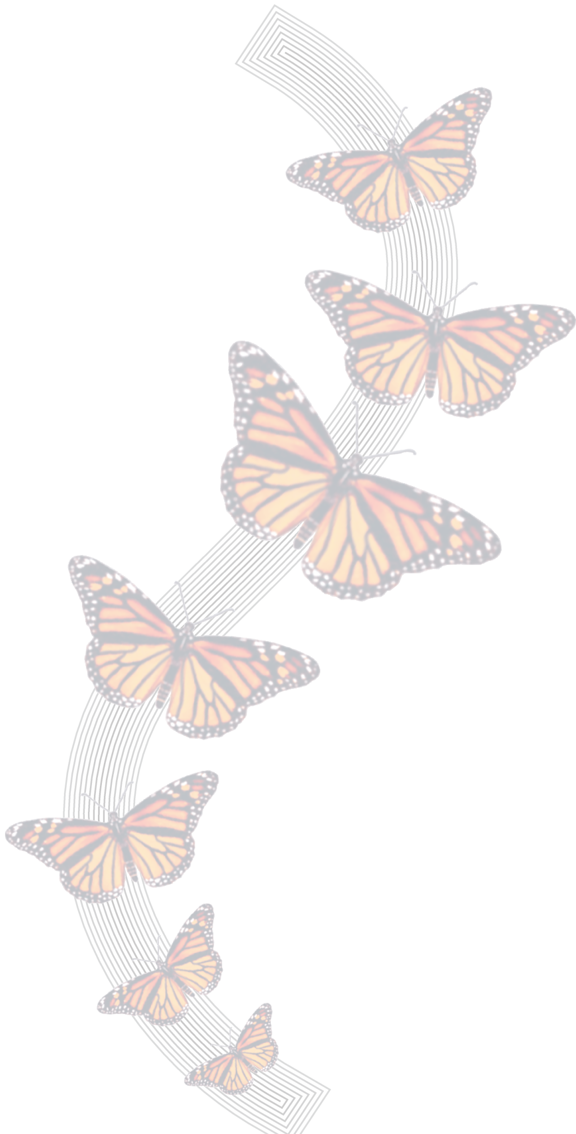
Mathematical Ideas:
History, Education, and Cognition



Vol. 2

Morelia, México, 2008

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International Group for the Psychology of Mathematics Education

Proceedings of the Joint Meeting of PME 32 and PME-NA XXX

Editors
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Cinvestav

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Research Reports
Aar-Ell

Morelia, México, July 17-21, 2008

Centro de Investigación y de Estudios Avanzados del IPN
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Aar - Ell

ACADEMIC IDENTITIES OF GEOMETRY STUDENTS

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University of Michigan

Geometry students are engaged in a balancing act. Simultaneously, they are responsible to act in ways that receive positive evaluation from their teacher and in ways that will deepen their understanding of geometric concepts. I view the geometry classroom as a place where the teacher and student come together to trade work done together for claims on the didactical contract (Herbst, 2006), that is, claims that they have, 'covered' part of the geometry curriculum. Though examining interviews with geometry students, I show that some students do classroom work with an eye towards receiving praise from the teacher, while other students do classroom work with an eye towards leaning mathematical content.

WHO IS THE GEOMETRY STUDENT?

This paper attempts to answer the questions, who is the geometry student? And how does the geometry student understand her place in the geometry classroom? I am interested in uncovering students' understandings of what it is that a geometry student does and the ways that students make meaning of geometry instruction.

This paper extends the work on 'doing school' (Chazan, 2000; Eckert, 1989; Herbst & Brach, 2006; Jackson, 1968; Lave, 1997; 2001; Powell, Farrar, & Cohen, 1985) by showing the different ways that students "do school" in geometry class and the obligations that students hold with respect to the geometry classroom. The academic identities discussed in this paper give a way of understanding what is meant by "doing school" in the particular context of the high school geometry classroom. Through these identities we understand what actions students see as available to them in instructional situations and what meanings they make of the tasks that are put before them.

THEORIZING IDENTITY

I will begin from two assumptions about the nature of identity with the aim of arriving at a conception of identity that will allow me to look at the ways that individuals' dispositions and classroom context combine to create the academic identities of geometry students. Two aspects of identity that are essential to this study are:

- Identities are experienced in practice
- Identities are dynamic and vary with context

[1] This work is supported by NSF grant REC-0133619 to P. Herbst. Opinions expressed are the author's sole responsibility and don't necessarily reflect the views of the Foundation. I would like to acknowledge and express my gratitude for help that I have received from Patricio Herbst while working on the project reported.

Below I will briefly expand on these two aspects of identity.

Identities are Experienced in Practice

Children are not born knowing how to be a student. Through their time in school they learn how to behave and what is expected of them. Students learn what they would like to get out of school and they learn what school would like to get out of them (Doyle, 1983). By the time students reach high school they are adept at reading their teachers and scanning the content offered to see what matches with their goals for the course. Each moment in a classroom serves to structure the next moment, so even events that feel novel are structured by every moment that came prior (Bourdieu, 1990).

Identities are Dynamic and Vary with Context

A helpful construct for understanding dynamic identities is figured worlds. Holland et al (1998, p. 52) define figured worlds as: “A socially and culturally constructed realm of interpretation in which particular characters and actors are recognized, significance is assigned to certain acts, and particular outcomes are valued over others.” People acting within figured worlds act “as if” such-and-such a thing was true. For instance, in some classrooms, students and teachers act “as if” completing worksheets corresponded to mastery of a subject. Different students will take on different identities with different stances toward the ‘as if’ scenario. Because different students understand the figured world in different ways these students feel that different actions are appropriate when faced with a task.

A consequence of this view of identity is that we need to have a clear picture of the context of the geometry classroom. One way that researchers have understood the context of the classrooms is to study students’ engagement in instructional tasks. These studies aim to understand how students make sense of the tasks they are given and how students attempt to complete these tasks. Students’ actions in response to a task, and the common meanings that students make of a task, show a membership in a figured world that understands tasks in a certain way. While this has proved fruitful in the past, it only takes into account one layer of interaction, that of the task. As described below, for this study it is even more fruitful to view instruction as consisting of three layers, the task, the situation, and the contract.

INTERPRETATIONS OF INSTRUCTION IN MATHEMATICS CLASSROOM

Economy of Symbolic Goods

Bourdieu (1990; 1998) explains economies of symbolic goods through the example of gift exchange. In many cases, the giving of a gift is taken as a spontaneous act of good will on the behalf of the gift giver. In return, the gift receiver shows gratitude for the gift (whatever it might be), and the interaction appears to be complete.

Bourdieu argues that the interaction is not complete, but a new cycle of giving has begun. The gift receiver is now obligated to take the role of the gift giver. But it is important for the reciprocal gift to not appear as a response to the initial gift, but as a spontaneous act of good will. The second gift would lose its value if it were seen as fulfilling an obligation.

This camouflaging of obligation is what Bourdieu refers to as “misrecognition.” For the economy of symbolic goods to function, both parties must “misrecognize” the gift exchange by acting as if each gift is a unique action (not one in a long string of gifts between the two), and that each gift is not part of an obligation that the two parties have to each other. The other side of “misrecognition” is recognition. To recognize the exchange would be to say that the gift is not valuable because it is not a unique action, but only the fulfillment of an obligation.

Within the symbolic economy of the classroom, teachers and students are obliged to trade classroom work for claims on the didactical contract (described below). This economy gives a way to account for the work that the teachers and students do in classrooms. I claim that teachers and student act in a way that is similar to the gift exchange example above; teachers and students misrecognize the exchange by acting as if they are doing classroom work because of a spontaneous good will that they feel for each other and the mathematics. Students arrive in class with the intent to learn geometry, and teachers interpret students’ work as evidence that students are learning mathematics. To keep the economy functioning, teachers and students refrain from recognizing that they interact because they are obligated to. When the symbolic layer of the interaction is taken away, the work done by students is shown to be due to an assignment from the teacher, and the teacher evaluates students’ work because she is compelled to give grades.

Both teachers and students are continuously balancing the tension of misrecognition and recognition. Just as there is a need to misrecognize the obligations, there is a need to recognize the constraints that these obligations entail. Students need to present their work in a way that is understood by the teacher (instead of say, only working through problems in their head) and the teacher must be clear in her expectations (so that say, students know that they are expected to present written proof their work).

I hypothesize that some students are attuned to the misrecognition of their work as learning geometry, while other students are more attuned to the recognition of their work as actions performed in response to the teacher’s directions and subject to the teacher’s evaluation.

Contract, Situation, Task

This economy of symbolic goods is not complete without understanding more about the objects of the trade (classroom work and claims on the didactical contract) and the ‘marketplaces’ in which this trade occurs (instructional situations). See figure 1.

Tasks, situations, and contract, as developed by Herbst (Herbst, 2003, 2006; Herbst & Brach, 2006) provide a frame for a three-tiered analysis of classroom interactions. Tasks are segments of classroom work comprised of problems or questions chosen by the teacher, along with the resources, material and cognitive, that students deploy to participate in those activities (Doyle, 1983, 1988). The doing and completion of tasks also has value as regards the entitlement to claim that part of the contract has or has not been accomplished. To facilitate this exchange, tasks exist in instructional situations, such as reviewing homework, doing proofs, teaching theorems, etc. These situations frame the exchange that gives value to the work that the teacher and students are engaged in. The situation provides an answer to the question, "what are we doing?" and provides a frame for participants to understand what they are supposed to do and what they may lay claim on by doing it.

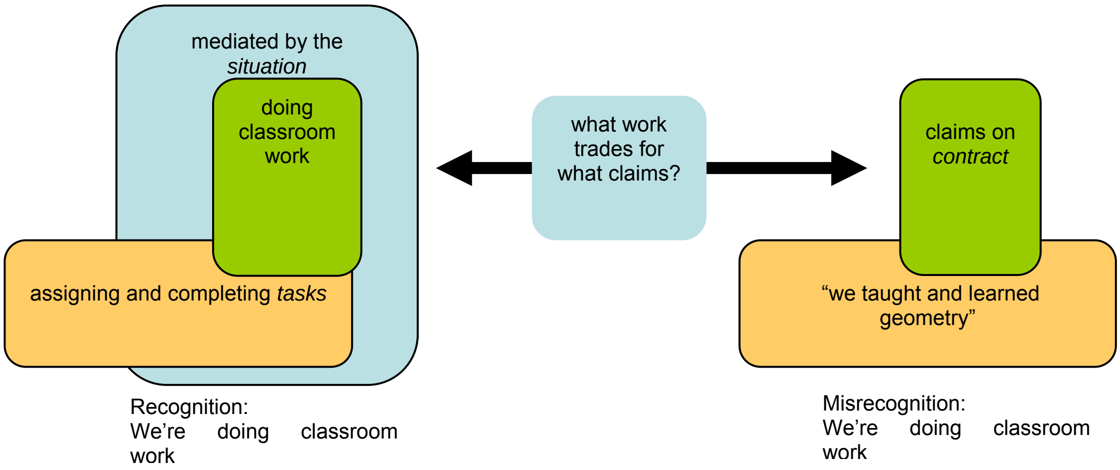
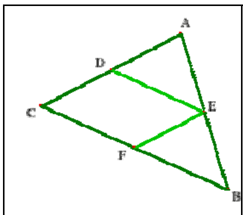


Figure 1. Symbolic Economy of the Classroom.

This economy reveals the need for students and teachers to simultaneously recognize and misrecognize the value of that work. The student profiles detailed below show how different students hold different implicit conceptions of the contract and economy that lead to students enacting different academic identities.

DATA

The data analyzed in this paper are interviews with 14 honors geometry students from two classes. The interviews asked students to think about how they would go about completing three tasks. The first task was a word problem titled the "antwalk problem" (see figure 2), the second task was a concise statement of a theorem that the students were asked to prove, and the third was a proof exercise as seen in the students' textbook, with "given" and "prove" statements. After being shown a task the students were asked questions such as, "How likely is it that your teacher would assign this problem?" and "Would she expect a proof in response?"



