. However, if we want to understand better what goes into teaching mathematics effectively, the challenge is to identify the ways in which preservice teachers' knowledge of mathematics, or lack of it, is evident in their teaching. No one type of knowledge functions in isolation in teaching and thus, research in the field of teachers' knowledge should focus on understanding the relationship between the different kinds of their knowledge. Teachers need not only to have comprehensive understanding of mathematics but at the same time they must know how to use their understanding to help students learn mathematics. The identification of the relationship between SMK and PCK will help teacher educators to assess teacher preparation programmes, and to improve them where necessary.

THE STUDY

The study reported here was carried out in the context of my ongoing doctoral study which is centred on understanding the relationship between Cypriot preservice teachers' SMK and PCK to teaching. My approach to investigating this relationship involved a mixed-methods methodology.

My study entailed four data collection methods. First, a questionnaire was designed to examine Cypriot preservice teachers' SMK of mathematics. 104, final year university students, following a teacher preparation programme, completed the questionnaire. It aimed to collect information about the participants' experience with mathematics, their beliefs about mathematics and its teaching, and their substantive and syntactic knowledge of it. As a part of the questionnaire the participants were asked to answer to ten mathematics items that assessed their SMK. For the purpose of my study participants' total score for those ten items was used to identify groups with 'high', 'medium', and 'low' scores. Seven participants from each of these three groups were then interviewed.

The aim of the interview questions was first to clarify the questionnaire data and second to gather some information about the interviewees' PCK of mathematics. The interview questions proposed two hypothetical scenarios that were relevant to teaching mathematics, representing real classroom situations which a teacher might encounter while teaching mathematics. The interview tasks provided information about what teachers know and believe about mathematics, and also about the knowledge and skills that they draw on in making teaching decisions.

While these interview tasks represented real situations in the mathematics classroom, their context remained hypothetical, and did not provide a sense of what teachers actually do in the classroom and how their knowledge of mathematics influences their teaching decisions in classroom where they interact with their students. This kind of information was provided by observing participants teaching mathematics in the classroom. Five of the interviewees were chosen to be observed while teaching mathematics. In Cyprus a large part of the teacher preparation programme (a four year university course) is spent in teaching in schools under the guidance of a school based mentor.

For the observations I used a framework that emerged from observing several lessons that were taught by preservice teachers in England (Rowland et al, 2004). This framework is called the *Knowledge Quartet* and is a tool that can be used in order to describe the ways in which SMK and PCK are related to teaching.

Finally, the data from the questionnaire, interview and observations were compared with data from the analysis of mathematics textbooks in Cyprus. Textbook analysis provided information on what policy makers expect preservice teachers to understand and teach. In other words, it provided information on what mathematics policy makers consider desirable knowledge for teachers. However, what is considered desirable knowledge for teachers is often different from the knowledge that teachers put into practice. A comparison of these two kinds of knowledge can be proven to be very helpful in modifying and improving teacher preparation programmes.

The combination of four methods and their integration during the interpretation phase provided stronger inferences and produced more complete understanding about the relationship between participants' content knowledge and teaching. In the remainder of this paper I will focus on just one aspect of the study described here and demonstrate how the Knowledge Quartet, a framework developed for the analysis of mathematics lessons taught by preservice teachers, was used.

THE KNOWLEDGE QUARTET

The Knowledge Quartet (Rowland et al, 2004) is a theoretical framework developed to describe, analyse, and develop the mathematics content knowledge (SMK and PCK) of preservice elementary school teachers in the UK, and identified ways in which their SMK and PCK is related to their teaching. This framework can be used as a tool for classifying ways that preservice teachers' SMK and PCK come into play in the classroom.

The Knowledge Quartet consists of four dimensions, namely, foundation, transformation, connection and contingency. Foundation consists of trainees' knowledge, beliefs and understanding of mathematics. The second category (transformation) concerns knowledge-in-action as demonstrated in the act of teaching itself and it includes the kind of representation and examples used by teachers, as well as, teachers' explanations and questions asked to students. The third category, connection, includes the links made between different lessons, between different mathematical ideas and between the different parts of a lesson. It also includes the sequencing of activities for instruction, and an awareness of possible students' difficulties and obstacles with different mathematical topics and tasks. Finally, the fourth category, contingency, concerns teachers' readiness to respond to students' questions, to respond appropriately to students' wrong answers and to deviate for their lesson plan. In other words it concerns teachers' readiness to react to situations that are almost impossible to plan for.

The Knowledge Quartet is currently used as a framework for lesson observation and for mathematics learning development within a PGCE programme at the University of Cambridge. Below, I demonstrate how this framework was used for the analysis of a lesson on fraction taught by Rita.

THE CASE OF RITA: TEACHING FRACTIONS

Rita was a student teacher in the last term of her studies in the Cypriot four year university degree for primary school teacher preparation. She was classified in the group with 'medium' SMK score based on her responses to the ten mathematics items included in the questionnaire.

Rita's forty-minute mathematics lesson was observed during her final teaching placement. This was the sixth lesson that I had observed her teach: she had been teaching the class for four weeks. The focus of the lesson was on fractions (in particular on 'a third'). The class was a mixed ability class and the children were in the second grade of the primary school (pupil age seven years) in a typical public primary school in Cyprus.

The Knowledge Quartet was used for the analysis of her lesson. This analysis helped me to understand better the ways in which Rita's lack of knowledge was evident in her teaching.

Foundation

I raise for discussion here two aspects of Rita's lesson that fall within the scope of 'foundation' knowledge. These are Rita's beliefs about mathematics and her understanding of what is meant by 'fraction'.

Rita's choice of activities and examples points to certain beliefs about mathematics teaching and learning. She seemed to believe that mathematics concepts should be developed on the basis of problems present in real life contexts. So, she incorporated applications from students' everyday life in her teaching. For example she used problems such as:

- Maria decorated her room with twelve balloons. Four were red and the rest were blue. What part of the balloons was red and what part was blue?

- George had twelve chocolate bars and he wants to give a third to his little brother. How many chocolate bars will George's brother get?

Rita appeared to consider that students' demonstration of good reasoning should be valued more. She frequently asked her students to explain their answers and to justify their thinking.

Rita believed that it was important to promote students' understanding of the logical relationship among fractions of shape and fractions of numbers (discrete sets). Although, she seemed to have this connection in mind she was not successful in clarifying the link. This will be discussed further later. I will argue that her inability to clarify this connection was due to her limited understanding of the meaning of fractions of discrete sets and in particular her failure to know, and to recognise, two different division structures. Her limited understanding was evident throughout the whole lesson and in the explanations and representations used in her lesson.

Transformation

Rita's explanations and her choice of representations merit some comments and discussion. At the beginning of her lesson Rita defined fractions in terms of equal parts, and she continued to develop this by using regions of shapes. For example she drew a circle, she divided it into three equal parts, shaded one of them, and asked the students to tell her which part of the circle was shaded. She then continued her lesson by asking her students to find fractions of a whole number. However, she did not define fractions in terms of equal parts in the case of fractions of numbers. This can be seen in the following extract from lesson (the transcript is translated from Greek).

Rita: I have six cubes. How many cubes is a third?

Student: Two cubes. A cube from each triad [meaning group of three].

Rita: Well done! So when we have a number of cubes and we want to divide them into thirds we need to divide them into triads. If we want a third, this in turn means that we get a cube from each triad. Here we have two triads and we get one from each. This means that we will get two cubes (*she used cubes to represent her thinking*).

The extract above reveals that Rita explains fractions of discrete sets in terms of the quotative division structure i.e. she divided six into group of threes. In the case of division there are two key problem structures, namely partition and quotition. The first structure, partition, is exemplified by the problem:

- Andri has twelve pounds. She shares them with Maria and Eleni. How many pounds will each get?

The second structure, quotition, can be exemplified by the problem:

- Andri has twelve pounds and gives three each to some friends. How many friends will get the money?

Each problem structure involves some pounds and some children, but the 'answer unit' is pounds in one case, but number of children in the other. Rita's reference to the size of groups (triads) points to a quotition structure. Yet an 'equal parts' account of fractions, in keeping with Rita's initial definition of fractions, and her division of the shape (the circle), would require the partition division structure. Indeed, while a quotative account of a unit of fraction of discrete set is possible. It would be rather non-standard. It seems that Rita was not clear about the two division structures. Rita's adherence to quotition was also evident in the representation she used (Figure 1) for the balloon problem mentioned above.

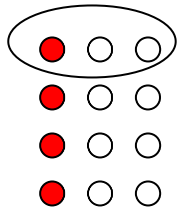

Figure 1. Representation - Balloon problem.

Rita drew the figure shown above, and explained, 'Four out the twelve balloons are red (she wrote $\frac{4}{12}$ on the board)…Since I divided the balloons into triads and there is one red in each triad this means that the red balloons are the third'

Both the examples above illustrate how Rita's incomplete understanding prevented her from connecting the model she used to explain a third of a circle with her explanation of what is meant by fraction of numbers.

Connection

Rita began her lesson by reminding her students what they did in their previous lesson (when the focus was on a half) and tried to help them make connections with a third. Another important connection to be established in this lesson was that between fractions of shapes and fractions of numbers. Rita seemed to have this connection in mind and she included activities in her lesson where students were finding fractions of numbers and activities where students were finding fractions of shapes. However, she did not attempt to make any crucial link between fractions of shapes and fractions of numbers.

An aspect of a mathematics lesson that falls within the scope of the connection category of the Knowledge Quartet is the sequencing of activities in the lesson. There is evidence in the literature to suggest that the fraction of shapes model should take pedagogical priority over the fractions of numbers model (Dickson, Brown, and Gibson, 1984). Thus, Rita correctly chose to begin her lesson with fraction of shapes, and then continued to fraction of numbers. However, because of her adherence to quotition in her account of fractions of numbers, she missed the opportunity to make a connection between the two models in her planning and teaching.

Contingency

Rita's reaction to students' responses and explanations merits some comments. At one point in the lesson, a student appeared to be thinking of fractions of numbers in terms of equal groups i.e. partition. This student explained her approach to finding a third of twelve chocolates.

Student: It is four chocolates.

Rita: How do you know?

Students: We need a third… four plus four plus four is twelve. This means that a third is four chocolates.

Rita: So you divided twelve into triads. You had four triads. You get a chocolate from each triad and this means that you have four chocolates.

Here the child wanted to point out that she divided the chocolates into three equal groups of four chocolates in order to find a third of the number of chocolates. Rita seemed not to recognise that the child's explanation referred to the partition structure of division. Instead, in response, Rita reiterated her previous quotative account, in terms of triads. It appears that Rita was not clear about the two division structures, and that is why she drew on the language and concepts of quotition in responding to the child's thinking. Her failure, to know, and recognise, these two structures let her to misinterpret the child's explanation. This episode offered Rita a good opportunity to rethink her explanations, but she did not take advantage of. She appeared not to be aware of the mathematical significance, for the whole class, of the child's approach.

CONCLUSION

In this paper I have described how the Knowledge Quartet was used as a tool for analysing mathematics lessons in order to understand the relationship between Cypriot preservice teachers' mathematics content knowledge and their teaching. Rita's case shows that teachers' incomplete understanding of mathematics limits their ability to teach it effectively. In my presentation I will demonstrate further the use of the Knowledge Quartet in the analysis of mathematics lessons, and give some suggestions for the improvement of teacher preparation programmes in Cyprus.

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VOICES OF NON-IMMIGRANT STUDENTS IN THE MULTIETHNIC MATHEMATICS CLASSROOM

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Research studies on multiethnic mathematics classrooms tend to focus on the perspective of the immigrant student (or the language / ethnic "minority" student) and less on that of the "local" student (or the "non-minority" student). However, the representations that both groups of students have on each other are crucial towards the construction of learning opportunities in diverse mathematics classrooms, which are quite common in many countries. In this paper, we reflect on three aspects – group work, mathematics talk and alternative approaches to problems– as viewed by non-immigrant students of a multiethnic mathematics classroom.

INTRODUCTION

Research studies on multiethnic mathematics classrooms tend to share a focus on conflict, that is, whether looking at issues of power, of discourse, of participation (e.g., Setati, 2005; Vithal, 2003), the concept of the classroom as a conflictive place for learning is often present. Much of this research has centered on analysis of classroom interactions, for example, looking at small groups of students as they solve problems and how these interactions are influenced by who the students are, how they are positioned (by other students as well as by the teacher) or how they interpret the norms of mathematical practice. These studies tend to focus on the perspective of the immigrant student (or the language / ethnic "minority" student) and less on that of the "local" student (or the "non-minority" student). What do "local" students think about being in multiethnic mathematics classrooms? This is the question that we address in this paper. The representations that both local and immigrant students have on each other are crucial towards the construction of learning opportunities in diverse mathematics classrooms, which are quite common in many countries.

CONTEXT

The research reported here took place in a mathematics classroom at a public high school in a low-income neighborhood in Barcelona, Spain. The high school student population was largely Moroccan (60%) and, in particular, 14 of the 28 students in the classroom were immigrants (seven Moroccan, three Dominican, two Pakistani and two Bangladeshi), nine of them being first generation immigrants. By "immigrant" we mean first or second generation immigrant (there are practically no third generation immigrants in our local context yet); by "local" we mean individuals who can trace their origins back for generations to Spain.

We chose this classroom because the teacher promoted a reform-based approach to the teaching of mathematics, thus allowing us to explore three aspects of mathematical practice that tend to be associated with reform oriented approaches: group work, mathematics talk (hence, language aspects), and alternative approaches to problems. The teacher had collaborated with the first author on a prior research project and was very welcoming of this new project. He had over ten years of experience in urban settings and enjoyed working in schools such as the one where this study took place. We point out, however, that up to that class the students had not experienced a problem-solving approach in the teaching of mathematics. Also, we want to note that all the immigrant students in this classroom had quite a good command of at least one of the official languages (Catalan and Spanish). This is important to note because of the role that language plays in small group work on problem-solving tasks.

METHOD

The main source of data for this paper comes from interviews with twelve 15 and 16 year-old non-immigrant students (seven females and five males). Each student was individually interviewed once for about 30 to 50 minutes during the second and third month of the school year. Interviews were tape-recorded and transcribed. Additional data were collected during informal conversations with immigrant and non-immigrant students after some of the lessons that were observed. The students ranged over different levels of achievement from high to low (as determined by results in mathematics classroom assessments).

The first author conducted all interviews (in Catalan or Spanish depending on the student's preference) using an interview protocol that covered four general areas: 1) their background with special attention to their schooling history; 2) their mathematics experience as learners; 3) their mathematics experience in their current multiethnic classroom; and 4) their perceptions of learning obstacles as well as opportunities in relation to their being in a multiethnic mathematics classroom. The use of open-ended questions such as "How do you collaborate with your immigrant peers?" provided students with an active role in the research process. This format afforded them the opportunity to formulate their own answers, to seek clarifications from our comments and questions, and to pursue ideas that they felt as relevant and had not been directly introduced.

The data analysis followed Glaser and Strauss's (1967) constant comparative method. The different pieces of data (field notes and transcriptions of the tapes) were looked at and codified. This process led to the development of themes. We organized data related to points 3 and 4 by means of three initial themes –'Group work', 'Mathematics talk', and 'Alternative approaches to problems'– that appeared to be highly related to both learning obstacles and opportunities in the multiethnic mathematics classroom, as seen by most of the twelve students that were interviewed.

SOME RESULTS

In this paper we focus mostly on the third and fourth areas of the interview, by looking at these local students' references to the three themes we mentioned earlier:

group work, mathematics talk, and alternative approaches to problems. Although our emphasis is on the local students in this study, we bring data from immigrant students in that same classroom to underscore the need to listen to both groups.

Group Work

The teacher deliberately used linguistically and ethnically heterogeneous groups. He did so to encourage students to interact across ethnic and linguistic groups, as he had noticed that otherwise students tended to stay within "their" groups. To us, group work is potentially a way to develop joint mathematical practice, although we are aware that different interpretations of norms, students' status in the class, and teacher's role in orchestrating group work play a key role in limiting or promoting productive functioning of groups. Alrø and Skovsmose (2002) have argued the benefits, in terms of the communication and the learning, of classroom environments of group work where the teacher plays the role of a consultant.

The local students tended to view group work as a strategy that the teacher used to help the immigrant students, but saw little value towards their own learning:

> Marc [high achiever]: Most times we work in small groups, and this is quite a problem for us, at least for me, though I try. We are mixed groups. But you know, what immigrants say they do and what they actually do is not always the same. It is very difficult to talk about math with them, it is difficult to complete the tasks, though I try.

> Maria [low achiever]: To learn mathematics with immigrant students? To tell you the truth, it's very tiring... they need lots of help. I try to help them as much as I can, but when I need to get concentrated on the task, they must wait for my help.

From Maria's perspective, the help given to immigrants can become excessive under certain circumstances. She experiences the ideas of 'being concentrated in the task' and 'helping the immigrants' during group work as contradicting. What is interesting to note is that while the local students do not seem to value working in groups as a way to enhance their own learning of mathematics, neither do the immigrant students:

> Khadija [Moroccan arrived the year before, high achiever]: I do not like to work in the math groups because I cannot concentrate; everybody talks and I cannot think. Here they do it this way, but it can be done different ways, with more silence.

> Harim [Moroccan, born in Barcelona, low achiever]: In the afternoons (at the mosque) we listen to the math explanations, and in the mornings (at school) we listen to other students. The teacher is there but he is not there because we cannot ask him. It's rather funny.

We are aware that just referring to group work in mathematics is rather vague, as we do not really describe how this group work was orchestrated, nor do we talk about norms of participation. In this classroom, students were given a task that was briefly introduced by the teacher. Students were said to solve the task in the context of their groups and to write down the main mathematical questions that had orientated their discussions. Neither the immigrant students, nor the local ones were used to working in groups in a mathematics classroom. Some of the immigrant children, like Harim, had access to other schooling experiences, such as the mosque. For him, the learning of mathematics took place there, at the mosque. One of the local students, Mireia, gave us her perspective on the immigrant students going to these other schools:

> Mireia [high achiever]: They go to those schools in the evening and they are a bit confused about how things work in this school.

She did not elaborate on what she meant by being a bit confused. Is it because the expectations and norms at the two schools are different and thus these students need to navigate two systems?

Mathematics Talk

Closely related to group work is the idea that students have to engage in conversations about mathematics with each other. Moschkovich (1999), raises the point that although an emphasis on discourse could make English Language Learners (ELLs) (in her study she is focusing on Latino ELLs in the U.S.) vulnerable to be assessed as deficient in terms of their language skills, it could also be seen as an invitation for ELLs to engage in meaningful, context-based conversations about mathematics, thus providing them with more opportunities to enhance their learning of language and mathematics.

> The new emphasis on mathematical discourse and the new forms of student participation point to the need for Latino students to have the opportunity to engage in mathematical discussions with their peers, with the teacher, and with their whole class (Moschkovich, 1999, p. 10).

In our case, the teacher's use of linguistically diverse small groups was intended to follow Moschkovich's suggestion. Yet, the local students focused mostly on "language" as a problem:

> Eduard [low achiever]: I always work with Imram, he helps me with the mathematics and I help him with the language. They really need it. Sometimes Imram makes small changes in the words and big changes in the wording, and then the others do not understand him. Then I explain what he means. I don't mind helping him but sometimes I help him much more with the language than he helps me with the math. Sometimes it is kind of wasting my time.

Imram had arrived from Pakistan six years ago and spoke Catalan and Spanish quite fluently. The teacher explained that Eduard and Imram often did not agree with each other's ideas in mathematics. Yet, as members of the same working group, one of the expectations was to reach some consensus to bring up to the whole class discussion. The teacher added that Imram did not expect Eduard to help him with language and that he would get tense when Eduard "explained [to the class] what he [Imram] means" as he sometimes misrepresented what Imram had said.

Although it is the case that the immigrant students had a good command of at least one of the official languages for social communication, it is less clear what their command of academic language was, as this one takes longer to develop (Cummins, 2000). As Khisty (2006) writes,

> Academic discourse competence … is acquired through active participation in the community that uses that discourse…. Without the academic discourse or language, students are systematically excluded or marginalized from classroom curricula and activities (p. 436).

It is very likely that by having the students work in small groups, the teacher was indeed aiming at developing students' academic language. Which students? We argue that it is not clear to us that only immigrant students need experiences in which to develop academic language. Yet it is possible, like Helena's quote below suggests, that the local students did not see the use of small groups as a technique to develop their own competence in mathematical communication, but more as a way to once again, help the immigrant students—in this case with language development.

> Helena [high achiever]: They put us in small groups and they say that this way we will learn more mathematics, but the real reason is that they do it so that those from outside get a chance to practice our language. I don't think this is right because I think that these decisions should be based on the mathematics.

Helena talks about group work as being "an occasion" for immigrant students to practice the use of the local languages. In her opinion, the attention paid to the language interferes with the students' mathematical learning. During the interview, when Helena was asked to clarify what she meant by "decisions should be based on the mathematics", she said: "They practice our language, and mathematics happens in the meantime. Shouldn't we practice the mathematics?" Why is it that practices of mathematics talk are not seen as compatible with the learning of the language?

Alternative Approaches to Problems

For the teacher, as well as for us, one of the richness of working in multiethnic classrooms is the possibility for multiple approaches to problems to emerge. But to which extent are all these different approaches seen as part of valid mathematics? By pointing out the cultural aspect of certain methods (e.g., by attributing them to an immigrant student) are we contributing to developing an idea of "their" mathematics and "our" mathematics? The second author in her work with preservice elementary teachers in the U.S. asked them to reflect on the article "Mathematical notations and procedures of recent immigrant students" (Perkins & Flores, 2002). Several comments pointed to a belief that immigrant students needed to learn the way arithmetic is done in the U.S. As one of them wrote, "this is nice but they need to learn to do things the U.S. way."

The local students in the study in Barcelona had similar comments, though some of them did show an interest towards these methods:

> Laia [high achiever]: Last week Afzal solved an equation by drawing a kind of diagram. It was interesting, though I missed some details because I was still finishing the task... I often wonder if he feels out of place with our math... we cannot learn everything, our math is already too much!

Pau [low achiever]: Their comments help us make sense of the situations before starting solving the problems, but, anyway, we cannot always start making sense of it like they do. Our math is what it is. And theirs… it is fine, but sometimes it just doesn't fit in.

Overall, however, there was a shared concern among the local students that they had enough work to do to learn "their own" mathematics and that it was up to the immigrant students to adapt to the local ways of doing mathematics.

Maria [low achiever]: We are not in the classroom to learn their mathematics but to learn ours. That's what the exams are about. (…) I am not expected to learn Murshed's way of subtracting.

Sergi [low achiever]: I learn from what others say and do, but you see, in the case of the immigrants, you must be very careful for your own benefit. They learned some mathematics differently and you must know what to learn from them (…). Most of them easily learn our ways… There is a Chinese girl, she was in my class last year. One day, she drew part of a circle with her compass and then she said that she had drawn an angle. The teacher said it was okay because the angle has not to do with the directions of the two intersecting lines but with the idea of amplitude. That girl, she was always listening to us... I often think that she must have many other examples, like the one of the angle, but I'm not going to use the compass when drawing and angle.

Only Sergi referred to the opportunities foregone in making the choice of not integrating different meanings. The year before, he had widened his notion of angle by listening to an immigrant student, and he was aware of having learned from her, though he believed that the local mathematical practices are more appropriate than the immigrants'. How has Sergi developed such a clear perception of which "mathematical contents are not to be learned"?

FINAL REMARKS

The immigrants' classroom practices are seldom used as a starting point for a reflection on the local practices. The tendency to favor some practices over others is often based on pre-existing representations rather than on concrete classroom experiences. In our interviews, many arguments point to the language problem though it is not the case in this classroom. Some local students did provide examples of knowledge learned from immigrants, but they concluded introducing a notion of conflict: they listen to their immigrant peers, they are interested in their practices, they even take responsibility for their level of understanding when helping them, but they do not contemplate the option of carrying on prolonged mathematical conversations with them or adopting their mathematical practices.

How do local students develop the perceptions they have? For example, they viewed working in groups with immigrant students as something that they needed to do to help them but did not seem to envision it as a learning opportunity for themselves. Further research should explore what factors may influence these perceptions. What role do teachers and parents play in this process? In Civil and Planas (2004), we have the case of a local student who refers to his father not really wanting him to work with the immigrant students for fear that he may be wasting his time. How can teachers really promote and build on the diverse approaches to mathematics that students bring to class without creating this image of ours versus theirs, which seems to lead to situations of conflict in the learning environment?

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A SOCIOCULTURAL PERSPECTIVE ON THE FIRST YEAR OF TEACHING SECONDARY MATHEMATICS

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The purpose of this study was to investigate the experiences of a group of four beginning secondary mathematics teachers in their first year of teaching. Each teacher was interviewed individually for approximately 30 minutes in the middle of the school year. The interview responses were analysed in terms of the participants' views of themselves as teachers and the factors which they identified as influencing their classroom practices. The study indicates that the culture of the school and the practices of more experienced teachers were important factors in the beginning teachers' perception of themselves as effective teachers.

INTRODUCTION

The process of learning to teach is situated in numerous settings including university courses, school-based practicum experiences, and ultimately in the schools where teachers work. In this paper we adopt a sociocultural perspective to investigate the experiences of four beginning secondary mathematics teachers in the early months of their teaching careers. A similar theoretical position has been taken in previous studies of novice secondary mathematics teachers. For example, Ensor (2001) followed a group of seven pre-service teachers from their year-long university course into the first year of teaching and noted the importance of contextualizing learning about teaching. Goos and Bennison (2006) adopted a community of practice model to investigate the development of an online discussion board for pre-service and beginning teachers. We consider the context of teachers' work to examine the crucial role the school environment plays in shaping how beginning teachers learn their craft through increasing participation in the practices of the teaching profession.

CONTEXT

The present study is part of a larger one which took place over a two-year period. We tracked a group of ten pre-service teachers through their final year of teacher training and interviewed them individually on four separate occasions about their experiences (Cavanagh & Prescott, 2007). We explored how their participation in university studies and the school-based practicum influenced their formation as teachers. In this paper, we report on research which we subsequently undertook with four of the original participants as they commenced full-time work in schools. Our aim was to investigate the beginning teachers' perceptions of themselves as teachers and how their classroom practices were shaped by interactions with others at the school, especially within the mathematics faculty.

The study took place at a time when a reform-oriented syllabus which specifically promoted the use of problem-solving and mathematical investigations was introduced.

Through its Working Mathematically strand, the syllabus described five interrelated student processes: questioning, applying strategies, communicating, reasoning and reflecting. Teachers were encouraged to cover the syllabus content by developing and using a variety of activities that allowed students to engage with the five processes.

SOCIOCULTURAL PERSPECTIVES

Sociocultural perspectives on mathematics education have become increasingly prominent and offer a useful means of analyzing teachers' professional growth (Lerman, 2000). These theories emphasise that learning occurs at specific times in particular locations through interactions with others. Sociocultural theories posit that the contexts in which learning takes place are critical because they help to shape the learning that occurs within them (Franke & Kazemi, 2001). Hence the social, historical, cultural and physical settings play an integral part in what is learned and how it is learned. Learning is described as a social activity which happens in communities of practice (Wenger, 1998) where people engage in some collective activity or shared enterprise. Learning is said to occur through increasing participation in joint activities aligned to common goals, purposes, means and ends (Lave & Wenger, 1991).

Rather than describe learning as a process of passively receiving new knowledge, sociocultural theorists concentrate on the active involvement of individuals and groups in particular tasks as the means by which learning takes place (e.g., Schön (1983) described the acquisition of professional knowledge as "knowing-in-action"). Learning is seen therefore not as a purely theoretical exercise but rather as participation in specific activities which helps to shape the individual by means of the new insights and skills which are gained. Hence learning is closely connected to "transforming who we are and what we can do" (Wenger, 1998, p. 215).

In sociocultural terms, learning to teach can therefore be understood in terms of increasing participation in the activity of teaching in order to gain insights into the practices of teachers (Adler, 1998). Beginning teachers advance their knowledge and skills in the teaching profession by direct interaction with other teachers in specific tasks which are focused on teaching secondary mathematics. But learning also takes place through less formal processes such as listening to and casually observing one's colleagues as they go about their normal daily routines (Stein & Brown, 1997). Thus learning to teach secondary mathematics can be seen to occur primarily in the company of other mathematics teachers in schools.

Each individual brings a different perspective and level of experience to the task of teaching. Rogoff (1995, p.146) refers to *guided participation* to emphasise the two-way nature of the interactions of beginning and more experienced teachers and the mutual involvement of each individual in them. At the same time, research suggests that beginning teachers often learn what is valued and practised by their more experienced colleagues (Stein, Silver & Smith, 1998).

The purpose of the present study was to investigate the perceptions of beginning teachers about their classroom practice. We were particularly interested in the extent to which the first-year teachers were able to implement the Working Mathematically strand of the new syllabus and the factors which they identified as influencing their classroom practices.

METHOD

Our original study focused on ten pre-service teachers as they undertook a one-year Graduate Diploma of Education program taught by the authors in two universities and comprising units in mathematics education curriculum, teaching methodology and supervised professional experience. Those who obtained full-time employment in the same city as the university were invited to take part in the second phase of the research which is reported here. Four teachers agreed to participate and each one was interviewed using a semi-structured protocol for approximately 30 minutes half-way through their first year of teaching. The interviews were audio-taped and transcribed.

Analysis of the data began with a thorough reading of the transcripts during which patterns in the responses for each individual participant were identified. The ideas which emerged from each transcript were categorised into themes for each participant. These were then cross-checked against the other transcripts so that a set of common themes began to emerge. These were classified according to whether they related to issues of self-perception, working mathematically, or classroom practices. The findings are summarised in the following sections.

RESULTS

From a sociocultural perspective, a teacher's professional identity is closely related to the ways in which each one participates in the activity of teaching in association with colleagues (Wenger, 1998). Identity is not fixed but rather evolves in response to the school environment and particularly in how teachers perceive the reactions of co-workers, pupils, parents, and so on. Moreover, each teacher's view of him or herself acts as a prism through which future interactions are perceived and through which learning about teaching occurs. It also provides a basis for making decisions in the classroom as lessons progress. All of the participants in our study spoke about how they quickly became aware of "school politics" and the accepted classroom practices of fellow teachers, which were invariably quite traditional and not really in the spirit of the new syllabus. So, despite their desire to follow a working mathematically approach, the beginning teachers conformed to the traditional practices of their colleagues.

SELF-PERCEPTIONS

Many studies (e.g., Adams & Krockover, 1998; Kardos & Johnson, 2007) have found that beginning teachers find their first year of teaching stressful, chaotic, a roller coaster ride, and emotionally draining as they find themselves in a situation where they move from one 'crisis' to the next. Our teachers were no exception.

It was very easy for the beginning teacher to obsess about the relatively small number of difficult students they dealt with and lose sight of their achievements.

> Low points. Every day, one out of 60 kids doesn't do it. Or, you know, has been rude or whatever and I was concentrating more on that one person than the whole lot. It took me a while to sort of think: You know, there were 60 today and just one was out of whack. But it was very draining [Neroli].

Relating to other teachers

When new teachers are expected to be independent from the start, as opposed to being sheltered as novices, they find the experience a solitary one where they mostly plan and teach alone (Kardos & Johnson, 2007). Beginning teachers aim to fit in so they make sure they 'toe the line' by conforming to the status quo. To this end they find it hard to ask for help because they do not want to appear to be floundering and besides, the other teachers all look so busy. Alongside this feeling of trying to be independent, is the view of many experienced teachers that beginning teachers must learn from their own mistakes and too much help is not good for their survival in the long term.

> I guess one thing would be good if the mentor came in and observed my lessons … Not necessarily to grade me but to say I can see some difficulties, here are some things you can do [Stephen].

The tension between looking for support and being regarded as an effective teacher was problematic for the beginning teachers.

WORKING MATHEMATICALLY

The emphasis in the new syllabus is Working Mathematically (indeed it can be seen as central to the whole syllabus) but that requires a less textbook oriented approach to teaching mathematics and the emphasis in the university course supports this. In contrast, schools are much more textbook oriented. Beginning teachers also see 'fitting in' as conforming to the style of teaching exhibited by their more experienced colleagues so working mathematically becomes problematic.

The beginning teachers recognised the need for a balance between the traditional textbook approach and the working mathematically approach to mathematics teaching but were also sure they were not yet getting the balance right. There is an unresolved tension in that the beginning teachers saw the value of working mathematically but perceived it as taking longer and creating classroom management issues. They said they would postpone working mathematically until they felt more confident in the classroom.

> I know [less able students] need [working mathematically] the most but I just fear that if I do this that they won't listen or they'll muck up [Stephen].

> Once I establish a good relationship and ... good communications with them ... then I'll be popping up interesting questions [John].

Many schools are poorly resourced and rely solely on textbooks – one school used the textbook rather than the syllabus as its programming document making anything but a textbook oriented approach much harder. Because the beginning teachers were on probation, they felt they had to keep a tight rein on their students and they felt that a textbook oriented style of mathematic teaching made this easier.

In addition, the beginning teachers were also aware that their students were not used to a working mathematically approach so classroom management issues were more likely to arise in those lessons. They were fearful of trying something new in case it did not work.

> You're on probation and you've got a teaching certificate to get so you don't want to be taking too many risks. The teachers often walk past my classroom so you want to keep the class reasonably quiet [Stephen].

The beginning teachers recognised their lack of experience in choosing examples that covered the syllabus and were pitched at an appropriate level for their students. Textbooks were therefore seen as a reliable classroom resource.

> I have the support of the textbooks so I'll be focussing on my ability to explain things, try and keep the right level, use the right words etc. [John].

Only one of the beginning teachers was happy with the possibility to undertake working mathematically in the classroom. During the practicum he had not been allowed to prepare less textbook oriented lessons but now had support from the school and the freedom of his own classes and was enjoying the experience. The support had come from the availability of resources and from a variety of people, including a mentor, the head of department and the principal.

> I guess I had a number of philosophical differences with my supervising teacher; just, totally different approach. So I was finally able to do what I wanted to do. I didn't have to worry so much, you know, I could, if I wanted to do a lesson and have a discussion for the most of, you know, most of it or whatever then that was my decision to do, and I didn't feel like I was having to please somebody else [Peter].

Despite the encouragement of his mentors, this beginning teacher was still using the more traditional approach because he saw this as more closely conforming to the culture of the school. In general, the culture of the school inhibited the beginning teachers' ability to use successful working mathematically activities.

CLASSROOM PRACTICES

The beginning teachers wanted to be effective in the classroom and saw this as being able to deal with classroom management issues and being a good communicator so that explanations are clear. They saw their discipline issues as emanating from their inability to cater for the range of abilities in their classes. This also included being able to take into account the specific different needs of different classes and the amount of work they should be covering in each lesson. The full teaching load made lesson preparation and reflection seem a luxury rather than an essential component of good teaching practice so time management skills were vital.

Examination pressures

Even though the beginning teachers acknowledged that the textbook approach was not necessarily the best way to learn, they felt pressure from their colleagues to keep up with parallel classes so all classes covered the material required for examinations.

> You've got to try to teach the material so that they can have some opportunity to do well in the exam … I'm trying to push them through the work so at least they've seen it. … So long as you've taught it, that's OK. But whether they've learned it or not is immaterial [Stephen].

It was not a case of making sure the students understood the work, rather it was a case of 'covering' the material in class so you could sign off the register. The beginning teachers also knew that the examination questions were usually procedural and that they would be unlikely to test conceptual understanding (despite working mathematically being central to the syllabus).

In addition, the beginning teachers felt pressure from students who did not see the value of working mathematically. The students' experience of mathematics lessons was almost exclusively instrumentalist and the working mathematically approach required greater effort on their part but many just wanted to know how to do the examination questions. The resistance from students discouraged the beginning teachers from pursuing that style of lesson, and those who tried and felt their lessons were poor were fearful of trying again.

> They said why are we doing this? We don't need this. They rebelled [John].

Time constraints

A common complaint expressed by the beginning teachers was the inordinate amount of time required on activities they regarded as peripheral to the work of a teacher. They wanted to spend more time preparing quality lessons, but were consumed by administrative matters and extra-curricular tasks such as playground duty. Classroom management problems also meant that many hours were needed to follow up recalcitrant students.

> I've got about eighteen lessons a week but in addition to that I've got things like sports… so that's time I can't use for anything else. I've got assembly … pastoral care, those kind of other things [John].

Classroom management issues also included dealing with students who needed extra help. The inability to deal with the various levels of ability in the classroom meant that lunchtimes were used for helping students to catch up with their work and to improve understanding.

> I see a lot of students benefiting from individual attention which I can't give them during a normal lesson and I tell them 'Look you have to come for a lunchtime because I want to cover this area in detail with you' [John].

Where there was a lack of resources in the school, there was also a feeling of having to 'reinvent the wheel' in creating worksheets and developing classroom activities. When time was at a premium, as was often the case, the beginning teachers felt they relied too much on the textbook but they saw no other option. Observations of experienced colleagues who seemed to be able to prepare lessons at will, only served to emphasise the time pressures experienced by the beginning teachers.

As well as a lack of time to prepare, there was virtually no time to reflect on lessons just taught. As a result the beginning teachers did not feel they were making sufficient progress in becoming effective in the classroom – mistakes were repeated more often than they would have liked.

> I often don't have time to reflect on what worked and what didn't. I do sometimes but not half as much as I would like to … it's basically trying to survive to the next lesson [John].

CONCLUSION

We sought to examine how the school environment can influence beginning teachers' self-perceptions and classroom practice, particularly in the context of a reform-oriented syllabus that emphasised mathematical problem solving. Our analysis indicated that the culture of the school and the practices of more experienced teachers have a powerful impact in the development of beginning teachers. In particular, the new teachers wanted to be seen as effective in the classroom by their colleagues and their students. Therefore, they emulated their peers and adopted what they perceived as the safer option of relying on the textbook.

This study indicates that beginning teachers could benefit from fewer extra-curricular activities and a lighter teaching load. A more structured system of mentoring rather the present ad hoc practice would provide an avenue of support and a forum for professional development. Mentoring could include an opportunity to share a class so the beginning teacher could participate in and reflect on good teaching practices within the culture of the school.

While each beginning teacher recognised that they had not yet adopted the Working Mathematically strand of the syllabus, they all indicated a desire to align their teaching practice more closely to it. Whether these beginning teachers actually achieve this goal or are entrenched in the current system is a matter for further study.

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THE EMERGENCE OF STOCHASTIC CAUSALITY

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This paper describes a range of students' (age 15) expressions of the interplay between causality and variation, which we relate to dimensions of complex causality implicit in mastering the concept of distribution. The results indicate support for our conjecture that it is possible to harness causality in order to help students to invent new ideas of "stochastic causality", a sense of control over random processes through the careful design of a simulation, which places emphasis upon deterministic as well as stochastic behaviour. This result stands apart from mainstream research, which tends to separate the determined and stochastic worlds.

THE BACKGROUND

Prodromou and Pratt (2006) define two epistemological perspectives on distribution: the data-centric perspective that identifies with an aggregated set of actual outputs; the modelling perspective that associates a set of possible outcomes with probabilities. In Prodromou and Pratt (2006), we conjectured that, given appropriately phenomenalised tools (Pratt, 1998), students would be able to bridge the data-centric and modelling perspectives on distribution. In this vision, features such as average and spread can be construed as parameters of a modelling distribution, which, alongside randomness, act as agents to shape the variation apparent in the data-centric distribution. In this study, we report on the use of such agents as on-screen instantiations that blur the distinction between controls and representations, identified as a key characteristic of designing for abstraction (Pratt, Noss, Jones & Prodromou, submitted).

It seemed though that there was a paradox at the heart of our work. On the one hand, the work of Pratt (1998) makes a prima facie case that technologically-based environments may have the potential to construct stochastic meanings out of causality. He built, for example, on the work of Noss and Hoyles (1996) to show how knowledge about randomness emerged in the form of heuristics, *situated abstractions,* that were deeply shaped by the setting. These situated abstractions had a feeling of determinism about them in the way they expressed causal factor and consequence, where the causal factor took the form of a phenomenalised piece of mathematical knowledge.

On the other hand, such a deterministic emphasis may reinforce the centralised mindset and militate against the construction of distribution as an emergent phenomenon (Prodromou, 2004) that may be needed to bridge the data-centric and modelling perspectives on distribution.

Grotzer and Perkins (2000) have provided a means of clarifying this apparent paradox. They have proposed a taxonomy that organises increasing complexity of causal explanation. The taxonomy comprised causal explanations organised in four dimensions of causal complexity was characterized: 1) *Underlying Causality* involving the causal mechanisms that are applied to an explanation in the form of surface generalizations, which portray the regularity in a generalised way; 2) *Relational Causality* pertaining to the patterns of interactions between causes and effects; 3) *Probabilistic Causality* referring to the level of certainty in the causal relationship. 4) *Emergent Causality* dealing with the compounding of many causes and effects that yield novel, and not always easily anticipated, outcomes.

In this study, we portray the co-ordination of different expressions by which students forged new conceptions about the interplay of causality and variation, and we relate those expressions to the above dimensions of causality through a tentative sketch of the forms of causality implicit in mastering the concept of distribution.

To that end, we designed a virtual environment to support students in discriminating and moving smoothly between data as a series of random outcomes at the micro level, and the shape of distribution as an emergent phenomenon at the macro level. We intended to use that environment as a window on the evolution of students' thinking-in-change (Noss & Hoyles, 1996) about the two perspectives on distribution. Through that window, we explored how students might co-ordinate the data-centric and modelling perspectives on distribution.

APPROACH OF THE STUDY

We used a design research methodology (Cobb et al. 2003), resulting in the Basketball simulation described below (Figure 1).

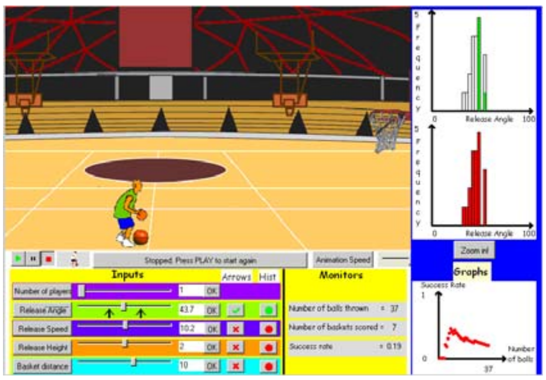

Figure 1. The Basketball simulation.

The animation of the basketball player's throws was controlled using the sliders for release angle, speed, height and distance. Initially, the path of the ball was completely determined by the settings of the sliders. However, the students were able to switch on the error, in which case the angle was selected from a distribution of values, centered on the position of the slider. When the error was switched on, two arrows appeared. The students were able to move these arrows to increase or decrease the spread of angles around that centre. The simulation also allowed the students to explore various types of graphs relating the values of the parameters to frequencies and frequencies of success. The students were able to inspect a linegraph of the success rate as well as a histogram of the frequency of successful throws or throws in general against release angle (or release speed, or height, or distance). In the first instance, the students were challenged to throw successfully the ball into the basket. Once the preliminary task was completed, they were asked whether they felt that the animation was realistic. Some discussion about the realism of the simulation normally introduced notions such as skill-levels and the use of error buttons. The subsequent task was to model a real but not perfect basketball player.

In this paper, we concentrate on the work of just two students, Anna and James, in order to illustrate the ideas that emerged from the analysis of the whole group of thirty four students involved across the four iterations. Their on-screen activity was captured on video-tape. We transcribed those sections to generate plain accounts of the sessions. Subsequently, we analysed the plain accounts in order to infer explanations for the students' actions and articulations.

ARROWS AS AGENTS OF ANTICIPATED BEHAVIOUR

Having already found how to throw the ball into the basket, Anna and James were first challenged to simulate a real basketball player (one who was not successful on *every* throw). James and Anna chose to switch the arrows on for two of the variables (Figure 2):

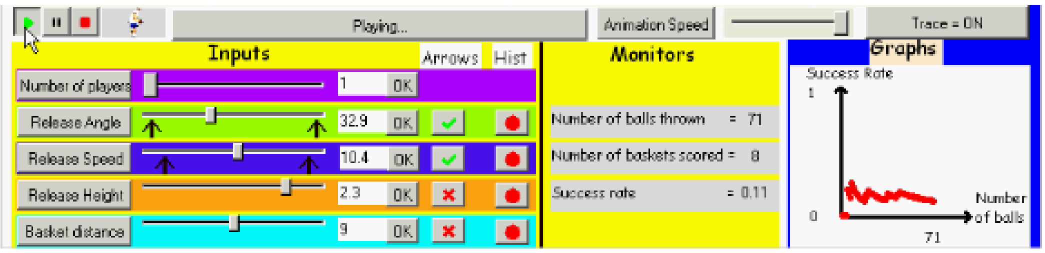

Figure 2. In order to simulate a real basketball player, Anna and James set up about a task by switching the arrows on for release angle and speed.

1. Interviewer: How can we make him behave as a real basketball player?
2. Anna: Not as much variation on the …
3. James: But after … after yeah … keep the variation a lot smaller compared to a … as far on the slider … but also … ehm … also after like certain … I don't know like 20 shots … Make the arrows wider apart a bit, because he will be getting tired because he is in a real match.

Shortly after, James and Anna introduced variability through the arrows to all the variables, moving the arrows very close to each other (Figure 3).

4. Interviewer: What do you expect now?
5. Anna: The success rate would be higher than it was before.

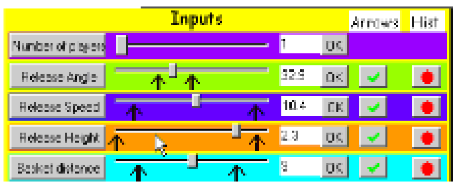

Figure 3. James and Anna introduced the arrows to many variables.

Anna expected the success rate to be higher. To her surprise, she encountered a success rate which was lower than the one they had before. We asked them to explore how they can set arrows on for all the variables, but at the same time have a high success rate. Anna began to question her assumption.

6. Anna: No, I don't think you should have the arrows for everything…because that's too much variation.
7. James: It's got more options to miss.

Arrows were set on for just three variables, release angle, release speed and distance to the basket (Figure 4).

8. James: The arrows are very close to the … where the bar is, so we won't have a lot of variation.
9. James: Oh, no! Lots of variation.

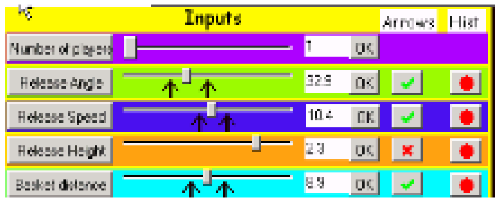

Figure 4. The students moved all the arrows very close to each other.

Their challenge seemed to have become on of balancing a conflict: On the one hand variation had been reduced by moving the arrows close to one another, but on the other hand more variation had been introduced by switching on the arrows for three or more variables.

Eventually Anna and James managed to obtain an amount of variation they liked, and we asked them about the graphs (Figure 5).

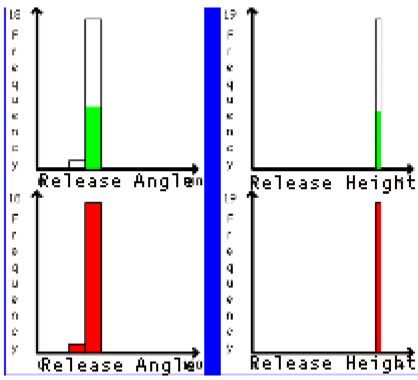

Figure 5. The students' understanding of variability in the current context affected their understanding of the histograms of frequency of successes against any variable.

10. Interviewer: Do you think that now he is behaving like in real life?
11. James: Yeah…It's realistic.
12. Interviewer: What will the graphs look like?
13. Interviewer: We use arrows but there is only one bar on this graph (referring to the graphs for release height). Why don't we've got two or three bars?
14. James: Because, they've not got a lot of space to change … like … cause the setting is just between those …

15. Interviewer: Look. There is another bar (referring to the graph for release angle).
16. James: It's choosing a different option between the two arrows … that's how we got more bars, but there are only two bars, because the distance between the arrows is small.

Thus, James had a breakthrough. He related the appearance of the two bars on the graphs to the distance between the arrows. When there was no gap, there was only one bar. When there was a gap, a choice needed to be made and so more than one bar appeared. A causal link between the distance of the two arrows and the number of bars seemed to be emerging. We note also how, at this point, James transfers the agency from the arrows to the computer (line 16).

STOCHASTIC CAUSALITY ARTICULATED AS SITUATED ABSTRACTIONS

We wish to discuss how Anna and James' work (and that of others in the sample) shows, despite the fact that the knowledge domain is that of random variation, a harnessing of causality in two interrelated ways. In this first section, we discuss the role of situated abstractions, whereas later we refer to the taxonomy of causal complexity.

Anna and James frequently articulated the situated abstraction, "the closer the arrows are, the higher success rate we have". This heuristic was often stated openly and could easily be detected as guiding their activity. To their surprise however, when switching the arrows on for several variables, even when the arrows were very close to each other, they encountered a disappointingly low success rate. They constructed a competing situated abstraction, "the more variables which have the arrows switched on, the greater is the variation". Anna and James' activity became one of managing that conflict, a process that the expert might see in terms of understanding the effect of the accumulation of errors.

James in particular coordinated the two conflicting situated abstractions through the new heuristic, "the more options to miss, the less likely they will score". Essentially, James and Anna discovered and exploited for explanatory power a causal link between the gap within the arrows and the success rate. This causal link proved to have further power when they tried to make sense of the graphs. The number of options to miss "determined" the number of bars in the graphs. (We make no apology for the use of the term 'determined' here, although we recognize that we use the term in our sense of stochastic causality by admitting the quotation marks.)

STOCHASTIC CAUSALITY WITHIN CAUSAL COMPLEXITY

We began with this study with some excitement about how we might characterise the notion of stochastic causality, and we recognise that this is work in progress, Nevertheless, Anna and James show us not only the important role of situated abstractions but also the place of such thinking in a taxonomy of complex causality.

With the arrows switched off, Anna and James found of course a direct causal link between the sliders and the path of the ball. Once the arrows were switched on, the causal link seemed to disappear. Eventually though, new causal links were established between the arrows and the aggregation of throws, seen either in terms of success rate or bars in the histograms. At this aggregated global level, causality can be harnessed to articulate the relationship between the parameters of the model (average, spread) and the shape of the distribution. Anna and James saw how the handles and arrows have a stochastic causal control over the data-centric perspective on distribution. (In contrast, the handles and arrows have a direct deterministic causal control over the modelling distribution.)

How does that emerging sense of stochastic causality and distribution relate to Grotzer and Perkins' taxonomy (2000) of causal complexity? At the outset, James (line 3) resorted to surface generalizations such as the tiredness of the basketball player. Subsequently, they appeared to move from the sliders as directly impacting upon the flight of the ball, through the messy meaninglessness of variation, to a situated but emergent causality that generates the data-centric distribution. As indicated by the more/less structure of their situated abstraction, Anna and James connected, again in a situated way, the spread of the histogram to the gap between the arrows, which we see as an important step in appreciating the accumulation of errors that lie at the local level but nevertheless constitute at the macro level the emergent phenomenon.

FINAL COMMENTS

We recognise though that the claim that thinking-in-change about distribution can centre around the role of causality is to some extent out of line with conventional thinking, which tends to make clear and distinct separations between the stochastic and the deterministic (for example, Piaget and Inhelder, 1975, portray randomness as inconceivable within operational thinking, at least until resolved by the invention of probability at a later stage of development). It is quite feasible, indeed we would suggest likely, that the role of causality is directly linked to the virtual nature of the setting for this study. By using phenomenalised quasi-concrete objects (Papert, 1996) such as on-screen sliders, we empowered students to conceive of the data-centric distribution as generated in a quasi-causal way out of the modelling distribution. Such an approach seems to allow the students to embrace determinism, rather than see the stochastic world only in contrast to the deterministic.

This paper reports on the connection made between the handles and arrows and the data-centric perspective on distribution. Although it can be argued that the handles and arrows are instantiations of parameters within the modeling distribution, we acknowledge that in this aspect of the research, we have underplayed the role of the modeling distribution. We note for example that James was consistently using the word "option" instead of other words, such as "chance" or "probability" to express the randomly selected throws by the computer. Elsewhere, we will report on how students also made connections with the modeling distribution itself and in some cases used the language of probability.

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STRUCTURAL DETERMINISM AS HINDRANCE TO TEACHERS' LEARNING: IMPLICATIONS FOR TEACHER EDUCATION

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In this paper, I use Maturana and Varela's (e.g., 1992) theoretical construct of structural determinism as a lens to better understand and discuss specific events of teachers' learning or non-learning in various situations. Through excerpts from the literature and data from one of my projects, I illustrate teachers' personal orientations that guide their potential learning. These interpretations have implications for teacher educators, who need to become more than facilitators or guides in order to trigger learning opportunities for/in teachers.

INTRODUCTION – TEACHERS' LEARNING ISSUES

This paper is partly theoretical, offering a perspective for thinking about mathematics teachers' knowledge and learning, and partly practical, using data from the research literature and my own studies to illustrate and make sense of the points highlighted. The discussion is framed around an intention to theorize and develop greater understandings of *teachers'* learning. One central aspects of conducting research in teacher education is to study phenomena of teachers' teaching and knowledge of how to teach (what, why and how teachers know and act), that is, to better understand who they are and what they know as learners of mathematics teaching and as teachers of mathematics. These understandings can lead to enriched conceptions of teachers' learning/knowledge which can in turn contribute to the constant endeavour to improve the teacher education practices that mathematics teachers are immersed in.

One of last year's conference research forums was "Learning through teaching: Development of teachers' knowledge in practice," by Roza Leikin and Rina Zazkis (2007) and collaborators, and focused on teachers' opportunities to learn in and from their everyday teaching practices. One of the contributors, Martin A. Simon (2007), offered us a different view on teachers' learning, talking about the possible *limitations* of teachers' learning from their practices. Using the insightful but catchy phrase "we see what we understand," he talked about how teachers can be limited from their own knowledge in what they can learn from their practices in schools. Simon's interesting contributions and explanations reminded me of some related issues found in the literature on mathematics teacher education.

One came from the study of Grant, Hiebert and Wierne (1998) who offered videos of reform-oriented mathematics teaching practices to teachers for them to see examples of such practices, make sense of them, and draw out some principles for their own practices. It was found that a number of teachers, who saw "mathematics as a series of procedural rules and hold the teacher responsible for students acquiring these rules" (p. 233), were not able to see significant differences between these practices and their own and tended to focus on technical aspects of the lesson and the material used. For Grant et al., this showed how offering videos of innovative practices were not a panacea for teachers' professional development since teachers needed to be able to appreciate what was in the video to see these differences and innovations.

A similar issue arose in Fernandez's (2005) research report on a collaborative lesson-study environment for elementary teachers. Playing the role of a facilitator, she realized after a while that the mathematical tasks teachers were exploring were not as insightful and provocative to the teachers as she thought they might have been and noted that "often the exchanges that took place did not push the teachers' thinking as far as they could have or sometimes even took them in unproductive directions" (p. 278). Fernandez then wondered if the teachers' mathematical knowledge was limiting to some extent what they were able to draw or not draw from the tasks themselves.

As well, my own research report at last year's conference (Proulx, 2007a) described secondary teachers' attempts to make sense of student solutions for a rate of change question in which conventions came into play. On analysing a solution where a student had inversed the roles of Δx and Δy in calculating slope, the teachers concluded that the student "did not understand anything about variation" and that nothing more could be drawn from such a solution by a teacher – it was just wrong. It appeared that the teachers were conflating the (arbitrary) order of the rate of change with its conceptual understanding. Teachers' own understanding of rate of change oriented their interpretation of the solution, leading them to not perceive some of the student possible understandings of variations in a graph. The same happened for the order in the coordinates of the Cartesian plan, where teachers did not see (x, y) as an arbitrary convention and interpreted students' solution in relation to it.

Here, one could point to teachers' lack of knowledge or inabilities to make sense of differences and worthwhile mathematics in these studies. However, what we need to understand more deeply is the rationale for and the mechanisms that are operating in these situations. The theory of cognition of Maturana and Varela (e.g., 1992), especially their concept of *structural determinism*, can shed light on some of these issues and help make sense of them. Below, I outline aspects of their theory and use it to interpret the above situations. Then, I use excerpts from my own research to illustrate additional ways of understanding the phenomenon of teachers' learning, and I raise some implications for the role of teacher educators in these situations.

MATURANA & VARELA'S THEORY AND STRUCTURAL DETERMINISM

Maturana and Varela's (e.g., 1992) theory of cognition is grounded in biological and evolutionary perspectives on human knowledge and processes of meaning making. Fundamental to this theory, and rooted in Charles Darwin's theory of evolution, are notions of *structural coupling* and *structural determinism*. Darwin used the concept of "fitting" to make sense of the process of survival of species. Hence, for species to survive, it must continuously adapt to its environment, to *fit* within it. If not, it would perish. The concept of fitting is, however, not a static one in which the environment stayed the same and only the species evolved and continued to adapt. Darwin explained that species and environment co-evolve, and Maturana and Varela added that they *co-adapt* to each other, meaning that each influences the other in the course of evolution. This idea of co-evolution/co-adaptation is key in regard to the origin of changes or adaptations of the species to its environment. Maturana and Varela call this *structural coupling*, as both environment and organism interact with one another and experience a mutual history of evolutionary changes and transformations. Both undergo changes in their structure in the process of evolution, which makes them "adapted" and compatible with each other.

> Every ontogeny occurs within an environment […] it will become clear to us that the interactions (as long as they are recurrent) between [organism] and environment will consist of reciprocal perturbations. […] The results will be a history of mutual congruent structural changes as long as the [organism] and its containing environment do not disintegrate: there will be a *structural coupling* (1992, p. 75, emphasis in the original).

Here, the environment does not act as a selector, but mainly as a "trigger" for the species to evolve – as much as species act as "triggers" for the environment to evolve in return. Maturana and Varela explain that events and changes are occasioned by the environment, but they are determined by the species' structure.

> Therefore, we have used the expression "to trigger" an effect. In this way we refer to the fact that the changes that result from the interaction between the living being and its environment are brought about by the disturbing agent but *determined by the structure of the disturbed system*. The same holds true for the environment: the living being is a source of perturbations and not of instructions (1992, p. 96, emphasis in the original).

Maturana and Varela call this phenomenon *structural determinism*, meaning that it is the structure of the organism that allows for changes to occur. These changes are "triggered" by the interaction of the organism with its environment. They give this example: A car that hits a tree will be destroyed, whereas this would not happen to an army tank. The changes do not reside inside of the "trigger" (inside the tree), rather they come about from the organism interacting with the "trigger." The "triggers" from the environment are essential but do not determine the changes. In short: changes in the organism are dependent on, but not determined by, the environment.

INTERPRETING DATA IN LIGHT OF STRUCTURAL DETERMINISM

If one uses structural determinism to make sense of the learning process, one understands that the response of the learner is dependent on the environment he/she is put in (e.g., video watching, mathematical tasks, students answers), *but* is determined by the learner's own way of making sense and interpreting. Thus, the response to a stimulus is not in the stimulus *per se* but is in the person that responds to it.

In the case of Grant et al. (1998) study, one could interpret that it is not that the teachers did not see the differences between the reform-oriented practices presented in the videos and their own classroom practices, but that for them these were not present as differences. In order to notice or appreciate these potential differences, the teachers needed to be aware of the possibility of these differences and be able to understand what they were. The same can be said of the teachers in my study (Proulx, 2007a), who did not distinguish between the usage of mathematical conventions and mathematical understanding. The teachers needed to be aware of issues of conventions in these situations to make sense of them in the student solution, which did not appear to be the case. As well, in Fernandez's (2005) study, it is not that the teachers did not see the mathematics in the problem, but that the mathematics that *she* saw in it was simply not present for these teachers. Simon's point that "we see what we understand" is of relevance to these potential learning situations: teachers' knowledge, their structure, did not allow them to "see" these distinctions (that were apparent to the teacher educators presenting the learning opportunities).

In fact, *we* stating that there are differences in the videos or mathematical aspects in the tasks offered is also representative of our own "blindness": we simply do not realize that it is *we* who sees them and that someone else could not see these distinctions. Or, simply, we make the assumption that these properties are present in and of themselves in the tasks and that these would determine teachers' reactions – leading to our conclusion that "Hum! They did not see that." It is not that they did not see, but simply that there was nothing for them to see.

This said, as well as our knowledge/structure can lead us to "not see" some aspects, our knowledge/structure can orient us to focus on other aspects; or, in a sense, to "*only* see" some aspects that we are oriented towards. To use Maturana and Varela's words, our structure leads us toward ways of understanding the world: what we understand is a function of our knowledge and is influenced by it. I further illustrate this idea through reporting on other data excerpts from one of my research projects.

THE OTHER SIDE OF THE STORY: TEACHERS' ORIENTATIONS

Recently, I set up a year-long professional development initiative for secondary mathematics teachers (Proulx, 2007b). The six teachers that participated in the project wanted to improve and rejuvenate their teaching practices. They felt their mathematical knowledge was too focused on procedural knowledge and that this impacted their teaching practices and ways of making sense of mathematics. Some of them expressed the following: "Why is it that we are not able to solve by reasoning? It is because we have not been educated to reason in mathematics. Me, I did copy, paste, repeat, and let's go. And I had 95% in mathematics!" or "I never understood why it worked. When students ask me why, I simply say that this is how it is!" Thus, the in-service sessions were structured around the study of school mathematics concepts and focused on sense-making and mathematical reasoning, rather than an application of procedures, for teachers to explore these issues. Through the work on specific mathematical tasks during the year, interesting characteristics of teachers' orientations arose. I underline these to illustrate how their knowledge, as structure determined beings, oriented them to engage in these tasks in particular ways.

Engaging in tasks at the procedural level

Through the sessions, teachers often appeared to approach mathematical problems in a technical fashion. When a problem was offered to them, their initial approach was often to look for the procedure to apply; almost as a reflex. It appeared as if this was their natural orientation to engage in mathematical problems. For example, in the following problem (Figure 1), teachers looked for and attempted to apply the area of the square formula, even if this approach was quite inefficient here.

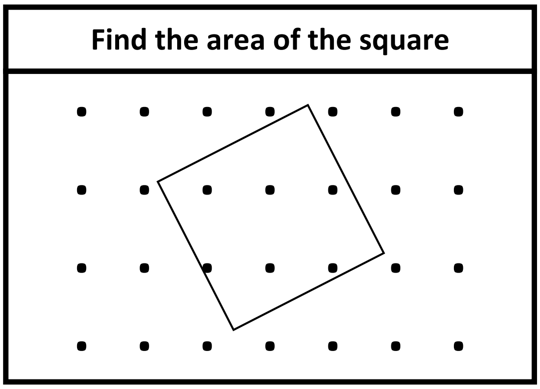

Figure 1. Problem on the area of a square.

It appeared as if these teachers' structure, a structure heavily focused on mathematical procedures, oriented them to work on problems through procedures. But, throughout the year, because the in-service program was focusing on aspects beyond procedures, the teachers became more and more aware of the possibility of entering differently into problems. Hence, these first attempts, which often resulted in a blank outcome, had an interesting effect on teachers as they themselves started to realize that there was something else to understand and work through, and attempted to probe from a different angle. In fact, the teachers often expressed out loud that it was their own procedural orientation that had lead them along this way and that they needed to work at getting away from this orientation. As sessions went on, the possibility of working differently on problems became more present to them (i.e., part of their structure) and they were able to explore these avenues.

Looking for techniques in mathematics

The teachers frequently expressed that their students had difficulties in specific mathematical domains and that they had been looking for precise mathematical techniques to communicate them for helping avoid errors and solve problems better. The issue was that some of these looked-for techniques appeared to not make much sense. For example, during a session focused on the creation of algebraic equations from word problems, the teachers continuously tried to find a specific mathematical technique to easily create the algebraic equation. This technique, for them, would help to avoid students' mistakes. Many options were offered as techniques: using other letters than *x* or *y*; underlining key words; creating an intermediary step where students would need to write down what each unknown represents; writing down the relations in a table; and so on. All of these approaches had at least some value, but the teachers' intentions was to find "the one" and to use it as an algorithm they could present to their students, with precise steps to follow in order to obtain the correct answer. After a while, they realized that these techniques were insufficient since one still had to make sense of the problem and the relation between the data in order to write down the equation and therefore that it could not be reduced to a simple technique to apply. Here, it appeared that the teachers' procedural orientation towards mathematics was leading them to look for even more procedures in mathematics.

Technical reading of the curriculum

This procedural orientation often brought the teachers to interpret the mathematical topics of the curriculum as expectations for working on techniques, algorithms and formulas. For example, when we worked on volume of solids, for the teachers this topic meant giving and demonstrating the volume formulas to students; that is, seeing volume as being only about its formulas. The same was true for analytical geometry, understood as a number of formulas of distance, middle points, etc., or for fractions, seen as a request to learn the addition, subtraction, multiplication and division algorithms, and so on. As Bauersfeld (1977) explains, some teachers develop technical "eyes" to read and interpret the notions and topics of the curriculum. In this case, the curriculum notions are read as requests to work on procedures. As studies have shown (e.g., Putnam, Heaton, Prawat, & Remillard, 1992; Ross, McDougall & Hogaboam-Gray, 2002), curricular changes alone seldom affect changes in teaching practices, something that can be explained by teachers' orientation to reading these topics: teachers read what they understand, their structure orients their reading.

IMPLICATIONS FOR EDUCATION: DEFINING THE LEARNING SPACE

Given the examples offered above, we might reread Simon's comment of "we see what we understand" to also mean "we only see" aspects one is oriented toward. This is in fact what Maturana and Varela explain: as structure determined beings, our structure orients the sense we can make of a situation (enabling or orienting). Thus, tasks, situations or contexts do not possess "the learning" for teachers; rather this learning is determined by teachers' own structure as they interact with these tasks and situations. But, what implications does this have for the educational act or educational initiatives? If teachers can only see what they already understand, it could mean that nothing new can be worked on with them since it needs to be already known by them. If this is so, this would mean that our structure not only determines what we learn, but also restricts us and stops us from learning. This is clearly not so, and Maturana offers some explanation of this process, which places an importance on the outside environment to provoke reactions/learning from the learner.

Maturana (e.g., 1987; Maturana & Mpodozis, 1999) makes a distinction between our structure and our actions, which he terms "conducts." The conducts demonstrate, and are permitted by, one's structure/knowledge. These conducts are determined by the structure, but are also dependant on the environment in which they are enacted. It is in the interaction with the environment that one's conducts arise, in its structural coupling with it. So, these conducts are coupled with, and embedded in, the environment within which they are made possible. Our conduct is a product of both. And, it is in this space that lays the potential of learning and of change, where the environment acts as a trigger on the learner's conduct.

By emerging from the coupling of one's structure and environment, the triggered conduct is "new" or created from this coupling. Thus, this conduct triggers back in return. It triggers back one's structure, by having emerged from a world of possibilities in the coupling of structure and environment. This emergent possibility, this conduct, influences (read, trigger) the structure itself and offers it new possibilities; possibilities for change, for learning. Thus, this generated conduct in return affects one's own structure. [As well, in this structural coupling, the conduct triggers changes in the environment. However, I am not addressing this here.]

This is how our structure evolves, through a continuous interplay between our structure (that determines possible conducts) and the conduct that emerges from the interaction/coupling with the environment. Our structure triggers conduct, and this conduct, by carrying aspects of the environment with it and therefore being un-thought of and possessing a new character, triggers our structure in return. It is a circular, never-ending, loop of structural change. One can infer that a new environment has the potential to trigger new conducts which in return can trigger changes in one's structure. Therefore, what is present in the environment is of fundamental value to generate these new conducts and in return to generate the structural transformations. The potential for education, for learning, lays here, where the environment the learner is placed in can trigger some conduct as the learner is in constant interaction/coupling with this environment.

FINAL REMARKS: SEEING TEACHER EDUCATORS AS TRIGGERS

From this understanding of the learning space, the environment in which the teacher is put in is of fundamental importance to trigger learning. One needs to see, in the teacher education environment, the presence of specific tasks as well as the presence of the teacher educator. The teacher educator acts as a trigger for teachers and has the opportunity to open new possibilities for teachers, new ways of making sense and of understanding (however, not implying he or she possess the "truth"). It is by bringing or throwing "something" into the learning environment that the teacher educator can create something and potentially trigger teachers' learning. This, therefore, calls for the teacher educator to be very active in the educational process, where his or her actions are to be seen as triggers for teachers' learning. This view challenges the view of an educator as mere facilitator or guide (Kieren, 1996). It calls for a teacher educator that puts oneself and act vigorously in this learning space to trigger and provoke something in teachers. On this, I conclude by citing one of Fernandez's (2005) remarks as to the importance of the role of teacher educators:

> This learning [of teachers] was no doubt possible because lesson study created a rich learning environment for these teachers in very much the same way that rich classroom tasks like those employed in reform classrooms set up opportunities for students to learn. However, although students learn a lot from working on such tasks, nevertheless a teacher who can push, solidify, and sometimes redirect their thinking is critical. Similarly, the teachers described here could have benefited from having a "teacher of teachers" help them make the most out of their lesson study work. (p. 284)

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PARENTAL ENGAGEMENT IN THEIR CHILDREN'S MATHEMATICAL EDUCATION: MOMENTS OF INCLUSION, MOMENTS OF EXCLUSION[1]

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In this paper we use the framework of communities of practice (CoP) to analyze the interaction of low-income Latino parents and the school system. In particular we use the concept of boundary practices (Wenger, 1998) to address the moments of inclusion or exclusion of parents in their children's mathematics education. The view of learning of mathematics as participation in particular communities brings to the forefront current power imbalances. There is an imminent need for mathematical practices that include mathematics as a cultural activity that draws on the resources of members of different communities, including parents.

INTRODUCTION

Parental engagement in children's mathematics learning must be understood as a product of the historical and socio-political context of schools influenced by educational policies, research, and teaching practices, among other factors such as race or income level of students' families (Calabrese Barton, Drake, Perez, St. Louis, & George, 2004). For over ten years we have conducted research on this topic in working-class Latino communities. The following quote is from an immigrant Mexican mother reflecting on her experiences with her children's school in the U.S. She shared it with us during one of our mathematics get-togethers in which not only we explored mathematics but we also engaged in conversations with the parents about their children's schooling with a particular focus on their mathematics teaching and learning.

> Esperanza: You count but you don't count, you are there but you are not, you symbolically go, but…

Her remark talks to her experience of parental involvement in which schools often control the form and aspects in which parents are invited to participate. It underscores the fact that parents' presence does not mean they have access as legitimate participants, that is, opportunity to participate in defining the enterprise, creation of mutual relationships, and in the negotiation of meanings. This access is often an uphill battle for those from disenfranchised communities. In this paper we analyze the interaction of parents and the school system using the framework of communities of practice (CoP); in particular we use the concept of boundary practices (Wenger, 1998) to address particular moments of inclusion or exclusion in children's mathematics education and their implications.

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THEORETICAL TOOLS

The use of the lens of legitimate peripheral participation (Lave & Wenger, 1991) considers central aspects of mathematics learning. It defines learning as participation which includes the whole individual in interaction with the world (Wenger, 1998). In our case, we define the community of practice as the classroom participants and mathematics learning as students' trajectories of participation in the particular mathematical practices.

Communities of practice are continuously in connection with other individuals and communities. The term "boundary practices" refers to those relations of a community with the outside world (Wenger, 1998). These practices have the purpose of simultaneously establishing connections and boundaries of a CoP. Thus, they create contradictions of inclusion and exclusion to a particular community. A fundamental component of culturally relevant education (Ladson-Billings, 1995) is boundary practices that include the funds of knowledge of diverse communities and address power differentials. Opening the boundaries of mathematics learning has the potential to break its historical encapsulation and making schools democratic spaces. Boundary practices in disenfranchised communities may become "third spaces in which alternative and competing discourses and positionings transform conflict and difference into rich zones of collaboration and learning" (Gutierrez, Baquedano-Lopez, & Tejeda, 1999; pp. 286-287). Parental engagement in mathematics education can transform current power structures that disenfranchise Latino students and their communities.

CONTEXT AND METHOD

The research presented here is part of a larger study that took place in a fifth grade classroom at an urban elementary school in the southwest United States, in which ninety percent of the students are of Latino background and almost seventy percent of the students receive free or reduced lunch (an indicator of poverty level). All the students have some understanding of English and Spanish, however several of them predominantly use one of the two languages. The participants in the study were nineteen fifth-graders and the classroom teacher. We developed in-depth case studies for five of the students, and so the parents of these students are also participants in the study. All the case study participants are either Mexican immigrants or are of Mexican descent. The case-study students were selected based on the teacher's knowledge to include diversity in gender (2 boys; 3 girls), mathematical proficiency, and language fluency in English and in Spanish. In this paper, we explore how boundary practices present contradictions of inclusion and exclusion of Latino parents from a legitimate peripheral participation in their children's mathematics learning. We used ethnographic tools for the data collection which took place in three sites: the classroom, students' households, and two after-school programs. We also include a classroom observation of some parents to a mathematics lesson. During this visit children reviewed an exploration of the surface area of rectangular prisms. All these interactions were in English and Spanish. The investigation began with a connection to the everyday significance of surface area. Then the adults in collaboration with the children drew three-dimensional prisms, traced the faces, and described their observations. Some other activities were mathematics games and mathematics discussions around number equations. The classroom observation concluded with a discussion guided by the observations of the mothers and one grandmother about their children's mathematical learning experiences. The data are the video transcripts, field notes of classroom observations, and interview transcripts. The analysis is based on grounded theory, a process that explores emergent themes (Charmaz, 2001). The different sources were used to triangulate the information and build thick descriptions.

ARE PARENTS LEGITIMATE PERIPHERAL PARTICIPANTS?

We present the case studies of two mothers, Lorena[2] (her daughter Yessenia) and Monica (her daughter Maite), who share a similar view of their school experiences and attitude towards mathematics.

Lorena and Monica went to school in northern Mexico. After repeated retentions they both finished their elementary education in Special Education. Lorena mentions that the cause of her difficulties is her lack of memory, in other words, she soon forgets what she learns in school. This explanation was the teacher's complaint and the reason given for her repeated retentions. Lorena has a strong view of learning as an individual endeavor, which is grounded on personal capabilities. Monica explains her struggles in school with similar arguments; she believes she is a slow learner. Mathematics especially tints Monica's schooling experience with a grim tone. She describes her mathematics learning experiences as a headache throughout all her schooling years. The belief in her internal inability to learn and her memories in school kept Monica out of school as an adolescent and still keep her away from schools as an adult.

Homework as a Boundary Practice

Homework has been one boundary practice that connects the classroom community with these families. At home, these women feel limited in their ability to help their children directly. They often feel the limitation of their short educational history as well as the fact that most of the homework is in English. Monica explains she does not understand what Maite currently learns at school, even if the work is in Spanish. Monica does not help Maite with her homework anymore; she only checks it for completion when she arrives home from the after-school. She mentions her children in previous years would get upset because they wanted to solve it in the same way the school teaches them. She explains to the researcher:

> Monica: Sometimes the girls get upset because I teach them my own way. "That's not the way the teacher teaches me" and they get upset because I try to teach them, but it is the same, anyway you get the same answer, but they want to do it as they teach them in school

[2] All names of the participants are pseudonyms.

Researcher: Do you remember a specific example when this happened or what were you doing?

Monica: Oh yeah, one day, Maite got really mad because I strongly wanted to teach them, she, because she wanted me to help her, and I was helping her but she was not happy the way I was teaching her, because she says, "I want you to teach me the way the teacher teaches us," "that's not possible," I told her "because the teacher has some thoughts and I have others, and I learned in one way and she learned in another, and it's not the same. But you get, it's the same when it comes to the answer, it's going to be the same." But, then she got upset and so I better let her do it alone. (parent interview, February, 2006).

Maite was upset because she did not receive the help she needed from her mother. She felt more comfortable with the school's method rather than her mother's method. In this conflict, a learning opportunity that considers multiple ways to solve a problem is lost, in addition to ignoring families' funds of knowledge. Although this difference in the algorithms is not uncommon, especially when their education was in a different country, children are frequently expected to mediate between the asymmetrical relations between home and the school's knowledge (Civil & Andrade, 2002; Civil, Planas, & Quintos, 2005; Civil & Quintos, 2006). Based on these experiences with her children's homework, as well as Monica's educational history, she does not participate in these practices of mathematics learning with her children anymore. Nonetheless, she resists the exclusion from these school related practices. Monica uses her own experiences to alert her children about their opportunities and choices. She also tells them they do not have the right to steal her mother's dream. She said, "I tell them that I am their mother and they do not have the right to steal my dreams" (parent interview, March, 2006).

Lorena is also proactive against exclusion. She encourages Yessenia to ask for help. She also believes her central means of support is to maintain a close relationship with her daughter and to hold high expectations. Next we present the microgenetic analysis of an interaction between Lorena and Yessenia. This mother and daughter attend an after-school program at the school site that focuses on mathematics. While the overt purpose of the program is to help children with their mathematics homework, Lorena's personal purpose for attending is to reinforce her close relationship with Yessenia. The connection with the classroom curriculum is mainly sustained through Yessenia. She brought questions or shared topics discussed in class. Yessenia asked for help with her mathematics homework. Yessenia brought the following question from her textbook:

"Can you show 0.02 using only tenths place-value blocks? Explain."

The place value blocks equivalencies with decimals established were the following: A small block (cube) represents one hundredth; a "long" (ten blocks) represents one tenth, and the flat (one hundred cubes) represents the ones. The tutor explained to Yessenia the decimals using place value, but the tutor was not familiar with the use of place-value blocks with decimals and was not using them to explain the question. Yessenia built on her classroom learning experiences and shared her knowledge of these representations using drawings of the blocks with the tutor. She drew base ten blocks trying to make sense of the decimals. She explained to the tutor that with two longs or two tenths she could not represent two-hundredths because they were smaller. Yessenia was unsure of her statement so she also represented the decimals drawing money. This time she explained to the tutor "I have two pennies and that [long] is two dimes." During this interaction, Lorena's participation consisted of watching Yessenia's efforts and the non-verbal cues of the tutor to evaluate Yessenia's explanation (field notes, February, 2006).

In this interaction there are two central resources that set the elements of boundary for Lorena, the language of interaction and the mathematical meanings in negotiation. Lorena becomes an outsider as soon as Yessenia reads the question in English. Although Yessenia is now in a bilingual classroom, her educational history did not support her development of academic Spanish. In this way the history of a practice that included only English situates Lorena as an outsider when she tries to participate in Yessenia's mathematics learning. This is especially contradictory when Lorena's goal for participating in the after-school program is to reinforce her close relationship with Yessenia. In this manner, the language choice for homework and instruction influences the access of parents to their children's mathematical learning.

The second structuring resource in these interactions connects to the negotiation of mathematical meanings. Lorena's schooling experiences taught her that only some children are innately good in mathematics while others are not born to become members of mathematical learning communities in school. In the example discussed above, Lorena is situated as an outsider not only due to language issues but also because she did not remember learning decimals at school and views school mathematics as a subject matter disconnected from her common experience. This practice evokes her personal history of exclusion in her mathematics education experiences. In contrast, Yessenia's experience with learning mathematics is one that focuses on creating meaning (e.g., from the abstract numbers of two-hundredths and two-tenths to the place value blocks and her experiences with money). In her classroom, mathematics is treated as a language or tool to create meaning. Yessenia turned to these connections with world experiences (e.g., money) and was able to make sense of the decimal numbers. Yessenia, therefore, did not conceive of mathematics as a series of procedures or rules to be memorized or practiced. This position is radically different in that it empowers her over the mathematics. Yessenia's approach to mathematics contests the relationship towards mathematics existent in many educational settings.

Classroom Observation as a Boundary Practice

We conclude with the analysis of a classroom observation that functions as a third space in which parents, teacher and children expanded their forms of interaction. In this activity the participants dialogue and share their perspectives. The hybrid linguistic practices and mathematical meanings allow parents to participate in this practice. This boundary practice impacts participants' views on mathematics, the community, and themselves.

The bilingual community welcomes the mothers and grandmother as legitimate peripheral participants. While the teacher uses both languages, each child chooses the language to interact with their family. This language is based on an emphasis on a learning environment in which children feel safe and celebrate mistakes. The negotiation of mathematical meanings is based on experiences that underscore participants' previous knowledge of surface area and supports it through a collaborative community and concrete tools. The teacher invites the adults and children to start thinking of surface area in their everyday experiences and then she connects the concrete prism, geometrical representations in two-dimensions, and the formula of surface area. In this way, mathematics in this classroom becomes a human practice that supports sense-making and opens the boundaries for them to be legitimate peripheral participants. In an interview after this experience, Lorena redefines her view of mathematics connecting to her conversations with the teacher about mathematics as a communicative competence to create meanings. Furthermore, she also reconsiders her ability as a learner. She shifts from a deficit view of herself to a critical analysis of her learning experiences. She describes herself using a new lens that included the analysis of the educational system in which she participated.

Lorena: I know that I am intelligent because now I can see it, but before, I don't know what happened to me, maybe I said to myself, "I'm not going to learn, I'm not going to learn" and maybe because of that I didn't learn, maybe I could have gone further, further than elementary school. …Yes, they teach [mathematics] differently too. They teach them differently because I only studied the times tables… I studied them and that was all, but they didn't explain and now they do explain, if you don't understand in one way they explain to you in a different way until you understand.

CONCLUSION

The data presented in these case studies suggest that the nature of the community of practice as well as the socio-historical development of its members play a critical role in the types of relationships established with parents. These examples of boundary practices decenter the notion of parental involvement and focus on the organization and history of the community, as well as the identities of the participants. Language practices and mathematics education practices play a defining role in these interactions. Mathematics practices do not negotiate meanings exclusively, but are element of boundary. They include elements of membership as well as elements of identity. Finally, our data indicate that a culturally relevant pedagogy includes an egalitarian dialogue with parents and between parents and children. The systematic exclusion of the funds of knowledge of Latino families, including bilingual mathematical communicative competencies, and mathematics as a knowledge base, will continue to exclude Latino students and their families.

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RELATIVE MOTION, GRAPHS AND THE HETEROGLOSSIC TRANSFORMATION OF MEANINGS: A SEMIOTIC ANALYSIS

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In this paper, we deal with students' transformation of meanings related to their understanding of Cartesian graphs in the context of a problem of relative motion. The investigation of the students' transformation of meanings is carried out in the course of a process that we term objectification, i.e., a social process related to the manner in which students become progressively conversant, through personal deeds and interpretations, with the cultural logic of mathematical entities. We provide a multi-semiotic analysis of the work done by one Grade 10 group of students and their teacher, and track the evolution of meanings through an intense activity mediated by multiple voices, gestures and mathematical signs.

INTRODUCTION AND THEORETICAL FRAMEWORK

A graph is a complex mathematical sign. It serves to depict, in specific ways, certain states of affairs. Instead of being merely a reproduction of these, a graph supposes a selection of elements: what it depicts is relationships between them. This is why the making of a graph of an elementary phenomenon, such as the motion of an object, is like putting a piece of the world on paper (or electronic medium). But because they are not copies of the phenomena that they depict or represent, making and interpreting mathematical graphs is not a trivial endeavour. A Cartesian graph rests on a sophisticated syntax and a complex manner of conveying meanings.

The investigation of the difficulties surrounding students' understanding of graphs has been an active research area in mathematics education, since the pioneering work of Clement (1989) and Disessa, et al. (1991), informed by Cognitive Science and Constructivism, up to the recent work of e.g. Arzarello and Robutti (2004), Ferrara (2006), Nemirovsky (2003), and Roth (2004), inspired by embodied psychology. This paper wants to contribute to the research on graphs by looking into students' processes of graph understanding. We are interested in particular in researching the way in which students attempt to make sense of graphs related to problems of relative motion–an area little investigated thus far.

Our research draws on a Vygotskian sociocultural perspective in which mathematical thinking is considered a cultural and historically constituted form of reflection and action, embedded in social praxes and mediated by language, interaction, signs and artifacts (Radford, 2006a). A Cartesian graph is an artifact for dealing with and thinking of cultural realities in a mathematical manner. But as mentioned previously this artifact is not transparent: it bears the imprint and sediments of the cognitive activity of previous generations which have become compressed into very dense meanings that students have to "unpack", so to speak, through their personal meanings and deeds.

This process of "unpacking" is the socially and culturally subjective situated encounter of a unique and specific student with a historical conceptual object–something that we have previously termed *objectification* (Radford, 2002, 2006a, 2006b). The construct of *objectification* refers to an active, creative, imaginative and interpretative social process of gradually becoming aware of something and oneself (Radford, 2003). Within this context, understanding the making and meaning of a graph, the way it conveys information, the potentialities it carries for enriching and acting upon our world, rests on processes of objectification mediated by one's voice, others' voices and historical voices (Boero, Pedemonte, & Robotti, 1997). Objectification is indeed a multi-voiced–or what Bakhtin used to call heteroglossic–encounter (Radford, 2000) between an "I", an "Other" and (historical and new) "Knowledge".