Imagine two ants walking around this triangle.
Ant Jill goes AE, EF, FC, CD, DE, EB.
Ant Jack goes BC, CA, AB.
When they reach B, each of them argues to have walked more than the other one. Who is right and why?

Figure 2. Antwalk Problem.

METHODS

I analyzed the data using an open coding scheme to look for patterns in the responses that the students gave to the interview questions. A partial list of interview codes is given below (table 1). Due to space constraints only the analysis of the antwalk problem is given here. The complete analysis includes all three problems, as well as students' understanding of why proof is part of the geometry class, students' attitudes toward measurement, and students' interactions with diagrams.

Topic	Response Code	Description of response
Interpretation of "antwalk" problem	See the theorem (1)	The student quickly sees that the problem hinges on the fact that D, E, F are midpoints.
	Make it work (2)	The student notices superficial traits of the task such as "you need to compare lengths." The student attempts to interpret the task as one they would see in their class.
	Dismiss (3)	The student does not believe that this problem is appropriate for a geometry class because it is not similar to other problems that they are given by their teacher.

Table 1. Codes for responses to antwalk problem

The codes were used to inspect the data to find implicit references to the economy of symbolic goods and the ways that students either recognize or misrecognize the value of their work.

For each coding topic, numbers were assigned to the codes (1-3), these numbers correspond to the location of the response on the recognition/misrecognition continuum. The sum of the codes, which I refer to as 'coding score,' was found, and this was used as a way to numerically represent the students position along the continuum. A low score means that the student is attuned to the mathematics and a high score means that the student is attuned to the teacher's evaluation.

In addition to coding for stances with respect to the symbolic economy, for each interview I counted the number of times that the student referred to the teacher (that is, I counted the words 'teacher,' 'Ms./ Mrs. X' and 'she,' her,' 'they,' when this pronouns pointed to the teacher). The number of occurrences of references to the teacher varied from 0 to 86. This number was divided into the total number of words in the analyzed text. This ratio, which I have called 'teacher token', is a measure of the extent of the teacher's influence on students' instructional decisions. The higher the ratio, the lower the number of times the student mentioned the teacher; the lower the ratio, the higher the number of times the student mentioned the teacher.

student	Max	Marcus	Cabe	Andra	Luke	Yakim	Craig	Karen	Erin	Erie	Yuri	Jade	Hamid	Betsy
teacher token	>700	700	229	228	228	212	192	191	165	120	112	111	110	51
coding score	4	3	4	4.5	9	5	7.5	7	6	9	7.5	6	9	7

Table 2. Teacher tokens and coding score

From this table we can see that as the number of 'teacher tokens' increase in the interview transcript, the more students' responses reflect a teacher centric view of classroom interaction. This continuum was used to cluster students and combine them to form profiles; each profile describes a kind of student who sees the system of exchange in a particular light.

There are a few outliers, such as Luke, Jade and Karen, who do not fit well in the continuum, or whose interview responses did not fit into the coding scheme. These students were not included in making the profiles that are given in the results section.

RESULTS

In the following section I discuss differences among students' academic identities. To highlight the differences between the student profiles, below is a comparison of reactions to the antwalk problem. The reactions showcase how different stances toward the didactical contract appear in student actions and interpretations.

In general, students agreed that the antwalk problem is not they type of problem that they encounter in their geometry class. Matthew, representing the 'misrecognition' end of the continuum, sees the antwalk problem and immediately looks for the mathematical relations that exist in the problem. He notices that if the points on the sides of the triangle are midpoints then he would be able to add the number of segments that each antwalk and compare the result.

> Mathew: I think a proof would probably work easiest to solve this problem
>
> Interviewer: How's that, excuse me?
>
> Mathew: Because you could, you could say like, if, you could find out if like EF were to, were the median or like F was midpoint of CB and D was the midpoint of CA and E was the midpoint of AB so you could find out the distance each one walked, of each segment and then add them up to see who would walk the farthest.

The antwalk problem does not give enough information to answer the question that it poses; so Matthew notes that if he made an assumption then he would be able to answer the question.

This is very different to the reaction of Peter, who represents the recognition end of the continuum. Peter also notices that the problem does not give enough information to solve the problem but instead of assuming the missing information he rejected the task as undoable.

> Peter: Well, I'd probably think about knowing like, knowing that I can't guesstimate, or estimate at least like what these lengths are, like I'd think well I'd know that's approximately half but you don't know if it's perfectly made to match the answer so

Peter first mentions that he cannot estimate the answer, even though he can approximate the relative lengths in the figure. Peter goes on to say that even if he did feel that he could estimate, it would not be prudent because his estimate might not match 'the answer,' presumably held by the teacher.

Students on the recognition end of the continuum also reject this problem for another reason. Peter does not believe that his teacher would give a word problem about ants.

> Peter: I don't think she would use it, cause she uses more geometry stuff, like she would probably use that but she would say more like AE plus EF plus FC plus CB plus DE plus EB is greater than or less than BC plus CA plus AB, she would put it in more geometry form

This view of the problem is not based on mathematics like Matthew's reason, but based instead on an understanding of the teacher and the problems that the teacher chooses for the class. Peter is disposed to only spend time on tasks that will have value in the eyes of the teacher. The antwalk problem is not one worth his time.

June, a student in the middle of the continuum, is much less sure of her answers to the interview questions than her peers at either end. She is hesitant about whether or not the antwalk problem is one that she would be given. But, she says, if she were given the problem she would most likely be asked to produce a proof as an answer.

> Interviewer: Okay, how likely it is that if you would receive a problem like this, you would be expected that the answer would come in the form of a proof?
>
> June: Oh. Um…that's…ahh…I guess that's pretty likely actually if we were to get that.
>
> Interviewer: Okay. So even though it doesn't say do a proof it doesn't say do a proof you might be expected to do a proof
>
> June: Exactly, cause that's how we're used to figuring stuff out

June's first response is very hesitant. She says that students would do a proof if they were given that problem. She doesn't explicitly say that she would not receive the antwalk problem, but she will not endorse it either. Her response to the second question seems to be free from the context of the antwalk problem; no matter what problems students are given, students do a proof.

These three profiles of Peter, June, and Matthew showcase three different ways that students can engage with proof tasks in the geometry classroom. This analysis shows three unique responses to the antwalk problems, depending on the student's position along the recognition/misrecognition continuum. Depending on how the student is disposed to interpret the economy of symbolic goods of the geometry classroom she will be more or less likely to honor the value in her work based on the evaluation of the teacher, or based on the mathematics that she sees as available to learn. This analysis highlights three segments of the continuum between recognition and misrecognition and the actions of students who occupy that segment.

CONCLUSION

This paper highlights the different ways that students can experience the geometry classroom and the obligations that students hold to the classroom. The three students profiled show how different stances to the economy of symbolic goods manifest in students views of classroom work.

A key point to note is that for understanding the figured world of the geometry classroom, it does not matter if these students act in ways that are complementary with the views they express in these interviews. What we learn from these interviews is what stances that are available to students, regardless of whether or not students actually take up these stances. We learn what are the issues that one could take a stance on. We learn how students make sense of the figured world of the classroom – even if that is not consistent with the meaning that a teacher an observer would make of the classroom.

References

Bourdieu, P. (1990). *The Logic of Practice*. Stanford, Ca: Stanford University Press.

Bourdieu, P. (1998). *Practical Reason: On the Theory of Action*. Cambridge, Ma: Polity Press.

Chazan, D. (2000). *Beyond Formulas in Mathematics and Teaching*. New York: Teachers College Press.

Doyle, W. (1983). Academic Work. *Review of Educational Research, 53*(2), 159-1999.

Doyle, W. (1988). Work in Mathematics Classes: The Context of Students' Thinking During Instruction. *Educational Psychologist, 23*(2), 167-180.

Eckert, P. (1989). *Jocks and Burnouts*. New York: Teachers College Press.

Herbst, P. (2003). Using Novel Tasks in Teaching Mathematics: Three Tensions Affecting the Work of the Teacher. *American Educational Research Journal, 40*(1), 197-238.

Herbst, P. (2006). Teaching Geometry with Problems: Negotiating Instructional situations and mathematics tasks. *Journal for Research in Mathematics Education, 37*(4), 313-347.

Herbst, P., & Brach, C. (2006). Proving and Doing Proofs in High School Geometry Classes: What is it that is Going on for Students? *Cognition and Instruction, 24*(1), 73-122.

Holland, D., Lachicotte, W. J., Skinner, D., & Cain, C. (1998). *Identity and Agency in Cultural Worlds,* Cambridge, Ma: Harvard University Press.

Jackson, P. (1968). *Life in Classrooms*. New Your: Teachers College Press.

Lave, J. (1997). The Culture of Acquisition and the Practice of Understanding. In D. Kirshner & J. A. Whitson (Eds.), *Situated Cognition: Social, Semiotic, and Psychological Perspectives* (pp. 63-82). Mahwah, NJ: Erlbaum.

Pope, D. C. (2001). *Doing School: How we are Creating a Generation of Sense out, Materialistic, and Indicate Students*. New Haven: Yale University Press.

Powell, A. G., Farrar, E., & Cohen, D. K. (1985). *The Shopping Mall High School*. Boston: Houghton Mifflin Company.

DEVELOPING CRITICAL THINKING IN PROBABILITY SESSION

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This article presents a session on probability which incorporates elements of Critical Thinking (CT). This session is part of an in-depth study that comprises fifteen math sessions of similar constitution. The purpose of this research is to determine if teaching methods that encourage complex thinking can improve students' CT, within the framework of probability session. This study involved fifty five subjects. Analysis of interviews conducted with the students and an analysis of their submitted work indicated that students' analytical capabilities greatly improved. These results show that if teachers consistently and methodically encourage CT in their classes, by applying Mathematic theory to real-life problems, encouraging debates, and planning investigative sessions, the students are likely to develop critical and analytical thinking skills as a result.

INTRODUCTION

In the field of education, it is generally agreed upon that Critical Thinking (CT) capabilities are crucial to one's success in the modern world, where making rational decisions is increasingly becoming a part of everyday life. Students must learn to test reliability, raise doubts, and investigate situations and alternatives, both in school and in everyday life.

As will be discussed, as well as acquiring CT, it is important to assess students' application of their CT in different contexts. Many studies investigate CT in general, or in fields other than Mathematics, but few discuss CT in Mathematics. This study will explore CT in the context of a probability session.

THEORETICAL BACKGROUND

This research is based on three key elements: CT taxonomy that includes CT skills (Ennis, 1987); The Learning unit "probability in daily life" (Liberman & Tversky 2002), The Infusion approach between subject matter and thinking skills (Swartz, 1992).

Critical Thinking skills by Ennis (1987)

Ennis defines CT as "reasonable reflective thinking focused on deciding what to believe or do." In light of this definition, he developed a CT taxonomy that relates to skills that include not only the intellectual aspect but the behavioural aspect as well. In addition, Ennis's (1987) taxonomy includes skills, dispositions and abilities. Ennis claims that CT is a reflective (by critically thinking, one's own thinking activity is examined) and practical activity aiming for a moderate action or belief. There are five key concepts and characteristics defining CT according to Ennis: practical, reflective, moderate, belief and action.

Learning unit "probability in the daily life" (Liberman and Tversky 2002)

In this learning unit, which is a part of the formal syllabus of the Ministry of Education, the student is required to analyse problems, raise questions and think critically about the data and the information. The purpose of the learning unit is not to be satisfied with a numerical answer but to examine the data and its validity. In cases where there is no single numerical answer, the students are required to know what questions to ask and how to analyse the problem qualitatively, not only quantitatively. Along with being provided with statistical instruments, students are redirected to their intuitive mechanisms to help them estimate probabilities in daily life. Simultaneously, students examine the logical premises of these intuitions, along with misjudgements of their application. Here, the key concepts are: probability rules, conditional probability and Baye’s theorem, statistical relation, causal relation and judgment by representative.

The Infusion approach (Swartz, 1992)

There are two main approaches to fostering CT: the *general skills* approach which is characterized by designing special courses for instructing CT skills, and the *infusion approach* which is characterized by providing these skills through teaching the set learning material. According to Swartz, the Infusion approach aims for specific instruction of special CT skills during the course of different subjects. *According to this approach there is a need to reprocess the set material in order to combine it with thinking skills.*

This report is a description of an initial study, a snap shot that focused on one session and demonstrates the entire study. In this report, we will show how the mathematical content of "probability in daily life” was combined with CT skills from Ennis' taxonomy, reprocessed the curriculum, tested different learning units and evaluated the subjects' CT skills. Moreover, one of the overall research purposes is to examine the effect of the Infusion approach on the development of critical thinking skills through probability sessions. The comprehensive research purpose will be to examine the effect of learning by the Infusion approach using the Cornell questioners (a quantitative test) and quantitative means.

METHODOLOGY

In this article we ask how can CT skills be incorporated into a structured Mathematics session, such as a probability one?

Setting and population

Fifty five children between the ages of fifteen and sixteen participated in extra curriculum program aimed to enhance students from different cultural backgrounds and socio-economical levels. An instructional experiment was conducted in which probability sessions were combined with CT skills. The experiment constituted fifteen sessions of 90 minutes each, during the course of an academic year, in which the teacher was also one of the researchers.

Data collection

Data collection was conducted by way of triangulation:

- Personal interviews - conducted randomly. Five students were interviewed at the end of a session and one week after. The personal interviews were conducted in order to reveal a change in the students' attitudes throughout the academic year.
- The students' products were collected: exams, in-class papers and homework.
- All sessions were documented and analysed -the sessions were recorded and transcribed. The teacher kept a journal (log) on every session. Data was processed by means of qualitative methods which enabled to follow the students' patterns of thinking and interpretation with regards to the learned materiel in different contexts.

The teaching experiment

A probability unit comprised of fifteen sessions of ninety minutes each was taught. The probability unit combines CT skills with the mathematical content of "probability in daily life".

This new probability unit is a processed unit that includes questions taken from daily life situations, newspapers and surveys, and combines CT skills.

Each of the fifteen sessions that comprise the probability unit has a fixed structure: A generic (general) question written on the blackboard; the student's reference to the question and a discussion over the question using probability and statistical instruments and; an open discussion that combines practicing the CT skills. Table 1 depict an example for a session.

The mathematical subjects learned during these fifteen sessions were: Introduction to set theory, probability rules, building a 3D table, conditional probability and Baye’s theorem, statistical relation and causal relation, Simpson's paradox, and judgment by representative.

The following CT skills were incorporated in all fifteen sessions: A clear search for a thesis or question, the evaluation of reliable sources, identifying variables, “thinking out of the box,” and a search for alternatives.

Case study

Hereinafter a detailed description of a session, following the description, the session will be analysed by the following skills: referring to information sources, raising questions, identifying variables, suggesting alternatives and inference.

The session subject was statistical relation and causal relation. The session's aim was to teach how to determine the existence of causal relation.

Mathematical concepts used in the session: determining how a third factor can affect a statistical relation between two existing factors, including Simpson's paradox (the combination of A and B seeming to cause reversal of “success”).

CT skills practiced: evaluating source reliability, identifying variables, suggesting alternatives and inference.

Session plan:

Phase A - At the beginning of the session the teacher presented a short article about a research that indicates of a relation between calcium and vitamin D, and dental health. The research is taken from a daily Israeli newspaper that translated the article from "The American Journal of Medicine". The teacher writes a question on the blackboard. The students are requested to address the question; *Phase B* - Discussion in small groups about the article and the question. *Phase C* - Open class discussion. During the discussion the teacher asks the students different questions to foster the students' thinking skills and curiosity and to encourage them to ask their own questions. *Phase D* - The teacher refers to the questions raised by the students and encourage CT while instilling new mathematical knowledge - the identification of and finding a causal relation by a third factor and finding a statistical relation between C, and A and B, Simpson's paradox.

The discussion conducted in class	The practiced skills
The article presented with the class was "Calcium and vitamin D contribute to dental health" and claimed that the consumption of food additives of Calcium and vitamin D can help protecting the teeth. The data was taken from a research conducted in a dentistry school in a Boston university and published in "The American Journal of Madison". In this research one hundred and forty fife people participated, at the age of thirty five and above. Part of them took Calcium and vitamin D and the rest of them took placebo. 27% Of the placebo group lost at least 1 tooth in comparison to 13% of the Calcium and vitamin D group. The generic question on the blackboard was: *Is calcium good for your teeth?*	In paragraph 1 we encounter skills such as "searching for the question"- a fundamental skill. First there is a need to clarify the starting point for the interaction with the student. We also need to clarify to ourselves what is the thesis and what is the main question before we approach decision making. The paragraph also demonstrates relevance to daily life.
1. **Teacher:** Last week I visited a friend who is a dentist. When we set to the table she served a variety of cheese and told me she read in the newspaper calcium was good for our teeth and presented me with the article. What should I check before I decide whether I should increase the amount of calcium I consume? Should I eat Calcium or not? What do you think?*	In paragraph 2 the students are taking a step back, we refer to "identifying information source and evaluating the source's reliability" skill. This step is crucial, as it helps us to assess the quality and the validity of the article discussed. This skill was practiced in past lessons. See paragraph that summarizes the article.

Table 1. Classroom discussion over an article and the infusion of CT skills, Part 1

The discussion conducted in class	The practiced skills
2. **Student 1**: Where is the article taken from? Can we see the article for ourselves?*	
3. **S2**: Is the article's source reliable? How can we check it?*	
4. **S3:** Where is the article taken from? What is its source?	
5. **S1**: Should I answer the identification of the sources question?	
6. **T:** Not yet. We are focusing on searching for questions. Please think of other questions.	In paragraph 6 we encounter "searching for the question" skill again. We will continue searching for the main question through practicing the "variables identification" skill.
7. **S3**: What relation does the article discuss?	
8. **T:** A very good question. Before you look for the relation, what do you need to do?	Raising the search for alternatives. Posing questions enables the practice of this skill.
9. **S2**: To identify the variables!!!	P(A) , P(B), N(S)
10. **T**: Right. First, we ask what the variables are. Then we refer to the relation between them.	Paragraph 10 deals with identifying the variables and understanding them by a 2D table and a conditional probability formula $P(\bar{A}/B)=\frac{P(\bar{A}\cap B)}{P(B)}\Rightarrow$
11. **S3**: Do you mean a statistical connection?	
12. **S4**: What a silly question. It's obvious.	
13. **S3**: What's so obvious?	
14. **S4**: The connection is obvious - statistical relation between the vitamin and healthy teeth.	The mathematical part P(A/B)≠ P(A/B). Calculations according to sets and supplementary sets.
15. **S3**: How do you know?	
16. **T**: <u>There are no silly ideas or silly questions</u> in this class. In fact, student 3's question is excellent. Student 4, please try and think why student 3's question is a good one. Try to follow student 3's line of thought, remembering our discussion last week.	In paragraph 16 the teacher builds the students' self esteem by encouraging them to express their ideas and opinions (even if they are not always correct or relevant). She prevents any intolerance of other students. The method of instruction that aims at fostering the confidence and the trust of the students in their CT abilities and skills is, according to Ennis "referring to other peoples points of view" and "being sensitive towards other peoples' feelings."
17. **S4**: If there is a connection, then it must be a statistical relation, right?	
18. **T**: Did you calculate the existence of P(A/B) ≠P(A/B)?	
19. **S4**: You can infer it from the title that suggests that a relation exists between taking vitamins and healthy teeth.	
20. **S3**: According to the data from the article, you can find a statistical relation (the student specifies the calculation).[SF]	
21. **T:** Very good. An excellent inference. I want you to keep thinking of other questions.	
22. **S4**: Can you give a reasonable explanation for the relation we found?	

Table 1. Classroom discussion over an article and the infusion of CT skills, Part 2

	The discussion conducted in class	The practiced skills
23.	**S2**: I know! We can ask: suggest at least 2 other factors that might cause the described effect.	In paragraph 23 the student is referring to other sets and finding the connection between them.
24.	**S5**: The question is what causes what?	
25.	**S6**: Does vitamin D contribute to healthy teeth?	
26.	**T**: What do you think?	
27.	**S6**: Vitamins contribute to healthy teeth.	
28.	**T**: How can you be sure?	
29.	**S6:** Umm…	
30.	**S4**: Does deficiency in vitamin D cause damage to the teeth?	
31.	**S3:** Are there other factors, such as genetics!?	Paragraph 31 depicts the skill of "Searching for alternatives".
32.	**T**: Very good. What did student 3 just do?	
33.	**S1**: He suggested an alternative!!	
34.	**T**: How can we check it? Do you have any suggestions? Can you make a connection between this problem and the material we have learned in the past few lessons? Can you offer an experiment that would solve the problem?	Paragraph 35 refers to a controlled experiment or an observational experiment. An additional grouping and finding the connection between the variables by Bayes theorem or a 2 dimensional table.
35.	**S3**: Of course. An observational experiment.	

Table 1. Classroom discussion over an article and the infusion of CT skills, Part 3

Analysis according to CT skills

With the Infusion approach, students practice their CT while acquiring technical probability skills. In this session, the following five skills are exercised:

Raising questions - asking question about the article and probing on the main question about the connection between Calcium and vitamin D contribute to dental health (paragraph 1, see Table 1, Par 1).

*Referring to information sources and evaluating the source's reliability - the article went through a number of interpretations. It was published in an Israeli newspaper, which translated it from an American journal, which, in turn, published a research that had been conducted in a dentistry school in a university located in Boston with its name unmentioned. All the above raised the students scepticism (paragraph 2, see Table 1, Par 1).

Identification of variables - students identified the variables: Calcium, vitamin D, dental health (paragraph 6, see Table 1, Par 2). Following these skills, another skill, searching for alternatives (paragraph 31, see Table 1, Par 3), was presented.

In class we spoke about suggesting alternatives, not taking things for granted but examining what had been said and suggesting other explanations. At this stage, we combined the Mathematical aspect of the session - the connection reversal (a third factor that reverses the conclusion made before hand). We found the connection between the tree events (A, B and C). Another skill that was practiced is inference, in light of the alternatives suggested.

Hence, the skills that were practiced in the described session are: raising questions, evaluating the source's reliability, identifying variables, suggesting alternatives and inference.

In order to understand and monitor the student's attitudes toward CT as manifested by the skills specified above, interviews were conducted after the above session. In these interviews, the student expressed their acknowledgement regarding the importance of CT. Moreover, students are aware of the infusion of instructional strategies that advances CT skills. An example of two of the interviews is followed.

Student 4 was interviewed and was asked to define CT. His answer was:

> I think CT is important when you study Mathematics, when you study other topics and when you read the paper, but it is most important when you deal with real life situations, and you need the right instruments in order to do so (deal with these situations).

When student 2 was asked about important components during the last few classes and the present class, she answered:

> Now I understand 'variables identification' and it helps in everyday life. The issue of "intermediate factor" and the meaning of "reversing the connection" are also very important. Besides," she added with a grin, "now I'm more skeptical about what they write in the paper.

These initial findings suggest that infusion of CT into the formal curriculum in mathematics equips students with CT skills that are applicable to wider disciplines.

CLOSING REMARKS

The small scale research described here constitutes a small step in the direction of developing additional learning units within the traditional curriculum. Current research is exploring additional means of CT evaluation, including: the Cornell CT scale (Ennis, 1987), questionnaires of varied approaches, and a comprehensive test composed for future research.