The distinctive historically and culturally mediated nature of human cognition is such that, in the objectification of mathematical knowledge, recourse is made to body (e.g. kinesthetic actions, gestures), signs (e.g. mathematical symbols, graphs, written and spoken words), and artifacts of different sorts (rulers, calculators and so on). All these signs and artifacts used to objectify knowledge we call *semiotic means of objectification* (Radford, 2003). In the practical investigation of students' understanding of graphs, we will hence pay attention to the students' discourse, gestures and symbols as they attempt to make sense of a graph.

METHODOLOGY: A MULTI-SEMIOTIC DATA ANALYSIS

Data Collection

Our data, which comes from a 5-year longitudinal research program, was collected during classroom lessons that are part of the regular school mathematics program in a French-Language school in Ontario. In these lessons, designed by the teacher and our research team, the students spend substantial periods of time working together in small groups of 3 or 4. At some points, the teacher (who interacts continuously with the different groups during the small group-work phase) conducts general discussions allowing the students to expose, compare and contest their different solutions. To collect data, we use four or five video cameras, each filming one small group of students.

Data Analysis

To investigate the students' processes of knowledge objectification we conduct a *multi-semiotic data analysis*. Once the videotapes are fully transcribed, we identify salient episodes of the activities. Focusing on the selected episodes, we refine the video analysis with the support of both the transcripts and the students' written material. In particular, we carry out a low motion and a frame-by-frame fine-grained video microanalysis to study the role of gestures and words.

The data that will be discussed here comes from a Grade 10 lesson about the interpretation of a graph in a technological environment based on a graphic calculator TI 83+ and a probe–a Calculator Based Ranger or CBR (a wave sending-receiving mechanism that measures the distance between itself and a target). The students were already familiar with the calculator graph environment and the CBR. In previous activities, they had dealt with a fixed CBR and one moving object. In the activity that we will discuss here, the students were provided with a graph and a story. The graph showed the relationship between the elapsed time (horizontal axis) and the distance between two moving children (vertical axis) as measured by the CBR (see Figure 1). The students had to suggest interpretations for the graph and, in the second part of the lesson (not reported here), to test it using the CBR. Here is the story: "Two students, Pierre and Marthe, are one meter away from each other. They start walking in a straight line. Marthe walks behind Pierre and carries a calculator plugged into a CBR. We know that their walk lasted 7 seconds. The graph obtained from the calculator and the CBR is reproduced below."

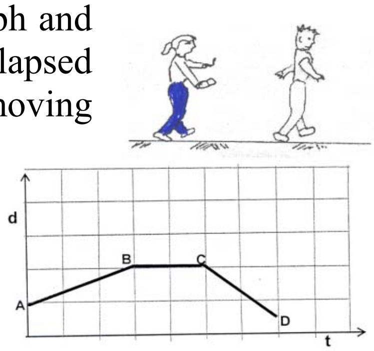


Figure 1. The Pierre and Marthe story, story illustration, and the graph given to the students.

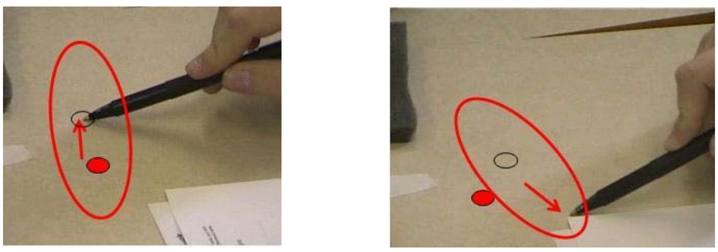
RESULTS AND DISCUSSION

We will focus on one 3-student group and present some excerpts of the students' processes of understanding, with interpretative commentaries on the progressive manner in which objectification was accomplished. The students were Maribel (M), María (MJ) and Carla (C). The students discussed the problem for a few minutes. In Line 1 (L1), Maribel gives a summary of the group discussion:

1 M: (*Moving the pen on the desk, she says*) He moves away from Marthe for 3 seconds and then (*moving the pen further along the desk; Picture 1*), he stops, so he might have like dropped something for 2 seconds, and (*moving the pain back this time; Picture 2*) he returns towards Marthe.

2 C: Well, even though he moves away, but he returns back to…I don't know.

3 MJ: Well, if she walks with him, so, it [the graph] doesn't really make sense!



Pictures 1 (left) and 2 (right). Maribel moving the pen on the desk to signify Pierre stops (segment BC) and Pierre returns towards Marthe.

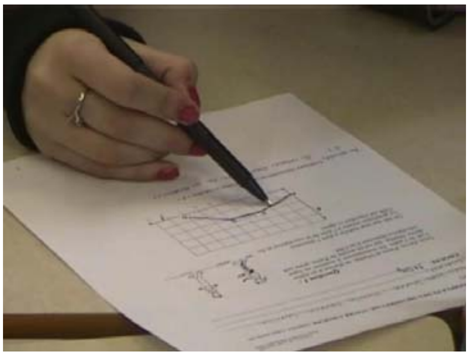
The students' first interpretation rests on the idea of "absolute motion". The segments AB, BC, and CD are interpreted as Pierre moving away, stopping, and coming back. Although the students' current interpretation is not yet aligned with the expected mathematical interpretation, we can see that the students' current interpretation has been forged through a complex coordination of perceptual, kinaesthetic, symbolic, and verbal elements. Maribel's dynamic pointing gestures are not merely redundant mechanisms of communication, but key embodied means of knowledge objectification. Through these gestures and their synchronic link with movement verbs ("to move away", "to come back"), Maribel offers an attempt at making sense of the graph. It is at the end of this episode that María reminds her group-mates that Marthe is moving too, so that, according to the current interpretation, the graph "doesn't really make sense!" Twenty seconds later, Maribel offers a refined interpretation that tries to address the issue raised by María:

4 M: Well technically, he walks faster than Marthe… right?

5 MJ: She walks with him, so it could be that […] She is walking with him, so he can walk faster than her (*she moves the pen on segment AB; see Picture 3*). [He] stops (*pointing to points B and C*)…

6 M: No, there (*referring to the points B and C*) they are at the same distance…

7 C: (*After a silent pause, she says with disappointment*) Aaaaah!



Picture 3. MJ moves the pen from A to B, meaning Pierre's motion (L5).

The graph interpretation has changed: In L4, Maribel introduces the two-variable comparative expression "X walks faster than Y". In L5, María reformulates Maribel's idea in her own words while producing a more sophisticated interpretation. Indeed, L5 contains three ideas: (1) Marthe walks with Pierre; (2) Pierre walks faster than her, and (3) Pierre stops. Although improved, the interpretation, as the students realize, is not free of contradictions. Even if, at the discursive level, Marthe is said to be walking (L5), segment AB is still understood as referring to Pierre's motion (see Picture 3). However, segment BC is interpreted not in terms of *motion* but of *distance* (L6). Moreover, it is interpreted as the distance between Pierre and Marthe. So, while segment AB is about Pierre's motion, segment BC predicates something about both children. The oddity of the interpretation leads to a tension that is voiced by Carla in Line 7 with an agonizing "Aaaaah!" The partial objectification bears an untenable incongruity.

The students continued discussing and arrived at a new interpretation: Pierre and Marthe maintained a distance of 1 meter apart throughout, but they could not agree on whether or not this interpretation was better than, or even compatible with, Maribel's interpretation (L4). Having reached an impasse, the students decided to call the teacher (T). When he arrived, María explained her idea, followed by Maribel's opposition: It is this opposition that is expressed in L8:

8 M: No, like this (*moving the pen along segment AB*) would explain why like, he goes faster, so it could be that he walks faster than her…

9 T: Then if one is walking faster than the other, will the distance between them always be the same?

10 M: No, (*while moving the pen along AB, she says)* so he moves away from the CBR and then….What happens here (*pointing to segment BC*), like?

11 MJ: He takes a brake.

12 T: So, is the CBR also moving?

13 M: Yes.

In L9, the teacher rephrases in a hypothetical form the first part of Maribel's utterance (L8) to conclude that, under the assumption that Pierre goes faster, the distance cannot be constant. Although inconclusive from a logical point of view, the teacher's strategy helps move the students' discourse to a new conceptual level. Maribel's L10 utterance shows, indeed, that the focus is no longer on relative speed but on an emergent idea of relative distance. The gesture is the same as María's in Picture 3, but its content is different. However, as shown in L10, the students still have difficulties providing a coherent global interpretation of the graph. How to interpret BC within the new relative motion context? Drawing on Maribel's utterance (L10), the teacher suggests a link between Marthe and the CBR, but the idea does not pay off as expected. He then tries something different:

14 T: OK. A question that might help you… A here (*he writes 0) at the intersection of the axes and moves the pen along the segment 0A*) What does A represent on the graph? (*he moves the pen several times between 0 and A; see Picture 4*)

15 MJ: Marthe.

16 T: This here is 0? We'll only talk about the distance. OK? (*He moves the pen again as in Line 14*)

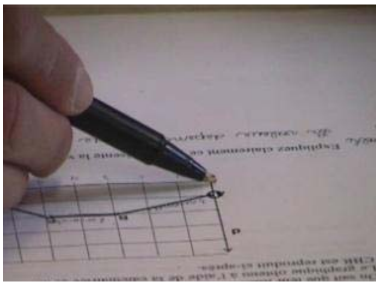
17 MJ: 1 meter.

18 T: It represents 1 meter, right? … 1 meter in relation to what?

19 M: The CBR.

20 T: OK. So, does it represent the distance between the two persons?

21 M: So this (moving the pen along the segments) would be Pierre's movement and the CBR is 0.



Picture 4. The teachers moves the pen back and forth between 0 and A.

22 MJ: (*Interrupting*) First he moves more…

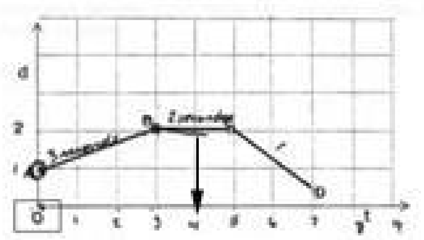
Capitalizing on the emerging idea of relative distance, the teacher's strategy now becomes to call the students' attention to the meaning of a particular segment–the segment 0A. He captures their attention in three related ways: writing (by writing 0 and encircling the point A); gesturing (by moving the pen between 0 and A back and forth); and verbally (L14). In L15, point A is associated with Marthe. In L16, he formulates the question in a more accurate way, and takes advantage of the answer to further emphasize the idea of the relative meaning of the distance. Line 21 includes the awareness that the CBR has to be taken into account, while L22 is the beginning of an attempt at incorporating the new significations into a more comprehensive account of the meaning of the graph.

The students thus entered into a new phase of knowledge objectification. They continued discussing in an intense way. Here is an excerpt:

23 C: He moves away from her, he stops then comes closer.

24 M: But she follows him… So, he goes faster than she does, after, they keep the same distance apart.

In L23, Carla still advocates an interpretation of the graph that suggests a fragile understanding of relativity of motion. In the first part, she makes explicit reference to Marthe ("He moves away from her"), but in the second and third part of the utterance, Marthe remains implicit. In L24, Maribel offers an explanation that overcomes this ambiguity. Even though the segment AB is expressed in terms of rapidity, the previously reached awareness of the effect of rapidity in the increment of distance makes the interpretation of BC coherent. The recapitulation of the students' efforts is made by Maribel, who, before the group start writing their interpretation, says: "Maybe he [Pierre] was at 1 meter (*pointing to A*) and then he went faster; so now he is at a distance of 2 meters (*moving the pen in a vertical direction from BC to a point on the time axis, see Picture 5and Picture 6*); and then they were constant and then (*referring to CD*) they slowed down. Would that make sense?"



Picture 5 and Picture 6. Maribel makes a vertical gesture that goes from BC to the time axis (we have indicated this gesture by an arrow). This gesture is a generalization of the teacher's gesture (Picture 4).

The students succeeded in refining their objectification, although some edges still remained to be polished. In the interpretation of CD, Maribel did not specify in which manner they slowed down. Was it Pierre who slowed down? Was the reduction of distance the effect of Marthe increasing her speed? Was it something else? These questions were discussed in the final general classroom discussion. In writing their answer, this group, however, realized that something important was missing. Naturally, writing requires one to make explicit, and thereby objectify, relationships that may remain implicit at the level of speech and gestures. Maribel's activity sheet contains the following answer: "Pierre moves away from Marthe by walking faster for 3 seconds. He is now 2 meters away from her. They walk at the same speed for two seconds. Pierre slows down for two seconds so he gets closer to Marthe".

CONCLUDING REMARKS

In this paper, we dealt with the students' transformation of meanings related to their understanding of graphs. The investigation of the transformation of meanings was carried out in the course of a process that we have called *objectification*, i.e., a social process related to the manner in which students become progressively aware, through personal deeds and interpretations, of the cultural logic of mathematical entities–in this case, the complex mathematical meanings that lie at the base of the ways in which Cartesian graphs are used to describe some phenomena and convey meanings.

Our data suggests that one of the most important difficulties in understanding the graph was overcoming an interpretation based on a phenomenological reading of the segments in terms of absolute motion, and attaining one that put emphasis on relative relations. Instead of representing the state of an object in reference to a fixed point, points in the second case represented and came to signify relationships between them and a moving point. As we saw, the logic of interpreting a Cartesian representation of relative motion became progressively apparent for the students through intense activity mediated by multiple voices, gestures and mathematical signs. The phenomenological interpretation of the graph was replaced by one centred on relative distances. Crucial in this endeavour was the teacher's intervention. The teacher was indeed able to create a successful *zone of proximal development* that afforded the evolution of meanings both at the discursive and gestural levels. Thus, after his intervention, in the same way that words became more and more refined, so too did gestures: while the students' first gestures were about Pierre's motion, their last gestures were related to distances in a meaningful relational way.

We want to submit that the successful creation of a *zone of proximal development* was due to the teacher's ability to find a common conceptual ground for the evolution of the students' meanings. The teacher brought out the students' meanings from behind, as it were, and helped them push their meanings beyond their initial locations. The coordination of words with the sequence of similar gestures and signs in the Cartesian graph (Pictures 4) helped the students understand the meaning of the segment 0A in the context of the problem. The segment 0A entered the universe of discourse and gesture, and its length started being considered as the initial distance between Pierre and Marthe at the beginning of their walk. Without teaching the meaning directly, the teacher's interactional analysis of the meaning of segment 0A was understood and generalized by the students in a creative way (Picture 5).

Borrowing a term from M. M. Bakhtin, we want to call the transformative process undergone by the students' meanings as *heteroglossic*, in that heteroglossia, as we intend the term here, refers to a locus where differing views and forces first collide, but under the auspices of one or more voices (the teacher's or those of other students'), they momentarily become resolved at a new cultural-conceptual level, awaiting nonetheless new forms of divergence and resistance.

Endnote

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REFUTATIONS AND THE LOGIC OF PRACTICE

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When arguments are refuted in mathematics classrooms, the ways in which they are refuted can reveal something about the logic of practice evolving in the classroom, as well as the epistemology that guides the teachers' teaching. We provide four examples that illustrate refutations related to the logic of practice, in which sufficiency and relevance are grounds for refutation, as opposed to falsehood.

INTRODUCTION

At first glance, refutations may seem to have little to do with teaching proof. Proofs are concerned with showing what conclusions follow from a set of premises, whereas refutations only tell use what conclusions *do not* follow. There are, of course, special cases, like proof by contradiction and contraposition, in which one seeks to refute one statement in order to prove its negation. However, we are not concerned with these cases here. Instead we are interested in the kind of refutations that appear in the proving process through which proofs evolve in mathematics classrooms, but which are not evident in the finished proof.

We are interested in these refutations in relation to what Toulmin (1958) calls the "logic of practice" which underlies proving processes in classrooms. That is, the logic upon which arguments are based in actuality, rather than the logic upon which one might like them to be based. As mathematics classrooms are contexts for learning, arguments in them are based on a logic in transition, from the everyday logic the students bring to the class to a mathematical logic accepted by the teacher. When arguments are refuted, the ways in which they are refuted can reveal something about the logic of practice as well as the teachers' purpose in engaging in argument in the first place and what epistemology guides her teaching.

BACKGROUND

As Balacheff (2002/2004) notes, the field of mathematics education includes approaches based on a number of distinct epistemologies. The role seen for refutations depends on epistemological factors. For example, for those whose focus is on the logical correctness of formal texts called "proofs", refutations do not play a role except perhaps in the special cases of proofs by contradiction and contraposition. Others' epistemology is based to some extent on Lakatos' (1976) view of mathematics, in which mathematics does not proceed by a process of proving theorems conclusively and then moving on, but rather through a cycle of proofs and refutations, with proofs being always provisional and refutations providing the mechanism for the improvement of theorems and their proofs. For those with this epistemology, proof is inextricably linked to refutations, and approaches to teaching proof from this perspective include an exploration of refutations as an essential element (e.g., Balacheff 1988, 1991, Sekiguchi 1991). Another epistemology for which proofs are essential is that founded on the concept of "cognitive unity" in which argumentation processes which may include refutations provide the basis for proof development (e.g., Boero, Garuti, Lemut & Mariotti 1996). Studying the role of refutations in classroom proving processes is important if one takes on an epistemology that gives an important role to refutations (e.g., one based on Lakatos or cognitive unity) but also for descriptive and comparative work looking at current teachers' practices, as a way to reveal the implicit epistemologies guiding teaching. It is such an interest in teaching practices that inspires our work. In classrooms we observe a *proving process* through which teacher and students produce a proof, and which can include refutations in important ways. In this paper we will describe a number of examples of refutations embedded in proving processes, their roles in those processes and what these roles suggest about the teaching practices and implicit epistemologies underlying them.

One of Toulmin's (1958) aims is to describe the layout of arguments in a way that is independent of the field in which they occur. In this paper we diagram arguments using a method derived from Toulmin's basic layout for an argument (see Figure 1). In this layout an argument is considered to consist of *data*, which lead to a *conclusion*, through the support of a *warrant*.

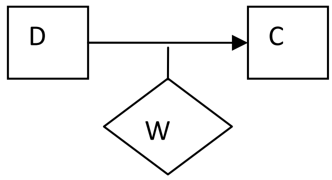

Figure 1. Toulmin's basic layout.

Toulmin does not consider refutations within this structure because he is considering arguments as they are once the assertion is established, not the process of their coming to be. However, Toulmin's first chapter deals extensively with refutations in order to explore how arguments in different fields are based on different criteria. There he gives examples of arguments in which an assertion is made which is true in one field but which can be refuted in another field.

Looking at Toulmin's basic layout three ways in which an argument can involve a refutation immediately suggest themselves. The data of the argument can be refuted, leaving the conclusion in doubt. The warrant of the argument can be refuted, again leaving the conclusion in doubt. Or the conclusion itself can be refuted, implying that either the data or the warrant is invalid, but not saying which. In the language of Lakatos (1976 pp. 10-11) the first two are local counterexamples, while the latter type is a global counterexample. Sekiguchi (1991) provides examples in a classroom context of several types of refutations within this framework. However, as we noted above, we are more interested in refutations where the focus of the refutation is not the data, conclusion or warrant, but rather the logic underlying the argument.

REFUTATION IN CLASSROOM ARGUMENTS

In our research* we have examined classroom arguments at upper elementary and junior high school, in Canada, Germany and France. In these contexts refutations sometimes occur, but in different forms and with different functions. Here we provide four examples along with discussion of the insight each gives us into the logic of practice, and the teacher's epistemology.

Refutation of a conclusion implied by a question

The conclusion that is refuted may not always be stated directly. In classrooms a common exchange is for the teacher to ask a question with the intent of pointing out an error. For example, in this exchange grade 5 students have been trying to develop a formula for how many squares there are in an *n* by *n* grid. They have been working with a concrete model in which three pyramids made of linking cubes are joined to make a roughly box shaped solid made up of $n\frac{1}{2}$, *n* by (n+1) layers. Here they are considering a 10 by 10 grid for which the solid has 10.5, 10 by 11 layers. They have multiplied these three numbers to find the total number of linking cubes used: 1155. (DAR is a guest teacher. For more background and details see Zack 2002, Zack & Reid 2003, 2004, and Reid 2002).

DAR: Right. So, 1155 is what you get if you multiply those three numbers.
Is that [1155] how many squares there are in a 10 by 10 [grid]? Q

Several voices: No A

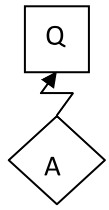

Figure 2

Here the question "Is that [1155] how many squares there are in a 10 by 10 [grid]?" implies the conclusion "1155 is … how many squares there are in a 10 by 10 [grid]" which the answer "No" refutes. This answer requires no further support as the students and DAR are all aware that there are 385 squares in a 10 by 10 grid. The jagged arrow in the layout (see Figure 2) indicates a refutation.

In terms of the final structure of the argument the statement "1155 is … how many squares there are in a 10 by 10 [grid]" plays no role, as it is false. Even its negation "1155 is not … how many squares there are in a 10 by 10 [grid]" is not important to the final argument. However, in the proving process it is an important statement, as the students have arrived at a point where they might expect 1155 to be the answer (as DAR has guided them to this result ostensibly to find a formula that works) but at the same time they know from counting previously the correct answer is 385. This tension offers a motivation for further exploration of why the product of the three numbers in question is not the expected answer.

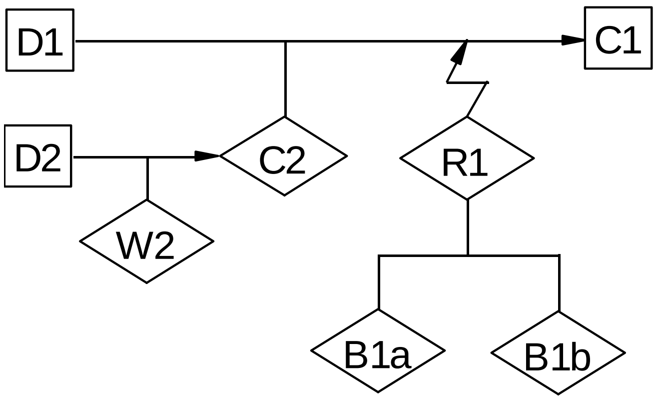

Figure 3. Refutation of sufficiency (B1a and B1b are backings supporting the warrant).

Refutation of the sufficiency of a warrant, while accepting the data and the conclusion

In the previous example, no warrant was offered to justify the connection between the data and the conclusion. In this example, a warrant is offered, but it is not the warrant that is refuted, but the sufficiency of it to establish the connection between the data and the conclusion. The example comes from a grade 9 class which is trying to explain why if two diagonals of a quadrilateral meet at their midpoints and are perpendicular, then the quadrilateral must be a rhombus.

Kaylee:	umm, I said cause if they meet, if	
	they meet at the midpoint, they meet at the midpoint and they're ninety degree angles	D1
	then umm, then, it would have to be — the [???] — then the sides, they have to be ummm, like	
	outsides have to be equal lengths.	C1
T:	Why?	
Kaylee:	Be, ummm, because they meet at,	
	they all meet at ninety degree angles and at the midpoint,	D1
	but the segments are different lengths, then	D2
	it can't be a square,	C2
	because squares they have to be the same length -	W2
T:	//K, but//	
Kaylee:	//so it has to be a rhombus//	C1
T:	Could it be a rectangle? Could it be a parallelogram?	
S:	If it — If none —	
T:	Cause there are other ones,	R1
	like the rectangle one met at the midpoint. It didn't meet at a ninety degree angle though.	B1a
	And then the rhombus we covered.	
	The one that made a kite met at a ninety degree angle, it didn't meet at the midpoint.	B1b
	You're on the start, but I'm not sure that you've clinched it, I'm not sure you've got that final part, but you've got — you're three quarters of the way there my dear.	

Kaylee's warrant is a correct statement. Figures with perpendicular bisecting diagonals are not generally squares, as squares have the additional characteristic that their diagonals are the same length. However, the teacher's objection is not to the truth of Kaylee's warrant but to its sufficiency. As the teacher notes, there are other quadrilaterals that have not been considered and excluded. Although she excludes rectangles and kites from consideration at the same time she uses them to back up her refutation, her point is made: other quadrilaterals, other than squares and rhombuses, exist, and so excluding squares is not sufficient to guarantee the shape must be a rhombus. Here the refutation is directed at the warrant, but does not refute it (as it is correct). Instead it suggests that the warrant is insufficient in the logic the teacher expects mathematical arguments to follow. By offering an argument of her own refuting the sufficiency of Kaylee's warrant the teacher provides that students with a hint as to the logic she would accept as mathematical.

Refutation of the relevance of data offered in support of a conclusion

In the previous case the refutation addressed the sufficiency of the warrant, but it is also possible to refute the relevance of the data offered. This example also comes from the grade 5 class looking for a formula for how many squares there are in an *n* by *n* grid. The students have suggested that by dividing 1155 by three, they can get the correct answer of 385.

DAR:	Could we have somebody … suggest a reason why we might want to divide by three – Mona?	 C1
Mona:	Because there's three numbers	D1
DAR:	Because there's three numbers. That's a good reason	
Mona:	I guess	
DAR:	OK – It's not a great reason, but it's a good reason. [Calls on another student]	 R1 R1

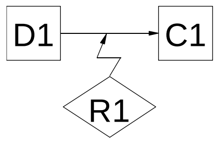

Figure 4. Refutation of relevance.

Here the teacher's refutation is an implicit one. His qualified support ("not a great reason but it's a good reason") and shift of attention away from Mona's response communicates to the class that there is something wrong with what she has said, without specifying exactly what. Neither the data nor the conclusion is refuted (as the class knows them both to be true statements), and there is no suggestion that the lack of a warrant in Mona's argument is the issue (as would be suggested by the teacher asking "Why would we divide by three when there are three numbers?"). Instead the focus is on the relationship between the facts, not on the facts themselves. It is the unspoken logic of the argument that is refuted. Mona has made a link between two statements, but not in a way that wins acknowledgment from the teacher. Note that here the teacher's refutation is based only on his authority (one of Sekiguchi's 1991 categories) and unlike the teacher's refutation in the previous case it does not offer any guidance for what might be an acceptable link. Instead he has the students guess until they come up with something acceptable.

A complex refutation

This example follows immediately after the previous one. The students have been working with a concrete model in which three pyramids are joined to make a roughly box shaped solid made up of $n\frac{1}{2}$, n by $(n+1)$ layers.

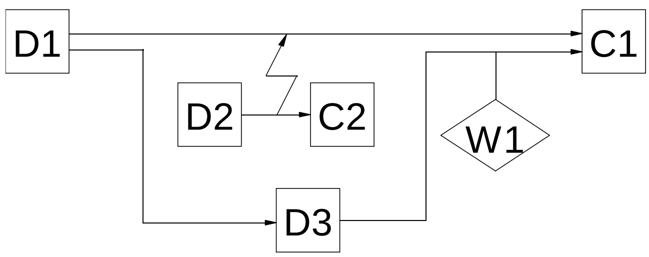

Figure 5. A complex refutation (C1 is the conclusion from Figure 4.).

Elaine:	Because there's three of those triangle thingies in there	D1
Maya:	But then why wouldn't you divide it by three and a half, because there's a half?	C2 D2
DAR:	OK how many triangles did we put together to make this thing?	D1
Several voices:	Three	D1
Maya:	And then there's the half	
DAR:	… So then we put together three of them and suddenly we had three times too many, — so that would be a good reason to divide by three if you've got three times too many of something.	D1 D3 W1

Here Elaine is trying to answer the question "why might we divide by three?" The data she uses to justify this refers to the three pyramids ("triangle thingies") in the box. But she does not offer a warrant to support the connection of this data to the conclusion. Maya's refutation consists of a parallel argument, which also makes reference to elements visible in the box (the three pyramids and the half layer). Her refutation is again on the level of the logic of the argument. By making an argument on the basis of a coincidence or analogy (one half is just as much a property of the box as three is) that leads to a false conclusion, she refutes Elaine's use of similar reasons. DAR then supports Elaine and in so doing implicitly refutes Maya's refutation. He provides a warrant for Elaine's original argument, in the process supplying a linking piece of data that shifts the logic of the argument from analogy (three thingies, so divide by three) to deduction (three times too many, so divide by three).

Maya's refutation offers a challenge not only to Elaine's argument but also to the teacher's practice. He refuted Mona's argument (above) by simply asserting his authority. If Maya had not refuted Elaine's argument, he might have used his authority again to endorse it and moved on. In order to refute Maya's refutation, he had to recast Elaine's argument into a more complete (and less refutable form) including reference to a new piece of data (D3: we had three times too many) and a warrant (W1) to support the drawing of the conclusion from it. This made the kind of logic he considered acceptable much more explicit.

CONCLUSION

These four examples illustrate some of the insights an examination of refutation in proving processes can provide, both into the nature and evolution of the logic of practice operating and into teaching practices and epistemologies related to proof.

By drawing attention to the insufficiency of a warrant or data (as in the second and third examples) or forcing the teacher to be more explicit about his implicit criteria for acceptable arguments (as in the fourth example) refutations provide hints as to what is the teacher's accepted logic. These hints are of value to students learning to shift from everyday arguments to mathematical arguments, as well as to us as researchers interested in this process.

We can also get insight into teaching practice from refutations. The first example, of a teacher using a refutation to provide motivation for further exploration, suggests an epistemology compatible with a Lakatosian view of mathematics as improving though confronting conclusions with counterexamples. The third and fourth examples reveal a teacher relying on authority as a means of refutation, suggesting an approach to teaching proving that relies on examples and non-examples as much as or more than direct modelling.

We believe that such research can provide insight into actual practice of teaching proof, which is necessary to any program of reform, as well as any comparison of approaches.

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LEARNING MATHEMATICS WITH TEXTBOOKS

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The mathematics textbook is an instrument for teaching and learning mathematics. Whereas a number of studies have examined the use of mathematics textbooks by teachers there is a dearth of research into the use of mathematics textbooks by students. In this paper results of an empirical investigation of the use of mathematics textbooks by students are presented. Firstly, a method to collect data on student's use of mathematics textbooks is introduced. This method was developed in accordance with the model of textbook use presented at PME 30. It is explicated, that this method is capable to explore the actual use of the mathematics textbook by students, and to find a way of recording the use of the mathematics textbook whenever and wherever students use it. Secondly, results from the study are presented. The results outlined in this paper focus on typical self-directed uses of the mathematics textbook by students.

INTRODUCTION

From a socio-cultural perspective the mathematics textbooks can be regarded as an artefact in the broad sense of the term. It is historically developed, culturally formed, produced for certain ends and used with particular intentions.

Valverde et al. (2002) believe that the structure of mathematics textbooks is likely to have an impact on actual classroom instruction. They argue, that the form and structure of textbooks advance a distinct pedagogical model and thus embody a plan for the particular succession of educational opportunities (cf. Valverde et al., 2002). The pedagogical model only becomes effective when the textbook is actually used. This points to the important fact, that the mathematics textbooks can not be looked at in isolation. It is involved in the activities of teaching and learning mathematics and therefore must be looked at as an element within these activities. In order to develop a better understanding of the role of the mathematics textbooks within the activities of teaching and learning mathematics an activity theoretical model was developed and introduced at PME 30 (Rezat, 2006a):

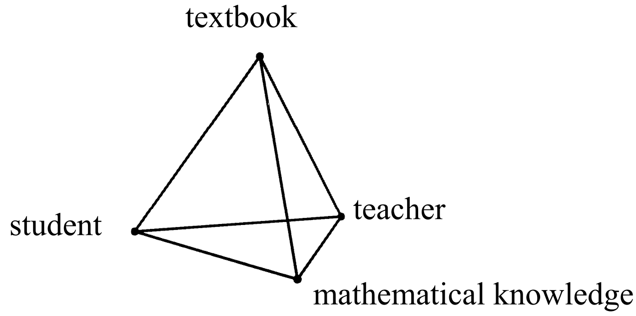


Figure 1. Tetrahedron model of textbook use.

Within the activities that are represented by the faces of the tetrahedron-model the mathematics textbook acts as an instrument. From an ergonomic perspective it is argued that artefacts have an impact on these activities, because on the one hand they offer particular ways of utilization and on the other hand the modalities of the artefacts impose constraints on their users (cf. Rabardel, 1995, 2002). Thus, the mathematics textbook has an impact on the activity of learning mathematics that is represented by the didactical triangle on the bottom of the tetrahedron.

Whereas a number of studies have examined the role of ICT in terms of tool use (cf. Lerman, 2006) the role of the mathematics textbook as an instrument for teaching and learning has not gained much attention. This becomes even more striking reading Howson's comment on the role of mathematics textbooks compared to that of ICT: "despite the obvious powers of the new technology it must be accepted that its role in the vast majority of the world's classrooms pales into insignificance when compared with that of textbooks and other written materials." (Howson, 1995)

At least two groups of users are associated with the mathematics textbook: teachers and students. So far, a number of studies have examined the use of the mathematics textbook by teachers (e.g. Bromme & Hömberg, 1981; Haggarty & Pepin, 2002; Hopf, 1980; Johansson, 2006; Pepin & Haggarty, 2001; Remillard, 2005; Stodolsky, 1989; Woodward & Elliott, 1990) whereas there is a dearth of research into the use of mathematics textbooks by students (Love & Pimm, 1996). This is striking, because as pointed out by Kang and Kilpatrick (1992), textbook authors regard the student as the main reader of the textbook.

In order to develop an idea of student's utilizations of mathematics textbooks a qualitative investigation of the use of mathematics textbooks by students has been carried out in two German secondary schools. The central research-question was how students use their textbook to learn mathematics.

METHODOLOGY AND THEORETICAL FRAMEWORK

The difficulty of obtaining data on students working from textbooks is one reason that Love and Pimm (1996) put forward in order to explain the dearth of research into student's use of texts. Therefore, developing an appropriate methodology to collect data on student's use of mathematics textbooks can be regarded as a major issue in this field.

The method to collect data on student's use of mathematics textbooks was developed within the framework of the activity theoretical model of textbook use. According to this model the use of mathematics textbooks is situated within an activity system constituted by the student, the teacher, the mathematics textbook, and mathematics itself. First of all, this implies that a method to investigate the use of mathematics textbooks by students has to take all four vertices of the tetrahedron-model into consideration.

In addition, three criteria were established for an appropriate methodology to collect data on student's use of mathematics textbooks:

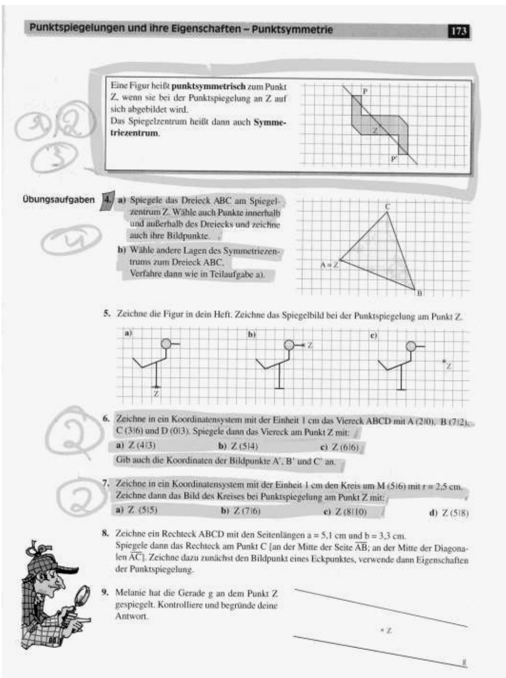
1. The actual use of the mathematics textbook by students should be recorded.
2. Biases caused by the researcher, by the situation or by social desirability should be minimized.
3. The use of the textbook should be recorded whenever and wherever students use it.

Criterion 1 leads to the rejection of quantitative methods and of methods that are likely to reveal only the conscious uses of the textbook. Experimental settings and artificial situations are refused due to criterion 2. Approaches that are solely based on observation are discarded because of criterion 3.

The methodological framework that was developed according to the three criteria combines observation and a special type of questioning.

First of all, the students are asked to highlight every part they used in the textbook. Additionally, they are asked to write down in a small booklet the reason why they used the part they highlighted. This idea was developed in order to get the most precise information about what the students actually use and why they use it by keeping the situation of textbook use as natural as possible. Nevertheless, highlighting sections in a textbook is not the natural way to use the textbooks and therefore a bias on the data cannot be totally excluded.

Provided that the students take their task seriously, this method enables to collect data on the use of the textbooks whenever and wherever students use it and therefore meets criterion 3. Fig. 2 shows an example of one page out of a student's mathematics textbooks with the used sections highlighted and the related comments:



Punktspiegelungen und ihre Eigenschaften – Punktsymmetrie 173

Eine Figur heißt **punktsymmetrisch** zum Punkt Z, wenn sie bei der Punktspiegelung an Z auf sich abgebildet wird.
Das Spiegelzentrum heißt dann auch **Symmetriezentrum**.

1 2 3

Übungsaufgaben **4.** a) Spiegele das Dreieck ABC am Spiegelzentrum Z. Wähle auch Punkte innerhalb und außerhalb des Dreiecks und zeichne auch ihre Bildpunkte.
b) Wähle andere Lagen des Symmetriezentrums zum Dreieck ABC. Verfahre dann wie in Teilaufgabe a).

4

5. Zeichne die Figur in dein Heft. Zeichne das Spiegelbild bei der Punktspiegelung am Punkt Z.

a) b) c)

2 **6.** Zeichne in ein Koordinatensystem mit der Einheit 1 cm das Viereck ABCD mit A (2|0), B (7|2), C (3|6) und D (0|3). Spiegele dann das Viereck am Punkt Z mit:
a) Z (4|3) b) Z (5|4) c) Z (6|6)
Gib auch die Koordinaten der Bildpunkte A', B' und C' an.

2 **7.** Zeichne in ein Koordinatensystem mit der Einheit 1 cm den Kreis um M (5|6) mit r = 2,5 cm. Zeichne dann das Bild des Kreises bei Punktspiegelung am Punkt Z mit:
a) Z (5|5) b) Z (7|6) c) Z (8|10) d) Z (5|8)

8. Zeichne ein Rechteck ABCD mit den Seitenlängen a = 5,1 cm und b = 3,3 cm. Spiegele dann das Rechteck am Punkt C [an der Mitte der Seite $\overline{AB}$; an der Mitte der Diagonalen $\overline{AC}$]. Zeichne dazu zunächst den Bildpunkt eines Eckpunktes, verwende dann Eigenschaften der Punktspiegelung.

9. Melanie hat die Gerade g an dem Punkt Z gespiegelt. Kontrolliere und begründe deine Antwort.

(1) I needed it to solve a task (at school).

(2) I just read it, because I wanted to understand something.

(3) I needed it to do my homework.

(4) Just for fun.

(Translation: SR)

Figure 2. A page from a student's mathematics textbook
with highlighted used sections and related comments.

In addition the lessons are observed and field notes are taken. In the classroom the use of the textbook of both the students and the teacher is taken into account. This is important for several reasons.

First of all, the methodological triangulation provides a measure for the validity of the data. Collecting data on how the textbook has been used in the classroom makes it possible to compare the markings and comments of the students with the field notes. The degree of correspondence between these two sources relating to the use of the textbook in the classroom indicates how serious the student took his task.

Secondly, the observation provides an insight into the way the teacher mediates textbook use. It makes a difference if the students only use the textbook when they are told to by the teacher or if they use it of their own accord. This difference will become apparent through classroom observation.

Finally, the observation allows to gain access to unconscious uses of the textbook. Again, the researcher can compare his field notes with the notes of the students and see whether he observed uses of the textbook that the students did not record.

While the method of highlighting and taking notes especially satisfies criterion 3 and at the same time aims at both, providing a precise record of the actual use of the textbook by students (criterion 1) as well as keeping biases low (criterion 2), the intention of the observation is threefold. On the one hand the idea is to lower biases that might be caused by the method of highlighting (criterion 2) and on the other hand it is meant to provide a deeper insight into the actual use of the textbooks by students (criterion 1), because it enables the researcher to observe unconscious uses of the textbook. Additionally, it provides a measure of the validity of the student's data.

The coding process followed the ideas of Grounded Theory by Strauss and Corbin (Strauss, 1987; Strauss & Corbin, 1990). Categories were established in the process of analysing the data. Each highlighted section in the textbook was categorized according to the kind of block it represented (introductory tasks, exposition, worked example, kernels, exercises) (cf. Rezat, 2006b), the reason why it was used, and finally whether the use of the section was mediated by the teacher or not.

In order to understand the role of the mathematics textbook as an instrument within the activity system represented by the tetrahedron model Rabardel's (1995, 2002) theory of the instrument was used. This theory has proven fruitful to provide insights into the use of new technologies as instruments for learning mathematics (cf. Monaghan, 2007). According to Rabardel an instrument is a psychological entity that consists of an artefact component and a scheme component. In using the artefact with particular intentions the subject develops utilization schemes which are shaped by both, the artefact and the subject. Vergnaud (1998) suggests that schemes are characterized by two operational invariants: theorems-in-action and concepts-in-action. The used notions 'theorems' and 'concept' already indicate that operational invariants relate closely to mathematical knowledge. In order to describe operational invariants of utilization schemes that are related to the use of instruments it is suggested to generalize Vergnaud's notion of theorems-in-action and concepts-in-action to the notion of beliefs-in-action. Since beliefs are supposed to guide human behaviour the notion of beliefs-in-action offers an appropriate means to characterize utilization schemes.

THE STUDY

Data was collected for a period of three weeks in two 6th grade and two 12th grade classes in two German secondary schools. Within the German three partite school system, these schools are considered to be for high achieving students. All four classes were taught by different teachers.

RESULTS

The study shows that students do not only use the mathematics textbook when they are told to by the teacher, but that they use it self-directed. The focus of the analysis lies on these uses of the textbooks that students perform in addition to teacher mediated uses of the textbook. Actually, teacher mediated utilizations of the textbook cannot be regarded as student's genuine ways of utilizing the textbook, but in the end reflect the way the teacher utilizes the textbook.

The analysis of the data revealed that students did not only develop individual utilization schemes of the mathematics textbook, but that some schemes emerged repeatedly. It is argued that these schemes that could be found in different groups of students of different ages seem to reflect typical schemes of utilizing a mathematics textbook within a given culture. It is suggested to call these schemes cultural-historical-utilization-schemes (CHUS). CHUS were found in conjunction with five different activities: solving tasks and problems, to look up something, practising, performing follow-up coursework, anticipation.

In order to solve tasks and problems students rely heavily on worked examples. This scheme can be characterized by the rule-of-action: Whenever I have to solve tasks and problems from the book I study the worked examples. This scheme is supported by the belief-in-action: Worked examples help to solve tasks and problems.

Students also use their textbooks whenever they have to look up something. Two CHUS were identified in conjunction with this activity: The lexical-scheme and the kernels-scheme. Students that exhibit the lexical-scheme whenever they look up something in the textbook utilize the index or the table of contents. The lexical-scheme is characterized by the rule of action: If I don't know the meaning of a term then I look it up in the index or in the table of contents of my mathematics textbook. Students that show the kernels-scheme skim through the kernels of the book until they find what they are looking for.

When students use their mathematics textbook for practising they show two different CHUS. They either recapitulate tasks and exercises from the book that the teacher mediated before or they pick tasks and exercises that are adjacent to teacher-mediated exercises. The related CHUS are the recapitulation-scheme and the periphery-scheme, respectively. The recapitulation-scheme is determined by the belief-in-action: Teacher mediated exercises are the most important for learning mathematics. The periphery-scheme is characterized by two beliefs: 1) Adjacent exercises are similar. 2) Effective practising means to do tasks and exercises that are similar to teacher-mediated exercises.

Three different CHUS of performing follow-up course work were identified: the mediated-follow-up-course-work-scheme, the independent-follow-up-course-work-scheme and the learning-kernels-scheme. Students that show the mediated-scheme depend on the teacher to tell them which sections from the textbook they can use in order to perform follow-up course work. In this way their choice of the parts they use from textbook is mediated. Students that show the independent-scheme make the choice on their own. The learning-kernels-scheme is characterized by the restriction to a specific block. Students only look at the boxed or shaded kernels in the book when they perform follow-up-course work.

Anticipation refers to activities of students using their mathematics textbooks to anticipate the unfolding of a subject matter in the mathematics class. It was observed that in a period of guided discovery students use their mathematics textbook to look for the result that they are supposed to discover themselves. The corresponding CHUS is characterized by the rule of action: If I want to know what the teacher wants us to discover, then I look it up in the book. This rule is supported by the belief, that the course of the mathematics class is mirrored in the textbook.

CONCLUSIONS

The CHUS that were introduced in the previous section can be regarded as typical utilizations of the mathematics textbooks by students in Germany. Some of them are connected to specific blocks of the textbook, e.g. the kernels-scheme, the learning-kernels-scheme, the worked-examples-scheme, the periphery-scheme. Since these blocks are characteristic for mathematics textbooks not only in Germany but all over the TIMSS-world (cf. Howson, 1995) the hypothesis can be put forward that CHUS are typical ways to utilize mathematics textbooks in general.

The activities the mathematics textbook is involved do not only give an insight into student’s utilizations of mathematics textbooks but they also give an idea of what learning mathematics is about for students. Looking at the activity of learning mathematics from this perspective shows that learning mathematics for students comprises solving tasks and problems, to look up something in the textbook, practising, performing follow-up coursework, and anticipating the development of concepts with the help of the textbook. The CHUS show how the textbook is implemented as an instrument within these activities.

From an ergonomical perspective it can be argued that the CHUS are partly induced by the textbook itself. Firstly, this raises the question wether these CHUS are desirable schemes or if other schemes are regarded as preferable. After all, this is a question of how mathematics textbooks should be structured in order promote the development of utilization schemes, that correspond to a recent understanding of learning mathematics.

Knowing about CHUS also has implications for teaching mathematics. It raises the following questions:

- How does the teacher have to implement the mathematics textbook into his teaching to be in accordance with student's ways of utilizing the textbook?
- How can the teacher promote the development of student's utilization schemes of the mathematics textbook?

The anticipation-scheme also points to the issue, that the teacher has to be more aware of the instruments that are available to the students. The anticipation-scheme is an example of how student's utilization-schemes of the textbook might foil the intended course of the lesson.

The CHUS presented in this paper are not associated with specific mathematical content. They only give an idea how the mathematics textbook is implemented in the process of learning mathematics in general. Further research will be needed in order to understand the role of the mathematics textbook regarding the acquisition of specific mathematical concepts.

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THE TEACHER IN THE MATHEMATICAL-ARGUMENTATION PROCESSES WITHIN ELEMENTARY SCHOOL CLASRROOMS

Mirela Rigo, Teresa Rojano, and François Pluvinage

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In mathematical education, theoretical and empirical studies have been undertaken on proofs, based on evidence from controlled scenarios. Those studies are used as a point of departure for this article in order to analyze the different practices of mathematical justification that teachers spontaneously bring into play in their ordinary classes. The article is furthermore supplemented by a characterization of other forms of argumentation that teachers naturally resort to in the classroom setting, based on techniques that coincide with those used in oral and written rhetoric. In addition to the how, we are interested in distinguishing the topics with reference to which teachers attempt to convince their students. This article is part of a broader research study that focuses on teacher beliefs and classroom practices.

BACKGROUND AND THEORETICAL PERSPECTIVE

This report is situated at the intersection of five lines of research, dealing with the following subjects: teacher practices (especially in ordinary classes, see Hersant & Perrin-Glorian, 2005), beliefs, proportional reasoning, mathematical proofs and rhetorical arguments. Some of the notions and results found in the foregoing research will be used in this article as the basis for integrating the methodology and definition of the study's interpretative framework.

The notion of belief has frequently been employed in reflections on teacher practices (Thompson, 1992; Schoenfeld, 2000, inter alia), possibly because it is a matter of a hypothesis that enables understanding and providing plausible explanations for teacher decisions and actions within the classroom setting. One methodological resource that enables revealing possible teacher beliefs consists of identifying routine teaching practices exhibited while performing educational duties. The afore-mentioned is supported by the idea that "beliefs lead the individual holding them to respond under different circumstances in a manner that is consistent with the belief" (Villoro, 2002, pg. 71). Two additional findings in studies on teacher beliefs should be highlighted here as well: the inconsistencies that teachers usually demonstrate between the beliefs they profess and their educational practices (Thompson, Ibid. pg. 138), and derived from the foregoing, the need for a "methodological triangulation" in data compilation (Schoenfeld, Ibid.; Stake, 1995).

Some of the pertinent contributions to this article include the stratification of mathematical reasoning produced by students at different educational levels. Of particular interest here are the categorizations that only include mathematical reasons –such as, for instance, that proposed by Balacheff, 2000- as well as another type of study, such as that undertaken by Simon (1996, pg. 23), among many others, which deals with justifications based on the motives and interests of the subject making the argument (Levels 0 and 1).

While other inputs to this article include studies on rhetorical argumentation, particularly those alluded to below. A theoretical contribution dealing with the forms of colloquial arguments, based on 'natural logic' –carried out from the standpoint of mathematics education- can be found in Duval (1995). The touchstones of this type of studies are Aristotle's *Rhetoric,* with the analysis of dialectic argumentations, and Socrates' maieutics as expressed in several of Plato's dialogues (such as in *Theætetus* and *Menon)*. The latter consists of targeted questioning that has a precise aim and is based on an asymmetrical dialogue between the person expressing him/herself and the person listening. Perelman re-broaches these studies and states that "The objective of any argumentation is to prompt or increase adherence to the thesis presented for its approval" (1989, pg. 92) and it is concerned with "…the verisimilar and the probable" (Ibid. pg. 30). After Perelman, many other authors also contributed to the topic. Those that are important to this study include Austin and his speech acts, 'illocutionary acts' undertaken by saying something. Searle (1969) reintroduced the latter work and characterizes the illocutionary force of speech acts, which are manifest when the person delivering speech asserts, states, expresses a state of mind, directs or compromises the listener in some manner. Whereas López Eire (2001) takes a look back at the techniques of oral discourse –tone and linguistic marks, repetition of linguistic elements, reorganization of discourse to suit the audience, time management- and describes some of its advantages, such as mnemonic aids for the listener and its communal nature.

Based on ideas contained in the works cited, below we present some notions defined according to the empirical characteristics of this research, to its objectives and questions. For purposes of this article, a mathematical-argumentation process is considered as a succession of interactions –between the teacher and his/her students- that are thematically organized into a coherent whole by way of which arguments or proofs are generated (Krummheuer, 1995). Mathematical and logical inferences –not always rigorously applied- are involved in those arguments and proofs and their aim is to provide reasons that support the truth of a mathematical statement (Balacheff, 2000). The mathematical-argumentation process also involves adherence resources that are prototypical of oral rhetoric –of Socrates' maieutics as well- and of written rhetoric. Mathematical-argumentation processes moreover are for the purpose of explaining and having students understand the justifications (Hanna, 1996, pg. 903), and to prompt epistemic attitudes among such students –certainty and/or conviction, Hersh, 1993- as to the truth –or verisimilitude- of the mathematical statement and the validity of the argument (Duval, 1995; Perelman, 1989).

The notion of a 'mathematical-argumentation process' is a hybrid, which results from the mixture that stems from the concept of mathematical proof and the idea of argumentation.

METHODOLOGY AND DATA COLLECTION

This is an ethnographic case study of the instrumental type (Stake, 1995) and of a longitudinal nature, undertaken of three primary education centers (two public schools and one private school) in Mexico City. Data collection and interpretation consisted of the actions listed below, although such actions were not necessarily carried out sequentially. i) Design and application, within the context of a pilot study, of a school-type test for third and sixth grade students, dealing with proportionality problems and of a questionnaire for their teachers. ii) Interviews of the teachers from the three schools, dealing with their beliefs concerning mathematics and its teaching. iii) Observation of sixth grade classes on the subject of proportionality (ten observations of one teacher from each school, undertaken throughout the school year). The classes were video-taped using two video-cameras, one of which focused on the teacher and the other on the students. iv) Transcription of all video tapes taken. v) Analysis of the data collected in the video tapes as compared with the data from written records. Given the nature of the study, project researchers limited their classroom involvement solely to observation. During the classes observed, the teacher usually uses the official mathematics textbook as a guide. The didactic proposal of such official mathematics textbooks focuses on solving exercises and problems. Consequently in this study, a classroom argumentation consists of a process of social interactivity among teacher and students, in which reasons are presented in order to sustain the solution of problems raised in the classroom.

INTERPRETATION AND RESULTS

As previously mentioned, the results of this paper are part of a broader research study that deals with teacher beliefs and classroom practices, focusing on the topic of mathematical justification and classroom argumentation. In order to reveal the possible beliefs of the teachers who took part in the study, the activities listed below were developed, in approximate terms. i) The argumentation processes that arose in each of the classes observed were distinguished. ii) The mathematical argument contained in the classes observed was analyzed (considering the types of inferences involved, the semiotic representation systems used and the manner in which the truth was handled) and the rhetorical techniques and resources used by the teacher were acknowledged. iii) Teacher argumentation schemes that were repeated throughout the course were identified. iv) Possible teacher beliefs and values were suggested, dealing with the manner in which teachers argued and presented justifications in mathematics classes.

Partial results of the research are submitted in this paper. An extract of the class given by one of the teachers observed (Dioni) was chosen so as to illustrate and explain the mathematical and rhetorical resources that she usually resorts to while undertaking her educational duties. Dioni is a teacher who has 25 years of teaching experience and a positive attitude toward in-class mathematical argumentation. The latter was made patent by, inter alia, the disposition of her former students (in the pilot study) to provide reasons for the solutions to problems raised when they were not deliberately asked to do so.

The extract chosen (Episode 2, Solution 2, of Lesson 80 of the official Mathematics Textbook) contains a mathematical-argumentation process in which the teacher plays a leading role. Said extract is illustrative in view of the 'conceptual disagreement' that arises between the teacher and her students –something that rarely happens in her class. The circumstance probably forced the teacher to muster the patterns, standards and rituals –both mathematical and rhetorical- that she usually follows with her students when presenting arguments. It also makes it possible to identify what she seeks in a mathematical argumentation process, just how she attempts to persuade her students and the topics of which she wants to convince them.

A transcription of the extract is not included here due to space limitations. However a Table can be found in the Annex to this article, in which the middle column contains the steps of the mathematical-argumentation process, the right-hand column contains the mathematical contents and the left the rhetorical resources employed by the teacher. In the remainder of this paper, we have devoted ourselves to analyzing the mathematical and rhetorical adherence resources that appear in the argumentation cited.

Mathematical Argument. The argument contained in the extract is of the deductive type and is the result of a symbolic formula instantiation process (Reid, 2002) (see Steps 2, 3, 5, 7 and 14, plus Steps 8 and 13 of the Annex Table). This type of general solution is frequently used in Dioni's class, for which she usually resorts to use of symbolic languages.

In Step 9 (S9) of the argumentation process (Annex Table) the teacher believes that the mathematical argument is complete and that the conclusion can be derived. She is nonetheless quick to realize that the students are unable to arrive at the conclusion and assumes that this is due to the lack of a framework of reference with which to interpret the velocity and speed in the physical-arithmetic register introduced through the formula (S3). As a result she tries to 'complete' the proof (using an Euclidian-type proof, Duval, 1995), giving her students hints that will enable them to signify the terms in the register: she incorporates Table 4 (S10), shows the students how to read the physical-arithmetic statements contained therein (S13) and performs a few treatments on the register (S5 and S13). When she confirms that despite it all her students continue to believe that the fastest swimmer is the one 'who swam the greatest distance', without taking time into consideration (as done in daily contexts, in which people usually disregard the weight unit in product pricing provided through external ratios, of the $/Lb. type), she decides to conclude the process by providing the answer justification that she had expected her students to give her.

From the teacher's point of view, the argumentation process contains sufficient reasons (Villoro, 2002). Yet her students do not appear to be of the same opinion – possibly because they continue to interpret speed based on the tabular register and not on the physical-arithmetic register. In order to supplement and/or reinforce the possible certainties acquired during the logical and mathematical procedures, she resorts to other devices, based on procedures that pertain to oral and written rhetoric.

Rhetoric Techniques. The longitudinal study of Dioni's teaching practices enabled the researchers to identify the rhetorical adherence schemes that the teacher likes to resort to most frequently and to draft several plausible explanations for their introduction and usage. Foremost among the argumentations analyzed are persuasion techniques based on …:

a) Maieutics: The teacher's pedagogical method presents marked features of the 'art of maieutics', associated with Socrates' dialogues. She offers her students classroom space for them to construct their answer and justifications (S2). The latter enables the students to take active part in the dialogue that she guides and to benefit from learning the mathematical truths and reasons that they themselves 'have discovered' –with her help (See Plato as expressed by Socrates in the dialogue *Theætetus*, 1981, 150d) (S9, S11, S14). This assuredly aids in constructing their mathematical certainties and convictions. *b) The Illocutionary Acts* of the Teacher (Searle, 1969): foremost are the 'expressive acts', in which she implicitly sends messages to her students. An example of this is telling the students that use of the formula is the best strategy possible (S4) or that it is important to technically master the register introduced (S5 and S13). There are also the 'directive acts', aimed at use of the specific formula (S2, S6) or at the commitment students must make to reaching truthful results and their justification (S9, S11, S14). *c) Group Recitation* (López Eire, 2001): a resource to which the teacher may associate mnemonic ends, particularly as applied to familiarization with the physical-arithmetic register introduced and memorization of the formula (S1, S5, S8, S13). *d) Repetition of mathematical statements* (Ibid.): employed perhaps to underscore the truth of a result (S14). *e) Redundancy of reasons* (Ibid.): a technique that works like the sum of evidence, and which is prototypical of civil trials (S13).

And now, precisely what are the teacher's expectations as regards mathematical-argumentation processes? Moreover, what does the teacher want to convince her students of with her mathematical and rhetorical adherence techniques?

By using the argumentation, Dioni intends her students to attain the true result using the register introduced –had this not been the case, she would not have waited 20 minutes before providing the answer. She also uses the argument to persuade them that they should and can obtain the result, they should develop reasons that support the answer or that they at least be able to understand the reasons (S2, 6, 9, 11, 14) and that they be certain of the need for the conclusion obtained (S15).

Given the teacher's routine teaching practices however, she gives the impression that she seeks to convince the students of aspects and values that go well beyond the specific exercise, aspects and interests that are naturally in harmony with her personal beliefs and that are part of her long-term didactic agenda.

As the school cycle elapses, the researchers were able to detect several patterns of the teacher's didactic practices. Foremost was her propensity for use general formulae, her tendency to use arithmetic-algebraic language and to training her students in operational techniques associated with those representation systems. Those standards set the criteria for discovering some of her deep convictions and bringing to light the topics of which she seeks to convince her students on the long-run. An example of the latter is the importance of general formulae in mathematics and the validity of general and parsimonious solutions. Identification of this network of beliefs in the teacher makes it possible to in turn understand and explain, albeit only partially, some of her decisions and manners of proceeding with the solution of Lesson 80: restating the problem so as to adjust it to application of the general formula (S2), emphatic approval of the introduction of the general rule (S4), the moments of the technique introduced (S5 and S13), the actions aimed at memorization of the formula and the amount of time she devotes to the solution.

FINAL REMARKS

Very varied interests are interwoven within the classroom setting. There are institutional interests, collective interests and personal interests –those of the teacher as well as of the students. It is the type of scenario that "gives rise to social or affective phenomena that are meaningful enough to displace the debate outside of the cognitive field" (Brousseau, cit. in Balacheff, 2000, pg. 18). And it is precisely because of this that in the research reported we have introduced an interpretative framework that assumes heterogeneous mathematical argumentation processes in the classroom. An additional reason is the consideration that the epistemic positions generated may be derived from rational and supra-rational sources of certainty. This is not a matter of recommending such types of argumentation, rather a simple pointing out of an event that may be occurring fairly frequently in mathematics classrooms, and perhaps not just limited to primary levels of education. We hope that analysis of the cases reported may lead to the emergence of relevant meanings for the topic of mathematical argumentation in ordinary classes. It is furthermore our hope that this research may in some way help teachers to become more aware of their own teaching practices as regards argumentation and mathematical proof or perhaps aid teacher trainers to help teachers achieve that very end.

Endnote

We do appreciate the technical support Nayelly Dávila gave to this work.

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Annex. Table: Mathematical-argumentation process. Lesson 80. Episode 2.

<table>
<tr><th>Oral and Written Rhetoric Resources</th><th>Steps (S) of the Mathematical-Argumentation Process</th><th>Mathematical Contents</th></tr>
<tr><td>Group Recitation.</td><td>1. Who swam faster?
<table>
<tr><td></td><td>Distancia</td><td colspan="3">Tiempo</td></tr>
<tr><td></td><td></td><td>Horas</td><td>min</td><td>seg</td></tr>
<tr><td>Am</td><td>100 m.</td><td>0</td><td>2</td><td>0</td></tr>
<tr><td>Be</td><td>50 m.</td><td>0</td><td>0</td><td>50</td></tr>
<tr><td>Ca</td><td>150 m.</td><td>0</td><td>2</td><td>51</td></tr>
<tr><td>Da</td><td>1500 m.</td><td>0</td><td>40</td><td>0</td></tr>
</table>
Table 1 (of the Textbook) (T1)</td><td>Text of the problem in the official Mathematics Textbook (O-W)</td></tr>
<tr><td>Directive Action. Maieutics.</td><td>2. “What can we do to find out how many seconds it took Am ….?” (T)</td><td>Reduction to a single time unit (O)</td></tr>
<tr><td></td><td>3. “Dividing the distance by the time” (S)</td><td>Introduction of the general formula (O)</td></tr>
<tr><td>Expressive Action. Repetition</td><td>4. The T emphatically accepts the formula and paraphrases it</td><td></td></tr>
<tr><td>Group Recitation Expressive Action.</td><td>5. Conversion of min. to sec. of data in T1
<table>
<tr><td>50 metros</td><td>50 segundos</td><td></td></tr>
<tr><td>100 metros</td><td>2 min</td><td>120</td></tr>
<tr><td>150 metros</td><td>2 min 51 seg</td><td>171</td></tr>
<tr><td>1500 metros</td><td>40 min</td><td>2400</td></tr>
</table>
Table 2 (S at the blackboard) (T2)</td><td>Moment of Technique (to apply the formula) (O-W)</td></tr>
<tr><td>Directive Action.</td><td>6. The T encourages the S to solve the problem using the general formula</td><td></td></tr>
<tr><td></td><td>7. Application of the formula on data in T 2</td><td>Instantiation of the formula</td></tr>
<tr><td>Group Recitation. Assertive, Reiterative and Expressive Action.</td><td>8. Help from the T: “…meters per second”
<table>
<tr><td>50 m.</td><td>50 seg</td><td></td><td>1 m por seg</td></tr>
<tr><td>100 m.</td><td>2 min</td><td>120</td><td>0.83 m. por segundo</td></tr>
<tr><td>150 m.</td><td>2 min 51 seg</td><td>171</td><td>.87 m x segundos</td></tr>
<tr><td>1500 m.</td><td>40 min</td><td>2400</td><td>.6 por seg.</td></tr>
</table>
Table 3 (S at the blackboard) (T3)</td><td>Interpretation of the formula results in terms of velocity (O-W)</td></tr>
<tr><td>Maieutics. Directive Action.</td><td>9. The T requests an answer and justification. Incorrect answer (S)</td><td></td></tr>
<tr><td></td><td>10. Subjects are included in T4
<table>
<tr><td>Beto</td><td>50 metros</td><td>50 seg</td><td></td><td>1 m por seg</td></tr>
<tr><td>Amalia</td><td>100 metros</td><td>2 min</td><td>120</td><td>0.83 metros por segundo</td></tr>
<tr><td>Catalina</td><td>150 metros</td><td>2 min 51 seg</td><td>171</td><td>.87 m x segundos</td></tr>
<tr><td>Darío</td><td>1500 metros</td><td>40 min</td><td>2400</td><td>.6 por seg.</td></tr>
</table>
Table 4 (S at the blackboard) (T4)</td><td>T3 is completed at the blackboard (O-W)</td></tr>
<tr><td>Maieutics.</td><td>11. The T asks for an answer and justification. Incorrect answer. (S)</td><td></td></tr>
<tr><td>Directive Action.</td><td>12. By show of hands, the group majority votes in favor of the incorrect answer</td><td></td></tr>
<tr><td>Group Recitation. Redundancy.</td><td>13. Reading the text of T 4 and conversion of m. to cm. under the guidance of the T.</td><td>Interpretation of formula results</td></tr>
<tr><td>Maieutics. Repetition of the Correct Answer.</td><td>14. The T asks for an answer. Incorrect answer (S). The T poses the question again. Correct answer (S). The T requests it be repeated several times.</td><td>Conclusion (O)</td></tr>
<tr><td>Interrogative Action.</td><td>15. Aid from the T: “Is a meter smaller than 60 cm.?” (T)</td><td>Justification of the conclusion (speed) (O)</td></tr>
</table>

T: Teacher participation; S: Student; Ti: table of the extract;
O: Oral discourse; W: Written discourse; S: Steps of the process.

SOCIOCULTURAL INTIMATIONS ON THE DEVELOPMENT OF GENERALIZATION AMONG MIDDLE SCHOOL LEARNERS: RESULTS FROM A THREE-YEAR STUDY*

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Drawn from data obtained over 3 years, the research study addresses how shared content and understanding of generalization emerged and evolved in a middle school class using a socially shared symbolic system in a community of minds framework.

BACKGROUND AND RESEARCH QUESTION

Radford's (2008) characterization of algebraic generalization involves the explicit stipulation of a direct expression containing a variable(s) which represents a range of values that makes sense within what the knower perceives to be an appropriate structure. Hence, especially for beginning learners, two symbolic requirements have to be dealt with simultaneously: understanding what constitutes a direct expression and how it is established and justified, and; extending their static notion of a variable as representing a single value to a more dynamic view in which a variable could be endowed with multiple values. Fulfilling such requirements is central in algebraic thinking since they comprise the conceptual primitives that lead to further growth in higher-order algebraic processes (for e.g., abstraction, modeling). Recent emergent and semiotic investigations provide empirical evidence of the important and influent- ial role of the social context in the development of particular mathematical processes among learners. This paper has a similar intent, i.e., explain how social factors, in particular, the external affordance of mind-sharing processes, enable learners to develop the representational complexities involved in forming algebraic generalizations of patterns. Certainly, students on their own could establish generalizations resulting from developments in certain core domains of thought. For e.g., growth in additive thinking assists learners establish general recursive relationships in a pattern. But how does one account for, say, multiplicative thinking and analogical reasoning? While we agree that some learners are capable of inventing them on their own, most other learners develop them through social interaction as they enter a community of minds. Having such a "common but variable mind-space" (Nelson, 2005, p. 32) assists learners to first acknowledge that there are different ways of thinking about patterns. The community also provides sources in which to make sense of other valid generalizing strategies and, especially, "access to previously inaccessible sources of knowledge" (Nelson et al, 2003, p. 25). Thus, the value accorded to, for e.g., non-recursive forms of generalizing is sufficiently rooted on cultural grounds.

In this report, we wed a sociocultural dimension to reported psychological accounts of generalization among beginning learners at the middle-school level (cf. selected articles in *ZDM* 40(1)). We address the following question: *How do shared content and understanding of algebraic generalization emerge and evolve in the context of a community of minds?* We assume at the outset that constructing algebraic generalizations are interpretive experiences among learners, i.e., they are drawn externally from social interaction and the appropriation of tools which help shape learners' perceptions of actions and symbols needed in making a valid generalization.

THEORETICAL FRAMEWORK: BUILDING ON SSSS AND COMMUNITY OF MINDS

Establishing a socially and culturally acceptable generalization in patterning activity involves the use of a domain-specific language. We articulate what "this" language means to us since differing research programs characterize it in different ways. We share the functional perspective of Nelson and Kessler Shaw (2002) in which language mirrors shared symbols and symbol systems whose meanings have been drawn socially and culturally. Words and conventional concepts are related to each other in that they are shared expressions of use that are meaningful within, and are valuable to, communities. For Nelson and Kessler Shaw (2002), shared symbols involve "the social importation of the cultural meaning into the child's communicat-ive schemes that provides entrée into societal … meaning systems" (p. 36). Figure 1 illustrates the authors' proposed socially shared symbolic system (SSSS). In SSSS, language evolves from the outside - their social world - to more defined meaning systems that are shared in and by the relevant community. The words children develop also assist them to learn shared modes of discourse through years of interact- ion with the members. Language systems are, thus, "essentially social" as "they emerge from within a social matrix where the meanings incorporated in them are shared between individuals" (Nelson & Shaw, 2002, p. 28). In the *social circle,* children's sense of knowing occurs in an interpreted social world. They learn approp- riate sociocultural norms for communicating and ways of responding to social stimuli. In the *shared circle*, shared attention with others becomes their basis in constructing words that indicate shared reference to particular objects, events, or aspects. In the *symbolic circle*, those words transform into concepts and are interpreted within word-image relat- ionships indicating a shift in understanding words as referring to particular objects or contexts to denoting any objects or contexts of that type. They also develop various intent- ional meanings of words that they do not even have to experience or see within their surrounding contexts. Naming and categorizing assist them to acquire meanings within systems of meanings that enable them to understand how the same word can refer to different concepts. Ultimately, reality is constructed through language that evolves as a shared, continuous, and context-general phenomenon. In the *system circle*, language consists of conventional symbol systems where words are seen no longer as referring to or denoting but have meanings or senses in a "system of related terms within a domain" (Nelson & Kessler Shaw, 2002, p. 47). An SSSS evolves within a community of minds whose members "are related by common purposes and understandings"

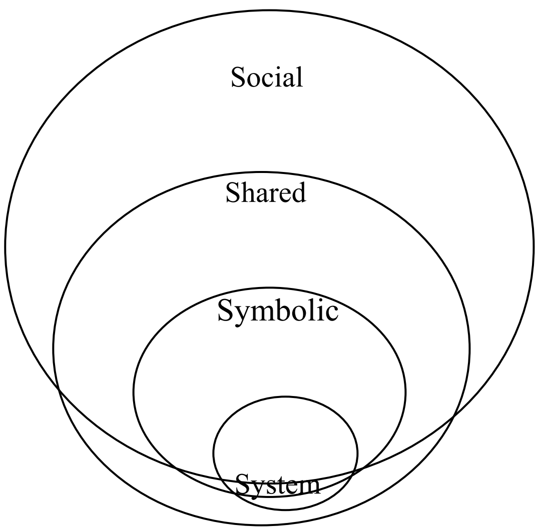


Figure 1. The Socially Shared Symbolic System (SSSS) (Nelson & Shaw, 2002).

(Nelson, 2007, p. 212). They hold differing perspectives which explains the need to communicate. The influence of the community underscores the significance of the "cultural context of [their] encounter with the world of mind and language" (ibid.).

METHODOLOGY

The 3-year study on generalization began in Fall 2005 with 29 11-year-old sixth grade students. Pre-clinical interviews on patterning tasks were conducted with all the students, followed by 12 weeks of teaching experiments (integers; building formulas; patterning), and ending with post-clinical interviews with 12 students. In Fall 2006, pre-interviews were conducted with the 12 students with a focus on generalizing tasks that involved increasing and decreasing patterns. Two sequences of teaching experiments were implemented (integers; patterning) followed by post-interviews with 8 of the 12 students. In Fall 2007, the 12 students were then mixed with 22 new 7th and 8th graders. Pre-interviews were done with all 12 students and 6 7th graders from the same class with tasks focusing on simple and non-simple generalizations. Two sequences of teaching experiments were then implemented (integers and polynomials; patterning) over 12 weeks followed by post-interviews with all 18 students. Grounded categories of generalizing schemes were established year to year; older recurring categories were noted, verified, and used in the teaching experiments.

RESULTS AND DISCUSSION

We draw on selected classroom episode data, individual interviews, and samples of written work from Fall 2005 to Fall 2007 in which the participants were engaged in various aspects of generalizing patterns. Supported by data, we develop an account of their evolving content and construct of generalization from circle to circle.

I. Social Circle. The interaction below was taken from a Fall 2007 session in which the class was being introduced to patterning activity through the classic Tile Patio task (Figure 2). Since there were more new students than old in the 2007 class, the older students assumed the role of explaining direct formulas as a way of assisting the new students to learn useful criteria relevant to establishing formulas.

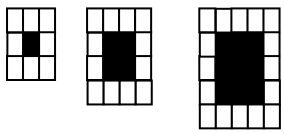
	(1) Make a table for the number of white tiles for several patio numbers. (2) Find a direct formula that expresses the total number of white tiles in terms of patio number. Explain how you arrived at your formula.

Figure 2. Tile Patio Task in Compressed Form.

Teacher (T): What does direct formula mean?

Dave: A formula to find the [*white tiles*] in this problem [*given*] the patio stage number. If you're given the patio number, you can find the number of tiles.

T: Okay. Meaning to say when you say direct formula, you need to find a formula that expresses the number of white tiles for any given patio number. So it should be like what? So how does it look then? … How do you know direct formula when you see one or when you're confronted with it?

Tere: Like you have to prove it. Like you have to do it.

T: Like how do you know one when you see one?

Tere: Coz it has variables in it.

T: So how? Describe those variables for me. How many variables do you see at least?

Tere: Two.

T: And where do you see those variables?

Tere: In the equation and other one at the end.

T: So there's an equation for one. … There should be one variable on the left side of the equation and there should be another one on the other side of the equation. [*T makes a gesture by extending both his arms to indicate both sides in an equation*.] Does that make sense? … If I have a recursive formula like add 2, is that a direct formula?

Emy: No.

T: Why not?

Emy: Because it doesn't have variables.

T: Well, for one it doesn't have variables, so that's one way to think about it. That's what you call a recursive formula. But I want you to come up with a direct formula. Alright. Give me an example of a direct formula just so we know.

Ford: n = 3n [*T starts writing the partial formula*.]

T: Same variable?

Ford: n = 3w plus 8. [*T writes* $n = 3w + 8$.]

T: Alright. This is an example of a direct formula. So there's a variable here on this side [*referring to left*] and a variable on the other side. What's another direct formula?

Dave: w = n times 4 plus 4. [*T writes* $w = n \times 4 + 4$.]

T: Dave, is there another way to express n times 4?

Dave: n four.

T: So what's another way to say n four plus 4? [*T writes* $w = n4 + 4$.] If you want to be hip about it, what's another way to write n4?

Che: 4n.

T: 4n. Normally in algebra we start with the coefficient and then followed that up with a variable.

In the above conversation, the teacher and five 8th graders from the old group surfaced criteria relevant to establishing a direct formula. The discussion was instructive for the new group because the criteria provided the content in which to construct a socially-accepted generalization and served as the common basis of communicative exchanges among the members. The interchange oriented the students to: (1) the meaning of a direct formula; (2) the elements that comprise a direct formula (equation form, two different variables); (3) the difference between a direct and a recursive formula, and; (4) the shape of a direct formula. Consequently, the new cohort was socially oriented to a shared practice of generalizing which helped them avoid less efficient ways of expressing a generalization. Among the old group, the powerful force of such normative-socializing discussions influenced the transformation of individual solutions from situated, verbal descriptions of generalization to more parsed, symbolic forms. For example, the first two columns in Figure 3 illustrate the two generalizations of Dung, 8th grader, for the Square Tiles task stated in the third column before and after a similar discussion took place when he was in sixth grade. Among the new group, knowledge of the appropriate content of a direct formula has allowed them to see the relative inefficiency of most recursive generalizations. Based on assessments from the written work of all the students and the post-interviews done with a selected subgroup, none of them in fact produced recursive generalizations.

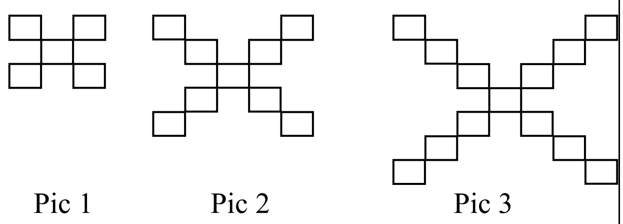
Picture n = n = how many numbers are on each side. n x 4 = n n + 1 = how many tiles are in picture n	S = n x 4 +1	

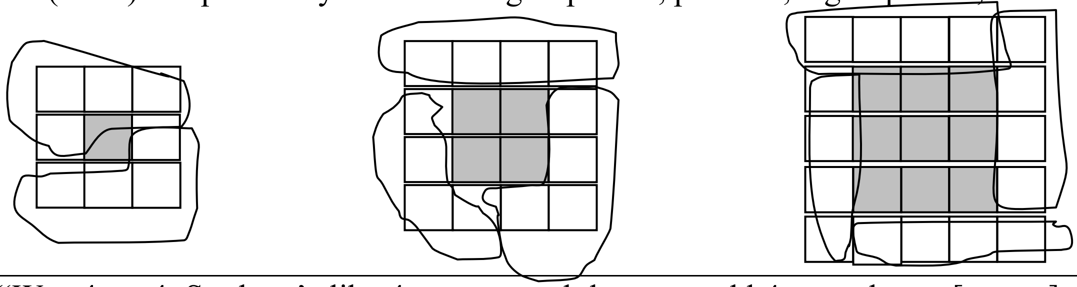
Figure 3. Dung's Two Generalizations on the Square Tiles Task on the Right.

<u>II. Shared Circle</u>. While the social circle focuses on ways in which sociocultural norms orient individuals to particular ways of talking and thinking about generalizat-ions, the shared circle foregrounds the necessity of shared attention to and imitation of someone else's actions prior to entrance into the symbolic circle. In the above conversation, lines 4 to 16, 18 to 23, 25 to 27, and 30 to 37 explicitly described various aspects of an algebraic generalization. Especially with the new group, they learned that a direct formula differed from a recursive one in terms of structure and mode of expression and that generalizations needed to take the particular shape of the former. They learned that a direct formula had at least two different variables appear- ing on either side of the equation with a note that an expression of the form n x a had to be rewritten as an. Overall, the content of their initial generalizations reflected a system of use rather than a system of meaning as a result of initially attending to aspects of generalization that other learners have pointed out for them. Consequently, for some learners, their beginning notion of a generalization and/or variables used to generalize had a referential and situated nature (versus symbolic). For example, Dung's generalization in the first column in Figure 3 used a variable in a referential or situated context because it made sense within the context of that generalization. An interesting situation drawn from Radford (2001) is worth noting in which a student-group failed to see the equivalence of two variable-based generalizations for the same pattern because the generalizations elicited for them two different actions and, thus, were referential or situated.

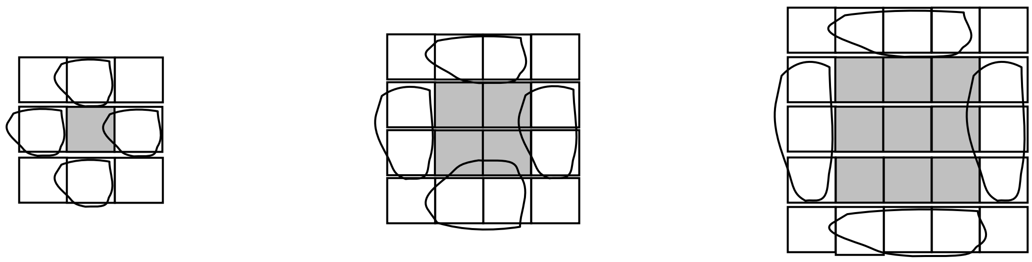
<u>III. Symbolic Circle</u>. After the discussion in section I above took place, the students worked in pairs to construct a direct formula. During the follow-up class discussion, three 8th graders from the old group presented three different formulas (see Figure 4). Lack of space prevents us from outlining the actual interchange, but we have empiric-ally verified the long-term effect of that particular interchange on the students on

succeeding written evaluations and post-interviews in which the students successfully developed different direct formulas for the same pattern. In this circle, generalization and variables used in a direct formula have transitioned from being referential to symbolic denoting conceptual knowledge. Overall, the students saw variables as dynamic placeholders in an algebraic generalization that externally captured what they perceived to be a reasonable structure. Also, equivalence became crucial in this phase in which two or more direct formulas referred to the same pattern very much like their experience in using different words to convey the same concept. Thus, equivalence carried with it students' shared assumptions about patterns. For example, in the case of linear patterns, the students would first agree on the nature of a pattern sequence before establishing an algebraic generalization. Valid equivalence indicated shared perception that two conceptual representations were related in some way.

Dina: "4(n + 1) coz patio #1 you have two groups of 4, patio #2, 3 groups of 4, etc."



Che: "W = 4n + 4. So there's like 4 squares and then you add 4 to each one [corner]. … And for patio #2, 2 x 4 is 8, so 1, 2, 3, …, 8, then you add 4. [There are] 4 groups of n."



Dave: "T = 2(n+2) + 2n. The top part, 2 + 1 = 3. Then I multiplied by 2, the bottom, so that's 6. And the 2n, so here's 1 [row 2 column 1 square] and 1 here [row 2 column 3 square], etc."

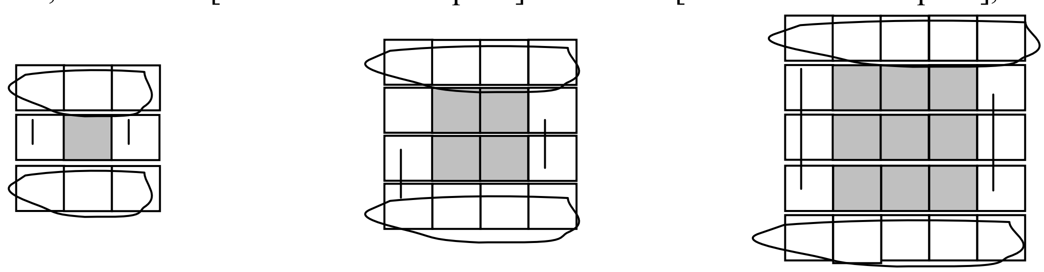


Figure 4. Algebraic Generalizations of Three 8^{th} Graders on the Task in Figure 2.

<u>IV. System Circle</u>. In a Fall 2007 session on linear pattern generalization, the old group of 8^{th} graders shared with the new group a numerical strategy (i.e., finite differences) for establishing a direct formula. Naturally, the new group applied the strategy in a variety of task situations. Consequently, both numerical and visual strategies resulted in the perception of two shared senses of algebraic generalization, namely, as a concept and as a process. When the class was presented with geometric patterns, some students who

constructed a direct formula numerically in the form y = ax + b saw generalization as a process of producing values for *a* and *b*, while other students who established their formulas visually perceived a generalization as a concept that captured the structural features in such patterns in terms of what stayed the same and what changed leading to several direct formulas which were then justified and assessed for equivalence. Figure 4 illustrates three concept-based generalization. The interchange below took place in Fall 2005 during a teaching experiment session in which the members in the same group shared their thinking with the class how they thought about the Adjacent Triangles task in Figure 5.

38 T: So how did you establish your formula? How did you come up with 2 x n + 1?

39 Jake: Coz you do 2 times the box plus 1. [Jake interpreted a triangle as a box.]

40 T: I don't get that.

41 Jake: This one [referring to figure 6] you times it by 2. For the 15th figure, you times 2

42 and then add 1. 30 plus 1, 31.

43 Che: Ahm you know figure number, like figure number 6. You times 6 by 2 equals 12

44 and then you add 1 equals 13 and you get the number of toothpicks.

45 Dung: Coz when you count it, you only count 1 extra [at the beginning] and then you

46 kinda … keep adding the 2 extra sides [gestures the adding of two toothpicks].

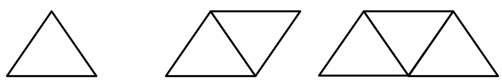


Fig 1 Fig 2 Fig 3

Figure 5. Adjacent Triangles Task.

We note the many occurrences in classroom interactions in which the concept-versus-process sense of generalization became a crucial point of discussion among the students. By the end of the teaching experi- ment on linear patterning in Fall 2007, some students were still struggling with the concept sense on the basis of inconsistencies in their responses on questions that targeted structural invariance and changes in a pattern. Figure 6 summarizes the construct and content of the students' generalizat- ions involving linear patterns in terms of their favored coding process. The character- izations were shared patterns of thinking across several linear generalization tasks.

Coding Process	Construct of Generalization	Content of Generalization
Numerical	Process Sense leading to a single form of a direct expression (weak concept)	Symbolic use of variables, but variables are seen as generators of sequences
Visual	Concept Sense leading to multiple forms of direct expression (strong process)	Symbolic use of variables, but variables are seen in relation to a structure

Figure 6. Construct and Content of Generalization Involving Linear Patterns.