The general educational implications derived from this research can and should be used to lever the intellectual development of the student beyond the technical content of the course, by creating learning environments which foster CT, which will, in turn, encourage him to inquire the issue at hand, evaluate the information and react to it as a critical thinker. It is important to note, that in addition to the skills mentioned above, in the course of this session the students also gain intellectual skills such as conceptual thinking and class culture that (climate) foster CT. Students practice critical thinking by probability, while the presented article constitutes the basis for practicing critical thinking skills together with the subject of probability. In this session, the following skills are practiced: referring to information sources (paragraph 2), encouraging open-mindedness and mental flexibility (all questions), a change in attitude (paragraph 28, see Table 1, Par 3) and searching for alternatives (paragraph 31, see Table 1, Par 3). A very important intellectual skill is the fostering of cognitive determination –to be able to express one's attitude and present an opinion that is supported by facts (paragraph 17-20, see Table 1, Par 2). In this session, students are shown to be searching for the truth, they are open minded and are self confident. In other words, they practice critical thinking skills.

Research limitations

This case study presents one session which is designed in a fixed pattern - a generic question, a discussion over the question, the practice of statistical relation, introduction to causal relation and experiencing the use of CT skills such as: raising questions, evaluating the source's reliability, identifying variables, suggesting alternatives and inference. On the basis of the interviews conducted and questioners that were qualitatively analyzed, it is unknown, at this stage, whether these skills had been acquired. Skill acquisition will be evaluated later on, using quantitative measures – the Cornell Critical Thinking Scale and the CCTDI scale. This case study raises encouraging evidence and a further investigation in this direction is needed.

References

Ennis, R. H. (1987). A taxonomy of critical thinking: Dispositions and abilities. In J. B. Baron, & R. J. Sterngerg (Eds.), *Teaching for Thinking* (pp. 9-26). New York: Freeman.

Liberman, V., & Tversky, A. (2002). *Probability Thinking in Daily Life*. Tel-Aviv, Israel: The Open University (in Hebrew).

Swartz, R. (1992). Critical thinking, the curriculum, and the problem of transfer. In D. Perkins, J. Bishop, & J. Lochhead (Eds.), *Thinking: The Second International Conference* (pp. 261-284). Hillsdale, NJ: Erlbaum.

INVESTIGATING THE TECHNOLOGICAL PEDAGOGICAL CONTENT KNOWLEDGE: A CASE OF DERIVATIVE AT A POINT[2]

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This paper emerged from our attempts to help pre-service mathematics teachers integrate technology into their instruction. We are convinced of the usefulness of the idea of technological pedagogical content knowledge (TPCK), which, we argue, provides a framework to diagnose pre-service teachers' difficulties and to identify the areas in need of development for a successful integration. We also argue that such diagnoses and identifications need to take the mathematical content into serious consideration, hence placing a strong emphasis on the content dimension of TPCK. These arguments are exemplified through the analysis of a pre-service mathematics teacher's microteachings with and without the use of technology in the context of teaching derivative at a point.

INTRODUCTION

Recently, the question of what teachers of mathematics need to know in order to appropriately integrate technology into their teaching has received much attention (see e.g. ISTE (2000) as cited by Mishra & Koehler, 2006). ISTE (2000) proposes technology standards for teachers when integrating technology in various subjects. In the literature, a theoretical framework called 'Technological Pedagogical Content Knowledge (TPCK)' is proposed to investigate the nature of knowledge to be able to integrate technology into the instruction.

This framework is crucial in the sense that merely knowing how to use technology is not the same as knowing how to teach with it. TPCK framework was originally derived from the idea of 'Pedagogical Content Knowledge (PCK)' which was proposed by Shulman (1986, 1987).

Shulman (1987) emphasises that what is missing in describing teachers' knowledge is the 'subject matter for teaching' and proposes PCK as an important domain of teachers' knowledge. Shulman (1987) argues that pedagogical content knowledge is the category 'most likely to distinguish the understanding of the content specialist from that of the pedagogue' (p. 8).

Given that technology has gained widespread use in learning and teaching, Pierson (2001) has added technology component to PCK framework and described TPCK as a combination of three types of knowledge: (a) content knowledge, (b) pedagogical knowledge, that is, the structure, organization, management, and teaching strategies for how particular subject matter is taught, (c) technological knowledge including the basic operational skills of technologies.

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Mishra & Koehler (2006) illustrate TPCK as an intersection of these three knowledge categories: technological, pedagogical and content (see Figure1). They further define the intersection of pairs of different categories of knowledge: pedagogical content knowledge (PCK), technological content knowledge (TCK) and technological pedagogical knowledge (TPK). TCK is the knowledge of the relationship between technology and content e.g. understanding the kinds of representations that softwares offer for a concept. In that sense, "teachers need to know not just the subject matter they teach but also the manner in which the subject matter can be changed by the application of technology" (p. 1028). TPK is "the knowledge of pedagogical strategies and the ability to apply those strategies for use of technologies" (p. 1028) e.g. having students use Powerpoint to share their ideas with their peers where necessary.

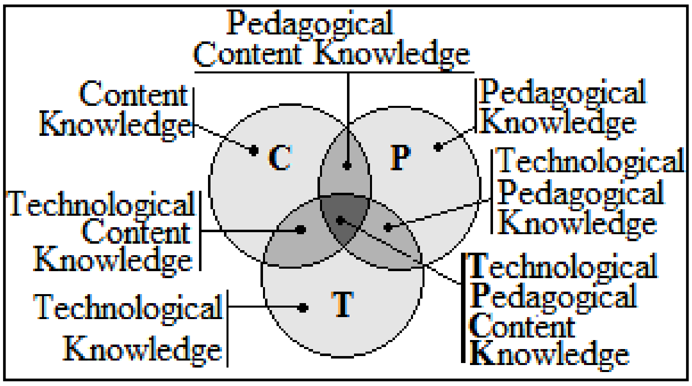

Figure 1. Representation of TPCK.

Up until now, only a few researchers (e.g. Pierson, 2001; Niess, 2005; Suharwoto, 2006) have examined the components of TPCK who adopt Grossman's (1990) four components of PCK to define the components of TPCK. Although they have provided a framework for TPCK, their works fall short in providing sufficient details regarding the content dimension of TPCK. In this paper, we aim to bring the content dimension into play and use the idea of TPCK as a framework to analyse the difficulties faced during teacher candidates' integration of technology into the instruction and also identify the areas which need development for a successful integration. Hence we will argue that TPCK framework has the power of not only diagnosing these difficulties and the areas in need of improvement but also guiding the design of courses concerned with the integration of technology into instruction as part of mathematics teacher education programs. We exemplify our arguments with a pre-service teacher's microteachings, in which concept of derivative at a point was delivered with and without the use of technology.

As will be clear throughout the paper, content is central to PCK, TCK and hence TPCK, we begin with a consideration of the content itself, namely derivative at a point.

THEORETICAL FRAMEWORK

Derivative concept is a mathematical model of instantaneous rate of change which is the limit of the function that describes the average rate of change. A graphical interpretation of the idea of rate of change engenders another aspect: slope of the tangent at a point. Mathematical meaning of derivative leads us to the three aspects of derivative which have also been investigated in the literature as the areas of student difficulties: derivative-rate of change relationship (Orton, 1983; Heid, 1988), derivative-slope relationship or so-called graphical interpretation (Amit & Vinner, 1990) and derivative-limit relationship (Orton, 1983; Hähkiöniemi, 2005).

When investigating the delivery of derivative at a point with the use of technology from the lenses of TPCK, we will focus on the content dimension considering these three aspects and will focus on TCK and PCK of derivative. We will also briefly analyse TPK since it might implicitly determine how the content is delivered using technology. In our framework TCK of derivative is defined as the knowledge of how the derivative concept (in three aspects described above) can be represented using the technological tools e.g. an understanding of how the idea of rate of change can be represented graphically and numerically by a technological tool. However, knowing how the derivative concept is represented using technological tools is one thing but using the technology for effective teaching is quite another. Teachers should also have PCK of derivative and combine it with general TPK. In terms of PCK, we will focus on only one of its components: knowledge of strategies and representations for teaching (Shulman, 1987; Grossman, 1990).

In this paper, we will make an attempt to answer the research question: "How can pre-service teachers' difficulties with technology integration be explained from the lenses of the components of TPCK framework, namely TPK, TCK, PCK?".

METHODOLOGY

This case study is a part of a wider study which sets out to investigate the development of pre-service secondary mathematics teachers' TPCK during a mathematics teacher education program in Turkey. The data was collected during the period of pre-service teachers' micro-teaching activities in which the participants used technology as a tool for teaching. Twenty pre-service teachers taught various topics. Four pre-service teachers taught the concept of derivative at a point. This study will focus on one of these four pre-service teachers. After the first micro-teaching sessions, a workshop was conducted in which a Turkish version of Graphic Calculus software was used and hands-on activities of technological content for various topics were practiced.

The potential of the software in terms of providing multiple representations and links between them were discussed. After the workshop pre-service teachers taught the same topics again but this time using the software. Pre-service teachers' content knowledge of derivative was assessed before and after the workshop by their written responses to questions on the three aspects of derivative described in the theoretical framework. Their PCK of derivative was investigated by analysing their lesson plans, teaching notes, observations of their teaching and interviews during which they reflected on their planning and teaching.

In what follows, we present data from one pre-service teacher's (called Sena) microteaching videos, lesson plans and interviews which will be examined with reference to the three components of the TPCK framework, namely PCK, TCK, TPK.

RESULTS

As noted earlier content knowledge is central to TPCK, therefore we first consider Sena's content knowledge of derivative. Then we analyse her micro-teachings from the perspectives of TPK, TCK and PCK.

Content knowledge (CK) of derivative at a point

Sena's content knowledge of derivative for the three aspects was first assessed after her first micro-teaching experience just before the workshop. Second assessment of content knowledge was carried out after her second micro-teaching lesson during which she used the software. The analysis of her responses to the derivative questions indicated that Sena's content knowledge of derivative was enriched after the second micro-teaching for all three aspects of derivative. For instance, before the first micro-teaching she explained the role of the limit to define the derivative concept algebraically as in the formal definition. However, after her second micro-teaching, she made an intuitive explanation of the limiting process which she related to graphical meaning of derivative at a point. Similarly, her understanding of instantaneous rate of change improved. After the second micro-teaching, she correctly solved the questions which required interpretation of derivative as instantaneous rate of change in real-world contexts. She was able explain the graphical meaning of derivative which she could not before her second micro-teaching. Despite this improvement in her understanding of three aspects of derivative, she could not relate these aspects in a coherent way. For instance, when explaining the graphical meaning of derivative she first assumed that the slope of the tangent gives the derivative at a point and then constructed the slope of the secants and took its limit. In other words, she did not use the idea of rate of change to conclude that the instantaneous rate of change gives the relationship between derivative at a point and slope of tangent at that point.

Technological Pedagogical Knowledge (TPK)

Although we focus on the content dimension of TPCK, TPK should not be dismissed since it might implicitly determine how the content is delivered using technology. Analysis of Sena's reflections on her teaching provided insights into her TPK. For instance, in her micro-teaching, Sena used technology for only teacher-demonstration without having students to try and discover the ideas for themselves using their own computers. The reason for that, as she reported in the interview, is concerned with the role of the teacher in the classroom. She intentionally preferred this approach believing that technology should not weaken her authority as a teacher by providing the solutions for students. Coping with the changing roles of a teacher with the existence of a new media in the classroom is crucial in terms of TPK and that affects delivery of the content. She also reflected on the contributions of technology to her teaching and emphasised that one can focus on the more difficult questions with the availability of technology. That is why, as she stated, she used a function which has an asymptotic behaviour at a point at which she examined the derivative.