CONCLUSION

A sociocultural orientation to understanding the development of generalization among middle-school children foregrounds the fundamental role played by the social environment in molding the content and construct of generalization among individual learners. While the sociogenetic view is not a novel perspective, there is very little agreement on, and not enough data that support, effective social mechanisms that assist middle school students in transitioning from, say, recursive to closed formulas in patterning activity. In this paper, we do not deny the creative capacity of learners in making such a transition happen on their own, however, our concern is the greater majority who benefit from experiencing "cultural conceptual meanings" (Nelson & Kessler Shaw, 2002, p. 54) from others. Establishing and justifying algebraic general- izations of patterns require knowledge of a domain-specific meaning system, and the specificity could be traced to the social sphere, in particular, in an SSSS that enables such processes to evolve in a genetic-like fashion. Papert (2002) astutely points out how algebra, unlike language, "was not allowed to align itself (to genetic tools already there)" since "it was made by mathematicians for their own purposes while language developed without the intervention of linguists" (p. 582). The existence of contributing mathematicians in the successful evolution of algebra as a discipline of thought always and already conveys a social foundation. Using an SSSS model, we endeavored to articulate how shared content and understanding of algebraic generalization emerged and evolved from among a community of minds initially by means of norms that became meaningful only when they were shared among the members. Of course, individual learners could hold personal norms that guide the content of their private or inner speech or thought but, as our data indicate, norms tend to develop in sophistication only when they are shared. Further, shared ways of responding to a generalization task enable growth in conceptualization and increased competence and flexibility in multiple representational skills. Even the notion of equivalence, viewed as a psychological phenomenon, has a social basis. The students' experience with equivalence required them to compare their individual views with those of others and to accept the view that others have different ways of looking at the same pattern. But certainly the end goal was still shared understanding.

We close with some thoughts for further research. More data are needed to assess how students employ generalization as symbol systems. Further, what implications could be drawn from the two senses of generalization (and there could be more) insofar as they relate to mathematical activity and task construction? Further, there is a perennial interest in how teachers can effectively set up the social space so that meaningful, purposeful, intent-driven discourse on generalization takes place among their students enabling growth in their SSSS.

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NATURE AND CONTENT OF GENERALIZATION OF 7TH- AND 8TH-GRADERS ON A TASK THAT INVOLVES FREE CONSTRUCTION OF PATTERNS*

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We report results drawn from 7th- and 8th-graders who freely-constructed patterns from given figural entities and then described and generalized their patterns. Data from 22 pre-tests and 17 post-tests are categorized in four ways in terms of: the type of pattern generated; nature of the instances constructed; identification of attributes of the pattern; and type of generalization developed.

> How otherwise can one ask for, say, the number of matchsticks … in an "arbitrary" item of the sequence? The situation would presumably be much more open if one asked simply "How can you continue?", or "What can you change and vary in the given figures?" … I rather want to hint to possible further directions for research … a plea for "free" generalization tasks not restricted by pre-given purposes. (Dörfler, 2008, p. 153)

In this report, we take to task Dörfler's (2008) suggestion of allowing learners to freely construct patterns and then for them to develop a generalization. Of course, free construction is still subject to constraints such as prior knowledge. For e.g., considering the 8th grade students whose work is analyzed in this study, their prior knowledge consists of two years of pre-algebra content, including two sequences of teaching experiments on linear pattern generalization, one in Fall 2005 and another one in Fall 2006. Hence, we expected that their constructed patterns would be, at the very least, linear in nature. While their patterns could also be nonlinear (say, quadratic) or (nontrivial) recursive, the *free* aspect involves learner-dependent actions of constructing and exploring attributes within and relationships among objects in an emerging pattern of their making. Prior to the free construction activity, which was conducted in Fall 2007, we had no knowledge of their capacity to establish and justify generalizations on the basis of an evolving pattern sequence of objects whose properties and structure co-emerge with their construction.

The basic research questions that we seek to resolve in this report are as follows: *(1) What kinds of patterns do students generate on a free construction task? What to them are indications that the stages they generate form a pattern? (2) To what extent does construction lead to a meaningful interpretation? By meaningful interpretation, we mean a justifiable algebraically-useful generalizat-ion or some other clear description that could sufficiently explain both the known and unknown cues of the constructed pattern.* Two important issues that are pursued in the first question involve assessing students' capacity for structure sense and identifying mechanisms that engender a structural sense of patterns. By *structure sense*, we mean their ability to cognitively perceive a representation of a pattern that is structured in some way. Such a structure is a function of several parts: the stated instances that initially define the pattern; the identification of a reasonable attribute that enables a description of individual instances, and; the establishment of a relationship that allows a connection to be made about the attributes of the known instances, including a statement of a probable inference that could be projected onto the unknown instances of the pattern. An issue we tackle in the second question deals with various semiotic resources that students use to capture and justify what they perceive to be the essence of their generalizat-ions. For example, do they recourse to making recursive statements on an unfami-liar pattern? What aspects in their pattern do they consider important and useful that enable them to establish a meaningful interpretation?

THEORETICAL FRAMEWORK

This study is grounded on two epistemologically-related ideas, namely: the construction of example spaces in which the focus is on structures, and; a philosophical account of patterns – as structures – which provides our basic characterizat-ion of the nature and content of student-generated generalizations involving free construction of patterns. Watson and Mason (2005) define an *example* to be "anything from which a learner might generalize" (p. 3). They see constructing "example spaces" as central to learning mathematics because such spaces involve a structural task, that of determining and evaluating aspects that stay the same and that change. The idea of developing example spaces is rooted in a number of prominent figures, in particular, the constructivist Vico (1990) who saw construct-ion and knowledge as co-emergent, with Watson and Mason (2005) expressly articulating a most significant telling about "the aim of learning," which is, "to construct meaning for ourselves, not to attain external, preexisting meanings, while conforming to social practices" (p. 8). While there are a variety of example types in mathematics that are implemented with purposes other than a structural one, in this paper we were interested in those "classes [that are] used as raw material for inductive mathematical reasoning … and then examined for patterns" (p. 3). Such a process of exemplification is indicative of a kind of constructive thinking that, according to Dienes (1963), "aims at a set of requirements and attempts to build a structure which will meet them" (p. 95; quoted in Watson & Mason, 2005, p. 8). Finally, Watson and Mason (2005) characterize mathematical thinking around exemplifying acts to be "individual and situational" with "perceptions of generality" as being "individual" and perceptions of examples as possibly "members of structured spaces" (pp. 50-51).

Beyond the promotion of constructive thinking in, and the meaningfulness of, learner-generated examples spaces is a fundamental philosophical view of mathematics in which "the primary subject-matter is not the individual mathem-atical objects but rather the structures in which they are arranged" (Resnik, 1997, p. 201). When the participants in our study have been asked to freely construct a pattern from objects consisting of pattern blocks, we were working under the assumption that those objects are "themselves atoms, structureless points, or positions in structures … with no identity or distinguishing features outside a structure" (ibid.), hence, our interest in structures that provide some meaning to the configured objects. Resnik

(1997) distinguishes between pattern recognition and knowledge of a pattern. In the former, learners are able to compare an instance from a non-instance; in the latter, they provide descriptions of the pattern. In this study, we focus on the latter and, in fact, subsume the former when a description of the generalization has been sufficiently justified. The nature of this description is expressed by Resnik (1997) in the following manner:

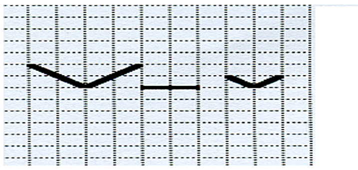
> Seeing a pattern is more a matter of *seeing that* certain of its instances fit or satisfy its defining conditions. To abstract a pattern from instances is neither to intuit nor to see it; rather it is a process by which we arrive at a description of the pattern by alternatively positing related positions and checking their fit against putative instances. (p. 225).

Hence, we view the nature and content of the students' generalizations within the context of a simultaneous process of theory development and data fitting or, in other words, complete abduction (see Rivera, 2008).

METHODOLOGY

Fourteen 8th graders (mean age of 13; 9 females; 5 males) and twelve 7th graders (mean age of 12; 8 females; 4 males) were part of an algebra class in an urban school in Northern California that participated in a yearlong study on algebraic thinking in which the second author was the instructor. In this paper, we analyzed the work of the 8th graders and the 7th graders separately on a free construction pattern task shown in Figure 1. The task was administered twice, once before and another after a two-week teaching experiment on linear pattern generalization in Fall 2007 as a homework assignment. In asking the students to construct patterns, we thought that classroom activity under pressure of time and influence of others could significantly prevent them from owning the task. Free from such constraints, we wanted to determine the nature of their constructions under more normal circumstances such as the home environment, thus, allowing them more time to explore, reflect, and understand their own thinking processes relevant to this task. In terms of prior content knowledge, the 8th graders were members of a class that participated earlier in a two-year study on patterning and generalization. The second author co-taught the class with two other mathematics teachers in those two years. We note that the students have not been exposed to such tasks in the earlier years of the study which focused on the generalization and justification of patterns with stated initial conditions. In fact, the research objective then was to trace the development of algebraic generalization (Radford, 2006). In Fall 2007, while the teaching experiments still focused on the same objective, we added the free construction of pattern task to further analyze the students' ability to perform generalizations on a non-routine situation. In the case of the 7th graders, they had one year of pre-algebra instruction with a different mathematics teacher and have not had instruction on patterning and generalization. Thus the work of the 8th and 7th graders was analyzed separately. As far as data analysis matters, we first individually analyzed the written work of the two student-groups on the pretest and developed codes. We then compared codings and reanalyzed the pretest results and finally analyzed the posttest using the established codes, adding a few new ones when they became necessary.

From the following three figures below, pick at least two figures to create a pattern sequence of five stages. Use the attached grid paper to draw your four additional stages.



(1) What stays the same and what changes in your pattern?
(2) Obtain a generalization for your pattern either by describing it in words or by constructing a formula. How do you know that your generalization works?

Figure 1. Free Construction Task in Compressed Form.

RESULTS

In the first two sections below, we address the research questions we stated in the introduction in relation to the data we collected and analyzed. In the third section, we illustrate pattern development and justification on this type of task through the work of one 7th-grader who established a quadratic pattern with no formal instruction in that content area. First, we note that a separate analysis of the 7th and 8th grade results revealed more similarities than differences, so the overall results are combined in the tables. Significant differences between the two groups are pointed out in text and through examples.

Question 1. *What kinds of patterns do students generate on a free construction task? What to them are indications that the stages they generate form a pattern?* Table 1 summarizes the kinds of patterns the students generated and the number in each category for the pre- and post-test on the task in Figure 1. Examples are included to provide the reader a sense of types of patterns we classified into each category. LP was the most common type of pattern, with over 50% creating a pattern of that type on both the pre- and post-test. In fact, some type of linear pattern was the most common structure constructed in both the pre-test (64%) and the post-test (81%). We use the terms linear and recursive on the basis of whether the students' generalizations were either an increasing linear direct formula or a verbal description conveying for them a recursion that is not linear. In an LP or RP, a succeeding cue is formed by concatenating a figure (or figures) on a part (or parts) of a preceding cue. In an LS or RS, a basic unit is first constructed using either the first stage (a 1-cycle) or the first n stages (an n-cycle). Then copies of the same basic unit appear at the end of subsequent cycles. For example, in the LS example in Table 1, stage 2 has two copies of the first stage, stage 3 has three copies of the first stage, etc. The RS example in Table 1 is a 3-cycle so that stage 3n, $n \geq 2$, consists of n copies of the 3-cycle. The category RO is a non-monotonic sequence that oscillates among a few terms. The category NLP is constructed the same way as an LP but the overall pattern is not linear.

Table 2 classifies the nature of the instances that initially define a constructed pattern as consistent or inconsistent; by consistent we mean we can predict with relative ease the succeeding stages in the pattern because the same constructive action is preserved from stage to stage. Note that at least 80% of both 7th and 8th grade students constructed a consistent pattern on both test administrations. Table 3 classifies the attribute identification of the pattern or the nature of what stays the same and what changes from stage to stage. We classified students' responses as either a surface or deep description. A surface description focuses on overall perception of shape or feature, while a deep description pays attention to specific properties that are invariant, changing, and other perceived particularities of the pattern. Note that both the 7th and 8th grade students improved in this category from pre-test to post-test, with a bit more improvement for 8th graders. Further, while both groups had 50% deep description on the pre-test, the 7th and 8th graders had 77% and 88% deep description on the post-test, respectively. Also, the number of surface descriptions declined dramatically from pre- to post-test.

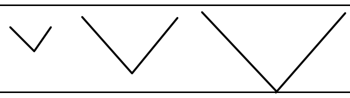
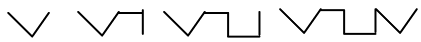
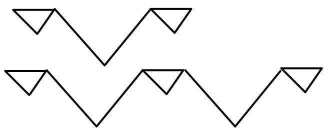
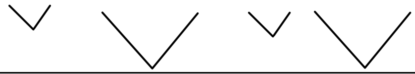
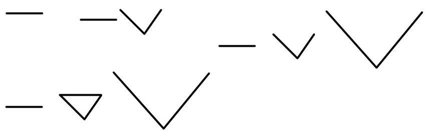
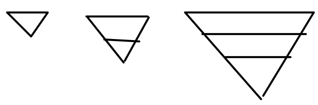
Category	**Example** **1 2 3 4**	**Pre-Test N = 22**	**Post-Test N = 17**
Linear by Parts (LP)		12	9
Linear by Shape (LS)		2	4
Recursive by Shape (RS)		4	2
Recursive by Parts (RP)		2	0
Recursive Oscillating (RO)		1	0
Not a Pattern		1	1
Nonlinear by Part (NLP) (for e.g., quadratic by part)	 (On a grid, count units of length and ignore difference of diagonal lengths)	0	1

Table 1. Kinds of Patterns Generated on Task

Question 2. *(2) To what extent does construction lead to a meaningful interpretation, i.e., a justifiable algebraically-useful generalization or some other clear description that could sufficiently explain both the known and unknown cues of the constructed pattern?* This was one aspect of the work that showed marked differences between 8th and 7th graders. On the pre-test, 65% of the 8th graders constructed a correct direct formula for their pattern, while only 20% of the 7th graders were able to do so. Both groups improved after instruction; on the post-test, 88% of the 8th graders and 67% of the 7th graders constructed a correct direct formula. The overall results for the type of generalization generated are shown in Table 4. We note that almost all consistently constructed linear and nonlinear patterns oftentimes produced linear and nonlinear direct formulas, while all consistently constructed recursive patterns resulted in verbal descriptions.

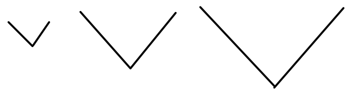
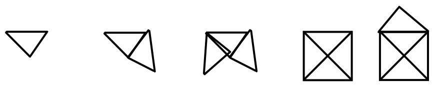
Category	**Example** **1 2 3**	**Pre-test** **N = 22**	**Post-test** **N = 17**
Consistent		19	15
Inconsistent		3	2

Table 2. Nature of Instances Defining the Pattern

Category	**Example**	**Pre-test**	**Post-test**
Surface	The thing you just did stays the same and the "V" or "___" you add changes.	9	1
Deep	The original pattern stays the same. It adds two squares each time to the V.	11	14
Ambiguous	You are adding two each time.	2	1
No response		0	1

Table 3. Attribute Identification

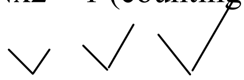
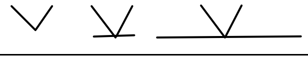
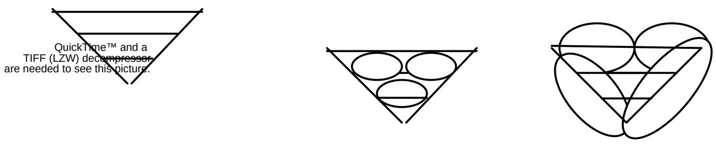
Category	**Example**	**Pre-test**	**Post-test**
Correct direct formula	P = n x 2 + 1	10	13
Incorrect formula	Nx2 – 1 (counting segments) 	5	2
Words	First you make the large "V" and The[n] add a "__"on the bottom. Next, you just add "__" on both sides of the previous "__" 	5	1
No response		2	0

Table 4. Type of Generalization

A Particular Example. 7th-grader Diana's work exemplifies a good example of a complex patterning process in relation to the free construction task. Diana's post-test pattern is shown in Table 1 under NLP. Although her pattern is nonlinear, it is growing by parts. We point out that Diana ignored the differences in lengths of the diagonal and horizontal line segments, a fact that applies to a significant number of students who saw segment length as unimportant. Despite that fact, Diana clearly identified an underlying structure in her growing pattern by first extending each side of the triangle by adding one additional segment and then closing the triangular shape formed. Diana then circled in two colors to distinguish the groupings she was counting, one the interior horizontal segments, and the other, the outer segments on the perimeter of the triangle, both of which she counted as groups of the stage number. In her written description of what to her stayed the same and what changed, she wrote:

1. Number 1 [*the original triangle*] will stay in all of them. The x(x-1) is for the lines
2. in the middle of the triangle. The +4x is for the triangle borders. It's really short
3. for 2x + 2(x). But it was pretty much the same.

To illustrate (see Figure 2), Diana counted the horizontal segments in the interior of the growing triangle as: 2 groups of 1 segment in stage 2; 3 groups of 2 segments in stage 3; 4 groups of 3 segments in stage 4; 5 groups of 4 segments in stage 5 *leading to the expression n(n – 1).* Then she counted the segments on the perimeter of the growing triangle in two parts. Part A pertains to the two diagonal sides of the growing triangle: 2 groups of 1 segment in stage 1; 2 groups of 2 segments in stage 2; 2 groups of 3 segments in stage 3 *leading to 2n*. Part B pertains to the base of the growing triangle: 2 groups of 1 segment in stage 1; 2 groups of 3 segments in stage 3; 2 groups of 4 segments in stage 4 *leading to 2n.* Clearly she conceptualized her pattern as number of groups of the stage number which matched the way in which she circled the parts of the figures. Finally Diana simplified her pattern of L = n(n-1) + 4n to L = n^2 + 3n.



[3 groups of 2] [4 groups of 3]

Figure 2. Stage 3 of Diana's Nonlinear Pattern.

DISCUSSION

The students' performance on the free construction task shows the frequency in which they perceived patterns to be linear and recursive, with one student, Diana, successfully demonstrating a capacity for quadratic patterning. Diana exemplifies how the students perceived direct expressions as resulting from a visual process. None of them in fact completely abduced numerically such as using a table of values to generate a formula. We also find it interesting that the 8th graders who had almost three years of design-driven instruction on patterning and generalizat-ion still constructed linear and recursive patterns which to us was an indication that their conception of patterns was not narrowly limited to linear and monotonic structures. Certainly, there is value in recursive patterning since it could be the basis in which to further develop more complex mathematical concepts or modeling processes that involve, say, piecewise functions.

We note the powerful role of visual cognition in their developing capacity for, and competence in, various dimensions of structure sense. To begin with, the structures of the constructed pattern cues from pre to post became simpler which to us is a consequent manifestation of the Gestalt law of pragnanz in which simplicity is that which drives our perceptual organization. Also, we infer that the students have developed "visual templates" – a useful term we appropriate from Resnik (1997) and Giaquinto (2007) – which helped them in their construction. Giaquinto (2007) notes the significant function of visual experience and theoretical knowledge in cognitively grasping simple and complex mathematical structures. In the case of the students, their visual experiences on linear patterning have somehow assisted them in the design of their patterns of choice and the need to account for invariance and change with the use of a finite number of initial cues.

Our third remark concerns the many ways in which the students interpreted the question of what stays the same and what changes in a pattern. In Table 3, we introduced a third category, Ambiguous, to surface how some of them considered the two questions as one question which we interpreted in the context of "action." For example, the ambiguous response "You are adding two each time" may have meant the following: What stays the same is the act of constantly adding and what changes are the two figures being added onto a cue.

Our fourth remark is related to the third remark. In Diana's case, as in others, there is strong evidence of structure sense. However, we note that what Diana identified as staying the same (see line 1) was not used in formulating a general-ization. Rather, she focused on the interior of each growing triangle and the outer perimeter to establish an algebraic generalization using a visual approach. In the case of those students who constructed simpler patterns, this disconnect was also evident. For example, in the LS pattern in Table 1, triangles are drawn and noted to be growing by the stage number, but the generalization reflected the perimeter (total number of segments, $y = 3x$). Thus, while the students were able to construct example spaces with a consistent underlying structure, the explicit analysis of invariance and change did not seem as valuable. This finding is explored in more depth as we analyze clinical interviews on similar free construction tasks.

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COGNITIVE TENDENCIES IN THE PROCESS OF CONSTRUCTING MEANINGS IN NUMBERS, VARIABLES AND LINEAR FUNCTIONS

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This article reports on the construction of number, variable and linear function meanings when facing continuous variation problems. Cognitive tendencies were identified among students who demonstrated centration phenomena and extended the meanings of parameters "m" and "b" of linear function y=mx+b to negative values.

During the process of constructing the meanings of algebraic objects such as numbers, variables and linear functions by way of an algebra teaching approach, which uses continuous variation word problems dealing with real life situations (Rubio, Del Valle and Del Castillo, 2006) two of the cognitive tendencies pointed out by Filloy, 1991, (he distinguished 11) have been revealed. One of the foregoing tendencies is that of "centration on readings made at language strata that fail to enable solution of the problem situation, which consists –in the particular case of this study- of forgetting the initial conditions supplied as data in continuous variation word problems (in t=0, b=cte), focusing solely on the increase in the dependent variable (Δy=mx) of the function: y=mx+b. The second tendency is "endowing intermediate senses", in this case when the students gives "senses of use" to the negative number (Gallardo, 2002), such as: a) subtrahend (when the negative is used as a subtraction); b) relative number (symmetrical numbers); c) ordered number (when a negative number is greater than another number); d) "negative parameter" (when the admission is made that m and b can take on negative values in linear function y=mx+b).

Authors such as Moschkovich, 1999, Mevarech and Kramarsky, 1997, have studied linear function y=mx+b within a graphic setting. Bardini and Stacey, 2006, explored the student conceptions for parameters "m" and "b" within the same setting, indicating that "b" is omitted both in their verbal and symbolic descriptions. Unlike the foregoing, our point of departure was using continuous variation word problems, in which we identified cognitive tendencies exhibited during resolution of those same problems, tendencies that are linked to the objects of number, variable and function.

THE STUDY

The objective of the study was to inquire into the issues that prevent students from following the process of building meanings for the mathematical objects of number, variable and linear function, and to see just how to overcome those issues.

Experimental settings were developed, in which the goal was to account for the process in which meanings are built for the previously mentioned algebraic objects among 14 to

15 year old students. The students were taught for a period of 14 weeks and the didactic implementation consisted of the following: a) a numerical approach that fosters progress of analytical capabilities and algebraic thought (for instance, see Rubio, 2002; Rubio & Del Valle, 2004); and b) a method for solving families of equations.

The settings consisted of a case study dealing with 4 students at three levels of knowledge, all four of whom were selected from a school group of 50 students. The staging was further based on a diagnostic questionnaire dealing with continuous variation word problems. This article will solely report on the performance of one low-level student who had previously solved continuous variation problems in which "b" was positive. The "centration tendency" arose and was overcome by way of a teaching sequence that enabled the student to correctly identify the meanings of each of the elements involved in the numerical operations that she spontaneously undertook. The tendency –not necessarily a characteristic exhibited solely among low-strata students- was also observed among the mid and high-strata during the process of solving the sundry continuous variation problems put to the students in the case study.

Below the reader will find segments of the interview carried out with the student Ana, and such segments demonstrate the cognitive tendencies of "*centration*" (for b<0) and of *"production of intermediate senses of use"*.

The vegetable problem: *"A bag of vegetables is removed from the freezer of a refrigerator whose temperature is 18°C below zero. The bag is placed in a microwave oven to heat; the timer is at 0 seconds. If the microwave has the effect of increasing the heat of the bag of vegetables at a rate of 2°C per second until reaching a temperature of 21° C: a) determine several moments in time at which the bag of vegetables registers a temperature higher than 5 ºC below zero, but less than 1ºC below zero; b) if the time changes, what happens to the temperature of the bag of vegetables? and, c) write an algebraic expression that relates the temperature of the bag of vegetables with time".*

Ana writes –18ºC (she represents "below zero" temperature as the relative number

–18, endowing the negative with that intermediate "sense of use").

She writes:

$$\begin{array}{r} 2 \\ \underline{\times 13} \\ 26^{\circ}\mathrm{C} \end{array}$$

Ana: This is the final temperature …that is to say the temperature of the bag ….

She remains silent, hesitating for a few seconds. She resorts once again to reading the text of the problem and answers:

Ana: No!… 26 minus 18 (Please note that at this point she overcomes the centration upon considering the initial temperature value of –18ºC)

Ana subtracts 18ºC from 26ºC and obtains 8ºC, and writes:

$$\begin{array}{r} 2 \\ \underline{\times 13} \\ -\ 26^{\circ}\mathrm{C} \\ \underline{18} \\ 8^{\circ}\mathrm{C} \end{array} \quad \text{is the final temperature of the bag.}$$

Once the centration tendency related to variable is overcome, she tries to answer the question: "*...determine several moments in time at which the bag of vegetables registers a temperature higher than 5 ºC below zero, but less than 1ºC below zero…"*. She writes:

$$\begin{array}{r} 2 \\ \underline{\times 10} \\ -\ 20^{\circ}\mathrm{C} \\ \underline{18} \\ 2^{\circ}\mathrm{C} \end{array} \qquad\qquad \begin{array}{r} 2 \\ \underline{\times 5} \\ -\ 10\ ^{\circ}\mathrm{C} \\ \underline{18} \\ -8\ ^{\circ}\mathrm{C} \end{array}$$

Clearly she produces another "negative sense of use" --as an isolated number, resulting from the operation 10–18 = –8.

> Ana: I don't understand …it has to be greater than 5º below zero and less than 1º below zero.
>
> Interviewer: Look at your data …what are your constants?
>
> Ana: 18ºC below zero.
>
> Interviewer: When is it colder? When you are at 18ºC below zero, when you're at 15ºC below zero or when it's 8ºC below zero?
>
> Ana: When it's 8º or 15º below zero …

With this answer, the student is comparing values that will lead her to obtain temperatures within the interval requested in the text of the problem, which is why the interviewer proposes the following to her:

> Interviewer: Why don't you organize this in a table …and you can see what happens to your bag of vegetables…?

She progressively writes tables 1, 2 and 3:

Time	Temperature
13 sec.	8ºC
10 sec.	2ºC
5 sec.	–8ºC

Table 1

Time	Temperature
5 sec.	-8ºC
10 sec.	2ºC
13 sec.	8ºC

Table 2

x	y
Time	Temperature
4 sec.	–10ºC
5 sec.	–8ºC
10 sec.	2ºC
13 sec.	8ºC

Table 3

During this process, one can see that the student produces another "*intermediate sense of use*" related to the order of negative integers. Said sense is produced once she has organized the values of the independent "time" variable into one of the columns of table 2 and the values of the dependent "temperature" variable, some of which correspond to below zero temperatures, into another column of table 2. After she has built table 3, she is told the following:

Interviewer: Now let's see if you can put all of that into a two-variable equation.

She focuses her attention on one of the operations 2×13=26ºC and on 26ºC–18= 8ºC, which enabled her to fill in one of the lines in the tables, in the last of which she associates the first column with literal "x" and the second one with literal "y", then writes:

$$\begin{array}{rl} 2 & \\ \times\underline{13} & \text{"This is x"} \\ -\ 26^{\circ}\text{C} & \text{"this is 2x"} \\ \underline{18} & \\ 8^{\circ}\text{C} & \text{this is 2x-18, the "Temperature" of the bag of vegetables} \end{array}$$

Finally she writes:

$$y=2x-18$$

One can see that the student is able to give meaning to each of the elements and operations that make up the equation, in which she assigns "the subtrahend sense of use" to the negative number.

In order to provide proof of the generalization processes linked to the variation of parameters "m" and "b" of functional relation y=mx+b, the student is presented with the bag of vegetables problem for a second time. She assigns a value of 10 sec. to the "time" variable and multiplies it by the speed of change (2ºC/sec) to obtain the temperature rise corresponding to said time. She then subtracts the 18ºC from that result (20ºC), identifies the "–18ºC" as the subtrahend of a binary operation --in other words she has not yet to give it "the relative number sense of use". In order for the student to produce no the latter sense of use, the interviewer undertakes the following teaching sequence:

Interviewer: So you subtract it? What happens to the temperature of the bag of vegetables? Is it being subtracted? Exactly what temperature do you have when you set the time?

Ana: Minus 18º ….18º below zero …Celsius…

Interviewer: Yes…the vegetables are at–18ºC when you take then out … What happens to them as time elapses?

Ana: They start to heat …

Interviewer: So then, why do you say that you subtract?

Ana: …because…I assumed that you had to subtract the amount that I had …

Interviewer: Why?

Ana: …I don't know …

Interviewer: Let's suppose that it's not at …minus 18 degrees Celsius…let's suppose it's at 3ºC…Would you subtract it then?

Ana: At 3 or –3?

Interviewer: … At three.

Ana: No…. I'd add it.

Interviewer: And, why would you subtract it from –18?

Ana: I don't know …maybe I thought that …since I had –18…that it had to be subtracted …

Interviewer: But…why? Aren't they starting to heat? Let's see, at –18…now they're at …they're heating …the temperature rises …Right?

Ana: Yes.

Interviewer: If it's at 3ºC… it's heating …and the temperature rises by 20ºC…In this case would you add or subtract?

Ana: I would add …because it already is 3 degrees …and over time another 20 are taken on. So…if…time elapses and goes by …. the 20 …and 3 that I already had …I add it …

Interviewer: Is there any difference in the operation that you would do with the –18 ºC?

Ana: I suppose I'd have to do the same thing …that is to say …add...right?

The student writes:

$$\begin{array}{r} 20\text{ ºC} \\ +\ \underline{18}\text{ ºC} \end{array}$$

Interviewer: But who's adding ….18ºC…?

Ana: No…to minus 18 I'm adding 20…

Interviewer: And where is the minus part in the minus 18?...

When the interviewer says the latter, the student erases the foregoing addition, and writes:

$$\begin{array}{r} -18 \\ \underline{+20} \\ -2 \end{array}$$

Interviewer: Are you sure? –18 + 20 is –2?

Ana: No

She then writes 2 instead of –2, leaving the expression:

$$\begin{array}{r} -18 \\ +\ \underline{20} \\ 2\text{ºC} \end{array}$$

It is important to note that she is able to make "the relative number sense of use". when operating with the –18 in order to obtain the temperature of the bag of vegetables (–18ºC+20ºC = 2ºC). She has hence been able to go from a subtrahend that she used to obtain the temperature of the bag of vegetables (20ºC–18ºC = 2ºC) to a usage in which she operates with a relative number.

When she produces this "sense of use", she carries out similar operations (7.5×2=15ºC and –18ºC+15ºC = –3º) using other moments in time to determine the temperature values of the bag of vegetables that meet the condition specified in the word problem. Finally, she represents the functional relation between the temperature of the bag and the time in a table. She is asked to obtain the algebraic representation of the temperature-time functional relation and accomplishes this task using the numerical operations as references to obtain temperature values in the table, subsequently using linear function y=mx+b when faced with isomorphic problems.

Let's see:

Interviewer: Could you write an algebraic expression for me that determines… the relation between temperature and time?

She writes "x" and "y", at the top of the table 3 columns for time and temperature, then writes the function:

$$y= -18+2x$$

Interviewer: Very good! Here the 2 was constant and the –18 was the value that you told me it had when the time is zero. Now, let's suppose that it heats up faster, for example if the bag of vegetables now heats at a rate of 4ºC per second, what would the expression be?

Ana writes:

$$y=4x-18$$

Interviewer: Exactly! Now, let's assume that we have a bag of vegetables and we put it into a more powerful freezer. That is to say, it will freeze at a rate of 3ºC per second. How would you represent that?

Ana: (does not answer).

Interviewer: In this case it's heating at a rate of 2ºC per second (I indicates the equation y=2x–18), here it's heating at a rate of 4ºC (I points to the equation: y=4x–18), but now it's going to cool off. How would you represent that?

Ana: …It would be… "y" is equal to –3… "x" …(writes y=–3x).

Interviewer: Perfect! What else?

Ana: Then you would add ... –18.

Interviewer: Right! That at zero time the temperature is –18.

The student adds –18 to the –3x that she had written, leaving the equation with two variables:

$$y = -3x-18$$

Interviewer: And what would happen if it freezes even faster, for instance at a rate of 5ºC per second?

She immediately writes:

$$y = -5x-18$$

Interviewer: I want you to find an expression for when it heats at a rate of 2ºC per second, but with an initial temperature of 10ºC below zero.

The student has no trouble writing:

$$y = 2x-10$$

Once Ana is able to write the foregoing linear functions that relate temperature and time, we once again have confirmation that she is able to overcome the *centration tendency* (for b<0). In short, the student is able to relate parameters "m" and "b" to the numerical values that correspond to each case, for both negative numbers ("negative parameter") and positive numbers, meaning that she has begun to build "the parameter sense of use of a literal". These new "senses of use" are indications that the student has been able to broaden the numerical domain to integers.

Conclusions

This study on building meanings for the algebraic objects of number, variable and function by way of continuous variation problems indicates how important it is to identify the tendencies demonstrated by students during the process of solving problem situations. Once those tendencies have been identified it becomes possible to adequately design teaching sequences that enable building the meanings of those algebraic objects. This research furnishes several pieces of evidence indicating that students tend to ignore the initial conditions (in t=0, b=cte) provided as data in continuous variation word problems, focusing solely on the increase of the dependent variable ($\Delta y=mx$) obtained as of the product of the rate of change (m), given as data, multiplied by a value of independent variable (x), and not in the final value of dependent variable (y) that requires adding the increase indicated to the initial value ($y=mx+b$), which is somewhat similar to that found by Stacey within the graphic setting.

One hypothesis concerning the "centration tendency" (which was overcome by way of a teaching sequence) may be due to the fact that the text of the word problem does not explicitly state the difference between the "length of the increase" and the "final temperature of the bag of vegetables". A didactic consequence of the foregoing would be to provide within the text of the word problem –at least in the first few problems- certain clues that enable the reader to identify the dependent variable as the result of the sum of its initial value plus "the increase of the dependent variable" obtained as a result of the product of the speed of change at each moment of time (mx). The other cognitive tendency is expressed when the student produces *intermediate senses of use* for the negative as a subtrahend, relative number, isolated number, ordered number and negative parameter. All of this contributed the extension of the natural-number domain to the integers and, as such, to development of competence in usage of increasingly abstract strata of algebraic language.

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THREE-DIMENSIONAL VISUALIZATION: CHILDREN'S NON-CONVENTIONAL VERBAL REPRESENTATIONS

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In this article, we share a framework for the development of 3-D spatial visualization. We believe that learners should be exposed to five representations, namely, 3-D models, conventional 2-D graphic models that resemble a 3-D figure, abstract 2-D representations that bear little resemblance to the 3-D figure, verbal representations, and dynamic computer simulations of 3-D figures. We focus on children's use of the verbal representation as they move among the other representations paying particular attention to non-conventional language that may have implications for the way they interpret 3-D figures.

INTRODUCTION

This paper, written in the voice of researcher, Jacqueline, demonstrates how children use standard language in their descriptions of spatial positions within 3-D objects. Our findings show non-conventional use of descriptive terms such as, "in front" or "behind," and "vertical" and "horizontal." We also observed a lack of correspondence in children's descriptive language when 2-D images are viewed and then described in the horizontal plane (as in a book) versus in the vertical plane (as on a screen).

This study is part of an ongoing research and program-development collaboration to help elementary children develop 3-D visualization skills. Activities for classroom use were developed by Retha van Niekerk (1997) in a rural South African setting. Her research provided empirical evidence that explicit attention should be paid to develop children's use of descriptive spatial language. She conjectured that indigenous language and unevenly developed first language among her native South African subjects contributed to their lack of ability to use spatial language consistently. During her two-year visit to Texas in 2002, I worked closely with van Niekerk as a co-instructor for two academic-year in-service teacher courses offered by the Rice University School Mathematics Project (RUSMP), *Geometry for Elementary Teachers* and *Geometry for Middle School Teachers*. In the sections of the courses devoted to developing 3-D visualization, Retha's activities and findings were shared with participant teachers. Later, co-writer, Irma, who participated in the elementary teachers' course, was invited to become a Master Teacher for RUSMP and co-taught later sections of the same geometry courses with me. Currently, we all collaboratively develop instructional materials that integrate and extend Retha's original activities. A new component to the instructional sequence is the dynamic software interface (Lecluse, 2005).

In the next section, we provide background for the theoretical framework that guides our view of 3-dimensional visualization, followed by the methodology that supports our study. Finally we present three exemplars including data, discussion and conclusions that focus on children's unconventional use of verbal language in this context.

THEORETICAL FRAMEWORK

The spatial operation capacity (SOC) model that guides the study is based on the research work of Yakimanskaya (1991) and van Niekerk (1997). Children should be exposed to activities that require them to act on a variety of physical and mental objects and transformations to develop the skills necessary for solving spatial problems (Yakimanskaya, 1991).

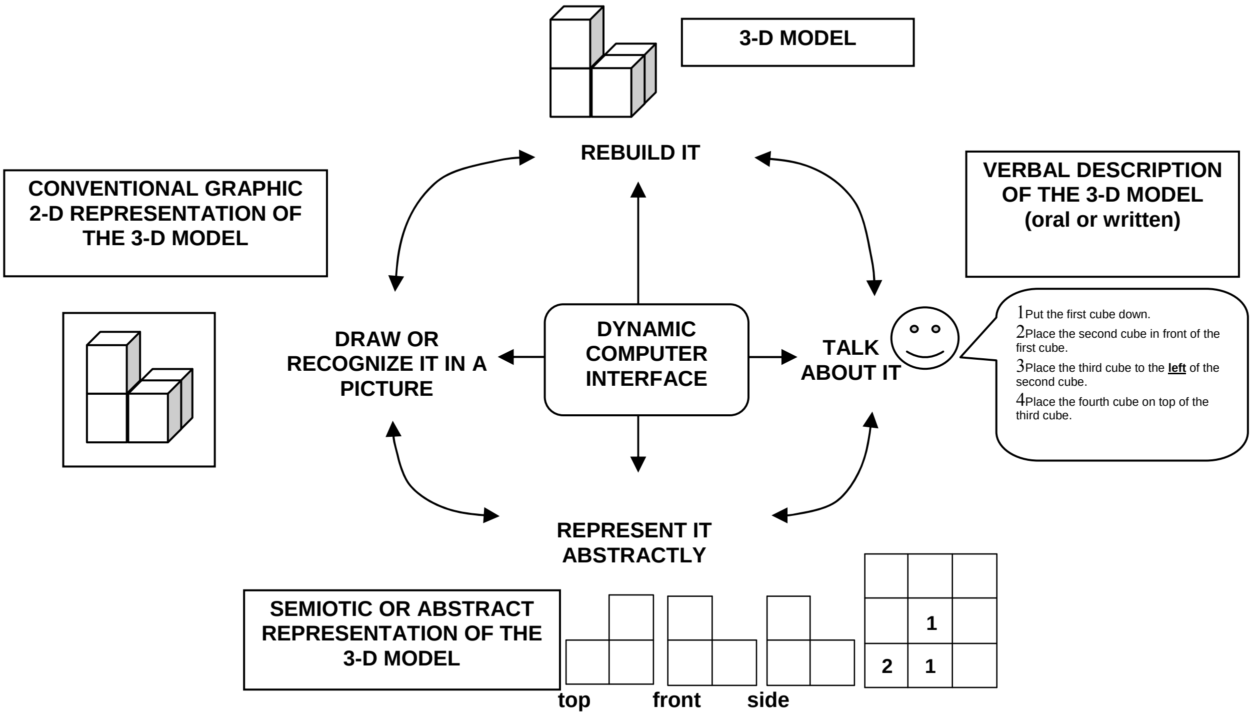


Figure 1. Multiple representations within 3-D visualization.

The instructional design based on the SOC model (see Figure 1) uses:

- *full-scale* models (or scaled-down models) of large objects which can be handled by the child;
- *conventional-graphic* models that are two-dimensional graphic representations which bear resemblance to the real three-dimensional objects; and
- *semiotic* models which are abstract, symbolic representations which usually do not bear any resemblance to the actual objects. Examples include view and floor-plan diagrams.

We now include the *dynamic computer interface* representation which is a tool that was not available at the time the above framework was developed. We believe that children should develop competence using all visual representation modes in addition to *verbal* descriptions regardless of the representation given in any particular problem using physical and mental processes. Our instructional activities have children moving among these different representations initially using wooden cubes and Soma pieces (Weisstein, 1999) progressing to 2-dimensional conventional graphic and semiotic models (Freudenthal, 1991), verbal representations integrated with a dynamic computer simulation, Geocadabra (Lecluse, 2005).

METHODOLOGY

We use the design research methodology of Cobb, Confrey, diSessa, Lehrer, and Schauble (2003) through iterations of experiment preparation, teaching phase and retrospective analysis. The theoretical intent is to support and account for young children's development of spatial reasoning using the SOC framework modified by a dynamic computer interface. The ongoing study is conducted in a dual-language urban elementary school within one of the largest public school districts in the mid-southwestern United States. A researcher (myself), teacher-researcher (Irma) and another teacher who has participated in RUSMP's *Geometry for Elementary Teachers* course, work with a third-grade group and then a fourth-grade group of children weekly (one hour per group) during the after-school program. At the beginning of the fall semester, English and Spanish parent/guardian and student consent-to-participate forms were sent home to parents of all after-school third and fourth graders. All respondents were accepted into the program.

With respect to the classroom ecology, Irma had taught mathematics and science to all fourth-grade participants during their entire third-grade year. Due to staffing changes for the current third-grade class, she teaches all core subjects to half of the school's third-grade class. Consequently, some of the third-grade participants in the after-school SOC program are not her students during the day. However, all participants became attuned to her behavioral and communal expectations very quickly during the first month of the research program. She expects all students to develop independence by asking each other for help or support before asking the teacher and to treat each other respectfully. She develops and supports a dominant social constructivist approach. Students are comfortable expressing their understandings knowing they are safe to express their confusion or frustration in front of their peers. They are expected to explain and provide justification for their mathematical conclusions. Irma rarely gives away answers or explanations. She serves as their facilitator while they construct meaning and representations for themselves. This environment supports our ability to learn how these students arrive at their understanding of 3-dimensional representations.

Irma and I conducted one-on-one pre-interviews using Soma-figure representations to establish students' baseline knowledge during the first two weeks with 14 fourth-grade and 11 third-grade students. We focused on students' ability to build various figures using loose 2-centimeter cubes from 3-D, conventional 2-D pictures and verbal (either oral or written) stimuli. Student responses were recorded on a check sheet and unusual observations were noted. Data from subsequent teaching experiments that reflected students’ learning processes and provided insight into students' thinking came from observation field notes, video recordings and student interviews. Some of these results will be discussed in this paper.

We based our hypothetical learning trajectories on activities developed by van Niekerk in her dissertation study (1997) and subsequent activities using the dynamic simulation software (Lecluse, 2005). During teaching experiments and retrospective analyses, we adapted forthcoming lessons if we believed alternative interventions were appropriate.

SAMPLE DATA AND ANALYSES

Pre-program data

During pre-program interviews, we noted some of the unconventional ways children responded to tasks with verbal stimuli (see Figure 2). Using the example item, we asked students to build a figure using unmarked, loose cubes while we read verbal directions to them and while we read with them. Then we asked each child if s/he preferred us to read to or with her/him for interview tasks I, II and III.

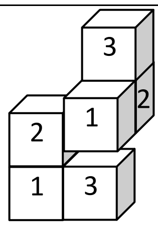
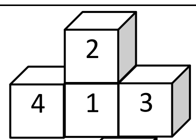
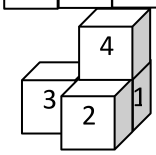
Example item 1. Put the first block down. 2. Place the second block **behind** the first block. 3. Place the third block **on top of** the second block.  	I 1. Put the first block down. 2. Place the second block on top of the first block. 3. Place the third block **to the right** of the first block.
II 1. Put the first block down. 2. Place the second block on top of the first block. 3. Place the third block **to the right** of the first block. 4. Place the fourth block **to the left** of the first block.   	III 1. Put the first block down. 2. Place the second block **in front** of the first block. 3. Place the third block **to the left** of the first block. 4. Place the fourth block on top of the first block.

Figure 2. Pre-interview verbal stimuli.

If a child was unable to build I or II successfully, then s/he was not given the opportunity to build subsequent figures in the section. Of the 11 third-grade children and 14 fourth-grade children we interviewed, one child from each grade level was unable to reach task III. Of those who attempted task III, 7 out of 10 third-graders and 3 out of 13 fourth-grade students consistently placed the second block behind the first. Of note, when building the figure for the example item which stated, “Place the second block **behind** the first block,” children who erred on task III were also consistent in placing the second block in front of the first.

Pre-interview retrospective analysis

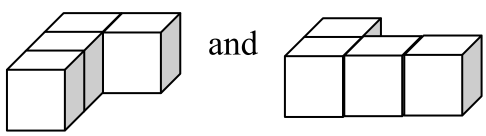
We do not speculate on why fewer children in fourth grade used descriptive language incorrectly than those in third grade. However, we believe we can correct the “in front"/"behind" confusion through discussion about conventional language use with ongoing practice opportunities. van Niekerk (1997) noted similar terminology issues in her earlier work. She refers to children's "use of different deictic terms for the same position in space." The child who sees himself in the referent position will refer to the face corresponding to his own front as the front (as in a translation of position). The child who uses common convention places himself facing the figure and claims its nearest face to be the front (as in a reflection across a line between the child and the figure.) By modeling a line of three children standing one in front of each other, first facing away from the whole group and then turning to face the group, the class understood the need for a convention when the child in the front then became the child at the back. By replacing the line with inanimate blocks, indicated only by different colored discs on their top faces, the class used the new convention to name the block in the front as the closest and most visible and the one in the back as partially hidden from view.

Lesson 1 data

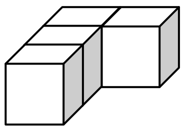
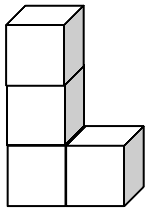
This lesson shows how children may transfer positional language previously learned in 2-D contexts to this new 3-D context. In our first lesson following the pre-interview, we adapted the "4 kubers" (4 cubes) problem (van Niekerk, 1996) as follows. The teacher tells the class a story of a new outer-space community in which the inhabitants will live on a distant planet in houses made of four prefabricated rooms. The rooms may be in any configuration but the must connect face-to-face completely. The houses will be delivered in finished form to their sites on a space vehicle and each home owner will tell the pilot where and in which direction to place his/her house. The first task is to build all the different 4-cube house combinations (without regard for windows or doors). The teacher asks students to think of some combinations to show to the class. After two or three share, the class moves back to work in pairs to discover other arrangements. During the activity, we asked individual children to describe particular structures to develop their verbal competence and to reinforce the new positional "in front" and "behind" language convention. We were surprised to discover additional language inconsistencies.

Field notes excerpt (Lesson 1, September 18)

> During the lesson introduction, some children were able to show why some houses were the same even though "they pointed in different directions" [they were rotations of each other in the horizontal plane.] However, during their investigation, most children, including those who seemed to understand that rotation in the horizontal plane resulted in "the same" house, believed these figures were different. Some used "horizontal" and "vertical" in their attempts to distinguish identical "L" figures, for example:



They also used "horizontal" and "vertical" to distinguish the 1-level from the 3-level "L."

 and 

When asked to explain the difference between the two "vertical" figures, [fictitiously named] Jason (third grade) said, "vertical down" and "vertical up." Fourth graders used geographic terminology, such as, "northwest" to distinguish among identical figures. We asked, "Which direction is north?" A number of hands pointed at the ceiling.

Lesson 1 retrospective analysis

We had hypothesized that this lesson would help the children become used to newly-developed conventional descriptive language. When the "vertical"/"horizontal" and geographical language inconsistencies arose, we realized that over time we may need to address many [yet-undiscovered] taken-for-granted expressions. At this early stage in the program, we decided to focus on reinforcing only the "in front/behind" concept and save the new expressions for later lessons when transformations and their associated verbal descriptions would be addressed.

Over the next seven lessons, activities focused on moving between conventional 2-D representations and 3-D figures using Soma figures singly and in combinations of two, and on becoming familiar with the dynamic software application. The next example, taken from lesson 9, demonstrates how fictitiously-named third grader, Eliot, used non-conventional verbal descriptions while working with the computer interface.

Lesson 9 field notes excerpt (November 13)

Students worked independently on laptop computers using the Geocadabra activities manual (van Niekerk, 2007) I had translated from Afrikaans. Evidence of positional language inconsistencies appeared when [fictitiously named] Eliot demonstrated his thinking about Task 2 [see Figure 3] using the digital projector during whole-class discussion.

Look at the figure.

When you place a mirror against the structure, you see the structure's reflection in the mirror.

The heavy line shows you where the mirror is placed.

How do you think the structure will look when you see it in the mirror?

Write the numbers for the figure and its mirror image in the grid to the right.

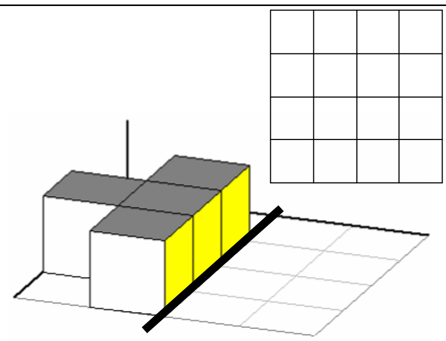


Figure 3. Abbreviated Task 2 from the Geocadabra manual.

Eliot performed the task correctly but some concerns about his positional language arose when he explained his moves to the class. Table 1 shows transcriptions and reproductions of screenshots from the video clip taken during Eliot's explanation.

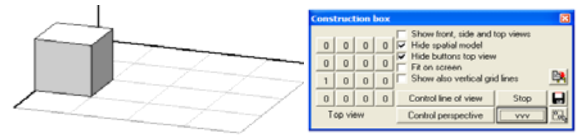
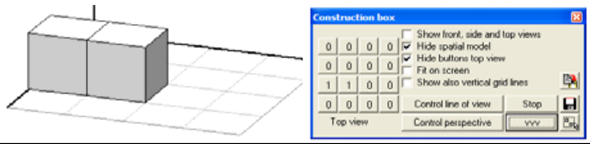
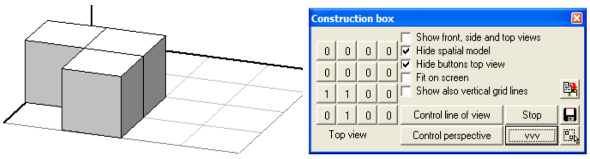
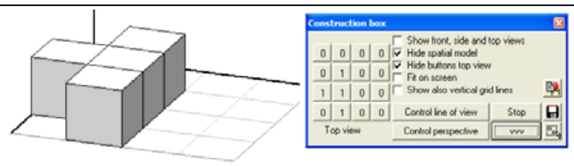
	Eliot: It had one right there and then in the back of that it had another one. . .
	Eliot: ... and on the side of that there was another one.
	Eliot: Under that one there was another one. . . on top of that one it had another one.
	Irma: When you're saying "on top," are you referring to the grid or your building? Do you mean on top right here [pointing at the construction box grid] or this block on top of the other [pointing at the fourth-placed block]? Eliot: I am meaning, on top of the grid. Irma: Oh, so you say the one that's on top [pointing to the uppermost row of the construction grid]. OK, but how is it here [pointing to the fourth-placed block]? Eliot: Side to side ... on the right or on the left. Irma: Okay, it's in front or behind? Eliot: Yes

Table 1. Transcription of Eliot's explanation to the class under Irma's facilitation

Retrospective analysis of Eliot's explanation

Eliot demonstrated mastery of the goal to move between the 2-D conventional stimulus in the book and the abstract representation in the form of the construction grid on the computer. However, his verbal descriptions were mathematically imprecise. He appeared to see the 2-D picture and its corresponding conventional computer image from the right side instead of from the front. He continued to confuse "in front" and "in back" using a translational orientation rather than a face-to-face reflectional orientation (lines 1-2). Lines 4-5 indicated his side-to-side positioning of the second and third blocks. In line 6, he referred to positions in the construction grid, using "under" for the third block and then "on top" for the fourth block. Irma clarified this with her question (lines 9-16). When prompted to describe the third and fourth blocks' positions in the conventional representation, he moved back to seeing the figure from the side instead of the conventional frontal position (line 23). Eliot affirmed Irma's correction (lines 25) but she did not pick up his confusion until our post-lesson reflection conversation.

There also appeared to be a conflict of orientation when correlating 2-D conventional objects (the picture in the book and the computer picture) in which the blocks stack vertically upon a 2-D grid lying on a horizontal plane. Although Eliot confused the figure's front and back, he used these terms together with right and left directionality. However, when he referred to positions on the construction grid he used "on top" and "under." His words seem to point to his view of the construction grid as a vertical plane even though he placed the blocks in the correct grid positions.

CONCLUSION

These examples underscore the importance of teachers paying careful attention to children's descriptive language in the domain of 3-D visualization. We believe that even if children appear to be mastering transitions among the other spatial representations, they should practice using the verbal representation frequently to develop conventional fluency. We speculate that student performance on standardized test items that use verbal visualization terms (for example, top, side and front views) may be compromised by unconventional language use rather than lack of visual cognition.

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MANIPULATIVE MATERIAL AND REPRESENTATIONAL MATERIAL

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The aim of this paper is to discuss the importance of manipulative material and representational material in the conceptual additive field development of learning. Two classes were used, one with 35 children, the other one with 34, in the age group of 8 to 13 years and attending the 3rd year of primary school. A pre-test, an intervention of teaching and post-tests were applied. The Theory of Conceptual Fields underpinned the development of two education strategies: one of them based on the use of manipulative material and the other one on representational material with the diagrams of Vergnaud. The results show that there are no significant differences in the performance of the two classes. The two strategies show advantages in the creation of meanings to the children.

INTRODUCTION

One of the justifications given for school failure in mathematics is the necessity and the difficulty of giving a practical sense to mathematical concepts. This study presents reflections of an inquiry that took into account the use of manipulative materials and representational materials (diagrams of Vergnaud), in the expansion of the conceptual additive field with children in the age group of 8 to 10 years.

In this research the Theory of Conceptual Fields was used as a theoretical reference. Vergnaud (1982, 1996) has a development objective of the conceptual additive field, which is classified in six categories. In addition to that, each problem that exist in a given category has its resolution consists of the relational calculus and numerical calculus, where the relational calculus is shaped by the use of the diagrams fixed by Vergnaud (1982, 1991, 1996) and the numerical calculus is shaped by the mathematical operations that the child carries out in the resolution.

Based on this theoretical reference, was built a sequence of teaching aiming for expansion of the conceptual additive field. In this form, two strategies of teaching are developed. One consists of the use of the diagrams of Vergnaud that is based on the relational calculation, in other words, on the use of the reasoning for the understanding of the existing problems. The other one consists of the use of the manipulative material, which helps in the operational calculations understanding wrapped in the resolution of the problems.

The present article discuss the following issues: - Is there a difference in the creation of meanings for children between the two applied strategies? - The use of

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manipulative material or representational material facilitates the learning of a child of the age group under consideration?

While talking about questions of this nature, the roads followed take us to the divergence of opinions regarding the importance of the use of manipulative material or representative material in the creation of meanings for children and in the problems of learning when we use abstract strategies.

Generally, the primary school teachers' expectations of the use of manipulative materials is in hope that the difficulties of teaching could be reduced by the support of the materials. Nevertheless, we should be wary of the indiscriminate use of this type of material. By relying on the thought that only material can solve problems of learning of mathematical concepts, we sometimes avoid analysing the truthfulness of this assertion. It is not a decision in favour or against the use of this type of material, but it is referent to an analysis of the performance of children if we use a strategy based on the abstract material and one based on the manipulative material.

THE CONCEPTUAL FIELD THEORY AND THE ADDITIVE STRUCTURES

The conceptual field theory of Vergnaud (1982, 1990, 1991, 1998) supplies elements for the analysis of the students' difficulties and it constitutes a powerful tool for the construction of situations (tasks). The reason is that it presents a coherent picture for the study of the development and of the learning of complex competences.

It is worth pointing out that for Vergnaud (1982), knowledge must be seen within conceptual fields, a power that develops throughout a long period through experience, maturing and learning. Learning is, par excellence, a responsibility of the school and it is a factor that contributes to the construction of knowledge of student through acts by teachers (their choices, projection and development of educational experiments).

A conceptual field can be defined as a set of problems or situations, in which analysis and treatment apply to several types of concepts, proceedings and symbolic representations; those exist in close relations to one another (Vergnaud, 1982, 1990). This theory affirms that the acquisition of knowledge happens through already known situations, and that knowledge has local characteristics. Consequently, all the concepts have a limited power of validity, which varies in accordance with the experience and the cognitive development of the subject. From this perspective, according to Vergnaud's (1996) theory, the construction of a concept covers a set of tripod that are called symbolically of (S, I and R); where S is a set of situations, in other words, tasks that make the concept significant, I is a set of invariants (objects, properties and relations) and R is a set of symbolic representations that can be used to punctuate and to represent the invariants.

In the sense of establishing relations between concept and situation, Vergnaud (ibid) finds support in the ideas of Piaget, relating the abbreviations (S, I, R) to the basic elements of symbolic function, where S refers to the reality or referent, and I and R refer to the properties and the representations. These representations are seen as the interaction between two aspects of the thought: I the meaning and R the significant.

The cases of addition and subtraction are examples of concepts where it does not make sense to study them separately, but it does make sense to study them it within a Conceptual Field, the field of Additive Structures.

Due to the great diversity of concepts involved in this Conceptual Field, they are part of student knowledge acquired in a medium or long term. The situations found in the Additive Structures can be analysed like simple problems of relations between the whole and its parts (problems of **composition**), like problems where we connect an initial state with an end through a transformation (problems of **transformation**), or like comparative problems, where we have a referee, a referent and a relation between them (problems of **comparison**). We can still have the problems that are called by Magina et al. (2001) "mixed problems". They are the ones which two of the previous categories are present. Vergnaud (1982) classifies them in three other categories: **composition of two transformations**: in this category two transformations are given and one looks for a third one, which will be determined through a composition; **transformation tying two static relations**: in this category two static relations exist, and we look for a third one which is produced from the transformation of two static given relations; **composition of two static relations**: in this category there is a composition of the static given relations.

For the resolution of the situations classified in six categories Vergnaud (1991) presents diagrams that are used in the relational calculation (see table 1 for an illustration of the schemes used by Vergnaud for the construction of his diagrams). The construction of this table was based on the explanations provided in Vergnaud (1991, p. 165).

Schemes	Symbols	Meaning
Rectangle	□	A natural number.
Circle	○	A relative number.
Vertical or horizontal key	} ⏞	Composition of elements of same nature.
Vertical or horizontal arrow	↑ →	A transformation or a relation; the composition of elements of different nature.

Table 1. Schemes used by Vergnaud in his diagrams

In Table 1 we present the diagrams of Vergnaud that demonstrate the reasoning covered in the three principals[2] categories presented above. The other categories

[2] This denomination of three principals is being given for a better understanding that the other relations are produced from this three.

(three) have diagrams formed from the grouping of the diagrams presented in the Figure 1.

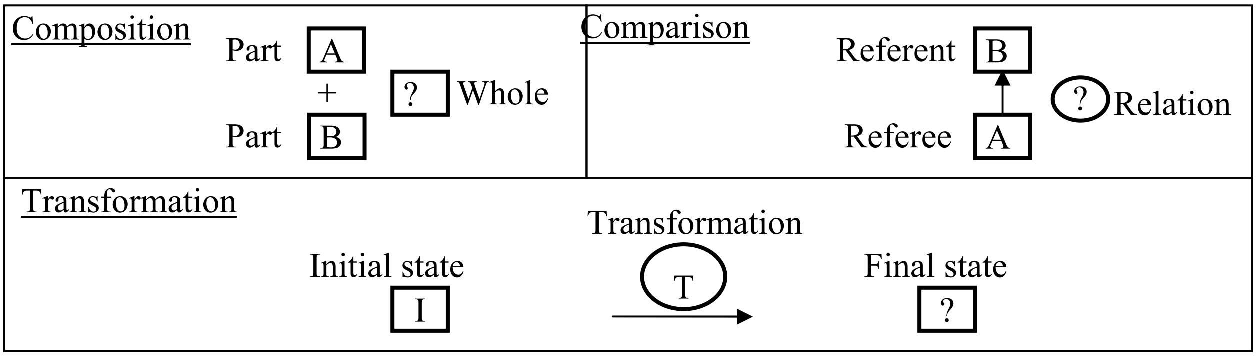

Figure 1. Diagrams of Vergnaud of the three principals relations of base.

MANIPULATIVE MATERIAL

I understand for manipulative material, a prepared or selected material, from materials in the context of the three-dimensional space and that, like the principal aim, orientates the learning of determined mathematical concepts by students.

Schliemann, Carraher and Carraher (1989) developed a research in Brazil which became a landmark for discussions and studies, and which consider many other questions that go beyond the classroom, like culture and the situations of the child on a daily basis. For these researchers, "[...] we do not need objects in the classroom, but situations in which the resolution of a problem implicates the use of the mathematic-logical principals to be taught " (p. 179). They still assert that if manipulative material has no relation with the daily life of a child, it might in fact become a representation abstract of the material of mathematical principles.

For these researchers manipulative materials, such as small rods, sticks of ice lolly, lids of bottles, boxes of matches, seeds and many others considered "concrete" objects can start to be "abstract" when treated like mere substitutes of children's fingers or formal calculations usually used. We need to be prepared to do a reflection regarding the use of this concrete material in classroom.

In the present research I use three manipulative materials: golden material, the picture of value and place and the abacus of small glasses. The abacus of small glasses is an adaptation of the abacus of sticks.

There are basic differences in the advantages of using each one of the chosen materials. While the golden material allows students to visualize groupings formed by units included in the whole. The other two materials, picture of value and place and abacus of small glasses, allow the visualization of the groupings and exchanges in terms of separate units.

The basic difference between the picture of value and place and the abacus of small glasses is that in the first one the student has the possibility of seeing the placing of the pieces that will be solved and consequently the formation of the whole, whereas in the second the student would be very unlikely to be able to handle the units to solve the operations asked for without forming the pieces that they worked with.

The abacus of small glasses presents an advantage over the golden material, which is the possibility of handling the individual form of the units that are being worked with.

METODOLOGICS PROCEEDINGS

This research was classified as semi-experimental, "[...] in which the independent variable is manipulated by the researcher, operating with groups of subjects chosen without his or her control" (Fiorentini & Lorenzato, 2006, p. 105). In this case, the strategy of teaching is the independent variable and the performance of the children is the dependent variable. The goal of the study is of an interventionist nature, involving the application of two diagnostic instruments (pre-test and post-test). This allowed the collection of qualitative and quantitative data that made possible an analysis of the process of intervention.

This research involved 69 children from two classes of the 3rd year of primary public school. In the class GE1, with 35 children, the strategy of teaching based on the diagrams of Vergnaud was used. In the class GE2, with 34 children, the strategy of teaching based on the manipulative material was used.

Six meetings were carried out. Two involving the application of diagnostic instruments (pre and post-tests) and four for the process of intervention. Each lasted 2 hours.

The teaching intervention was developed collectively in the classes (GE1 and GE2), and the problems were handed out printed on A4 paper. A standard procedure of application was followed: in the 1st meeting the problems were distributed to teams of four to five children, one problem at a time; in the 2nd and 3rd meetings, the problems were given to individual children to solve, and handed out one at a time; and in the 4th meeting the problems were handed out on a single sheet of paper at a time and were solved by pairs of children.

During the intervention meetings, some time was given in order that the children could solve the proposed problems and then the class would discuss the solution followed by and explanation from the researcher. At the end of each meeting an activity was given as homework, for discussion by the group at the beginning of the following meeting. In total 34 problems were worked out.

The diagnostic instrument was of pencil and paper type and consisted of 20 problems from the Additive Field answered once in each separate classroom. The same instrument was applied in the pre-test and in the post-test.

The answers in the pre and post-tests were marked as correct or wrong, so that the performance of the children varied from zero to twenty.

ANALYSIS OF RESULTS

Of the 35 children of the GE1 class, 77,1% were girls and the average age of the class was 8,9 years, with the ages varying between 8 and 11 years. While of the 34 children of the GE2 class, 55,9 % were boys, on average 9,6 years of age, with ages varying from 8 to 13 years.

Table 1 shows the average performance of the classes, on the pre and post-tests, analysing the results in the categories given by Vergnaud (1982).

	GE1 Pre (%)	**GE1 Post (%)**		**GE2 Pre (%)**	**GE2 Post (%)**
Composition	76,4	80,0		72,1	78,0
Transformation	59,3	68,6		51,8	59,6
Comparison	51,8	62,9		35,7	46,2
Mixed	2,9	25,7		11,8	20,6

Table 1. Pre and post-tests Children's performance on the following categories

In the problems of composition two basic structures are listed - the parts and the whole - and it is expected that the children will demonstrate good performance in this category since the problems are presented with little complexity in structure. Further, they are problems faced daily by the children. In the problems of composition, the children from the two classes started with more than 70 % correct. GE1 reached 80 % correct and GE2 78 %. The class that worked with the diagrams of Vergnaud, showed a growth of 3,6 %, and that class that used the manipulative material had a growth of 5,9 %. Nevertheless this difference is not statistically significant.

The structure of the problems of transformation is composed of an initial state, of the final state and of a transformation between them. In general the children, of the age group researched, showed a good performance on the problems that looked for a final state or a transformation, but inferior performance on those problems looking for an initial state. That complexity seems to justify the low rate of performance in this category. Nevertheless, after the intervention, the children who used the representative materials (GE1) presented a growth of 9,3 %, being a little higher than the GE2 with growth of 7,8 %.

The problems of comparison involve a referent, a referee and a relation between them. Among the three principal categories defined by Vergnaud it is the most complex. Seven problems of comparison were put in the instrument. We see that the children of GE1 grow 11,1% from the pre-test to the post-test and GE2 grows 10,5%.

Only one problem classified as mixed by Magina et al. (2001) was placed in the instrument. Based on Vergnaud (1982) it is a problem of the composition of two transformations category. The mixed problems are the most complex for the children of the age group included in the study. That can be observed in the low performance of the children of the two classes. The children of GE1 (representative material) began with an average success rate inferior to that of the children of GE2 (manipulative material), nevertheless they grow 22,8 %, whereas the GE2 children started with a higher average but only grew 8,8 %.

It is possible to observe in the results above that there is a similar growth between two groups (GE1 and GE2). In spite of the group GE1 (representative material) starting from a higher level (57,3 %) than the GE2 (manipulative material) (48,2 %), they present similar performance in post tests, with GE1 having a slight increase in the percentage correct (66,7 %) versus the GE2 (56,6 %). This result is shown in Figure 2. Which shows the percentage of correct answers on the pre-test and in the post-test by class.

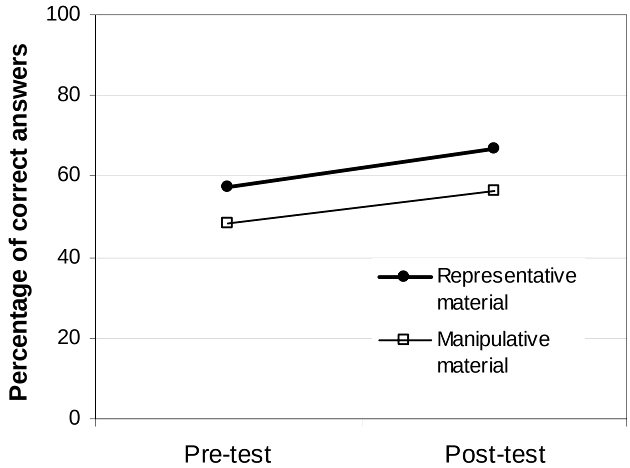

Figure 2. Percentage of correct answers in the pre test and post-test by group.

These results reveal another flaw in the "myth" held by many primary school teachers regarding the use of manipulative material. A strategy based on abstract schemes was applied, and other one based on the use of manipulative material, yet the results point to a similarity in the performance of the children.

We can infer that there are no differences in the creation of meanings for the children between the two applied strategies, since they present similar growth in the performance on the pre and post tests. Further the class that worked with the diagrams of Vergnaud (representative material) presented a result a little better than that of those who worked with the manipulative material.

I do not want exaggerate the importance of the use of manipulative material, but rather to initiate a discussion concerning the creation of meaning that can be woken in a child, when this material is used or when using abstract strategies.

It is possible to awaken in the child the creation of meanings from the use of these two strategies of teaching. As a researcher in the classroom, the experience was that: the use of manipulative material makes learning easier in terms of understanding the algorithms of addition and subtraction, in the groupings and in the exchanges that were done inside the operations. Children of this age group commonly have difficulty of this type. With representative material it was possible to follow the growth in their understanding of problems. The children shifted gradually from asking the question "Teacher, is this problem addition or subtraction?", to "Teacher, is this diagram going to be of change or of transformation?".

Moreover we believe that the two strategies make possible the creation of meanings for the children of the age group involved in the research.

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WHEN STUDENTS DISAGREE: ENGAGEMENT AND UNDERSTANDING IN AN URBAN MIDDLE SCHOOL MATH CLASS

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We provide a description of the mathematical activity of students in a classroom, highlighting the different types of engagement that are enlisted, the mathematical understanding of two students (using the Pirie/Kieren model) and their role within their respective groups. Our goal is to better understand some of the many factors that can influence the way in which students react to criticism by their peers.

INTRODUCTION AND FRAMEWORK

In their research involving urban middle school students, Schorr, Epstein, Warner, and Arias (in press) and Epstein et al. (2007) note that students may be willing to abandon what they know are mathematical truths in order to avoid appearing weak or wrong in front of their peers. In particular, they discuss the case of Dana—a young student who defended a solution that she knew might be incorrect as another student, Shay, pointed out an error in her work in the presence of her group members. Rather than admit that she might have made a mistake, she vehemently defended her incorrect solution. In a follow-up interview (shortly after), Dana conceded that she becomes uncomfortable "…when people try to prove *me* wrong." Immediately prior to Shay's comment, Dana was interested in understanding the concept. After Shay's public criticism, her position changed to one that focused on avoiding looking foolish, or publicly losing face.

Dana's response is consistent with the findings of Dance (2002), Anderson (2000), and Devine, (1996), that students are often hypersensitive to situations in which their emotional safety, status, or wellbeing may be challenged. The main goal of this paper is to provide a "prequel" to the episode above between Dana and Shay (which is one part of a larger year long study involving several math classrooms), noting in particular the interplay of engagement and mathematical understanding, using the Pirie-Kieren (1994) model for the growth of mathematical understanding.

The Pirie-Kieren model (1994) provides a framework for analyzing the growth of understanding, via a number of layers through which students move both forward and backward. Pirie (1988) discussed the idea of using categories in characterizing the growth of understanding, observing understanding as a whole dynamic process and not as a single or multi-valued acquisition, nor as a linear combination of knowledge categories. Pirie & Kieren (1994) illustrate eight potential layers or distinct modes within the growth of understanding for a specific person, on any specific topic.

The middle school years can be a particularly stressful time for students. There is no doubt that most middle school students have a variety of issues and concerns that compete for their attention, and high among those are social issues involving things like peer group acceptance. It is therefore not surprising that actions instrumental to attaining acceptance, or avoiding confrontation, are likely to be allocated a large share of any student's attention. Indeed, in interviews during our study students often mentioned these issues.

Eccles and Midgley (1989) note that middle school students *need* an environment that provides a 'zone of comfort'. In our own work, we also note the importance of providing students with what we term "an emotionally safe environment". In such an environment, the students are free to question ideas, and openly discuss (mis)understandings without risk or fear of embarrassment or humiliation. Oftentimes, students in such an environment work in small groups and engage in mathematical discourse that includes efforts to prove and justify contentions to peers and the teacher. Implicit in this discourse is a socio-mathematical norm that permits and even encourages students to challenge the ideas put forth by fellow students (Cobb, Wood, & Yackel, 1993; Franke, Kazemi, & Battey, 2007). Of course, the teacher is of key importance in shaping the emotional safety of the classroom and the nature of the discourse that takes place. "How teachers and students talk with one another in the social context of the classroom is critical to what students learn about mathematics and about themselves as doers of mathematics" (Franke et al., 2007, p. 230). Nonetheless, given an emotionally safe environment, *and* a teacher who actively seeks to instill classroom norms that encourage productive and meaningful mathematical exploration and discourse, different students will engage with mathematical problems in different ways, ultimately impacting their overall understanding and interactions with each other. Further, despite the best intentions of the teacher, students may feel threatened or uncomfortable when their work is criticized by their peers. Our goal in this paper is to better understand the differing modes of engagement, how they impact learning, and how they unfold when students criticize each other's work.

To this end, we have identified several different types of *student engagement structures* (see Goldin, Epstein, & Schorr, 2007). Certain structures contribute directly to mathematical engagement, while others, at times, impede it. We see most or all of these structures as present within individuals and becoming operative under given sets of circumstances. We have identified at least seven engagement structures ranging from extreme engagement, akin to what Csikszentmihalyi (1990) describes as "flow" (complete immersion in a task or activity), to complete disengagement, where students do what Kohl (1994) describes as "not learning". For the purposes of this report, we focus on three types of engagement structures: A. *Check This Out*: In this structure, the student is highly engaged in the task, often to the exclusion of other events that may be occurring within the context of the group or the classroom; B. *Get The Job Done*: This structure involves a person's sense of obligation to fulfill his part of a work "contract." The student is much more aware of anything that helps or hinders progress toward that goal; and, C. *Don't Disrespect Me*: This structure involves the person's experience of a perceived challenge or threat to his or her wellbeing, status, dignity, or safety. Resistance to the challenge raises the conflict to a level above that of the original mathematical task. The need to maintain "face" supersedes the mathematical issues.

Different structures vary in the degree of engagement that they recruit. The greatest engagement of the structures described above occurs in the "Check This Out" structure. Less intense engagement is seen in the "Get The Job Done" structure. As part of our analysis, we also consider what psychologists may refer to as "figure" and "ground" (popularized by the Danish psychologist Rubin, 2001). Figure, as we use it here, refers to the primary focus of attention, whereas ground refers to that which is present in the background. In our work, we have found that, at times, the mathematics may be figure and other aspects of the context may be ground, and vice versa.

METHODS

Subjects: The 8th grade classroom that is the focus of this research consisted of 20 students, 93% African American and 7% Hispanic. The school, classified as "low income," is in the largest city in the state of New Jersey. This class was homogeneous and designated as a low ability class (lowest in the grade level). The teacher encouraged what we describe above as an "emotionally safe" learning environment for students, a necessary condition for inclusion in the larger study.

Procedure: In the larger study, classes were observed in each of four "cycles," with each cycle spanning a period of two consecutive days. The first cycle occurred approximately one month into the school year and subsequent cycles occurred later on. Prior to the start of a cycle, an interview was conducted with the teacher to ascertain her plans for the lesson and what she expected to happen. A follow-up interview (using a stimulated recall protocol) with the teacher and several students took place after each cycle (for more details see Epstein et al., 2007). Classroom interactions for each of the classes were videotaped using three separate cameras and all student and teacher interviews were videotaped using one camera. Transcripts were created from videotapes and student work was collected.

Analysis: A team of researchers from the fields of mathematics education, social psychology, mathematics, and cognitive science reviewed and analyzed the results. All videos were viewed through four distinct, yet overlapping lenses: the mathematical (cognitive) lens; the affective lens, particularly with regards to engagement; teacher interventions (including actions, behaviors, etc.); and social interactions. In all cases, the structures that we have identified were created after observing the data, rather than ahead of time.

For this paper, we analyze data from one class during cycle one over the span of two days (about 43 minutes each day). The students were working in groups of three to five (groups formed according to the typical seating arrangements-no roles or assigned tasks were given to any students) on the following task:

Farmer Joe has a cow named Bessie. He bought 100 feet of fencing. He needs you to help him create a rectangular fenced in space with the maximum area for Bessie to graze. Bullet 1: Draw a diagram with the length and the width to show the maximum area. Bullet 2: Explain how you found the maximum area. Bullet 3: How many poles would you have for this area if you need 1 pole every 5 feet?

RESULTS AND DISCUSSION

We share results by comparing and contrasting Dana and Shay with respect to engagement structures and the role they assumed within their respective groups. We then offer a more complete description involving their respective mathematical behaviors.

Student Name	Engagement Structure	Role Within Group
Shay (Day 1: beginning to almost end of class)	Check This Out	Worked alone for most of session; occasionally shared ideas with group mates & teacher; at times, asked group mates to help supply him with paper & a calculator.
Shay (Day 1: last 5 minutes & 1st half of Day 2)	Check This Out	Took on role of leader, assigned roles to group mates; asked group mates to supply him with tools (yardstick, scrap paper); recorded group's solution on chart paper to share with the class.
Shay (middle of Day 2)	Check This Out	Shared ideas with teacher and group mates; critiqued Dana's group's work.
Dana (1st half of Day 1)	Check This Out	Asked questions about mathematical ideas; made requests for tools (calculator & scrap paper).
Dana (last half of Day 1 & 1st half of Day 2)	Get The Job Done	Monitored every group member's progress to make sure they were all on task; made requests for tools (i.e. chart paper); recorded group's solution on chart paper to share with the class.
Dana (middle of Day 2)	Check This Out	Shared mathematical ideas; asked questions about Shay's group's solution.

Table 1. Engagement structures and roles within the group

Shay: Shay, a young male student who is described by his teacher as being a bright and “street wise” student, began investigating the task alone while the other three members of his group worked together, spending a considerable amount of time talking about other non-mathematical issues (socially related). Thirteen minutes later, Shay announced to his group, in an apparent breakthrough, “You could do a lot of ‘em, it could be like…wait…” Realizing that there could be many rectangles with a perimeter of 100 feet, he encouraged his group members to investigate some possibilities while he continued to quietly work, primarily alone, for most of the class. For the most part, he interacted with the others only when the teacher asked questions and/or when the teacher encouraged them to share ideas with each other.

During this time, Shay expressed some difficulty in figuring out the length and width of potential rectangles. We suggest that he was functioning in the *image making layer* of the Pirie & Kieren model (2004) in that he still needed to draw specific rectangles in order to create an image of what each potential solution could be, but was still tied to the action of drawing in order to figure out the length of the sides in any one specific case. As he continued to work, he began to develop strategies for building rectangles with a perimeter of 100. He developed a method for finding the dimensions of rectangles, which involved finding the width for a specific length. He added the side length to itself (for the length of two sides), subtracted the sum from 100, and divided the difference by two (to find the width). At this point, it appeared as though he had reached a *don't need* boundary and had an image of how he could construct rectangles with a set perimeter of 100. He was, at this time, working in the *image having layer* because he was no longer tied to the action of drawing the rectangle.

At the very end of the first session, Shay realized that the other group members were still not sure about how to find the area of these rectangles. He also realized that it would take a long time to construct all *possible* rectangles with a perimeter of 100 feet (using only whole numbers). He stated, "you can keep going but it can go all day". He now, for what appeared to be very practical reasons, assumed the role of leader in his group, noting, "Move, I know what to do, look…what number, look…we gonna start, I'm gonna do the lower numbers, you do the higher numbers…" The other three members of his group began working independently on the different rectangles that he assigned them.

As he and the members of his group continued to work on the problem, Shay realized that he could find every (whole number) rectangle by increasing the length by one and decreasing the width by one. He now had an image of the construction of these rectangles and was no longer tied to the action of drawing each one to figure out the length and width. He continued with this strategy until he came to the conclusion that the rectangle with the maximum area had dimensions of 26 by 24 (because he didn't consider the 25 by 25 square to be a rectangle). At this point, Shay also became invested in seeing to it that his group mates understood his method.

Dana: Dana, a young female student, described by her teacher as being popular and eager to please, but also "tough", worked on the task with 3-4 other students. In the beginning of the session, Dana attempted to understand the task by asking questions and trying to figure out the meaning of area. Despite some misconceptions (especially relating to area) Dana spontaneously took on the role of group leader, often telling her group mates what to do and when to do it. Her overall approach was to find the area of a rectangle that appeared to meet the conditions of the problem. After finding one such rectangle, Dana directed the group to consider another part of the problem task (related to the number of poles).

After a short time, Dana told the group, “Yes. We need eight poles, so for the second bullet umm…two poles…two poles…” “So that’s…stop (to another group member) so that’s something that’s asking for bullet one [bullet refers to the different parts of the problem task], and bullet two, and bullet three. So, is everyone caught up yet (addressing the other four members of her group)? Ya got bullet three (as she monitored their work)?” Dana was concerned when someone in her group was off task-perhaps because it might get in the way of completing all of the parts of the problem.

We suggest that Dana did not have an image of how to create different rectangles with a constant perimeter when she finished the task. She used, as her final solution, the first rectangle (40 by 10) she constructed that had a perimeter of 100. She didn’t explore any other possibilities. She noted that in order to find the maximum area, you needed to “multiply length times width” without actually doing so. She never progressed past the *image making layer*, and unlike Shay, didn’t reach a *don’t need boundary* (where she was no longer tied to the action of drawing the rectangle).

Engagement structures: Based upon supporting evidence from the data, we conclude that Shay spent most (if not all) of his time in the “Check This Out” structure, where the mathematics was figure, and the other aspects of the classroom context (social, for example) were ground. It was only after he was sure of his method that he monitored the progress of his group, and even then, the mathematics was central.

Dana, on the other hand, monitored the progress of her group throughout the entire session, trying to enlist their attention when it seemed to fade. While she was open to the ideas of her peers, she remained the sole arbiter of what ideas would be pursued and what ideas would not. We suggest that for Dana, the primary engagement structure was “Get The Job Done” since she appeared to be mainly interested in being sure that she had answered all parts of the question rather than formulating a more complete solution for each. In Dana’s case, the mathematics was ground, and monitoring her group’s activity was figure.

Critiquing each other’s work: In both cases, Shay and Dana recorded the solutions for their respective groups, and therefore had considerable ownership of the work. After all of the groups had recorded their solutions, the teacher asked the students to walk around and review the work of the other groups, noting on small pieces of paper any questions that they had about another group’s work.

During this time, Dana noticed a major difference in the way that she had solved the problem and the way that Shay’s group had solved it, and she raised questions about Shay’s solution to members of her group, trying intently to understand it:

Dana	But I don't know how they got this, how they got maximum area… (inaudible)… I don't get that But I want to know how do they get the answer. I want to know how they get this…they got this. I don't understand how they got this.
Von	You add it up. You do it and see what you get.

Dana	I'm talking about this, and why did they do this (referring to the dimensions of the largest rectangle)?
Dana	Look, they said length times the width here; they multiplied the length and the width.
Von	Could we just agree on something?
Will	No, just write, "We don't understand how they got the maximum area." Write that.
Von	Write that? "We don't understand how y'all got your maximum area" (referring to Shay's group).
Dana	Ok, yes I do. I got it. *They are right*. 'Cause they said they got 26, they multiplied 26, so they multiplied this one for, um…Could say good job because they did. First, I didn't understand it.

Analysis of the data (interview, classroom, field notes) leads us to believe that at this point in time Dana moved into the "Check This Out" structure, as she tried intently to understand Shay's group's solution, and no longer needed to monitor her group. It now appeared that for Dana, the mathematics became figure. Shay, on the other hand, remained in the "Check This Out" structure as he openly explored each group's solution. However, as reported in Schorr et al. (in press) and Epstein et al. (2007), Dana went into the "Don't Disrespect Me" structure when she heard Shay criticize her work. She was adamant in her response to Shay, and acted defensively regarding the accuracy of her solution.

CONCLUSIONS

Our analysis is intended to highlight several points. First, we note that within the context of a single classroom, different types of engagement structures can simultaneously be enlisted. Further, an individual may move into and out of several different types of structures depending on the circumstance.

In Dana's case, one might speculate that she had much to lose by admitting that she had, potentially, led her group down an incorrect (or at least incomplete) path. Further, because she had a somewhat limited understanding of the math, she may have felt that she could not explain her ideas sufficiently well to others, if challenged. Finally, she knew, to some extent that Shay's ideas *were* correct, and hers might have some flaws. Taken together, these factors contributed to Dana's reaction to Shay.

By considering the confluence of many factors (at the very least, mathematical understanding, role within group, and perceived ability to defend one's work), we can better understand how students may respond to their peer's criticism, even in the context of an emotionally safe environment.

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CLASSROOM MATHEMATICAL NORMS IN AUSTRALIAN LESSONS: COMPARISON WITH JAPANESE LESSONS

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This study is a part of the international research project, the Learner's Perspective Study (LPS). LPS selected three experienced 8th-grade mathematics teachers from each country. They collected the videotapes of ten consecutive lessons of each teacher, interview transcripts of the teachers and students, and students' written work. The current study focused on classroom mathematical norms, and analysed the Australian LPS data qualitatively, in comparison with the Japanese LPS data. The results showed that the Australian lessons emphasized different classroom mathematical norms from the Japanese ones. Cross-national comparison revealed close relations among classroom mathematical norms, instructional structures, and curricula.

INTRODUCTION

TIMSS 1999 Video Study (Hiebert et al., 2003) has not only given important insights into international differences of students' mathematical performance, but also stimulated new research on international differences of mathematics lessons. The *Learner's Perspective Study* (*LPS*), an international research project coordinated by David Clarke, has attempted to complement that Video Study by selecting a limited number of experienced teachers in each country, and conducting in-depth analysis of consecutive lessons of each teacher (Clarke, 2004). This paper focuses on a normative aspect of classroom mathematical activities, *classroom mathematical norms*, which was not dealt with in the analysis of Hiebert et al. (2003). Though classroom mathematical norms are often not explicitly taught by teachers nor written in textbooks, they are crucial when the learning process of mathematics is conceived as mathematical activities.

Classroom mathematical norms are knowledge "about" doing mathematics; therefore, they belong to the domain of metaknowledge in mathematics. It is hypothesized that beginning teachers are often occupied with covering curriculum content, paying attention to mathematical knowledge and skills: Competent teachers, as selected by the LPS design, would invest more time and effort in teaching metaknowledge. The major questions that guided this analysis are, what classroom mathematical norms would surface in the lessons? How would the teacher introduce, negotiate or utilize those norms during the lessons? This paper investigates those questions in the Australian data, and compares the results with that of the Japanese data.

THEORETICAL FRAMEWORK

Social sciences study patterns, norms, regularities, rules, or laws appearing in human activities (cultures), so that they can explain and understand human activities.

Ethnomethodologists had studied people's "rule" use in social situations, and claimed that people, rules, and situations were mutually shaped in practice, in their terminology, "reflexively" related to each other (Mehan & Wood, 1975, pp. 75-76). Even if a norm is taken as shared among people, it cannot prescribe their actions. Norms are cultural knowledge that may help people to accomplish something in some specific situations:

> Members of society do not simply follow internalized norms or rules in the manner of 'cultural dopes' … but rather practically analyse situations in terms of the relevance of such norms and rules. (Francis & Hester, 2004, p. 206)

Cobb and his colleagues have been developing the most sophisticated arguments on studying norms in the mathematics classroom (e.g., Yackel & Cobb, 1996). They began their research with a constructivist framework that conceived learning mathematics as active construction process by individuals. Adopting the frameworks of symbolic interactionism and ethnomethodology, they have come to incorporate into it a sociological framework. They investigated how students developed beliefs and values on mathematics. Their focus of analysis was on classroom processes of the "inquiry mathematics" tradition, where children actively participated in exploring, explaining, justifying, and arguing mathematics. For the analysis, they introduced the notion of "norms" of classroom process as a device to interpret classroom processes and clarify how children's beliefs and values developed. They identified several classroom social norms working in their project classroom.

Cobb and his colleagues then proposed norms specific to mathematical learning, "sociomathematical" norms, distinguished from the above social norms (Yackel & Cobb, 1996, p. 461). They contended that mathematical activity has norms as constituent, and that norms are reflexively related to beliefs and values of mathematical activities. Sociomathematical norms are interactively constructed in each mathematics classroom, and may be different from one classroom to another (p. 474). Since those norms are specific to classroom mathematical activities, *classroom mathematical norms* seem to be more appropriate and understandable term than "sociomathematical" norms; therefore, in this paper I use the former (for a detailed discussion, see Sekiguchi, 2006).

RESEARCH PROCESSES

Unlike TIMSS 1999 Video Study, the LPS project did not select eighth-grade teachers randomly. The project selected only three teachers for each country, who were considered "competent" by local educators. In addition, it videotaped ten consecutive lessons for each teacher using three cameras (teacher camera, student camera, and whole class camera), and interviewed students by the stimulated-recall method using videotapes of the lessons.

An analysis of classroom mathematical norms was reported in Sekiguchi (2005, 2006) for the Japanese LPS data. This paper reports an analysis on the Australian LPS data, Sites A1, A2, and A3. The videotapes of the lessons, their transcripts, the interview data, and students' written work were analyzed with the grounded-theory approach (Glaser & Strauss, 1967). To let classroom mathematical norms emerge from the data, any piece of the data that appeared to indicate beliefs on how to work on mathematics was coded, and the normative aspects behind those beliefs were repeatedly analyzed. At the same time, a cross-national comparison was performed on the Australian and Japanese LPS data. Its purpose was twofold: to understand Australian practices from the international perspective, and to maximize the diversity in data to stimulate theory generation for research of classroom mathematical norms (Glaser & Strauss, p. 58).

In the following, I provide a brief description of the lesson sequence, and report classroom mathematical norms for each Site. After that I compare them with those of the Japanese data. I refer to lesson data by indicating site name, lesson number, and the time from the start of the lesson. For example, [A1L05, 39:50] means that the data came from the 5th lesson of Site A1, and occurred at 39 min. and 50 sec. from its start.

CLASSROOM MATHEMATICAL NORMS IN SITE A1 [L04-L13]

At Site A1 the teacher first introduced the concept of pi (π) through an activity of measuring diameters and circumferences of the given circles, and calculating the ratios of the circumferences to their diameters [L04]. The teacher introduced the formulas "C = πD" and "C = 2πr" [L05]. The class then worked on several problems to find perimeters of circles, semicircles and quarters of circles [L06-L10]. The teacher moved on to the concept of area of geometric figures, discussed how to find the areas of hands, rectangles, squares, and triangles [L11-L13].

Norm A1-1: You should follow the steps

The teacher several times reminded the students of writing their answers following a fixed format, "the four steps."

> … when you set out your work could you please go through those four steps that I've talked to you about before with your setting out. You write down what it is you know, you write down what rule it is that you're going to use, and you do have a choice of two in this case…you substitute and you evaluate…[calls a student's name]. And always estimate the answer once you've finished. Double check what you've done, … [A1L05, 39:50]

The teacher and students often used the format in class. For instance, at L09 student S wrote his answer to a problem-Find the length of circumference of a circle with radius 5 m-on his notebook as Figure 1.

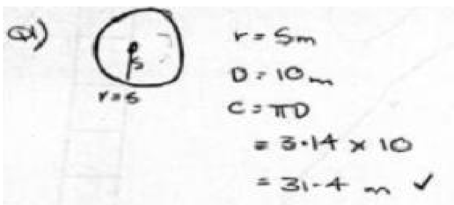

Figure 1. S's writing.

Norm A1-2: You should find a way to figure out measurements of geometric figures, other than actually measuring them

In L04 the teacher held an activity of measuring diameters and circumferences using strings, but in L07 she discouraged students using them. In L07 the teacher (T) asked students to find the length of perimeter of a quarter circle with radius 6 mm. Student M was sitting next to another student A. A suggested to M, "It's easy to work out if you just use a string"[A1L07, 24:20]. Then, M called the teacher:

M: Miss, I know an easier way to get that.

T: Oh, the first one or the second one?

M: Both of them.

T: Really?

M: A string.

Referring to the "string" activity, M proposed here to measure the perimeter of the quarter circle on the board by using strings.

T String? With string? There's always the string, absolutely. But I haven't drawn it to scale so what would you do? If you were using string what would you do? You would have to do?

M: Use a ruler…

T: And?

M: To measure the length.

T: Would you measure the one on the board?

A: You'd just guess that, 'cause it's not accurate, it's not right.

Agreeing to A, the teacher emphasized that because the drawing on the board was not the right size, measuring it by string would not be accepted.

In L09 the teacher introduced the idea of measuring the area of a figure by counting the number of squares that the figure covers. In A1L11 she began to discuss how to find the area of four-by-ten rectangle, then reviewed the area formula of rectangle. In A1L12 she assigned a task to find the areas of triangles, and walked around the students.

T: [to students E and S] So what have you guys done? You've drawn the grid. You've resorted to drawing the grid.

E Yes.

T Well, that's one way. [A1L12, 38:20]

The teacher temporarily accepted the idea of drawing grids and counting the number of squares, but it was not discussed further in the next lesson in the class.

CLASSROOM MATHEMATICAL NORMS IN SITE A2 [L05-L14]

At Site 4 the teacher first gave a task of measuring lengths and angles and introduced a property of vertical angles [L05-L06]. He also discussed how to construct angles by compasses [L07]. Then the teacher discussed properties of alternative, corresponding, and cointerior angles when parallel lines were crossed by a transversal, and assigned many exercises about those properties [L08-L11]. The teacher moved on to activities of finding angle sums of polygons [L12], and making 3-dimentional shapes using nets [L13-L14]. Note that L05-L06, L09-L10, and L13-L14 were double-period classes.

Norm A2-1: You should find a way to figure out measurements of geometric figures, other than actually measuring them

The same norm as Norm A1-2 appeared here also. At L05 the teacher held an activity of measuring lengths and angles, but he soon moved on to problems to find the measures of angles by using the properties they studied, and discouraged the use of protractors.

D: Can we have a protractor?

T: Nah, 'cause they won't be perfect in the book, they won't, and you'll get confused. [A2L09/10, 43:00]

Also, at L12 the teacher held an activity of finding angle sums of polygons by cutting and arranging the angle parts of the polygons drawn on a paper. Once the formula of the angle sum of polygon was introduced, only algebraic solutions using it were encouraged.

Norm A2-2: You should not worry about the proofs

Throughout the lessons proofs were not discussed, though the textbook contained problems to give reasons using geometric properties. Actually, the teacher avoided getting into proofs.

> [to all] Alright, shh. With question two, I just want you to find out what the angles are rather than getting into the proofs and so forth 'cause you're going to get more confused at this stage. It's alright. So just find out what the angles are. [A2L09/10, 41:20]

The teacher's intention to encourage informal reasoning in the geometry problems was evident in his interview also:

> Even when the guys are going on about complementary and supplementary- and they were getting … they were saying it *wrong* and I thought- I was sort of thinking 'well just forget *complementary and supplementary* cause … you know … there's an easier way of remembering this rather than trying to remember complementary means this supplementary means this- you know look at it as a … as *diagrammatic*- what is it and … what's it going to be?' [A04, T's interview 3]

CLASSROOM MATHEMATICAL NORMS IN SITE A3 [L06-L15]

At Site A3 the teacher discussed rounding of decimal numbers, computation of percentages [L06-L09], and percentages of an amount [L10-L13], reviewed the previous content [L14], and gave a test [L15].

Norm A3-1: You are allowed to use a calculator to solve percentage problems

Every problem given in the class was to find a value by calculations. The teacher allowed and encouraged the students to use calculators freely to solve those problems.

Norm A3-2: You should write down the process, not just the answer

Some students pushed buttons of their calculators, and put down just the result on their notebooks. The teacher often asked the students to put down also the steps of finding it:

> Most people have been working through those steps. Listen to me very carefully. Thank you. I want you to put down this step, please. If you are using a calculator, I do not want to see just an answer written down. I want to see the question, I want to see the setting out, and then I want to see the answer. Not just answers written down. Okay, I just don't want to see question 'a' fifteen percent. I want to see the question set out please. [A3L08, 13:15]

For instance, student C wrote his answer to the question "Finding 20% of 150" on his notebook as Figure 2 [A3L10].

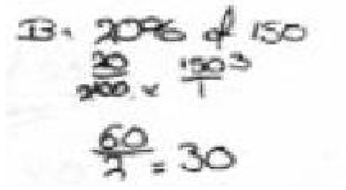

Figure 2. C's writing.

On the test sheet at L15, the teacher emphasized the above norms by putting instructions at the beginning: "You can use your calculator," and "Remember to show working out."

COMPARISON OF THE AUSTRALIAN AND JAPANESE LPS DATA

The classroom mathematical norms emphasized in Australian LPS lessons seem to be different than those (Sekiguchi, 2005, 2006) in Japanese lessons. This section looks into those differences and tries to illuminate the contexts behind them.

Curriculum differences and classroom mathematical norms

From its definition, a classroom mathematical norm is supposed to be applicable across several mathematical content domains, beyond a particular mathematical task or problem. Therefore, content differences seem to contribute to the different occurrence of classroom mathematical norms.

The Japanese lessons contained several topics of algebra: linear functions, simultaneous equations, and equations on proportions. As a result, classroom mathematical norms on algebra such as "*students are expected to solve simultaneous equations and explain their solutions by using algebraic operations, and only those solutions and explanations would become acceptable*" were evident. In contrast, the Australian lessons did not go into algebraic argument. At Site A1 algebraic expressions were used when finding the measurements of geometric figures, but algebraic operations of those expressions were little discussed. Site A2 was mostly about geometry. Even when the teacher discussed the formula of angle sum of polygon, he avoided algebraic expressions: "Actually, we'll do it in big letters, big words, rather than algebra" [A2L12, 38:40]. At A3 the lessons mainly dealt with arithmetic problems, percentage.

Also, the Japanese geometry lessons at Site J3 contained writing of mathematical proofs; therefore, a classroom mathematical norm of proof "*in a mathematical proof you cannot write what you have not yet shown to be true*" appeared (Sekiguchi, 2006). On the other hand, the Australian geometry lessons at Site A2 did not go into mathematical proofs. Hence, the norm of proof did not appear.

Though classroom mathematical norms are not about individual learning, they facilitate the development of personal mathematical orientations consistent with those norms. This is the internalization of classroom mathematical norms within individual students. Experienced mathematics teachers in regular classrooms like the teachers of the LPS seem often to be able to anticipate the emergence of a norm, and pay explicit attention to its development and internalization. Then, not only mathematical content but also classroom mathematical norms can be considered to be objects of learning, and shape a mathematical curriculum.

Instructional structures and classroom mathematical norms

Unlike TIMSS, LPS is able to investigate instructional patterns *spanning several lessons*. From the lesson sequence described earlier for each Australian Site, their instruction seems to have contained a common pattern:

[1] The teacher assigns a task to the students through the whiteboard or a worksheet.
[2] The students engage in an activity about the task.
[3] The teacher introduces some mathematical “rules” (formulas, or properties) emerged from the activity.
[4] The teacher discusses how to use those rules using typical problems.
[5] The teacher assigns many exercises, and helps the students to solve them individually.

The lesson sequences of A1 and A2 exhibited this whole pattern very closely. In the data of Site A3 the process [1]-[2] did not appear, and only [3]-[4]-[5] sequence appeared. This seems to be because the A3 lessons began after the concept of percentage and some of the related rules had been already introduced.

All the classroom mathematical norms pointed out above were introduced and emphasized in the process [4]-[5]. This seems to explain why those norms focused on steps of writing solutions, or using of the previously introduced “rules.”

In contrast, the Japanese lessons of LPS had rather different instructional structures. Commonly found was a whole-class problem-solving activity, which also often spanned more than one lesson. It consisted of 4 phases (cf. Stiegler & Hiebert, 1999):

Phase 1: The teacher presents one problem.
Phase 2: Students first try to solve it individually. Then, they may work with neighbors, or in small groups.
Phase 3: Some students present their solutions on the blackboard, and the class discusses the presented solutions.
Phase 4: The teacher summarizes important points.

In this problem solving process, the teachers encouraged students to produce and discuss diverse ideas. Through phases 3 and 4, various classroom mathematical norms surfaced: efficiency, important ideas, valid writing, and awareness of meaning (Sekiguchi, 2006). These norms guided or regulated the variation of mathematical ideas in the discussion and summary.

CONCLUDING REMARKS

Australian lessons emphasized different classroom mathematical norms from Japanese ones. The cross-national comparison of consecutive lesson data revealed close relations among classroom mathematical norms, instructional structures, and curricula. The comparative analysis of classroom mathematical norms attempted here was, however, still sketchy. More detailed comparison of them is necessary, together with comparison of instructional structures, and curricula (cf. Clarke et al., 2006).

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THE PRESCRIPTIVE ROLE OF THEORY OF CONCEPTUAL CHANGE IN THE TEACHING AND LEARNING OF MATHEMATICS

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The role of theory in mathematics education differs in different research traditions (Bishop, 1992). Broadly speaking, one of the remarkable philosophical or theoretical stances depends on the descriptive role of theory or prescriptive one. The purpose of this paper is to suggest the prescriptive role of theory of conceptual change (TCC). For attaining this purpose, this research report consists of two main parts: in the first place we discuss about the two theoretical stances not only in mathematics education research generally but also those of conceptual change particularly; in the second place we attempt to illustrate the TCC in the teaching and learning of irrational numbers with the help of preliminary analysis.

TWO THEORETICAL STANCES IN MATHEMATICS EDUCATION RESEARCH

The use of the words "descriptive" and "prescriptive"

Research in mathematics education needs three components, i.e. *enquiry*, *evidence*, and *theory*, as qualified by Bishop (1992). Our main concern here is about theory, because two other components cannot be available without theory. We focus on two theoretical stances in mathematics education research: so-called the *descriptive* and *prescriptive* theory. In short, what is common in the use of the words "descriptive" and "prescriptive" is the difference between "*be* (*sein*)" and "*ought to* (*sollen*)". Furthermore according to Bishop (1992) and other previous works, these are concerning with role of theory in different ways as follows:

> *The descriptive role of theory* can be languages to encode and read a practice, so the immediate concern is with an explanation of how or why something happened and prediction of what will happen in the similar conditions;
> *The prescriptive role of theory* can be sets of statements that could be used to guide the deliberate enactment of a practice, so the focus is on idealized or desired situations to which educational reality should aim.

As a matter of fact, these two theoretical stances can stand out as being very important on relationship between research and practice in mathematics education (Bishop, 1992; Mason & Waywood, 1996; Malara & Zan, 2002; Silver & Herbst, 2007). In Bishop (1992), for example, he invented two notions, namely "*what is*" and "*what might be*" in order to characterize three different research traditions in mathematics education as the Pedagogue tradition, the Empirical tradition, and the Scholastic-philosopher tradition. These two notions can correspond closely to "descriptive" and "prescriptive". The difference between these two extremes appears as the tension, as Bishop (1992) notes:

> The 'what is' researcher will accuse the 'what might be' researcher of ignoring the real situation, of not knowing where to start from, of proselytizing rather than doing proper research, and perhaps of exhibiting intellectual arrogance. The 'what is' researcher can equally be accused of upholding unstated educational values, of an obsession with gathering so-called objective facts, of ivory-tower remoteness, and of a lack of commitment to any educational ideals (p. 715).

On the other hand, he also remarked that "both can learn from the other, of course, and the appropriate balance must be struck by every researcher, depending on the their social and political situation" (ibid., p. 715). In Silver & Herbst (2007), they made clear distinction between the prescriptive and descriptive theory of mediating connections between research and practice in different ways. They also implied the relationship between two characteristics of theory:

> The distinction between theories that prescribe practice on the one hand and theories to understand practice is not as sharp as it may seem. […] That is, even the most descriptive approaches of research to practice include a prescription of what that practice should be that allows it to be visible and isolated from the rest of experience (Silver & Herbst, 2007, p. 53).

The relationship between two philosophical or theoretical stances can stand out as being very important when conducting any research in mathematics education (Bishop, 1992). In this way we notice that the relationship between two stances may be opposed one against the other or that they may be interrelated each other. The use of the words "descriptive" and "prescriptive" in this paper refers to *different facets of the same theory*, though not all theories in mathematics education. We shall have more to talk about this point in discussions on the role of theory of conceptual change (from now on, TCC).

The descriptive and prescriptive role of TCC

The TCC as a *foreground theory* (in the sense of Mason & Waywood (1996), namely, consisting of the studies aimed at locating, specifying, and refining theories about what does and can happen within and without educational institutions) has been developed in the various domains (diSessa, 2006). The conception of TCC originated from an explanatory framework for the history of science, in particular Thomas Kuhn's account of theory change. It was mainly used to explain knowledge acquisition, in particular for characterization of drastic reorganization of existing knowledge in processes of learning. Furthermore it has been widely used to explain learner's understanding in a series of developmental studies referring to science education (e.g., West & Pines, 1985). Although some theoretical models have been proposed to explain learner's conceptual change, there are some theoretical problems to solve: for example, most of the models have focused on individual's cognitive aspect, but ignored many other aspects (social, affective, etc.); no models take into account the intermediate states of the process of conceptual change; there is no general consensus on what is understood under the label "prior knowledge", and so on (Limón, 2001). Recently, we found that there are similar discussions concerning above points in mathematics education (Verschaffel & Vosniadou, 2004; Vosniadou, 2006).

Let us review such earlier researches from two different facets of the TCC. Most of the TCC that have been developed, be they implicit or explicit, can play a descriptive role much more than prescriptive one. In this theoretical stance it is important to address the research questions, for example, "what is the explanatory power of the conceptual change approach in the context of mathematics learning?", "Is conceptual change a 'cold cognition' approach, or does it take into account factors other that cognitive?" (cited from Vosniadou, 2006, p. 158). And then these questions might be answered respectively as follows: "the conceptual change theory could explain the development of learners' understanding of mathematical concepts and of the common difficulties that they encounter in learning them" (Tirosh & Tsamir, 2004, p. 536), and "although originally a cognitive-oriented approach, the conceptual change framework can take into account motivational, affective, situational and other factors, such as epistemological beliefs, that influence learning" (Vosniadou, 2006, p. 158).

In contrast to such descriptive role, it may be pointed out that Confrey's initial study (1980, 1981) on the TCC can play a prescriptive role. She applied a conceptual change theory of knowledge to curriculum inquiry in mathematics education in prescriptive way:

> A curricular conceptual change problem arises from a disruption in learning by students which is attributable to the curriculum, occurs for a large percentage of students; and which entails student difficulty in *extending*, *modifying*, *replacing* or *supplementing* a concept. Moreover, the disruption must be fundamentally linked to a concept, rather than to either instrument limitations, or language or reading difficulties, sequencing timing or a lack of prerequisite skills (Confrey, 1980, p. 71).

To examine a curriculum for conceptual change problems, she suggested some methods for curriculum design: for example, look for places where a large number of students experience difficulty; choose a significant concept and analyze its development in the curriculum; look for a series of places where students falter slightly; examine the history of the subject matter for prolonged conceptual change struggles, etc. (ibid., pp. 72-88).

What has to be noted is that this overview of earlier researches involves our interpretation because most of researches do not always manifest their own philosophical or theoretical stances. Broadly speaking, although the TCC in mathematic education field has been pursued in recent years as the approach to mathematics learning and teaching (e.g., Verschaffel & Vosniadou, 2004; Vosniadou, 2006), there is little argument oriented to prescriptive role of TCC, but also the interplay between two extremes. The authors do not think that future research on the TCC will have to draw special orientation to the descriptive role of theory, nor that these two theoretical stances are independent each other, but rather that, let quote again, as Bishop (1992) notes "both can learn from the other, of course, and the appropriate balance must be struck by every researcher, depending on their social and political situation”(p. 715). Therefore it is important to keep in mind that we should avoid concentrating to only one role of TCC. Our research position in this paper is, however, to take the prescriptive role of TCC more explicitly, because of the need for designing the teaching situation for conceptual change. This is not to say that we intend to ignore another role of TCC. We will attempt to illustrate the TCC in the teaching and learning of irrational numbers with the help of preliminary analysis. This job, conversely, can highlight our research position as a result.

AN ILLUSTRATION OF THE TCC IN THE TEACHING AND LEARNING OF IRRATIONAL NUMBERS

Preliminary analysis

As have mentioned above, we should like to explore a further possibility of the prescriptive role of TCC. The research on the TCC as a *background theory* (in the sense of Mason & Waywood (1996), namely, it serves as a backdrop to teaching and/or research, and often remains in the background) needs preliminary analysis for the following reasons. For example, because all observation is a *theory-laden* undertaking (Hanson, 1958), we do see *as* mathematics classroom (not see *that*…), therefore in this sense the preliminary analysis can play a role as the basis for designing mathematics classrooms. Or, since we cannot observe learner’s status of knowing in a direct way, in this sense the preliminary analysis can assess learner’s status of knowing based upon a theoretical framework. As the theoretical model can describe the process of conceptual change, the preliminary analysis can have both descriptive and prescriptive facets of the TCC. Thus the preliminary analysis takes place in three elements as follows:

- A priori components in the problematic situation
- A framework of the process of conceptual change
- Didactical implications

Let us consider the teaching and learning of irrational numbers as an example. What we will see in the following sections can be explained according to the three elements concretely. At first a priori components of learner’s status of knowing can be identified by considering the particular mathematical concept (in this case, number concept). Secondly these components can be organized relevantly in characterizing the process of change. Finally didactical implications are discussed.

A priori components in the problematic situation

One of the main objectives of the primary school mathematics is to educate knowledge of “quantity” (Steen, 1990; Hirabayashi, 1994). In terms of this objective, number concepts also can be introduced and applied in relation to the “quantity”. On the other hand, in the (lower) secondary school mathematics, since it is not necessarily approaching the “quantity” as well as in the primary school mathematics, its treatment should be something different from the primary school level. The teaching and learning of irrational numbers is possible situation that brings on the "problematic" situation, in the sense of Tirosh & Tsamir (2004), because of relating to the extension of number concepts and concept of infinity.

Irrational numbers are usually introduced as square root numbers in terms of the practical need to express the concrete quantity (length) as well as the teaching situations at the primary school level. For examples, it has been often taken the instructional way for finding out the length of the diagonal of the square, or the side of square having the double area of a given square. Here the introduction of irrational numbers, be it implicit or explicit, assumes the following points.

- The existence of incommensurable magnitudes
- The correspondence between numbers and points of a straight line

On the first point, the "incommensurability" can be an essential aspect of irrationals, since "an irrational number represents the length of a segment incommensurable with the unit"(Courant & Robbins, 1996, p. 60). But if we took *measures* small enough size for the practical purposes, there are no incommensurable magnitudes in reality. From the epistemological point of view, the concept of incommensurability did originate not from the practical source but from the theoretical one (Szabó, 1969). The Euclidean algorithm, though it be in operative way, can play a role of a direct proof of incommensurability. As the concept of incommensurability in itself involves infinite processes, in the operative activity the meaning of conclusion of "it continues endlessly" needs to construct under the thought-experiment (see more detail in Shinno (2007)).

On the second point, Courant & Robbins (1996) state: "if we demand that there should be a mutual correspondence between numbers on the one hand and points of a straight line on the other, it is necessary to introduce irrational numbers"(p. 60). For instance, Sironic & Zazkis (2007) have investigated into understanding of irrational numbers, by focusing on the representation of them as points on a number line, and they reveal that "that the vast majority of participants perceive the number line as a *rational* number line"(p. 483). One of the remarkable suggestions from this result is "confusion between irrational numbers and their decimal approximation and overwhelming reliance on the latter"(p. 477). This can relate to the ontological view of numbers. Because even if we could construct an irrational length on the number line, "nothing our 'intuition' can help us to 'see' the irrational points as distinct from the rational ones" (Courant & Robbins, 1996, p. 60).

Based on the above considerations and author's previous work (Shinno, 2007), we can identify at least three a priori components (two components and one meta-component) of learner's status of knowing in the teaching and learning of irrational numbers as follows:

S: Symbolic notation of number
R: Rigor on the infinity
A: Attitude towards knowledge of number

S is the issue concerning not only with the formality of the symbolism $\sqrt{}$ as a mathematical language, but with the decimal notation and/or infinite decimals. **R** is the issue of one essential nature of irrational numbers, concerning the measuring the incommensurable magnitudes. **S** is not being independently, but depends on the degree of **R**. As a meta-component, **A** is the issue of an ontological view of number", concerning how to approach "quantity and its difference between at the primary and secondary school levels.

A tentative framework of process of conceptual change

With the help of a priori components above, though it be tentative, we attempt to characterize learner's status of knowing in the process of change at least the following three phases: *pre-status*, *post-status*, and *intermediate status*.

*Pre-status: (**S**-operational, **R**-implicit, **A**-real)*

The "number" that learners have already learnt before introducing irrationals can be represented as fractions and finite decimals. The familiar notation (i.e., place value system of decimal notation) cannot represent the quantity (magnitude) in question precisely. This un-correspondence between symbolic notation and notion of number can bring on the "problematic" situation. As long as the symbolic notation of number is considered as representing some quantity (magnitude), it can be, in Sfard (1991)'s sense, *operational*. Also the decimal representations can reinforce the operational notion of number that subordinated to that of quantity (magnitude). In this phase, the degree of rigor on infinity can be rather empirical/implicit, if the concept of infinity has never been dealt at least as a teaching content. And the *reality* of numbers for the learner consists in the "concrete". Underlying such ontological view of number, what we can point out here is the attitude as a real/practical toward knowledge of number.

*Post-status: (**S**-structural, **R**-explicit, **A**-ideal)*

$\sqrt{2}$ as a mathematical language implies arithmetic operations, rather than represents certain quantity (magnitude), so it is a computational object that can be processed formally. This symbolic notation of number can be, in Sfard's sense, *structural*. In this phase, the degree of rigor on infinity can be more logical/explicit, as the concept of incommensurability in itself involves the infinite process. If we consider the logical rigor intervening in reflecting the action of "measuring" and in reasoning the potential infinite process under the thought-experiment, the negational or impossible situation can lead to advance the degree of rigor on the infinity. Furthermore the *reality* of numbers that the learner can form through the teaching and learning of irrational numbers consists not in the "concrete", but in their thought or reasoning. Underlying such ontological view of number, what we can point out here is the attitude as an ideal/theoretical toward knowledge of number.

*Intermediate status: (**S**-structural, **R**-implicit, **A**-real)*

In the course of the teaching and learning of irrational numbers, irrationals can be used as a computational object like the use of literal symbol in algebra. In educational reality it is likely that the computations with irrationals can be foreground and performed without reflecting on the incommensurability. This phase represents an intermediate status (in other word, "quasi" post-status) that the symbolic notation can appear to be *superficially* structural. It means that even if the computational aspect of irrational number has done successfully, there is a pitfall of the essential aspect of them. We need teaching and learning activity that can lead learners to become aware of incommensurability for their conceptual change.

Didactical implications

The phases of pre/post status themselves presupposed above show desired situations at which the teaching and learning of irrational numbers should aim. It means that such possible situations can bring on the conceptual change. The phase of intermediate status shows another didactical implication. If we use the Euclidean algorithm as a didactic tool, it can enable learners to become aware of incommensurability (e.g., Iwasaki, 2004). However if such a kind of setting does not take into consideration, it is unlikely to achieve the conceptual change in the teaching and learning of irrational numbers. This implication constitutes a prescriptive way to identify and propose an expected mathematical activity. Perhaps designing for conceptual change in the teaching and learning of mathematics may not be achieved only in existing treatment of the teaching contents. This is the pedagogical problem concerning not only particular subject matter but also school mathematics as a whole, which takes deep into epistemological and didactical considerations.

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COMPARING JAPANESE AND FINNISH 6TH AND 8TH GRADERS' WAYS TO APPLY AND CONSTRUCT DEFINITIONS

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Defining mathematical concepts is an important theme. Many researchers have recognized the importance of developing students' understanding of the idea of mathematical defining and attempted to propose methods for helping students to achieve a more sophisticated understanding of the defining (e.g. Zaslavsky & Shir 2005). However, little close attention, except the studies like Fujita and Jones (1997) and Ouvrier-Buffet (2006a & 2006b), has been given to how defining skills are actually related to other skills like for instance hierarchical classifications in students' reasoning. The purpose of this paper is to clarify Japanese and Finnish students' ways of defining geometric concepts. Although, both Finland and Japan have been the top countries in international comparisons, such as the OECD Programme for International Student Assessment (PISA), in addition to many similarities, there are also significant differences between Japanese and Finnish students' geometrical thinking.

INTRODUCTION

Defining mathematical concepts is an important theme. Many researchers have indicated the importance and necessary of the defining skills for learning geometry (e.g., Vinner 1991; Govender & de Villiers 2002; Matsuo, 2004) Just the definitions give a mathematical justification for the existence of hierarchical concept structures and class inclusions between concepts belonging to different abstract levels. Many researchers have recognized the importance of developing students' understanding of the idea of mathematical defining and attempted to propose methods for helping students to achieve a more sophisticated understanding of the connection between the definitions of concepts and the mutual relationships between the concepts (van Hiele, 1984; Matsuo, 2000 & 2006; Silfverberg 1999 & 2000; Keiser 2000). However, little close attention has been given to how these processes are actually related in students reasoning (however, see Fujita & Jones 1997 & Ouvrier-Buffet 2006a, 2006b).

The purpose of this article is to clarify Japanese and Finnish students' understanding of defining geometric concepts and its relation to understanding class inclusion relations between these figures. Specifically, we present some results about the connection between students' understanding of the defining process and hierarchical categorization of geometric figures. We also shortly discuss some possible linguistic factors behind the differences found between Japanese and Finnish students' geometric reasoning.

DESIGN OF THE STUDY

Framework of the study

Defining is a multifaceted process. From one point of view it is a mathematical task and there are some specific mathematical characteristics and norms of a definition

(Leikin & Winicki-Landman & 2000; Govender & de Villiers 2002). On the other hand it is a psychological process (Linchevsky et al. 1992). Thirdly defining is also a linguistic process. From the linguistic point of view definitions have especial, subject and context dependent structure (c.f. Morgan 2005). In mathematics, definitions have a central role when we are introducing new concepts to the learner. In traditional teaching, based on a deductive approach, the definition of a concept acts as a starting point for clarifying the meaning of the concept. In an inductive teaching method, the meaning of the concept is learned part by part but the final goal usually is however to get the concept meaning condensed into a form of definition. The common way that textbooks tell a reader what is meant by a certain concept is to present the definition of the concept and a few examples of the concept or to present these in reverse (a few examples and then the definition).

It is clear that understanding the idea of mathematical defining should be more than just passively remembering the names and the definitions of the concepts. From a constructivist point of view students should form new concepts by themselves, name them, try to discover necessary and sufficient conditions for the concepts and tentatively try to define the concepts. Over time, the learner gradually learns to know that there are some demands which are set for the form of a mathematical definition. The definition is at least in principal arbitrary, it includes only previously defined concepts, and a set of the defining conditions is normally minimal and so on. In many cases there are also several optional possibilities to define concepts (Leikin & Winicki-Landman 2000; Govender & de Villiers 2002).

Learning the idea of mathematical defining takes time and is affected by many psychological factors. Many researches have shown that the definitions, which students write, don't for instance normally fill the minimality demand (Linchevsky et al. 1992, Silfverberg 1999). The younger the students are the more probably they write lexicon-type definitions which are descriptions of all the knowledge they have about the concept which they try to define (Silfverberg 1999). It is also shown that one important part of the learning process of defining is the growing skill to handle both the figural and the conceptual aspects of the concept in constructing the definitions (Mariotti & Fischbein 1997). The concept's definition is generally based on the so called concept image (Vinner 1991) and the class membership is in many cases judged on the basis of phenomena like family resemblance described in prototype theoretical research tradition (c.f. the classical one Rosch 1978).

The tests

In the whole research project, three different tests were used each of which focuses on different points of view of pupils' geometrical thinking. However, in this paper, we restrict our analysis only to the data obtained from the Test3 (a defining and class inclusion test), so we firstly provide some details about this test.

In Test 3, we firstly asked pupils to select which one(s) of the following seven statements

A: A rectangle is one of the geometrical figures;

B: A rectangle is a quadrangle, which has four right angles;

C: A rectangle looks like a stretched square;

D: A rectangle has four right angles, four straight sides, two equal diagonals and two equal and parallel opposite sides;

E: A rectangle is a quadrangle, which has four right angles and which adjacent sides are of different length;

F: A rectangle is a quadrangle, whose opposite sides are parallel;

G: A rectangle is a parallelogram, which has one right angle;

could be used as a definition of a rectangle (question Q1). After that, students were asked to decide which of these alternatives would be the best definition for a rectangle (Q2). In this test, we also showed pupils a picture of a typical rhombus and a picture of a typical parallelogram and asked them to write a definition for a (slightly slanted) rhombus (Q3) and for a typical (long and slightly slanted) parallelogram (Q5). Finally, we showed a picture of a square and a picture of a rectangle without naming the figures and asked if the figure representing the square was a rhombus (Q4-1) and a figure representing a rectangle was a parallelogram (Q6-1) and reasons for both of pupil's decision (Q4-2 and Q6-2).

Test 3 included three components each focusing on a different aspect of the defining and classification process of geometric concepts. The first component (Q1 and Q2) measured how well a person understood kinds of linguistic-mathematical form, which definitions should have, including the necessity and sufficiency of the conditions, and minimality demands (Winicki-Landman & Leikin 2000, 17; Linchevsky et al. 1992, p. 49). The second component (Q4-1, Q4-2 and Q6-1, Q6-2) measured especially the understanding about the status and task of the definitions in a mathematical system. In our questionnaire, this component was especially emphasizing the primary role of the definitions in determining the meaning content of the concept and through that also the class inclusions between concepts. The third component (Q3 and Q5) measured a respondents' own skill in producing formally correct and workable definitions.

The tasks included in each of the tests were carefully content validated by both of the authors, one from Japan and the other from Finland, in order to be sure that the tasks were as suitable as possible for the mathematics curriculum of both countries. After joint discussions, some revisions were made to each of three sub-tests.

The participants

The tests were administered to 68 6th graders (35 boys and 33 girls) and 94 8th graders in Finland and 89 6th graders (41 boys and 48 girls) and 63 8th graders (28 boys and 35 girls) in Japan. Students had 15 minutes to work on each three sub-test.

METHODS OF ANALYSIS

The variable "Total" = T measuring the total understanding of the different aspects of the defining and class inclusion was scored as follows:

Q1: If the selection of the possible definitions for a rectangle included the correct choice B, score +1 and/or included the choice G, score +1, and did not include E, score +1

Q2: The selection for the best choice either the formally correct alternative B or G, score +1

Q4-1: A square considered as a rhombus, score +1

Q4-2: The reason for the above class inclusion was based on the definition of a rhombus, score +1

Q6-1: A rectangle considered as a parallelogram, score +1

Q6-2: The reason for the above class inclusion was based on the definition of a parallelogram, score +1

The qualitative part of the analysis consisted of the content analysis of the definitions. Definitions were categorized into the seven classes ***a***: a naïve definition; ***b***: an insufficient characterization; ***c***: a sufficient list of attributes, some attribute(s) incorrect; ***d***: a sufficient but not minimal list of attributes, all correct; ***e***: a formally correct and minimal definition, including no connection to any upper level concept; ***f***: a formally correct but not minimal definition, including a connection to upper-level concept; ***g***: a formally correct and minimal definition including a connection to an appropriate upper level concept,. If the definition was missing, we coded it as '*none*' and if we could not categorize it to any of the categories a-g we coded it as '*other*'.

RESULTS AND DISCUSSION

The following table shows the total score T of Q1 & Q2 & Q4 & Q6 measuring the level of understanding of the formal demands and tasks of mathematical definitions for sixth (6) and eight (8) graders in Japan (J) and Finland (F).

Group/ T	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
J6	7.9	31.5	24.7	15.7	6.7	9.0	4.5	0	0
J8	7.9	11.1	14.3	11.1	7.9	16.0	22.2	9.5	0
F6	33.8	8.8	20.6	5.9	30.9	0	0	0	0
F8	22.3	1.1	47.9	2.1	26.6	0	0	0	0

Table 1. Total score (%) in Japan and Finland

In Japan the number of students who obtained at least five scores significantly increased from elementary to secondary school ($\overline{T}_6 = 2.27$, $\sigma_6 = 1.59$, $\overline{T}_8 = 3.84$, $\sigma_8 = 2.22$, $t = 4.808$, $p = .000$). Because Japanese eighth grade students systematically learn the relations among triangles or quadrilaterals and their dependence on the definitions of the figures the eighth grade students have become to be able to understand how the standard definitions of quadrilaterals imply the inclusion between squares and rhombi, rectangles and parallelograms. On the other hand, in Finland there is no statistically significant difference between elementary and secondary school students ($\overline{T}_6$ = 1.91, σ_6= 1.66, $\overline{T}_8$ = 2.10, σ_8= 1.41, t = 0.769, p = .222). An explanation for this can be the fact that in Finland lower and secondary school students not do much practicing of defining geometric figures and the inclusion relations among geometric figures are learnt mostly by heart. The means of the Finnish and Japanese 6th graders did not differ statistically significantly (t = 0.002, p = .499). However, the Japanese 8th graders obtained statistically significant better results in the variable Total than the Finnish 8th graders (t = 6.139, p = .000).

There were differences between Japan and Finland for the variance of scores. At elementary level, many Japanese students got one or two scores, on the other hand many students got zero or two or four scores in Finland. Japanese elementary school students usually do not give correct answers to contents which have not been learnt in curriculum. However, some Finnish elementary school students seem to answer correctly also these items by intuition. At the secondary level, almost a half of the students got a score five, six or seven in Japan, on the other hand all students scored below five in grade 8 in Finland. As described above, in Japan the results show the effects of systematic teaching and learning of definitions and of the structure implied in the definitions to the system of geometric figures.

Secondly we qualitatively analysed the definitions which students wrote for a rhombus and for a parallelogram coding the answers to the categories *a, b*, …, *g* explained before plus the categories '*none*' and '*other*' . The following table shows the quality of students' definitions of a rhombus in Japan and Finland.

	a	*B*	*C*	*D*	*e*	*f*	*g*	*None*	*other*
J6	16.9	16.9	25.9	34.8	2.2	0	0	1.1	2.2
J8	3.2	30.2	11.1	34.9	3.2	3.2	7.9	6.3	0
F6	19.1	51.4	5.9	5.9	5.9	2.9	1.5	7.4	0
F8	14.9	37.2	7.4	18.1	11.7	4.3	2.1	4.3	0

Table 2. Categories of students' definitions of a rhombus in Japan and Finland (%)

Most typically the definitions given by Japanese students were encoded in the category *d* and the definitions given by Finnish students in the category *b*. Many of the definitions of a rhombus given by Japanese students included some characteristics of parallelograms. Neither Japanese nor Finnish students seemed to have learnt what distinguishes the definition of a geometric figure from a pure collection of the properties of the same figure. Moreover in Japan the numbers of students who are assigned *b* (insufficient characterization) increase from elementary to secondary school, on the other hand the numbers of students who are assigned *b* decrease in Finland. Japanese secondary school students came to know many kinds of properties, thus they used them when they described *a* figure. For Finnish students, a rhombus is a less familiar concept and they clearly tried to form a definition by stating the most salient visual properties of the given example.

Third the following table shows students' ways of defining a parallelogram in Japan and Finland.

	a	*b*	*C*	*D*	*e*	*f*	*g*	*none*	*other*
J6	10.1	75.3	4.6	2.2	3.4	0	0	2.2	2.2
J8	3.2	66.7	0	14.3	3.2	0	6.3	6.3	0
F6	22.1	39.4	13.2	8.8	9.2	0	2.9	4.4	0
F8	13.8	21.3	29.8	24.5	2.1	2.1	3.2	3.2	0

Table 3. Categories of students' definitions of a parallelogram in Japan and Finland (%)

Most of the Japanese students and Finnish 6th graders were assigned the category *b* (an insufficient characterization) in the item Q5. In both countries, many of the students' definitions of parallelograms did not refer to the qualifier "two pairs of". In general Japanese people do not distinguish between 'quantifying' nouns and 'non-quantifying' nouns in their spoken language. They think that it is not so important to distinguish one from more than one. Also many native Finnish speakers interpret the expression "the opposite sides are parallel" so that it implicitly means that both pairs of opposite sides are parallel. In Finland the categories *c* and *d* are more frequent at grade 8 than at grade 6. In Silfverberg's (1999) previous study the same result was reported. The so called long list definitions become more frequent as students' knowledge of geometrical figures increases.

The transition of the number of students from elementary to secondary in Q3 in Finland is similar to the transition of the number of students from elementary to secondary in Q5 in Japan. The reason for this is probably the fact that in the Finnish curriculum the parallelogram is considered to be a much more important concept than the concept rhombus and therefore students know it much better than the concept rhombus. In Japan, the curricular status of the concept rhombus is stronger than in Finland.

Fourth we discuss the interdependence between understanding inclusion relations among geometric figures and defining the figures. First the results of Q4-1, 4-2, 6-1 and 6-2 showed that almost all of the students who understood the inclusion among two figures could also describe the reason for of it based on their definitions both at the elementary and the secondary school level in both countries Japan and Finland. Both in Japan and Finland students who accepted the class inclusion among two figures in Q4-1 and Q6-1 did not however, usually select the appropriate definitions for a rectangle in the item Q1 (selection of the suitable definition for a rectangle) either at the elementary or at the secondary level. On the other hand, in the item Q2 many students who accepted the class inclusion among two figures in Q4-1 and Q6-1 selected the choice *d* (list definition) at both elementary and secondary level in both Japan and Finland. The understanding or remembering the fact that there holds a class inclusion relation between two concepts seemed not to guarantee that a student has a developed idea of the meaning of defining. Contrary to our findings, in the van Hiele theory the understanding of class inclusions and the understanding the definitions are normally placed at the same van Hiele level.

Furthermore in the Q3 (defining a rhombus) the category of answers of many secondary and elementary school students who understood the class inclusion among two figures was *d* in Japan, but *b* in Finland. In Japan the definitions which students wrote for a rhombus in many cases included characteristics of parallelograms. Similarly in the Q5 (defining a parallelogram) the category of answers of many secondary school students who understood the class inclusion among two figures was *b* (insufficient characterization) in Japan, but *c* (a sufficient list of attributes, some attribute(s) incorrect) or *d* (a sufficient list of attributes, all correct, not minimal) in Finland. Both in Japan and Finland many students wrote merely that “the opposite sides are parallel” instead of “two pairs of opposite sides are parallel”.

To summarize, we found that there are some similarities and differences between Japanese and Finnish 6th and 8th graders’ ways to apply and construct definitions of geometric concepts. Student's understanding of defining geometric concepts relates to the understanding of the class inclusion relations in both countries. However, understanding the fact that there holds a class inclusion relation seemed not to guarantee that a student has a developed idea of the meaning of defining. Moreover a student's understanding of defining geometric concepts depends at least to some extent on the curriculum or mathematical contents for teaching in schools.

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EXTRAPOLATING RULES: HOW DO CHILDREN DEVELOP SEQUENCES?

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We report results on students' strategies to continue various patterns. The analysis of the answers given by a large sample of students to open questions seems to convey that the option for a specific continuation is influenced by the degree of the semantics understanding and the capacity to decode the syntax of a pattern.

RESEARCH CONTEXT AND OUR WORKING HYPOTHESIS

Questioning young children's intuitions about infinity, Singer and Voica (2003) found out that students' arguments about the infinity of the sets of numbers is mostly based on a processional perception. Primary students can bring convincing arguments for the infinity of the set of natural numbers using a spatial rhythmic perception: after learning to count, children seem to be able to develop sequences of natural numbers indefinitely long (Singer & Voica, submitted) without special learning/training because this core knowledge is very adequate to the recursive property of the mind. The innate ability of iterating allows for imitation in young ages and further evolves in identifying and developing patterns. The role of recursion is essential for mathematics, and, as Chomsky (1980) emphasized it, for language. Language and mathematics both have computational properties and redundancy (Singer, 2007b). While the internal architecture of language supports many debates, there is an agreement that a core property of the faculty of language is recursion, attributed to narrow syntax; this takes a finite set of elements (words, sentences) and yields an array of discrete expressions (Hauser, Chomsky & Finch, 2002), which can be considered potentially infinite. Similarly, from a set of a few digits, infinitely many natural numbers are generated through a recursive procedure given, essentially, by the Peano's axioms. The evolution of the Indo-European written languages, the ideograms of the Chinese or Japanese, and recent evidence from brain scans seem to converge to the idea that strings of words and strings of objects/ figures are similarly perceived (and processed) by the human mind (although in different areas of the brain).

New specific questions arise starting from here: do children activate recursion when they continue sequences? Is recursion the most natural way to develop a sequence? To what extent the contexts of problems (algebraic or geometric; continuous or discrete; one-dimensional or multi-dimensional) influence the processional perception?

METHODOLOGY

Samples. To answer these issues, we addressed students questions that asked the continuation of sequences of numbers and sequences of geometric figures. As progressing with the research, three types of samples were involved. First, 31 students from 1st to 4th grades (6 to 10 years old), randomly selected, answered to a series of questionnaires. As a second round, we used a representative statistical sample composed of 3,837 students in 4th grade (10-11 years old, 51.1% boys and 48.9% girls, 41.7% from rural schools and 58.3% from urban schools). The third sample consisted in 55,214 students from 7th to 10th grades (13 to 17 years old) who participated in a multiple-choice contest. The findings presented in this paper are mostly based on the second sample, the others being used to either configure the hypothesis or check the conclusions.

Tools. A series of questionnaires and interviews were used during the research. The questionnaires for the first sample contained numerical and geometrical sequences grouped in items following various criteria (for example, patterns with a dominant development on one direction, as in figure 1A, versus patterns with a bi-dimensional development, as in figure 1B). In the same time, the students were asked to express their preference for only one of the patterns of the same item. Later on, during the interviews, we discussed with the students about their answers and we proposed them to continue other new sequences.

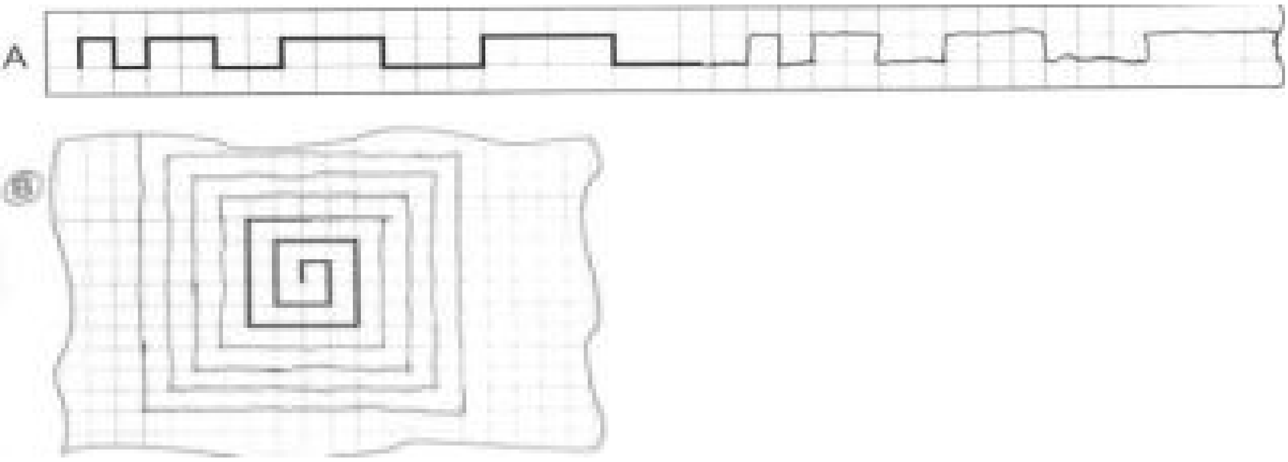


Figure 1. The answers given by a grade 2 student (8 years old). The pattern A was continued with a translation, while for B the student used a "spiral" development. He argued that he preferred the sequence B because he "had to go around".

For the second sample, we processed the results of a questionnaire composed of 8 open questions that supposed the free continuation of various patterns. The students in the third sample had to answer multiple-choice questions with a single correct answer.

Methods. The patterns used in the various stages of the study were classified according to a set of criteria. One criterion focuses the mathematical domain and refers to algebraic versus geometric, another highlights the 'nature' of the pattern and refers to discrete versus continuous. For example, the patterns S3.1 and S3.2 from figure 3 are geometric, and the pattern S3.3 in figure 3 is algebraic; the patterns in figure 1 are continuous geometric, while the pattern S3.2 in figure 3 is discrete.

For all the open questions in this research, we have taken into account any answer given by the students, without considering that a certain continuation is more correct than another. In processing the data, we clustered the students' answers in some well defined categories. In some cases, where, to some extent, the drawings were just nearby the defined category, we considered the dominant modality through which students seemed to arrive at proposing a continuation for that sequence. Thus, for example, if a continuation as the one in figure 1.A did not completely mirror the initial pattern, but the student's intention for making a translation seemed obvious, we clustered it in the translation category.

SAMPLE DATA AND RESULTS

The questionnaires and the interviews applied to the students in the first sample showed that they continue various patterns in a different way.

For example, in Figure 1 we can see how Stefan (2nd grade, 8 years old) continued two geometric patterns: for the first, he used a translation, while for the second he developed the spiral. In the next parts of the paper, we refer to figure 1.A as an example of a *continuation through translation*, and to figure 1.B as an example of a *continuation through development*.

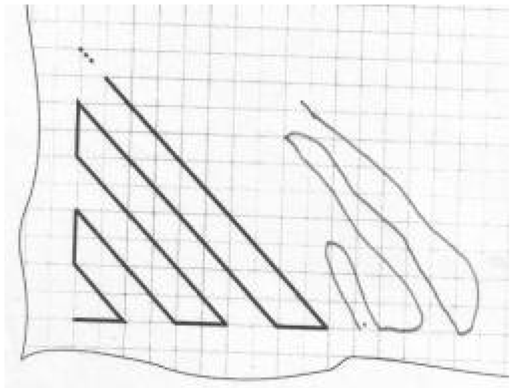
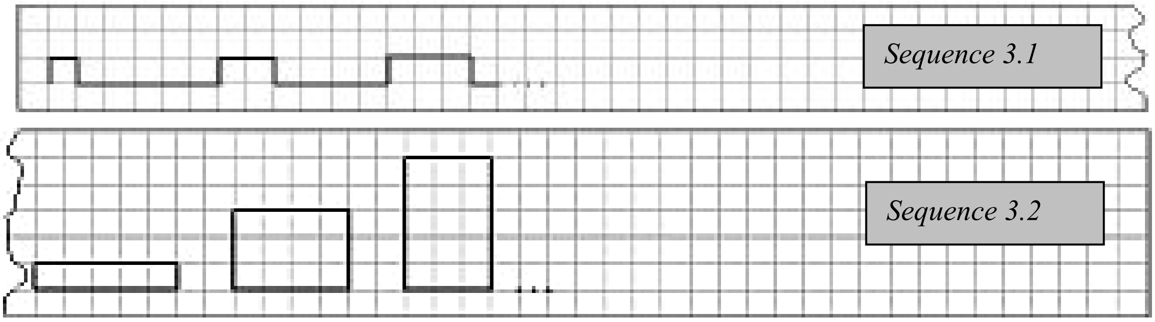


Figure 2. The continuation through translation proposed by Denise (3rd grade).

These types of continuations also appeared for other patterns. Thus, for example, Denise (3rd grade) continued the shape from figure 2 through translation. We also met continuations through translation for the pattern 1.B, which we did not reproduce here because of the lack of space.

Analyzing the results of the first sample, we noticed two facts. First, there was a variety of continuations for a same pattern. Second, in general there were not significant differences between the way in which children continued numerical sequences and the way they continued discrete geometrical sequences. More precisely, a child that proposed a certain way of continuation (through development or through translation) for a discrete sequence did propose the same method for all the discrete types of sequences.

In order to check this conclusion, the second sample was presented with different types of patterns that can be expressed through the same rule, as for example, the ones in figure 3. We notice that these three items are of different natures: the first pattern is geometric continuous, the second is geometric discrete and the third is algebraic. However, all these three patterns have a common characteristic, that is the lengths of the horizontal superior elements of the first pattern, the heights of the rectangles in the second pattern, and the terms of even ranks of the third sequence increase, while the lengths of the horizontal inferior elements of the first pattern, the widths of the rectangles of the second pattern, and the terms of the odd ranks of the third sequence decrease. For these three patterns, a continuation through development means a completion of the type: 10, 1, 9, 3, 8, 5, 7, 7, 6, 9, ..., and a continuation through translation means the completion of the type: 10, 1, 9, 3, 8, 5, 10, 1, 9, 3, 8, 5, ...



10, 1, 9, 3, 8, 5, . . . *Sequence 3.3*

Figure 3. Three of the sequences proposed to the second sample.

Following the results of the first sample, we would expect that the options for a modality or another of continuation be almost equally distributed, no matter the nature and domain of the sequence. Surprisingly, the data (given in table 1) did not confirm this hypothesis.

	S 3.1	*S 3.2*	*S 3.3*
continuation through development	35.47%	13.73%	23.00%
continuation through translation	34.19%	47.95%	2.20%

Table 1. The statistical results concerning the continuation of the patterns from Figure 3

In order to validate these data, we compared them with the results obtained on the same sample for the question regarding the continuation of the sequence in figure 5.

20, 11, 18, 10, 16, 9, ...

Figure 5. A numerical sequence proposed to the second sample.

The statistical results for this item: 27.4% – development, and 2.8% – translation, confirm the above conclusion.

The great differences among the results presented in Table 1 show that the 4th graders perceived very differently the three working tasks presented in figure 3. What could be the explanation? The sample to which we applied the test being a representative one, the data can be considered significant. To have a clearer view about the way in which the same student completed different items we used correlation matrices. Table 2 contains the correlation matrix for the items S3.1 and S3.2 showed in figure 3. In this table, "2-dimensional development" means that the student understood the interchange superior-inferior, length-width or even-odd, and continued the discovered alternative rules; "1-dimensional development" means a continuation in which the student paid attention to only one of the rules, neglecting the other; and "other" means either that the student did not answer or we could not classify the continuation in any of the categories.

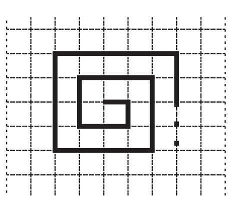
		Sequence 3.2					
		Transl.	*1-dim. dev.*	*2-dim. dev.*	*Symm.*	*Other*	*TOTAL*
Sequence 3.1	*Translation*	26.16%	2.05%	0.54%	3.25%	2.16%	**34.19**
	1-dim. dev.	2.99%	0.96%	0.59%	1.56%	0.46%	**6.59%**
	2-dim. dev.	10.29%	2.63%	10.55	9.30%	2.68%	**35.47%**
	Symmetry	0.70%	0.08%	0%	1.25%	0.08%	**2.11%**
	Other	7.79%	3.10%	2.03	3.28%	5.42%	**21.63%**
	TOTAL	**47.95%**	**8.83%**	**13.73%**	**18.66%**	**10.81%**	**100%**

Table 2. Correlations between the answers to S 3.1 and S 3.2

We notice that for S3.1 almost half of the students (47.95%) opted for using a translation to continue the pattern, while for S3.2 the scores are almost equally distributed between translation and 2-dimensional development. More precisely, the percent of students who continued S3.1 through development and 3.2 through translation is significantly bigger than the percent of students who proceeded the other way round (the cells colored in Table 2). Therefore, there is an asymmetry in processing sequences in different contexts.

To better understand the causes that generated these differences in students' options, we tried to answer the question: does the transfer between different contexts (geometric versus algebraic; continuous versus discrete) happen? To answer, we analyzed the results from the third sample. The questions we discuss below focus on the transfer from a geometric continuous context to an algebraic discrete context.

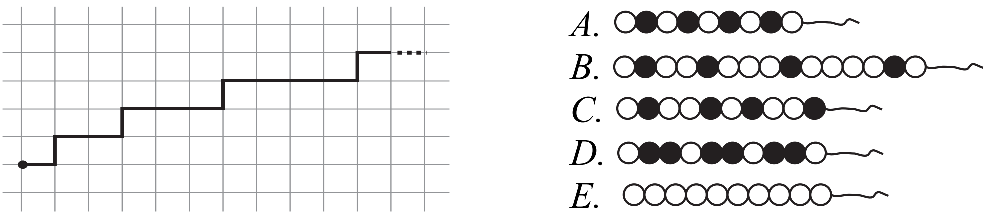
Problem 1: *Which of the next numeric strings best describes the pattern in the figure?*



A. *1, 1, 2, 2, 3, 3, 4, 4, ...*
B. *0, 1, 2, 3, 4, 5, 6, 7, ...*
C. *2, 2, 4, 4, 6, 6, 8, 8, ...*
D. *1, -1, -2, 2, 3, -3, -4*
E. *1, -2, 3, -4, 5, -6, 7, -8, ...*

The correct answer (D) was chosen by 35.17% of 9,041 students from the 9th grade, respectively by 36.76% of 8,660 students from the 10th grade. If we also consider the answer A, where the students perceived the 1-dimensional increase, but they did not see the change in the direction, we get a total of 70% reasonable answers.

Problem 2: *To remember the road, Tartarin put beans on a string. Which is the string he got?*



The problem needed a transfer from a geometric continuous pattern to a discrete pattern. This transfer is intermediated by the understanding of the (numerical) structure of the given pattern. The correct answer (B) was chosen by 82.25% of 22,213 students in grade 7, respectively by 83.90% of 15,300 students in grade 8.

We can conclude that, at least for this age, students can usually make the transfer from geometric to algebraic patterns and from continuous to discrete models. Other studies confirm this hypothesis for earlier ages (Singer and Voica, submitted). Therefore, this transfer seems to naturally happen. Consequently, the explanation for the great differences among the options of continuing the patterns S3.1, S3.2 and S3.3 must be found somewhere else.

EXPLAINING THE RESULTS

A continuation is a transfer that starts from the source-pattern. The departure from the source-pattern (so the continuation proposed by the student) seems to depend upon the degree of understanding of the pattern structure.

At a most elementary level, the student identifies *a shape* for the proposed model. Some children remain at this stage of understanding and propose developments that preserve an approximate shape of the pattern, as in figure 6. Other children cross further and try to understand *the structure* of the pattern.

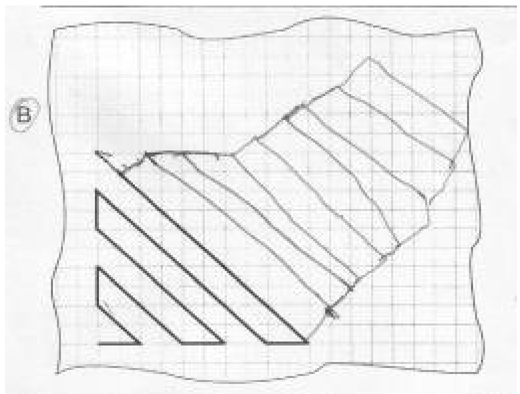


Figure 6. The way in which Elena (2nd grade) continued a geometric pattern.

Interviews show that a second level of understanding would be a search for finding a meaning for the identified shape. A superficial understanding of the structure could generate "the illusion of linearity" (De Bock, Van Dooren, Janssens & Verschaffel, 2002), which seems to lead to the continuation through translation. Recursion is somehow embodied in translation, compared to the continuations in which the student recognized only the general shape of the pattern: actually, the translation is generated by periodicity, which is the simplest type of recursion. Because recursion is intrinsic in the human mind, the passage from identifying a shape to translation development could involuntarily happen, blocking student's access to a deeper analysis of the pattern structure. However, when the student arrives at expressing the structure in a numerical shape, i.e. when, for instance, she/he codes the pattern 3.1 from figure 3 through an information of the type "1 small square up, 1 small square down; 2 up, 2 down, etc. ...", she/he is usually led to a developmental solution for the sequence.

A topological perception (Singer & Voica, 2003) seems to act in order to find meaning for the decoded shape, i.e. to find the "semantics of the shape". Subsequently, the semantics leads to the intuition of a general pattern that invites to reproduce the gestalt. In order to get a more accurate answer, after understanding the semantics, the child needs to arrive at the syntax of the shape, which means at decoding the intrinsic rules of the pattern. The syntax generates the identification of an algebraic or geometric structure of the pattern. Sometimes, when the structure of the pattern can be expressed numerically, the processional perception is activated. This seems to operate through searching rules of development of the numerical pattern. Therefore, the passage from the semantic of the shape to its syntax activates the recursiveness of mind, allowing a developmental continuation.

Counting represents one of the sources of recursion. It seems therefore reasonable to hypothesize that a pattern will be continued through development when the context of the problem favors the passages from continuous to discrete and from geometry to algebra. In this way, we can explain the fact that students frequently continue a pattern through development when they are able to express its structure algebraically.

Thus, for the continuous geometric patterns, the semantics is visible and offers support, while for the discrete geometric patterns, the semantics is hidden. In this last case, the child resorts to a minimal way of development that is periodicity, which is a 0-level recursion. To find the developmental rule, the child has to decode the syntax, in other words, she/he has to identify an internal structure of the given pattern.

CONCLUSIONS

In our research, 7 to 17 years old students witnessed an obvious ability to deal with various representations and to use adequate algebraic and geometrical tools to make transfers between them. Using patterns as support, a primary conclusion on which we have focused this article is the student's intuition about the subtle interface between deep and surface structure of mathematics concepts. The successful learner constructs meaning that uncovers the semantics, and (simultaneously or successively) decodes the syntax of the concept. This induces an important consequence on teaching and learning: students should be presented with opportunities for analyzing both semantics and syntax of mathematical concepts.

Secondly, we found that children are naturally able to make transfers between algebraic and geometric contexts, between continuous and discrete properties. This potentiality is, however, too little valued in teaching practice. A more transdisciplinary approach, in which the underlying structure of a math concept (Singer, 2001) is highlighted through crossings in-between algebra and geometry, could contribute to a better understanding of both algebra and geometry.

Finally, students' perception of math concepts is deeply influenced by the context. Consequently, effective teaching should favor concepts explorations in various contexts, with various passages, using multiple representations. Teaching for representational change (Singer, 2007a) allows an operational focus that moves beyond tasks and domains to uncover knowledge structures. More research needs to be done in order to see to what extent students' exposure to a variety of representations and transfers is effectively able to develop their fluent thinking.

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SUCCESS IN ALGEBRA

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The need for an elementary mathematics program that prepares students to be successful in more formal mathematics courses has been highlighted in the literature. This paper presents a study, conducted as part of the Measure Up (MU) Project, to investigate 5th and 6th grade students' preparedness to engage in a formal algebra course by assessing their understanding of the concept of variable. Sixty students took the Chelsea Diagnostic Mathematics Test: Algebra (CDMT1). Results showed a significant effect of the MU program. In addition, a small sample of students who had been trained in MU were interviewed using a sub-set of CDMT1 items in an attempt to trace connections between the program experience and students' reasoning.

INTRODUCTION

Much attention is paid to students' success in studying formal algebra. Algebra is foundational to all areas of mathematics, providing the language and structure used in other branches; it is a graduation requirement in most states that students demonstrate proficiency in algebra; and algebra serves as a "gatekeeper" course without which students limit their ability to choose certain education and career paths.

A number of studies focused on factors that impact children's learning as they move from the arithmetic of the elementary years to the algebraic thinking required for more formal mathematics in secondary years (Kieran, 1981; Williams & Cooper, 2001). The transition is not a smooth one for may children and researchers have tried to identify the source of their difficulty. Key areas in which students' misunderstanding impedes their progress involve the abstractions needed to work with operations, use of the equal sign and operational laws, and the concept of variable. Students who have learned computation as a series of steps often lack the abstractions needed to deal with operations. In elementary school arithmetic their use of the equal sign was often as a prompt to 'do something' (Kieran, 1981) rather than to represent a relationship between two quantities.

Students' misconceptions about variables as they enter middle school also stem from earlier experiences. In their study of sixth- and seventh-graders' conceptions of variable, Weinberg, Stephens, McNeil, Krill, Knuth, & Alibali (2004) found sixth-grade students exhibited a wide range of responses to questions involving variables, with frequent errors. One way students mistakenly think about a variable is that it must represent only a single-digit number, thus creating a one-to-one correspondence between the number of digits and the number of letters. Students' experience with arithmetic representations of verbal text often creates confusion in using algebraic notation to represent verbal text. For example, "Five more than 20" expressed as 25 may lead students to represent "10 more than h" as $10h$ (Weinberg, et al., 2004).

Measure Up Program

The research reported here is part of *Measure Up* (MU), a research and development project currently underway at the Curriculum Research & Development Group University Laboratory School of the University of Hawai⍰i. The aims of the project are: (1) to create a researched-based mathematics curriculum that is developmentally appropriate for children in grades 1 to 5 and (2) to assess the impact of that mathematics curriculum in promoting algebraic reasoning on students' later encounters with more formal algebra courses. The MU program is built on a theoretical framework connecting instruction and cognitive development proposed by D. B. Elkonin and V. V. Davydov (1966). Their research demonstrated that children, from the early grades, could engage in algebraic reasoning by solving problems involving quantitative relationships.

In MU, relationships between and among quantities are the bases for introducing variable. Children start their mathematics learning by exploring relationships of generalized quantities. They work with the measurable attributes of real objects (length, area, volume and mass) making comparisons that enable them to build a basic understanding of equality and non-equality and their properties. To describe these relationships without number, children use terms—shorter, longer, heavier, lighter, more than, less than, and equal to. This leads to representing the comparisons with relational statements, like $K > H$, that use letters to represent the quantities being compared. The statement is read so that the quantity is emphasized not the object. If they are comparing the volumes of two objects, students read the statement $K > H$ as, "volume K is greater than volume H." First-grade students are able to read and write these relational statements using letters and understand what they mean because the statements describe the results of physical actions that the children themselves have performed in the comparison process.

In subsequent experiences, letters representing quantities lead to the introduction of number. In MU, number is the expression of the relation between a chosen unit and a larger quantity. Children use letters in a variety of statements to represent the results of measuring a given quantity with a chosen unit. The following all express that when quantity B is measured using unit E the result is 4.

$$\frac{B}{E} = 4 \qquad\qquad 4E = B \qquad E \xrightarrow{4} B$$

These first uses of letters express generalized rather than specific quantities; they do not represent either a single unknown (number) or generalized numbers. When number lines are introduced, children begin to work with variables. Their experience with length helps them to use the number line in a way that has a deeper meaning for them. For example, the comparison of 7 and 3 is thought of as comparing the number of length units each is from 0. Since 7 is a greater distance from 0 than 3, $7 > 3$. This measurement context enables children to make generalizations even if specific numbers are not used. If any point is selected on the number line, let's call it n, then n

+ 1 would be found one unit to the right. If n is any point on the number line, then n – 2 would be found two units to the left of n. Students use expressions with variables to compare two quantities, for example, $n + 3$ and $n - 1$, without knowing specifically what n represents. In this case, students would think $n + 3 > n - 1$ because $n + 3$ would be further from 0, than $n - 1$, provided n is greater than or equal to 0.

When solving worded problems, particularly one in which the syntax can cause confusion, MU students' experience with variables helps them choose an equation to represent the problem situation. Consider the following problem:

Malia had some books on her shelf. She gave 5 books to Tanya. Now Malia has 8 books. How many books did Malia have in the beginning?

To solve this problem, MU students may first write a fact team, a set of four equations, either two addition and two subtraction or two multiplication and two division, in which the terms representing the whole and the parts are the same.

For example, using x as the whole and 8 and 5 as the parts, we can make an addition/subtraction fact team:

a) $x - 8 = 5$ b) $x - 5 = 8$

c) $x = 8 + 5$ d) $x = 5 + 8$

Any of the four equations will lead to a solution. However, students will usually choose the equation that matches their preferred method for solving the problem. For example, students who prefer to use an equation that most literally follows the chronology of the words, might choose equation b. Students who recognize that working backwards leads to an addition equation, would most likely choose equation d but might also choose equation c.

While in the above worded problem the variable represents a single unknown, some worded problems use variables as generalized unknowns (Küchemann, 1978). In the following problem students must operate on the variables without any notion of what numbers they might represent.

David and Jason are cooking v mass units of rice in one pot. Jason put k mass units into the pot. How many mass units of rice did David put into the pot?

Given this emphasis on the structure of mathematics through measurement and algebra students should build a foundation in grades 1 to 5 that leads them to be better prepared to deal with the level of generalization demanded by a study of formal algebra.

METHOD

Instrument

The Chelsea Diagnostic Mathematics Test: Algebra (CDMT1) (Hart, Brown, Kerslake, Küchemann & Ruddock, 1985) was administered to assess students' understanding of variable preparedness for learning Algebra I concepts. The two criteria used in designing the assessment are the structural complexity of the items and the meaning that can be given to the letter representations. Levels of difficulty are assigned to each problem and used in determining students' readiness to study algebra. At levels 1 and 2 students may be capable of using letters with assigned values or as objects but not as specific unknowns, whereas at levels 3 and 4 students are able to use letters as specific unknowns. The within pair differences of the levels relies on the complexity of the problem structure. The following are examples from the range of assessment items on the CDMT1:

> Level 1:
>
> 5. a. If $a + b = 43$
>
> then $a + b + 2 =$ _______.
>
> Level 4:
>
> 17. b. When are the following true – always, never, or sometimes?
>
> *Underline the correct answer.*
>
> $L + M + N = L + P + N$ Always Never Sometimes (when), ______

(Hart, et al., 1985, p. 106 &109)

Although the CDMT1 was developed with 13–15 year olds who typically had more mathematics instruction than the 10–11 year olds in our study, we believed the test would still be a valid measure of how students interpret the use of letter representations.

Participants

Fifth- and sixth- grade students at the University Laboratory School (ULS) participated in the study. They are representative of the larger student population in the state, including students identified as having special needs. Students at ULS are chosen through a stratified random sampling method based on achievement (where applicable), ethnicity, and social economic status (SES).

The CDMT1 was administered at a time when the fifth-grade students (10) were in their last year of MU. Nine of them had been in MU since grade 1 and one student was newly admitted in grade 5. Of the 50 sixth graders, the nine students who had had MU in grades 1–5 were present on the day the test was given. All sixth grade students had begun a middle school mathematics program *Reshaping Mathematics for Understanding* (Slovin, Venenciano, Ishihara, & Beppu, 2003) that emphasizes motion geometry and the NCTM *Principles and Standards for School Mathematics* (2000). There were 19 MU students and 41 students without any MU learning experience, $N = 60$. Approximately half of the sample was female.

Variables

The dependent variable is the level of achievement on the CDMT1 (scores were measured by levels 1–4) and the primary independent variable of interest is MU program experience, measured by the number of years a student had been in the program. Other independent variables in the analysis are previous Stanford Achievement Test 9 mathematics and verbal scores, as reported on the Hawaii State Assessment; grade level; SES; and gender. The data were retrieved from the school admission school files. SES codes were based on parents' education level and occupation, as reported on a student's school application form, and determined by the school administration. SES and gender were treated as categorical variables.

ANALYSIS AND RESULTS

Descriptive statistics seem to indicate that the group who had no MU experience had higher mean scores on the SAT9 measures than the MU students, however, the MU students had a higher mean score on the CDMT1 (Table 1).

	No MU experience, N=41	MU students, N=19
Mean SAT9-math (SD)	7.0 (1.4)	6.4 (1.3)
Mean SAT9-verbal (SD)	6.7 (1.5)	5.6 (1.4)
Mean CDMT1 level (SD)	1.1 (0.5)	1.6 (0.8)

Table 1. Means and standard deviations of variables

A multiple regression model building technique was used. The maximum R^2 improvement (MAX–R) technique selects the variables with the most explanatory power and includes them first. Thus in Step 1 the model was built using the one variable with the highest R^2 and in Step 2 it used the top two variables.

Step 1 of the regression model building process selected the MU variable as the best 1-variable model found, R^2 = 0.14, F (1, 58) = 9.42, p = .0033. Step 2 selected the MU and SAT9-math score variables as the best 2-variable model found, R^2 = 0.26, F(2, 57) = 10.02, p < .0002, with both variables significant at p < 0.001.

In a post-hoc analysis, gender, grade level, and SAT9-verbal scores were controlled for and MU experience was tested for its explanatory power of achievement on the CDMT1. The results verified a significant positive MU effect. Gender, grade level and SAT9-verbal produced R^2 = 0.05, F (3, 56) = 1.07, p = 0.37 whereas the same three variables with in addition of the MU variable produced R^2 = 0.20, F (4, 55) = 3.4, p = .00148. The inclusion of the MU variable created a significant model that gave a better prediction of achievement on the CDMT1.

Discussion

Although the sample size was relatively small, this analysis suggests an impact of the MU mathematics program on students' ability to work with variables. *Measure Up* mathematics seems to provide students experiences that will prepare them to understand representations of mathematical generalizations necessary to begin a study of algebra I concepts.

In further analyzing the results, we were able to identify a sub-set of items where MU students performed disproportionately better than students who had not had MU experience, regardless of grade level. All of these items were at level 2 or above in difficulty, and the ratio of MU students who responded correctly on a particular item to the total number of students who responded correctly on the item ranged from .58 to .89.

This additional finding was important to us as curriculum researchers. In addition to the evidence indicating a potential impact of the MU program experience, we sought to trace explanatory paths between the mathematics content of the MU curriculum and the sub-set of Chelsea test items. We conducted post-assessment interviews with three fifth- and three sixth-grade students, randomly selected to represent low, middle and high achievement profiles.

Three Chelsea items from the sub-set comprised the interview. (See Table 2) With each item, students were given copies of their written assessment and asked whether they agreed or disagreed with their original response and to explain why. The interviewer also cautioned students that the selection of items had nothing to do with the correctness of the response.

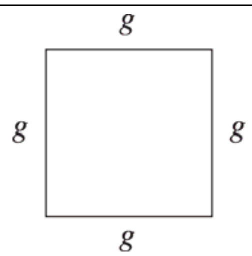
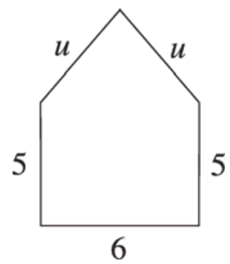
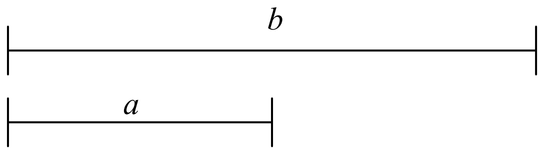
9. This square has sides of length g. So for its perimeter, we can write $p = 4g$. What can we write for the perimeter of each of these shapes? **c.** p = ____________	 
12. $a + 3a$ can be written more simply as $4a$. Write these more simply, where possible: **a.** $2a + 5a$ ___________ **b.** $2a + 5b$ ___________	

Table 2. Sample items from Chelsea Diagnostic Mathematics Test: Algebra

A preliminary analysis of student explanations reveals some patterns both with respect to students' incorrect and correct explanations. On item 9.c., which none of the fifth-graders answered correctly and two of the sixth-graders answered correctly, students failed to combine the numbers or combined the 2 fives but kept the side of six a separate term, indicating that the equation needed to represent the sides added to get the perimeter.

On items 12.a. and 12.b. all six students interviewed made some reference to the letters as representations of units. Not unexpectedly, all students answered 12.a. correctly either on the written assessment or in the interview. Brooke's explanation was typical of the reasoning, "It's like two units of *a*, if you're using units, and then add five [units of *a*] and you count them all." (BK student interview, 1/09/08) Only one student answered item 12.b. correctly on the written assessment while two others knew they could not combine *a* and *b* but had not realized until the interview that they could say a more simplified expression was not possible. However, the more significant pattern was that all students recognized that the two terms in the expression were different, that is, they could not be handled the same way as the terms in 12.a. Justin Y. referred to *a* and *b* as "different groups," and Stephanie noted that 12.a. was easier because the same unit was used in both parts. "In 12b you have to find *b,* then you can add that unit to the *a* unit. You have to find out how many *b*'s go into *a* or *a*'s go into *b*." (SR interview, 1/09/08) Alicia explained there was no way to write the expression more simply but also suggested conditions under which she could work with different units. She sketched two line segments as she spoke:



"If $2a = 1b$, then you could say it was $12a$ if *b* was twice the size of *a*….Without that information, you don't know how to change it." (AW, interview 1/09/08)

CONCLUSIONS

Qualitative data collected over the six years of the MU project have suggested children's strong understanding of algebraic concepts. Evidence for this comes from classroom observations and interviews with students. These quantitative findings corroborate the previous results. Additionally, the explanations students gave in post-assessment interviews indicate that the concept of unit and experiences with using units in measurement situations may play a role in developing student understanding of variable that warrants further investigation.

A limitation of the study is the small sample size. Until MU is disseminated to a wider population we will continue to administer the CDMT1 as ULS students exit the MU program. We are also expanding the study by using alternate forms of assessment and by continuing to track the progress of the MU students as they progress through the middle school program and enter an algebra I program.

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STUDENT LEARNING OF BASIS IN LINEAR ALGEBRA

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One of the earlier, challenging concepts in linear algebra at university is that of basis. Students are taught procedurally how to find a basis for a subspace using matrix manipulation, but may struggle with understanding the construct of basis, making further progress harder. In this research we sought to apply an APOS theory framework, in the context of Tall's three worlds of mathematics, to the learning of the concept of basis by a group of second year university students. The results suggest that an emphasis on matrix processes may not help students understand the concept, and embodied, visual ideas that could be valuable were usually lacking.

BACKGROUND

Over the past three decades, a number of researchers have considered difficulties related to first year university linear algebra courses (see e.g., Harel, 1997; Carlson, 1997; Hillel, 2000). This paper is concerned with student understanding of the fundamental linear algebra concept of basis, a construct describing a set of vectors that are linearly independent and span a specific (sub)space. It is relatively easy to present students with a method for finding a basis for the column space or nullspace of a set of vectors in a matrix *A* by reducing the matrix to echelon form, identifying the pivot columns and taking the corresponding columns of the original matrix, which are linearly independent, to form a basis for Col *A*, or those without a pivot for Nul *A*. Students often develop coping strategies to get through courses but are reluctant to address mathematical understanding and this failure has a compounding effect (Carlson, 1997).

This research used an action-process-object-schema (APOS) framework for the development of learning proposed by Dubinsky and others (Dubinsky & McDonald, 2001). It suggests an approach for basis different from the definition-theorem-proof and matrix calculation emphasis common in advanced mathematics; instead concepts are described in terms of a *genetic decomposition* (GD), comprising actions, process and objects in the order the learner should experience them. For example, students should not be presented with the concept of basis if they do not understand span, since basis is constructed from span and linear independence, each of which must be understood first. The framework for basis employed (see Figure 1) is built on APOS theory and Tall's embodied, symbolic and formal worlds of mathematics (Tall, 2007), with the aim of promoting versatility of mathematical thinking (Thomas, 2006). Basis, like many linear algebra concepts has embodied and symbolic forms; in fact several representations (Hillel, 2000). Thus in $\mathbb{R}^3$ it may be considered in a visual, embodied manner as a set of three non-coplanar vectors, symbolically as the column vectors of a row-reduced matrix with three pivots or v_1, v_2, v_3, or by a formal definition as a linearly independent set of vectors that span $\mathbb{R}^3$.

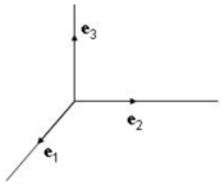
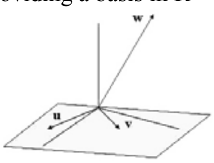
APOS	Embodied World	Symbolic World		Formal World
		Algebra	Matrix	
Action	See 3 specific non-coplanar vectors as a basis of R^3  e.g., The standard basis for R^3	Can find a basis for S, where S is the two-dimensional subspace of R^3 satisfying an equation e.g. $x+2y-z=0$. Can find the NS(A) which we call the general solution of the system $Ax=0$.	Can find a basis for the vector space in R^3 spanned by the vectors e.g. $S=\left\{\begin{bmatrix}1\\5\\9\end{bmatrix},\begin{bmatrix}2\\6\\10\end{bmatrix},\begin{bmatrix}3\\7\\11\end{bmatrix},\begin{bmatrix}4\\8\\12\end{bmatrix}\right\}$ Find the nullspace of a specific matrix A, and find a basis for NS(A). Find a basis for CS(A).	
Process	Can picture general orth-ogonal and orthonormal bases. Can see certain transformations (e.g, rotation, reflection) of a basis as also providing a basis in R^3 	Can describe a basis for any vector space in R^n.	Can generalise the method for finding a basis for Col A or Nul A to describe the resulting bases. Can find a basis for any vector space in R^n	Understands that linear independence ensures that there are not too many vectors in a basis, and spanning ensures that there are not too few.
Object	Can operate on a basis, with certain transformations (e.g, rotation, reflection) to provide another basis for R^3	A set of vectors $\{v_1, v_2,\ldots, v_n\}$ form a basis for all of R^n because they are linearly independent and span R^n	See the columns of an invertible $n\times n$ matrix forming a basis for all of R^n because they are linearly independent and span R^n	Understanding the definition of a basis: A basis for a subspace H of R^n is a linearly independent set in H that spans H.

Figure 1. A framework for the linear algebra concept of basis.

However, it seems that in linear algebra lectures at university students are often not given the time and opportunities to develop embodied notions of fundamental ideas, such as basis, because they may be considered too trivial, and this may lead to problems later as these concepts are built upon. This research sought to examine the hypothesis that students may be working with the concept of basis without developing aspects of the construct that could be very useful to them later.

METHOD

The research comprised a case study of two groups (A and B) of second year students from Auckland University studying a general mathematics course (40% linear algebra and 60% calculus) that is a prerequisite for commerce and economics courses, and is recommended for students with a less strong mathematics background. Although, there was no intention to make comparisons between the two groups of students, they were taught with different styles of teaching, one by the first author, emphasising embodiment and linking of the concepts (Group A), and the other with an emphasis on definitions and matrices (Group B). There were 16 students in Group A and 11 in Group B who volunteered to participate in the study. The lectures for Group A were designed around the proposed framework (Figure 1) to give students an overall experience of the concepts in the embodied, symbolic and formal worlds of mathematics. At the end of the linear algebra lessons students were given a set of 14 questions on a variety of concepts in linear algebra, which was designed to examine their embodied, symbolic and formal understanding, rather than procedural abilities (see examples in Figure 2). In addition, 8 students from group A and two from Group B were interviewed. Group B students, who did the same course in the previous semester with different lecturers, were given the same test 4 days before their final examination.