Technological Content Knowledge (TCK) of derivative at a point

In this section, an analysis of Sena's TCK of derivative will be reported. In terms of technological content, the software that was used provides graphical and numerical representations of derivative at a point which are dynamically linked as can be seen in Figure 2. An understanding of this technological content is required for the development of TCK, therefore TPCK. In the interview, Sena reported that she did not have any experience with using technology neither as a student nor as a teacher. During the interview, Sena was asked to perform the activities of software and explain three aspects of derivative and this revealed that she could explain these three aspects separately using the technology. However, analysis of her content knowledge as described above and her teaching as will be described in the next section indicated that she could not relate the notion of rate of change to graphical meaning of derivative. In the next sections, we will look at how this knowledge of TCK shapes her teaching.

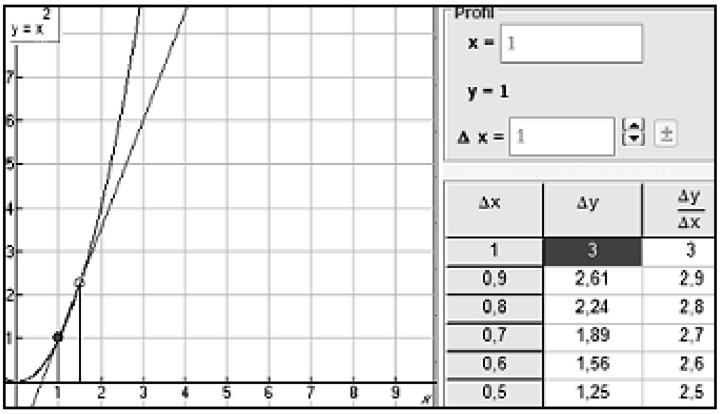

Figure 2. Technology content for derivative.

Pedagogical Content Knowledge (PCK) of derivative at a point

Sena's PCK was investigated by analysing her micro-teaching videos, lesson plans, teaching notes and interviews after her teaching. As described in the theoretical framework, we focus on only one component of PCK: knowledge of strategies and representations for teaching particular topics. In that sense, for Sena's first micro-teaching session her teaching strategy can be described as "introducing the concept by its formal definition followed by applications of definition with examples". She did not address any of the three aspects of derivative. For instance, she did not explain why limiting process was required when defining derivative at a point. She explained neither the graphical meaning of derivative nor the notion of rate of change. In the interview she said that she did not know about rate of change, therefore did not consider it at all in her teaching. However, she said she deliberately ignored derivative-slope relationship:

> Sena: students might have difficulties with analytical geometry therefore they may not understand the geometrical meaning of derivative…it shouldn't be given when introducing the concept. Students should first learn what the derivative means, that is how it is calculated (algebraically)

As can be understood from her response, although she takes students' difficulties with graphical meaning into account, she does not use any strategy to address this difficulty. This is not surprising considering that she does not know geometrical meaning of derivative herself. She strongly believes that the most important aspect of derivative for students is the algebraic rules of differentiation.

Analysis of the data indicated that Sena's PCK was enriched after the workshop. Different from her first micro-teaching session, she followed a strategy which places the notion of rate of change into the centre. She started her lesson by explaining the notions of dependent and independent variables, and verbally described the notion of rate of change as the ratio of the change in the dependent variable over the change in the independent variable. Then she used the 'diagram of rate of change' activity of the software which evaluates $\Delta y / \Delta x$ for the function $f(x) = x^2 + x$ (see Figure 3). She focused on values of rate of change around $x = 2$, first for [2,3] as shown in Figure 3, then for [2,2.1], [2,2.01], [2,2.001] and found the values of rates of change as 6, 5.1, 5.01 and 5.001. She mentioned that the values of rates of change approach to a number and this reveals the relationship between limit and derivative. She explained the derivative at 2 as the number to which the values of $\Delta y / \Delta x$ approach. However, she did not use the term 'instantaneous rate of change'. Up to this point, she did not explain the graphical meaning of rate of change by using the graphical representation of tangents approaching to the slope which dynamically progresses simultaneously with the table of values (see Figure 2).

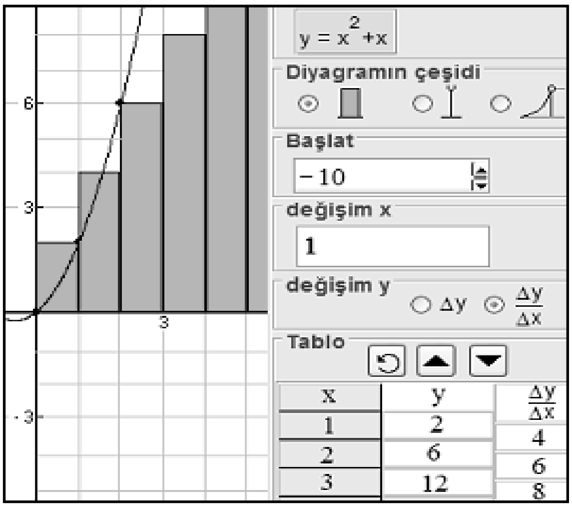

Figure 3. Rate of change activity.

After writing the formal definitions of left and right derivative, she explained them with an example, $f(x) = |x - 3|$, using the software. To explain why the left and right derivatives at 3 are different, she used the values of $\Delta y / \Delta x$ but she did not explain that the slopes of tangents from the left and right are different. When she was asked why she did not use the graphical representation of derivative to introduce the concept or to explain the left and right derivatives graphically, she stated that she planned to give the graphical meaning in the next lesson. She also stated that her students would have difficulties if she introduced the graphical meaning in the beginning.

DISCUSSION

The data of this study indicated that TPCK framework, without dismissing its content dimension, was useful in examining the difficulties faced during the integration of technology into instruction and also to identify the areas which need development for a successful integration.

The analysis of data from the lenses of TPCK framework revealed Sena's difficulties with technology integration in detailed and specific terms, namely as CK, PCK, TCK and TPK which all shaped her TPCK. The data also revealed the dynamics among these components; that is how they enrich or hinder the development of each other. In terms of CK, Sena's understanding of derivative in three aspects (derivative as limit, graphical meaning of derivative and derivative as rate of change) has enriched by her understanding of technological content, namely her TCK. However, her TCK falls short in relating the three aspects of derivative at a point in a coherent way. Sena's TCK affected her PCK in the sense of strategies and representations used. In her first micro-teaching session she did not address any of the three aspects of derivative believing that the most important aspect of derivative for students is the algebraic rules of differentiation. In her second micro-teaching during which she used the software, she used a numerical approach to emphasise the notion of rate of change and make use of intuitive understanding of limit. However, she did not explain the graphical interpretation of derivative although she used the activity of the software which has a potential for addressing the relationship between graphical meaning and notion of rate of change by providing graphical representation of tangents approaching to the slope which dynamically progresses simultaneously with the table of values of rate of change (see Figure 2). Therefore, as the data suggests, TPCK of derivative is not just mere understanding of TCK of derivative. In fact, we believe that technological content has also pedagogical underpinnings e.g. the software Sena used relates three aspects of derivative by the way the table of values of rate of change which is connected to the notion of secants approaching the tangent to a point. For the development of TPCK, one should interpret this pedagogical idea behind the technological content and also combine his/her TCK with PCK and TPK. As the data indicated, Sena's resistance to change her role as a teacher, as part of her TPK, is an obstacle for successful technology integration as she prefers her students not to use their computers since it might weaken her authority and control as a teacher. In summary, Sena needs to enrich her understanding of technological content and pedagogical idea behind this content which directly affects her TPCK for a successful integration of technology to teach derivative at a point.

The analysis of data under TPCK framework provides some implications for mathematics teacher education. First of all, having the power of diagnosing pre-service teachers' difficulties with integration of technology into instruction and areas which need development for a successful integration, TPCK framework can guide the design of courses concerned with technology integration as a part of mathematics teacher education programs. Secondly, since many pre-service and in-service teachers might not have learnt their content with technology, school mathematics should be revisited using various technological tools aiming to develop TCK. Third, as data indicated in this paper, knowing merely the technological content is not enough for effective teaching. Teachers also need to develop technological pedagogical content knowledge. This paper analysed TPCK of derivative and future studies should investigate TPCK with a particular focus on the content dimension for other mathematical concepts. This kind of research could be useful for teacher educators concerning what to teach in terms of TPCK and how to monitor their development of TPCK especially during the courses such as 'Instructional Technologies for Mathematics Teaching' or in-service training for technology. Future studies should also be conducted to investigate the development of TPCK considering the other components of PCK as described by Shulman (1987) and Grossman (1990). We, in this study, looked at a pre-service teacher's teaching using a single technological tool. Since the ability to choose a tool based on its fitness is an important aspect of TPCK (Mishra & Koehler, 2006), it would provide deeper insights to investigate TPCK in contexts where there is a wide repertory of technological tools available for teaching.

References

Amit, M. & Vinner, S. (1990). Some misconceptions in calculus: Anecdotes or the tip of an iceberg? In G. Booker, P. Cobb, & T. N. de Mendicuti (Eds.), *Proceedings of the 14th International Conference of the International Group for the Psychology of Mathematics Education*, 1, 3-10, Mexico: Cinvestav.

Grossman, P. L. (1990). *The Making of a Teacher: Teacher Knowledge and Teacher Education*. New York: Teachers College Press.

Hähkiöniemi, M. (2005). Is there a limit in the derivative? - Exploring students' understanding of the limit of the difference quotient. *CERME 4*, Sant Feliu de Guixos, Spain, 17–21 February 2005, 1758-1767. (http://ermeweb.free.fr/CERME4/).

Heid, K.M. (1988). Resequencing skills and concepts in applied calculus using the computer as a tool. *Journal for Research in Mathematics Education*, 19(1), 3-25.

International Society for Technologyin Education. (2000). *National educational technology standards for teachers*. Eugene. Available: http://cnets.iste.org

Mishra, P. & Koehler, M. J. (2006). Technological pedagogical content knowledge: A framework for teacher knowledge. *Teachers College Record*, 108(6), 1017-1054.

Niess, M. L. (2005). Preparing teachers to teach science and mathematics with technology: Developing a technology pedagogical content knowledge, *Teaching and Teacher Education*, 21, 509-523.

Orton, A. (1983). Students' understanding of differentiation. *Educational Studies in Mathematics*, 14, 235-250.

Pierson, M. E. (2001). Technology integration practice as a function of pedagogical expertise. *Journal of Research on Computing in Education*, 33(4), 413-429.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, 15, 4-14.

Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review*, 57, 122.

Suharwoto, G. (2006). Secondary mathematics preservice teachers' development of technology pedagogical content knowledge in subject-specific, technology-integrated teacher preparation program. *Unpublished PhD Thesis*, Oregon State University.

MEXICAN PRIMARY SCHOOL TEACHERS' MISCONCEPTIONS ON DECIMAL NUMBERS

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Using the methodology of Steinle and Stacey (2005) to detect and classify misconceptions on the order of decimal numbers, three workshops on decimal numbers were conducted with Mexican primary school in-service teachers. The results obtained are presented. Some teachers display some of the most common misconceptions: thinking that the shorter a decimal number is, the larger it is (thus 0.6>0.73); other teachers seem to apply partial rules and analogies with money.

BACKGROUND

Teachers' content knowledge

This paper reports partial results of a research project related to in-service primary school teachers' mathematical content knowledge as defined by Shulman (1986). Several authors have found that in- and pre-service teachers do not always master the mathematical contents they need to teach. Some authors who reviewed the PME Proceedings point out:

> Most of the studies over three decades of PME conferences, directly or indirectly, focused on the difficulties or deficiencies teachers exhibited for particular mathematics concepts or processes (Da Ponte & Chapman, 2006, p. 462).

In Mexico, primary school teachers are mainly trained in special colleges called Escuelas Normales. To enter these schools 12 previous years of schooling are required and the studies' program lasts four years. The curriculum in these colleges includes subjects such as history of education, pedagogy, psychology and didactics of all the disciplines they will have to teach in primary schools. It is believed that when future teachers enter the Escuelas Normales they master most of the topics they have studied in those 12 years of schooling, particularly in the case of mathematics. But, as the results of national and international evaluations show, this is not so. Mexican outcomes in international assessments of children's performance in mathematics are among the lowest (see e.g. OECD, 2002). Unfortunately, it has been detected that not only the students but also a fair amount Mexican in-service primary school teachers perform poorly in mathematics (see for example Sáiz, 2003), and other countries share this situation:

> Future and practicing teachers have become the object of much research. These studies may be categorized in three types. In the first type of study, teachers' content knowledge (CK) is tested, often revealing alarming weaknesses (Verschaffel et al., 2006, p. 68).