3. Describe the following terms in your own words.
Linear combination; Span; Linearly independent; Basis; Subspace

(b) Use your explanations in part (a) to design a concept map of the concepts. (You may also make use of the concepts of vector and scalar multiple, or any other appropriate ideas.)

14. Let $\mathbf{v_1} = \begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix}, \mathbf{v_1} = \begin{bmatrix} -3 \\ 5 \\ 7 \end{bmatrix}, \mathbf{v_1} = \begin{bmatrix} -4 \\ 5 \\ 6 \end{bmatrix}$, and $H = \text{Span}\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$. Note that $\mathbf{v_3} = 5\mathbf{v_1} + 3\mathbf{v_2}$, and show that $\text{Span}\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\} = \text{Span}\,\{\mathbf{v_1}, \mathbf{v_2}\}$. Then find a basis for the subspace H.

Some formatting has been changed.

Figure 2. The two test questions discussed here.

After the first test the first author offered group B students two tutorials on the central concepts of their linear algebra course (linear combinations, span, linear independence, basis and so on) based on the framework. A recent PhD graduate in mathematics also answered the test in order to generate answers likely based on formal world thinking for comparison purposes.

RESULTS

The first part of Question 14 proved to be difficult for most students, and those who did attempt it may have considered it 'obvious' and hence did not really know what to write, e.g. A4 (see Figure 3) and A9.

Span$\{v_1, v_2, v_3\}$ ~~Span~~ is the collection of all possible linear combinations of v_1, v_2, v_3. ~~v3~~ Span$\{v_1, v_2, 5v_1+3v_2\}$ is the same as $\{v_1, v_2\}$ because any linear combination $5v_1+3v_2$ form with v_1 and v_2 is already in the span $\{v_1, v_2\}$.

$\{v_1, v_2\}$ $\{v_1, v_3\}$ $\{v_2, v_3\}$

Figure 3. Student A4's answer to Q14.

The second part of the question, finding a basis for the subspace H, should follow immediately for those with a good understanding of the two criteria for basis, namely a set of spanning, linearly independent vectors. Certainly the PhD graduate immediately wrote "Basis for H = $\{v_1, v_2\}$" and listed in brackets two other possibilities $\{v_2, v_3\}$ and $\{v_1, v_3\}$. There was a facility of 37.0% for finding a correct basis in this part of the question, but the only other student who wrote down all three immediately was A4 (Figure 3). However, students A3, A9, A16 and B5 were also able to write down the basis as $\{v_1, v_2\}$ without any working, as Figure 4 shows. Of the remaining students, 9 from group A and 8 from Group B chose to row reduce the matrix comprising $[v_1, v_2, v_3]$. Of these only one student, A1, used the reduced matrix to try to write down a basis from it, but giving $\{(1, 0, 0), (–3, 1, 0)\}$.

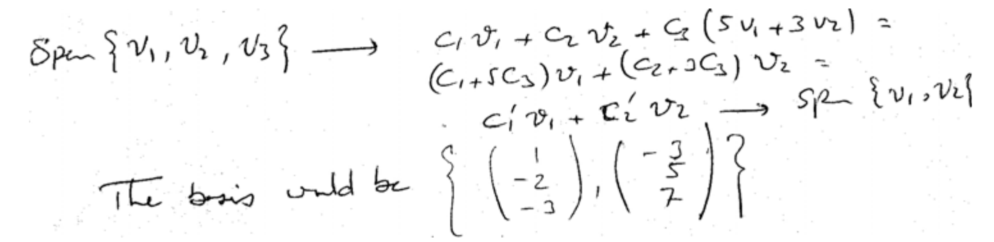


Figure 4. Student A16’s answer to Q14.

Students A2, A12, A15, B6, and B7 all appear to have used the reduction only to check for pivot columns and hence possibly the independence of v_1 and v_2, since they did not seem to use it in any other way. The working of two of these is shown in Figure 5, where A15 circles the pivot columns. Student A15 was the only one in the whole group who attempted to include any visual, embodied explanation here, probably using a mental image of orthonormal unit vectors. Unfortunately, the imagery does not incorporate the idea that v_3 is a linear combination of v_1 and v_2, and hence his conclusion that “$\therefore$The plane will be contained by the cube” is incorrect, as the inclusion goes both ways in this case.

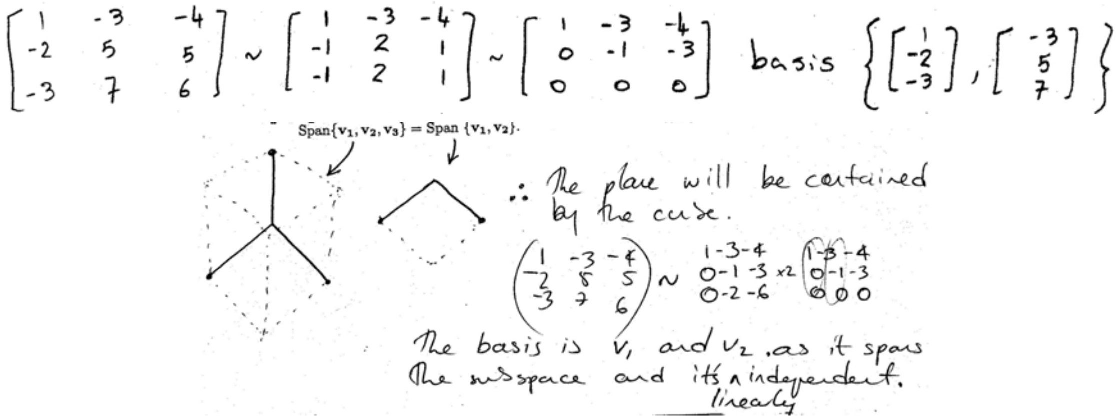


Figure 5. Student A2 (top) and A15’s answers to Q14.

However, 9 (33.3%) of the students appear to have carried out the procedure with little understanding of its relevance to the question. Students A6, A7, B2, B3, B8 and B11 all used the reduced matrix with back substitution to try and solve the equation Ax=0, where A is the matrix [v_1, v_2, v_3], hence finding a basis for the nullspace, but without mentioning the concept of nullspace. In Figure 6, for example we see the working of B8, who, along with B10 arrived at the answer {(–5, –3, 1)} for the nullspace basis, based on their solution of Ax=0. We can also see some of the confusion of B8, who seems to know from his working leading to ‘two pivots’ that two vectors are required to span the subspace H, but is then happy to have a basis with a single vector. A sizeable minority of the students carried out the reduction procedure with little apparent knowledge of its relevance, so we conclude that there was a tendency to employ a known matrix procedure of some relevance to basis, but that this was coupled with a lack of understanding of how to find the basis from it.

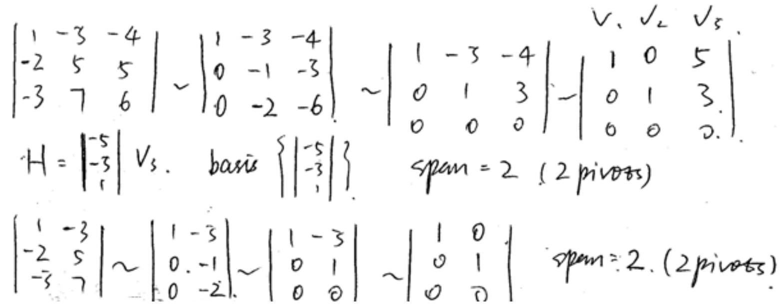


Figure 6. Student B8's answer to Q14.

Confirmation of this arose from the interview comments, with B8, for example, stating a predisposition to calculate: "When we read the question, I think I answer depends on what you asked. If you ask a basis I will try to solve it. If you ask can use the span of subspace I will try to figure out the relationship. I don't have to solve it."

It is informative to compare these results with what students thought the concept of basis was. In Question 3(a)(iv) we asked them to describe basis in their own words, and this was repeated in the interviews. Eleven of the Group A students correctly identified the two defining characteristics of a basis, giving the linear independence and spanning criteria (see example in Figure 5), but none of the Group B students was able to do so. In contrast to the 42.3% of correct answers, 4 Group A and 8 Group B (a total of 46.2%) did not give any response, likely because they were unable to (for example many left this part blank but completed other parts of the same question). Two students mentioned only the spanning condition (A2, B2) and two only the linear independence (B8 and B10), while three students referred to the result of calculations. The interviews probed a little further into student thinking on basis, by asking about their understanding of it.

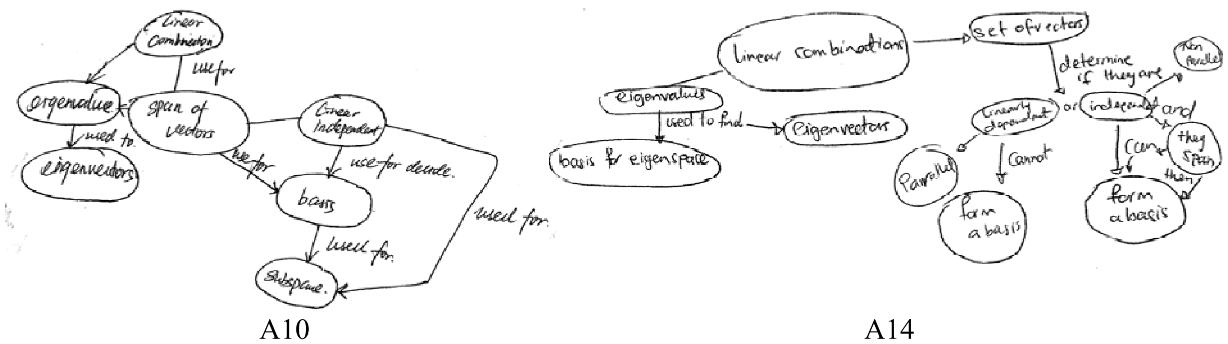
Student	Test response	Interview response
A5	No answer	This is the hardest for me to explain. For my understanding the basis is that the very basic vector that can span the space. So for example, if we get vector and we got echelon form…then those two vectors are the basic ones. So if you times the scalar you will get many of the vector that span the space.
A10	Correct; span and lin. indep.	How can you form a basis? That is the clearly one like two limitations, one is they must be linearly independent and second one it must be.. like form into like the rows like 3 vectors.. must be, have enough, to form into R^3 or R^4, something like that.
A13	No answer	I don't have a great understanding of a basis. Basically I know how to figure it out. It has to span; it has to be independent, but I couldn't really give you the definition of basis.
B10	No answer (1) and Lin indep only (2)	I remember basis also dependent on the picture of span because my understanding of span, span are not related with linearly independent, …but basis is something like span and those vectors not be linearly independent, that's my understanding of basis.

Table 1. A comparison of responses to the definition of basis

Table 1 shows how some of their responses relate to what they wrote in the test. The interviews confirmed the difficulty that students A5 and A13 were having with the concept of basis. A5 has only the span criterion, while A13 has both, but is unaware that these are both necessary and sufficient for the concept. This again shows the concept is not firmly established. B10 appears to contradict his test answer by saying that "basis is something like span and those vectors not be linearly independent", and again he is confused. A10, who answered correctly in the test gives the first criterion but then, instead of speaking of span, reverts to the symbolic matrix world to talk about "rows like 3 vectors". Later he spoke of how "…3 vectors can span subspace of a basis and you are talking about, like, subspace of a basis. I just I cannot easily link to each other", so he was not that clear, as he admits, and the expression 'subspace of a basis' indicates. Phrases like this, and, for example, 'the basis of the span' used by B8 show problems linking the concepts.

Concept maps

Seven of the students in Group B did not draw anything at all when asked for a concept map of the concepts in Q3(a), 3 drew a procedural map and 1 a more conceptual one, with a few verbs. A notable difference between the concept maps of the students was the linking of both span and linear independence with basis. Ten Group A and two Group B students drew this, and Figure 8 shows the maps of two of the Group A students. In contrast Figure 9 contains two examples of the five concept maps where either span (B10) or linear independence (A5) was not linked to basis.



A10 A14

Figure 8. Concept maps of basis with span and linear independence.

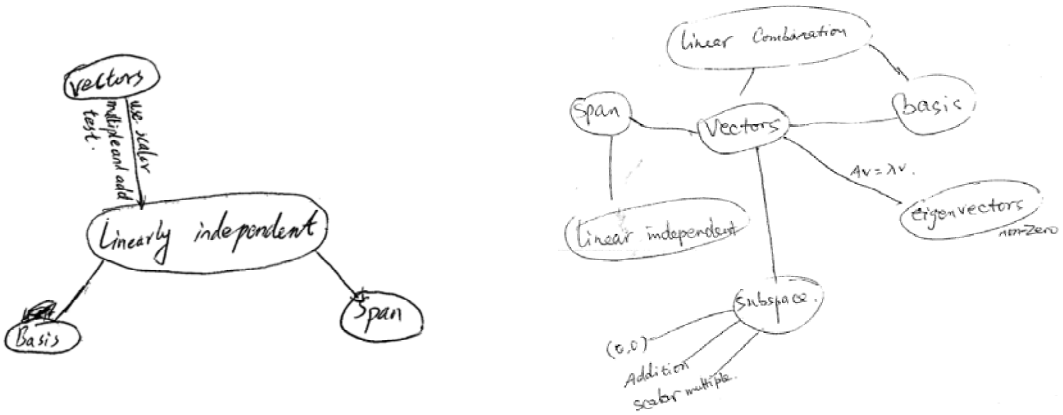


Figure 9. Concept maps of B10 and A5, showing missing links for basis.

CONCLUSIONS

In order to construct the concept of basis students need to build on a number of previous concepts. A GD (e.g. Czarnocha, Loch, Prabhu, & Vidakovic, 2001) of basis requires a combination of GD's of span and linear independence of vectors, a link that the students in this study often didn't make. The first GD includes a process conception of linear combination (the ability to generalise the addition of scalar multiples of any set of more than two vectors), an action view of span (being able to construct the set of all linear combinations of a given set of two or more vectors), a process view of span (able to generalise to the set of all linear combinations of a set of two or more vectors), an object perspective of span (seeing the set of all linear combinations of a set of two or more vectors as an object, such as a plane). This is difficult enough in itself, as has been previously shown (Stewart & Thomas, 2007a). However, the second GD, again based on linear combination, comprises an action view (being able to rearrange an equation to write one vector in a set of dependent vectors as a linear combination of the others), a process conception (generalising to see one can write any vector in a linearly dependent set as a linear combination of the others), and an object view (of a set of linearly independent vectors, v_i, as an entity where each can not be written as a linear combination of the others). This has also been shown to be a difficult concept for students, even more so than span (Stewart & Thomas, 2007b). Thus the common ground of these two GD's is the concept of a linear combination, and it appears that more teaching time could be spent establishing this key idea, agreeing with the PhD graduate's central concept map placement (Figure 10).

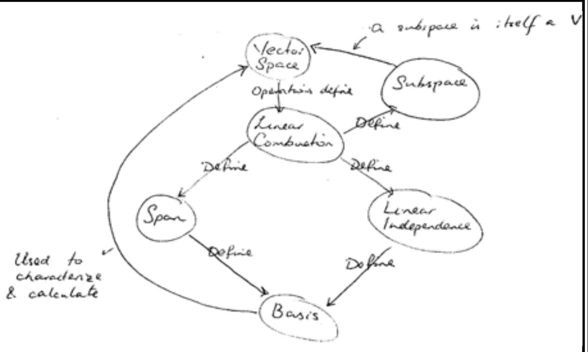


Figure 10. The PhD's concept map.

We have seen that a number of the students tended to prefer to work procedurally with symbolic world matrix representations. This is not too surprising since it is easier than grappling with the formal world ideas, and is the method emphasised in the course. For example, the course notes speak about how *to find* "a basis for the Nullspace $NS(A)$ of an $m \times n$ matrix A", "a basis for the column space $CS(A)$ of a matrix", and "the span of a set of vectors $v_1, v_2, \ldots v_n$, [by forming] the matrix $A = [v_1, v_2, \ldots v_n]$ with these vectors as its columns", and gives a symbolic, matrix method for each. In contrast the only mention of basis apart from matrices describes it in terms of span: "The span of any set of vectors…forms a subspace, W say. If these vectors are linearly independent, they form a basis for W". However, we found that students who thought that they should row reduce a matrix often did not know why, or what to do with the result, showing there may have been an overemphasis on matrix operations to the detriment of ideas. While we believe that an embodied perspective is highly valuable, and students can 'see' the span and subspace as geometric objects, basis, a set of vectors, can also be exemplified geometrically. Moreover lack of uniqueness, with an infinite number of possibilities, including isometries, as well as orthogonal/orthonormal bases, adds a layer of complication, especially for students who have come to expect a solution to be, more or less, unique. We found little use of an embodied perspective for basis, and when it was used it was in error.

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REPRESENTATION AS A COGNITIVE AND SOCIAL PRACTICE

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Representation has been increasingly viewed as central to mathematical activity. Yet, it is becoming obvious that students are having difficulty negotiating the various forms and functions of representations. The purpose of this paper is to draw on the literature to synthesize these various components in one scheme, and briefly discuss some implications for the teaching of representation. In the next four sections, I first present a brief background on representation, and then I use the two distinct lenses of individual cognition and social activity for looking at representation in mathematics. The two imply different roles for representation, but I find them to be complimentary as is explicated below. I conclude with some pedagogical suggestions as a result of this classification.

In the past decade representation has been increasingly viewed as a “useful tool for communicating both information and understanding” (NCTM, 2000). This awareness that representation is central to mathematics is clearly reflected in recent reform efforts and it is particularly obvious in the position of the National Council of Teachers of Mathematics [NCTM] who in their standards document elevated the status of representation to one of five process standards. It is now recommended that students “create and use representations to organize, record, and communicate mathematical ideas; select, apply, and translate among mathematical representations to solve problems; and use representations to model and interpret phenomena" (p.64). It is advocated that students should be fluent users of representations and instruction should support students in learning how to navigate mathematical concepts and problem solving through the use of a variety of representations.

But, despite these recommendations, it is becoming clear that students are having difficulty negotiating the various forms and functions of representations. Often students are asked to use multiple representations, but it is not clear to them whether all representations are needed during the entire problem solving process or what distinct roles different representations may play in this process. Similarly, for teachers, the use of multiple, conceptually-based representations is a new dimension to the teaching of mathematics and many have few instructional tools at their disposal to facilitate their students’ development of this practice.

The question we need to consider is what functions does representation have in mathematics, which can potentially be utilized in the mathematics classroom to make representation a more meaningful activity? In answering this question, we need to consider both the cognitive and social aspects of representation in mathematics activity. Indeed, current understanding of our ways of knowing recognizes a symbiosis in the social and cognitive domains of learning and an interaction between the two (e.g., Cobb & Yackel, 1995). Various research efforts from each of the two domains have identified some of the important functions of representation in mathematical activity. The purpose of this paper is to synthesize these various components, and briefly discuss some implications for the teaching of representation.

Background

A representation is a configuration that can represent something else (Goldin, 2002). For example, symbolic expressions, drawings, written words, graphical displays, numerals, diagrams are all *external representations* of mathematical concepts. And, as Goldin (2002) points, these examples illustrate that individual representations cannot be understood in isolation but, rather, belong to wider systems – representational systems. To put it more simply, representing is not only a matter of copying what one sees. Instead, it involves inventing or adapting conventions of a representational system for the purpose at hand (e.g., Grosslight, Unger & Smith, 1991). As such, "the term representation refers both to process and product – to the act of capturing a mathematical concept or relationship in some form and to the form itself" (NCTM, 2000). This dual nature of representation, though a common characteristic of mathematical constructs (Sfard, 1991), reflects, to some extent, the complexities associated with representations.

With respect to "representation as process", research in mathematics education at a theoretical level has suggested a number of processes for working with representations. For example, Janvier (1987) described translation – the cognitive process of moving among different representations of the same mathematical concept – as a way to better navigate one's way through problem solving.

With respect to "representation as product", empirical work has examined students' difficulties in understanding and using graphical representations (for a detailed review of this literature see Leinhardt, Zaslavsky, & Stein, 1990) and have pointed to a number of misconceptions and misunderstandings that characterize students' responses to the visual qualities of graphs. These difficulties of students are contrasted by evidence that individuals who are skilled in problem solving in fact rely on visual representations as tools that add information in this process (e.g., Ochs et al, 1994; Stylianou, 2002). Similar difficulties are found in students' attempts to use other types of conventional representations. One of the reasons underlying students' difficulties may be the fact that representations have often been taught and learned as if they were ends in themselves (Eisenberg & Dreyfus, 1994; Greeno & Hall, 1997; NCTM, 2000) – an approach that limits the power and utility of representations as tools for learning and doing mathematics.

The new emphasis on representation has brought to the surface the complexities of representation not only as an individual or cognitive practice as described above, but also as a social process, closely related to students' understanding of the concepts and situations being represented (Monk, 2003). From this social perspective, recent studies are beginning to address the complexities involved in negotiating individually constructed representations in the shared space of a group or a classroom as well as the teacher's role in facilitating these interactions (e.g., Silver et al., 2005). This work has emphasized the role of discourse and mechanisms, such as negotiation of meaning, by which taken-as-shared interpretations and uses of representation are established in classrooms (e.g., Cobb & Yackel, 1995; Sfard, 2000).

Clearly, students are having difficulty negotiating the various functions of representations. These difficulties suggest, in part, that the role that representation can play in the learning and problem-solving process is not clearly articulated. In the next two sections I aim to summarize these various roles of representation.

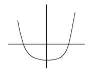
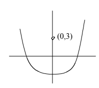
Representations and representing – as an individual cognitive activity

Representation is central to a person's understanding of a mathematical concept and a person's problem-solving activity. In particular, it has been argued that the use of a variety of representations in a flexible manner has the potential of making the learning of mathematics more meaningful and effective. Each representation has specific strengths, but it also has disadvantages, hence, their combined use can be a more effective tool by showing different facets of one mathematical idea (Cuoco, 2001; Kaput, 1992).

The ability to choose an appropriate representation of a mathematical concept and to "capitalize on the strengths of a given representation is an important component of understanding mathematical ideas" (Lesh et al, 1987, p. 56). Similarly, different representations may need to be constructed for specific purposes during the problem-solving process. A review of the literature on mathematical problem solving using the lens of representation suggests that representations can be used as tools to facilitate different subtasks of the process of problem solving. But, before discussing these different roles of representation in this process, it is useful perhaps to illustrate some of them with an example. Here, a mathematician was asked to solve a non-standard but elementary Calculus problem:

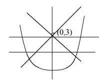
"Given the quadratic function $y=x^2-3$ and a family of linear functions $y=ax+3$, for which value of parameter a is the area of the region bounded by the parabola and by the line minimal?"

The figure below is a mathematician's protocol while solving the problem (detailed discussion of this task and the role of *visual* representations in problem solving can be found in Stylianou and Silver (2004)).

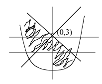
[reads] A quadratic function, x^2-3 and the linear functions, which value of the parameter α…" I will first draw a plot of the quadratic, […]okay, $y=x^2$ is a parabola which I will shift down 3 units, so that I will get a parabola that goes up like this…	*representation as a means to understand the problem* 
I have a family of linear functions whose graphs all go through this point, […] this is the point where coordinates uh, $x=0$ and $y=3$.	

> And… And as one varies the parameter α, one gets various lines, going through that point and an obvious candidate for the minimizer of the area is the horizontal line as such and one asks oneself as you tilt uh, this line a little bit by varying the parameter α, from $\alpha=0$ what happens to the area, […] is that you remove this bit here, that pie slice and you... Let's call this slice alpha. And you add on this other pie slice, beta, and it's, it's pretty clear that the pie slice beta is bigger than the pie slice alpha, so the area that gets added on above the horizontal line is bigger than the area that gets removed below the horizontal line, so you have in fact increased the area between the line and the parabola by varying the, the, by moving the parameter α from 0 to a negative number, and exactly the same thing happens when you move it from 0 to a positive number, so the answer is, that α should be 0.

representation as an exploration tool



representation as a recording tool



Representation as a monitoring/evaluating tool

The mathematician constructed a visual representation to understand the problem. Note that he constructed the visual representation in steps as he read the problem. By doing so, he reduced the cognitive load of trying to keep all the information in his mind. The representation on the paper was an extension of his working memory and allowed him to summarize the information provided – the representation is used as *a means to understand the problem*, its constraints and affordances.

Once he introduced a visual representation, he used it to explore the problem space. As he gradually varied the slope of the line, he focused his attention on the change of the area bounded by the parabola and the line, and conjectured that any deviation from the horizontal line added more area than the area it removed. Thus, when the line is horizontal, that is when $\alpha=0$, this area is minimal. Hence, the visual representation was used also as *an exploration tool*. However, he gradually shifted to a new representation; he introduced a symbolic notation (i.e., $\alpha=0$) that would better facilitate the second stage of his problem solving – the production of a deductive argument. He used this second representation mostly as *a recording tool* of the conjecture he had reached using the first representation. Finally, he returned to his initial representation and used it to evaluate the correctness of his conjecture, that is, used this representation (while also using language that implied that he combined the first and second representation) as an *evaluation or monitoring device*.

In a summary, the discussion above suggests four important roles for representation as part of individual problem solving activity. Each one of these roles has been identified in either theoretical or empirical work on problem solving or representation

- As a means to understand information – one uses the representation as a tool to combine different aspects of the problem and to see how these interact. This is an important function of representation, as, often, not all aspects of a problem situation are immediately obvious to the reader.

- As a recording tool – one uses a representation as a tool that combines all the information provided instead of trying to keep it "all in the mind". It provides a compact and efficient means to record these thoughts on an external medium.
- As tools that facilitate exploration of the concepts or problems at hand – one uses the representation as a flexible device that allows him to manipulate concepts and reveal further information and implications.
- As monitoring and evaluating devices to assess progress in problem solving. One may use a representation as a way to monitor her/his progress in problem solving and to make informed decisions when selecting subsequent goals and maintaining or revising current plans.

This discussion not only illustrates the different the roles representation but also how the meaning and utility of a representation can shift as problem solving purposes and difficulties change (Hall, 1989). Currently, common methods of teaching, do not tend to focus on these shifts. Rather, students are often instructed to represent a task in multiple ways, but are only instructed how to meaningfully manipulate the standard representational forms (usually symbols) and are encouraged to use these standard representations throughout their solutions. There is little discussion regarding a fluid use of representations in the problem-solving process and the advantages of the ability to translate (Janvier, 1987) among these different representations.

Representations and representing – as a social activity

Let us now view representation and the act of representing using a sociocultural perspective where learning is situated in everyday practice. Here, knowledge is seen as a construct of the social and cultural environment surrounding the individual, implying that the development of knowledge cannot be understood apart from the social context in which it occurs (Vygotsky, 1962). The learning of mathematics is viewed as situated in practice and in communities of practice. Consequently, as Greeno (1988) argues, "learning mathematics involves acquiring aspects of an intellectual practice, rather than just acquiring some information and skills" (p. 481). The critical element in this perspective is that learners acquire understanding of mathematics through their participation in mathematical practices – including the use of mathematical tools and taking part in mathematical discussions.

Representing as a mathematical practice takes meaning from context. Representations are used as tools to understand mathematical concepts and as means of communicating about these concepts. These uses of representations fit well with those found among mathematicians and scientist. For example, Hall and Stevens (1995) studied the activity of engineers working on design projects and highlighted the various roles different representations (sketches, graphs, sets of measurements) took on while the members of the design team communicated to each other different ideas and concerns they might have, and the central role these representations had during the negotiation of the final specifications of the proposed projects.

Representations as social activity serve two main purposes as proposed by Roth and McGinn's (1997): as rhetorical objects and as conscription devices.

Rhetorical objects in scientific communication; a representation can often be manipulated (e.g., in the case of a graph, one can change the scale) in a way that allows certain not-so-obvious patterns to emerge. In this case, the user of the representation makes choices during the presentation of her/his work in order to show an alternative perspective. For example, mathematicians often manipulate graphs for alternative perspectives or to allow new patterns to emerge.

Conscription devices mediating collective scientific activities (talking, or constructing facts). Here representations are central to interaction among scientists. They constitute a shared interactive space that facilitates communication, as these representations may be used as a common language tool. In the example presented by Hall and Stevens (1995), graphs and tables were used as a common communication code to explain reasoning and perspectives and to negotiate a plan.

This brings to the surface the role of the teacher as a facilitator of discussions which evolve around or make use of representations as rhetorical objects or as conscription devices. A "sharing" of solutions (in representations) does not suffice; the teacher needs to facilitate discussions that help students negotiate the meaning of various representations and advance common understandings.

Summary and pedagogical considerations

A summary related to the roles of representation described in the previous two sections is presented in the figure below:

Representations as Tools		
	How they help students	When they help students
Individual cognition		
• To understand information	Organize all given information	Initial stages of problem solving
• Recording tool	Reduce cognitive load	Throughout problem solving
• Facilitate exploration	Allow manipulation of given info	During exploration/analysis phase
• Monitoring devices	Detect wrong approaches	During self-assessment points
Social practice		
• Rhetorical objects	Present not-so-obvious perspectives and information	When reaching an impasse
• Conscription devices	Allow sharing of strategies and negotiation of new ideas	When looking to expand existing problem-solving repertoire

As shown in the figure above, a representation can take on different roles and its use depends on the purpose of that representation in particular. In other words, a representation may take on different roles depending on the context in which it is being used and on the user's needs at any given moment. One cannot classify one specific form of representations as ideal for serving one purpose – for example, either a visual representation or a symbolic representation may be used for exploration but one may be advantageous over the other for a specific problem. However, one needs to be aware of the purpose for which a representation is used at a given time.

Hence, representations should be presented and used in different ways. However, teachers often treat all representations alike. Teachers should be explicitly aware of the purposes of a representation they use and, further, should make the students explicitly aware of this. For example, it is important to make the students aware of the fact that certain notation is being used only as a way to summarize the information in the problem and make it readily accessible, while a diagram will be later on constructed for exploration; the compact character of the former and the flexible nature of the latter allow for these different uses.

Many of the examples of "multiple representations" found in textbooks, or used in instruction, do not communicate the distinct character and role of representation. Consider for example the presentation of linear functions in several algebra textbooks. Often the "big three" representations, that is, a table of values, a symbolic expression of the function, and a graph are offered on the same page without any explicit attempt to discuss the different uses each representation can afford. Similarly, when representations are used in classroom discussions, students need to be made aware of the role the representation can play in the discussion, and explicit ways in which the representation can help them advance their argument and their thinking.

In a summary, representations used in mathematics instruction often serve several purposes other than just presenting information in a new format. An awareness of the different roles that representations play and the ways the representation can be manipulated or used for the purpose at hand can de-mystify for students the request for use of multiple representations and can, ultimately, improve students' appreciation of the role of representation in mathematics.

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SCAFFOLDING SPECIAL NEEDS STUDENTS' LEARNING OF FRACTION EQUIVALENCE USING VIRTUAL MANIPULATIVES

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This collaborative action research project explored strategies for enhancing mathematics instruction for students with special needs using virtual manipulatives. Teachers and researchers employed qualitative action research methods using memos to systematically record the processes of planning, teaching, observing and reflecting. Results showed that teachers' opportunities for reflection and discussion influenced their instructional strategies. Teachers' observations of their students' learning indicated that affordances in the virtual manipulative applets enabled special needs learners to "off-load" information and focus more on mathematical processes and relationships among equivalent fractions.

THEORETICAL FRAMEWORK

Kaput (1992) noted that the impact of technological tools in mathematics learning and teaching is the ability to "off-load" routine tasks, such as computations, to compact information and providing greater efficiency in learning. More recently, Zbiek, Heid, Blume, and Dick (2007) highlighted affordances of technology tools in mathematical activity, based on their externalized and linked representations, dynamic actions, and built-in constraints. Pea (1987) defines cognitive technologies as "technologies that help transcend the limitation of the mind…in thinking, learning and problem solving activities" (p. 91). Technology tools in mathematics have the capability to graph, compute, visualize, simulate, and manipulate, while providing users with immediate feedback.

When students work with physical manipulatives, one major challenge is that the manipulation of multiple pieces creates an excessive cognitive load for learners. This causes them to lose sight of the mathematics concept that is the intent of the lesson and prevents them from connect the physical manipulations with mathematical ideas. Kaput's (1989) explanation for this disconnect was that the cognitive load imposed during the activities with physical manipulatives was too great, and that students struggled to maintain a record of their actions. Essentially, students are unable to track all of their actions with the physical manipulatives, and additionally, are not capable of connecting multiple actions with mathematical abstraction and symbol manipulation.

Recent research seems to indicate that the built in constraints in virtual manipulatives overcomes some of the limitations of physical manipulatives. For example, research using virtual manipulatives has shown benefits for ESL learners (Moyer, Niezgoda, & Stanley, 2005) and lower ability learners (Moyer & Suh, 2008). The topic of fraction equivalence was chosen for this project because it is an important prerequisite to understanding rational numbers and the addition, subtraction, multiplication and division of fractions. Developing visual models for fractions is crucial in building fraction understanding. Yet conventional instruction on fraction computation tends to be rule based. In particular, special needs learners often receive direct instruction on "how to" perform algorithmic procedures using mnemonic devices or steps to follow without having opportunities to construct conceptual understandings of mathematical processes. The research question that guided this collaborative action research was: How can teacher reflection and discussion be used to enhance instruction using virtual fraction applets with special needs students?

METHODOLOGY

Participants and Data Sources

The participants in this study were 19 fourth-grade students. Ten of the 19 were identified as special needs students and had Individual Education Plans (IEPs). Both a regular education teacher and a special education teacher worked collaboratively in this inclusive classroom. Students attended a Title One designated elementary school in a major metropolitan area with a diverse population of 600 students at the school: 51% Hispanics, 24% Asians, 16% Caucasians, 3% African Americans and 6% others, with over 50% receiving free and reduced lunch.

The qualitative data included teachers' and researchers' memos during the planning, teaching, observing and debriefing processes, students' written work, student interviews and classroom videotapes. Students' written work contained drawings, solution procedures, numeric notations and explanations. These qualitative data were examined and categorized along dimensions of students' strategies and sense making procedures. Student interviews, memos, and classroom videotapes were used to triangulate students' understanding of fraction equivalence.

Research Design

The study used qualitative methods of collaborative action research using memos. According to Miles and Hubermann, memos are essential techniques for qualitative analysis. They do not just report data; they are a powerful "sense-making" tool that ties together different pieces of data into a recognizable form (Miles & Hubermann, 1994, p. 72). Maxwell (1996) recommends regular writing of memos during qualitative analysis to facilitate analytic thinking, stimulate analytic insights, and capture one's thinking. Strauss and Corbin (1990) recommend that memo writing begins from the inception of a research project and continues until the final writing. To follow this design, researchers asked teachers to record their observations and reflections throughout the planning, teaching, observing and debriefing phases of the study. These memos were collected and analyzed for emerging themes.

Procedures

In the first phase of the study, teachers identified topics that were challenging to special needs learners based on previous state assessments. Based on these topics, teachers and researchers chose fraction equivalence as the focus of the study. To begin the collaborative planning, one researcher, a mathematics educator, engaged the classroom teachers and special educators in planning a lesson. This process involved three phases over two 3 hour sessions: 1) Collaborative planning phase, where novice, experienced, and special educators collaborated on planning the lesson; 2) Teaching and observation, where one teacher taught the lesson and the others observed and took notes; 3) Debriefing phase, where teachers reflected on the lesson design, tasks, student engagement and learning and discussed future steps.

Planning of the lesson

To begin the planning, teachers created a concept map that unpacked the concept of equivalent fractions. (See figure 1.)

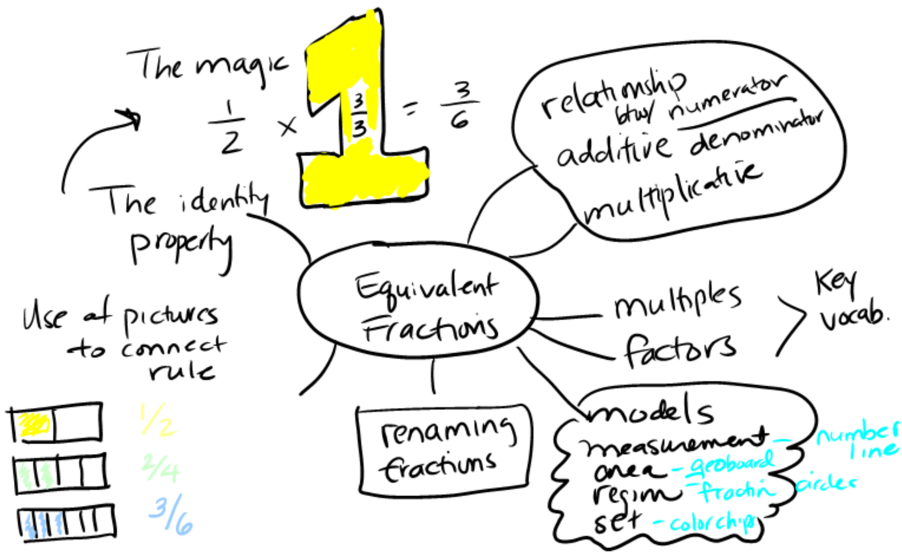

Figure 1. Concept map for fraction equivalence.

Some of the guiding questions crucial to the planning and teaching processes were:

- What is the important mathematical understanding that students need to learn?
- What are potential barriers and anticipated student responses?
- What conceptual supports and instructional strategies can best address our students' learning? How will we respond when students have difficulty?
- How will we know when each student has learned it?

Teachers took active notes and memos to self as they proceeded in the research.

Teaching of the lesson

Researchers used the virtual manipulative tool, Equivalent Fractions, from the National Library of Virtual Manipulatives (at http://matti.usu.edu). (See figure 2.)

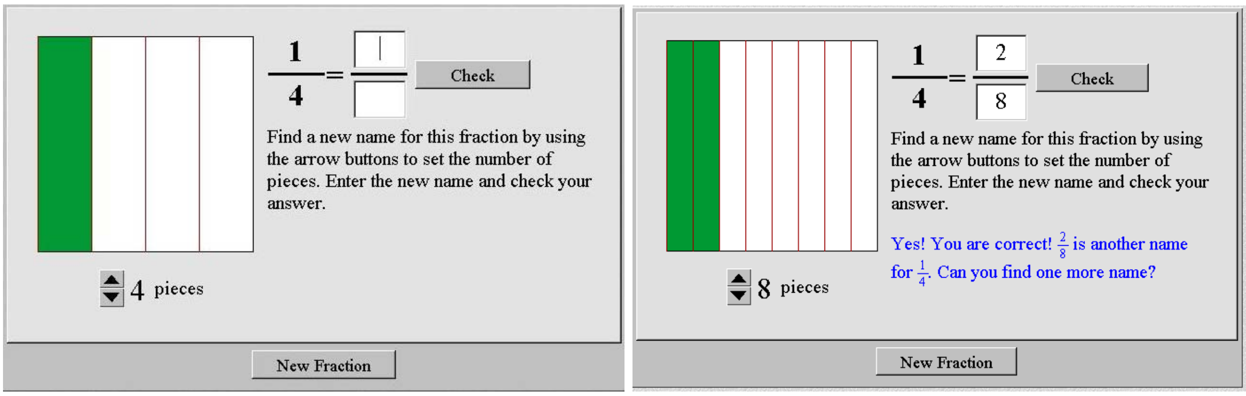

Figure 2. Virtual manipulative applet for fraction equivalence.

The objective for the 4th grade lesson on fractions was renaming fractions to find equivalent fractions. A subsequent lesson focused on adding and subtracting fractions with unlike denominators. The Fraction Equivalence applet allowed students to explore relationships among equivalent fractions. The applet presents students with a fraction region (circle or square) with parts shaded. The text on the applet directs students to: “Find a new name for this fraction by using the arrow buttons to set the number of pieces. Enter the new name and check your answer.” To do this, students click on arrow buttons below the region, which changes the number of parts. When students find an equivalent fraction, all lines turn red. Students then record a common denominator and corresponding numerator in the appropriate boxes, and check their answers by clicking the “check” button. Throughout this process, pictures are linked to numeric symbols that dynamically change with moves made by the student. To help students explore relationships among equivalent fractions, the applet prompts students to find several equivalent fractions. This applet was specifically designed to develop the concept of renaming fractions. Unlike physical manipulative fraction pieces, the virtual fraction applet allows students to equally divide a whole into 99 pieces, thereby generating multiple equivalent fraction names. In the final task, students were asked to create a rule for finding equivalent fractions.

RESULTS

Analysis of the Memos from Planning

In their memos on planning, teachers wrote that the concept map was extremely helpful in planning the lesson for the special needs students.

> Teacher 1: I have used concept maps to teach writing, but never thought of using them in mathematics. It helped me unpack the math and see the complex nature of fractions. It made me see how all of the other skills were linked to fraction equivalence. By unpacking the mathematics, I was able to see what skills I may

have to remediate with my special needs students in order for them to grasp this concept.

Another teacher was excited to see how the computer based virtual manipulative might give students more guidance with the concept.

Teacher 2: Using the virtual manipulatives as a learning support will be interesting to see, since many times, students get so distracted by the physical manipulatives. But with this virtual manipulative, students can't play around as much with the tools. They will have a chance to experiment without really goofing off.

A common theme from the memos was the importance of the mathematical discourse among teachers about sequencing the lesson, anticipating students' misconceptions, and preparing teaching strategies to overcome those misconceptions.

Teacher 3: Discussing the common misconceptions and errors that student made on previous assessments allowed for us to pinpoint what the problem was with our past teaching strategies.

Analysis of the Memos from Teaching and Observing

The teacher's role in extending students' thinking during this task was in engaging students to record a list of equivalent fractions, to determine patterns among the numbers, and to generate a rule. For example, using the applet on a SMARTBOARD, one student showed the class that 1/3=2/6=3/9=4/12. As this was recorded on the board, students' eyes started to widen and hands raised saying:

Student 1: Oh, oh, I know the rule! The denominators are going by a plus 3 pattern.

Student 2: It's like skip counting.

Student 3: It's the multiple of 3 like 3, 6, 9, 12…yeah.

As students shared their observations, some noticed the additive rule. To encourage students to explore the relationship more deeply, the teacher asked students to think about equivalent fractions for 2/3, so that they could see the multiplicative patterns in the numerators and the denominators. Students listed 2/3=4/6=6/9 and again they quickly saw the additive pattern and the multiples of two for the numerator and three for the denominator. Then the teacher posed the question: Are 2/3 and 20/30 equivalent fractions? What about 2/3 and 10/15?

Students used the fraction applet to explore this question with a partner. Other fractions were provided to encourage students to determine relationships beyond the additive rule. When students returned to the whole class discussion, several of them shared their discoveries. One student noted:

Student: The fractions 2/3 and 20/30 are equivalent because you multiply both numerator and denominator by ten. And in 2/3 = 10/15, you multiply both numerator and denominator by 5.

The teacher extended the previous student's comment by taking out a learning tool she called "the magic 1" and placed it next to the fraction 2/3. With an erasable marker, the teacher wrote 10/10 on the laminated "magic 1" next to 2/3 and said, Teacher: So are you saying that we are multiplying it by 10/10, otherwise know as 1? Turn to your partner and talk about this.

This discussion led to a lively conversation on how 10/10 and 5/5 equal one whole. The teacher connected this idea to the identity property of multiplication by asking:

Teacher: What happens when we multiply any number by one?

This discussion reinforced the idea that, no matter how you rename the fractions, as long as you multiply it by one or *n/n*, the result will be an equivalent fraction. To challenge the students, the teacher posed the question:

Teacher: What would the equivalent fraction be for 1/3 if the denominator was divided into 99 parts?"

This type of questioning encouraged students to extend their thinking by making conjectures and testing their rule or hypothesis.

Analysis of the Memos from the Debriefing

During the debriefing, teachers noted that students were able to generate a rule and stick with that rule to test other equivalent fractions. Among the different rules were:

1) The additive rule: Students tended to list a sequence of equivalent fractions:

1/2 = 2/4 = 3/6 = 4/8

Each denominator is increasing by 2s and the numerator is increasing by one.

2/3 = 4/6 = 6/9 = 8/12

Each denominator is increasing by 3s and the numerator is increasing by two

2) Doubling numerator and denominator

1/2= 2/4=4/8=8/16 : 2/3= 4/6= 8/12=16/24

Students doubled the numerator and denominator

Only a few students noticed the multiplicative relationship prior to the group discussion. Teachers noted, during the debriefing, that perhaps listing the fractions horizontally led students to seek a pattern going across the series of fractions leading to the additive rule. In a later class discussion, the teacher listed some of the equivalent factions in pairs to extend their discussion of the relationships. For example, she listed:

$$\frac{1}{2}=\frac{2}{4} \qquad \frac{1}{4}=\frac{2}{8}$$

$$\frac{1}{2}=\frac{3}{6} \qquad \frac{1}{4}=\frac{3}{12}$$

$$\frac{1}{2}=\frac{4}{8} \qquad \frac{1}{4}=\frac{4}{16}$$

Teachers reported that the use of the virtual manipulatives seemed to provide special needs students with greater access to the mathematics by allowing them the flexibility to make and test conjectures with the applet. Teachers agreed that the lesson gave students a better understanding of using “one” to find an equivalent fraction.

> Teacher 1: In the past years, we have used the idea of the “magic number 1” to show students that you can find equivalent fractions when you multiply both numerator and denominator by the same number such as, 3/3=1. So if you multiply 3/3 by 1/2 that equals 3/6, which is equivalent to ½, but before I think we taught it like a procedure to follow. I know for sure, my special education students just learned “the trick” without really understanding it and without having a mental image of it. After having used the virtual manipulatives and having to record the list and draw the pictures that they saw on the screen, I feel that students have a better understanding of the idea that multiplying it by n/n is not changing the fraction but simply renaming it.

The applet seemed to benefit special needs learners by giving them built in supports for the mathematical ideas that reduced their cognitive overload. Having the visual and numeric representations closely tied together on the screen allowed students to make direct connections between the images of fractions and the fraction symbols. The kinesthetic/tactile advantages of using the SMARTBOARD also enabled special needs students to be more involved in the manipulation of the fractions on the screen.

CONCLUSION

In this study teachers benefited through their collaborative reflection which impacted their teaching strategies for fraction concept development. Students benefited as their teachers’ reflective actions translated into instruction, and from the unique affordances of the virtual manipulative tools which were a particular support for the special needs students in the class. The use of the fraction applets allowed students to think and reason about relationships among equivalent fractions. Opportunities to work with student partners encouraged mathematical discourse. Unique features of the virtual tools enabled special needs students to off load the task of maintaining both pictorial images and symbolic notations as the images and notations changed in response to the students’ input. This allowed students to focus more on mathematical processes and relationships, enabling them to formulate a rule that made sense. Kaput (1992) stated that constraint-support structures built in to computer based learning environments “frees the student to focus on the connections between the actions on the two systems [notation and visuals], actions which otherwise have a tendency to consume all of the students cognitive resources even before translation can be carried out” (p.529). The potential of these tools, used in lessons where teachers and students are engaged in meaningful discussions about the mathematics, is important to explore for special needs learners. The linked representations in the virtual fraction environment offer meta-cognitive support, such as keeping record of users’ actions and numeric notations. This allows learners to use their cognitive capacity to observe and reflect on connections and relationships among the representations. This study suggests that integrating reflective planning with effective mathematical tools can benefit special needs students, especially when they are exploring concepts where stored images and notations are necessary for student learning.

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DEVELOPING MATHEMATICAL CONNECTIONS AND FOSTERING PROCEDURAL FLUENCY: ARE THEY IN TENSION?

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This is the report an implementation of a two week sequence of lessons based on open-ended tasks focused on the learning of subtraction by Grade 6 students. There was no immediate gain in procedural fluency as measured by a pre- and post-test, but there was marked improvement after an interval of four months. The approach worked well with high achieving students, but the teacher needed to take specific actions to encourage the middle group, and to support low achieving students.

INTRODUCTION

Teachers are urged by mathematics education researchers, among others, to offer students learning opportunities that are engaging, relevant, and which foster creativity and making connections. At the same time, jurisdictions and school principals in many countries insist that teachers ensure that students are able to perform well on mandated assessments, predominantly consisting of routine questions. On-going debates in the US and Australia, for example, suggest that some commentators feel that emphasising mathematical creativity, for example, detracts from the development of procedural fluency. The following describes one teacher's approach to addressing both aspects together, while at the same time seeking to support the specific learning needs of low achieving students.

ADDRESSING THREE CHALLENGES SIMULTANEOUSLY

Teachers pursue multiple and possibly even competing mathematical goals. One characterisation of the different goals was proposed by Christiansen and Walther (1986) who described learning in Dimension 1 as being "determined by the object for the activity, by the mathematical core of the given task" (p. 261), which includes the type of questions commonly used in mandated assessments, and incorporates both the elements *conceptual understanding* and *procedural fluency* proposed by Kilpatrick, Swafford, and Findell (2001). The emphasis here is on the latter of these. Christiansen and Walther (1986) suggested that learning in Dimension 2 "concerns the general aspects of problem solving, exploration, generalisation, description, reasoning, application, relational storing of knowledge" (p. 262), which includes what Kilpatrick et al. (2001) described as *strategic competence* and *adaptive reasoning*. The particular focus in the following is promoting creativity and making connections. Both developing procedural fluency and fostering the making of mathematical connections represent challenges for teachers. A third challenge is to accommodate the needs of low achieving learners at the same time.

Promoting creativity and making mathematical connections

The teacher of the students described below used a particular type of open-ended task to promote creativity and support the making of mathematical connections. Christiansen and Walther (1986) argued that opening up tasks can engage students in productive exploration of mathematical ideas, Fredericks, Blumfield, and Paris (2004) suggested that engagement is enhanced by increased student choice, and Pekhonen (1997) and Sullivan (1999), for example, argued that open-ended tasks encourage students to explore, make decisions, and seek patterns and connections. The students worked on a particular type of *content-specific open-ended* task which can be illustrated by an example:

> If the surface area of a closed rectangular prism is 94 sq cm, what might be the dimensions of the prism?

Solutions can be found by random trial and error, by systemic trials, by drawing diagrams, or by using elementary algebra. The task is different from a conventional closed question in that it does not rely on the recall of a formulae or algorithm, it allows students to explore aspects of surface area of rectangular prisms in their own way, it has some easy entry points, it encourages students to find their own strategy and method of recording, and the range of approaches used and solutions found can lead to an appreciation of their variety and relative efficiencies. The task addresses conventional mathematics content so there is less need for teachers to feel they are jeopardising students' performance on subsequent mathematics assessments. This task clearly has potential to foster learning in Dimension 2.

Fostering procedural fluency

Those responsible for policy expect teaching approaches to develop procedural fluency as measured by mandated assessments. Yet there is widespread concern that such assessments have the effect of constraining mathematics teaching. Nisbet (2004), for example, in Australia, and Williams and Ryan (2000) in the United Kingdom, studied the implementation of mandated assessments, and noted the effect of narrowing of teaching that results. Similarly, in commenting on the *No Child Left Behind* legislation in the United States, Menken (2006) argued that high-stakes tests have become the *de facto* curriculum policy in schools, and Goldhaber (2002) suggested that testing results in the test content being emphasised at the expense of other, perhaps more important, aspects of the curriculum.

At the same time, there is evidence that open-ended approaches can enhance students performance on such assessments. Stein and Lane (1995), for example, reported that the greatest gains on systemic assessments resulted when "tasks were both set up and implemented to encourage use of multiple solution strategies, multiple representation and explanations" (p. 50). Similarly, Hiebert and Wearne (1997) found that students in classrooms characterised by having fewer tasks but spending more time exploring and discussing each, gave longer responses and demonstrated higher levels of performance on mathematical assessments. Likewise, Boaler (2002) argued that, after working on an "open, project based mathematics curriculum" (p. 246), students "attained significantly higher grades on a range of assessments, including the national examination" (p. 246). In other words, while the understandable pressure created by mandated assessments seems to constrain teaching, it does not need to, and it is possible to address Dimension 1 goals while also fostering Dimension 2 learning.

Assisting low achieving students

There is also the ever present challenge for teachers, while extending the learning of all students, of supporting those who experience difficulty in learning mathematics. McGaw (2007) argued that Australia was one of a number of countries whose overall performance on international assessments was good, but which had a wide diversity in levels of performance. It has also been argued (e.g., Delpit, 1988) that open-ended approaches can exacerbate disadvantage experienced by particular groups of students.

Theoretically, it seems that open-ended tasks should be more accessible for students experiencing difficulty in that they can choose the level at which they engage with the task. In the case of the example above, students who have difficulty in comprehending the task can choose possible dimensions for themselves and try them out. Sullivan, Mousley, and Zevenbergen (2006) outlined a particular approach to supporting such students involving teachers posing *enabling prompts* that are variations on the initial task but which reduce the level of demand in some way, while allowing ready bridging to the original task once the enabling prompt has been addressed. Sullivan et al. found that such an approach is both feasible and effective in supporting the learning of low achieving students.

In summary, challenges arise because increasing student choice and posing tasks that may foster the building of connections, for example, are sometimes seen as jeopardising the development of procedural fluency, and teachers may feel a need to choose one focus or the other. It is also possible that open-ended approaches exacerbate the challenge of supporting students experiencing difficulty in learning mathematics. The study reported here provides a perspective on these issues.

TEACHING SUBTRACTION USING OPEN-ENDED TASKS

The data reported below are selected from the analysis of a sequence of lessons, the foci of which included whether content specific open-ended tasks:

- promote identifying patterns and making connections by students;
- support the development of procedural fluency; and
- require specific actions to support students experiencing difficulty.

The sequence of lessons was taught by a teacher who was part of a project examining barriers to successful learning of mathematics (see Sullivan et al. 2006). All teachers had participated in a developmental project, and at this phase teachers were creating their own lesson sequences within the guidelines of the overall project. Data included observations of teaching, teacher diaries and interviews, class tests, and student work products. This draws on the latter two sources for one iteration with one teacher.

The particular teacher used open-ended questions but focused on a specific aspect of mathematics, subtraction, allowing assessment of the development of the students' procedural fluency. The teacher, Mr Smith (all names in the report are pseudonyms), was experienced, highly professional, and had an engaging personality, especially when interacting with his class. He taught a Grade 6 class (about age 12) in a regional Australian primary school, serving a community with both middle and low income families. Mr Smith gave the following as a summary of his goals:

> To further developing understanding of subtraction …. To look more closely at assessing students' progress with single/double digit problems (no trading), double-digit problems with trading, from 100 and from 1000 subtraction problems with trading.

Assessment of the development of procedural fluency

Three questions from a pre-test and matching post-test were selected to allow consideration of the students' development of procedural fluency. The test had some open-ended items, such as "How many subtraction equations can you make using these numbers? Show examples". However, the procedural fluency of the students can be better determined by considering responses to the following assessment items.

Question 6 consisted of four conventional subtraction items, set out vertically, the easiest example being

$$\begin{array}{r} 533 \\ -\ 296 \\ \hline \end{array}$$

(The question was scored as correct only if all four answers were correct.)

Question 8 was "The Jones family completed a trip around Australia of 1389 km. When they arrived home the odometer read 40142.6 km. What might the reading have been before the trip began?

Question 10, headed "Missing Numbers", was set out like 5 $\square$ 2 – $\square$ $\square$ 4 = 68. There was no specific prompt nor were multiple responses sought explicitly.

The difficulty level of these items was within the expectations of the relevant curriculum guidelines. Table 1 presents the profile of responses of students who completed both the pre-test and the post–test. The symbols √ and × are used to represent "Correct" and "Incorrect" respectively.

	Pre × Post ×	Pre √ Post ×	Pre × Post √	Pre √ Post √
Question 6	4	2	1	13
Question 8	13	2	1	4
Question 10	10	2	3	5

Table 1. Comparison of pre- and post-test responses for 3 subtraction questions (n = 20)

From inspection, it does not appear that the two-week unit had much impact on the students' ability to complete such tasks. Most of the group were competent at skill items (Question 6) even at the start. Questions 8 and 10 were more difficult and nearly all students who could not do them at the start could not do them after the lessons as well. Even recognising the short time frame, overall it did not seem that the sequence of lessons had much impact of the students' procedural fluency.

Analysis of students' responses to a particular open-ended task

The lack of student improvement in procedural fluency was a concern for the teacher and the researchers since it appeared that the students had worked productively on the sequence of lessons, and all students seemed engaged in the tasks, most of which had potential to promote making connections. To examine possible reasons for the lack of obvious improvement in the assessment scores, the written work of selected students was examined. Three students (out of 4) who were incorrect on the each of each of questions 6, 8, 10 (Jenni, John, and Eric) were selected at random and termed "low achievers"; three of the (11) students who scored question 6 correct, but question 8 and 10 incorrect on both tests (Elaine, Sheryl, and Jeremy) were termed the "competent group"; and 3 (out of 5) of the students who completed all 3 questions correctly on both tests (Diane, Ellen, Becky) were termed the "high achievers". The following reports these students' responses to a particular task. It is stressed that this task is representative of those used in the two week unit, and the profile of responses of the students across the various tasks was similar.

Students were given a sheet divided into four parts, with a number in each part (respectively, 26, 982, 3193, 5.78). The students were invited to create subtraction questions that gave that number as the answer. All students in the class gave multiple responses, with most students giving more than 20 different possibilities indicating that the task did engage them in thinking about subtraction. To indicate the nature of their work, the following illustrates actual responses of the selected students.

Responses of the "low achievers" were as follows:

> Jenni gave more than 20 responses, most non trivial, using a pattern of responses with whole numbers mostly correctly (e.g., 3205 – 12 = 3193; 3206 – 13 = 3193), but extended the patterns to decimal numbers incorrectly (e.g., 5.79 – 1 = 5.78; 5.80 – 2 = 5.78, …)
>
> Josh gave 23 responses, most trivial (e.g., 3197 – 4), and gave similar responses to Jenni for the decimal part.
>
> Eric gave 9 responses, some non trivial (e.g., 200 – 174; 2000 – 1018). All were correct, although he did not attempt the decimal task.

Judging by the number of responses, these students were engaged in the task, they explored aspects of subtraction, although they did not appear to extend their thinking, and they seemed to reinforce a particular misconception. This is similar to the response of this group to all lessons analysed in that they did create examples and possible solutions, but these tended to be less sophisticated than the items on the test.

Responses of the "competent" group were as follows:

Elaine gave more than 20 responses. In the first two tasks she used trading even when it was not necessary (e.g., 990 – 8). Her response to the third task was simple and her responses to the decimal task were incorrect like Jenni's.

Sheryl gave more than 20 responses, all correct, with the exception of the decimals task in which the responses were also similar to Jenni's.

Jeremy gave a substantial number of correct responses to each of the tasks (e.g., 200 – 174 = 26; 1000 – 18 = 982; 4000 – 907 = 3193; 6.78 – 1.0 = 5.78).

The students created more complicated examples than the low achieving group, perhaps at the level of the test items, although two demonstrated the same misconception as the other group.

Responses of the "high achievers"

Diane gave 18 responses, some substantial (e.g., 333 – 307), with no errors.

Ellen gave 23 responses, many substantial (e.g., 7.94 – 2.16), with no errors.

Becky gave 15 responses, some substantial (e.g., 9.20 – 3.42), with no errors.

Each seemed to extend their thinking, and created responses using generalisable patterns. It can be inferred that this group benefited from the open-ended challenge.

Of the class overall, there were nine students who gave multiple incorrect responses, possibly reinforcing misconceptions, eight students who were predominantly correct but used straightforward examples, and seven students whose responses could be categorised as insightful (e.g., 10 – 4.22 = 5.78; 11 – 5.22 = 5.78). This suggests that the nine focus students are representative of the spread of responses overall.

As with other lessons in the sequence, observations indicated that the students were willing to engage with little prompting by the teacher. While the high achievers seemed to extend their thinking, the same cannot be said for the other groups, and so it would not be expected that there would be much improvement on an assessment of their skill level. The "low achievers" were most likely to choose examples within their level of competence, and in some cases were reinforcing misconceptions.

It is noted that while this task does not have the potential of the earlier example, it was created by the teacher, and it has a number of the relevant elements of open-ended questions, such as identifying patterns, and encouraging the making of connections. Some of the other examples created by the teacher were more interesting. For example, in one task Mr Smith told the class that he had completed five subtraction exercises which he wrote on the board horizontally with some correct and some incorrect, (e.g., 100 – 21 = 89), and also five calculations presented vertically, also with some incorrect. The class was asked to work out which were correct and which were not, and to advise Mr Smith on how to avoid the errors in the future. Along with other lessons, this has potential to enhance procedural fluency so the lack of improvement of the students on the post-test was puzzling.

Assessment of performance after an intervening period

The possibility that the sequence of lessons had an impact which was not detectable over the short time elapsed between test administrations was considered. To examine this, the students were presented again with items 6, 8 and 10 about 4 months after the teaching of the unit, first in a test format and then one day later on a worksheet.

The "low achievers" Josh and Eric scored all of the Question 6 items correct on the worksheet but not on the test, indicating improvement. On the four parts of question 6, Josh achieved respectively on the three test administrations, 3 out of 4 correct, 2 out of 4 correct, and then 3 out of 4 correct. Eric scored 0 out of 4 correct, 2 out of 4 correct, and 3 out of 4 correct. Jenni had Questions 6 and 8 correct on both the delayed post test and the worksheet. She had earlier got 0 out of 4 correct (for question 6) in both the pre-test and post-test. Thus all students improved and in Jenni's case the improvement was substantial.

In the "competent" group, Sheryl answered Question 6 correctly on both the delayed post test and the worksheet, and also got Question 10 correct on the worksheet. Elaine got Question 6 correct again, and was correct on both the test and worksheet for Question 10. Jeremy got all three questions correct in both forms. All the students improved, and Jeremy improved to the level of the "high achievers" on the post-test.

The "high achievers" maintained their accuracy. Diane and Ellen got all three questions correct in both forms of delayed assessment. Becky was absent.

This overall longer-term improvement of both the low achievers and competent groups is of interest because the teacher, Mr Smith, reported that there had been no explicit teaching of subtraction in the intervening period. He did report that he had adopted open-ended approaches frequently, and the associated pedagogies, such as class reviews and support for low achieving students, had become part of his daily routines. Even though it is not possible to identify the impetus for the improvement in the skill levels of these students, perhaps the nature of the experiences in the sequence of lessons created sufficient awareness of conceptual possibilities and skill development to support the potential for further growth to continue after the teaching period.

SUMMARY AND CONCLUSION

The analysis of selected student responses to a particular mathematics lesson presented above is representative of the work of the class overall over the two week sequence of lessons on a range of similar open-ended tasks. The tasks encouraged the student to create subtraction examples for themselves, to make decisions, and to build patterns of responses. There was potential for students to analyse the patterns and to form generalisations. There was no immediate improvement in the skill level of the students as measured on a post-test, although after an intervening period there was a marked improvement that could possibly be in part attributed to working on the open-ended tasks. The nature of this delayed improvement could be further explored.

The analysis overall suggests particular actions for teachers when using such open-ended tasks. These include the need to monitor the work of students who may be experiencing difficulty, to find ways to support them in their learning, and especially ensuring they do not reinforce misconceptions. It also highlights the particular need for teachers to foster reflection on learning to encourage students to search for patterns and generalisations and so take advantage of the opportunities in such tasks.

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TYPE OF TEACHER CONTENT KNOWLEDGE AND ITS IMPACT ON STUDENT PERFORMANCE IN STANDARDIZED TESTING

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The study examines whether and how the cognitive demand level of teachers' content knowledge contributes to student achievement. In the context of this study, the level of cognitive demand refers to the kind of teacher content knowledge and thinking processes required to successfully accomplish a task, in terms of knowledge of facts and procedures (level 1), knowledge of concepts and connections (level 2), or knowledge of models and generalizations (level 3). 105 middle grades mathematics teachers from a number of school districts in an urban area in the Southwestern U.S. were tested using the Teacher Knowledge Survey. The level of teachers' content knowledge was assessed and tested for correlation with student achievement in the state-mandated standardized test.

RESEARCH FOCUS

The purpose of this project was to conduct a focused study specifically tailored to measure the cognitive demand level of middle grades teachers' content knowledge and its impact on students' achievement in state-mandated standardized test. The term *cognitive demand* is used in this study to describe both learning and teaching opportunities:

- how much thinking is called for in the classroom – learning opportunity aspect;
- the kind of teacher knowledge needed to sustain high level thinking in the classroom – teaching opportunity aspect.

The study examined the following research questions:

- What *kind of teacher content knowledge* is critical for student success?
- To what extent does the *cognitive demand level* of teachers' knowledge impact student achievement and student learning?
- Do differences in levels of teachers' knowledge have different effects on students' success?

The study explored whether teachers' knowledge at the level of facts and procedures will have different effects on students' learning and achievement, relative to knowledge at the level of concepts and connections, or at the level of models and generalizations.

THEORETICAL CONSIDERATIONS

No one questions the idea that what a teacher knows is one of the most important influences on what is done in classrooms and ultimately on what students learn.

> However, there is no consensus on what critical knowledge is necessary to ensure that students learn mathematics (Fennema & Loef Franke, 1992, p. 147).

A body of existing research on teachers' knowledge of subject matter claims that U.S. teachers lack essential knowledge for teaching mathematics (Ball, 1991; Stigler & Hiebert, 1999; Ma, 1999). It is also well documented that teachers' intellectual resources affect student achievement (Coleman et al., 1966). A study by Hill et al. (2005) found that teachers' mathematical knowledge for teaching was significantly related to student achievement.

At the same time, studies on the relationship between student learning and teacher knowledge as measured by standardized tests such as the National Teachers Examination indicated no significant correlation (General Accounting Office, 1984). No important relationships were found between how many mathematics courses teachers had taken and student learning (School Mathematics Study Group, 1972; Eisenberg, 1977; Hill et al., 2005). However, Monk (1994) found that "courses in undergraduate math education contribute more to student gains than do courses in undergraduate math" (as cited in Wilson et al., 2001, p.8).

A majority of these studies are based on the following methodological approaches: (1) the educational production function approach, and (2) the process-product approach. The first approach argues that teacher preparation and experience are the best predictors of student achievement. In contrast, the process-product approach considers teaching behaviors associated with teacher-focused direct instruction (i.e., a teacher's verbal ability, clarity of presentation, wait time, feedback, and questioning strategies) as a predictor of student achievement. An exceptional attempt to combine these two approaches is the recent study by Hill et al. (2005) on the effects of teachers' mathematical knowledge for teaching on student achievement.

Steinberg, Haymore, & Marks (1985) claim that the teachers whose mathematical knowledge appeared to be connected and conceptual were also more conceptual in their teaching, while those without this type of knowledge were more rule-based. "When a teacher has a conceptual understanding of mathematics, it influences classroom instruction in a positive way" (Fennema & Loef Franke, 1992, p. 151).

Cognitive demand & mathematical tasks. One of the indicators of teachers' conceptual understanding of mathematics is an ability to engage students into meaningful discourse in the classroom through selecting worthwhile tasks that embody learning goals. Why are tasks important? Students learn from the kind of work they do during class, and the tasks they are asked to complete determines the kind of work they do (Doyle, 1988). Mathematical tasks are critical to students' learning and understanding because "tasks convey messages about what mathematics is and what doing mathematics entails" (NCTM, 1991, p. 24). "The tasks make all the difference" (Hiebert et al., 1997, p. 17). Tasks provide the context in which students think about mathematics and different tasks place different cognitive demands on students' learning (Doyle, 1983; Henningsen & Stein, 1997; Porter, 2004). Cognitive demands can be defined as the kind and level of thinking required of students in order to successfully engage with and solve the task (Stein et al., 2000, p.11). Such thinking processes range from memorization to the use of procedures and algorithms (with or without attention to concepts, understanding, or meaning), to complex thinking and reasoning strategies that would be typical of "doing mathematics" (e.g., conjecturing, justifying, or interpreting) (Henningsen & Stein, 1997, p. 529).

Given the importance of tasks, the next issue is: "What do teachers need to know to select or make up appropriate individual tasks and coherent sequences of tasks? The simple answer is that they need to have a good grasp of the important mathematical ideas and they need to be familiar with their students' thinking" (Hiebert et al., 1997, p. 34). Similarly, Grossman, Schoenfeld, & Lee (2005) posed a critical question: "What do teachers need to know about the subject they teach?" (p. 201), and provided a fairly straightforward answer: "Teachers should possess deep knowledge of the subject they teach" (ibid.). However, the more important question, "What kinds of knowledge are important for teaching?" (p. 202), remains unanswered.

As indicated by this review of the literature, researchers have substantially advanced the knowledge base and development of theory about teachers' knowledge. However, Wilson and colleagues (2001) argue that although knowledge of some forms of subject matter is important, the field needs more information about the specific kinds of subject matter knowledge that is critical for teaching.

METHODOLOGY

The study was conducted at an urban area in the Southwestern USA with more than 70% of teacher and student population of Hispanic origin. A review of local data in student mathematics achievement showed its' considerable decline in middle grades (from 81% in 5th grade to 56% in 9th grade, 2006 data). The research sample consisted of 105 middle grades teachers of mathematics with various backgrounds and teaching experiences. The research sample was randomly selected from 3 major local school districts.

The instrument was developed to assess teachers' knowledge based on different levels of cognitive demand (Fig. 1):

Level I: Knowledge of Facts and Procedures. This is a low level of knowledge and thinking for memorization of facts, definitions, formulas, properties, and rules; performing procedures and computations; making observations; conducting measurements; and, solving routine problems.

Level II: Knowledge of Concepts and Connections. This level includes, but is not limited to the following: understanding concepts; communicating "big ideas"; selecting and using appropriate problem solving strategies; explaining solutions; selecting and using multiple representations; making connections; transferring knowledge to a new situation; and, connecting two or more concepts to solve non-routine problems.

Level III: Knowledge of Models and Generalizations. This level requires a high level of teachers' knowledge and thinking for generalization of mathematical statements, designing mathematical models, making and testing conjectures, and proving theorems.

Example of Increasing Cognitive Demand

Level 1. **What is a rule for fraction division?**

Solve the following fraction division problem

$$1\frac{3}{4} \div \frac{1}{2} =$$

Level 2. **Solve the same problem in more than one way, for example, draw a model or illustrate the problem with manipulatives**

Make up a story for the fraction division problem

Level 3. **Is the following** $\frac{a}{b} \div \frac{c}{d} = \frac{ac}{bd}$ **ever true?**

Figure 1. Example of the Teacher Knowledge Survey item.

The instrument consisted of 33 items reflecting key standards and competencies for teachers' knowledge: Number Sense, Algebra, Geometry and Measurement, Probability and Statistics. These standards also reflect main objectives of the state-mandated test for students. Item development included the following steps: (1) selection of a test item reflecting a particular standard and competency, (2) identification of the level of cognitive demand to which the item belongs, and (3) development of test items that address the same standard and competency for the two other cognitive demand levels.

The instrument was validated and analyzed to determine if any particular items need to be modified, so they are comprehensible to middle school teachers. A randomly selected sub-group of teachers was asked to explicitly rate if the survey items are understandable using a Likert-type scale from 0 (not understandable at all) to 5 (understandable). During the pilot testing, teachers were asked open-ended questions and interviewed, in order to collect qualitative information about the items. Recommendations obtained from teachers incorporated in the revised instrument, which was randomly assigned to a smaller sample of 22 teachers before it is implemented in a larger population.

Participating teachers' average student passing scores in state-mandated standardized test was collected. A multiple regression test was applied to examine the correlation between the type of teacher content knowledge and student achievement.

RESULTS

A multiple regression test revealed a statistically significant correlation between teacher content knowledge at the Level I – knowledge of facts and procedures ($r=0.31$, $p<0.05$) and Level II – knowledge of concepts and connections ($r=0.27$, $p<0.05$) and student achievement. However, the study didn't detect any statistically significant correlation between teacher content knowledge at the Level III – knowledge of models and generalizations – and student achievement.

The study also explored the difference in the level of content knowledge between two groups of teachers:

1. teachers whose students perform above the state recognized level – 70% and up (for the purpose of the study, this group was called the group of 'successful' teachers, N=35)
2. teachers whose student perform below the state recognized level – 69% and down (for the purpose of the study, this group was called the group of 'failing' teachers, N=70).

The study didn't find a significant difference between two groups of teachers with regard to Level I and Level III knowledge (teachers' scores of 77% and 73% for Level I and 51% and 49% for Level III knowledge accordingly). However, the Level II teacher content knowledge revealed the true gap (fig. 2) between 'successful' and 'failing' teachers (teachers' scores of 55% and 41% accordingly). This level of teacher knowledge includes, but is not limited to in-depth understanding of mathematical concepts; selecting and using appropriate problem solving strategies; explaining solutions; selecting and using multiple representations; making connections; transferring knowledge to a new situation; and, connecting two or more concepts to solve non-routine problems.

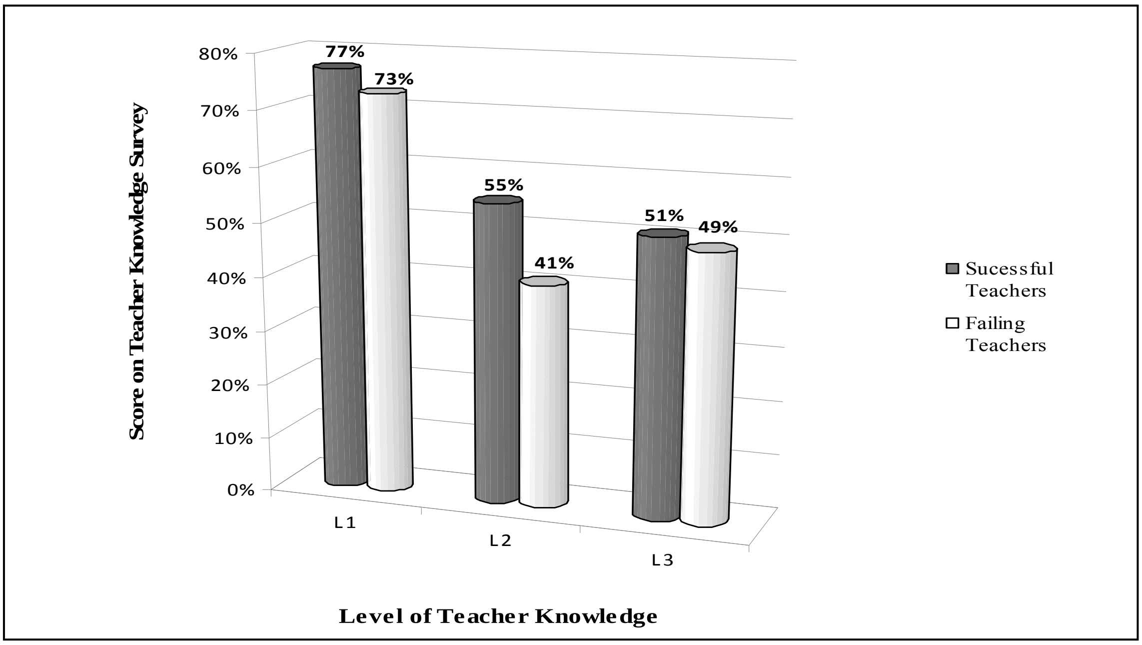

Figure 1. 'Successful' vs. 'Failing' Teachers' Content Knowledge Structure.

CONCLUSION

The conducted research was a focused study specifically tailored to measure the cognitive demand level of teachers' content knowledge and its impact on students' achievement. The results of the study attempts to find an answer to the question "What specific kind of teacher subject-matter knowledge is important for student achievement?"

The study showed that teachers' content knowledge at the level of facts and procedures as well as concepts and connections have statistically significant positive correlation with students' performance in the state-mandated standardized test. One of the key findings of the study is the fact that teacher knowledge of concepts and connections may serve as a good indicator of teacher success with regard to student achievement.

This project involved 105 middle school teachers and approximately 2400 middle grades students from local school districts with a predominantly low socioeconomic status Hispanic student population. We believe that the results of the study will inform educational community and deepen the knowledge base about predictors of student success in mathematics.

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THE COORDINATION OF COGNITIVE PROCESSES IN SOLVING GEOMETRIC PROBLEMS REQUIRING FORMAL PROOF

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We present here a study designed to analyse the cognitive processes relating to visualisation and reasoning as shown by trainee primary teachers when solving geometry problems. The purpose of this research is to understand and describe the role played by cognitive activity in order to enable the subjects to construct geometrical proofs in a pencil-and-paper environment.

INTRODUCTION

Two lines of research are currently being followed as mainstream areas of investigation in the processes of teaching and learning geometry. One of them concentrates on proofs and different ways of producing them (Harel & Sowder, 1998). The other is more concerned with the development of cognitive processes revealed by students when they solve geometry problems (Hershkowitz et al., 1996; Duval, 1998, Koleza & Kabani, 2006; among others). Duval (2007) claims that teaching and learning should be viewed from a cognitive standpoint, and following this idea our research aims to describe some of the cognitive processes employed by student teachers when they have to solve a problem that requires the construction of a formal mathematical proof.

Research projects, which concentrate on the learning of geometry and the cognitive processes involved, indicate how important the construction of a proof is. According to Soucy and Martín (2006), lecturers tend to believe that working with proofs in geometry helps to develop logical thinking processes and to produce coherent arguments to explain why a certain result is true. However, the dichotomy of students' cognitive activity on the one hand and those required by mathematical discipline on the other give rise to a series of difficulties which influence teaching and learning processes. The student must coordinate the various cognitive processes and representational registers either from a mathematical or from a cognitive viewpoint in order to construct proofs in problem-solving. How, then, can we relate visualisation and reasoning processes in solving geometry problems requiring a deductive proof?

The main aim of this paper is to generate knowledge which will help in understanding, as far as possible, the cognitive processes shown to be used by students when they solve geometry problems requiring the provision of a deductive proof. More specifically, we can state our goal as characterising of the coordination of the processes of visualisation and thought in solving geometry problems requiring a deductive proof.

THE THEORETICAL FRAMEWORK

Visualisation has become an important element in Geometry in recent years. According to Hershkowitz et al. (1996), visualisation is understood as the transfer of objects, concepts, phenomena, processes and representations towards some kind of visual representation or vice versa. This includes transfers from one kind of visual representation to another.

In agreement with Duval (1998), we define visualisation processes as follows:

- Perceptual apprehension is characterised as the simple identification of a configuration. It is the first step in the student's cognitive process.
- Discursive apprehension is the cognitive activity which produces a connection between the identified configuration and certain mathematical principles (definitions, theorems, axioms …). This association may be brought about in two ways, depending on the direction of the transfer, either from the thought process to the configuration or vice versa (change of anchorage).
- Operative apprehension consists of a visualisation process dependent on some mental or physical modification brought about by the problem-solver on the initial configuration, thereby enabling her to extract, introduce or manipulate various sub-configurations. Depending on the modification produced, we may distinguish two types: a figure-modified operative apprehension in which new geometrical elements are added to the initial configuration (new sub-configurations), and a reconfigurative operative apprehension in which the initial subconfigurations are manipulated like the pieces of a jigsaw puzzle.

Characterising the different apprehensions may help in analysing students' answers to geometry problems and in representing the activities carried out by the subjects.

Broadly speaking, we see reasoning as any process that enables us to derive new information from previous information, irrespective of whether the latter comes from the problem itself or from previous knowledge. We can distinguish two kinds of reasoning process related to discursive process (Duval, 1998): a) The natural discursive process. This process takes place spontaneously in any ordinary communicative act, through description, explanation and argumentation. b) The theoretical discursive process. This occurs through deduction. It may take place in a strictly symbolic register or in a natural register, but always through deduction.

This cognitive model can be completed by including the configurative process, which will enable us to understand certain end-products in solutions given to problems of geometry. We take the term "configurative process" to mean the coordination of discursive and operative apprehensions. This process should be seen as the set of mental operations undertaken by the student during the application of reason to geometry, and should be distinguished from those she uses during processes of communication. In other words, the configurative process is the sequence of coordinated operations (discursive/operative apprehensions) performed by the student when she is solving a geometry problem. Solving a geometry problem requires an interaction between the initial configuration and possible later modifications of it in accordance with mathematical statements, which makes it possible to identify the configurative processes involved.

In this context, and still following Duval's (1993) proposals, our definition of a deductive proof is centred exclusively on the structure of an inference: "A is true; A implies that B is true; therefore B is true" This ternary structure includes a premise "A is true", a reference to established knowledge "A => B", and a conclusion "B is true". The statement of the relation may take the form of a theorem, and axiom, a property, a definition etc. We now have a structure with which to identify the students' solutions.

METHODOLOGY

The data used in this research are the answers given by 55 in-training primary teachers to a set of geometry problems contained in a final examination in a pencil-and-paper environment. The choice of problems for the examination was made bearing in mind the purpose of the research project and the characteristics that would make them more interesting and productive as objects of study; that is to say, the presence or absence of configuration, the number of mathematical statements necessary to come to a solution, and the need for a deductive proof.

In order to carry out our analysis of the students' answers to geometry problems, we have looked for evidence of the ways in which their visualisation and reasoning processes interacted. To this end, we analysed both processes separately, identifying the operations performed by the students as described in the conceptual framework and functioning as an integral part of each answer, so that later we could observe how the two cognitive processes related to each other. We paid special attention to the configurative process as a type of reasoning used in solving problems of geometry.

In order to analyse the information provided by the students' answers, we divided these answers into fragments, each of which could be interpreted from the point of view of one of the two cognitive processes in the model. To facilitate the analysis and the localisation of each fragment, these were numbered.

RESULTS

The following is a transcription and analysis of a student's solution to problem 4, and is representative of the answers, which involve a configurative process by which the student obtains the key ideas necessary for a theoretical discursive process. The transcription has been divided into numbered sections to facilitate understanding of the analysis, and therefore each of our later comments refers to the number corresponding to the fragment concerned.

Problem 4

In Figure 1, □DEBF is a parallelogram and $\overline{AE} \equiv \overline{CF}$. Prove that □ABCD is a parallelogram.

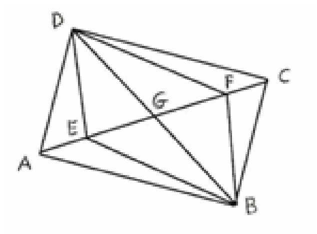

Figure 1.

Transcription of the answer given by student nº 40 to problem 4.

1) In a parallelogram, the diagonals bisect each other.

2) $\rightarrow \overline{GD} \equiv \overline{GB}$ y $\overline{EG} \equiv \overline{GF}$ in □DEBF.

3) Apply: opposite angles at the vertices are congruent.

4) $\rightarrow D\hat{G}A \equiv C\hat{G}B$

5) Considering the two triangles $A\overset{\Delta}{D}G$ and $G\overset{\Delta}{C}B$ (Fig. 2)

6) $m\,\overline{GA} = m\,\overline{AE} + m\,\overline{EG}$,

7) by hypothesis $\overline{AE} \equiv \overline{CF}$

8) $m\,\overline{GC} = m\,\overline{GF} + m\,\overline{FC}$.

9) therefore $m\,\overline{GA} = m\,\overline{GC}$

10) As $\overline{GA} \equiv \overline{GC}$, 11) $D\hat{G}A \equiv C\hat{G}B$, 12) $\overline{GD} \equiv \overline{GB}$

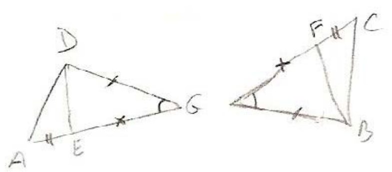

Figure 2.

13) By application of SAS,

14) the 2 triangles are congruent. Therefore $\overline{AD} \equiv \overline{CB}$;

15) By applying the same process to ∴DGC y ∴BGA (Fig. 3);

16) $\overline{DG} \equiv \overline{GB}$,

17) $\overline{GA} \equiv \overline{GC}$,

18) $D\hat{G}C \equiv A\hat{G}B$ .Opposite angles at the vertices are congruent.

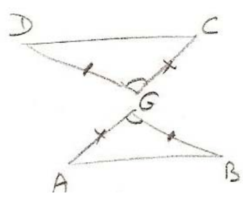

Figure 3.

19) SAS;

20) $\rightarrow$ we obtain $\overline{DC} \equiv \overline{AB}$;

21) We therefore have two pairs of opposite sides ≡ of the parallelogram □ADCB.

22) To show that each pair of angles are the same;

23) Comparing the 2 triangles (Fig. 3); 24) $\overline{DC} \equiv \overline{AB}$,

25) $\overline{DA} \equiv \overline{CB}$, 26) $\overline{AC}$ common side; 27) SSS; 28) They are congruent, so $\widehat{CAB} \equiv \widehat{DCA}$, $\widehat{CBA} \equiv \widehat{ADC}$, $\widehat{DAC} \equiv \widehat{ACB}$.

ANALYSIS

In Table 1 we give part of our analysis of the transcription of the student's answer, indicating in the left-hand column the visualisation process as decribed in the conceptual framework, and which are here identified. In the right-hand column we show the inferential steps taken by the student in the proof, in order to identify the reasoning process.