Our objective is to contribute in some way to overcome the low performance of Mexican pupils and their teachers, so we have designed a research project that attempts to answer the following questions:

- What mathematical concepts need to be reviewed in in-service primary school teacher's courses?
- Which tasks or problems may help to overcome common mathematical errors and misconceptions related to certain mathematical concepts?

These issues require the recollection of information about teachers' content knowledge. However, we believe, as other researchers (see for example Llinares, 2002), that it is ethically incorrect just to gather information from the teachers, and that it is necessary to recompense in some way the teachers who participate in our research; this is done by organising activities directed to correct misconceptions and/or to reflect about the teaching of mathematical topics. A collection of workshops has been designed for primary school teachers; it is called TAMBA: Talleres de Matemáticas Básicas (Basic Mathematics Workshops).

Research on Decimal Numbers

One of the curricular contents in which students of all levels have many difficulties is decimal numbers (for instance see Resnick et al., 1989). We as teachers in different levels (including the university level) have observed so, and this experience is concurrent with international research conducted on the topic:

> Most of the work on rational numbers represented as decimals is framed in terms of misconceptions, many of which are attributed attempting to assimilate decimal fractions to their existing natural number knowledge […] (Verschaffel et al., 2006).

In our research we have taken as a starting point the work by Stacey and Seinle (Steinle, V., Stacey, K., & Chambers, D., 2002; Steinle, 2004; Stacey, 2005; Steinle & Stacey, 2005). These authors have classified people's answers when comparing two decimal numbers in four coarse categories: L, S, A and U, and some refinements:

- Category L consists of considering for a variety of reasons that when comparing two decimal numbers the larger one is the longer. Thus, since 63 is longer than 8, 4.63 is considered larger than 4.8. (L stands for long). Some refined categories are: L1 interprets decimal part of number as whole number of parts of unspecified size, and L2 is as L1, but knows the 0 in 4.08 makes decimal part small, so that $4.7>4.08$.
- Category S (where S stands for short) consists of considering (again, for a variety of reasons) that the larger decimal number is the shorter. Thus, since 6 is shorter than 83, 2.6 is considered larger than 2.83. Some refined categories are: S1 assumes any number of hundredths larger than any number of thousands, so $5.736<5.62$, and S3 interprets decimal part as whole number and draws analogy with reciprocal or negative numbers, so $0.3>0.4$, like $1/3 > 1/4$ or $-3>-4$ ("reciprocal thinking").
- People in coarse code A are generally able to compare decimals. Within A, A2 people are correct on items with different initial decimal places; they may be following partial rules, drawing analogies with money, and having little understanding of place value.

- Category U contains all remaining people. Within U, U2 can "correctly" order decimals, but reverses answers, so than all are incorrect (e.g., may believe decimals are less than zero) (Steinle & Stacey, 2005).

Stacey's plenary speach in PME 2005 addresses the issue of which of these kinds of reasoning is more persistent with time and schooling. Among her results we wish to stress the following:

> Whereas persistence in the L codes decreases with age […], persistence in the S and A2 codes is higher amongst older students. This might be because the instruction that students receive is more successful in changing the naïve L ideas than S ideas but it is also likely to be because new learning and classroom practices in secondary schools incline students toward keeping S and A2 ideas […]. Whereas primary students in S codes have a better chance than L students to become experts, this is not the case in secondary school (Stacey, 2005, pp.29-31).

METHODOLOGY

In September 2005, October 2007 and November 2007 three workshops with in-service primary school teachers were held respectively in the towns of Xochimilco (a semi-rural area at the south of Mexico City), Monterrey (a prosperous industrial city at the north of the country) and Guanajuato (an industrial city at the centre of the country). They had the following characteristics:

- In Xochimilco, the workshop was organised by the head of the school district. The workshop was held for four hours (either in the morning or afternoon shift) during one monthly day when the children do not attend classes. The workshop that is reported here was compulsory for teachers of Grades 5 and 6 and covered several topics of school mathematics, among them decimal numbers. A total of 36 teachers attended the workshop in its two shifts.
- In Monterrey, a two-hour workshop on decimal numbers was held as a part of TAMBA and offered during the XL Annual Conference of the Mexican Mathematical Society (SMM). The Conferences of the SMM are customarily attended by many in- and pre-service teachers who receive a grant from the Ministry of Education; usually these grants are given to teachers who either ask for them or are have good results in exams designed for teachers (Carrera Magisterial. Once in the Conference, teachers can freely attend any of the sessions of the meeting, and within the session addressed to primary school teachers there are several simultaneous papers, courses and workshops from where to choose from. In 2007, a few teachers attended the Decimal Number Workshop. Data was gathered from 5 of them.
- In Guanajuato, the workshop was organised by the National Pedagogical University in its local Centre, as a part of a Symposium on the Teaching of Mathematics. In-service primary school teachers of public schools of the zone attended the Symposium, mainly because they are interested in topics of mathematics teaching. This workshop consisted of two three-hour

sessions in which several school mathematics topics were covered, among them decimal numbers, and was attended by 11 teachers.

In the three workshops, the time allotted for the decimal number topic was divided in three sections: a diagnostic test, whose results are the subject of this paper, a feedback on the test with explanation on the meaning of the decimal notation and the decimal-fraction link, and a reflection upon the difficulties of the teaching of the topic in primary school.

The diagnostic test was Steinle & Stacey's 30-item DCT3 (Steinle, 2005). It is shown in Figure 1. To the original test only reference letters were added.

Instructions: For each pair of numbers, EITHER circle the larger number OR write = between them								
a	4.8	4.63	k	2.681	2.94	U	3.92	3.4813
b	0.74	0.8	l	0.41	0.362	V	0.374	0.2165
c	2.6	2.83	m	5.62	5.736	W	7.942	7.63
d	0.6	0.73	n	0.426	0.37	X	0.62	0.827
e	1.86	1.87	o	0.3	0.4	Y	2.4	2.3
f	3.0	3	p	0.0	0	Z	0.8	0.80000
g	4.08	4.7	q	3.72	3.073	Aa	0.3	0.03
h	3.72	3.07	r	8.052	8.514	Ab	0.0004	0.4
i	17.35	17.353	s	4.4502	4.45	Ac	0	0.6
j	4.666	4.66	t	3.7	3.77777	Ad	0.7	0.00

Figure 1. Steinle & Stacey's decimal comparison test DCT3.

The test sheets were marked and each mistake was classified according to Steinle & Stacey's categories. Two parallel analyses were then conducted:

- Item-wise: on each item, totals were obtained for each of four possible answers (left side larger, right side larger, equal numbers or blank: no answer).
- Subject-wise: for each subject, totals were obtained for each of the possible categories. Then each teacher was classified in L, S, A or U with the following criterion: if the amount of correct answers was 27 or more, the teacher was classified as A. Otherwise, L or S were assigned when the great majority of mistakes corresponded to that category. U was assigned when a majority of mistakes were of the U kind or when there were both L and S mistakes in about the same proportion.

RESULTS AND DISCUSSION

In the item-wise analysis, the worse results (25 to 33 correct answers of the 52 subjects) were obtained in items j, t, i, and s, where many teachers made either the mistake of choosing the shorter number as the larger one (S mistakes) or answered that both numbers were equal: this is a mistake made by groups A2, S1 and S3, but not by L or A1. In a next group of items, items n, z, q, x, e, c, m, d, r, ac, o, u, w, and h obtained between 37 and 45 correct answers; the mistakes made in this group were mainly of the S coarse code. The best results (between 45 and 49 correct answers) were obtained in items aa, b, g, l, ab, f, k, a, p, v, y, and ad, where the mistakes were mainly of the L coarse code.

Except for items o, w, l, and k, in all items there was at least one blank. These can be interpreted as doubts: teachers not answering a question because they were unsure of the correct response. The items with more blanks were x, u, p (4 blanks each), j, n (5 each), and ac (6). Most noticeable in this group of items are items p (0.0 vs. 0) and ac (0 vs. 0.6); some teachers seem to wonder whether decimal numbers are *per se* smaller than whole numbers.

Among the mistakes made by the teachers who attended the workshops, the most frequent ones were the ones of the coarse S code. L mistakes only add up to 18% of the total amount. This is shown in Figure 2.

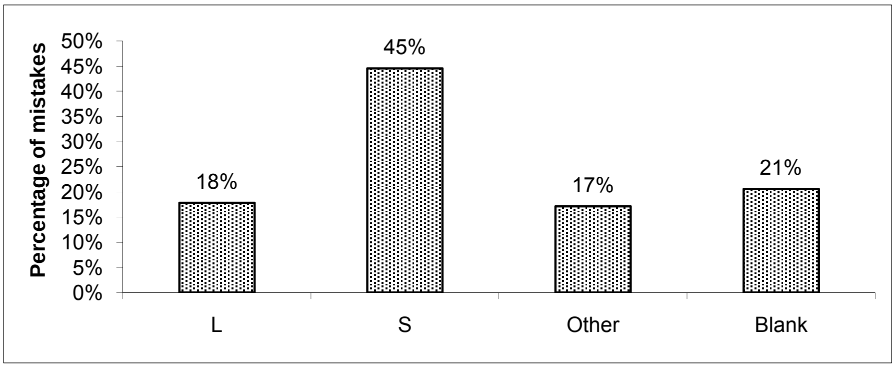

Figure 2. Distribution of mistakes made in DCT3.

However, this overall distribution changes when results are separated in the three workshops conducted. In order to be able to compare the three groups, the amount of mistakes in each category made by teachers of each workshop was divided by the number of teachers in the workshop, thus obtaining the quantity of mistakes of each kind per teacher in each group. The results are shown in Figure 3.

Several issues can be interpreted from this graph. The teachers of Xochimilco made as much as 7.4 mistakes per teacher, which compared to the 2.1 mistakes in Guanajuato and the 0.2 in Monterrey is a very large number. This means as much as 25% of mistaken answers in Xochimilco. Except for one, all of the blanks were in Xochimilco: teachers in this group seem to be the most unsure. Also noticeable is the fact that the larger percentage of S mistakes was obtained in Xochimilco (47%). In Guanajuato 61% of the mistakes were classified as “other”; they were the answer “both numbers are equal” in items j, t, i, and s: as commented above, this is a mistake made by groups A2, S1 and S3.

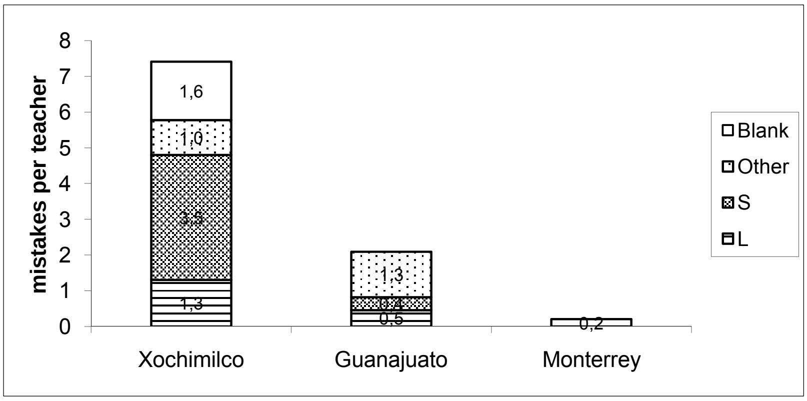

Figure 3. Mistakes per teacher in the three workshops.