Identification of the Visualisation Processes	Recognition of the Inferential Steps
1) The student associates statement 1 with the initial configuration (Discursive appreh.), by carrying out a change of anchorage from discursive to visual, revealed by the marks made on the configuration to show the congruent sides (Figure 2). 3) The student associates statement 2 with the configuration, by similarly applying a change of anchorage from discursive to visual, as revealed by the marks made on the configuration to show congruent angles (Figure 2). Discursive apprehension. 5) The student extracts both triangles (Fig. 2) from the initial configuration, we have called this reconfigurative operative apprehension. 6-8) Evidence here of Discursive appreh., since the student associates statement 3 with Figure 2. 13) The student associates the Side-Angle-Side axiom with Fig. 3. Discursive appreh. 15) The student extracts two triangles (Fig. 3) from the initial configuration. We have called this process Operative appreh. 19) The student associates the Side-Angle-Side axiom to Figure 3. Discursive appreh. 21), 22) The student associates statement 5 with the initial configuration. Discursive apprehension. 23) The student extracts two triangles from the initial configuration. We see this process as Operative apprehension. 27) The student associates the Side-Side-Side axiom. Discursive appreh.	1) Statement 1. In a parallelogram, the diagonals bisect each other. 2) Thesis for statement 1 3) Statement 2. Opposite angles at the vertices are congruent. 4) Thesis for statement 3. 6), 7) y 8) hypothesis for statement 3. Property of addition of segments. 9) Thesis for statement 3. 10), 11), 12) Hipótesis for statement 4. Axiom of congruence of triangles Side-Angle-Side (applied to Figure 2). 13) Statement 4. 14) Thesis for statement 4 and hypothesis for statement 5. Theorem of the characterisation of parallelograms 16), 17), 18) Hypothesis for Statement 4. 19) Statement 4 applied to Figure 3. 20) Thesis for statement 4 and hypothesis for statement 5. 21) Hypothesis for statement 5. 22) Hypothesis for statement 5 which should be tested to obtain the thesis required by the problem. 24), 25), 26) Hypothesis for statement 6. Axiom of the congruence triangles Side-Side-Side 27) Statement 6. 28) Thesis for statement 6 and hipótesis for statement 5, both necessary to obtain the conclusión required by the problem.

Table 1

The configuration is useful to the student as a tool to obtain information about the problem synoptically. In other words, it shows interrelated elements and facilitates a holistic view. Furthermore, the operations carried out by the student on the initial configuration (extracting triangles: ADG-GCB, DGC-AGB and ADC-CBA; making different marks on the figure; associating the theorem of the characterisation of parallelograms with the additive properties of segments and the SAS and SSS axioms) have enabled us to identify an interaction between the above-mentioned mathematical associations and the configurations obtained through modifications of the initial configuration. This answer reveals a degree of coordination between operative and discursive apprehensions, which we have termed "a configurative process". The configurative process in the solving of geometry problems requiring a deductive proof is characterised by the generation of a series of key ideas, the shortening of the process and the production of theoretical discursive, the coordination between the visualisation process and the discursive processes characteristic of the configurative process. This coordination enables the subject to obtain key ideas and use them to implement a theoretical discursive process, which in turn generates a deductive proof as the solution to the geometry problem.

DISCUSSION

Reiss et al. (2002) have listed certain skills which should be developed from a mathematical standpoint, such as a basic knowledge of mathematical facts and arguments, knowledge of proof methods and the evaluation of the accuracy of proofs, and lastly scientific reasoning. The development of these skills, however, should go hand-in-hand with specific cognitive development applicable to geometry problems requiring formal proof. It is precisely in this kind of problem where the configurative process and its development play a vital role in the assimilation of ideas capable of generating theoretical discursive.

The theoretical discursive process, and the subsequent deductive proof, depend partly on the mathematical statements which may arise from the coordination between the visualisation processes, in which the student faces certain difficulties: factors which trigger or hinder the visualisation of certain configurations, knowledge of the properties of these configurations (Reiss et al., 2002) and the constant two-way traffic between the configurative and the algebraic registers.

We have obtained evidence from our research data, which suggests that it is precisely the lack of this coordination which might explain why some students suffer a mental blockage when they try to find provable solutions to problems of geometry. In cases where the coordination factor has enabled students to solve the problem, their answers have prompted us to distinguish two different types of process:

- *Truncation*, where the coordination process provides the "idea" of how to solve the problem deductively. In these cases, the configurative process is cut short when the "idea" which will lead to the solution and proof has been found. It is this idea which generates the deductive thought-process.
- *Unproved conjecture*, where the configurative process enables students to solve the problem by accepting their own conjectures, which they arrive at by simple perception and then proceed to express the solution in natural language.

It may also be the case that the configurative process (potentially possible or actually carried out) does not lead to a solution. We have called this process a closed loop, in which the configurative process has bypassed the route to the solution and has therefore brought the reasoning process to a halt. One cause of such a blockage could be, as we have already indicated, a lack of coordination, but it could also be due to a defective sequence in the visualisation processes (Discursive Apprehension and Operative Apprehension).

CONCLUSIONS

In this paper we have described some of the cognitive processes which affect the solving of geometry problems requiring a deductive formal proof. Our analysis of the data has enabled us to generate a model of coordination which will help us to understand what cognitive steps students take and what difficulties hinder the process of coordination and therefore the possible working-out of the proof.

As Duval (2007) has stated, in most fields of mathematics formal proofs do not require the same fundamentals or developmental processes as in geometry because they do not involve the same representative registers, i.e. geometric figures and natural language. How then can we improve these learning processes? There seem to be two experimental directions for future investigation: in the first place, we should try to discover what kind of reasoning could be as important as accuracy in computation. Secondly, we should increase our awareness of different ways of working with natural language and with different configurations.

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STUDENTS' INTERPRETATIONS OF AUTHENTIC REPRESENTATIONS OF A FUNCTION IN THE WORKPLACE

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This paper comments on the complex role that workplace inscriptions play in the case of transfer. The data came from a study that focuses on students' interpretations of a number of inscriptions of a function such as a table, a metaphor, a formula and a graph. The students were doing their practicum in a telecommunication organization and they were interviewed while facing the above inscriptions. A semiotic analysis indicated the idiosyncracies of the workplace inscriptions and the analysis of one student's interpretations showed that a strong conceptual base and a familiarity with the work context are needed for making symbolic interpretations and transcending the boundaries from academic to workplace mathematics.

INTRODUCTION

Although there is a consensus among many researchers that exists a clear conceptual distinction between mathematics in school and mathematics in workplace (Pozzi, Noss & Hoyles, 1998; Evans, 1999) still an open issue is the problem of transferring the ideas and knowledge from the one context to the other. The purpose of most studies in the domain of mathematics in workplace is to explore and appreciate the culture of mathematics in different work practices. However, few studies focus on how students conceive work practice and investigate the process of building bridges between the academic community and the workplace. For example, Magajna (1999) examined students' understanding of the drawing a curve in a CAD environment, Jurdac and Sahin (2001) analysed and compared students' and plumbers' strategies in solving a specific problem, while Williams and Wake (2007) showed how cultural models such as metaphors and mathematical models facilitated students understanding. Evans (1999) recommended an approach for building bridges based on analysing the discourses involved as systems of signs and he highlighted the role of affective factors.

All the above studies emphasize the breath and depth of mathematical activity that encountered in the workplace settings even if this is not realized from the practitioners. In the first part of our research project we identified a variety of mathematical activities in different groups of practitioners in a Telecommunication Organization (Triantafillou and Potari, 2006). The development of those activities was mediated through tools such as artefacts and inscriptions. In the current paper, we focus on the students from a Technical Institute, who are doing their practicum in the above Organization. Our main research goals were to: analyse students' conceptualisations of authentic cultural inscriptions and interpret difficulties they face.

THEORETICAL FRAMEWORK

We adopt a socio cultural perspective where mathematics is embedded in the work context and is mediated through the tools and inscriptions (Daniels, 2001). By inscriptions we mean, any graphical and pictorial representation such as graphs, data tables, maps, diagrams but not words (Roth and Bowen, 2001).

Since our research is focused on the meaning that students attribute on mathematical representations in the workplace, a semiotic perspective has been used to provide us a more detailed framework to investigate the context that the students experience and their interpretations of this context. Noss, Bakker, Hoyles and Kent (2007) have acknowledged the potential use of the semiotic theories in connecting various domains of mathematical knowledge and they emphasize that these theories take into account of how the different types of mathematical signs are used at work.

The triadic model of Pierce recognises three basic components: the *object*, the *representamen* (standing for the object in some way) and the *interpetant* (the idea the representamen produces in the mind of the Interpreter). According to Presmeg (2006), the term “sign” is referring to the whole triad, or the interrelationship of the three components of object, representamen, and interpetant. The above view framed our study and the analysis of our data in this paper.

Our focus is on the way that the various inscriptions *(representamen)* are interpreted by the students and how this interpretation is related to the mathematical object. In our study the *mathematical object* is the concept of a function and its *representamens* are various representations of the function, which come from authentic workplace situations. There are many possibilities of interpretations of these representamens. One way of gaining a more detailed account of these interpretations is based on Deacon’s (1997) interpretations of Peirce’s hierarchical trichotomy: *iconic*, *indexical* and *symbolic*. We use this semiotic framework as described by Davis and McGowen (2001) in our attempt to classify students’ understanding of the phenomenon. The simplest of the three classes of “signs” is the class of *icons*, or likenesses. When a collection of marks or sounds is iconic for someone, it is so because it brings to mind for that person something else, which it resembles. An *index* is a “sign” that refers to something else by association. Finally a “sign” is a *symbol* if it is involved in conventional relationships with other “signs”. The hierarchical relation between the three classes means that as indexes organize icons in higher-order relationships, symbols organize indexes as well.

METHODOLOGY

The overall research project had two phases. In the first one, we identified a number of mathematical activities in which the technicians were involved in the context of the telecommunication organization. In the second phase the focus was on how the students who were doing their practicum in this organization reflected on their work experiences and on how they interpreted a number of representations associated with mathematics that had emerged in the first phase. In this paper we use data from the second phase.

The students

Five male undergraduate students of a technological educational institute, in their last year of their studies, participated in the project. Three were from the Department of Electronics, one from the Department of Electrology and one from the Department of Informatics. All of them were doing compulsory practicum in various departments of the Organization. In this phase, they had completed successfully the courses required for their degree and they were apprentices for six months mostly assisting a technician on some fieldwork assignment.

	No of interviews	Ethnographic observation
St1	6	3 (3 hours)
St2	5	5 (1.5 hour)
St3	5	-
St4	4	1 (2 hours)
St5	5	2 (2.5 hours)
TOTAL	25	11 (9 hours)

Table 1. Research tools – duration

The process

The whole process lasted eight months. The data came from semi-structured interviews and ethnographic observation when this was possible (Table 1). Each interview lasted approximately one hour. Initially the questions focused on students' backgrounds, on their work experiences and on the way that they associated previous school and academic experiences with the workplace. Later on the questions were based on the authentic representations and focused on students' descriptions of the representations, on the sources of these descriptions, on the comparisons they made among the different representations, and on their explanations and arguments.

Distance (Km)	Diameter (mm)
Up to 3	Φ 0.4
From 3 – 6	Φ 0.6
From 6 – 9	Φ 0.8
From 9 - 10	Φ 0.9

Figure 1. The table representation.

The authentic representations

In this paper we analyze students' interpretations of four different authentic representations of a linear function: a table, a metaphor, a formula and a graph. The function was the variation of the Electrical resistance (R) of a copper wire in terms of its length (L) and its cross – sectional area (s).

The table: It came from a technical book and gives information about the diameter of the copper wires that are used in a local subscriber line and the corresponding distance between the subscriber and the organization building (Figure 1). The symbol Φ represents the image of a circle with a diameter. The students were asked to describe the phenomenon that is represented in the table and to justify their claims.

The metaphor: Many technicians in their attempts to describe the phenomenon surprisingly used the following metaphor: “The flow of charge through wires is like the water in a canal; when you want to send water through a great distance, you must use a canal with a larger diameter to avoid the losses.” The students were asked to make connections between the metaphor and the function represented in the table.

The formula: The technicians used the formula $L = R \cdot 45 \cdot d^2$, where L stands for the length of the wire, R for its resistance and d for its diameter, in order to make measurement in underground networks. This elaborated formula is equivalent to the common formula $R = \rho \cdot L/s$, ρ represents the resistivity of the material that the wire is made of and s is the cross – sectional area of the wire. The students were asked to describe the elaborated formula, relate this formula to their previous knowledge and also compare it with the conventional one.

The graph: It is from a technical book (Figure 2). The graph shows the resistance versus the length for three different diameters of a wire (0.4mm, 0.6mm, 0.8mm). The students were asked to compare the three different drawings and relate them to the initial table and the phenomenon in general.

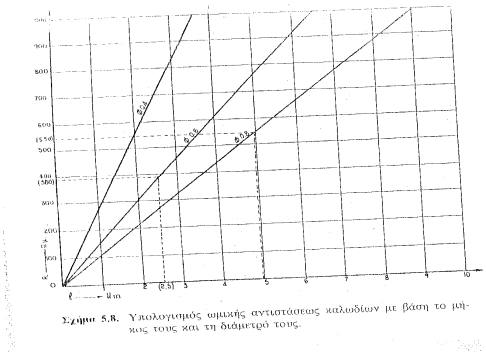

Figure 2. The graph representation.

ANALYSING THE INSCRIPTIONS

In this section, we attempt to indicate the particular characteristics of the inscriptions. The table is a non-transparent representamen of the function for an outsider as the resistance, which in school mathematics and science is the central variable, is hidden in this representation. The choice of presenting only the distance and the diameter in the table indicates a tendency that exists in the workplace to give immediate answers in a specific problem that is to choose the appropriate wire for a certain distance. Comparing also the genres of this workplace representation to the conventional table used in mathematics for representing a function we see different units of length measurement and a combination of verbal (words “up to”, “from”), arithmetic and figural symbols. Moreover, the function of the diameter in terms of the distance is a step function**,** a function that is not often met in school mathematics. In this case, there is a transformation of a well-known academic formula to a functional inscription embedded in the workplace expertise. The use of metaphor as a tool in workplace mathematics has been identified by a number of researchers (Williams and Wake, 2007). The metaphor is one form of analogical reasoning where a certain situation is related to another familiar for the user. In this particular case the canal is similar to the wire, the water is similar to the electrical current and the losses are similar to the resistance. The linguistic features of the elaborated formula differ from the conventional one. The elaborated formula is relevant to the work context as all the wires are made of the same material, so the resistivity takes a particular value and the wires are classified according to their diameter. Besides, taking into consideration that the wire makes a loop, the represented Length in the formula is double the indicated distance in the table. Finally the graph although looks like a common mathematical inscription its conventions are intimately tied to the meaning and the goals of the situation. For example, the represented length is actually the distance between the subscriber and the building of the organization because of the local loop of the wire. Another convention that is known to the practitioners is the need to change the diameter of the wire in certain distances. The reason for this is that the resistance of the total wire cannot be more than 1000 Ohm. Then the phenomenon would be represented by the graph of one function of R in terms of L, which is piecewise continuous, and the diameter d is a parameter which gets certain values.

STUDENTS' INTERPRETATIONS

Classifying the interpretations

From the analysis of the students' interpretations of the inscriptions of the linear function we framed our own conceptualization of what iconic, indexical and symbolic interpretation means in this particular context. So, iconic interpretation of the table would mean that the students could identify the elements that are represented in the table but without recognizing certain relations. An indexical interpretation of the table could refer to a global description of the underlied relations. In this case, the student recognizes that there is a relationship between the distance and the diameter but without being able to make it explicit. Moreover, this representamen reminds him formulas that he had met before but often without a complete meaning. A symbolic interpretation of this representation could refer to the mathematical relationship between the three variables and how the variation of the two affects the third. The metaphor was not used as a tool to develop understanding of the phenomenon but its interpretation remained at an iconic level although the technicians had used it as an indexical reference to indicate the structure of the relation. In the case of the formula, an iconic interpretation would mean that the students can identify only the symbols in the elaborated formula and cannot see any relationship between the two formulas. In an indexical interpretation the students recall images relevant to the particular inscription and compare the two formulas only in terms of their symbolic notation. The recognition of the algebraic relation among the variables in connection to the actual phenomenon indicates a symbolic interpretation. Finally concerning the graph, an iconic interpretation could mean that the students recognize only the symbols in the graph such as the three lines, the represented variables in the two axes and the different diameters. An indexical interpretation in this case means that the students recall prior knowledge, for example the concept of slope, to explain the different position of the three lines but without making a complete argument. A symbolic interpretation of the graph means that they can discover the way that the three variables - resistance, length and diameter - are correlated and connect this relation to the actual phenomenon.

The case of one student

We illustrate through our analysis of the interview data from one student the above characterization of the interpretations. The student originally made an indexical interpretation of the table by giving a global description of the relation: "When the distance becomes bigger, the diameter of the wire also increases". He justified his claim by referring to the quality of the signal and he used indexical references from the workplace or from his academic experiences:

> It seems logical to me. To have a wire for a short distance, a small diameter is ok. If you want to transmit a signal to a long distance, for example to a hundred kilometres, you have 100% quality of the signal. Then at five hundred kilometres the quality reduces to 95%. […] I keep in mind the wire. When it's for a short distance, we leave the wires without protective shields. When the distance increases, we see that they use more protections … plastic, then they become thicker and thicker.

The indexical references such as the quality of the signal and the image of a bunch of wires seem fragmental and not appropriate to the phenomenon. In this case, the table acts as a non-transparent inscription since the mathematical object is hidden. On the one hand, the student has not developed a rich work experience that could allow him to make relevant references to the phenomenon while on the other hand the conventions of the inscription prevented him from focusing his attention on the central variable, the resistance, and bring relevant academic knowledge to interpret the inscription.

The metaphor did not help him to develop an understanding of the situation. Initially, he made an iconic interpretation of the metaphor: "I can see that the water canal as something analog to the wire". However he did not manage to reveal the supported relation. He recognized that the losses indicated the existence of some noise in the signal. Although the metaphor seemed to be a meaningful tool for the technicians this was not the case for the student. As Kadnuz and Strasser (2004) argue there is a direction in the metaphor in the creation of meaning. In our case, the student could not transport properties from the metaphor to the function while the technicians used the metaphor as an explanation tool to a phenomenon which was already known.

The student described the formula in a conventional way when the one variable remains constant and the other two are connected in a proportional way: "The resistance is proportional to the length and we could say also that the length is proportional to the square of the diameter." We classify this interpretation as indexical since the student could not manage to relate the formula with the workplace situation and recognise that for certain values of length of the wire, the diameter of the wires needs to change. Besides the student could not relate the elaborated formula with any previous knowledge since his attention was on the icon of the symbols and not on the relations which they describe: "I haven't seen this formula before. […] The 45 is something that I have never met in a formula. […] It reminds me something; I think there is a relation among them. […] I can't remember exactly how it is." The student remembered the school formula after the researcher wrote down on a paper the known variables (R, L, s). The image of the symbols helped him to recognize and connect the elaborated formula with the one in school. The student related the two formulas in terms of their symbolic expression: "The constant (he means the number 45) we could say is 1/ρ no …" A memory image of the conventional formula without a complete mathematical meaning acted possibly as an obstacle for the student to recognize relations between the conventional and the elaborated formula. Moreover, the student seemed to treat the variables in the formulas as symbols to be manipulated technically and the function as a static quantity and not as a manipulative mathematical object that can be transformed in such a way to give a precise meaning to the actual phenomenon (Michelsen, 2005).

By referring to the graph, initially the student realized that the graphical relation of the resistance in terms of the length of the wire is a "diagonal line passing through the axes' origin" but he could not give a reason for this. In this case, he used the prototypical image of a diagonal. When he was asked to explain the different position of the three lines the student recalled the concept of slope: "Since the qualities are inversely proportional the bigger cross sectional area the lower the slope", but without making a complete argument. So, the student recalled mathematical images, the diagonal line and the slope, without being able to connect them meaningfully with the situation. Finally, when he was asked to relate the graph with the initial table, the student could not make some connections. Student's interpretations could be characterised as indexical. Although in this final part of the interview, the student had faced all the different inscriptions and had identified certain relations still he could not construct symbolic interpretations. On the one hand, his mathematical knowledge seemed to be fragmented and surface and on the other his working experience was limited. For example, he needed additional information such as the maximum values that the resistance could get and the financial policy of the organization to keep the cost low in order to conceptualise the situation. Roth and Bowen (2001) also have acknowledged that competencies with respect to graph interpretation are highly contextual and require familiarity with the phenomenon that the graph pertains.

CONCLUDING REMARKS

The cultural workplace inscriptions of a mathematical boundary object (Noss et al., 2006) are non-transparent for a newcomer. Their nature is shaped from the particular workplace collective motives and communal purposes (Williams and Wake, 2006). They have many silent variables and invisible mathematical models (table), conventions (graph, formula) and hidden mathematical references (metaphor). Student's interpretations of the given inscriptions were at best indexical since he was not able to provide a rich symbolic association of meanings relating them to the mathematical object and the particular phenomenon. He was not well prepared to develop dynamical functional notions and flexible conceptual attitudes to associate his prior knowledge with the inscriptions. Moreover, he was not familiar with the working context so he could not realize the conditions under which a phenomenon occurs. We consider these two elements essential in order to transcend the boundaries between his academic and workplace practice.

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STRUCTURE SENSE AND THE USE OF VARIABLE

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Students at university level need to be able to work with complex algebraic equations, and to apply definitions and properties when needed. Some authors claim that the solution of complex problems requires what they define as structure sense. We consider that a flexible use of variable is essential for solving complex algebraic equations. Research on the use of variable has focused on the solution of elementary algebra problems. In this paper the solution given by 36 university students to 3 complex algebraic equations is analyzed using the 3UV model. Results show that a flexible use of variable is required for success in the solution of these problems, but that it has to be accompanied by structure sense. Some characteristics that can be considered as part of structure sense are added to those previously defined.

INTRODUCTION

Research on teaching and learning of elementary algebra has underlined that the development of algebraic knowledge implies a solid understanding of the concept of variable (Phillip, 1992; Warren, 1999; Bills, 2001; Ursini and Trigueros, 2001, 2004, 2006; Trigueros and Ursini, 2001, 2003). However, so far, most of these studies have focused on the difficulties students face when solving simple algebraic problems. The kind of capabilities they need in order to succeed in the solution of more complex algebraic problems has not been researched in depth. This knowledge is needed if we want to help students develop a better understanding and use of algebra. In previous papers (Ursini and Trigueros, 2001, 2006; Trigueros and Ursini, 2003) we have already stressed the need of a flexible use of the concept of variable, a condition essential for proficient algebra users. Students should be able to differentiate between the various uses of variable and to integrate them in a single conceptual entity: the variable. The basic capabilities underlying an understanding of variable have been summarized as well. An elementary understanding of variable was described in terms of the following basic capabilities: to perform simple calculations and operations with literal symbols; to develop a comprehension of why these operations work; to foresee the consequences of using variables; to distinguish between the different uses of variable; to shift between the different uses of variable in a flexible way; to integrate the different uses of variable as facets of the same mathematical object.

Moreover, a theoretical framework, the 3UV model (3 Uses of Variable model), has been proposed as a basis to analyse students responses to algebraic problems, to compare students' performance at different school levels in terms of their difficulties with this concept, and to develop activities to teach the concept of variable (Trigueros & Ursini, 2001; Ursini & Trigueros, 2004, 2006). The 3UV model considers the three uses of variable that appear more frequently in elementary algebra: specific unknown, general number and variables in functional relationship. For each one of these uses of variable, aspects corresponding to different levels of abstraction at which it can be handled are stressed. These requirements are presented here in a schematic way:

- The understanding of *variable as unknown* requires to: recognize and identify in a problem situation the presence of something unknown that can be determined by considering the restrictions of the problem (U1); interpret the symbols that appear in equation, as representing specific values (U2); substitute to the variable the value or values that make the equation a true statement(U3); determine the unknown quantity that appears in equations or problems by performing the required algebraic and/or arithmetic operations (U4); symbolize the unknown quantities identified in a specific situation and use them to pose equations (U5).
- The understanding of *variable as a general number* implies to be able to: recognize patterns, perceive rules and methods in sequences and in families of problems (G1); interpret a symbol as representing a general, indeterminate entity that can assume any value (G2); deduce general rules and general methods in sequences and families of problems (G3); manipulate (simplify, develop) the symbolic variable (G4); symbolize general statements, rules or methods (G5).
- The understanding of *variables in functional relationships* (related variables) implies to be able to: recognize the correspondence between related variables independently of the representation used (F1); determine the values of the dependent variable given the value of the independent one (F2); determine the values of the independent variable given the value of the dependent one (F3); recognize the joint variation of the variables involved in a relation independently of the representation used (F4); determine the interval of variation of one variable given the interval of variation of the other one (F5); symbolize a functional relationship based on the analysis of the data of a problem (F6).

An understanding of variable implies the comprehension of all these aspects and the possibility to shift between them depending on the problem to be solved. This implies students need to develop the ability to *interpret* the variable in different ways, depending on the specific problematic situation, acquiring the flexibility to shift between its different uses and the capability of integrating them as facets of the same mathematical object. They need as well to develop the ability to *manipulate* symbolic variables in order to perform simple calculations. Both these abilities help students to acquire gradually a comprehension of why the operations work. The ability to *symbolize* rules and relationships in different problem situations leads them to foresee the consequences of using variables.

Starting algebra students should develop these basic capabilities that are related to a deep understanding of elementary algebra but, the solution of more difficult problems as, for example, complex algebraic equations, requires the possibility to consider specific definitions and properties of algebraic expressions. In consequence, is flexibility in the use of variable enough in order to solve complex algebraic equations? If not, what else is needed to succeed in the solution of these equations? Some authors consider that in order to cope with complex algebraic problems students need to have "structure sense" (Hoch & Dreyfus, 2006, 2005, 2004; Hoch, 2003; Linchevski & Livneh, 1999), we would like to consider its role too.

METHODOLOGY

We analyse the responses given by 36 students of a private university in Mexico City to three questions of a college algebra test. We use the 3UV model to analyse the three questions, to determine the role that a flexible use of variable plays in students' responses and to uncover other abilities which may be also necessary for success.

The three selected questions belong to a test all the Pre-Calculus students of a private university in Mexico City had to take:

(1) Solve the following equation: $2|x+3|^2-5|x+3|+3=0$,

(2) Solve the following equation: $2|x+3|^2+5|x+3|+3=0$

(3) Solve the following equation, considering only real solutions: $x^{1/2}-3x^{1/3}=3x^{1/6}-9$.

The solution of these equations was analyzed in terms of the 3UV model to highlight the uses of variable and the corresponding aspects involved.

Detailed responses of 36 students to both questions were analyzed, independently, by two researchers using the 3UV model and focusing on students' solution strategies to differentiate between successful and unsuccessful students. The analysis was discussed and negotiated to validate results. Results form the analysis of students responses were classified to determine the factors that seem to strongly contribute to students' success.

Analysis of equations (1) and (2)

To solve the first two equations a possible strategy is to: recognize the expression as a second degree equation (U1) and x as the unknown (U2); consider $|x+3|$ as a global entity which can be considered as the unknown of the equation and substituted by a new variable $y=|x+3|$ (U1, U5, F1); solve the resulting equation (U4); substitute the values obtained for the new variable y (F2); use the definition of absolute value and solve the resulting linear equations (U5); verify that the values obtained are solutions of the original equation (U3). Another possible strategy is to start by considering the definition for absolute value; to apply it to $|x+3|$ considering the two possible cases involved in the definition, (x+3) and –(x+3); to manipulate the two expressions obtained using algebraic properties (G5); to get a second degree equation where x is the unknown (U1, U2); to solve the equations (U4); to validate the solutions (U3) taking into account the restrictions imposed by the absolute value.

Analysis of equation (3)

The solution of the third equation requires recognizing that x is the unknown (U1, U2); that by manipulating it is possible to obtain $x^{1/2}=(x^{1/6})^3$ and $x^{1/3}=(x^{1/6})^2$ (G4 and properties of exponents); that $x^{1/6}$ can be considered as a global entity and as the unknown (U2); that it may be re-named y (G5, F1) in a new equation; the new equation has to be manipulated and solved using algebraic properties (G4, U4); after this the original variable has to be recuperated by substitution of the values obtained (U5, F2); the unknown's values have to be determined in each case (U4); as well, it should be verified that the obtained values are indeed solutions (U3). This equation can also be solved by using factorization (G4) to obtain the possible values for the unknown.

RESULTS

Even though these students were enrolled in a Pre-Calculus course at the university and, before it, they had taken several algebra courses at school, only a third of these students were able to solve correctly the three equations. Students who could not solve the equations (see transcripts below) showed evidence of difficulties in interpreting the role of the variable. This difficulty makes it impossible for them to interpret an algebraic expression such as |x+3| or $x^{1/6}$ as an entity which can be symbolized as a new variable or just considered as a variable as itself, even if they were able to manipulate the expressions:

Mario: $x^{1/2}-3x^{1/3}=3x^{1/6}-9$, $x^{1/2}-3x^{1/3}-3x^{1/6}+9=0$, $x^{1/6}(x^{2/6}-3)-3(x^{1/3}-3)=0$, (cannot continue)

Julián: $2|x+3|^2-5|x+3|+3=0$, $2(x^2+6x+9)-5|x+3|+3=0$, $2x^2+12x+18-5|x+3|+3=0$, $-5|x+3|=-2x^2-12x-21$, $x+3=(-2x^2-12x-21)/5$, $5x+15=-2x^2-12x-21$, $2x^2+17x+36=0$, $(2x+9)(x+4)=0$, $x_1= -9/2$, $x_2= -4$

None of these students identified the presence of an invariant in the equation which can be used as a new variable. They also showed some other difficulties common to many students, namely, the difficulty to operate with exponents, and to use the definition of an absolute value.

In the case of equations (1) and (2) students need to know and apply correctly the definition of an absolute value. Although many students knew the definition by heart, they were not able to use it in a specific situation as was Ana´s case:

Ana: 2|x+3|2-5|x+3|+3=0, |x| = ±x so |x+3|= x+3 if x≥0 and –(x+3) if x<0, then
2|x+3|2-5|x+3|+3=2(x2+6x+9)-5(x+3)+3=0, or to -(2x2+12x+18)+5(x+3)+3=0, ...

In the case of equation (3) many students, as has been shown with the work of Mario, had difficulties to manipulate the algebraic expression:

Isaac: x1/2-3x1/3=3x1/6-9, x1/2-3x1/3+9=3x1/6, x1/2+9=3x1/6+3x1/3, x1/2+9=3x3/6,
9= 3x3/6-x1/2, 9=3x, 9/3=x, 3=x

Enrique: x1/2-3x1/3=3x1/6-9, x1/2-3x1/3-3x1/2= -9, x2/6(x1/6-3x-3x1/6)= -9, x2/6= -9

Isaac's problems with manipulation of terms containing exponents and Enrique's incorrect use of the property *ab=0 if and only if a=0 or b=0*, exemplify also how students rote learning of some procedures made them over-generalize or apply algebraic properties incorrectly.

Difficulties with, manipulation problems did not let them discover an invariant which could be considered as a new variable, and most of these students were unable to solve this equation.

Many students were able to consider an expression as a variable only when the presence of the "new" variable was apparent, as in equations (1) and (2). Some of these students used the definition of absolute value correctly, but after manipulating the expression they did not go back to consider the restrictions imposed by the definition in order to select the values of the unknown that make the equality true, as was the case of Cristina:

Cristina: $2|x+3|^2+5|x+3|+3=0$, $2y^2+5y+3=0$, $y= [-(5) \pm \sqrt{}25-(4)(2)(3)]/4$

$y_1= -3/2$, $y_2= -1$, as $y= |x+3|$, $|x+3|=-3/2$ then $x+3=-3/2$, or $-x-3=-3/2$ and,

$x_1= -9/2$,

$x_2= -3/2$ and also $|x+3|=-1$ so $x+3=-1$ and $-x-3=-1$, then $x_3= -4$, $x_4= -2$

Cristina, was not able to identify the "new" variable when it was not evident and she could not solve equation (3).

However, there were students as Angelica, for example, who were able to manipulate equation (3) and to "see" an invariant which could be designated by a new variable, and she was able to solve the equation.

Angelica: $x^{1/2}-3x^{1/3}=3x^{1/6}-9$, $x^{1/2}-3x^{1/3}-3x^{1/6}+9=0$, $x^{1/3}(-3+x^{1/6})-3(x^{1/6}-3)=0$,

so $(x^{1/6}-3)=u$ and then $u(x^{1/3}-3)=0$, so $u=0$ or $(x^{1/3}-3)=0$ and $x=27$ or $3^6=x$

Some students seemed to lose track of the original goal:

Alonso: $2|x+3|^2-5|x+3|+3=0$, $a=2$ $b=-5$ $c= 3$, $x= [-(-5) {}^{+}_{-} \sqrt{}(-5)^2-(4)(2)(3)]/2(2)$,

$[5 {}^{+}_{-} \sqrt{}25-24]/4$, $5 {}^{+}_{-} \sqrt{}1]/4$, $x_1=5 {}^{+} \sqrt{}1]/4=6/4=3/2$, $x_2=5 {}^{-} \sqrt{}1]/4=4/4=1$

Alonso, for example, recognized the structure of the equation, but later on he seemed to forget that the unknown he was looking for was *x* and not $|x+3|$.

In the case of equation (3), there were students who started by manipulating the equation and, in doing so they were able to find a pattern in the expression or an expression that appeared repeatedly:

Ernesto: $x^{1/2}-3x^{1/3}=3x^{1/6}-9$, $x^{1/2}-3x^{1/3}-3x^{1/6}= -9$, $x^{1/2}-3x^{1/3}-3x^{1/6}-9=0$, $x^{1/3}(x^{1/6}-3)$,

$3(x^{1/6}-3)$, $x^{1/3}(x^{1/6}-3) - 3(x^{1/6}-3)=0$, $(x^{1/6}-3)(x^{1/3}-3)=0$, $x^{1/6}=3$, $x^{1/3}=3$, $x=27$, $x=3^6$

Many of these students had also problems when they had to apply the algebraic properties of expressions that are usually taught in algebra courses:

Paula: $x^{1/2}-3x^{1/3}=3x^{1/6}-9$, $x^{1/2}-3x^{1/3}-3x^{1/6}+9=0$, $x^{1/6}(x^{1/3}-3x^{1/6}-3+9)=0$,

$x^{1/6}(x^{1/3}-3x^{1/6}-3-9)=0$, $x^{1/6}(x^{1/3}-3x^{1/6}+6)=0$ and I make $x^{1/6}=x$, so $x^{1/6}(x^2-3x+6)=0$

Students who succeeded in solving the given equations were able to find the structure in all the cases. They did not start by manipulating the expressions blindly. They seemed to be able to analyze the expressions and discover their structure independently if it was apparent or not.

They were also able to use definitions and properties correctly whenever they were needed, and keep always in focus which was the unknown in the problem and the implications of the restrictions of the definitions and properties. All of them verified that the unknowns they found were solutions to the equations as we can see, for example, in the work of Rodrigo:

> Rodrigo: (1) $2|x+3|^2-5|x+3|+3=0$, $2y^2+5y+3=0$, $y= [5 \pm \sqrt{25-(4)(2)(3)}]/4$, $y_1= 3/2$, $y_2= 1$, as $y= |x+3|$, $|x+3|= (x+3)$ if $x\geq-3$ or $|x+3|= -x-3$ if $x<-3$, then $x_1= -3/2$, $x_2= -2$ and both are greater than -3, so both can be solutions, and $x_3= -9/2$, $x_4= -4$ and both are smaller than -3 so both are solutions and the solution set is $\{-9/2,-4,-2,-1/2\}$
>
> (2) 2|x+3|2+5|x+3|+3=0, 2y2+5y+3=0, y= [-(5) ± √25-(4)(2)(3)]/4, y1= -3/2, y2= -1, as y= |x+3|, it is not possible for y to be negative so the solution set is φ.

Rodrigo organized his work by giving priorities to different steps, he used first of all the global structure of the equation without taking into account that the variable he was working with in the first steps represented an absolute value, once he obtained the possible values of that variable he went back to the absolute value, worked with the resulting equations, only in the cases it was necessary, and verified that the solutions he found were indeed solutions of the equation.

As can be seen from the previous examples, it is evident that a flexible use of variable plays an important role in students' success at solving these complex equations. The possibility to differentiate when the role of the variable is that of an unknown, a general number or when there is a need to use related variables and the integration of all these uses and the different aspects intervening throughout the solution of the problem were necessary conditions for students to solve these equations correctly. However, this seems not to be enough for success. When solving complex problems, there are other issues that have to be considered at the same time as interpreting the role of the variable.

Students need to be capable of using properties and definitions correctly, that is, to know why they work and what are they useful for; to consider algebraic expressions as variables; and to keep their goal in mind while considering the restrictions on the solutions imposed by the conditions of the equation. Some of these abilities are contained in Hoch and Dreyfus (2006) definition of structure sense in algebra. Our results coincide with what these researchers have found, that is, that students who can display proficiency when working with elementary algebra problems, may have difficulties in applying the techniques to more complex problems. These authors have also shown that the presence of brackets helps the student see algebraic structure. In this study we found that this is true because brackets helped to make evident the presence of a new expression that could be considered as a global entity or a new variable. We can also add, from the results obtained, that flexibility in the use of variable, the need to give priority to some actions, and the ability to keep in mind the goal of the problem throughout the process of solution are a fundamental part of structure sense in algebra.

CONCLUDING REMARKS

Results of this study show that even capable students find it difficult to solve complex algebraic equations. The comparison between successful and unsuccessful students makes it possible to underline some factors that can explain success.

Some of these are a flexible use of variable; the possibility to differentiate and integrate these uses throughout the process of solution; the consideration of algebraic expressions as global entities; the capability to analyze problems from the start instead of starting to do blind manipulation; the possibility to uncover some structure in the equation, or to have structure sense; the correct use of definitions and the restrictions they impose on the solution of the problems; the possibility to foresee implications of definitions; the capability to keep in mind the different issues involved in a problem and to keep track of the role of each of them in the solution; the capability to give priority to the different actions that have to be performed in the solution of a problem.

These results also point out that, as teachers, we should focus more in how to help our students to improve these abilities. For instance, we could work with examples where analysis or classification of problems in terms of their structural properties is the goal of the activities, instead of always asking for the solution. This approach would avoid blind manipulation leading students to more reflection and analysis. We should also make more explicit in our classroom discourse, how the role of variable change and how different properties and definitions are used. Stressing the importance of validation should also help students to verify their solutions, but also very importantly, to discuss and reflect on the possibility that their solution set matches the restrictions imposed on the problem by the definitions used throughout the solution process.

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INSERVICE TEACHERS' JUDGEMENT OF PROOFS IN ENT

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Reform calls around the world state that proofs should be part of school mathematics at all levels. Turning these calls into a reality falls on the teachers' shoulders. This research focuses on secondary school teachers' knowledge about proofs, as expressed in the field of Elementary Number Theory. Our findings show that while teachers correctly identified symbolic proofs for given statements more than half of them failed to identify correct justifications which were not general.

BACKGROUND

Proof is at the heart of mathematics (Aigner & Ziegler, 1998; Thurston, 1994). Therefore, recent reforms in mathematics education recommend including proofs as a key component in school mathematics (Australian Educational Council, 1991; Israeli Ministry of Education, 2004; National Council of Teachers of Mathematics, 2000). Teachers are responsible to take this recommendation from power to action, and to influence their students' related mathematical knowledge (Hart Research Associates, 2005; Preseley & Gong, 2005). However, "teachers' subject matter conceptions have a significant impact on their instructional practices” (Knuth, 2002a, p. 63), and hence, analysing teachers' knowledge with respect to proof and proving is important.

A number of studies addressed in-service teachers' knowledge of proofs. For example, Knuth (2002b) examined 16 secondary school mathematics teachers' conceptions of proof, and concluded that while teachers recognise the roles that proof plays in mathematics, many demonstrate inadequate understandings of what constitutes proof.

A different research focus was taken by Dreyfus (2000), who investigated in-service teachers' reactions given justifications to one universal statement from the field of Elementary Number Theory (*ENT*). Dreyfus used symbolic, verbal, diagrammatic and numerical justifications. He found that teachers may fail to recognize some arguments as incipient proofs. Dreyfus identified three elements in a proof: generality, a chain of inferences, and a final conclusion.

In a previous study, Barkai, Tsamir, Tirosh & Dreyfus (2002) examined elementary school teachers’ justifications to *ENT* statements. They found that a substantial number of elementary school teachers applied inadequate methods to validate or refute various propositions. In addition, the same group of teachers rejected each others' justifications when they were asked to verify or refute them (Barkai, Tsamir & Tirosh, 2004). In the present study, we analysed in-service secondary school teachers' knowledge of proofs of statements in the field of *ENT*. More specifically, the study addresses the questions: Do secondary school teachers tend to identify the correctness of symbolic justifications? and How do they explain their judgments?

THE STUDY

Participants

A group of 50 in-service high school teachers, each teacher with at least 2 years of experience, participated in the study.

Tools and Process

The participants were asked to answer a questionnaire that addressed six statements from the field of *ENT*. The statements were chosen to include one of three predicates (always true, sometimes true or never true), and one of two quantifiers (universal or existential). The validity of the statements is determined by the combination of predicate and quantifier. Table 1 displays the six statements according to their quantifier and predicate.

Predicate / Quantifier	Always true	Sometimes true	Never true
Universal	The sum of every five consecutive numbers is divisible by 5.	The sum of every three consecutive numbers is divisible by 6.	The sum of every four consecutive numbers is divisible by 4.
Existential	There exists a sum of five consecutive numbers that is divisible by 5.	There exists a sum of three consecutive numbers that is divisible by 6.	There exists a sum of four consecutive numbers that is divisible by 4.

Table 1. Statements classification

The teachers were asked first to produce a proof (or a refutation) for each of the six statements, and did so correctly. Then, the teachers were presented with different, correct and incorrect justifications to each statement. The incorrect justifications were formulated so that in each, one of the elements of a proof was violated – the generality, the chain of inference, or the conclusion.

For each justification, the teachers were asked to determine if it proves the statement, and to explain their answer. The justifications were presented in three representations: numerical, verbal or symbolic. In this paper, we will limit the discussion to a sub-set of 16 symbolic justifications.

Three patterns of symbolic justifications were provided:

a) *Symbolic – correct*. For each statement, a general, sound symbolic justification was presented. It should be noted that such justifications present more than the sufficient conditions needed for proving existential statements or for refuting universal ones. In these justifications, the first element in a sequence of consecutive numbers was presented as the variable, and accordingly the next elements and their derived sum. A series of correct inferences was then presented, to reach a correct conclusion (see example in Figure 1).

The statement:	The sum of any five consecutive natural numbers is divisible by five.
Justification:	Let's denote the first number by x (x ε N). Thus the sum of the five consecutive numbers is: x+(x+1)+(x+2)+(x+3)+(x+4)=5x+10. 5x is divisible by 5, 10 is divisible by 5, so the sum 5x+10 is divisible by 5. Therefore the statement is true.

Figure 1. A correct justification.

b) *Symbolic – inference lapse*. For each statement, a justification that violates the correctness of the chain of inferences was presented. These justifications included a correct symbolic expression for the sum of the consecutive numbers, but the chain of inferences was not valid. In some cases the wrong inference was in transforming an expression into an equation, while others included a wrong interpretation of the expression that represent the sum (see example in Figure 2).

The statement:	There exist four consecutive numbers whose sum is divisible by four.
Justification:	Let's denote the first number by x. Thus the sum of the four consecutive numbers is: x+(x+1)+(x+2)+(x+3)=4x+6. 4x=(-6), and thus x=(-1.5). The solution of the equation is not a whole number and therefore the statement is not true.

Figure 2. A justification in which the chain of inferences is incorrect.

c) *Symbolic – generality lapse.* For each of the statements where the predicate is always true and where the predicate is never true, a symbolic justification which is correct only for a sub-set of cases was given: The first element in the series was presented as a multiple of 5 (or 4), and the rest of the elements were presented accordingly, and hence the generality of the justification was violated. The inferences that follow as well as the conclusion were correct (see example in Figure 3).

The statement:	The sum of any five consecutive numbers is divisible by five.
Justification:	Among any five consecutive numbers, there is one that is divisible by 5… : 5x, 5x+1, 5x+2, 5x+3, 5x+4 (5x is divisible by 5). The sum of this sequence is: 25x+10, and 25x+10 is divisible by 5 for any x. Therefore the statement is true.

Figure 3. A justification which is not general

Methodology

For each of the justifications, we first analysed whether the teachers provided a correct or an incorrect judgment. Then, we analysed the explanations teachers provide for their decisions.

FINDINGS

Table 2 presents the frequencies (percentage) of teachers' correct judgments with respect to the 16 symbolic justifications.

Table 2: frequencies (percentage) of correct judgments of justifications (N=50)

Predicate	Always true				Sometime true				Never true	
Quantifier	universal		existential		universal		existential		universal	existential
validation	Valid		Valid		Not valid		Valid		Not valid	Not valid
Justification pattern:	+	-	+	-	+	-	+	-	+	-
Symbolic - correct	100		92		92		82		100	
Symbolic – inference lapse		100*		100*		50**		96**	100**	86*
Symbolic – generality lapse		54	84		Not presented				68	42

* transforming expression into an equation ** wrong interpretation of the expression

Table 2. Frequencies (percentage) of correct judgments of justifications (N=50)

We will refer to the data presented in Table 2 according to the justifications pattern.

Symbolic – correct

Almost all the participating teachers correctly judged the correct symbolic justifications. In particular, all the teachers accepted the justifications for the cases in which a general proof is needed – universal statement / always true and existential statement / never true. For most statements where the provided justifications presented an over-proof and a numerical example would be sufficient, some teachers rejected the given justifications. We comment on teachers' explanations for these rejections, according to statements type:

Existential, always true – Some teachers (8%) erroneously rejected the justification for this statement, explaining that for proving a valid existential statement a numerical example is needed, rather than a general proof.

Existential, sometimes true – A relatively large percentage (18%) of the teachers rejected the symbolic justification in this case. Some of the teachers (the same group who rejected the previous justification) claimed that one supporting example is sufficient. These teachers were consistent with their judgment. Ten percent of the teachers rejected the justification on the grounds that the inferences in this case are problematic, since the focus of the justification was on odd numbers. Maybe the symbolic representation of the sum of the consecutive numbers led the teachers to assume that a general inquiry is in order: for what numbers will the sum be divisible by six (the part which was included in the statement), and also for what numbers will the sum not be divisible by six (a missing element in the provided justification).

Universal, sometimes true – The teachers' (8%) explanations for their reason to reject the justification were not clear enough in this case. They referred to the oddness and evenness of the numbers which can be substituted for x.

Symbolic – inference lapse

Only half of the teachers correctly judged the incorrect justification for the case universal statement / predicate sometimes true, and rejected this justification. This is opposed to all the other cases, where almost all the teachers correctly judged and reject the incorrect justifications that were provided.

Teachers' explanations for the (correct) rejection of the justification in the case universal statement / predicate sometimes true, included specific reference to the incorrect inference**.** The 50% of the teachers, who (wrongly) accepted this justification as proof, claimed that this is a general justification, and hence correct. Some of the teaches may have interpreted the wrong inference ("the sum is: $\frac{3x+3}{6} = \frac{1}{2}x + \frac{1}{2}$. We obtained a number that is not whole"), as a representative of not necessarily a whole number. We note that for the corresponding statement (existential statement / predicate sometimes true), all but two teachers correctly rejected the justification, and their explanations pointed out the wrong inferences. The noticeable difference may be due to the fact that the existential statement is valid, while the justification also includes a wrong conclusion.

All the teachers rejected the wrong symbolic justification presented for the cases existential and universal statements / predicate always true. However, for a similar justification (existential statement / predicate never true), not all the teachers rejected the wrong justification. Indeed, 14% of the teachers accepted the wrong justification, explaining that the transition to an equation was wrong, and yet accepted the justification saying that the resulting number was not whole.

All the teachers correctly rejected the justification in the case universal statement / predicate never true.

Symbolic – generality lapse

As can be seen from Table 2, a relatively low percentage of teachers correctly judged the justifications to these statements. The justifications are correct for the statements in which an example is sufficient, and false for the others. To better understand teachers’ reasoning when judging these justifications, we will focus on the explanations teachers provided for their judgments. Teachers’ explanations relate to the following four issues: (1) reference to the limited generality; (2) reference to the fact that the justification is presented symbolically, while ignoring the limited generality of the justification; (3) reference to the first sentence of the justification ("among every 5 (4) consecutive numbers, there is one that is divisible by 5 (4)"); (4) reference to the fact that one numerical example is sufficient. Table 3 presents teachers’ explanations with respect to the four categories of explanations.

The findings presented in Table 3 show that more teachers correctly judge statements for which numerical example are sufficient. Teachers, who used the second explanation, did so consistently for each of the statements. Hence, in statements where a proof must refer to all cases, the judgment was wrong, whereas for statements which needed only an example, the judgment was correct. This may be the reason for the different percentages of correct judgments found.

Predicate	Always true				Never true			
Quantifier	universal		existential		universal		existential	
validation	Valid		Valid		Not valid		Not valid	
Correct judgments	54		84		68		42	
(a) from the correct explanation	44		42		34		42	
(b) based on wrong explanation	10		42		34		--	
(1) limited generality**	--	44*	42*	14	34*	30	--	42
(2) ignore the limited generality	46	--	42*	--	34*	--	58	--
(3) first sentence	--	10*	--	--	--	--	--	--
(4) numerical example	--	--	--	2	--	2	--	--

* correct judgments, ** correct explanation for the justification

Table 3. Teachers' explanations for their judgments of partial generalization. (N=50)

Teachers, who used the first explanation, presented correct judgments and rejected the justifications for statements in which all cases must be considered. However, for the statements, which do not demand considering all cases, some teachers noticed the lack of generality, but presented a wrong judgment and rejected the justifications claiming that it is necessary to consider all cases.

Teachers, who used the third explanation, rejected the justification in the universal case for the wrong reason – they claimed that one needs first to prove the claim that among every five consecutive numbers we can find one that is divisible by five. In mathematics, a deductive proof relies on previously proved claims, and we do not prove them again. Hence, relying on a proved claim is not a reason to reject the justification. Note that such a claim was made only with respect to this justification.

CONCLUDING REMARKS

In this paper we examined whether teachers identify the correctness of symbolic justifications, and how they explain their judgments.

We recall that all the teachers correctly proved (or refuted) the given statements. Moreover, all these teachers identified the correct, symbolic justifications (proofs) for all six statements. Had we stopped here, we might have concluded that "teachers know symbolically presented proofs to ENT statements". But this was not exactly the case, because our study indicated two kinds of difficulties: The first involves the need to identify generality lapse in a symbolic justification. Only about half of the teachers identified the incorrectness of a symbolic justification that was not general.

Similar findings were reported by Vinner (1983) who analysed high-school students' judgements of such justifications, and claimed that proofs have ritualistic aspects. It may also be that symbols impart the feeling of generality. Knuth (2002b) reported that: "In determining the argument's validity, these teachers seemed to focus solely on the correctness of the algebraic manipulations rather than on the mathematical validity of the argument” (p. 392). When being presented with an algebraic justification, the teachers' focus was on the examination of each step, ignoring the need to evaluate the validity of argument as a whole.

The second difficulty we found involves the need to identify inference lapse in a symbolic justification of a universal, sometimes true statement (Figure 2). In the present case, half of the teachers accepted this wrong justification. Since the teachers correctly judged the parallel statement with the existential predicate, we assume that they interpreted the phrase "not a whole number" as "not necessarily a whole number", and on this ground accepted the judgment.

The everyday practise of teachers involves a constant evaluation of students’ justifications for statements. It is not unlikely that justifications of the kind presented to teachers in our study will emerge during interactions with students. Therefore, it is important that teachers will be able to identify correct and incorrect justifications. The findings presented here may serve as a starting point for an in-service training program. More research is needed to better understand and enhance teachers' knowledge of the many aspects of proof.

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GROWTH IN TEACHER KNOWLEDGE: INDIVIDUAL REFLECTION AND COMMUNITY PARTICIPATION

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In this paper I examine the interplay between personal reflection and participation in 'communities of practice' in the development of beginning teachers.The project on which I report has involved teachers in reflection on their mathematics teaching with a focus on both subject matter knowledge and pedagogical content knowledge. Evidence from this study suggests that their growth over three years has been influenced by individual reflection as well as by participation in communities of practice, with the interaction between the two dependent on individual contexts. In this paper I present some findings from the case studies of Amy and Kate.

INTRODUCTION

Barbara Jaworski (2001) posed the question "In terms of teacher education, do we see a teacher's growth of knowledge as a personal synthesis from experience or as deriving from interactions within social settings in which teachers work?" (p. 298). In summing up ideas from the PME research forum, 'Learning through teaching: Development of teachers' knowledge in practice' Ron Tzur (2007) suggested a number of issues for further discussion. One of these concerned the way in which teachers' continual engagement in expressing their ideas to others contribute to their learning through teaching. In this paper I try to throw some light on both these issues by examining the relationship between personal reflection and participation in communities of practice of teachers in my study.

My project is predicated on the view that well developed content knowledge, both subject matter knowledge and pedagogical content knowledge (Shulman, 1986), are necessary for mathematics teaching (Ball, 1998; Ma, 1999). The view was also taken that such knowledge might be developed through reflection *in* and *on* practice (Schön, 1983). A framework for lesson analysis - the '*knowledge quartet'* - has been used to support the participating teachers in reflecting on their mathematics teaching with a focus on mathematical content knowledge. The framework offers four dimensions as lenses through which to view teaching; *foundation, transformation, connection* and *contingency*. For details of these dimensions and the development of the framework see Rowland, Huckstep and Thwaites (2004).

MODELS OF PROFESSIONAL DEVELOPMENT

A model of *professional development through reflection* is often referred to in initial teacher education. Schön (1983) proposed that "skilful professional practice depends less on factual knowledge and rigid decision making models than on the capacity to reflect" (p. 16). Schön identifies two forms of reflection; *reflection-in-action* – the process of monitoring and adapting ones behaviour in context and *reflection-on-action* – after the event evaluation. Reflection may be seen as an individual activity however, this is not necessarily the case and it can be argued that reflection is more effective in developing practice when carried out within a 'community of practice'.

Lave and Wenger (1991) described the learning of professionals as 'enculturation'. Those new to the profession begin as 'peripheral participants' and grow into 'old stagers' as they absorb and are absorbed in the 'culture of practice'. Knowledge to be gained is situated in the practice of the community and can only be gained through participation. Lave (1988) suggests that beginning teachers learn the skills of teaching simultaneously with becoming part of a particular professional social group or *community of practice*.

Cobb, Yackle and Wood (1991) have promoted a socio-cultural view of teacher development that focuses on the situated nature of learning in which teachers' experiences in their classrooms determine their development. In later work they came to see learning as happening not in isolated classrooms but within a professional teaching community. Cobb and McClain (2001) explained how their thinking had progressed from seeing teacher learning as chiefly situated in classrooms and grounded in interactions with students, to recognising the importance of interactions with other professionals.

Jaworski (1998) described the development of teachers working on their own action research projects. She believed that this development could be explained by the cyclical processes of reflecting on teaching. In her later work she emphasised the importance of communities of practice in such development as they "provide opportunities for sharing experiences, synthesising from and explaining outcomes of research and developing critical frameworks related to practice." (Jaworski, 2001; p. 298). Lerman (2001) contended that reflection alone does not lead to learning but must involve others.

> "Reflection" per se, does not give us enough to serve as a process of learning. This is not to say that we don't reflect, only that for reflection to say something about how people learn involves others in one way or another. Reflective practice takes place in communities of practice, as groups of teachers in a school, teachers attending in-service courses, or other situations, and learning can be seen as increasing participation in that practice (p. 41).

It is the interrelationship between individual reflection and participation in communities of practice that I examine here in relation to the development of two of the beginning teachers in my study.

THE STUDY

The two teachers, from whose case studies I draw here, were both student teachers in the 2004-5 cohort of the early years (3-7 years) and primary (5-11 years) postgraduate pre-service teacher education course at the University of Cambridge. The study began with 12 participants reducing, as anticipated, to 9 in the second year, then 6 in the third year and finally 4 in the fourth and last year of the study.

All participants were observed teaching during the final placement of their training year, twice during their first year of teaching and three times during their second year of teaching. These lessons were all video-taped. In the training year the video-tapes were the basis for stimulated recall discussions using the *knowledge quartet* to focus on the mathematical content of the lesson.

During the first year of teaching, feedback using the *knowledge quartet* was given following the two observed lessons. Participants were then sent a DVD with a recording of their lesson, and a request to observe the lesson and write their reflections on it. In the second year of their teaching only minimal feedback was given following the lesson as I wanted to see how the teachers would independently make use of the *knowledge quartet* in their reflections. They were sent DVDs of their three lessons and wrote reflections on each of these, drawing on their previous training in using the *knowledge quartet*.

Participants also wrote regular reflections on their mathematics teaching which they sent to me. Group meetings were held to discuss the mathematics teaching and participation in the project of participants. These happened at both the end of the training year and the first year of teaching, and at the end of each term in the second year of teaching. In the third year of teaching, the fourth and final year of the project, each teacher will be interviewed individually once during each of the three terms. There will be group meetings twice during the year.

Case studies are being built from observations of teaching, discussions following observed lessons, contributions to group meetings, written reflections and individual interviews. Data informing this paper comes chiefly from transcripts of discussions following observed lessons and group interviews as well as from written reflections. This data has all been analysed using the qualitative data analysis software NVivo. A grounded theory approach (Glaser and Strauss, 1967) was used which led to the emergence of a hierarchical organisation of codes into a number of themes. Themes on which I draw here include ‘*working with others’*, ‘*reflection*’ and ‘*experience’*.

FINDINGS

All of the teachers in the project have claimed to find using the *knowledge quartet* useful in focusing their reflections. Their comments over the three years of the project suggest that they have learned through individual reflections ‘*on practice’* (Schön, 1983). Amy has shown herself to be a particularly reflective teacher. She teaches a reception class (4-5 years) and though she lacks confidence in her own mathematical subject knowledge, she is confident in her pedagogical knowledge and attributes this to some degree to using the *knowledge quartet* framework.

> I found it (using the *knowledge quartet*) does make you more reflective and it makes me, um from the *transformation* section, I think it makes me think of examples I am going to use or the images really carefully … and also planning, even what things I might say or do or extra little activities like bringing something that works. *(From the group meeting at the end of Amy’s first year in post)*

> You don't take your teaching for granted. You think about all the images, prompts or examples you'll need. You think as you are teaching of extra aids, how you are phrasing explanations. I think the Knowledge Quartet has pushed me to think from the other side and see more clearly how the children see and what they need. *(From written reflection at the end of Amy's second year in post)*

Amy's reflections may have helped her make better use of the interactions she has with children in her classroom by focusing on their mathematical conceptions (Simon 2007). This 'child-centred' approach to teaching has emerged as a clear strand in Amy's development.

Kate teaches a Year 1/2 class (5-7 years) and is also a very reflective teacher. Though less confident in her pedagogical content knowledge than Amy, Kate is confident in her own mathematics subject knowledge, having gained an advanced qualification in school mathematics.

> I think the categories are very useful … they kind of give you a way of thinking about what would be a kind of sensible remark to make about your maths lesson, and they evaluate it against certain things. *(From group meeting at the end of the first term of Kate's second year of teaching)*

> If I think about it in the car on the way home and think, if it wasn't very good, why wasn't it very good, was it the sort of concept behind what I told them to do or was it the resources they had to do it with. So that would be *transformation* and the first one would be *foundation*. What would have enabled them to understand that better than they did? *(From interview with Kate at the beginning of her third year of teaching)*

Having presented some evidence that Amy and Kate have both made use of the *knowledge quartet* to reflect on the content of their mathematics teaching, I turn now to consider how such reflections interact with their participation in communities of practice.

Amy has a close professional relationship with the other reception class teacher, who was also her mentor during her first year of teaching. They plan together, share ideas and discuss the effectiveness of their teaching.

> The other teacher in reception is totally like-minded with me, we use lots of practical objects and it is more like how I want to teach [than her mentor in her final teaching placement]… We have a good dialogue about how we want to teach and how we want to change things. *(From group meeting at the end of Amy's first year in post)*

> I have developed, especially through working with an excellent early years teacher, a good feel for the ways young children learn effectively. *(From Amy's written reflection half way through her second year in post*

Interactions with other professionals seem to have contributed to Amy's learning, as predicted by Cobb and McClain (2001). Amy has a great deal of respect for her colleague, though her strong views on effective pedagogy in the early years means she can be critical of other teachers whose approach does not align with her own.

> I teach the same as Daphne but I am in agreement with her and I share the planning and contribute my own ideas… I am critical when I hear of teachers using worksheets with young children. I am critical of maths teaching which is too much about abstract procedures and doesn't address what children can already do mathematically *(From written reflections at the end of Amy's second year in post)*

Amy's development has also been influenced by attendance at a continuing professional development course.

> We went on a course called 'Maths through Story' but now we have changed our maths teaching and have made it much more fun and made it much more through story with links to our topic work. (From the group meeting at the end of Amy's first year in post)

The sharing of this experience with the other reception teacher facilitated a critical review of their teaching, an outcome suggested by both Jaworski (2001) and Lerman (2001). However, attendance at courses does not seem to have been a continuing influence on Amy's development. By the beginning of her third year of teaching she viewed her interactions with colleagues in school and with pupils in the classroom as the most influential factors in her development.

> And a lot of it is from talking to colleagues as well. You do learn from discussions, you're having constant discussions and you learn that way as well but I can't say I've read a lot of Maths books… I can't say I've learnt more from that and you never get time out to go on courses so it is all from in the classroom. *(From the interview with Amy at the beginning of her third year of teaching)*

Amy's comment is supportive of Lave and Wenger's (1991) idea that learning takes place in communities of practice. Her view that 'it is all from in the classroom' is consistent with the situated learning view of Cobb, Yackle and Wood (1991).

Amy's development seems to be influenced by both individual reflections on her own practice and her participation within communities of practice, particularly her relationship with the other reception teacher. However these two influences are not separate, Amy's confidence in her own pedagogical knowledge and her positive relationships with colleagues have enabled her to share her reflections and her participation in the project with her colleagues.

> I have found talking the feedback (from me) over with my colleagues, means we have had more of a dialogue about maths and our teaching of maths in school, well in the lower school, with my colleagues and that seems really interesting and useful and it is always good to talk about other people's … about how you are teaching or about how you can move forwards. *(From group meeting at the end of Amy's first year in post)*

Kate has had a less constructive relationship with colleagues. In her school, planning for the year 1/2 classes is done by three teachers in rotation. This means that sometimes Kate teaches from the mathematics planning of another teacher and at other times she will plan for the team. Such planning presents little opportunity for the type of professional interaction seen as leading to teacher learning by Cobb and McClain (2001). In her first year of teaching she tried amending the planning in order to focus on what she considered to be the needs of her class. She was told that this was not acceptable because it meant she became 'out of step' with the other classes.

> I kept finding that I had taught things differently to my colleagues who were supposed to be teaching the same lessons as me. And I was thinking "Oh no I am getting left behind, I always do it wrong". *(From group meeting in the middle of Kate's second year of teaching)*

Kate tends to accept the advice of her more experienced colleagues in a way that might be expected of a *peripheral participant* in the *community of practice* as they become '*enculturated*' (Lave and Wenger, 1991).

> My colleague who previously taught year 3/4 at our school believes that we should only be teaching 'counting on' along the empty number line for subtraction because that is what the children will be taught in year three. *(From Kate's written reflections in the middle of her second year in post)*

> One of the teachers I am working with this year particularly hates it when there are lots of things going on at the same time and she has had a lot of input and sort of synthesised the plans. I think in some ways it is possibly a good thing … it has been a lot more easy to manage. I don't know whether it is good for the children. *(From group meeting at the end of Kate's second year in post)*

Kate's final sentence here suggests she does not simply take on knowledge situated in this community of practice, without critical reflection and adaptation. This is a recurring theme in Kate's reflections.

> Occasionally, not very often but occasionally, I see with the other people I am planning with, the other teachers that they don't seem to understand something that I think is quite necessary to understand, and they have either got it wrong or they don't seem to realise that it is important. *(From group meeting at the beginning of her second year in post)*

> As mentioned under *foundation*, I changed the plans for one day as I didn't agree with the idea behind our investigation. I haven't discussed this with my colleagues as I didn't want to be awkward. *(From written reflections at the beginning of Kate's second year of teaching)*

> Not very often (have deep conversations about the use of representations), no, not as often as we should because nobody wants to do the planning again. Um, I guess I would just use the other representation rather than discussing it with anybody. *(From interview with Kate at the beginning of her third year of teaching)*

As yet Kate lacks the confidence to have discussions that might take her learning forward through the kind of critical dialogue described by Jaworski (2001). This may develop as she becomes an '*old timer'*. When asked directly if she felt she had learned from other colleagues in school, Kate's response suggested that although she recognised the situated nature of her learning she felt any learning from colleagues has been incidental or even subversive.

> I have learned about the variety of resources that are available from being at school and I think from doing SATs (national tests) I have learned sort of what questions you'd expect to ask a year two … I am sure that I have learned from other people's ideas and things

> other people have planned and also things I have seen the reception teachers do as well … Not really (get a chance to observe other teachers) I did a few times in my first year, I walked through their classrooms and see what they have put out … I don't know a lot about what they have done, I look through their books to see what they have done and go and look at the things they have got out on their tables. *(From interview with Kate at the beginning of her third year of teaching)*

Kate's experience of learning within a community of practice appears very different to that of Amy. Both teachers have demonstrated a propensity to reflect critically on their practice and the practice of others. Amy's individual reflections have confirmed and progressed her thinking about effective mathematical pedagogy for young children and she feels able to share her reflections with colleagues. Amy is able to organise the teaching and learning in her classroom in a way that is consistent with her thinking and to share this thinking with her colleagues whom she respects. Planning for Amy is a collaborative process in which she feels her ideas are as valid as those of her colleagues. Her 'enculturation' is facilitated by the compatibility of her beliefs about teaching with those of her colleagues.

Kate however seems to feel insecure in relation to her colleagues. In the face of disparity of ideas, she tends to give way to what she sees as the greater expertise and experience of her colleagues. Planning for Kate is shared but not collaborative. She feels constrained to work from the plans and resources produced by other teachers even when not entirely comfortable with them. Kate has few opportunities in the school community to rehearse her reflections with others and to try out ideas resulting from them. Her 'enculturation' is hindered by unresolved disparity between her thinking and that of her colleagues. However, Kate does reflect critically on the planning and on her own teaching and within the confines of her own classroom acts on these reflections.

SUMMARY

Jaworski (2001) and Lerman (2001) suggested that reflection is most effective when shared with others. Amy has more opportunity than Kate for such sharing in her school community of practice. However, I would contend that reflective individuals, such as Amy and Kate, will learn and develop whatever the nature of the communities of practice they find themselves in. The ability to be critically reflective may be especially beneficial where communities of practice do not support learning in what might be considered a positive direction. Giving the teachers in my study a framework to focus and deepen their reflections has positively affected the way in which they have engaged with learning situations in their own contexts. Though not sufficiently confident to explicitly question the ideas and practice of others, Kate continues to do so in her reflections and in relation to her own practice. Teacher educators have limited opportunities to influence the learning situations of beginning teachers in their schools. It is however possible to help teachers become reflective practitioners who are able to engage more effectively in critical discussion within communities of practice in their schools and within the wider profession.

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INDIVIDUAL PREDICTORS OF GEOMETRICAL PROOF COMPETENCE

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We present results from a study with 341 German grade-9 students, trying to light up the interaction of some individual cognitive predictors of geometrical proof competence. As predictors we consider declarative and procedural aspects of basic geometry knowledge and problem-solving skills related to mathematics. The results show a very strong influence of declarative aspects of basic knowledge compared to the other predictors. We discuss possible theoretical explanations and their implications for the teaching of geometry and geometry proof.

INTRODUCTION

Mathematical proof is seen as one of the foundations of mathematical knowledge. Nevertheless, for many students this part of mathematics stays quite enigmatic during their time in secondary school though it is often integral part of the curriculum. This impression was confirmed by several empirical studies (e.g., Healy & Hoyles, 1998, Küchemann & Hoyles, 2002, Lin, 2005). As for many other non-algorithmic mathematical skills, like modelling or mathematical problem solving, generating correct mathematical proofs is a task that requires a lot of different skills and abilities. So reasons for the frequent failure of students writing mathematical proof are manifold. Apart from a firm understanding of the nature of mathematical arguments, students need good mathematical knowledge in the area of the proof problem, skills to control their own thinking during the solution process (meta-cognitive skills), and much more.

In the past aspects of these prerequisites were studied, and mostly found to have important influence on the proof performance of students (e.g., Healy & Hoyles 1998 for knowledge on the nature of proof; Weber 2001 and Lin 2005 for strategic knowledge). We call these skills and abilities "predictors", because to some extent they determine the students' competence to construct mathematical proofs. The aim of our research is to clarify from a cognitive perspective – still in a simple model – the interaction of these predictors. This includes comparison of their impact on proof competence, which may give information on the relative importance of single predictors. On a longer scale, these results may (and should) be exploited to construct reasonable interventions to help students deal with proof problems. Moreover, we see our research as an approach that may be adapted to other complex mathematical competencies.

GEOMETRICAL PROOF COMPETENCE AND ITS PREDICTORS

The research presented here focuses on proof in the context of geometry in lower secondary school, including theorems on angles in geometrical figures, and congruence. Geometry is the field traditionally used for the (exemplary) teaching of mathematical proof in Germany and in many other countries. It is not a priori clear if our results can be transferred to other areas of content. It would be an interesting continuation of our research to cover proofs in algebra, where slightly different effects regarding some questions were found in previous research (see e.g. Healy & Hoyles 1998).

Geometrical Proof Competence

Both theoretical considerations and empirical results suggest that the process of constructing a valid proof for a given assertion involves several mental activities. In a very rough model the central processes are (1) sort out, what is given (the premises and the hypothesis) and (2) organize known theorems and principles into a coherent set of arguments inferring the hypothesis from the premises.

Even though many proofs in secondary school geometry are usually linear in the sense that a single chain of deductive arguments can be built that leads directly from the assumptions to the hypothesis, the proving process does not follow a similar linear course. This process was characterized by Boero (1999) as a non-linear process of six phases which encompasses inductive and deductive phases. Arguments are usually not constructed one after the other as they are needed in the proof, but possible arguments are generated, selected and refined simultaneously.

Duval (2002) analysed this process of proof construction from a cognitive perspective and the most important distinction he made was the one between proofs that require only one known theorem or argument to be applied (one-step proofs) and proofs that require multiple steps, each representing the application of one theorem or argument (multi-step proofs). The basic difference between the two cases is – according to Duval – that for single-step proofs, the student only needs to consider the premises and the hypothesis in their roles as given in the problem. When constructing a multi-step proof, some propositions may change their status, meaning that what is the hypothesis of one proof step may be the premise for another one (dynamic status of propositions).

For single-step proofs, the student has to identify the relevant aspects of the problem (meaning of premises and hypothesis, given in figure or text) and recall possible arguments connected to either of them (dealing with propositions of fixed status). The premises of the theorem to be applied must be checked in the setting of the proof problem (micro-reasoning) and then arrange the relevant premises, the theorem resp. argument and its conclusion in the right order and present it (usually in written form).

For multi-step proofs additional abilities are required. First, as indicated by Duval (2002), in the search for possible arguments the student cannot only rely on premises and hypotheses given in the text, but also possible intermediate hypotheses must be taken into account. They arise as hypotheses of single proof steps and may be used as premises for further arguments (dealing with proposition of dynamic status). Apart from this, the proving process is much more complex than for single-step proofs, because several arguments have to be constructed simultaneously. Usually the proof process is guided by a proof idea or proof sketch, perhaps including some proof principle or some important relationship. Generation of a proof idea is a complex process relying on knowledge of mathematics concepts as well as on planning competencies much alike those required for mathematical problem solving. The generation of an overall proof idea, the planning and the coordination of the proof process are abilities that are particularly important for multi-step proofs.

These considerations lead to the assumption, that multi-step proofs impose qualitatively higher cognitive demands on students than single-step proofs. This assumption conforms to the findings of empirical studies, for example, it is reflected in the competence model for geometrical proof competence described by Heinze, Reiss and Rudolph (2005). This model proposes three levels of geometry problems. Level I consists of basic calculation items that ask the student, for example, to calculate specific angles in geometric figures. Mathematical theorems have to be applied in these items to justify the calculations, but no proof is required. Level II comprises one-step proof items and Level III multi-step proofs. These levels as distinct levels of competence could be confirmed in several empirical studies (Heinze, Reiss and Rudolph, 2005; Heinze, Reiss & Groß 2006).

Individual Predictors of Geometrical Proof Competence

As an individual predictor we consider skills, abilities or knowledge of individuals that will potentially improve their competency to master some kind of problem. Empirical research identified several classes of predictors with respect to mathematical problem solving: Knowledge of basic facts and procedures connected to the specific content, strategic knowledge (e.g. heuristics), and meta-cognitive skills (monitoring and control) as well as non-cognitive dispositions like motivation, emotions, and beliefs, and practices (c.f. Schoenfeld, 1992 for an overview). For (geometrical) proof competence we have to add methodological knowledge on the nature of mathematical arguments (Healy & Hoyles, 1998).

The necessity of strategic skills for the proving process was also pointed out by Weber (2001). Lin (2005) reports on possibilities to improve these skills for secondary school students. As for non-cognitive factors, the relation of proof performance and interest in mathematics was studied in Heinze, Reiss and Rudolph (2005). For the role of individually perceived social practices for argumentation we refer to Yackel and Cobb (1996). All these individual variables influence geometrical proof competence in a complex interplay. In this contribution we restrict ourselves to some cognitive predictors. One of the most obvious predictors for geometrical proof competence is the amount and structure of basic knowledge a student has at his or her disposal. In cognitive psychology there is a frequent differentiation in declarative knowledge that refers to knowledge on facts and concepts that can be explicated (e.g. verbally), and procedural knowledge which is often (but not always) more implicit and refers to the solution of well-known tasks (Anderson, 2004). Declarative knowledge required in proof problems may be manifold, concerning the actual mathematical content as well as meta-knowledge on the nature of proofs or admissible arguments. Nevertheless, in secondary school the aspects of meta-knowledge are usually more implicit knowledge (perhaps more than desired). In our study we restrict the investigation of declarative knowledge to the ability to recall basic definitions, theorems, and concepts. This ability is necessary for the explorative parts of the proving process (identification of possible arguments) as well as for structuring the problem situation (fixed or dynamic status).

For many tasks in school geometry the students acquire procedural knowledge during their schooling (e.g. simple calculations of angles in a geometrical figure). It consists of more or less specific procedures that solve certain classes of problems. These procedures are interconnected with declarative knowledge, in the sense that their understanding relies on it, but they also enrich it and contribute to the formation of meaning of basic concepts. What is most important for both calculation and proof problems is the ability to activate the right concepts (and procedures) in view of a given problem situation.

Some tasks, like many proof problems, cannot be solved by a general known procedure. They incorporate the construction of a basic idea or plan which is not of routine character. These tasks have the properties of true problems. From a cognitive perspective a problem consists of some initial state that should be transformed into some desired state, with the additional demand that the appropriate operators for this transformation are not known initially. Intensive research on problem solving revealed among others the importance of meta-cognitive skills (monitoring, control) and heuristics for the solution of these problems (e.g. Schoenfeld, 1992).

STUDYING THE INTERACTION OF PREDICTORS

In the study reported here we consider individual cognitive predictors of geometrical proof competence encompassing declarative and procedural aspects of basic knowledge and mathematics-related problem-solving skills.

Declarative basic knowledge refers to knowledge of basic concepts, definitions and theorems. The corresponding items consist of short propositions that have to be identified as true or false (e.g. *Two triangles are congruent, if they coincide in the sizes of their angles*). Procedural aspects of basic knowledge were implemented by the items on competence level I from the test of Heinze, Reiss and Rudolph (2005). For problem-solving, six new items were constructed resp. adopted from other studies (e.g. Lin, 2005). Each of these items emphasizes either the development of a central idea (concept, invariant) or a detailed argumentation for the solution. They are related to mathematical concepts, but not typical for German mathematics lessons. Finally, geometrical proof competence was operationalized by items on the competence levels II and III (one-step and multi-step items) as in the study of Heinze, Reiss and Rudolph (2005). An example item is given in Figure 1.

Prove $\alpha + \beta + \gamma = 180°$!

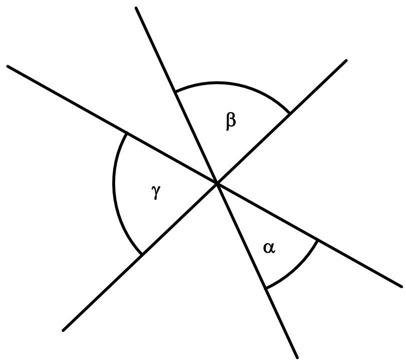

Figure 1. Example of a proof item – competence level II.

Design of the Study

We report data from a pilot study with N = 341 German grade 9 (about 15 years old) students from 14 classrooms. Germany has a tripartite school system and in only one of the three tracks (Gymnasium) proof is an integral part of the curriculum. Thus our sample is restricted to students from this high attaining school track. The data was collected in continuation of a project on proving in lower secondary school[1]. The survey consisted of two 45-minute tests. One was a geometry test containing eleven items from all three competence levels in the model of Heinze, Reiss and Rudolph (2005); three items covered level I, and four items for each of level II and III, which concentrated on content areas covered in grades 7 and 8. The second test contained six multiple-choice items on the knowledge of basic facts and six mathematics-related problem-solving items.

The geometry items on competence level I were treated as a predictor in this study, connected to procedural knowledge in the field of geometry. The score of all items on levels II and III is used as a measure for geometrical proof competence.

Results

Table 1 shows descriptive results for the single sub-tests. In both aspects of basic knowledge the results are quite good, with students solving an average of more than three quarter of the items correctly. This can be expected, since these items cover content from grades 7 and 8. The solution rates for geometry proofs and problem-solving items are much lower, reflecting the more complex nature of these tasks.
To check the competence model, the sample is split up into three achievement groups according to the score in the whole geometry test (levels I to III). As table 2 indicates, only the upper third is able to solve a considerable number of multi-step proof items (level III). While the lower third does not even succeed on single-step proof items, students from the middle third can solve them to some extent, but they do also fail on multi-step proofs. Similar findings are reported by Heinze, Reiss and Rudolph (2005) for grade 7 and 8.

[1] Funded by the German Research Foundation (DFG) in the priority program „Educational Quality of Schools“ and the Center for Teacher Education of the University of Munich.

Percentage of maximum score	*Basic knowledge, declarative*	*Basic knowledge, procedural (Level I)*	*Problem solving*	*Geometry proof (Levels II and III)*
Mean	79.6%	78.7%	27.4%	29.3%
Std. dev.	12.4%	20.1%	19.1%	20.9%

Table 1. Descriptive results

Percentage of maximum score	*Level I*	*Level II*	*Level III*	Overall geometry test
Lower third (N=112)	63.2%	15.9%	3.1%	24.2%
Middle third (N=106)	83.2%	39.6%	8.1%	40.1%
Upper third (N=123)	88.9%	70.2%	34.0%	62.2%
Whole sample (N=341)	78.7%	42.9%	15.9%	42.8%

Table 2. Proof performance by competence levels

In our design we consider three predictors of geometrical proof competence: declarative aspects of basic knowledge, procedural aspects of basic knowledge (items on level I) and problem-solving skills related to mathematics. The differences between the achievement groups from the analysis above are moderate for procedural aspects of basic knowledge and small for problem-solving skills.

Predictor	*Standardized regression coefficient β*	*Significance (p)*
Basic knowledge, declarative	0.497	<0.001
Basic knowledge, procedural	0.133	<0.01
Problem solving	0.158	<0.01
$R^2 = 0.416$, $F(3,337) = 81.872$, $p < 0.001$		

Table 3. Results of the regression analysis

To study the interaction of the predictors, we use a linear regression model explaining the geometrical proof competence by the three predictors mentioned above. The results are given in table 3.

The regression analysis shows that more than 40% of the variance of geometrical proof competence can be explained by three cognitive predictors covering knowledge of basic facts and procedures and problem solving. Considering the standardized regression coefficients it becomes clear that the influence of declarative aspects of basic knowledge is much stronger than the influence of procedural knowledge. This is as we expected. More astonishing, however, is the small influence of problem solving skills compared to declarative knowledge of basic facts.

DISCUSSION

Firstly, our study replicates the known result that (geometrical) proof problems form a great challenge for secondary school students, even more, since the items in our study build up on content from grades 7 and 8 that should be readily mastered in grade 9. The finding that only about one third of the grade 9 students is able to perform multi-step proofs and that about one third on the sample is hardly capable of constructing single-step proofs is in line with our previous studies for grades 7 and 8 mentioned before. Nevertheless, there is some variance in the results, there were also 19 students who achieved more than three quarter of the maximum proof score.

Beyond these descriptive results, we would like to identify reasons for the differences in students' performance. The approach used here is to analyse which impact certain predictors have on proof competence. The moderate differences in procedural aspects of basic knowledge within the sample and the small differences in problem-solving skills do not seem to be primary reasons for the differences in proof performance. The regression analysis indicates that differences in declarative aspects of basic knowledge are more essential and have a decisive impact on proof competence.

This finding is surprising on the one hand, as we would have expected to have a larger connection with problem-solving skills, given the analysis of the proving process that revealed many sub-problems with the nature of genuine problem-solving tasks. On the other hand, explorative processes and associative thinking are also important parts of the proving process. These processes do not primarily rely on meta-skills like problem solving, but they are closely connected to the nature of knowledge in the specific content area of the proof task. The corresponding items on declarative knowledge were originally designed to test if students can recall basic facts and definitions. Our assumption is that also for solving these items the structure of knowledge plays an important role. We assume that to construct a proof successfully, the students need basic knowledge that is well connected within itself and also with potential situations it is used in.

These results indicate a crucial role of sound content knowledge for the construction of proofs in geometry. It does not imply that other predictors have no influence. If a student does not understand the nature of mathematical proofs, or has no problem-solving strategies at hand, he or she will hardly be able to construct a proof in spite of the best geometric content knowledge. Nevertheless, the impact of these predictors seems to be smaller in our direct comparison.

For the teaching and learning of mathematics the implications are twofold: On the one hand, well connected content knowledge is an important prerequisite for proving (and presumably for other higher mathematical skills) and should therefore be one of our central aims. On the other hand, it is mediated by other influence factors like problem-solving skills or meta-knowledge, which should not be neglected.

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PUPILS' REASONING ON PROPORTIONALITY: SOLVING VERSUS CLASSIFYING MISSING-VALUE WORD PROBLEMS

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Upper primary school children often routinely apply proportional methods to missing-value problems, also when this is inappropriate. We tested whether this tendency can be broken if children do not need to produce computational answers to word problems. Seventy-five 6th graders were asked to classify 9 word problems with different underlying mathematical models and to solve a parallel version of these problems. Half of the children first did the solution and then the classification task, for the others the order was opposite. The results suggest a positive impact of a preceding classification task on students' later solutions, while solving the word problems first proved to negatively affect later classifications.

THEORETICAL AND EMPIRICAL BACKGROUND

Proportionality is an important mathematical topic that receives much attention throughout primary and secondary mathematics education. Typically, from fourth grade on, pupils are frequently confronted with missing-value proportionality problems such as: "12 eggs cost 2 euro. What is the price of 36 eggs?" (Kaput & West, 1994). Many pupils, however, also strongly tend to do proportional calculations in missing-value word problems for which this is questionable or clearly inappropriate. For example, more than 90% of 10–12-year old pupils answered *170 seconds* to the problem "John's best time to run 100 metres is 17 seconds. How long will it take him to run 1 kilometre?" (Verschaffel, De Corte, & Lasure, 1994) and several studies showed that most 12–16-year olds respond proportionally (*24 hours*) to the problem "Farmer Gus needs 8 hours to fertilise a square pasture with sides of 200 metres. How much time will he approximately need to fertilise a square pasture with sides of 600 metres?" (e.g., De Bock, Van Dooren, Janssens, & Verschaffel, 2007; Modestou, Gagatsis, & Pitta-Pantazi, 2004).

In-depth interviews (De Bock, Van Dooren, Janssens, & Verschaffel, 2002) in which pupils solved non-proportional geometry problems showed that pupils generally did not consciously and deliberately choose a proportional strategy. In a mature mathematical modelling approach (see, e.g., Verschaffel, Greer, & De Corte, 2000), essential steps would be: (1) understanding the problem, (2) selecting relevant relations and translate them into mathematical statements, (3) conducting the necessary calculations, (4) interpreting and evaluating the result. The pupils from De

Bock et al.'s (2002) study, however, seemed to bypass almost completely all steps except step 3. Their decision on the mathematical operations mainly was based on a routine-based recognition of the problem type, the actual calculating work received most time and attention, and after checking for basic calculation errors, the result was immediately communicated as the answer.