In another step of the subject-wise analysis, teachers were classified in one of the coarse categories. In Xochimilco 1 teacher was classified in the L category, 9 in S, 14 in "other" and 12 were A. In Guanajuato there were no L or S subjects, 2 were classified as "other", and 9 were A. Finally, all 5 subjects of the Monterrey workshop were A. It is interesting to compare these distributions with the results reported by Steinle et al. (2002) for Australian students in years 4-10 of primary and secondary school. Figure 4 shows this comparison.

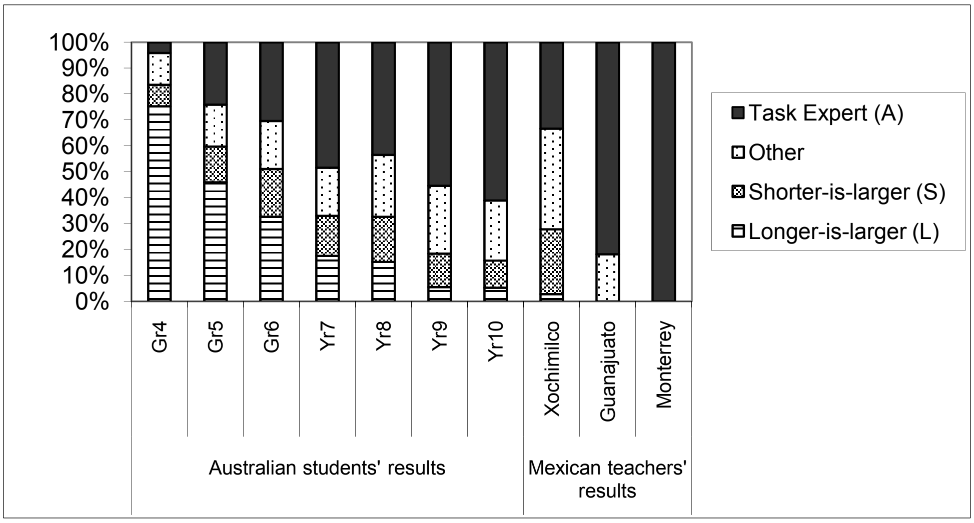

Figure 4. Classification of subjects. Comparison between groups.

Consistently with the results described above, the following results are noticeable:

- In Xochimilco there was a very large amount (25%) of S teachers; no group of Australian students reach such a percentage in this category. This is also the case with the "other" teachers; however, the percentage of L subjects (3%) is lower than that of any group of Australian students. Finally, the percentage of A subjects is barely larger to that of the Australian Grade 6 children.

- In Guanajuato and Monterrey, with no L or S subjects, the percentage of A subjects (respectively 82% and 100%) is significantly higher than the highest of all Australian students.

The observed differences among the results obtained by the three groups of teachers can be attributed to several differences among the type of teachers who attended each workshop. The teachers in Xochimilco were the less urbanised (although Xochimilco is close to Mexico City), and the workshop they attended was compulsory, whereas in Guanajuato and even more so in Monterrey the teachers who attended the workshops did so in a voluntary fashion and surging from both a personal interest in mathematics and its teaching, and a high level performance in teacher exams. Unfortunately, our experience with Mexican teachers takes us to suspect that the majority resembles more the case in Xochimilco than the other two.

Of course, the students of teachers who have so many misconceptions about decimal numbers are bound to repeat the misconceptions. As quoted above from Stacey (2005), the S and A2 misconceptions (as shown respectively by teachers in Xochimilco and Guanajuato) of primary school students tend to persist over time and schooling. It is unlikely that these children will overcome the misconceptions of their teachers. As for the teachers themselves, the frequency of S mistakes could be related to an incomplete learning in secondary school, and specially to a confusion originated in the learning of common fractions, negative numbers (reciprocal thinking) and rounding.

These results are of course worrying. Although it serves no consolation purposes, these Mexican results are in no way unique: "While some adults might have difficulty with problems involving decimal numbers, the fact that pre-service elementary teachers, in particular, have difficulty is a great concern" **(**Steinle, 2004, p. 2).

No single action can be taken to solve the problem. Educational authorities and curriculum designers should be aware of it, and emphasise the teaching of decimal numbers, not only in primary and secondary school but also in the Escuelas Normales. In-service teachers should be helped in as many ways as possible to be aware of the misconceptions and to overcome them.

References

Da Ponte, J. P. & Chapman, O. (2006). Mathematics teachers' knowledge and practices. In Gutiérrez, A. & Boero, P. (Eds.), *Handbook of Research on the Psychology of Mathematics Education: Past, Present and Future* (pp. 461-494). Rotterdam/Taipei: Sense Publishers.

Llinares, S. (2002). Participation and reification in learning to teach: The role of knowledge and beliefs. In Leder, G., Pehkonen, E., & Törner, G. (Eds.), *Beliefs: A Hidden Variable in Mathematics education?* (pp. 195-209), The Netherlands: Kluwer Academic Press.

OECD (2002). *PISA 2000 Technical Report.* Paris.

Resnick, L. B., Nesher, P., Leonard, F., Magone, M., Omanson, S., & Peled, I. (1989). Conceptual bases of arithmetic errors: The case of decimal fractions. *Journal of research in Mathematics Education. 20*(1), 8-27.

Sáiz, M. (2003). Primary teachers' conceptions about the concept of volume: the case of volume measurable objects. *Proceedings of the 27th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4. pp. 95-102). Honolulu: PME.

Shulman, L. S. (1986). Those who understand: knowledge growth in teaching. *Educational Researcher, 15*(2): 4-14.

Stacey, K. (2005). Travelling the road to expertise: A longitudinal study of learning. In H. L. Chick & J. L. Vincent (Eds.), *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 19-36).Melbourne: PME.

Steinle, V., Stacey, K., & Chambers, D. (2002). *Teaching and learning about decimals*. CDROM. Melbourne: The University of Melbourne.

Steinle, V. & Stacey, K. (2005). Analysing longitudinal data on students decimal understanding using relative risk and odd ratios. In H.L. Chick & J. L. Vincent (Eds.), *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 217-224).Melbourne: PME.

Steinle, V. (2004). *Changes with Age in Students' Misconceptions of Decimal Numbers.* Unpublished PhD, University of Melbourne, Melbourne.

Steinle, V. (2005). Personal communication.

Verschaffel, L., Greer, B., & Torbeyns, J. (in press). Numerical thinking. In Gutiérrez, A. & Boero, P. *Handbook of Research on the Psychology of Mathematics Education: Past, Present and Future* (pp. 51-82). Rotterdam/Taipei: Sense Publishers.

THE INTERPLAY OF SOCIAL INTERACTIONS, AFFECT, AND MATHEMATICAL THINKING IN URBAN STUDENTS' PROBLEM SOLVING

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From the detailed analysis of videotapes in an urban middle school classroom, taken as part of a larger study, we provide further interpretation of the notion of an "archetypal affective structure." This is a psychological construct that emerged from the analysis of other mathematics classrooms in this study, proposed as a way of describing a complex behavioral/social/affective interaction that can enhance or hinder a student's motivation to engage mathematically. We look closely at one such structure, labelled "Check This Out," and tentatively identify the concurrent and subsequent affect-related behaviors of students.

BACKGROUND AND THEORETICAL FRAMEWORK

The research reported here is part of a larger study investigating the occurrence and development of powerful affect around conceptually challenging mathematics. Its focus is on urban middle school classrooms serving low-income, predominantly minority communities. It extends earlier research that values close attention to children's mathematical thinking as they construct and justify their solutions (Davis & Maher, 1997), with the perspective that attending to issues of affect, context, social interactions, and culture in studying mathematical activity is essential to understanding how students gain confidence and motivation leading to success (Ball & Bass, 2003; Cobb & Yackel, 1998; Martin, 2000; Moschkovich, 2002).

By "conceptually challenging mathematics," we mean mathematical content that requires some development of new concepts or changes in existing ones. This frequently involves "figuring something out" within a problem situation that can be fraught with contextual distracters. Students may experience impasse (Schoenfeld, 1992), and their problem-solving efforts are likely to evoke discussions, explorations, and challenges to individuals' thinking. Research suggests that students may lose track of underlying mathematical concepts as they are caught up in surface characteristics of the problem, or as they become personally engaged in details of the context. According to Lubienski (2007), this phenomenon is particularly apparent among low SES students. Under such conditions, students may experience a variety of strong emotional feelings, leading to longer-term consequences for their engagement with mathematics. By the "affective domain," we refer to emotional feelings, attitudes, beliefs, and values in relation to mathematics (DeBellis & Goldin, 2006; Evans, Morgan, & Tsatsaroni, 2006; McLeod, 1994). "Powerful affect" refers to those patterns of affect and behavior that lead to interest, engagement, persistence, and mathematical success. It is not restricted to positive emotions, such as curiosity, pleasure, and satisfaction, but includes the effective management and uses of feelings such as bewilderment and frustration. It involves affective structures such as mathematical integrity, intimacy, and self-efficacy.

The earlier analysis of student affect, using data from classrooms included in the larger study, led to the detailed description of a construct called an *archetypal affective structure* (Epstein et al., 2007; Goldin, Epstein, & Schorr, 2007). As described in the latter reference, this is, "roughly speaking, a *behavioral/affective/ social constellation* within the individual." Relevant structural characteristics for this study include: "(1) a characteristic pattern of behavior, beginning in response to particular circumstances in the social environment, and culminating in a characteristic behavioral outcome, (2) a characteristic sequence of emotional feelings, or affective pathway, (3) information or meanings that may be encoded by the emotional feelings… (5) characteristic problem-solving strategies and heuristics for decision-making, (6) interactions with the individual's systems of beliefs and values, (7) interactions with the individual's structures of self-identity, integrity, and intimacy," as well as, "(10) expressions from which affect may be inferred that are socioculturally-dependent as well as idiosyncratic, which can serve some communicative function ..." (p. 261).

Several such structures were identified, with confirming evidence drawn from classroom videotapes, teachers' observations, and retrospective interviews with individual children. One of these structures, called "Check This Out," involves a student's realizing that solving a mathematical problem or understanding another person's solution strategy can have a payoff. Such a payoff might include the satisfaction of meeting the challenge of a complex mathematical task or investigating a situation that is relevant to the student's experience. The present article is concerned with identifying alternative pathways that we call "branches," the concurrent and subsequent affect-related behaviors of students when "Check This Out" was inferred to be operative.

RESEARCH QUESTIONS AND METHODS

In the exploratory sub-study on which this report is based, the guiding qualitative questions were as follows. (1) How, if at all, do the contexts of the problems posed in the lessons influence the students' understandings of the intended mathematical concepts? (2) How, if at all, do the contexts of the problems influence the students' engagement with the mathematical activity, in particular with regard to the "Check This Out" structure and its possible branches? (3) Can we construct a coherent "affective picture" of the class as a whole, including the observed impact of teacher interventions and descriptive information about individual students? This report focuses primarily on the second of these questions.

The class is one of three urban, low-SES, middle school mathematics classes (in two different districts), that were studied in depth over the school year. The student population is predominantly African-American and Hispanic. Data were collected during five cycles. For each cycle, data included videotapes of two consecutive lessons, pre- and post-interviews with the teacher, and videotaped, stimulated-recall interviews with four focus students. In addition, at least one videotaped interview was conducted with each of the other children in the class. Three cameras were used for each class session: two following the focus students, and the third stationary camera capturing an overall view of the class. Additional data included students' written work, observers' field notes and earlier analysis (Alston et al., 2007). The classroom and interview videotapes were transcribed. Each classroom tape, together with its transcription, was then analyzed using four different lenses, descriptive of: (a) the flow and development of mathematical ideas, (b) key affective events, where strong emotional feelings are inferred to occur, (c) social interactions among the students, and (d) significant interventions by the teacher. This analysis is still under way for the later cycles. The classroom teacher joined the research team subsequent to the school year and is participating in the analysis; he is a co-author of the present report.