Undoubtedly, some reasons for pupils' association of missing-value problems with proportional reasoning are found in the mathematics classroom. Most proportional reasoning tasks children encounter in upper elementary and lower secondary school have a missing-value format (Cramer, Post, & Currier, 1993) while at the same time, other types of word problems are rarely stated in a missing-value format. Moreover, a lot of attention is paid to the technically correct and fluent execution of the required procedures, without explicitly and systematically questioning whether – and to what extent – they are applicable. As such, it is not surprising that children start to expect that all missing-value problems require proportional methods.

In line with the above interpretations, the over-use of proportionality might be broken if pupils pay more attention to the initial steps of the earlier-mentioned modelling cycle, i.e., the understanding of the relevant aspects of the problem situation and their translation in mathematical terms. So, when pupils are engaged in a task with proportional and non-proportional word problems without the need to actually produce computational answers, they might be stimulated to engage in a qualitatively different kind of mathematical thinking, and develop a disposition towards differentiating proportional and non-proportional problems. This assumption was tested by administering a type of task that is rather uncommon in the mathematics classroom: the classification of a set of word problems.

Interest in the value of problem classification and reflection on the relatedness of problems is rather old. Polya (1957) indicated that when devising a plan to solve a mathematical problem, a useful heuristic is to think of related problems. Seminal work was also done by Kruketskii (1976), who indicated that high-ability students differ from low ability students on their skills to distinguish relevant information (related to mathematical structure) from irrelevant information (contextual details), to perceive rapidly and accurately the mathematical structure of problems, and to generalize across a wider range of mathematically similar problems. Studies that actually used problem classification tasks, however, are rare.

METHOD

Subjects, tasks and procedure

Seventy-five 6th graders – belonging to five classes in three different primary schools in a middle-sized Flemish city – completed a classification task and a solution task.

In the *solution task*, pupils got a traditional paper-and-pencil word problem test, containing 9 experimental word problems: 3 proportional, 3 additive, and 3 constant ones. These different types of word problems were already used and validated in previous research (Van Dooren, De Bock, Hessels, Janssens, & Verschaffel, 2005). *Proportional* problems are characterised by a multiplicative relationship between the variables, implying that a proportional strategy leads to the correct answer (e.g., *Johan and Herman both bought some roses. All roses are equally expensive, but Johan bought less roses. Johan bought 4 roses while Herman bought 20 roses. When you know that Johan had to pay 16 euro, how much did Herman have to pay?*). *Additive* problems have a constant difference between the two variables, so a correct approach is to add this difference to a third value (e.g., *Ellen and Kim are running around a track. They run equally fast, but Kim began earlier. When Ellen has run 5 laps, Kim has run 15 laps. When Ellen has run 30 laps, how many has Kim run?*). *Constant* problems have no relationship at all between the two variables. The value of the second variable does not change, so the correct answer is mentioned in the word problem (e.g., *Jan and Tom are planting tulips. They use the same kind of tulip bulbs, but Jan plants less tulips. Jan plants 6 tulips while Tom plants 18 tulips. When you know that Jan's tulips bloom after 24 weeks, how long will it take Tom's tulips to bloom?*). The word problems appeared in random order in the booklets, but a booklet never started with a proportional word problem to avoid that – from the start on – pupils would expect the test to be about proportional reasoning. For the same reason, we also included 6 buffer items in the test.

For the *classification task*, pupils were given a box containing an instruction sheet, a set of 9 cards (each containing one word problem), 9 envelopes and a pencil. Again, 3 of the word problems were proportional, 3 additive, and 3 constant. The instructions for pupils were kept somewhat vague because we wanted to see which criteria pupils would use spontaneously while classifying: "*This box contains 9 cards with word problems. You don't need to solve them. Rather, you need to figure out which word problems belong together. Try to make groups of problems that – in your view – have something in common. Put each group in an envelope, and write on the envelope what the word problems have in common. Use as many envelopes as necessary.*"

Both tasks were administered immediately after each other, but their order was manipulated. Half of the pupils got the solution task before the classification task (SC-condition, n = 38), the other half got the solution task after the classification task (CS-condition, n = 37). Because both tasks relied on 9 experimental word problems, two parallel problem sets were constructed, each containing 3 proportional, 3 additive, and 3 constant word problems. In both conditions, pupils who got Set I in the classification task got Set II in the solution task and vice versa, so that in principle, differences between both sets would be cancelled out.

Analysis

Pupils' responses to the problems in the *solution task* were classified as *correct* (C, correct answer was given), *proportional error* (P, proportional strategy applied to an additive or constant word problem) or *other error* (O, another solution procedure was followed). Evidently, for proportional problems, only two categories (C- and O-answers) were used.

For the *classification task*, the data are more complex. Two aspects of pupils' classifications were analysed. The first aspect concerns the quality of the classifications, the second the kind of justifications pupils provided (as written on their envelopes).

The first aspect involves the extent to which pupils' classifications took into account the different mathematical models underlying the word problems. For each pupil, scores were calculated using the following rules:

- First, the group with the largest number of proportional problems ("P-group") was identified. It acted as a reference group: If children would experience difficulties distinguishing proportional and non-proportional problems, they would probably consider some non-proportional problems as proportional, and thus include non-proportional problems in the P-group.
- Next, among the remaining problems, the "A-group" and "C-group" were identified (the groups with the largest number of additive and constant problems, respectively). When more than one group could be labelled as A- or C-group, the group having the highest score (see next point) was chosen.
- Every group (P, A, and C) got two scores: An uncorrected and a corrected score. We explain these for the P-group. (It is completely parallel for the A- and C-group.) The uncorrected score for the P-group (*Pu*) is the number of proportional problems in the P-group. The corrected score (*Pc*) is *Pu* minus the number of other problems in that group. If no A- or C-group could be distinguished, these scores were set to 0.

The second aspect was the quality of the justifications given by pupils. The justifications for the P-, A-, and C-group of every pupil were labelled using the following distinctions:

- Superficial: Referring to aspects unrelated to the mathematical model: problem contexts (e.g., "these are about plants – tulips and roses", "they all deal with cooking"), common words (e.g. "they both have the word when"), or numbers (e.g., "there is a 4 in the problems", "all numbers are even").
- Implicit: Referring to the mathematical model in the problems, but not unequivocally or explicitly to one particular model. For example, "the more pies – the more apples, the more you buy – the more you pay" does not per se refer to proportional situations, and "they all relate to the speed with which activities are done" does not grasp the additive character of situations.
- Explicit: Referring clearly and unambiguously to the (proportional, additive, or constant) mathematical model underlying the problems (e.g. referring to a proportional model "3 times this so 3 times that, and in the other problem both things are doubled", or referring to the additive model "one person has

more than the other, but the difference stays the same" or for the constant problems "these are tricky questions: nothing changes").

- Rest: There is no justification written, or it is totally incomprehensible. This label is also assigned when the particular group does not exist.

RESULTS

Solution task

	Proportional problems		Additive problems			Constant problems		
	C	O	C	P	O	C	P	O
SC-condition	2.61	0.39	0.65	2.08	0.26	0.24	2.08	0.68
CS-condition	2.76	0.24	1.11	1.86	0.03	1.00	1.70	0.30
Total	2.68	0.32	0.88	1.97	0.15	0.61	1.89	0.49

Table 1. Mean numbers of correct (C), proportional (P), and other (O) answers on the three proportional, additive and constant problems

Table 1 presents the answers to the solution task. A first observation is that the proportional problems elicited much more correct answers (2.68 out of 3 problems, on average) than the additive (0.88) and constant (0.61) problems. A repeated measures logistic regression analysis indicated that this difference was significant, $\chi^2(2) = 23.87$, $p < .0001$. For the additive and constant problems, almost 2 out of 3 answers were proportional, indicating that pupils strongly tended to apply proportional calculations to the two types of non-proportional problems.

More importantly, pupils in the CS-condition performed significantly better than pupils in the SC-condition, $\chi^2(1) = 10.72$, $p = .0011$. Even though the Problem Type × Condition interaction effect was not significant, $\chi^2(2) = 2.76$, $p = .2514$, the most pronounced differences occurred for the additive problems (1.11 correct answers in the CS-condition vs. 0.65 in the SC-condition) and constant problems (1.00 vs. 0.24), while the difference was much smaller for the proportional problems (2.76 vs. 2.61).

Table 1 further reveals two explanations for these better performances: First, CS-condition pupils applied less proportional strategies than SC-condition pupils (1.70 vs. 2.08 and 1.86 vs. 2.08 proportional errors for the constant and additive problems, respectively), $\chi^2(1) = 4.73$, $p = .0297$. Second, CS-condition pupils also made significantly less other errors than SC-condition pupils (0.30 vs. 0.68 and 0.03 vs. 0.26 for the constant and additive problems, respectively), $\chi^2(1) = 8.05$, $p = .0045$.

Classification task

Table 2 provides an overview of the different scores regarding the *quality of pupils' classifications*. First of all, this table reveals a high mean *Pu*-score of 2.37 (on a total of 3). Most pupils put at least 2 – many even all 3 – proportional problems in one single group. In contrast with the high *Pu*-score, the mean *Pc*-score is only 0.40, indicating that pupils frequently also put some (on average almost 2) additive and/or constant problems in the P-group, instead of putting them in separate groups.

	P-group		A-group		C-group	
	Pu	*Pc*	*Au*	*Ac*	*Cu*	*Cc*
SC-condition	2.34	0.42	1.76	1.18	1.58	1.05
CS-condition	2.41	0.38	1.70	1.51	1.84	1.49
Total	2.37	0.40	1.73	1.35	1.71	1.27

Table 2. Mean uncorrected (*Pu*, *Au*, *Cu*) and corrected (*Pc*, *Ac*, *Cc*) scores for the classification task

For the additive and constant problems, the uncorrected scores (*Au* and *Cu*) are 1.73 and 1.71, respectively. These scores are lower than the one for the proportional problems. This is inherent to our scoring rules, because we first determined a P-group (which often also included some additive and constant problems) so that, on average, less than 3 additive and constant problems were left to create A- and C-groups. But still, the size of the *Au*- and *Cu*-values indicates that many pupils did make separate groups for the additive and constant problems. As was the case for the proportional problems, also for the non-proportional problems the corrected scores (*Ac* and *Cc*) are somewhat lower than the uncorrected ones (*Au* and *Cu*), but the difference is not as pronounced as for the proportional problems (1.73 vs. 1.35 for the additive problems, and 1.71 vs. 1.27 for the constant problems, respectively). So, even though other problems were sometimes included in the A- and C-groups (i.e., on average about 0.40 word problems in each group), this happened less often than for the P-group (on average almost 2 word problems).

In sum, the results presented so far point out that most pupils created a group containing proportional word problems, but often also some additive and/or constant problems were included in this group, suggesting that not only in their problem solving but also in their classification activities, pupils had difficulties to distinguish all non-proportional word problems from the proportional ones. Nevertheless, there was evidence that pupils distinguished some non-proportional problems, and made separate groups of proportional, additive, and constant word problems, even though their classifications were often imperfect.

We also compared the classifications in the two conditions. As can be seen in Table 2, the scores for the proportional problems (*Pu* and *Pc*) hardly differ for the SC- and CS-condition (a finding that parallels what was found for the solution task). However, classification scores for the additive and constant problems are somewhat higher in the CS-condition than in the SC-condition (except for the *Au*-scores, which are approximately equal). So, whereas doing the classification task first has a beneficial impact on performance on the solution task (particularly on non-proportional problems), the reverse is not the case: Doing the solution task first does not improve performance on the classification task. On the contrary, it has a slightly negative impact on children's classifications.

	P-group				A-group				C-group			
	S	I	E	R	S	I	E	R	S	I	E	R
SC-condition	19	13	2	4	12	18	3	5	12	19	3	4
CS-condition	16	14	3	4	14	17	4	2	7	22	4	4
Total	35	27	5	8	26	35	7	7	19	41	7	8

Table 3. Number of superficial (S), implicit (I), explicit (E) and other (R) justifications given by pupils to the P-, A-, and C-groups

With respect to *quality of justifications*, Table 3 gives an overview of the various justifications for the P-, A-, and C-groups (for the explanation of the different labels, see the Analysis part in the Method section). A first observation is that explicit justifications are very rare for all three groups of word problems. They occurred in maximum 7 out of 75 cases. Second, many of the justifications are implicit, particularly for the C-group, but also for the other two groups. Third, also many superficial justifications were observed in all three groups. Of course, this does not necessarily imply that pupils actually *used* these superficial criteria while classifying. Their classifications were often in accordance with the underlying mathematical models, so children might have used criteria that acted tacitly, with superficial justifications occurring *post hoc*, in response to the instruction to provide justification for their classification. And fourth, the kinds of justifications are very comparable for the SC- and CS-condition. So, even though many pupils made appropriate – sometimes even perfect – classifications of the 9 word problems in terms of their underlying mathematical models, they were rarely capable to justify their classifications explicitly.

CONCLUSIONS

The current study assumed that – if pupils would work on an unfamiliar task not focused on producing computational answers but on reflecting on commonalities and differences within a set of word problems – they might engage in a deeper kind of mathematical thinking, and distinguish more easily between proportional and non-proportional problems, which might have a beneficial effect on their problem solving skills.

Taken as a whole, the results supported this assumption. On the solution task, pupils were prone to the over-use of proportional methods: Performances on proportional problems were very good, but almost 4 out of 6 non-proportional problems were solved proportionally, as observed in previous studies (De Bock et al., 2007). As expected, pupils' behaviour on the classification task, however, was different. Nearly all pupils classified the proportional problems in one group, but they typically also included a few non-proportional (additive and constant) word problems. Many pupils also made a group of additive problems and another group of constant problems. Most often, pupils did not provide adequate explicit justifications for their groupings, but justified them implicitly.

The difference between the two conditions provided convincing evidence for the potentially positive effect of the classification task. Pupils who received the solution task after the classification task performed significantly better on the solution task than those who immediately started with the solution task, suggesting that the classification task made them more aware of differences among the word problems, which pupils transferred to the solution task.

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THE CENTRAL ROLE OF THE TEACHER – EVEN IN STUDENT CENTRED PEDAGOGIES

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We describe a case study of an unusually 'student-centred' mathematics teacher whose students construct unusual - and positive -– mathematical dispositions and identities. We draw on a self-report 'teacher-centred pedagogic practices' scale, interview and classroom lesson analyses to identify her pedagogic practice and her reflections on these. Our analysis distinguishes 'mathematical' and 'social' strands in her narrative within lessons which ensure that whilst her practices are found to be engaging and ensure agency and choice she maintains firm control over the 'mathematical narrative'. In this sense her practice appears contradictory: Activity Theory suggests this as an objective contradiction between the learners' knowledge and the mathematical reformulation of it that the teacher mediates.

INTRODUCTION

The UK Economic and Social Research Council funded project, 'Keeping open the door to mathematically demanding courses in Further and Higher Education' is investigating the effectiveness of two different programmes of mathematics for post-16 students in England. This involves both case study research investigating classroom cultures and pedagogic practices and students' narratives of identity together with quantitative analysis of measures of value added to learning outcomes. In this paper we focus on teachers' classroom practices as we attempt to come to an understanding of how different practices impact on students' engagement with mathematics in such courses and their future studies.

THEORETICAL PERSPECTIVES

International studies such as TIMSS have analysed mathematics lessons in an attempt to characterise typical lesson structures and flows across different countries. Stigler and Hiebert (1999), for example, suggest that mathematics lessons within a particular country will tend to develop a common format that forms part of the national culture and may be difficult to alter. Our intention, however, was to look for diversity of practice within and across colleges as our ongoing analysis attempts to identify and understand practices that might better support students' learning. We have a particular focus on students in danger of marginalisation from mathematics, as ultimately we wish to inform policy makers and teachers as to how they might structure curricula and implement pedagogies to maximise participation and attainment.

It is clear that teachers' beliefs are fundamental in shaping aspects of their classroom practice, but practices also reflect the historical and social setting in which they are constructed and situated. For example, our case studies suggest that colleges position themselves differently in their “local market-place” for students, who at this level are able to shop-around amongst providers looking for those who are best able to provide for their needs. Some colleges, therefore, have open access attempting to provide for all learners regardless of their ability whereas other colleges position themselves as high achieving institutions and will take only the most able students. Such positioning we find acts in affording or constraining teachers’ practices at all levels including in their mathematics classrooms. Our concern here then, whilst recognising potential institutional and programme filtering (programme design being another important filter), is to focus on teachers’ pedagogic practices. We have therefore built on the work of Swan (2006), who in turn developed the work of Askew et al. (1997) to explore the practices of teachers with different orientations: transmissionist, discovery and connectionist. These different categories reflect varying degrees of emphasis on teaching (with a strong teacher focus) and learning (with a strong student focus).

As we report elsewhere (for example, Pampaka et al., 2007; Wake et al., 2007) there is evidence that teaching that is strongly student focused (connectionist) might better serve those students at risk of marginalisation from the study of mathematics. Our concern here, however, is to explore what such teaching might look like in practise and start to understand some of the key features of appropriate pedagogies.

‘TEACHER-CENTRICISM’

We constructed a new instrument based on Swan’s items (2006) but following a different analytical methodology resulting in a uni-dimensional scale of ‘teacher centrism’. A detailed account of the development of this instrument and the validation of the constructed measure is reported elsewhere (Pampaka et al., 2008). A total of 110 cases/teachers were used for the validation of the measure based on statements about classroom practice with analysis allowing the development of a scale in which both items and respondents could be mapped together (Figure 1). The histogram on the right hand side shows how the teachers were distributed and identifies those whose self-reported pedagogy is mainly student-centered at the bottom of the scale to those whose pedagogy is mainly ‘teacher-centered’ at the top. On the left hand side of the figure the items that constitute the scale are distributed, ranging from the easiest to report frequency of occurrence (bottom) to the most difficult in this sense (top).

The figure is best interpreted considering both teachers and items together: having located a teacher on the scale, it is likely that they are able to agree relatively easily with statements below their position but have difficulty agreeing with items above (with statements being ranked in increasing order of teacher-centrism). ‘Difficult’ statements such as B20 [“I encourage students to work more quickly”] are more easily endorsed by teachers that are located at the high end of the scale who are more teacher-centred in their practices than student-centred. As can be seen practices denoted by ‘easier’ items, such as B17 (reversed) [“Students (don’t) invent their own methods”] are likely to be endorsed by the vast majority of teachers.

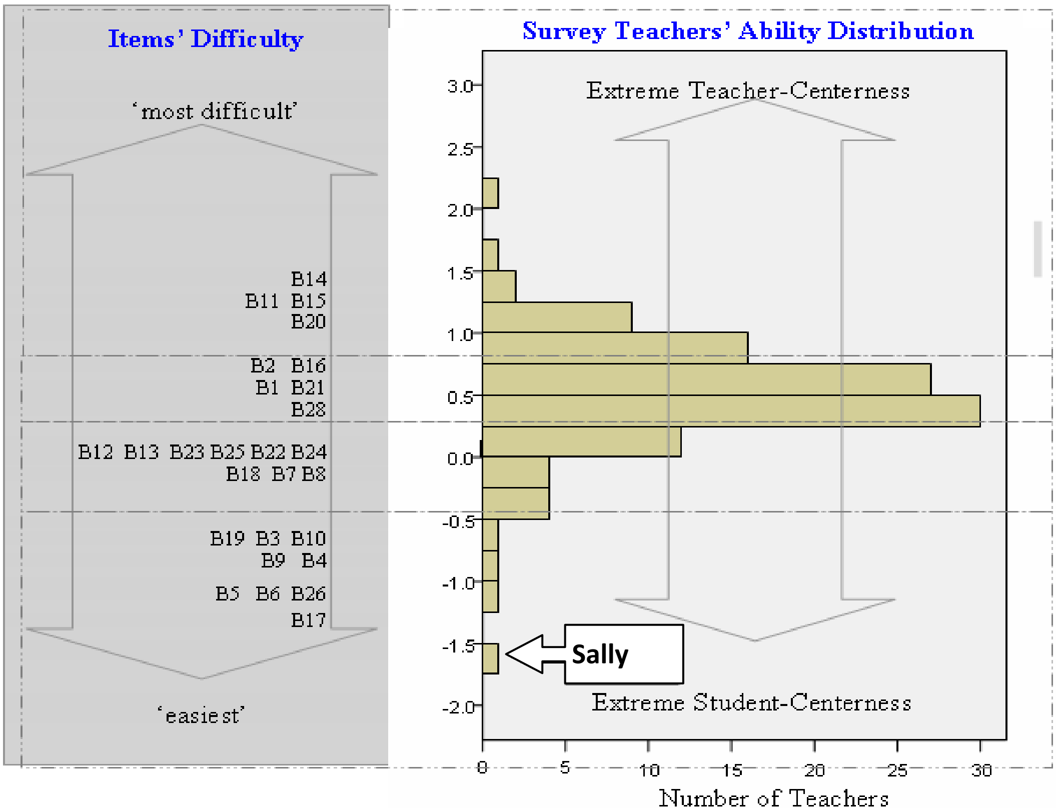

Figure 1. The 'teacher-centrism scale'.

In the brief space available here we focus on what a highly "connectionist" or student centred mathematics lesson might look like. We turn, therefore, to the teaching of Sally who works in England in one of our project case study colleges. As can be seen in Figure 1, Sally is the teacher who reports her practice as being most students centred lying at the very bottom of our measure of 'teacher-centrism'.

Sally has difficulty in self-reporting her practices as being teacher-centred. In just this brief extract from an interview with her, for example, she rejects the teacher centred statements "I tend to follow the textbook closely" [B14] and "I (don't) draw links between topics and move back and forth between topics" [B11] of our scale.

> from the teachers that I've met and talked to… it seems to me that one of the big differences is, I mean I don't sort of use textbooks…that makes me different to start with…in that sense. The reason why I don't use them is because…I want to get students to think about the maths, I want students to understand, I want students to connect ideas together, to see all those things that go together and I don't think a text book does that…

The following report of her practice is based on classroom observations over a number of Sally's lessons carried out in the ethnographic tradition with video and audio recordings being supplemented by observational notes and follow-up interviews with students and teacher. In total we observed eight of Sally's lessons totalling 670 minutes. Here we report just the first part of a sequence of teaching in which the class starts to explore transformations of the graphs of quadratics. As the descriptive account that follows shows her practice clearly demonstrates why she rejects other teacher-centred statements such as “Students (don’t) discuss their ideas” [B15] and “Students work through exercises” [B1].

NARRATIVES IN LESSONS AND OF MATHEMATICS

Although the instrument we use to characterise a teacher’s orientation focuses on distinctive pedagogic practices such as whether or not the teacher relies heavily on a text book, or students work collaboratively in small groups, we suggest that to better understand the students’ experience, and in particular, in terms of mathematics we need a different framework or lens through which to view a lesson. In an attempt to do this we turn to the construct of “narrative”, in the sense of Ricouer, and as developed in educational settings by Bruner (1996) and others. We conceptualise the teacher as “narrator” revealing a mathematical plot whilst drawing on a range of pedagogic practices in an attempt to engage his or her audience in different ways. This allows us to focus not only on the method of engagement chosen (often a dominant feature when making classroom observations, as Shulman (1986) pointed out) but also the structuring of the mathematics; in other words, the story that the teacher tells about this. Pietig (1997) argues that essentially every mathematical argument, however presented, has a pedagogic function vis-à-vis the intended audience. We attempt to build on this, suggesting that pedagogy must have narrative, and therefore that any effective mathematical pedagogy or argument must have a genre of narrative. In an attempt to characterise different genres we suggest that in classrooms a teacher’s narrative might be considered to have two different dimensions: a mathematical dimension based on the distinctive way in which the teacher unfolds their particular story about the mathematics at issue and a social dimension that comprises of two sub-dimensions involving (i) the social discourse between those involved and (ii) the different practices in which they engage. Further details of this framework and its use in describing lessons can be found in Wake et al. (2007).

STUDENT CENTRED PRACTICE

Sally’s classroom is spacious and the walls are covered with posters her classes have made in previous lessons: these are working documents rather than polished artefacts and students tell us they refer to them in later lessons using them as publicly displayed notes. Students sit in groups around tables and at the start of this lesson in addition to their own stationery equipment each has a small (mini-) whiteboard, pen and cloth. Sally’s classes use these regularly to respond to questions she asks: sometimes there is a public display of responses with students holding their mini-whiteboards in the air, at other times Sally will quickly sweep around the room monitoring students’ working on their mini-whiteboards whilst she decides how to take the lesson forward. The classroom environment is designed and arranged by Sally to allow for students to engage easily in pair, group and whole-class discourse and using a range of ‘sociable’ pedagogic practices.

The lesson starts with Sally showing the graph $y = x^2$ on the (interactive) whiteboard at the front of the room, asking for its equation: an initial suggestion of $y = x$ is met with the response from Sally, "nearly", and is quickly followed by the correct answer. She asks everyone to write on their mini-whiteboard a point that lies on the graph and selects some responses which she writes alongside the graph. There follows some discussion with two students contributing about how one would know whether or not a point would lie on the graph if it was not clear from inspection or there were no grid lines to assist. In a follow-up interview Sally considers this an important starting point as she is aware that students up until this point have often been used to a procedural method of plotting graphs of functions by developing a table of points.

Sally then shows a different graph ($y = (x - 2)^2$), and asks the group to write down some points on this and decide on its equation, checking that their chosen points fit this. Sally selects the points that one student had written on her mini-whiteboard and writes them alongside the graph at the front of the room. She asks students to check that these points fit their equation and if not try adjusting this as, "every point has to fit". After a short while students are asked to hold their boards up and Sally makes a selection ($y = 2x^2 + 4$, $y = 2x^2$, $y = x^2 – 4$) of their suggested equations to write alongside the graph at the board. Although one student suggests that the point (0, 4) fits the first suggested equation ($y = 2x^2 + 4$), another is spotted by Sally shaking his head and asked to explain his problem: he suggests that none of the other points fit this particular equation and therefore it is not valid. Sally then checks whether (2, 2) fits, verbalising each step of the calculations as she starts with $x = 2$ leading to $y = 12$, rather than the required $y = 2$. Discussion moves to the next selected equation with students now suggesting whether it is correct or not to the whole class and justifying their decision. Individual students quickly suggest why the remaining two suggestions should be discarded: Sally asks them to try again to think of an appropriate equation. Having circulated the room monitoring the work of students on their mini-whiteboards Sally is just about to move on when a student suggests he has a possible solution, $2y = x^2 + 4$, which as soon as he suggests to the class he dismisses. There is a comfortable atmosphere in the room as students are very willing to make mathematical suggestions and expect their peers to comment on the validity of the statements they make.

Sally now, controlling the mathematical narrative, shifts the students' attention asking them to consider her next "picture" which shows graphs of $y = x^2$ and $y = (x – 1)^2$ displayed on the same set of axes. She draws attention to the fact that the original $y = x^2$ passes through the point (2, 4), marking this clearly on the board and asks "whereabouts is that 4 (pointing to the vertical line segment she has drawn) on the graph that has moved across one?" A student responds by pointing out that it has moved across to $x = 3$. Sally points out that, "to get $y = 4$ when $x = 3$ you don't square the three but square the two". She now asks what happens when the graph is moved another one unit horizontally, and returns to the previous picture. One student suggests $y = x^2 – 4x + 4$ and, after recording this, Sally quickly sweeps the room and identifies that a number are suggesting $y = (x – 2)^2$ by writing this equation on their mini-whiteboards. Again this latter suggestion is checked using the original points that it must satisfy. The group quickly determines that this is a suitable equation. Sally's suggestion of checking the former of the two suggestions is rebuffed by a student who says that it is the same, explaining that multiplying out the brackets in the "completed square" version leads to $y = x^2 - 4x + 4$. Sally now asks students to give a possible equation for the graph she had used to prompt the current line of thinking (i.e. the graph of $y = (x - 1)^2$). Many of the students quickly suggest the correct equation and once again this is verified by examining some points that lie on the graph. One of the students, at this point, tries to explain why this is the equation: this attempt comes without prompting from Sally. Recognising that this is a difficult idea, particularly to articulate, she asks one or two others if they understand and if so to explain in their words. After a number have done so, one student asks if the graph of $y = x^2$ is moved one place to the left "should one be added on in the brackets". It is significant that this was going to be the next area of exploration suggested by Sally: she is able to bring up a ready drawn graph of the situation which she now asks students to consider by again checking whether points on the graph satisfy the equation. Here the control that Sally maintains over the mathematical narrative of the lesson is brought to the surface by the very fact that although a student appears to suggest the next logical line of enquiry it is clear that they have been carefully led to this by Sally's unfolding story about the mathematics. Perhaps, as might be expected, students are asked to explain why one needs to be added to the x before it is squared.

Sally moves on to ask students to write on their mini-whiteboards a possible equation for a quadratic which touches the x- axis and has a positive intercept on the y-axis but where neither axis is scaled: she demands that everyone on a table has a different possible equation. Following a public display, Sally writes three suggestions on the board, $y= (x - 3)^2$, $y= (x - 7)^2$, $y= (x - 4)^2$: she asks a student to explain whether or not he considers these correct and to explain why the number in the bracket is subtracted from x. Due to space restriction here we must leave further description of a lengthy teaching sequence spread over some two hours.

ANALYSIS AND DISCUSSION

This extract of just part of one of Sally's lessons exemplifies how central she is in orchestrating her student-centred classroom. The lesson contrasts markedly with many we have observed in case study colleges where the teacher, whilst equally central, relies on a transmission mode of teaching, telling students key results followed by them working through carefully graded exercises and eventually practising the types of questions they will meet in terminal assessment. The whole-class is almost always socially involved in the development of the mathematical narrative in Sally's lessons: this is fundamental in her planning of lessons, as our follow-up interviews have determined. In her planning it is the mathematics that takes the lead, as she considers, "why, what are the problems, what are the difficulties, what do they [students] need to know, what are the issues?" The mathematical narrative of Sally's lessons is determined by her consideration of these key questions whilst the social narrative which is reflected in the practices she uses to engage her students is subservient to this: having decided on the key features of her mathematics she brings together mathematical and pedagogic content knowledge to devise activities with which she provides a guided re-invention of the mathematics at issue, in the sense of Freudenthal and colleagues (e.g. Treffers, 1987).

We consider that there is a contradiction, then, between the lesson being ‘student-centred’ in some activity and ‘teacher centred’ at other times, with these different phases having different objects and being differently mediated. When Sally engages students in social participation (eg group work, making posters etc) she is working with their ideas, (mathematical and intuitive) and ensures that their agency is given expression. Her own role is one of monitoring / assessing and so on. She facilitates students in their attempts to solve problems or create explanations ‘in their own words’, although they may be adopting some of her mathematics too to the extent that they understand it. The *object* of such activity is the problem the students work on and it is understood that they are to ‘have a go’ with their own mathematical tools/concepts.

However, central to Sally’s teaching are episodes when she takes control of the key elements of the emerging mathematical narrative, when her students’ misconceptions are addressed. Sally interweaves this narrative with the students’ own mathematical productions as they emerge from their ‘sociable’ activity: however, she subtly ensures that priority is given in such episodes to the ‘correct’, or more advanced mathematics that she wants them to understand, and of which she is pretty much the arbiter/judge. The *object* of this activity, then, is effectively to construct some sense of the ‘more advanced mathematics’ of which Sally is the key mediator.

The contradiction between the lesson being ‘student-centred’ in some activity and ‘teacher centred’ in other activity is explained by considering that these episodes have different objects and are differently mediated. The contradiction can be considered to be resolved when – through practice – the object of the teacher-centred activity (the new mathematics to be learnt) has become operationalised by the students as a mediating tool in their own student-centered activity. What one sees when the students begin to play a substantial part in the teacher-centred activity (eg when the students begin to take over the work from Sally) is the beginning of this process. On the other hand this progression also continues as Sally withdraws her pedagogic mediation during student-centred activity (due to lack of space we have been unable to describe later episodes here where this is more evident). This process of operationalisation (or what Leont’ev called automisation) often concludes in Sally’s case with the students being required to write their own notes or poster of what they have learnt, eg with examples of the new mathematics.

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PROFESSIONAL DISCOURSE, IDENTITY AND COMPULSORY STANDARDISED MATHEMATICS ASSESSMENT

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Using a range of perspectives including Lacan's psychological theories of identity, Foucault's postmodern theories of identity within and through productive discourse and Mendick's socio-cultural focus on identity as continuous project in practice, this paper contemplates the ways in which teachers are (self)inscribed and (self)constituted within the standardised mathematics test process. Teachers of Year 5 and Year 7 students at one state primary school in Queensland, Australia recount their experiences of mandatory standardised state Numeracy Tests, exposing the paradoxes, tensions, and challenges the tests present for teachers, and demonstrating the ways in which the Tests play a significant role in teacher's identity work.

IDENTITY, MATHEMATICS, AND TEACHERS – AN INTRODUCTION

As is every member of human societies, teachers of mathematics are engaged in a continuous discursive process of *identification* both as individuals and as social beings. Teacher identity is formed through the teaching of mathematics. Miller and Marsh (2002) describe this process as follows:

> …the various discourses that define what it means to be a particular type of student or teacher in this particular moment…are rooted in the social, cultural, historical, and political contexts in which schools are situated...These discourses of schooling shape what and who schools, teachers, children, and families can become. (Miller Marsh, 2002, p. 460).

Identity can be examined and explained using a range of perspectives including psychological, post-modern and sociocultural theories. Identity theories are shaped by beliefs about where identity is situated and from whence it arises. Côte and Levine (2002) describe oppositional views of identity as either based in the individual or in social interactions as a 'structure-agency debate' which argues the degree to which individuals exercise control independent from social structure and how much social structure determines individual behaviour. From a cultural studies perspective, Holland, Lachicotte, Skinner and Cain (1998) view identity as a *self-in-practice* contending that identity arises within the nexus of sociocultural worlds and the world of the individual. They draw a distinction between *figurative* and *positional* identities, the former described as something generic, desired and imagined, and the latter more specific, located and relational.

Research which examines teacher identity is important for the ways in which it can help us to understand the complex and shifting demands of teachers as individuals acting within the social settings of classroom, school, community, and beyond. Schifter (1996) alerts us to the ways in which teacher identity is multifaceted and formed through professional narratives constructed in practice. Researchers such as Wenger (1998) and Boaler, Wiliam and Zevenbergen (2000) focus on the situated and social nature of identity in education as something individuals build within communities of practice. Hogden and Johnson (2004) and Van Zoest and Bohl (2005) incorporate *specific teacher knowledge*, enactment in *social situations*, and *cognitive engagement* into their theories and analysis of the development of mathematics teachers' identities.

Mendick (2006) is uncomfortable with the word *identity*, explaining that "identity' sounds too certain and singular, as if it already exists rather than being in a process of formation" (p. 23) preferring to speak of 'identity work' or 'identification' (from Hall, 1991). These terms capture the nuanced, mutable and 'lived' nature of identity as situated, as in constant process, as both psychic and relational, and as represented in narrative. Mendick believes that "'identity work' positions our choices as producing us, rather than being produced by us" (p. 23).

Lacan's psychoanalytic theories of subjectivity (1977, 2002) are useful for the ways in which they supplement Mendick's inclusion of the psychic. Lacan points to the desire of the *self* to be 'present' as a *secure identity*. Where many post-modern theorists perceive identity as the product of deliberative, social, conscious constructions of realities of self including 'self as *mathematics teacher* or *learner of mathematics*, Lacan was more interested in the work of the subconscious, believing that it is in the interplay of what he termed the Symbolic, Imaginary and Real psychic registers that identity hovers as something always in the making, something formed and form*ing* in its own seeking of itself.

I use a combination of theories in this paper to explore mathematics teacher identity, in particular the ways in which teacher identity is shaped by standardised mathematics assessment.

RESEARCH DESIGN

The paper uses data from a research project which began in 2005 and was extended in 2007. The administrators of a large suburban school in Queensland Australia were aware of challenges the Queensland Studies Authority (QSA) Numeracy Tests presented for teachers, students and parents and requested research assistance in gathering data and compiling a report to present to the QSA, believing a community-based, bottom up approach to ongoing development in education to be essential. Four Year 5 teachers and four Year 7 teachers were involved in the study in which they shared their views and experiences of the Numeracy Tests. Shortly after the test had been administered, the teachers talked about the test in small focus groups. Their conversations were recorded and transcribed. The thoughts expressed and issues raised by the teachers during these conversations were examined for the ways in which standardised mathematics assessment was implicated in mathematics teacher identity. In contemplating their experiences of the test, the teachers were defining and reflecting their identities as teachers. Some like Karen, Allie and Col had been teaching Years 5 and 7 students for many years and were familiar with the tests, but for teachers like Jemma and Ralph, this was a new experience. In the following analysis of selected excerpts from the teachers' conversations, I look at the ways in which the teachers were both *inscribed* from without and *became* from within, through the productive and signifying processes of their engagement with the test.

RESULTS

One of the first issues the teachers raised in their conversations was how they had prepared the children for the test. They spoke about preparation as something they believed they must do but also as problematic since the test was an unknown.

> Karen: You have to prepare them, but you don't want to scare them. You try to prepare the kids for what's in the test. As you prepare, you think, that in the past there has been a big measurement component, and operations with the calculator. And this year what is the emphasis? - spatial knowledge!...For us to prepare, because of where [the test] comes in the year, we haven't covered all the concepts so you try to pick up what possibly will be there and spatial knowledge [in the past] has only been fairly small [proportion of the test].

> Jemma: This was my first time doing the test. I didn't really know how to prepare them. In the few weeks before you do a lot of practice tests. I worked with number more than anything presuming that this is such a large part of it.

> Ralph: You want your kids to do the best that they can and you try to help them out and you do your preparation…it's good to do them in the multiple choice format that they have, just to familiarise the kids.

> Col: We talk about pressure on the kids but also on the teachers as well, because if we don't, you know… we've got to prepare for this because you're not giving the kids the opportunity to show their best.

> Dan: You've just got to make sure that you've covered everything, so the kids aren't surprised…it's a stress to make sure that you've covered everything, and you've covered it well enough…we did a lot of practice, especially the numeracy one, I found a really good book and it had quite a bit of multiplication, colouring in bubbles. I felt that it really helped them.

These teachers both acted and spoke of themselves as *trainers* whose job it was to ensure that the children were well-prepared without 'scaring' them. They attempted to anticipate the content of the test based on their knowledge of previous tests. They did their best to 'cover' all the mathematical concepts that were likely to be included. Because the test was not in a format that the teachers usually used to assess mathematics in their classrooms, the teachers trained their students for the test through practice so the children would be fit to sit the real test, and 'do their best'. The teachers regarded their attempts to prepare the children as being thwarted by the unseen writers of the test whose agendas were at odds with their own. Here we can discern not only Lacan's Symbolic psychic register at play, in which the "Big Other" of the test itself – its questions, its instructions and its statistical results – both allowed and constrained teacher actions, but also the Foucaultian notion of discourse as 'productive' – hence teachers are produced as coaches, kept in the dark about the content of the test, shut out of the process of test design, and distinctly separated from governmental assessment authorities.

The test was as much a test of the teachers' mathematical skills as it was the children's. Both the Year 5 and Year 7 teachers found that they were sometimes struggling to determine the correct answers for test questions, as these excerpts show.

Allie: Even in the angles [one] as adults we all had huge discussions about that didn't we?. You know the one with the angles.

Dan: It was worded really strangely, and the kids got really confused… they had to do the "insides" (angles).

Kate: Some of it was ambiguous.

Jemma: I found a lot of the questions were trick questions rather than just doing straight operations. I was thinking that these twelve-year-olds were asked to do some horrendous problem solving.

Ralph: I enjoy maths but I didn't enjoy all those spatial questions, probably because I hadn't covered as much of that type of stuff.… It's testing comprehension rather than mathematical skill.

Lee: If we teachers don't know, how are the children expected to know?"

A Lacanian analysis looks to the ways in which the teachers reacted when confounded by some questions in the test. Their need for clarification had led to discussions among the staff to establish the 'right' answers. Teachers, they seemed to be saying, should be experts. That which is too difficult for the teacher must be too difficult for children. To maintain (self)perceptions of mathematical competence, some teachers had looked to external causes such as 'strange wording' 'horrendous problem solving' or 'ambiguity' to explain their own difficulties in answering the questions. Although it may have crossed their minds, none of the teachers suggested that their own mathematical content knowledge may have been insufficient, since his would have undermined the security of their identities as 'experts'.

The test altered the teachers' mathematics programs, and was thus viewed as an intrusion into their everyday role in deciding what mathematics the children should learn and how they should learn it.

Col: To me it rearranges my teaching format…I used to do chunks of things, so you know in the fourth term might have been the measurement or it might have been heavy in that, and therefore you've changed your whole way of teaching because you've got to do bits of everything.

Allie: In a way this term has been totally modified because you've been trying to teach the kids the things that they need to know for the test that's in August... You're disadvantaging them by not doing it…It has really altered, you know, almost like a whole term of nothingness…

As *teachers* they seemed to believe that they were conferred the responsibility and the freedom to design and implement mathematics programs as they saw fit. The test created dissonances by reconfiguring this function. Teaching children 'things that they need to know for the test' – teaching *to* the test, in other words - was rationalised as 'ensuring the children were not disadvantaged', even though in so doing the teachers' best laid programs were reduced to 'nothingness' or 'bits of everything'.

The test ran counter to what the teachers said they viewed as best practice in teaching and assessment of mathematics.

Yolanda: I find it frustrating that we test them that way when it's not what we teach, we teach them to talk and discuss, to find information, that's how they work now….. There's no way you can have it all in your head.

Karen: When we are teaching, you get a mark for the formula and a mark for the process, and a mark for the answer. The answer is the least important part, whereas understanding the process is more important.

Contemporary approaches to teaching and learning mathematics espoused in current curricula were reflected in the teachers' descriptions of their teaching. Speaking as those who know best how mathematics should be taught and learned, the teachers were able to criticise the test for its undermining of exemplary practice. In this way the test enabled the teachers to define and clarify their pedagogical positions.

The teachers also spoke of themselves as intermediaries between the children, the test authorities, and parents. They were particularly concerned about the reaction of the parents and saw their identities as 'competent teachers' under threat if the marks were not what the parents expected. They had developed a self-protective strategy for such a contingency – the role of the mediator. Both children and parents were told by the teachers that the test 'means nothing'.

Karen: Parents will come in and say "Can you explain this to me?" The biggest most important thing to say to them is [their child] may have got very close to the correct answer, they may have actually known what they were doing, but if they colour in the wrong bubble, they are wrong… He understands, he knew to divide, but he made a 'boof head' mistake.

The teachers saw their own assessment of the children's mathematics as more accurate, indeed more 'truthful' than that of the test. This placed them in a difficult position when the test results were at odds with their professional judgement.

Dan: I have a much better idea [of children's mathematical abilities] myself, of what I see every day …it's not a true reflection. I suppose it's a good way…the next reporting system to go back to parents to show where their kids are up to… I don't think it's totally accurate so is that a good thing?

Karen: Sometimes it is a shock. If I think it's no reflection on the child's ability I'll just tell them that…I had a kid with full-on 'flu sitting the test. So when he got his test back he was crying…he was so upset and he was so bright. I said, "Don't worry about this test, mate."

As professionals, the teachers were caught between seeing the merits of the test as a 'good way' to report children's achievement to parents because of its supposed objectivity, and their view of the test as an invalid form of assessment. They trusted their judgements about the children because they were based on professional observations of children's mathematical skills that the test could not 'see'. However, the underlying notion that a truth about the children's mathematical skills was there to be discerned and judged by a trained and experienced professional was reinforced rather than disturbed by the test.

Identifying as the 'good' mathematics teacher was another significant theme.

Jemma: That's a concern of mine that [the results] will come back and they will be down in the dark zone (referring to the shaded part on the scale of results that indicates results below the average or national benchmark). I think I do a good job, but I don't actually know. Maybe I'm not doing a good job. We know that there are some teachers around that really aren't good teachers and this test would probably pick that up…if my kids don't do well I would have thought, "What am I doing wrong?"

Karen: It's not meant to judge the teacher…Those kids bring so much to the test anyway, so if it was a really bad teacher, those kids could still do well.

Ralph: I guess the pressure comes in the results. I was confident that I prepared them well for it, but I haven't had to sit down next to somebody and compared my class and seen whether my kids were above, average or below compared to other classes.

Jemma's distinction between 'thinking' and 'knowing' that she is a good teacher, is telling. It was insufficient for her to think she was a good teacher; she needed some external measure such as the test to tell her for sure. Only then would she really 'know'. The test therefore served as a reliable gauge of performance. For this reason it was worrying for teachers when the children's marks on the test differed from the marks they received in classroom-based assessment tasks because it created a tension between *thinking* and *knowing* they were good teachers. Any discrepancy in the 'truths' that the assessment results told about the child were also difficult for children and parents to reconcile. The teachers were then faced with a choice between either explaining away the test results as invalid, or admitting they were not good teachers. For the teachers it was a relief when the test results were consistent with their own judgements since their desire to be seen as good teachers was fulfilled. Allie's remarks illustrate this very clearly.

Allie: [The test results] came out pretty much exactly the same as the record cards. It's nice when that happens…Normally they look much worse. Maybe I'm getting better at it.

Allie's comment, "Maybe I'm getting better at it," suggests that she believes that a good teacher is one who is able to make judgements about children's mathematical abilities that closely align with the official, objective, standardised and therefore more 'truthful' judgement of the test. Ralph took 'good teaching' a step further:

Ralph: Another point is, when it comes to the federal government's interest in performance-based pay… a national standardised test, something like that, will be a guide …

Karen: Can you imagine how we would be teaching? We would be teaching straight multiple choice. Oh my God, it would be so boring.

DISCUSSION AND CONCLUSION

From the teachers' conversations there is evidence that the test challenged as well as reinforced teachers' perceptions of themselves as teachers of mathematics. The teachers spoke of the format of the test as undermining the ways that they usually taught mathematics. Standardised testing has long been criticised for the way it channels teachers' actions, particularly where classroom practice becomes dominated by 'teaching to the test'. There is clear evidence that this cohort of teachers treated the test not only as a measure of the children's capabilities in mathematics, but also of their own teaching of the subject. Even though school policy dictated that a child's progress should be monitored and reported using a suite of assessment results gathered over time and using 'authentic' assessment methods, the capacity of the test to define, shape and reflect teacher identity subverted the school's attempts to offset what teachers described as the untrustworthiness of the test results.

'Teaching to the test' can be seen as a response that is tightly tied to teacher identity. This analysis shows that the 'identity work' involved in teachers' engagement with the test is complex. Caught between multiple views of themselves as teachers of mathematics within the defining apparatus of the test, the teachers struggled to reconcile their beliefs about best practice involving processes, understanding and discussion rather than correct answers, the test as an authoritative judge of children's mathematical capabilities, their desire to be seen as 'good' teachers as shown by positive test results, and the need/wish to support children to do their best. In the end, despite its disruption to teachers' mathematics programs and the difficulties it created for teachers in mediating between parents and children, the test was vested with authority as much from teachers' choice to go along with it for the part it played in their 'identification' as teachers of mathematics, as from the intrinsic power it wielded as an externally imposed form of assessment. In Mendick's (postmodern) view, such a choice *produced* the teachers rather than being produced by them, in other words, the teachers were 'choosing' to behave only in the teacherly ways that the test allowed and/or demanded, such as trainers and mediators between child, test and parents. From a Lacanian perspective, the choice was tied to the teachers' desire for a secure identity. While they may have railed against aspects of the test for the troubles it caused, in so doing, the teachers were producing themselves as professionals, as mathematically competent experts, and as 'good' mathematics teachers. The test symbolised a commanding authority, made them 'real' as mathematics teachers, and at the same time opened spaces for their visualising themselves as ideal teachers in ideal mathematics classrooms.

This research illustrates how teacher identity is as much about the social and psychical aspects of relationship, interaction and techniques of power as it is about local needs, specific events or specialised knowledge suggesting that the test is only marginally concerned with the teaching and learning of mathematics and primarily serves, intentionally or not, to establish and maintain an accepted 'order of mathematical identity' of which teachers' identities, in a continuous process of formation, are a (self)recognised and recognisable part.

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HOW DO MATHEMATICS TEACHERS DEVELOP TEACHING CONCEPTIONS: KNOWLEDGE, PRACTICE AND COMMUNITY

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This paper explores the developmental processes about the teaching conceptions of practice teachers in mathematics. Six cases of practice teachers in different secondary schools, their mentors and students were involved in the final year study of a three-year longitudinal research project. The case study method, including classroom observations and pre and post-lesson interviews, was used as the major approach to investigate the development processes and values about their teaching conceptions. We preliminarily addressed some reasons that led the teaching conceptions of practice teachers to change, adjust or maintain. We thought that there appeared to be some implications in activating the developments of practice teachers' teaching conceptions, then to develop their professional competencies.

INTRODUCTION

Student teachers of secondary mathematics in Taiwan study both mathematical and educational courses in the university departments, followed by a paid placement of teaching practice at a junior or senior high school as practice teachers. Some experienced school teachers are assigned to be their mentors. It is necessary for teachers to pursue professional growth in the process of teacher education continually; however the stage of teaching practice is a very important period for practice teachers to learn professional competencies. Wenger (1998) proposed a social theory of learning, and the primary focus of this theory viewed learning as social participation. He indicated that "a social theory of learning must therefore integrate the components necessary to characterize social participation as a process of learning and of knowing" (p. 4) and "these components include *meaning*, *practice*, *community* and *identity*" (p. 5). When practicing in school, we view such school as a kind of scale-down society. In schools, there are not only many experienced teachers, but varied social status and identities. Thus, the circumstance of school shapes a large-sized learning *community of practice* (COP) (Wenger, 1998) naturally; and the classroom including mentor and students shapes another miniature learning COP. So, practice teachers learn to develop their professional competencies of teaching in such two-level COP simultaneously. Mentors may play important roles to improve practice teachers' "mathematical power" and "pedagogical power" (Cooney, 1994). Wang & Chin (2007) found that mentors could intervene in practice teachers' teaching to enhance their professional power when mentors thought practice teachers were lack of varied professional knowledge.

Simon (1994) asserted that "the Learning Cycle consists of an exploration stage, a concept identification stage, and an application stage which triggers a new exploration stage" (p. 76) and proposed a six-learning-cycle model. On the other hand, Tzur (2001) distinguished the professional development of mathematics education participants into four levels including learning mathematics, learning to teach mathematics, learning to teach teachers, and learning to mentor teacher educators. Because practice teachers are unexperienced in teaching, they must learn to reinforce their profession of teaching; that is, they mainly learn how to develop their mathematical and pedagogical power. Proulx (2007) asserted that professional development intervention in practice could offer secondary mathematics teachers learning opportunities to experience and explore school mathematics at a conceptual level. Lerman (1994) described critical incidents as ones that could provide insight into classroom learning and the role of the teacher, challenge our opinions, beliefs and notions of what learning and teaching mathematics are about, as well as offer a kind of shock or surprise to the observer or participant. In the light of this, critical incidents can be conceived from teaching aspects, because the incidents might invoke the conflicts and challenges of practice teachers' beliefs and values, as well as their thinking about professional identities from teacher's stand to make the best teaching decisions.

Sullivan (1999) indicated that teaching is a kind of complicated activity involving student's cognitive process, motivation and learning, as well as teacher's designing of teaching activities and framing of classroom norms; at the same time, he thought that teachers could be aware of the teaching questions and possible methods to solve them in specific contexts, and be able to make teaching decisions. When teachers decide and choose how to design lesson activities, and think about when to address the critical questions, they have held important guiding principles of teaching in their hearts to lead them to make final decisions. These guides or principles leading classroom actions are conceived as the pedagogical values of mathematics teachers (Bishop, Seah & Chin, 2003; Chin, Leu & Lin, 2001; Gudmundsdöttir, 1990). So, we assert that teaching conceptions mean that practice teachers reveal their thoughts about the mathematical contents and pedagogical strategies when they are taking the teaching of specific topics. Such thoughts are involved in mathematics teachers' value judgments and choices of pedagogies. So, teaching conception is a kind of objective and intended mental action based on a variety of knowledge, and it is adjusted in terms of the topics, learners, learning circumstances and learning contexts. But some researchers showed that it is difficult to change the mathematics conceptions of teachers (Cooney, 1999; Lerman, 1999; Thompson, 1992), because of their own pre-experience of learning, social identification, or lack of teaching competencies (Chen, 2002). We thought that mentors could play the role of instructor to tutor practice teachers' teaching, and then influence their conception development of teaching. We intend to explore the reasons about the developmental processes of practice teachers' teaching conceptions and what roles mentors play.

Boaler (2002) described the relationship of practice, knowledge and identity as the three major conceptions of the learning theorem based on COP. The framework describes that students learn mathematical knowledge through classroom practice, in which the practice and the arising of knowledge are highly correlative. At the same time, they would develop their learning identities when engaging in the practice of mathematical learning; and the connection of identity and knowledge is then constructed through disciplinary relationship. Chang (2005) further extended the relationship of student teacher's teaching identity, practice and conception paralleled to that of Boaler's framework (as Figure 1). We use the framework to explore the development about the relationship of teachers' teaching identities, classroom teaching practices, and teachers' conceptions of mathematics and pedagogy. Thus, mathematics teachers undertake their teaching practices according to their teaching conceptions, and at the same time, they may renew such conceptions through classroom teaching practices, which in turn enhance or extend the educational knowledge and theorems that they have already known and thus might develop their own styles of mathematics teaching (that is teaching identities). We intend to investigate the effects of practice teachers' teaching conception development in varied practice circumstances and the relationship of teaching conception and practice.

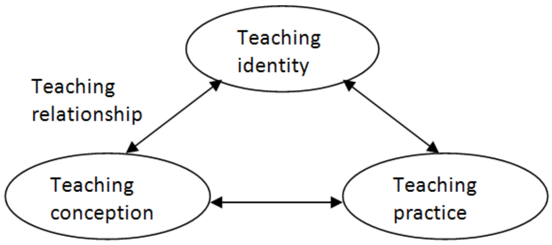

Figure 1. The relationship of student teacher's teaching identity, conception and practice (Chang, 2005, p.165).

RESEARCH METHODS

In this study, we adopt the case study method, including classroom observations and pre and post-lesson interviews, as the major approach of inquiry to investigate the development processes and the values of mathematics practice teachers about teaching conceptions. The systematic induction process and the constant comparisons method based on grounded theory (Strauss & Corbin, 1998) were used to process data and confirm evidence characterized by the method of the present study. Six practice teachers (A_i, i=1~6), their mentors (M_i, i=1~6) and students (S) participated in the 2005 academic year as the third of this three-year longitudinal case studies on the beliefs and values of pre-service teachers for secondary mathematics. The second author of this paper was the university tutor of the six practice-teacher cases. We as both the researcher and tutor visit every A_i twice during the academic year (2005.9-2006.6), one in the first semester (2005.9-2006.1) and the other in the second semester (2006.3-2006.6), observing one lesson of A_i's classroom teaching with M_i and interviewing A_i. Post-lesson interviews are used to clarify the critical issues or events emerging from the observations, and could also be used to explore the conceptions of and underlying values for the A_i's teaching decision-making. During the interviews, we respected all of practice teachers' perspectives and there was no attempt to correct their statements and opinions. And then all classroom observations and post-lesson interviews were tape recorded and transcribed later.

RESEARCH RESULTS AND DISCUSSIONS

According to the data collected through classroom observations and interviews, we initially find some reasons leading the changing, adjusting or maintaining of practice teacher's teaching conceptions. Those include the personality and learning willingness, the teaching circumstance practice teachers engaged in and the teaching resources they could utilize, as well as their professional competencies and future goals settled. It is impossible for us to report all 6 cases in detail, but an outline of our discoveries for the developmental process categories and underlying reasons about practice teachers' teaching conceptions is given in table 1. We will describe in detail the transcripts and interpretations of 3 representative cases about their conception development processes which include the forms of maintaining and changing/adjusting.

Category	Maintaining (A_1, A_2, A_4)			Changing/Adjusting (A_3, A_5, A_6)		
Case	A_1	A_2	A_4	A_3	A_5	A_6
Reason	Mentor's attitudes Teaching circumstance	Teaching circumstance Professional competencies Mentor's attitudes	Mentor's attitudes	Mentor's attitudes Learning willingness	Personality Mentor's attitudes	Professional competencies

Table 1. Preliminary categories of the development processes of practice teachers' teaching conceptions observed

Case of A_2

A_2 took a paid placement of teaching practice in a form like "co-mentoring" (Jaworski & Watson, 1994) which is different from the form of traditional internship. So, she could observe varied teaching of mentors, interact with students and actually teach. Because A_2's mentor (M_2) designed the mentoring strategies carefully and the teaching resources matched with A_2's teaching needs sufficiently, so, A_2 had more opportunities to centralize the learning focus in teaching and to teach. At the same time, A_2 possessed more resources in teaching practice, and she could learn the teaching styles and skills of varied mentors. In the situation, A_2 can reveal her thoughts and conceptions about teaching fully. We found that A_2 maintained her teaching conceptions during the 2 teaching observations.

A_2 had much more own autonomy in teaching, and could plan the teaching contents. In the first visit, we found that A_2 emphasized the use of some activities or examples related with real life to develop the mathematical conceptions, which connected her own thoughts with the suggestions of mentors. She indicated that "I integrate some mentors' thoughts in teaching which are useful for me about my own conceptions, in other words, I reorganize those to be my own teaching activities". At the same time, she said that "my professional competencies are still insufficient, so I must refer to the teaching contents of other teachers". In the second visit, we found that A_2 tried to integrate other teachers' suggestions in her teaching designs much more. We found that teaching circumstance, professional competencies and mentor's attitudes caused the maintaining of A_2's teaching conceptions.

Case of A_5

A_5's mentor (M_5) wished her not to copy her materials when A_5 was preparing her teaching materials. We found that although A_5 designed materials by herself, however, she would first observe M_5's teaching before she actually taught, so, A_5 almost copied M_5's material contents to design her teaching contents. In the first visit, we found that A_5's teaching about mathematical conceptions was not problematic, and we thought that it was related with observing the same topic of M_5's another class before teaching. At the same time, M_5 would discuss the teaching contents with A_5 before her teaching, so, there was less difference between M_5 and A_5's teaching contents. So, we viewed A_5's teaching as 'copying' M_5's teaching in the period.

In the second visit, M_5 changed her mentoring strategies to give A_5 more freedom and let A_5 teach directly without observing M_5's teaching. So, we found that A_5 changed much more in the period. A_5 seemly could ask students some questions to lead their thoughts actively after M_5 guided A_5 how to question and lead students' conception development. A_5 gradually trended to connect her own teaching conceptions with M_5's. Because A_5's personality, she accepted the opinions of other persons easily, so, she had fewer self-opinions in learning to teach. So, A_5 copied M_5's teaching conceptions in the first period, although M_5 asked A_5 not to teach according to her materials fully, however, M_5 didn't give many suggestions. But, in the second period, M_5 asked A_5 to prepare her teaching materials by herself, so, A_5 began to connect her own thoughts with M_5's methods, not only to copy M_5's materials. We thought that A_5 had already changed from the copying to trying to add her own designs and thoughts of teaching. At the same time, she changed her teaching conceptions from listening to students' questions passively to questioning students actively.

Case of A_6

Although A_6 would observe M_6's teaching before her own teaching, M_6 gave A_6 much more freedom and hoped that A_6 could try to teach and learn how to teach first, and then discussed with her in pre and post-lessons. In the first visit, we found that A_6 designed her teaching materials according to M_6's materials, and then taught some lessons using those materials. But she didn't observe all M_6's lessons of the same topics which she would teach. During the A_6's teaching, she would choose the contents of teaching materials, examples and exercises in terms of the discrepancies of students' competencies. Because A_6 emphasized the interactions with students and their learning very much, so she would change the designs of materials and the order of conception developments, and she could intend to approach with students through many 'affective vocabularies' and represented the mathematical symbols using colloquial words. We thought that those are A_6's original teaching conceptions.

We observed that M_6 intervened in A_6's teaching when she was teaching, and most of the interventions occurred when A_6 had difficulty in developing students' mathematics conceptions or solving mathematics or pedagogical questions. When A_6 felt that she was lacking of teaching competency and her teaching made students confused, she would begin to think whether her designs and materials of teaching were inappropriate or not. In the second visit, we found that A_6 paid much attention to the suggestions of M_6 in the designs of teaching contents, and she copied M_6's materials to design her own materials massively. When A_6's teaching could not match with the expectation of M_6, it would yield enormous pressures for her to adjust her teaching methods and conceptions. We thought that A_6 had already adjusted from her original ideas to accepting M_6's suggestions fully because of her deficiency of professional knowledge. At the same time, she adjusted her teaching conceptions from using many affective vocabularies to close students to reducing those words according to M_6's suggestions.

RESEARCH CONCLUSIONS AND IMPLICATIONS

Designing the contents of teacher education programs appropriately

We know that the traditional teacher education programs in Taiwan focused on the development of theories, and fewer programs were connected with real teaching. And teacher educators could perhaps not understand student teachers' conceptions about teaching fully, and didn't improve varied opportunities to motivate student teachers' learning of teaching and to correct their learning attitudes and willingness. In the study, we found that teaching contexts in COP and the accumulation of practice experiences could influence the developments of the cases' teaching conceptions significantly. So, we think that teacher educators could use the exemplar teaching critical incidents as practice materials to connect with pedagogical theories, to stimulate student teachers' thoughts and to challenge their perspectives. But student teachers still need to possess solid mathematics knowledge except having correct learning attitudes, high learning willingness and accumulated practice experiences. Therefore, teacher educators should think how to reinforce the mathematics knowledge of student teachers, to modify their learning attitudes, to enhance their learning willingness, and to assist them to collect practical experiences in teacher education programs.

Emphasizing the responsibility and strategies of mentoring

We found that mentors' attitudes and mentoring strategies play important roles in the transformation of practice teachers' teaching conceptions. Mentors must comprehend the sublime responsibilities in developing their professional competencies. Practice teachers engaging in the different contexts of COP will be influenced by the varied learning forms of COP and mentoring perspectives of mentors. Mentors can adopt the varied mentoring strategies to improve their professional development by asking practice teachers to observe their own mentors' teaching, discussing some critical teaching incidents after lessons with them, and letting practice teachers actually teach some topics. For example, first, mentors may invite other mentors to observe practice teachers' teaching in their own classes, and then analysis practice teachers' teaching each other. Secondly, mentors could encourage practice teachers to observe other mentors or mathematics teachers' teaching, and then share and interchange substantial ideas of and about teaching mathematics with them. Thirdly, mentors could arrange practice teachers to teach other mentors or mathematics teachers' classes, and then involve other school teachers in a form of co-mentoring. Finally, mentors could invite university tutors to engage in mentoring jointly.

Providing appropriate teaching resources of schools

In the study, we identified the reasons about the transformative situations of practice teachers' teaching conceptions were that what teaching circumstances they engaged in and whether they had enough applicable teaching resources or not. For example, A_2 who possesses more abundant teaching resources is relatively active in learning to teach; A_6 increases her own teaching experiences through teaching observations and discussion with other practice teachers; M_5's experience of being as a practice teacher influences her mentoring strategies and methods to mentor A_5; and because of lots of administrative works in the practice school and lack of the opportunities of teaching, A_1's learning of teaching was influenced. We also find that the amount of teaching will influence the transformative forms of their teaching knowledge. For example, A_3, A_5 and A_6 have more opportunities to correct their contents and methods of teaching because of constant accumulations of teaching experiences, and to compare and implement varied pedagogical methods. So, their changes of teaching conceptions are more significant during two periods. Thus, practice teachers who have exposed many pedagogical theories from teacher education institutes must possess much more teaching opportunities to accumulate their practice teaching experiences.

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MIDDLE YEARS TRANSITION, ENGAGEMENT & ACHIEVEMENT IN MATHEMATICS: THE MYTEAM PROJECT

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To date, research has not established evidence-based connections between engagement and students' achievement in mathematics, nor has it provided clear causes for disengagement in the crucial middle-school years. This paper presents the theoretical background and a proposed program of research of a large-scale project that aims to improve academic engagement in mathematics through a mix of multivariate quantitative and qualitative methods- giving rise to the development of an innovative intervention to test our findings.

INTRODUCTION

Recently released results of international studies, such as TIMSS and PISA (Thomson, Cressell & De Bortoli, 2003; Thomson & Fleming, 2004) reveal that generally, Australian students are doing quite well in numeracy overall. However, when results are compared to the previous round of these tests (1994/5), Australian achievement levels remained static while the levels of countries that performed equal to or below Australia a decade ago, now perform equal to or above Australia (Thomson et al., 2004). In the case of NSW students, the national reporting of numeracy benchmarks (MCEETYA, 2005) not only indicate the occurrence of a declining trend in middle-school students' numeracy results, they show that New South Wales [NSW] Year 7 students performed below the national average. This "dip" in results during the middle years is more pronounced in the transition from primary to secondary school (Lokan, Greenwood & Cresswell, 2001). Furthermore, the dip in Year 7 results across Australia increased from 2002 to 2003 with NSW students showing the biggest dip of any state (78.2% of students achieved the numeracy benchmark in 2002 and only 73.9% in 2003).

While the popular belief that social and emotional turbulence associated with the onset of adolescence may be to blame for lowering of performance has some merit, it is also likely that reasons specific to mathematics exist. A number of studies have tried to explain this worsening trend in Australian middle-school numeracy performances, but the findings are often contradictory. For instance, Hollingsworth, Lokan & McCrae (2003) in their report on the TIMSS Video Study, propose that there is a culture of shallow learning and an absence of higher order thinking in Australian Year 8 mathematics classrooms. However, the latest PISA results indicate that generally Australian 15-year-old students performed well on problems requiring creative problem solving skills. Another study by Boaler, Wiliam and Zevenbergen (2000) pointed to US and UK secondary students' dislike of mathematics as a possible factor contributing to poor numeracy performances in the middle years. Conversely, TIMSS-R results reveal that Japanese students demonstrate a significantly stronger dislike for mathematics than Australian students, but continue to out-perform their counterparts in the majority of countries (Gates & Vistro-Yu, 2003). While success in mathematics seems to be a critical factor in determining student engagement, it is obviously not the only factor.

While exploring possible reasons for the pronounced 'dip' in numeracy performance with the transition from primary to secondary school, Siemon, Virgona and Corneille (2001) found that much of the content in Year 7 mathematics textbooks was treated in primary school. This repetition of content means that the progress of many Year 7 students is unnecessarily delayed and could potentially lead to disengagement if students become bored. While student engagement is undoubtedly another factor to be considered in studies concerned with the underachievement of middle-school students in numeracy, as yet research has provided no clear guidelines as to its causes or to the role it plays in student achievement.

In his discussion of issues in middle-school mathematics, Doig (2005) identifies potential problems with the nature of mathematics. He refers to numerous studies indicating that the increased complexity of many mathematical concepts introduced in the middle-years (e.g., fractions and algebra) is a major source of difficulty for students. Similarly, Doig identifies potential difficulties originating from teachers' pedagogy, teacher content knowledge, and the organisation of schools as well as the inadequacy of teacher professional development. However, Wang and Lin (2005) argue differences in educational systems or teaching pedagogy between the top performing Asian countries and their non-Asian counterparts are not totally to blame for the gap in performance.

Furthermore, digital technologies are rapidly emerging as an important factor in student engagement and motivation (e.g., Freebody, 2005), yet there is a lack of solid research data regarding the relationship of these technologies to the quality of teaching and learning, particularly in the context of transitions between primary and secondary classrooms (Smith, Higgins, Wall & Miller, 2005).

THEORETICAL FRAMEWORK

In the past, researchers studying student motivation have conceptualised it in various ways; some taking a cognitive perspective (e.g., Pintrich & DeGroot, 1990), an affective (e.g., Pajares, 1996), a social perspective or combining aspects from two different perspectives (e.g., a cognitive behavioural approach). Hence, research on student motivation has generally focused on separate facets of motivation such as motivational goals (a cognitive perspective) or self-efficacy (an affective perspective) and failed to provide a more realistic, multi-faceted perspective of motivation and engagement. More recently, the necessity of studying motivation as a multi-dimensional construct has been emphasised by researchers (e.g., Athanasiou & Philippou, 2006). Such a multi-dimensional conceptualisation is provided by

Martin's (2007) Motivation and Engagement Wheel and is the theoretical framework we have adopted for our current program of research (see Figure 1).

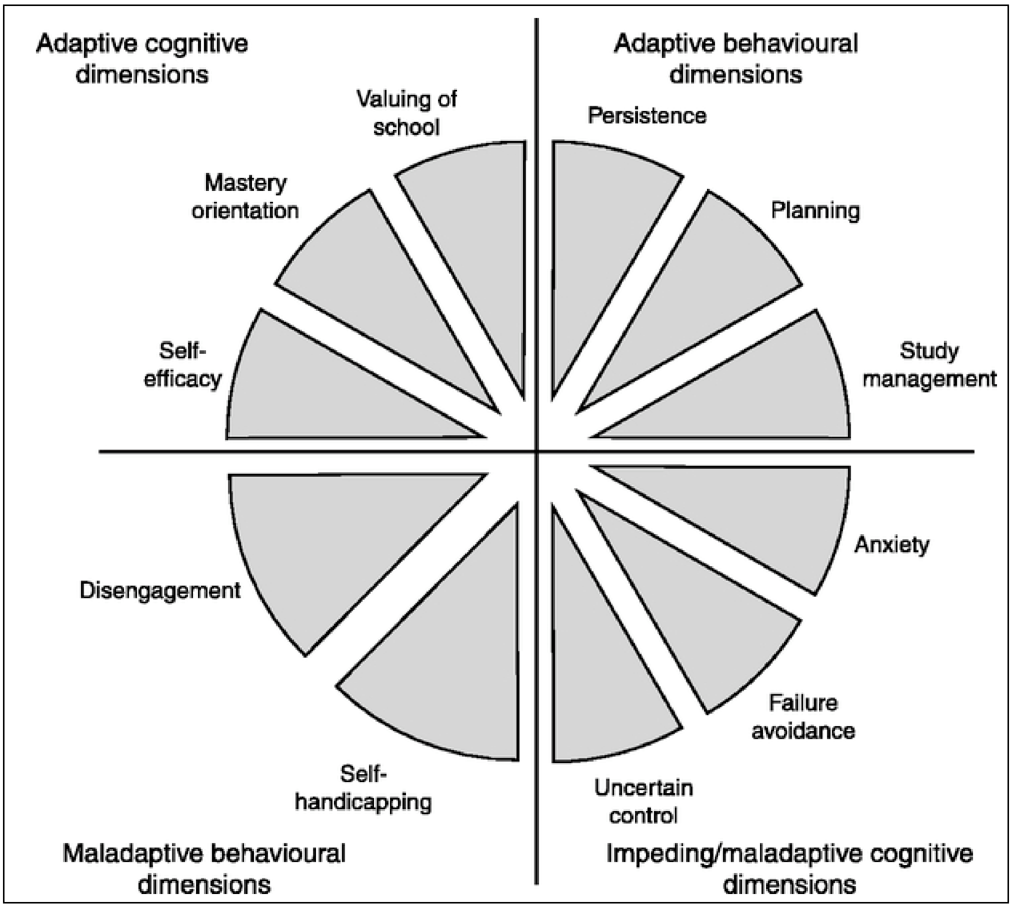

Figure 1. Motivation and Engagement Wheel from Martin (2007).

The Wheel is conceptualised in terms of two levels: a higher-order level comprising (a) adaptive cognitions (self-efficacy, valuing of school, mastery orientation), (b) adaptive behaviours (planning, study management, persistence), (c) impeding cognitions (anxiety, failure avoidance, uncertain control), and (d) maladaptive behaviours (self-handicapping, disengagement). Each of these higher-level dimensions are operationalised in terms of 11 lower level constructs. For example, persistence, planning and task management comprise the Adaptive behavioural dimensions and self-handicapping and disengagement comprise the Maladaptive behavioural dimensions (for a full explanation of the Wheel and its parts, see Martin, 2007). Together, the 11 constructs form the foundation for measuring student motivation and engagement in educational settings. Importantly, the wheel conceptualisation encompasses a number of recent developments in motivation and engagement theory and research, providing a comprehensive multi-dimensional perspective on motivation and engagement. The resultant framework is intended for use by educational practitioners, parents and high school students and will assist their understanding of motivation and engagement. Greater understanding will allow interventions to better target specific facets of motivation and be more meaningful to students participating in such interventions. Hence, the Wheel has potentially important implications for educational practice and motivation research such as is intended in our large and longitudinal program of research.

RESEARCH AGENDA AND METHODOLOGY

The current research project responds to the mounting evidence that student disengagement, particularly in mathematics, is related to academic underachievement, lower participation and retention rates at school, and lower global self-esteem. To date, research has not established evidence-based connections between engagement and students' achievement in mathematics over critical transition points, nor has it provided clear causes for disengagement in the crucial middle school years.

There are four gaps in the current body of research. First, little research tracks students in the significant transition between primary and high school. Second, and implicit in the first gap, is that much of the transition research derives from cross-sectional designs. Hence, it is critical to assess change using a longitudinal design where the same student is tracked through primary and into high school. Third, little research engages multidimensional instrumentation in a domain specific way in the course of transition analyses. Fourth, whilst some research may assess 'primary to high school' transition and other research assesses 'year to year' transition, very little research strives to assess both transitions amongst the same students and in so doing denies a more encompassing understanding of multiple transitions in young students' academic lives.

The project's key aim is to improve academic engagement in mathematics. Combining the disciplines of psychology and mathematics education, it will detail how the experiences and practices of students and teachers in the critical middle years transition into high school uniquely affect student outcomes and aspirations.

Accordingly, the major questions for our program of research are:

1. What is the cause of disengagement in mathematics among middle-years students?
    a. To what extent does class climate, school environment and teacher pedagogy impact on individual students' levels of disengagement in mathematics?
    b. Why is the transition from primary to secondary school so critical to student disengagement in mathematics?
2. What are the major causal factors to the 'dip' (a precursor to disengagement) in mathematical achievement of middle-years students?
3. What are the characteristics of effective pedagogy that enhance students' mathematics-related aspirations at school and beyond?
    a. Do recent shifts in pedagogy as a result of increased use of technology have a sustained positive impact on mathematics-related engagement levels?
4. To what extent and in what ways can a program of teacher intervention positively impact on student engagement levels in mathematics?

To convincingly answer these questions a multifaceted project has been designed that integrates large-scale, longitudinal, quantitative studies with rigorous qualitative methods, both informing an intervention study to test the findings in practice. The project, developed in partnership with a particular school system, comprises a series of five studies across three years (commencing 2008), involving 47 primary and secondary schools.

Study 1 (Project year 1): Cross-Sectional Construct Validity Study

The critical first step in our program of research involves a thorough psychometric assessment of the identified factors of mathematics achievement, attitude, motivation, engagement, aspirations and perceptions of class environment. The primary instrument for this assessment is a composite survey consisting of established scales from the Motivation and Engagement Scale (Martin, 2007), Attitude to School Survey (Khoo & Ainley, 2005), Self Description Questionnaire (Marsh, 1990), Classroom Environment Survey (Fraser, 1990), as well as standardised test results for mathematics and numeracy. Emphasis will be given to evaluating the instruments within a construct validation framework (e.g., Martin, 2007). Students from at least 30 primary schools (Years 5 & 6) and 10 secondary schools (Years 7 & 8) will complete the survey, giving an initial sample size of approximately 4800 students.

Study 2 (Project years 1 to 3): Transition and Causal Modelling Study

The survey will be repeated with the Year 5, 6, 7 and 8 students from the same schools in the second and third years of the study. The longitudinal data enables two key questions to be addressed. The first relates to the nature of transitions across key school-based stages (e.g., from primary to high school and from year to year). The second relates to the causal ordering of important constructs: Does mathematics engagement ‘cause’ mathematics performance or vice versa?

Study 3 (Years 2 to 3): Longitudinal Qualitative Study

The ongoing annual measurement will provide very rich and detailed data about individual students who have evinced significant changes (for better or for worse) in their mathematics engagement and achievement. Using the second Survey results, students will be selected on the basis of their upward or downward shifts between primary school (Year 6) and high school (Year 7). These students will then be tracked from Year 7 into Year 8 and interviews conducted to elicit student reactions to specific issues such as teacher-student relationships, textbooks, syllabus content, homework, memories of primary school mathematics and the learning environment. Additionally, the teachers of these students will be interviewed with particular focus on understanding the individual-, classroom-, pedagogical-, and school-level factors that are relevant to the identified shifts.

Study 4 (Project year 3): Observational Study

There is existing evidence that a good proportion of the variance in achievement is explained at the class/teacher level (Martin, 2007). Study 4 is an opportunity to focus quite specifically on the pedagogy of mathematics classrooms identified as having high levels of student engagement. The top five ranked primary and high school classrooms will be the focus of case studies (hence 10 teachers/classrooms). Approaches to teaching and learning of each case study teacher will be measured in three ways: a pedagogy survey (Anderson & Bobis, 2005), classroom observations (video-taped) of two mathematics lessons for each teacher (20 sessions in total), and interviews with teachers before and after the lesson observations (Anderson et al., 2005).

Study 5 (Project year 3): Intervention to Test Findings

The real test of the validity of the findings emanating from our program of research is to show that using the salient findings from Studies 1 to 4, we can effect change in engagement and achievement through intervention. Study 5 comprises an educational intervention program using design research methodology, which is particularly suited to classroom interventions as it “explains why designs work and suggests how they may be adapted to new circumstances” (Cobb, Confrey, diSessa, Lehrer & Schauble, 2003, p.8). A cluster of schools with classrooms reflecting relatively low results on engagement measures will be targeted. This will involve one high school (four Year 7 teachers) and three primary schools (six Year 6 teachers) in a professional development program that considers the theoretical foundations of the research project and plans interventions derived from the findings, involving both pedagogical and organisational strategies.

IMPLICATIONS OF THE RESEARCH

This paper has presented, albeit briefly, the status of research and theory surrounding student motivation, engagement and achievement regarding mathematics in the middle years. It has also outlined our views on what research is now needed to establish evidence-based connections between these facets of learning during a crucial transition period in students’ educational journeys. It has also presented an encompassing multi-dimensional model-the Motivation and Engagement Wheel-as the theoretical framework guiding our ambitious program of research.

Due to its multi-dimensional approach, the project’s findings will have implications across multiple levels-at the individual, class, school & community levels. Potentially, it will generate new knowledge associated with disengagement, under-participation and underperformance in mathematics and provide practical intervention programs addressing issues that underpin an individual’s potential to successfully function at school, work and in social situations.

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HOW DO MATHEMATICIANS LEARN MATHEMATICS?

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In this paper, I present preliminary results of a study investigating how experts make sense of unfamiliar *mathematical phenomena. Specifically, 10 mathematics graduate students and professors thought aloud and responded to questions as they read and tried to understand an unfamiliar, but accessible, mathematical proof in the domain of geometric topology. Interview data was coded for experts' descriptions of their own understanding of the mathematical phenomena presented, and for instances in which experts systematically linked different types of knowledge to make sense of new information. While experts' pre-existing and post-interview understandings of the topics presented varied, general strategies for connecting specific types of information to make sense of the proof were identified across participants.*

INTRODUCTION

Typically, studies of expert mathematical activity are based on experts' descriptions of their own mathematical practices, experiences and discoveries, or observations of highly performing advanced students and experts as they work with mathematics that they have already considered in depth (Tall, 1991; Vinner, 1991; Dubinsky, 1992; Sfard, 1992; Schoenfeld, 1985; Wilensky, 1991; Lakoff & Nunez, 2000). Little work has been done, however, on understanding how experts think about and make sense of an *unfamiliar mathematical idea as it is first introduced*.

Just as it cannot be taken for granted that mathematics as a field of study is reflective of mathematics as a cognitive activity (Papert, 1980; Tall, 1991; Lakoff & Nunez, 2000); it cannot be taken for granted that the ways in which one describes, understands, or even misunderstands an idea with which they have considerable formal experience is indicative of the processes by which that idea was first acquired. If we are to describe expert *acquisition* and *building* of ideas, we must take all aspects of knowledge into account: not only the structure, description and use of knowledge that experts already possess, but also how experts acquire and build new mathematical knowledge. With this study, I attempt to investigate how this process of knowledge acquisition and construction occurs as an expert engages with a common tool of his discipline – a mathematical proof.

THEORETICAL FRAMEWORK

This study aims to describe the nature of mathematical knowledge – both its *structure*, and how that knowledge is built and enacted in authentic expert *practice*. Therefore, this section will concentrate on major theories of expert knowledge, expert practice, and how these theories inform the design and analysis of the study.

The Structure of Expert Mathematical Knowledge

It is well established that the knowledge possessed by novice and expert mathematicians In novices and students, this collection of experiences, images, features and examples is described as a *concept image* (Vinner, 1991) or "informal knowledge" (Schoenfeld, 1985). As these individuals move to expertise, however, these collections of experiences are described as *encapsulated* or *reified* into a mathematical object with associated processes (Tall, 1991; Sfard, 1992). This distinction – between possessing mathematical knowledge that is better described as encapsulated and object-like entities versus a collective store of experiences and informal understandings – is cited as a hallmark of expertise and advanced mathematical thinking. We cannot assume, however, that every new mathematical idea that an expert encounters is automatically understood as such a formal abstraction. While the encapsulated view of mathematical expertise is particularly well-suited for analysing expert practice as it related to well-known mathematical ideas with which one has had considerable experience, it does not necessarily illuminate the process by which such encapsulation occurs, or how new information about an idea is incorporated into an existing organization of knowledge.

Instead, I use the notion of understanding as *connection* (Skemp, 1976; Papert, 1993) and expert knowledge as *dense connection* between mathematical rules, examples, images, everyday experience, and other resources an expert encounters during study. These rules, examples, and so forth can vary from person to person as a result of one's background knowledge, experiences, their own interpretation of the mathematics as presented in disciplinary materials (Wilensky, 1991). This is not to discount the usefulness of the encapsulated view of expert knowledge – indeed, , but rather to illuminate the *mechanism* by which that encapsulation occurs – in other words, to find out how mathematical ideas unknown to an expert can eventually come to be understood by them in a formal way.

Expert Mathematical Practice

If expertise is characterized by encapsulated or densely connected knowledge that can be deconstructed and reconstructed in a number of ways (Tall xxx), then it is not only the structure of knowledge, but also the act of identifying, manipulating, and coordinating that knowledge that is an important component of expertise. For example, Schoenfeld (1985) showed that experts are more likely to monitor their own progress when solving problems, and that experts often employ general-purpose problem solving heuristics unknown to novices. Sierpinska (1994) notes that a distinction should be made between one's *resources for understanding* and *acts of understanding*, in which such resources are put to use in order to solve a problem or make sense of some mathematical idea. Duffin and Simpson (2000) refer to one's ability to not only *have*, but to *build* and *enact* knowledge.

In the context of describing knowledge and understanding as connections between existing and acquired knowledge resources, it makes sense to also investigate the *strategies* by which such connections occur. How do experts select and coordinate existing and new pieces of a given mathematical phenomenon in order to make sense of, or create, new mathematics? Are the combinations of elements used to build, as opposed to enacting knowledge, different?

Research Questions

1) What is the nature of knowledge acquired by experts as they encounter and make sense of a new or unfamiliar mathematical idea?
2) How is this newly acquired knowledge employed by experts to build an adequate understanding of some new or unfamiliar mathematical idea?

METHODS

Participants

10 participants, including 8 professors (assistant, associate, and full) and 2 advanced graduate students from a variety of 4-year universities in the Midwest participated. Preliminary analysis of 3 interviews is included in this paper. Participants were identified primarily through university directory listings, and contacted via email to see if they would agree to be interviewed. In the email, participants were told that I was interested in how experts reason about mathematics, and that they would be provided an unfamiliar proof and asked to discuss the ideas presented within.

Protocol

Students and professors who wished to participate were given semi-structured clinical interviews using a think-aloud protocol (Ericsson & Simon 1993; Chi, 1997; Clement, 2000). Each was provided with the same mathematics research paper (Stanford, 1998) – not directly related to any of the interviewees' specific fields of research – selected for its relative accessibility in terms of complexity and vocabulary. They were asked to read the paper and try to understand it such that they would be able to teach it to a colleague. They were also asked to describe what they understood of the mathematical ideas presented as they read, if this did not come up naturally in the course of the interview. Interview data was videotaped, transcribed, and coded using the TAMSAnalyzer software. The coding system is discussed below.

Proof

The research paper provided to participants (Stanford, 1998) concerns *links,* which can be thought of informally as arrangements of circles of rope that are entwined with one another, and the conditions under which those circles can be pulled apart. If a link has the property that when any single circle is removed from the arrangement, the rest can be pulled apart, that link is said to be *Brunnian*. If, as a result of the entwining of circles, one circle passes over (or under) a different circle, this is called a *crossing*. If part of one circle passes over another circle and is rearranged so that it then passes under the other circle, this is called *changing crossings*. Finally, if all circles in a link are arranged such that there are *n* distinct collections of crossings that, when changed, make the loops fall apart, the link is said to be *n-trivial.* A *trivial* link is one for which all circles can be pulled arbitrarily far away from one another (or, can be reduced to a point without touching one another). The proof establishes a systematic relationship between the properties that make a link *Brunnian* and *n-trivial*, such that any Brunnian link can be described as *(n-1)-trivial*.

DATA

Two sets of codes were used. First, any descriptions of mathematical knowledge were classified as one of six distinct categories: a *parent*, *fragment*, *example*, *construction*, *prototype*, or *definition*. Although these categories were derived for this study specifically and thus may be an artifact of the structure and content of the task provided to participants, I believe that it is applicable to additional domains of mathematics. To illustrate this, each category description includes a real example obtained from interview data, and a hypothetical example to illustrate how each category would apply to possible descriptions of even numbers.

Parent. Aspects of some mathematical idea that are inherited from more familiar experiences or understandings related to the idea under consideration.

> Ted: Some guy I knew in grad school did some sort of knot theory things, and had I don't know, lots of little lines that were supposed to represent little loops and he'd move them around and see if he could make them look more complicated or less complicated. And so I'm thinking it's somehow related to that, but I don't have a good sense.

> Even numbers as a type of integer: "Well, I know that even numbers are a kind of number, so I can perform operations like adding and subtracting with them."

Fragment. Components, pieces, or relations that comprise the building blocks of a mathematical idea or object; or ways to divide the idea into smaller, easier to manage pieces.

> Ana: "the rest is trivial whatever that means, and I'm assuming that means the disjointed circles."

> Groups of two as fragments of an even number: "They are made up of groups of 2."

Example. Specific instantiations of the idea being considered that are immediately available to an individual, either via recall or because it is provided.

> Mike: "Yea, the Borromean rings should be... I know enough... they have that property… that when you unlink a component, you get a trivial knot."

> The number "10" as an instance of an even number: "I know that 10 is one."

Prototype. Special instantiations of the idea being considered that are assumed to be representative of a more than one single example or instance of the idea.

> Joe: so, so my definition was sort of, construct a canonical example and say this is, any Brunnian link is isotopic to this brunnian link, so.. it's like a representative of equivalence classes of brunnian links, so…

> The last digit of a number as an indicator of evenness: "every number that ends with 0"

Construction. Ad-hoc example, usually developed by combining fragments, examples, and/or prototypes in some way.

> Mark: Okay. And so if I got something like that [forms circle with one finger] and [interlocks with other finger] something interchanging here, if I remove one of the links the other two come apart, then that's what they're talking about .

> An even number as constructed from fragments: "They are made of groups of 2, and 6 is three groups of 2, so 6 is even".

Definition. Complete descriptions of the behavior, structure, or properties of the focal mathematical idea, which accounts for all instances of the idea.

> Joe: …he's saying if we have n-components of brunnian… whenever [turning page] you look at… whenever you throw away one of the components you have something trivial.

> A formal definition of an even number: "Any integer multiplied by 2."

Second, experts' responses to interviewer questions and their think-aloud statements while reading the proof were coded as *questions*, *solutions*, or *explanations*. For example, if an expert simply states that she does not understand some aspect of the proof, that statement would be coded as a *question.* On the other hand, if the expert is not immediately familiar with some aspect of the proof, but is able to use definitions, examples, or other features described within the proof to arrive at an explanation of that aspect, that statement would be coded as a *solution*. Finally, if an expert asserts that she was already familiar with phenomena described in the proof and simply describes that existing understanding, that statement would be coded as an *explanation*. While there are some differences, these codes map particularly well to previous theories of mathematical understanding - notably, Duffin and Simpson's (2000) classification of understanding as *building*, *enacting*, and *having* understanding. Each question, solution, and explanation could contain any number of parents, fragments, examples, and so forth.

Question. Participant does not understand some aspect of the proof.

> Joe: Okay. [takes pencil] Okay, so they're saying something about… n-triviality, I've never heard of that…
>
> Interviewer: Do you have any idea of what that might mean?
>
> Joe: Not a clue.

Solution. Participant is unfamiliar with some aspect of the proof, but is able to use other components of the proof such as definitions, examples, and so forth to arrive at an explanation.

> So he's saying here are these crossings, these are in one set and these two are in another set. But then what does it mean to change them? [pause] So suppose I picked um one corresponding to A, what am I supposed to do what does that mean to change them? I wonder if it means to go from an up crossing to a down crossing, so let's try. […] oh yeah, see I do think I'm right, because that circle is disengaged by changing these two crossings okay. So changing crossings means going from up crossing to down crossing. Until two pages later when we'll realize that that's wrong. (Ana)

Explanation. Participant is familiar with some aspect of the proof, and readily describes their understanding of that aspect.

> [reads] Note that n-trivial implies n-1 trivial for n > 0. [done reading] Which of course, if you remove one link and it's trivial, and then you remove another link, well it's already trivial. You're expanding on your triviality, you're feeling… really trivial. (Greg)

RESULTS

First, experts varied dramatically in the amount of background knowledge they possessed and employed when solving problems, and this affected the resources that they had available to make sense of the proof. However, the *types* of knowledge employed for different expert statements were relatively consistent: for example, experts are much more likely to refer to examples, constructions, or prototypes when working on a solution than when asking a question or explaining an already understood component of the proof. Similarly, experts are more likely to refer to multiple definitions within explanations rather than within questions or solutions – a sign, perhaps, of mature understanding in which an expert is able to, as expected "…link together large portions of knowledge into sequences of deductive argument" (Tall, 1991, p. 4).

Statement Types

While all participants exhibited a relatively even distribution of statement types – that is, they all explained, asked questions, and found solutions during the interviews – the number of elements of the proof that they brought together within each statement type varied. Notably, for all three participants that have been analysed, more elements of the proof were brought together in solutions than in any other type of statement, and less elements were brought together for explanations. This may indicate that experts build the most knowledge – that is, they make the most connections between knowledge elements – when they do not fully understand all of the components needed for a given mathematical idea. Similarly, the small number of elements involved in a given explanation may reflect the object-like, unitary nature of well-understood (or, depending on perspective, well-connected) knowledge commonly described in studies of expert performance with ideas with which they have had a great deal of experience.

	Ana	Mark	Joe	All
Explanations	42%	35%	43%	41%
Questions	32%	29%	28%	30%
Solutions	26%	35%	29%	41%
Elmts. per Q	2.31	2.00	2.41	2.24
Elmts. per S	2.60	3.16	3.20	2.99
Elmts. per E	1.33	1.60	2.21	1.75

Table 1. Statement types and elements per statement type by participant

Patterns Within Statement Types

In addition to patterns in the number of elements mentioned in a given type of expert statement, there were also patterns in the *type* of elements mentioned in various types of statements. Below, I discuss two of these patterns: the high frequency of direct connections between fragments (F) and definitions (D) or between more than one definition (DD) within explanations (DF~E; DD~E), and the high number of embodiments (E; examples, constructions, or prototypes) that accompany connections made between fragments and definitions within solutions (FED), and the low frequency of questions in these categories asked by Mark – a function, perhaps, of his increased use of parent references to make sense of unknown elements within the proof.

Patterns	Statements	Ana	Mark	Joe	All
DD~E	Explanations	50%	0%	18%	12%
	Questions	8%	0%	29%	19%
	Solutions	0%	17%	27%	19%
FD~E	Explanations	50%	60%	27%	46%
	Questions	25%	0%	43%	23%
	Solutions	10%	17%	33%	20%
FED	Explanations	6%	0%	9%	5%
	Questions	0%	0%	0%	0%
	Solutions	20%	33%	13%	22%

Table 2. Frequency of multiple definitions (DD~E), definitions and fragments (FD~E), and definitions and fragments with an embodiment (FED) within statements

Embodiments (E) – items coded as examples, constructions, or prototypes – were much more likely to occur within *solutions*: statements in which experts did not readily understand an idea presented in the proof, but were able to build an understanding using other pieces of the idea. When embodiments did occur, they did so within the context of a solution. It might be that, when participants did not readily understand how two definitions, or a definition and a fragment were related, they used embodiments to help form those connections, which later manifested as well-understood explanations without those scaffolding embodiments (DD~E and FD~E).

CONCLUSION

In this paper, I presented a framework, coding system and data illustrating how experts acquire and synthesize knowledge to make sense of new and unfamiliar mathematical concepts. Preliminary results suggest that though expert knowledge is often described as encapsulated or object-like, it may be experts' familiarity and systemic interaction with a given idea – rather than an experts' status as an expert – that results in this organized structure. Instead, expertise may lie in the ways that experts combine and scaffold their knowledge in order to identify, acquire, and build dense connections between components of a mathematical idea.

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ASPECTS OF THE CONCEPT OF A VARIABLE IN IMAGINARY DIALOGUES WRITTEN BY STUDENTS

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An algorithmic approach is presented to introduce the concept of a variable to students of age 12 to 14. In the learning environment a simple programming language was introduced without using a computer. The students wrote imaginary dialogues in which one can detect various aspects of the concept of a variable. This allows to investigate the potential of the learning environment for enhancing versatile thinking about the concept of a variable. Furthermore, facets of the students' concepts of a variable are elaborated.

INTRODUCTION

The concept of a variable

The importance of understanding mathematical concepts, in particular the concept of a variable, has been the subject of various research projects. Graham and Thomas (2000) formulated in this context:

> To do well in mathematics one needs versatile thinking which enables both procedural skills and understanding of concepts.

The concept of a variable itself has many aspects and is difficult to grasp for students, compare Schoenfeld & Arcavi (1988). When looking at a variable, often it is seen either as a specific unknown, as a general number or in a functional relationship. Ursini & Trigueros (1997) work out a decomposition of a variable, where they explain which aspects are used in which context. Malle (1993) gives two categorizations of variables within elementary algebra. In the first categorization variables can be seen under the *object aspect*, as an unknown, not specified number, under the *substitution aspect*, as a placeholder for a number, and under the *calculus aspect*, as a sign without meaning that can be manipulated by rules. In the second categorization Malle distinguishes between the *single* and the *area aspect*[1]. In the single aspect a variable represents a fixed number, while in the area aspect, either the variable represents a set of numbers (*concurrent aspect*) or a single but changeable number (*changing aspect*). Several learning and research environments use programming languages and graphic calculators to introduce variables to students. For example the language LOGO is used by Papert (1980), Noss (1986), Ursini (1991), BASIC is used by Reggiani (1991) and Tall & Thomas (1991), and graphic calculators by Graham & Thomas (2000). In that way, computer memories are taken as a model for variables. Tall and Thomas (1991) also let students work with inanimated objects.

[1] Note that the English translation of the aspects' names is given by the author.

Mathematical writing

> Writing about a mathematical idea can be an elaborative process which has the potential to enhance understanding. (Shield & Galbraith, 1998)

Within the last years mathematical writing by students has been investigated, compare Borasi & Rose (1989), Clarke, Waywood & Stephens (1993), Gallin & Ruf (1998), and Shield & Galbraith (1998). One can categorize the writings as journal writing and expository writing (Shield & Galbraith, 1998). In a four years long study Clarke, Waywood & Stephens (1993) analysed the journals of students from class 7 to 12. They found three different modes in which the students wrote their journal: Recount, Summary and Dialogue, where Dialogue means an *internal dialogue*. They report:

> In the Dialogue mode, students begin to focus on the "ideas" being presented.

and conclude:

> In this mode, students are able to identify and analyze their difficulties, suggesting reasons why they are thinking in a certain way.

The teachers in the study by Clarke, Waywood & Stephens encouraged the students to write their journals in Dialogue mode.

LEARNING ENVIRONMENT

The aim of the learning environment is to offer students first experiences with the concept of a variable. It is based on algorithms that have to be executed in an enactive way. The students were given a simple programming language which is executed without a computer. Each programme is based on an algorithm that prescribes how to move a robot on an integer grid. The robot stands always on intercept points. The programming language is similar to LOGO, but there are some differences:

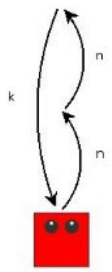
- the programming and execution are performed without digital media
- the language forces the use of variables in every command for moving the robot
- labelled matchboxes are used for the variables
- a robot in the form of a wooden cube is moved on a coordinate grid drawn on a sheet of paper instead of a computer turtle on a screen

Within the language variables have to be given values. The variables are realized by labelled matchboxes which contain a certain number of matches. A matchbox for example labelled with "n" could be assigned by the command: $n \leftarrow 2$. When executing this line of the programme it has to be ensured that two matches are inside of the box with name "n". The basic commands forward(n), right(), and left() can be used to move the robot as well as two other commands that put the robot on certain coordinates. A for-loop is the only construct beside these commands. Since the robot moves on an integer grid, the commands right() and left() always turn the robot by 90°.

Several variables are used at an early stage. One of the beginning tasks has been to ask which variable assignments are possible if the robot moves in a certain way. After having a similar example in the classroom teaching the following task was given as homework:

> Two students work on the following task: The robot was programmed such that one after another:
>
> 1. two variable boxes with name "a" and "b" get filled
> 2. the command forward(a) is executed three times
> 3. the command left() is executed two times
> 4. and the command forward(b) is executed two times
>
> After the execution the robot stands on the intercept point where it started (only turned around). What could have been in the variable boxes a and b?
>
> Write down a dialogue between the two students in which they try to solve the task. One student can solve it easier than the other.

Later the students had to move between different presentations as in Figure 1 ("vorwärts" and "links" are the German words for "forward" and "left").



n	k
1	2
2	4
3	6

$\boxed{n} + \boxed{n} = \boxed{k}$

$2 \cdot n = k$

```
n <- 2
k <- 4
vorwärts(n)
vorwärts(n)
links()
links()
vorwärts(k)
```

Figure 1.

Of these five presentations only one was given to the students and others had to be found. The programme had to be executed with the matchboxes and the robot in order to confirm the results.

METHOD

The project was carried out in two classes in two different grammar schools (Gymnasien) in Bremen, Germany, in 2007, one of grade six with 30 students at the end of their school year and one of grade seven with 31 students in the beginning of their school year. Each project was a three weeks long series of lessons within the normal classroom environment, but the author replaced the teacher in this period.

In total, the students of grade six did four sets of homework in form of a journal, where they had to write their journal while solving a task, explaining what they do. In the last week they wrote an expository text in the classroom, most of them followed the suggestion to use the form of an imaginary dialogue between two students.

The students of grade seven did three sets of homework, two of them in form of an *imaginary dialogue*, where one protagonist understands the task better than the other. The first task for this class was the one named in the preceding section. For the third task the students had to write imaginary dialogues in which they explained programmes that had for-loops. At the end of the series, all students wrote one more imaginary dialogue in quiet time to explain the tasks from the previous lesson.

Note that there are substantial differences between imaginary dialogues and the Dialogue mode mentioned by Clarke, Waywood & Stephens (1993), namely:

- Imaginary dialogues are explicit dialogues between two imagined students instead of having an internal dialogue of the writer.
- The students' task was to write such a dialogue instead of being free to decide how to write their journal entry.

The hypothesis underlying this research is:

A learning environment with the aim to introduce a mathematical concept has the potential to enhance versatile thinking if it allows students to experience the concept under various aspects.

Thus, the following research question is examined in the sequel:

Which aspects of a variable occur in the students' imaginary dialogues?

As a theoretical framework, the analysis is based on the categories of Malle (1993).

FINDINGS

All examples in this section are taken from the students of the class of grade seven. All imaginary dialogues were originally in German[2]. Before we get to the findings, a part of the quiet time work of a boy is quoted here to be able to refer to it at several places. Let us call him A. He calls his protagonists "student 1" (S1) and "student 2" (S2).

1 S2: The robot moves $1 \cdot n$ forward, turns around and moves $2 \cdot k$ forward. Then he stands on the starting field. Thus, when he moves $1 \cdot n$ and $2 \cdot k$ back, it must be $n = 2 \cdot k$.

2 S1: I see, right!

3 S2: k must be the double of n. Since when he goes 1 times something and 2 times back and when he stands then on the starting field, the number he goes once must be the double of the number he goes twice in order that both courses of movement $1 \cdot n$ and $2 \cdot k$ have the same length.

4 S1: So, for example in the table for n we can write 2 and for k 1?

5 S2: Yes, for example. And what is the term then?

6 S1: $k = 2 \cdot n$

7 S2: No!

8 S1: Why not? Next to n we write 2 and next to k 1?!

9 S2: Yes, but he doesn't moves forward 2 times n, but only once. Therefore n is bigger than k, because he moves k twice.

10 S1: Like this: $2 \cdot k = n$

[2] In the following we use the shorthand form "dialogue" for "imaginary dialogue".

11 S2: Yes, because in the programme it is written that in n we put two matches and in k one. But he does not move $2 \cdot n$ and $1 \cdot k$, these are only the numbers which are in the particular matchboxes. With this term we want to find out exactly these numbers. For example $2 \cdot k$ means that the robot moves two times what is in matchbox k.

Students assume role of students with learning difficulties

We can observe that students put themselves into the position of students with learning difficulties. For example a girl writes in her dialogue:

Lilly: For me it is completely different. I don't even understand the question. No one can solve this!

Tina: Then I'll try to explain it to you. Okay?

Lilly: Well, please try it. But I think you won't be lucky with me.

Another example is by a girl of the same class:

Sammy: Why did you say you understood everything when I asked you?

Tom: I felt embarrassed in front of the others.

Some weaker students identify with the weaker person in the dialogue. A boy uses his name for the asking student and the name of another boy in class for the other. Within another text a girl names an invented girl as "smart girl" and herself as "dumb girl" before the dialogue begins.

Learning obstacles are in the focus of the dialogues

Within the dialogues we can observe that learning obstacles are pointed out by the students. For example, one girl names in her quiet time work a problem that was asked several times in the lessons before:

S2: What didn't you understand?

S1: Well, in the table there is written n=1 and k=2, but in the picture the robot walks 2 steps n and 1 x k!

In the lines 5 to 8 of the dialogue of A he let "student 1" run into a "trap" by extracting the wrong equation from the table. In the beginning this was a major difficulty of about half of the students in class. It is interesting that he did not write likewise in his first set of homework. Note that in the lesson before the quiet time work the better students explained the tasks to the weaker ones. So, we can presume that A writes down his observations of the difficulties that the weaker students had.

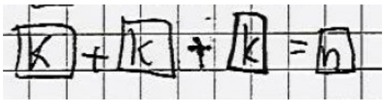
All aspects of a variable occur

One of the main results is that all aspects of the concept of a variable mentioned by Malle (1993) occurred. We will give examples for each aspect.

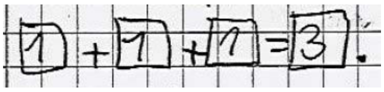
Object aspect: A girl wrote in her quiet time work: "Thus, k is the half of n." and in the first set of homework of a boy we can read: "You calculate a number, that is a, times 3. You calculate the result : 2, then you have b."

Substitution aspect: In one quiet time work a boy draws pictures within his dialogue, where we can see how he substitute variables by numbers:

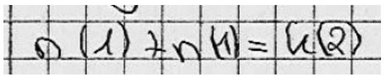
Franz: An example: If the matchbox diagram looks like this



then you insert for each k a 1 and for n a 3 as it is in the table. Then it should look like this:



A girl draws within her dialogue another picture where she invents her own notation:



Since she does not know functions yet, we interpret this notation as a substitution.

Calculus aspect: A girl manipulates an equation in her quiet time work: “You know that n = 3 · k. That means n : 3 = k.” Note that the manipulation rules of equations were not taught yet.

Looking at the second categorisation by Malle, i.e. single, concurrent and changing aspect, we also find examples for each of it:

Single aspect: A boy seems to like big numbers and writes in his first homework:

1. Well, we’ll do it like this. First we must fill the matchbox a, for example with 7 million matches. Okay?”
2. Yes.
1. Then we move three times 7 million steps forward. Okay?”

Concurrent aspect: Besides the single aspect the students grasp that a variable can have several values. A boy writes for example at the end of his explanation: “That way you can check if this works not only for these numbers.” In the quiet time work of A, we can read in the lines 4 and 5, that many numbers are possible because of the answer “Yes, for example.” Filling the tables with different numbers also shows that different values are possible.

Changing aspect: The changing aspect was mostly visible when using the for-loop as in the third set of homework. A girl writes “i is now 1 because it is the first round” and later “i is now two because it is the second round.” In the first set of homework of a different girl we can read: “‘a’ was 1! Now ‘a’ is 2.”

Additional aspect: a variable as a shell

Within the substitution aspect of the first categorization the variable was seen as a placeholder. The learning environment triggered also a different aspect where a variable is seen as a shell. We refer to it as the *shell aspect*. The difference is that a placeholder vanishes when a number is assigned to it, a shell is just a cover or a box for the number but it is still there. If a student sees a variable only as a placeholder or blank space, in the equation x + x = 4 he could substitute the first x by a different number than the second x. Regarding x as a shell, this is not possible. In the students' dialogues we can find the shell aspect for example in line 11 of the quiet time work of A or in a dialogue by a girl: "Then it should be in n half as much as in k."

Further observations

If we order the occurrence of the aspects by frequency, in the first set of homework we have (starting with the most frequent aspect): shell aspect, substitution aspect, object aspect, calculus aspect, while in the quiet time work it is: object aspect, substitution aspect, shell aspect, calculus aspect. Furthermore, the area aspects occurred more often in the quiet time work than before. Looking at the modes of representation: the enactive aspect, in the terms of describing enactive actions, can be found in the first set of homework more often than the symbolic, while in the quiet time work it is the other way round. The iconic aspect did not occur very often in the dialogues.

Some students seem to put their own interests into the dialogues, like the girl writing about a tube of lipstick instead of a robot or a boy, who let two soccer players talk about the homework of the daughter of one of them.

It should be mentioned that in the middle of the series about half of the students did not understand the task of the different representations of a variable, while in the quiet time work only two students remained with significant difficulties and all the others could explain the task well.

SUMMARY

According to the mentioned hypothesis, the learning environment seems to have the potential for enhancing versatile thinking about the concept of a variable, since it allows students to experience the concept under various aspects.

Imaginary dialogues were useful as a method to gain an insight into the students' perceptions of the concept of a variable and in the difficulties in their learning processes.

The students themselves mentioned that it was helpful to them to read the dialogues of the others, that they first discovered while writing what they had not understood and that they understood the solution path well that way.

Finally, the shell aspect as an additional aspect of the concept of a variable became apparent within the dialogues, where a variable remains as a shell if a number is assigned to it.

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GROUP COMPOSITION: INFLUENCES OF OPTIMISM AND LACK THEREOF

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Lesson video and video-stimulated post-lesson interviews were used to study the role of optimism in collaborative problem solving in a Grade 5/6 classroom for the purpose of informing group composition. This study focuses on the activity of two students who differed on the personal characteristic 'optimistic orientation'. It examines how the presence or absence of an optimistic orientation to failures (Seligman, 1995) contributed to these students' interactions with their groups and opportunities for collaborative creation of new knowledge. One group collaborated to develop mathematical knowledge that was new to each group member and the other group did not. These findings raise questions about how to group students who are not yet optimistic to enable collaborative activity, and how to build optimism.

INTRODUCTION

The composition of groups that are likely to support the 'collaborative' generation of knowledge that is new to the group is an area of study that needs attention now that research has raised our awareness of interaction as a crucial aspect of mathematics learning (e.g., Dreyfus, Hershkowitz, & Schwarz, 2001). This study uses previous research findings to inform the composition of groups in a Grade 5/6 classroom, and researcher observation of video data in that classroom, and reflection on interview data, to refine the groups formed to increase opportunities for collaboration. It draws attention to the nature of optimistic indicators that should assist teachers to identifying whether or not a student is optimistic and illuminates how differences in the personal characteristic 'optimism' can influence whether students learn groups.

THEORETICAL FRAMEWORK

The term 'collaboration' has been used to describe 'peer tutoring' situations (e.g., Wood & Yackel, 1991) involving an 'expert other' (Vygotsky, 1978) who is a peer who knows and explains. This term has also been used to describe group interactions in which new knowledge is developed without the continual presence of an expert other (Williams, 2002, 2007). In this paper, the second description of collaboration is used. The purposes associated with group formation differ depending on whether peer tutoring or the collaborative development of new knowledge is the goal. Group composition for peer tutoring requires an expert other as peer tutor (e.g., Webb, 1991). Group composition for collaboration requires all members of the group to be unfamiliar with the mathematics under focus. By working together outside their present understanding in overlapping Zones of Proximal Development (ZPD) (Vygotsky, 1978; Brown, 1994), these students can collaborate as long as their ZPDs continue to overlap. For ZPDs to continue to overlap, students need to be able to think at a similar pace (Williams, 2005). One of the groups in this study shows what can occur when ZPDs do not overlap.

Conditions for collaboration during mathematical problem solving include discovering a mathematical complexity that is unfamiliar to all group members and spontaneously deciding to explore it (Williams, 2002). It is accompanied by intense interest, and high positive affect (Brown, 1994; Kieran, & Guzmàn, 2003; Williams, 2002) or 'flow' (Csikszentmihalyi, 1992). Conditions for flow during mathematical problem solving (Williams, 2002) include: a) discovery of a complexity that is unfamiliar to all group members; b) spontaneous interest in exploring it; and c) similar paces of thinking to maintaining overlapping ZPDs. Some students are inclined to take part in such activity and others are not. The two students whose activity is examined in this study illustrate these differences. Williams (2005) found that the personal characteristic 'optimism' (Seligman, 1995) was associated with this inclination when students were individually involved in creating new knowledge. This raised questions about the role of optimism during collaborative problem solving. This is the focus of the broader research study from which this data is drawn.

An optimistic child perceives failure as temporary, specific, and external, and success as permanent, pervasive, and personal. The inclination to explore unknown territory rather than remain within the confines of what is already known (i.e., the inclination to collaborate) is associated with optimism because exploring what is unknown (present failure) is consistent with the perception that 'not knowing' is temporary and 'knowing' can result from personal effort [Failure as Temporary; Success as Personal]. This involves identifying what can and cannot be changed and making decisions about changes that are likely to increase chances of future success [Failure as Specific]. The research question that focuses this paper is: Does optimism and/or absence of optimism influence opportunities for collaboration?

RESEARCH DESIGN

Context

The Fours Task is the final task in a series of three tasks undertaken over six lessons across the school year in a Grade 5/6 classroom in a government elementary school in Australia. The task spanned one eighty-minute lesson in which two cycles of five to ten minutes of small group activity were followed by whole class reporting sessions. Students were asked to improve their speed in generating numbers after they had: worked individually on a task for three minutes; shared their results; and checked each other's answers. The task required students to make each of the whole numbers from one to twenty inclusive using four of the digit four and as many of the following operations and symbols as necessary "+ + - - x / ÷ √ . ( ) ^{2}". The researcher and teacher team-taught with the researcher as the primary implementer of the task. Groups were given tiles with fours, operations, and symbols on them. Transparent tiles were used by students, on an overhead projector during reporting sessions to enabled students to communicate in visual images and language (Ericsson & Simons, 1980). This also contributed to the data collected.

During the first reporting session, groups could focus on any of the following: two numbers they had generated; something they had found, something that was not working that other groups might be able to help with; a 'big picture idea' that helped generate numbers faster; or anything else they had found that they thought could be useful to other groups. The task was accessible to students with varying understandings of whole number operations because numbers could be generated with simple operations, or a wide variety of permutations and combinations could be used when students were familiar with many operations and symbols. Groups were expected to learn more about operations and symbols and how to use them from the reports of other groups and this was expected to increase their opportunities to create new sums. Trying to generate sums fast was also expected to elicit generalisations.

Focus Students

Patrick, a high performing student who displayed frequent indicators of optimism in his interviews, enacted optimism in his classroom activity. He reported learning by reflecting on the mistakes of others and on what had not been completed. He identified possible variables he could control, and adjusted them to increase the likelihood of success [Failure as Temporary; Failure as Specific; Success as Personal]. Patrick's group was not altered, by the researcher throughout the tasks because this group collaborated well. One student was absent during this task.

Sam was the highest performing student on tests and was perceived as 'very good at mathematics' by class members and his teacher. He described the tasks in this study as boring and stated that he did not learn anything new. He described learning as listening to the teacher, reading books, and searching the internet [Success as External], not as self-generated knowledge [Success as Internal]. To the surprise of his teacher, Sam's understandings in relation to the tasks were, and remained instrumental (Skemp, 1976).

Criteria for Group Composition

Criteria for group formation were informed by research literature. The teacher was provided with these criteria to form the original groups. These groups were refined from task to task after observation of group interactions in the lesson videos and interviews. In this way, there was opportunity to increase the collaborative nature of interactions, and test emerging ideas about the role of optimism in collaboration.

The criteria (with their purposes in brackets) are now provided: a) students thinking at the same pace grouped together (enable overlapping ZPDs); b) never less girls than boys in a group (increase likelihood of girls participating); c) every group to have a positive, encouraging member with more influence than any negative members of that group; d) separate behaviour problems to buffer against negative activity (to retain task focus); e) separate friendship groups (to decrease previously constructed interactions that might not be intellectual in nature). An excerpt of an email from the researcher to the teacher prior to Task 2 illustrates the refining process:

> Callum and Amit played around a lot. I think Amit would contribute if we added a serious eager boy … [like] Jarrod. … [and] Elsa might have more to contribute [in her group] if Jarrod were not dominating.

This excerpt addressed several criteria: participation of girls, and separating and focusing off task students. Adding Jarrod to the group was expected to provide conscientious participation from a boy sufficient to focus Amit, and transferring Jarrod from Elsa's group was expected to increase her participation opportunities.

During The Fours Task, the criteria used to form Sam's group were modified to try to give him further opportunities to create new mathematical ideas because the teacher was surprised he had not done so. He was placed in an all boys group in case he was uncomfortable working with girls, and two boys (Jarrod and Wesley) who had demonstrated they thought at a fast pace and were willing to collaborate were included in the group. Although the fourth boy, Donald had previously dominated activity in another group and taken it off track, it was considered that Jarrod would be focused enough and sufficiently dominant to keep this group on track.

Research Method

To enable study of group interactions, group reports to the class, and individual student learning resulting from these interactions, the Learners' Perspective Study methodology (Clarke, 2006) was adapted to capture the private talk of at least three groups, the physical activity of all groups, interim reporting by groups, and student reconstruction of their thinking in video stimulated interviews. Four cameras were used and group written work was collected. Three cameras captured the groups and the fourth camera captured the reporting sessions at the blackboard and overhead. A mixed image was generated with a group at centre screen and the reporting sessions as an insert in the corner. Post-lesson video-stimulated individual interviews were undertaken with four students after each lesson. Students were selected from at least two groups each lesson based on the positioning of video cameras, and the interactions in that group. In these interviews, students controlled the video remote and found and discussed parts of the lesson they considered important. Indicators of optimism were captured through questions like: "How do you learn something like that? and "How are you going in maths and how do you decide that?

RESULTS AND ANALYSIS

This section reports the individual work undertaken by Patrick and Sam and the types of interactions they took part in during group work. These activities in combination show the effects of Patrick's optimism and Sam's lack of optimism on collaboration.

Initial Three Minutes Individual Work on the Fours Task

Patrick purposely chose to use operations and symbols that were more challenging for him: "I went looking for hard one's first like decimals and stuff and times". He generated more than half of his sums by retaining underlying structures and changing the positions of operations. He progressively increased the number of unfamiliar symbols and operations he used in a sum. Patrick was willing to move into unknown territory to develop new mathematical ideas. Table 1 shows the sums generated by Patrick [Column 2] and Sam [Column 3] in descending in the order matching the order they were generated in. The rows in Table 1 group these sums according to how they were generated, indicate their accuracy, and summarise the activity.

Types of Sums and Answers Generated	Patrick	Sam
Systematic Generation	$4^2 + 4 - 4/4 = 19$ $4^2 - 4 + 4/4 = 13$ $4^2 - 4 - 4/4 = 11$ $4 \times 4 + 4/4 = 17$ $4 \times 4 - 4/4 = 15$	$4/4 + 4 + 4 = 9$ $4/4 + 4 - 4 = 1$ $\sqrt{4} + 4 + 4 - 4 = 6$ $\sqrt{4} \times 4 + 4 + 4 = 20$
Other Sums Generated	$\overset{8}{(4 + 4)} - \overset{16}{4 \times 4} = 8$ $.4 \times 4 + 4 + 4 = 13.6$ (queried whether allowed)	$4 + 4 + 4 - 4 = 8$ $4/4 + \sqrt{4} + 4 = 7$ $4/4 + 4 \times 4 = 17$ $4 \times 4 + 4 - 4 = 5$
Incorrect Calculations	$(4 + 4) - 4 \times 4 = 8$	$4 \times 4 + 4 - 4 = 5$ $\sqrt{4} \times 4 + 4 + 4 = 20$
Summary	7 Generated, 6 Correct 1 Incorrect calculation, Systems used more effectively	8 Generated, 6 Correct, 2 Incorrect calculations, Systems used to some extent

Table 1. Sums Generating Numbers in 3 Mins Individual Time on Fours Task

Sam generating eight sums quickly, stopped early, looked around, and appeared surprised that others were generating longer lists. He then covered his work but did not generate more sums. Although Sam's number fact recall was faster than Patrick's, the sums Sam generated, and his less sustained use of patterns to generate sums showed less evidence of experimentation. Unlike Patrick, Sam did not progressively increase the number of harder operations he used in the same sum, and did not try decimals or brackets. Sam was not inclined to explore. These findings support the indicators of lack of optimism Sam displayed in his interview.

Group Interactions

Patrick contributed to the development of new ideas in various ways. For example, when Gina generated a sum and Eliza queried it, Patrick looked at what could be changed thus eliminating the need to start again and demonstrating a strategy: "Put something in the middle like a plus or something" [Failure as Specific]. He was the first to begin to package parts of sums together as mathematical objects (e.g., '4/4': "Well four on four is just one *whole*!" and '- 4 + 4': "We don't really need these … they cancel each other out"). Subsequently, group members referred to -4 + 4 as 'zero'. Patrick's optimistic orientation to failure increased his group's opportunities of recognising mathematical structure within the sums, and strategies to use.

After individual work and sharing time, Sam used ideas gained from Jarrod's and Wesley's more extensive lists to increase his own list instead of working with others as expected. New sums he generated included: '$4^2 - \sqrt{4} + 4 - 4 = 16$; $4\sqrt{4} + 4 - 4 = 8$; $(4/4 + 4) \times 4 = 20$'. Seeing the work of more expert others enabled Sam to use: the index 2, a product of a square root, and brackets. This showed Sam's lack of inclination to move from what was known into unknown mathematical territory even though he wanted to generate a long list. It would appear that the indicator of lack of optimism 'Success as External' that Sam displayed in his interview contributed to his lack of inclination to create new ideas. Sam perceived learning as occurring through expert others and not through creating ideas when there was no expert, thus he was not inclined to collaborate as he did not see this as an option for learning.

When the group were meant to be exploring the task with the intention of finding fast ways to proceed, Sam used his own sheet, including his later generated sums, to focus discussion and refused Jarrod's sheet that he offered in this discussion. Sam monopolised the group time in explaining to Donald how to get answers to sums and attempts at collaborative interactions were inhibited. For example, Wesley's attempt to put forward an argument for why the decimal point could not be used was not 'taken up'. This group did not create new knowledge around this idea but rather reported that it was not possible to make whole numbers with a decimal point in this sum without adequate justification. They gave one example containing additions and subtractions to support their argument even though Wesley and Jarrod had given stronger justifications for other arguments in previous tasks.

Patrick's group developed several big ideas including: -4 + 4 could be used when wanting to obtain a small answer; brackets can change the size of the answer; and it may not matter whether multiplications or divisions are done first because the answers seem to be the same in these cases. Through collaboration, this group came to realise that the order of operations was important because different answers were achieved when these sums were calculated in different orders. Patrick's optimistic activity was crucial to these outcomes.

DISCUSSION AND CONCLUSIONS

The composition of Sam's group that did not fit with the optimal criteria formulated: Donald's pace of understanding was slower than that of other group members, and the dominating influence was not a student who encouraged new ideas even though it was considered that such a student had been included (Jarrod). Peer tutoring took the time that could have been used for collaboration about potential uses of the decimal point. Although Wesley made attempts to move the focus beyond the secure understanding of the group by focusing on this (Csikszentmihalyi, 1992; Williams, 2002), Sam and Donald inhibited these attempts. Jarrod was unable to fulfil a role of encouraging collaboration because Donald and Sam in combination used the time to focus within the present understanding of three group members. Donald worked in his ZPD with the assistance of an expert other (Sam) who willingly took on this role. Thus, the absence of overlapping ZPDs for group members was a contributing factor to inhibiting collaboration. In addition, Sam's lack of optimism inhibited the usually collaborative interactions of Wesley and Jarrod. Because Sam was not inclined to challenge himself (as evidenced through his individual work), he decreased opportunities for the group to work outside his own understanding by using the time available for collaboration for other purposes and refusing to focus on more creative work than his own (included on Jarrod's sheet). In comparison, in the other group, Patrick's activity continually set up the conditions for flow that contributed his group's frequent collaboration and the development of mathematical ideas. The optimism of other students in his group supported such interaction.

This study of the activity of two students in two groups is sufficient to raise issues for further study. The findings demonstrate that lack of optimism can inhibit collaborative activity and so further research is needed to find how to best compose groups when some students are not yet optimistic. This study also highlights the need to undertake research on developing optimism in our students to study the effects of such outcomes on problem solving capacity. Seligman's (1995) findings that optimism building occurs by engineering flow situations provide a fruitful area for further study. Research presently being undertaken contributes to this area of study (see Williams, 2008).

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CLASSROOM TEACHERS' REACTIONS TO CURRICULUM REFORMS IN MATHEMATICS

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Teachers' reactions to curriculum reforms are important because of their immediate influence on students' experiences of mathematics. Reactions to reforms and their effects on teaching practices were documented from 22 experienced classroom teachers in 18 elementary schools who voluntarily participated in a written survey and structured interview. While all teachers displayed a strong commitment to mathematics teaching and learning, the evidence indicated many reforms had not penetrated schools and classrooms sufficiently to bring about long lasting changes to teaching practices. Furthermore, most teachers had not participated in any significant mathematics education since their initial training and resented top-down reforms that were often introduced with little or no professional development.

Although teachers hold the key to reform in mathematics education (Battista, 1994), few studies have examined teachers' reactions to the reforms they have encountered and their long term effects on teachers' pedagogical practices in the classroom. Since the 1960s numerous reforms in mathematics have been enacted in many countries but each has been largely unsuccessful (Handal & Herrington, 2003), with many teachers continuing to use traditional pedagogies (Sparks & Hirsh, 2006). Most reforms have been imposed by education authorities through a top-down approach (Kyeleve & Williams, 1996) which does not take into account teachers' pedagogical beliefs and knowledge (Knapp & Petersen, 1995). Furthermore, the pace of reform often mitigates against change, with many teachers suffering from reform fatigue and burn out (Hargreaves & Evans, 1997) while others have become cynical and resistant to reform efforts (Handal & Herrington, 2003). Although some teachers ignore reforms altogether, many respond to reform initiatives superficially (Fullan, 1993) by incorporating some of the more easily assimilated practices into their pedagogical repertoire (Hargreaves, 1994), but without really fully understanding the underlying theories, principles and rationale for the changes (Fullan & Stiegelbaure, 1991).

Failure of curriculum reforms in mathematics is a significant problem worldwide, with a lack of congruence between the intent of curriculum innovations and teachers' pedagogical knowledge, beliefs and practises the most cited reason for the poor history (Cuban, 1993). Curriculum change is complex (Handal & Herrington, 2003) often requiring substantial shifts in teachers' beliefs about the nature of mathematics, effective pedagogical practices, understandings of how students learn mathematics and skills in assessing students' learning (Timperley, Wilson, Barrar & Fung, 2007). Furthermore, successful implementation of mathematical reform initiatives, particularly those based on a constructivist paradigm, is hampered by limitations in the knowledge and competencies of aging teaching workforces in many countries

(Timperley et al., 2007). Many teachers were educated during the traditional mathematics-as-computation era (Gregg, 1995) in which mathematics was regarded as sets of transmitted facts and procedures. In Australia mathematics was introduced into elementary schools in 1966, so many of the older teachers were only taught arithmetic during their own elementary schooling (Keeves & Stacey, 1999).

Very little is known about teachers' experiences of curriculum reforms as they are enacted at the classroom level and professional learning (PL) associated with them. Teacher PL is seen the panacea of reform efforts, with the pace and scope of curriculum reforms in mathematics dependant upon the teachers continuing PL (Organisation for Economic Co-operation and Development, [OECD] 2004). However, some teachers do not engage in any PL beyond their initial teacher training that in many cases was undertaken quite some time ago (OECD, 2004). In the recent Programme for International Student Assessment (PISA) studies, Australian teachers' rates of participation in PL were ranked third alongside USA, UK and Sweden (McKenzie & Santiago, 2004). However, teacher participation in PL varies within and between schools and is often informal, episodic and fragmented (Skilbeck & Connell, 2004). Many teachers may not have had sufficient time (Snow-Renner & Lauer, 2005) or opportunities (Borko, 2003) to develop the requisite mathematical knowledge, skills and understanding to enact reforms successfully. Furthermore, time allocations and attendance at PL activities do not of themselves guarantee changes in teacher practices as change takes place slowly over time (Timperley et al., 2007).

THE PRESENT STUDY

This paper focuses on 22 elementary teachers in South Australia with over 10 years experience in teaching mathematics who completed a written survey and volunteered to be interviewed as part of a longitudinal study of curriculum reforms in mathematics (see Yates, 2006a; 2006b). Several reforms have been enacted since the 1960s, with some mandated by the State and Commonwealth Governments as well as more local changes. Teachers' reactions to the reforms are gauged from their written responses to open-ended survey items and oral responses to a structured interview.

Aims of the study

1. To explore experienced teachers' reactions to curriculum reforms in mathematics
2. To investigate teachers' reports of the effects of the reforms on their current classroom teaching practices in mathematics.

METHOD

Participants

Seventeen female and 5 male teachers in 18 government schools, ranging from 36 to 62 years (mean 52.45 years) participated. Almost half had been teaching mathematics for more than 30 years, with only three teaching less than 16 years. One teacher had completed a 2 year teacher education course, while 20 had a Bachelor degree. Only 1 teacher had studied mathematics at the tertiary level.

The Written Survey

All teachers provided written comments to three open ended items in the survey:

Reform in maths is inevitable and teachers just have to learn to go along with it

While there have been many reforms, the most effective way to teach maths is to show students what to do and give them time and opportunities for practice

How do you think curriculum reforms in maths should be introduced?

The Interview

All teachers were all asked the same set of six questions in individual interviews:

Which reforms in mathematics have you experienced?

What do you think about the reforms you have experienced?

Have you changed your teaching methods to accommodate changes suggested by the reforms?

Why (or why not) have you made the changes?

How do you think the changes to mathematics teaching and learning should be introduced?

What methods do you think are most successful in helping children to gain a good grounding in mathematics?

Procedure

Surveys were administered to teachers in their schools by reply-paid post. Interviews were conducted individually with each teacher in his/her school and audio-recorded.

RESULTS

Written comments to the open-ended items were matched with the transcribed interviews using each teacher's self assigned four letter ID code and the responses collated. Results are presented in the topic order of the interview questions.

Mathematics reforms experienced by teachers

Teachers listed more than 20 maths reforms and curriculum initiatives which had made some impact on their teaching and PL over their careers. At face value, many interviewees seem to have participated in an impressive array of reforms, but what is largely missing from their accounts is the extent and degree of involvement in PL activities that have a major focus on mathematics. A good deal of ongoing teacher education seems to be confined to system-driven reforms, many of which have a much broader curriculum focus. Few teachers have been involved in tertiary maths courses or rigorous school-based programs that might have some longer lasting impact on teaching practices. However, some older teachers had experienced major shifts in thinking about maths teaching in which they had moved from traditional approaches to New Maths, problem solving, constructivism and hands-on learning.

Teachers' reactions to reforms

There was a considerable mixture of positive and negative comments about the merits of maths reforms experienced by teachers. Most had some favourable comments about the reforms and a few whole-heartedly endorsed them: *Overall the reforms have been good. They have made us rethink our thinking and try different things* (ALEX). However, a small number of teachers expressed their frustration about the constancy of change and cyclical nature of the process which *goes in waves and cycles* (ACKA), while two teachers indicated they had learned nothing new. It appears from the tenor of some comments that a number of teachers do not have access to the kind of ongoing PL programs that would improve their knowledge and skills. The general consensus was that teachers need opportunities to be engaged in school-based PL to ensure that new knowledge is more firmly embedded in classroom practices.

Changes in teaching methods to accommodate reform initiatives

One teacher claimed to have no involvement in PL activities and continued *to work the same way I did 15 years ago* (MILL), although 19 teachers indicated that they had changed their teaching practices as a result of their exposure to maths reforms. Typically these involved a shift towards more student-centred approaches to maths with an emphasis on individualised instruction, problem solving and hands-on activities. MORA stated: *I use a much larger range of materials. More practical hands-on now than before. More problem solving so that the students apply their skills in different ways. More group work.* Some teachers were prepared to modify their practices to comply with mandated curriculum requirements but were not prepared to jettison tried and true ways of working with children. HARD remarked *I take bits from different reforms and modify the things I like.*

Why (or why not) have changes been made?

Only one interviewee indicated not making changes to teaching practices *I've not found anything better. Neither have I been exposed to very much* (MILL). Teachers cited a combination of reasons for changing their practices including (a) exposure to new ideas (b) conforming with policy guidelines (c) benefits to children although they were not so enthused with reforms as to abandon traditional ways of doing things; rather, as HARM pointed out, they tend to *pick and choose the best from the changes.*

Exposure to new ideas: Several teachers claimed maths reforms had brought them into contact with new ideas and practices: *I have learnt how kids learn and I now apply that across the curriculum* stated MORA whilst HARD commented that there were now more effective teaching tools and highlighted *the value of a shift from "rote" to contextual learning*.

Conforming to policy: A few teachers suggested changes in their practices were driven by the need to conform to departmental guidelines and school policies rather than an overriding conviction of the merits of maths reforms. *Teachers had to change to suit reforms* according to TEAG and MCKE, while STRA suggested that there was *pressure from principals to accommodate the school's policy and philosophy*.

***Benefits to children*:** A sizeable proportion of teachers reasoned that their changes to teaching practices were done in the best interests of students. For ACKA this meant *making it more relevant to daily life and children's experiences*; and for NUNN it meant students *being competent with what they are doing and understanding what they are doing*.

How should curriculum changes in mathematics be introduced?

Teachers described a mix of system-led and school-based ways in which changes should be introduced, but there was a particular emphasis on the need for:

Rationality and coherence: Maths reforms need to be well thought through at the system and school level before teachers are willing to adopt them. *Work out what they want us to do and stick with it! Constant innovation is too frustrating. Give teachers solid Training & Development and stick with it* (TEAG**).**

Whole school reform: PL in maths should be part of an ongoing programme that has a whole school focus. *There needs to be a team focus. Whole school involvement. Need to be immersed in it— sharing it.* (PAPA and LOWB)

***Appropriately resourced PL*:** Some informants claimed PL programmes were often under-funded, inaccessible to classroom teachers and lacked relevant resources. *Teachers are already overloaded, busy and stressed. They need a lot of support. PL is often out of hours and is expensive* (SHIE). *We need sufficient resources. Inservicing with 'grab and take' resources - books etc. Things to develop and modify for themselves* (MOYL).

Release time for PL: In view of the intensification of teachers' work, the provision of release time for teachers to learn about new mathematical knowledge is viewed as critically important. One teacher (MATT) remarked on the need for *continued professional learning for older teachers*.

Direct links to classroom practices: Whilst some teachers acknowledged the benefits of PL programs that introduced them to new ideas, most were adamant they should have direct applicability to classroom practice. *Teachers need continued practice and resources for what the children are doing. Need support in management of the teaching process, managing the broad curriculum. Ideas, ways of programming, structure* (COOK).

Collaborative approaches to PL: Many teachers appreciate opportunities to work in teams, to learn from others and to share ideas with colleagues. MCKE described a successful team approach to professional development: *where we were given lots of examples which you could trial and experience them yourself. Then as a team, we planned a unit together, trialled it and then discussed it. So you learn by doing as well as by experiencing what other people are doing.* Furthermore, DAWS noted *the* *value of skilled mentors to work with over long periods of time … a consistent person to help you along and to give advice.*

Professional associations to support teacher development**:** At least one teacher spoke of the value of professional groups such as the Primary Maths Association.

Successful methods to help children gain a good grounding in mathematics

Some teachers attached particular importance to hands-on methods, self-discovery, process maths, use of concrete materials, working in groups, problem solving and contextualised learning whilst others emphasised the need for explicit instruction, rote learning and basic skills. Most teachers felt that a combination of 'traditional' and 'progressive' approaches was needed to ensure a proper grounding in maths. STRA reasoned *because children learn differently, a variety of approaches are necessary to maximise the child's learning, including rote learning.* MOYL claimed *children in the formative years needed hands-on learning, but drill and practice was necessary to reinforce concepts at all levels.* There seemed to be broad agreement that successful strategies build on what children know, provide opportunities for them to apply what they know, and challenge them in a variety of ways. Assisting students to develop a positive attitude towards maths seems important. Other supportive factors include: *teacher aide support* (ALEX); *using mathematical language* (HAND); *lots of talking about what maths is* (MILL); *motivated and confident teachers who make maths fun* (DAWS); *peers supporting each other* (HARM).

DISCUSSION

Teachers' written and oral responses provide a rich source of information about their reactions to curriculum reforms they had encountered and the effects of the reforms on their teaching practices. The stereotypical picture of teachers in the later phases of their teaching careers is that they suffer from reform fatigue and burn out and lack the enthusiasm and motivation to change their ways (Hargreaves & Evans, 1997). As evident from the data, few informants in this project projected themselves in this light. Although some were sceptical about the benefits of mathematics reform, the majority displayed a strong commitment to teaching and learning and a willingness to change their practices where they could see the tangible benefits to students. Collectively these experienced elementary school teachers possess a vast repertoire of knowledge about mathematics education that should be acknowledged and utilised by schools for the long-term benefit of students and the education system.

Notwithstanding the disposition of teachers to embrace new ideas, data from the survey and interviews suggests that many mathematics reforms have not penetrated schools and classrooms sufficiently to bring about long-lasting changes in teaching practices. Handal and Herrington (2003) suggest that a combination of curriculum, instructional and organisational factors may explain teachers' resistance to mathematics reforms. Many teachers resented top-down curriculum reforms that were introduced with little consultation (Kyeleve & Williams, 1996) and with scant regard for existing practices (Knapp & Petersen, 1995). They were especially critical of staff development initiatives “that take the form of something that is done to teachers rather than with them, still less by them” (Fullan & Hargreaves, 1991, p. 17). With respect to instructional factors teachers claimed they have accumulated knowledge of what works for students in particular contexts and are reluctant to reject tried and true practices for new ideas - especially if they see these as passing fads. Whilst this argument may have some currency, it is quite evident that in keeping with several countries (OECD, 2004), many teachers in this study have not participated in any significant mathematics education since their teacher training days. Although most have encountered a plethora of mathematics reforms, the extent of their engagement with new ideas and practices is often limited to one-off PL activities or short-term programs. Follow through support systems are often lacking with the result that teachers do not have the time to absorb mathematical ideas and new methodologies, let alone incorporate them into their teaching practices. Furthermore, teachers expressed concerns about a lack of time and resources to support their PL, the ill-directed and inequitable nature of training and development programs, and a lack of connection between theory and practice in many reforms.

In the circumstances, it is not surprising that many teachers tended to pay lip-service to reforms, sidelining where possible what does not fit with their ideas of ‘good’ teaching and integrating what they see as useful into their toolkit of teaching practices (Hargreaves, 1994). This capacity to resist all but the most rigidly mandated reforms is nurtured by a culture of individualism in many schools (Fullan & Hargreaves, 1991). Whilst teaching in isolation gives teachers a certain degree of autonomy and discretion to exercise their judgement in the interests of students, it also shuts them off from meaningful feedback from other teachers. As Fullan and Hargreaves (1991, p. 40) point out, a culture of isolation works to ‘institutionalize conservatism’ and unless schools can find ways of cracking open this privatism there is little prospect of widespread adoption of mathematics reforms.

The value of this study is that it gives voice to the learning experiences of teachers who are longstanding recipient consumers of multiple mathematics reform agendas. How teachers respond to reforms is clearly an important issue because they, more than educational administrators and policy makers, have the most immediate influence on students’ experience of mathematics. The methodologies they employ, the perspective they bring to the classroom and their own attitudes to the subject can switch students on or turn them off maths. Although it may be tempting to push the responsibility for changing student attitudes to mathematics back to individual teachers, the task requires a far more systematic and collaborative response from the education system and from within and across schools.

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MAKING GENERALISATIONS IN GEOMETRY: STUDENTS' VIEWS ON THE PROCESS. A CASE STUDY

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The paper presents results of the research, which was focused on studying students' abilities make generalisations in geometry. Students' activities in a classroom were analysed through the evaluation of their inquiry work on different tasks that required using deductive reasoning and non-routine approach to carry out possible generalisations. Cognitive processes regarding to different geometrical structures were described and analysed in detail. The special emphasis was given to identify students' obstacles while making generalisations.

Generalising activity has traditionally been given significant attention both in schools and in research. Within the literature, different types of generalisation are distinguished, e.g. empirical and theoretical generalisations (Davydov, 1990). At the same time there are many papers dealing with various aspects of generalising process in the different branches of mathematics education. Radford (2001) identifies three levels of generalisation in algebra (factual, contextual and symbolic generalizations). Ainley et al (2003) note that the importance of generalising as an algebraic activity is widely recognised within research on the learning and teaching of algebra. Undoubtedly generalising activities in geometry are very important in research on the learning and teaching of geometry as well. Moreover, taking into account the great role of visualisation and perception in the learning geometry, investigation students' abilities make generalisations of different concepts, definitions, properties and ways of their development are of significant interest for researchers in mathematics education.

DESCRIPTION OF THE STUDY

We would like to put into consideration some types of generalisation, which, on the one hand, can be successfully used in stimulation students' inquiry activities while learning geometry, on the other hand, they are good didactical tools for investigation students' abilities to generalise. The following three types of generalisation were considered in the research: 1. Generalisation of definitions of different geometrical objects; 2. Generalisation of geometrical object's properties by giving up one or some features; 3. Creative generalisation.

All types of generalisation above are disposed in the order of increasing cognitive difficulties students encounter in generalising process. The aim of the study was to evaluate students' abilities in generalisation and analyse possible ways for further development. Actually, in the first case we paid attention to students' skills to determine which of geometrical objects was more general than another, what argumentation students used to explain it. In the second case our main aim was to investigate and identify the ways students establish essential features of the property and differ them from non-essential ones while generalising, i.e. if they give up one or some features of the property of geometrical object whether it always leads to correct generalisation of that property. In the case of the third type of generalisation, in opposite to the second type, students had to change either some features of the property or geometrical object itself instead of giving them up. Creative generalisations were the most complex ones for students to work on and teacher's help was an acceptable, but not necessary condition.

During the teaching year several experienced school teachers observed prospective candidates (9^{th} and 10^{th} Year, 15 and 16 years old respectively) for study the author's course on geometry of a triangle. All students, who were involved in the selection process, had their learning profile on mathematics. When the selection procedure was over, two groups of students were organised. The first group consisted of 30 students with average mathematics abilities, in the second one there were 20 gifted in mathematics students. For the groups formation we used the following criteria. We regarded a student as gifted in mathematics (not necessarily talented or genius), if he/she had successfully shown himself/herself during the year before at least in two positions out of the following three ones: 1. Deep understanding advanced theoretical material given by a teacher or studied on his/her own; 2. Solving/proving difficult problems; 3.Posing original and new for himself/herself problems. Students, who did not fit in the mentioned above conditions, however, having overall satisfactory marks in mathematics formed the other group. All students had taken an extended course on elementary geometry before, however, no any part of the course was aimed specially on problem posing skills. Teaching programme of the course consisted of six modules of theoretical material with solving of 54 problems, 9 problems per each module. In the end of the course 24 tasks on generalisation (4 of the first type, 8 of the second type and 12 of the third type respectively) were proposed for both groups of students.

In the paper we consider 4 tasks (1 task of the first type, 2 tasks of the second type and 1 task of the third type) with detailed analysis of differences in the strategies and thinking processes between students of two groups as well as some individual peculiarities within each group of students. It is important to note that the observation part of the study was carried out in three stages. We took the following order: at the first stage tasks of the first type were proposed to the students of both groups, after discussion and some teacher's explanations tasks of the second type were considered. At first students had been asked to solve them, and after that they proposed possible generalisations and tried to prove their conjectures. In the last stage the most complicated tasks with creative generalisations were in the focus of students' attention. For problem solving activities for the tasks of the second and third types a sufficient period of time was given.

ANALYSIS OF THE PROPOSED GENERALISATIONS

We would like to stress that each task of a certain type had its own priorities in the research. Tasks of the first type were intentionally similar in their content and format in order that students had possibility for training and discussion of their results with teacher's help on this stage if necessary. Tasks of the second type were different, from simple to hard ones, for a problem itself and its generalised conjecture. The similar situation was with the third type tasks. Moreover, being important part of the research, the students' work on the second type tasks was preparation to strengthen their activity and improve their understanding on the last stage, where tasks on creative generalisation were the key tools. Also, tasks complexity was taken into account according to Williams & Clarke (1997) Framework of Complexity. Following the study we start with a first type task. A number of the task shows this task was from the first stage of the observation part and its consideration was the second at this stage.

Task 1.2

> Two triangles, isosceles and equilateral, are given. Which of these geometrical objects is more general than another? Give the reasons.

Students with average mathematics abilities distinguished all features of definition of each geometrical object, after that they compared every feature of these objects separately, finding out which feature gives a more general case. Their actions are shown in the Table 1.

Isosceles triangle	Equilateral triangle
1. triangle 2. two sides are equal	1. triangle 2. three sides are equal, i.e. two sides are equal and the third side is equal to two others

Table 1. Average mathematics abilities students' actions

According to the written above, they used the following strategy for comparing two geometrical objects in the context of their possible one to another generalisation.

Strategy 1

A geometrical object is more general than another, if its definition fits under conditions of the other object's definition.

However, most of the gifted in mathematics students used another strategy in the task:

Strategy 2

If definition of a geometrical object does not fit under conditions of the other object's definition, then that other object should be more general one.

It is interesting to note that the second strategy looks more complicated than the first

one because two parts of the statement relate to the different geometrical objects just as the first strategy consists of two parts of the statement for the same geometrical object. We observed that most of gifted in mathematics students chose Strategy 2 due to their abilities to work on the task analysing several features of the same object or even of the different objects simultaneously: all angles of an equilateral triangle are equal to 60°, but it is not necessary for an isosceles triangle, all sides of an equilateral triangle are equal to each other and, again, it is not a necessary condition for an isosceles triangle, etc. Full results of this task are given in Table 2 below.

Groups of students	Using 1st strategy	Using 2nd strategy
Students with average abilities in mathematics	26 students	-
Gifted in mathematics students	7 students	17 students

Table 2. Using different strategies by students in both groups

It is interesting to note that 4 students in the first group couldn’t propose anything, at the same time in the second group 3 students proposed the first strategy only, 13 ones proposed the second strategy only and 4 students did both of them. Thus, gifted in mathematics students used both strategies in the task with their preference to Strategy 2 just as students with average mathematics abilities took into consideration only Strategy 1. At the end of discussion students had been asked to provide their answers for this task in arbitrary form. Gifted in mathematics students used both forms of the answers (symbolical and graphical ones, see Figure 1 and Figure 2 respectively).

equilateral triangle $\Rightarrow$ isosceles triangle

isosceles triangle $\not\Rightarrow$ equilateral triangle

Figure 1. Symbolical form of the answers.

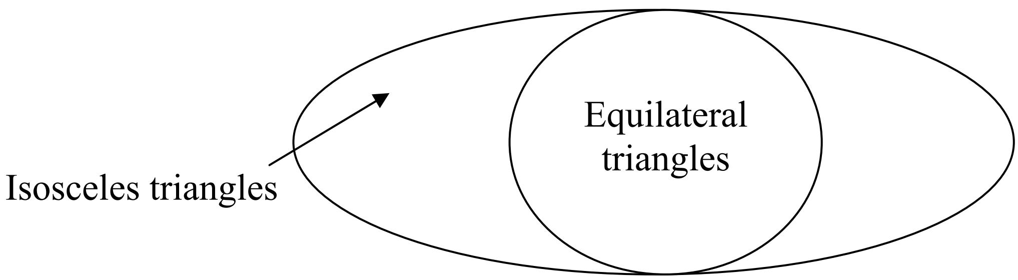

Figure 2. Graphical form of the answers.

However, most of the students with average mathematics abilities gave their answers in a verbal form (orally) or its written version, i.e. “an isosceles triangle is a more general geometrical object than an equilateral triangle”.

In the second type tasks we paid great attention to using visual thinking in generalising process. Consider the following task on the basis of Pompeiu’s property:

Task 2.5

If an arbitrary point P lies on the plane of equilateral triangle ABC, then a triangle can always be constructed from the segments PA, PB, PC, taking into account the case of a degenerated triangle. In what way could you generalise this property?

Definitely the possible generalisation of the property could be quite clearly revealed in the words *'point P lies on the plane of equilateral triangle ABC'*, but after analysis of students' drawings we were surprised to conclude that a clear indication of that generalisation disappeared (Figure 3) and students in both groups experienced difficulties at this stage. It was a surprise for students that a drawing of the task was a visual help for solution only, not for generalising process. Most of students in both groups were aware that solution of any task should contribute to more or less extent to finding the ways for its generalisation and further solution of generalised conjecture. However, quite often they couldn't argue that idea clearly in different tasks.

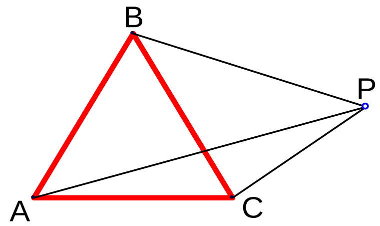

Figure 3. Students' drawing for Task 2.5.

As a hint we proposed the following drawing (Figure 4) for students, who hadn't made correct generalisation of Pompeiu's property after making its solution (there were such students in both groups, 13 and 2 students respectively).

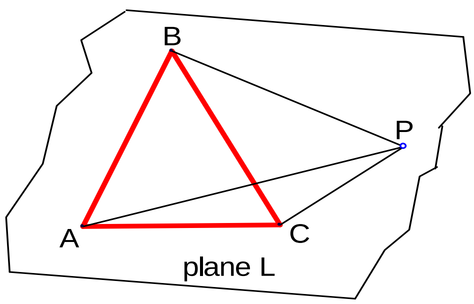

Figure 4. A hint-drawing for Task 2.5.

There was an opposite reaction to this drawing in groups. The rest of the gifted in mathematics students grasped a generalised interpretation of the drawing immediately. However, most of 13 students with average mathematics abilities characterised this drawing as inconvenient for generalisation. Also, gifted students separated essential and non-essential features (in their understanding) of the tasks. They suggested that essential ones should be considered and could be changed for generalisation, but non-essential features should remain unchangeable. Some of students explained it in an interesting way:

> An equilateral triangle is an essential feature of the problem,…, without it statement of the problem is not true, hence, this feature is a non-essential one for possible generalisation. (It was said when the problem had been solved – note of the author.) But, moving point P in all directions, it is unclear whether the property will remain the same. Therefore, the position of point P is an essential feature for generalisation. (Student X)

However, students with average abilities in mathematics carried out generalising process with consideration all features of the properties, giving them up one after another. They didn't distinguish essential and non-essential features in the tasks and tried to solve all conjectures constructed without taking into account that some of them could be incorrect. Moreover, some students didn't understand how a certain feature could be given up for generalising. On the other hand, several times we observed that some of the gifted students moved in the direction of particular cases instead of generalisation. Below is such an example of "generalisation":

> We can always choose the point P in order that the sides PA, PB, and PC of a triangle, their lengths, of course, wouldn't be three successive terms of an arithmetic progression. (Student Y)

The following task was given on the basis of Carnot's property:

Task 2.7

> Let *ABC* be a triangle that has been drawn by you. Find out how the sum of the lengths of perpendiculars dropped from the circumcentre to the triangle's sides depends on circumradius and inradius. How could you generalise this property?

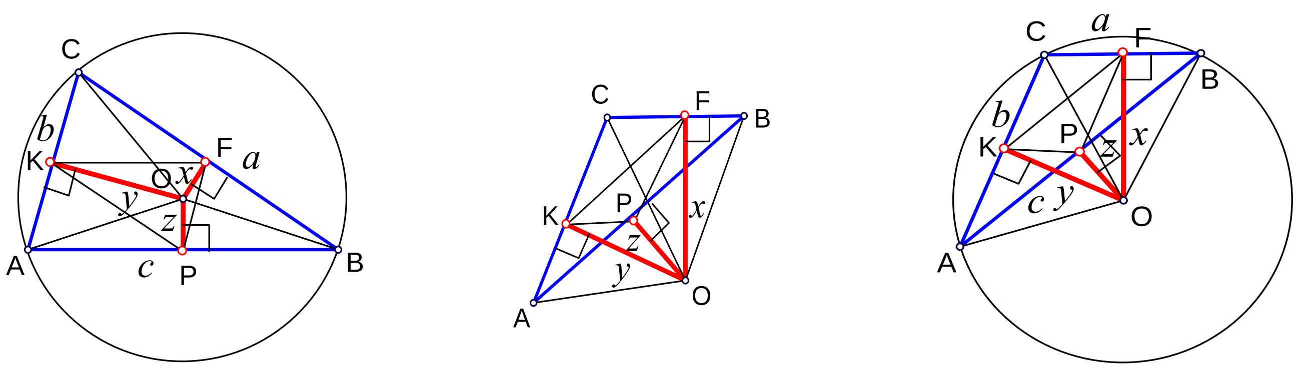

Figure 5. Three possible drawings for Task 2.7.

We would like to emphasise two peculiarities of this task. The first one was a student's choice of a triangle. The second peculiarity was a relationship between sum of lengths of perpendiculars and radiuses that could be expressed in an explicit way in the task, but we didn't define that intentionally. The reason was we tried to trace links between students' generalisations and their solutions. A task drawing for the case of a triangle with acute angles is given on the left side of Figure 5. We observed that most of gifted students didn't pay attention to the case of a triangle with an obtuse angle (a task drawing is given on the right side of Figure 5) and considered two other types of a triangle as students with average abilities did. However, there was attempt to generalise a circumcentre location. In this case general conjecture was the following (a task drawing is given in the middle of Figure 5):

> It seems to me it would be interesting to consider how sum of lengths of perpendiculars dropped from an arbitrary point (in the plane of this triangle – note of the author) to the triangle's sides looks like. (Student Z)

Gifted in mathematics students had advantage in making generalisations of the second type because it was often connected with solution and other group of students couldn't solve some problems from the tasks. At the same time difference in students' abilities to generalise was not so significant. In both groups students made similar generalisations for most of tasks, only approaches were different.

At the solution stage the following task on creative generalisations seemed even easier than Task 2.5 and Task 2.7.

Task 3.8

> All angle bisectors of a triangle intersect in one point. How could you generalise this property? Give as many conjectures as you can.

Indeed, it is not a hard problem for solution, but our aim was to stimulate students' creative approaches for possible generalisations. This is a nice example where a plenty of different properties are hidden behind the simplicity of the statement (Yevdokimov, 2007). Many of them can be found through generalising process. However, the strategy of using essential and non-essential features as most of gifted students did in the second type tasks on generalisation couldn't bring them to the desired result here.

Following Sierpinska (2003) we observed students' difficulties in achieving a balance between visual and analytic thinking while making generalisations. Most of gifted in mathematics students distinguished, though intuitively, visual and analytic generalisation. In their understanding visual generalisation could be related to a geometrical object and to some of its properties as well, but analytic generalisation – only to properties of a geometrical object. Therefore, they proposed to consider a geometrical object and its different features separately and clarify in which way an object itself could be changed. Of course, it is necessary to note if a geometrical object is characterised by the only feature then changing of an object or changing of its feature leads to the same result. Gifted students used both kinds of generalisation in their work, though analytic generalisation caused much more difficulties for them due to its more complex structure. In the case of visual generalisation they changed a geometrical object immediately. It is interesting to note that in the tasks on creative generalisations students with average abilities in mathematics preferred to make visual generalisations.

CONCLUDING REMARKS

We noticed that students with average mathematics abilities experienced difficulties in the tasks with creative generalisations because they needed to analyse changing of some geometrical objects and/or their features simultaneously (in a task as well as in the suggested conjecture), e.g. Lemoine point and the point of intersection of angle bisectors in Task 3.8, etc. Also, we often observed that solutions of the suggested conjectures were not perceived by students as generalised ones for the tasks even among the gifted students. In other words, some well known properties from the third type tasks were not understood as particular cases of generalisations already made. However, gifted students much more used generalisations in their argumentation because they easier perceived giving certain features up and creating new features instead them. We would like to emphasise significant individual difference in students' abilities for generalisation in both groups. Nevertheless significant difference between the groups in abilities to generalise occurred in the work with the third type tasks only. As for solving and proving of the suggested conjectures the final result was more predictable: the group of gifted students had great advantage in such activities. However, we would like to stress that students' abilities to make generalisation of any statement without its preliminary investigation and solution were not considered in detail at the study. Also, in learning geometry we have to pay much more attention to the needs of students with average mathematics abilities. They are not so bad in constructive actions and making suggestions. Undoubtedly that further work in this direction can bring a number of such students nearer to potentially gifted in mathematics students, at least in comparison with their abilities in mathematics.

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ASPECTS OF TEACHER KNOWLEDGE FOR CALCULUS TEACHING

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This is a case study of an experienced teacher concerning her knowledge about calculus teaching. Specifically, we investigated aspects of this knowledge, and the way in which these affected this teacher's practice and professional development. The data collected from a number of observations of this teacher's calculus teaching, informal discussions and a semi-structured interview with her. The analysis of the data revealed two elements of teacher knowledge that played an important role in her calculus teaching development. One was the specialised calculus knowledge she developed through a course on calculus teaching. The other was the knowledge of students' understanding of calculus which she developed through the above-mentioned course and her actual practice.

INTRODUCTION

A number of studies have investigated teacher knowledge which is necessary for mathematics teaching through teachers' actual practice. There is an agreement among researchers that this knowledge is rooted in the mathematical demands of teaching itself and it differs from the knowledge that a teacher has acquired in the formal education (Ball & Bass, 2000; Cooney & Wiegel, 2003). Moreover, theoretical frameworks have been developed that describe the nature of this knowledge which share similarities and differences. These frameworks have mainly emerged from the analysis of mathematics teaching in primary school. However, a rather small number of studies have focused on mathematics secondary school teacher knowledge. Even and Tirosh (1995) studied teachers' subject matter knowledge and knowledge about students in the case of function concept. An, Kulm and Wu (2004) considered a network of pedagogical content knowledge and investigated the differences in teachers' pedagogical content knowledge between middle school mathematics teachers in China and the United States. Chinnappan and Lawson (2005) examined the connectedness of secondary school teachers' geometric knowledge by using concept maps as a tool for their analysis. In this paper we attempt to develop further our understanding of secondary mathematics teachers' knowledge and its development in the area of calculus.

THEORETICAL BACKGROUND

An initial characterization of teacher knowledge comes from Shulman's work (Shulman, 1986; 1987). Two of the categories identified in his work were the subject matter content knowledge and the pedagogical content knowledge. Although these categories are not specific to mathematics teaching, many researchers in mathematics education have used them as a framework for their work. Ball and Bass (2000) used the term mathematics knowledge for teaching to capture the complex relationship between mathematics content knowledge and teaching. They also attempted to refine Shulman's categories (Ball, Hill & Bass, 2005) by distinguishing two domains in the subject matter knowledge: the common knowledge of mathematics and the specialized mathematical knowledge. They defined as common knowledge for a primary school teacher the knowledge that any well-educated adult should have, while specialized mathematical knowledge as the knowledge that only teachers need to have. In a recent paper (Ball, Thames & Phelps, submitted) they clarified further the above two domains and they elaborated the meaning of pedagogical content knowledge by identifying three domains: the knowledge of content and teaching, the knowledge of content and students and the knowledge of content and curriculum. Rowland, Huckstep and Thwaites (2005) also formed a framework, the knowledge quartet, to describe mathematics content knowledge of prospective primary school teachers. They distinguished four categories: the foundation, the transformation, the connection and the contingency. By drawing elements from the above frameworks we studied the nature of mathematics teacher knowledge in Calculus and the factors that influence its development. The study involved the investigation of nine teachers' calculus teaching. In a previous paper (Potari et al., 2007), we identified aspects of teachers' mathematical knowledge for teaching. In this paper, we focus on one particular case of a teacher investigating: the nature of common and specialized mathematical knowledge in the case of calculus; the way in which the above knowledge affect the pedagogical content knowledge; and teacher's reflections on the process of developing knowledge for teaching.

METHODOLOGY

The teacher in our study, Stefanie, was an experienced mathematics teacher who had been teaching mathematics in school for eight years, two in lower secondary and six in upper secondary. She had a mathematics degree and she had just graduated from a two–year Masters programme in mathematics education. The authors came to know her as one of the students in their courses when she was studying for her Masters degree. One of these courses was on calculus teaching focusing on students' difficulties, on conceptual issues in calculus and on teaching approaches aimed to promote students' conceptual understanding. This teacher had showed a particular interest on issues concerning calculus teaching and she was keen to participate in the study.

The data comprised classroom observations, informal discussions before and after teaching and an audiotaped semi-structured interview. The researchers observed three teaching sessions on calculus and kept field notes. The interview focused on teacher's experience concerning mathematics and mathematics teaching; her views about teaching and learning mathematics in general and calculus in particular; and her interpretations of specific pedagogical actions that were identified during the researchers' observations. The analysis of the classroom data aimed at identifying elements of teachers' knowledge as they emerged from her practice. The analysis of the interview aimed to verify consistencies or inconsistencies between the teacher's actions and views and explore this teacher's awareness of the process of developing her knowledge for teaching.

RESULTS

We present below issues concerning teacher's knowledge that resulted from the analysis of one episode about teaching continuity and also teacher's reflections about the process of developing this knowledge.

Teacher's knowledge emerging from teaching: Stephanie encouraged certain norms of classroom communication. She expected these would facilitate students to participate actively by explaining their strategies, making conjectures, communicating their observations, comparing and evaluating their solutions and responding to their peer's questions. These norms often acted as opportunities for the development of students' understanding and extended the mathematical communication at a metacognitive level.

In introducing a new concept or a new theorem Stephanie often started from problems that the students were asked to consider. These problems were often everyday life problems or familiar exercises that could lead them to conjecture a definition or a statement based on their intuitions. In the interview, she said that she liked to do this introduction both for herself and for the students. She often did this introduction by using a worksheet. The worksheet supported the "discovery" of the new concept or theorem and its further clarification through a number of tasks and questions. For example, to introduce the concept of a continuous function the teacher asked the students to work on two tasks. In the first task, she asked the students to sketch the relation between the height of an airplane during a flight from a city A to a city B and the time of the flight. In the second, she gave to the students a table with prices of posting a parcel in relation to its weight and asked them to graph this function. Then, she asked them to compare the two graphs and identify their differences. The students observed that the first graph did not break while the second was cut in a number of points. These observations introduced the term "continuity". The teacher asked the students to consider the way in which this term was used in everyday life. She discussed the students' contributions, she used a dictionary to identify how this term was defined and she related this definition to the two graphs. She continued this inquiry which was based on students' intuitions, by giving them three graphs (Figure 1) and asking them to decide about their continuity.

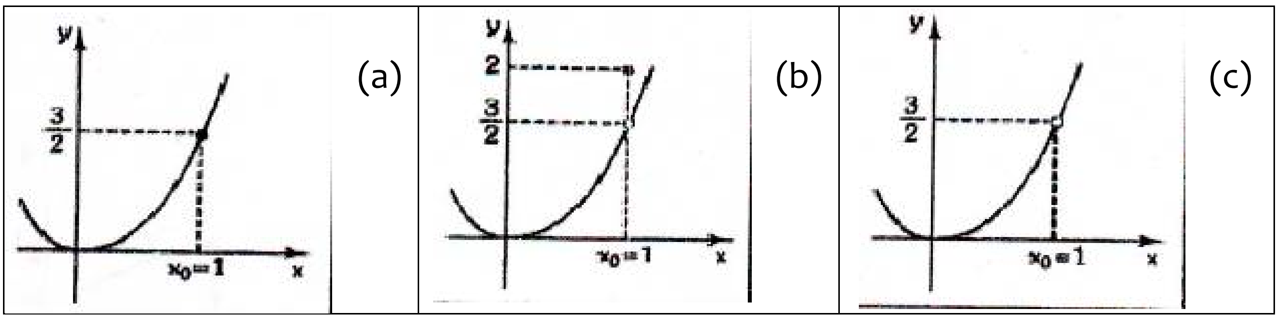

Figure 1. Graphs given to the students to compare in terms of their continuity.

The discussion focused on graphs b and c. The teacher posed the question: “What is the 3/2 for x equals one?” This question triggered off a debate about the limit of the function when x tends to 1. One student claimed that “this limit is 3/2” and another student wondered “why doesn’t it equal to 2?” The teacher reminded to the students that “for the limit we are not interested in what the function does at this point but in the values of the function near to this point”. Discussing about graph c most students believed that it was not continuous. Only one student considered this function continuous. The teacher tried to resolve the conflict by asking the students to identify the differences between graph c and the graph of the discontinuous function emerged in the beginning of the lesson (the weight-cost relation). This attempt was unsuccessful, so the teacher used another graph of a continuous function with domain the union of two intervals (a,b) and (c,d) where b<c. Most students accepted this graph as a graph of a continuous function. In the next task the students were asked to conjecture what is the definition of the continuity of a function at a point, by observing in three different graphs (figure 2) the relation between the limit of the functions when x tends to xo and the value f(xo).

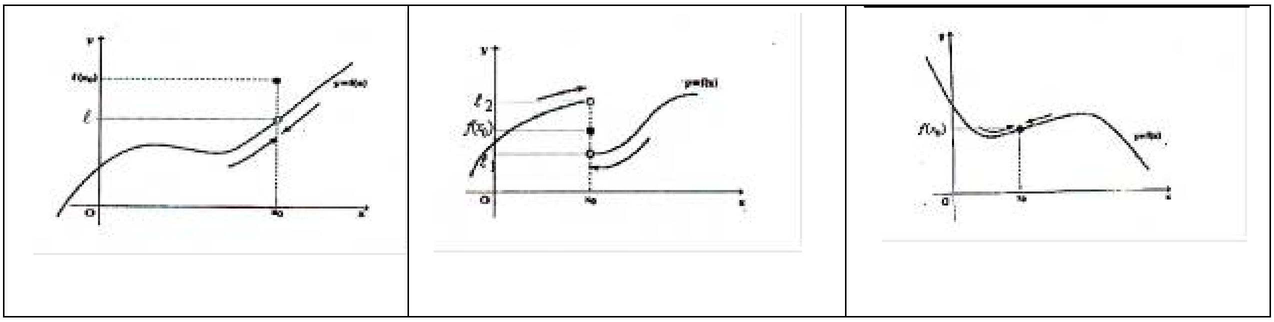

Figure 2. Graphs given to the students to derive the definition of continuity.

As it appears in the teaching session discussed above, the teacher incorporated different pedagogical tools in her teaching. She used everyday situations, graphs and linguistic analysis of mathematical terms to support a “discovery” teaching approach. The aim of this approach was to build on students’ spontaneous ideas and transform them to formal mathematical knowledge. To encourage this transformation she discussed with the students about prior knowledge that was required for approaching the new mathematical concept. For example, she focused on students’ understanding of the concept of limit and she discussed further with them about this concept. This approach helped students move towards the formal definition of continuity.

Developing knowledge for teaching: Stephanie considered that her undergraduate studies did not have any impact on her teaching: “Nothing comes to my mind from the university. I remember more from the school when I was a student. I do not know why.” During the first years of teaching she based her teaching on her own experiences as a student at school while later she started to learn from her students. Her experience in the Masters program, after six years of classroom teaching, was crucial to her professional development:

> I do not mean that what I learned there I used it exactly in the same way in my classroom, but I took ideas. Now, I pay more attention on students' mistakes, I take more into account their different responses, I see these things differently now than before. I think that I will see more changes in the next years. I have a lot of ideas which I wish to implement if I find the time.

The above extract indicates that the teacher started to focus on students' mathematical thinking and to integrate this knowledge into her teaching. She also appears to consider teaching as a continuous process and that she had to build her own tools and materials under the constraints of the school context.

By considering calculus teaching in school she claimed that the focus was mostly on procedural aspects:

> As other colleagues also admit, we do things superficially, we do exercises and we do not do theory, we do not ask theoretical things from our students. During the first years of my teaching I was doing things in this way. I was telling the students what I should by following the textbook without understanding some things. Later on I started to realize a number of things. Now, I have a better knowledge of students' difficulties and I am in a better situation.

She added that she transformed her calculus teaching towards a more conceptual approach because of the following reasons:

> I learned in the course of calculus teaching a lot of details about calculus. For example, concerning the continuity of function I did not know all the things during my first years of teaching and even now I do not say that I know in depth all the things I teach in calculus. I also learned about the students' mistakes.

Comparing calculus to other mathematical areas in the curriculum, she considered that teaching calculus is more difficult than teaching geometry:

1. St.: It is more difficult to teach calculus because there are these small details that if you do not know, the books do not help you. In geometry the theorems are there and you can study them
2. R: Which are these details?
3. St.: All these that we did in the course on Calculus Education in the Masters program.
4. R: Give us an example of "details" in Calculus.
5. St.: For example, in the continuity students can have in mind the image of a continuous line but this image does not hold in the cases where the domain of the function is not an interval. These things you may not realize by reading various mathematical textbooks if you do not discuss them with someone.
6. R: Isn't this the case in geometry? What about the effect of a shape on how the students will approach a problem? Or what is the role of the prototypical cases? Aren't these "details" for the teacher?
7. St.: It is something that you can learn from the classroom. You understand from your experience. In calculus you need someone to "open your eyes". It is not only me but also teachers with many years of teaching that do not know these things.

From the above dialogue, the teacher seems to be aware that for teaching calculus you need a special knowledge which can not be found in books [1, 5] and it does not emerge only from the classroom experience [7]. This knowledge seems to have two components; one consists of a specialized mathematical knowledge and the other of the knowledge of students' understanding [5]. She also believes that such knowledge can be developed in appropriate courses [3, 7]. However, concerning geometry teaching the teacher initially does not see analogous "details" with those she recognized in calculus [1]. Later in the discussion she consider that the necessary 'details' for teaching and learning geometry can be learned from classroom experience [7].

CONCLUSIONS

Stephanie emphasized conceptual aspects of mathematics in her teaching. In particular, she designed tasks beyond those included in the textbook that encouraged students to get involved in mathematical inquiry; she used critical examples that covered the different features of the concept; she thought on her feet alternative examples when the students could not overcome some of their difficulties. Concerning her mathematical knowledge about calculus, she distinguished two domains: the knowledge she developed during her undergraduate studies and the knowledge developed during her graduate studies. The first type of knowledge did not help her in teaching calculus while the second offered her a meaningful context for reconsidering and continuously developing her teaching actions. By talking about the nature of the second type of knowledge she referred to a number of "details" that helped her to deepen her mathematical understanding and often are not included to a typical calculus course. She said that she acquired this knowledge in a course about calculus teaching. These details seem to indicate what it is called specialized content knowledge (Ball et all, 2005, Ball et all, submitted). Specifically, this knowledge is the specialized knowledge in calculus that a teacher needs to develop in order to promote students' conceptual understanding. As it appears from the fact that the teacher does not recognise details of a similar type in the case of teaching geometry, this knowledge seems to be content specific. On the other hand, the knowledge that a secondary school teacher has acquired from her undergraduate studies is possibly the common mathematical knowledge that seems to be not adequate for the teacher to develop her teaching practice. Another dimension of teacher knowledge that the teacher considered crucial on her professional development was the knowledge about students' mathematical understanding. This knowledge developed both from the teacher's classroom experience and from her graduate studies. As it appears from the study, the combination of specialised mathematical knowledge and the knowledge about students' understanding concerning the concept of continuity allowed the teacher to transform students' spontaneous ideas about this concept to formal mathematical knowledge through appropriate teaching situation. Summing up, we could claim that the knowledge of the teacher in our study that seemed important for teaching calculus was the specialized knowledge in calculus and the knowledge of students' understanding. These two domains of knowledge in relation to teacher's general pedagogical knowledge lead to the development of teacher's pedagogical content knowledge that can be the source for teaching that promotes conceptual understanding. They are content specific and are not developed only from usual courses in undergraduate studies in Mathematics or from classroom experience. They are part of what Rowland et al (2005) call foundation knowledge and more research is needed to integrate these two domains in teacher education.

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