After the initial analyses, we sought to identify evidence indicative of archetypal affective structures, in particular the "Check This Out" structure and its branches. Four branches identified in earlier analysis were used to create a coding scheme for student **(S)** mathematical behavior during the transcribed episodes, as follows: **(S1)** comparing and integrating the ideas of others with the student's own; **(S2)** moving toward the practical task of completing the activity; **(S3)** defending one's own solution or that of a peer with little reference to the mathematics involved; and **(S4)** retaining one's own solution, possibly passively, despite recognized logical contradictions in it. At this early stage in our research, we make no claim regarding the reliability of coding. Our results are preliminary and conjectural, though intended to lay the groundwork for future, larger-scale investigation.

We report here on data from class sessions and interviews with students during the first cycle. The segments occurred in the final third of the first videotaped lesson. The students were working in small groups and engaged in whole-class discussion. The lesson was based on an investigation from "Variables and Patterns," a unit of *Connected Mathematics 2* (Lappan et al., 2006).

RESULTS

During these segments, the students were completing a series of questions based on their earlier investigation of three sets of data they had entered into a four-column table, representing distances travelled at 50, 55, and 60 mph after 0 through 6 hours. The data were to be represented graphically on a single coordinate grid.

After discussing the graphs constructed by different students, the class appeared to agree that there should be three linear representations, with the steeper line representing the faster speed regardless of the scale that the student had established (See Figure 1). The teacher, Mr. P., asked the students to continue in their small groups to complete the next task in the investigation:

> C. Do the following for each of the three average speeds: 1. Look for patterns relating distance and time in the table and graph. Write a rule in words for calculating the distance traveled in any given time. 2. Write an equation for your rule, using letters to represent

the variables. 3. Describe how the pattern of change shows up in the table, graph, and equation (Lappan et al*.,* p. 51).

The segments reported here illustrate the interaction of the four lenses of our analysis. The codes referring to the "Check This Out" branches are indicated in bold face type. As Segment 1 begins, three boys are responding to the task. Juan [students' names are fictitious], a thoughtful Latino boy who has recently joined the class, is looking closely at the numbers in the table. Ryan, a soft-spoken focus student whose family has emigrated from the Dominican Republic, appears to be trying to make sense of the task itself. Denzel, a somewhat volatile African-American boy assigned to special education but included in Mr. P.'s class at his mother's request, seems to be struggling to comprehend. His questions suggest his need to follow the thinking of his partners.

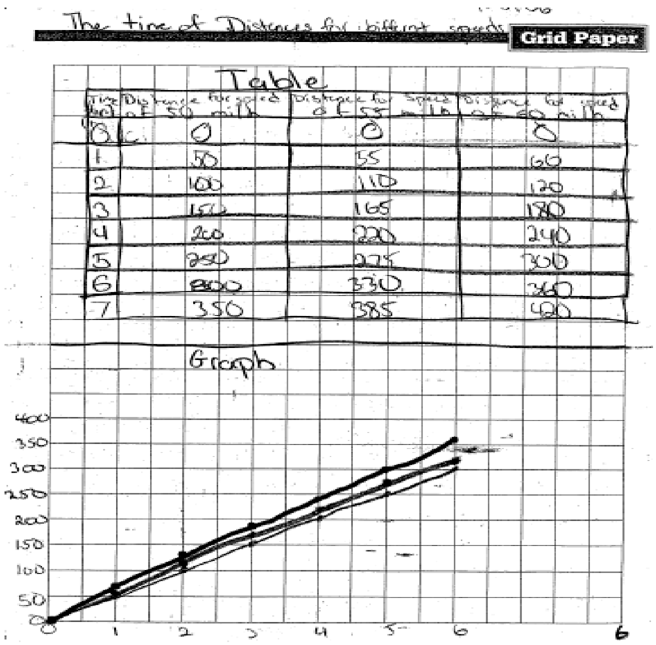

Figure 1. Nammi's table and graph, which the class agreed best represented the data.

Segment 1

In this segment, we observe students behaving as suggested by the "Check This Out" structure and two of its branches. This provides evidence of students' interactions as they attempt to integrate each other's ideas into their own **(S2)** and move towards the practical task of completing the activity **(S1)**.

(38:00)

Juan: What are you trying to look for? Ryan: Let me see. C 1. Write a rule for calculating the distance at any given time.

Denzel: What? Number what? C? C 1?

Ryan: Yeah.

Juan: Look for patterns?

Denzel to Ryan: What did you write? What did you start writing?

Juan (pointing to the three rows of the table that indicated speeds): It goes by 55. It goes by 50, 55, 60.

Ryan: They keep increasing. It says at any given time **(S2)**.

Juan: Wait.

Denzel: What? What happened? What happened? Ryan? What happened?

Juan (again tapping his pencil – this time horizontally along the first and second rows): It goes by 5's – then it goes by 10's. Ryan: I get it – it increases!

Juan: This one …5, 10, 15, 20, 25, 30...

Denzel: This one counts by …

Juan: It goes 50, 55, 60 and then 100, 110, 120 ... and then by 15 – 150, 165, 180 ... Then it goes up by 20 – 200, 220, 240.

Ryan: They keep increasing **(S1)**!

Ryan's written work includes: "That the rule if it's in any time diagonly it skip counting/increasing (sic)" **(S1,S2)**.

Others in the class recognize and repeat the pattern noted by the boys. Mr. P. asks the class to explain why the increase is 15 in the 3rd row of data rather than 5, as in the 1st row. The responses of three girls: Tyanna, an assertive and enthusiastic African-American focus student; Jana, a reserved African-American focus student; and Nammi, an assertive and confident African-American girl, provide another example of **S1**.

(40:19)

Tyanna (pointing to the data table on the overhead): Cover those zeroes! Then it's 5, 10, 15, 20, 25.

Mr. P: Why did you think it went up by 15 here instead of 5 here? And 10 here?

Jana: Because it's going by 5's!

Nammi: Because you are timing the distance (for one hour) by the hours **(S1)**.

Analyses of the complete transcripts document the students' mathematical focus shifting between surface characteristics of the table of values and the underlying concepts relating distance, time and speed. This shifting focus actually emerged in the first few minutes of the first session when, in answer to Mr. P.'s question about the topic, the first response was: "… multiples of 55, and 50 and 60 and you put them in the graph in the right way…" and the response of a second student was: "…miles per hour and the distance that you go." Mr. P. summarizes the class discussion so far, eliciting explanations that might connect the horizontal pattern to vertical patterns that indicate distance travelled from 0 to 6 hours.

Segment 2

Interactions among the students in this segment; particularly Juan, Ryan and Nammi, provide further evidence of **S1**. Behaviors exhibited by Ken, a somewhat moody African-American boy, and Ryan also show evidence of **S3**, defending one's own solution or that of a peer, and **S4,** retaining one's own solution, despite recognized logical contradictions. When Mr. P asks the class how far one would go in 10 hours at 50 miles per hour, Ryan and Juan both raise their hands.

(45:56)

Juan: It's this right here… you would look for the one that is going up by 10s.

Mr. P.: What's going up by 10s **(S1)**?

Ken: Look, 55 and 60, the third row, see how it's 110 to 120 **(S3)**.

Mr. P.: Oh, so you're looking at the pattern and which one goes up by 10s. Okay, so what about that?

Juan: So because you keep going, and you go up to 10, you would get the answer…

Ryan: 210. 210 - because if you skip counted by 50s 10 times it would give you 210 **(S1, S3)**.

Mr. P.: Okay what do you guys think about that?

Our analysis suggests that Ryan had counted by 10s horizontally, beginning with 120 on the third row of the table and ending with 210 as the tenth number in the sequence. Several students shout out their disagreement. One student offers 550 as an alternative solution. Our analysis indicates that she multiplied 5 by 10, and then skip-counted horizontally ten times to reach 550.

(50:35)

Nammi: Um, if you skip count 50 by ten times, it's going to be 500.

Ryan: Oh – because now I see it but before I think 50 x 10 will give you 210 **(S1, S3)**.

Nammi: No, because 50 x 10 is the same as skip counting by 50s. Ryan: Oh **(S4)**.

Our analysis suggests the potential challenges to Ryan as an English language learner that perhaps contribute to his difficulty in articulating and defending his thinking, possibly explaining his passive retention of his own answer.

(51:41)

Juan (points to the chart to concur with Nammi): What I'm trying to say is 1 x 50 is 50, and then 2x50 is 100, and then 3x50 is 150, and then you keep going **(S1)**.

Ken (returning to the horizontal pattern): You know, like, you see where 50 miles, miles, (sigh) 55 and 60 are right? How it's 220, 240? Like it's 200, 220, 240, it skipped 20 **(S3)**.

When Mr. P asks Ken why he thinks this is the case, Ken turns away from the discussion and begins a conversation unrelated to the mathematics with his neighbor **(S4)**.

In this segment, we observe student behavior that we interpret as defending their solutions **(S3)** and/or retaining particular solutions despite contradictory information **(S4)**. Although there was a high level of student involvement, at times this was not focused on the underlying mathematical concepts and occasionally it led to disengagement, such as that noted for Ken above.

At the conclusion of the lesson, Mr. P. poses a "real-world" question, "What if I told you that I wanted to drive (at 50 miles per hour)… to Florida which takes around 20 hours? How many miles…would I go?" The entire class becomes engaged in the discussion. Some noted the distance would be 50 x 20 or 1000. Denzel responds, describing an airplane trip to Las Vegas where he went 100 miles per hour. Several students note that 100 miles per hour would lead to a speeding ticket at which point Ken re-enters the discussion and explains that in Germany, where he was born, there were highways where you could legally go that fast. When Mr. P. brings the discussion back to the question, Nammi responds by stating, "Um, I figure it's going to be the time times speed equals distance" **(S1)**. Mr. P. concludes the class with the following homework assignment: "Write me a story about you going somewhere at a certain speed and tell me how long it's going to take you and how far you go."

The homework paper, eagerly submitted on Day 2 by Van, an African American, male focus student, evidences considerable work on his part. However, the task as he interpreted it, (confirmed during his interview) was to use the "Map Quest" function of the internet to obtain the estimated time and distance for his family trip to Myrtle Beach. He had written: "I got this information by looking at the 11 hours, 14 minutes, comparing them both and dividing 11 hours into 671.90 miles." He had not, however, calculated this on his paper. When asked during the interview to complete the calculation, he first divided 671.90 into 11 and obtained .02. When prompted to divide 11 into 671 by the interviewer, he agreed that 60 made more sense than .02 as his speed.

Ryan, in his interview, said that he had only completed half of the assignment. When questioned further, he described a trip to the Dominican Republic: "… The flight was like 14 hours because we went there in the afternoon and we arrived there like the next day. And I woke up from my sleep and we deported from the plane to the Dominican airport. … I was waiting for a taxi for like 30 minutes." There appeared to be no thought of mathematizing the situation, though Ryan responded willingly to direct questions from the interviewer about probable speed and the resulting distance.

CONCLUSIONS AND IMPLICATIONS

The classroom transcripts, along with the students' work and retrospective interviews, provide considerable documentation of the affective structure "Check This Out." The episodes presented in this paper, representative of the entire set of data for the cycle, document both the difficulty and the value of students' constructing bridges between the exploration of obvious but superficial patterns, the real-life characteristics of problems, and the underlying mathematical ideas. We see instances of dialogue and expression that can be interpreted as evidence for the four branches of "Check This Out" identified earlier. Based on the current data, we suggest the value of incorporating a 5th branch into the coding: **(S5)** diversion from the mathematical task to focus on personal or surface characteristics of the situation.

In our continuing analysis, we plan to systematically document and study occurrences of the "Check This Out" structure and other proposed archetypal affective structures in the complete data set from the larger study described above. Understanding of these preliminary findings would be enhanced by future studies replicating this research initiative in a wider variety of urban and other classroom contexts.

This analysis also provides evidence of the Herculean task facing teachers as they support urban students in this endeavor, and the complexity of the interactions taking place. Continuing analysis includes the development of codes to document various characteristics of teacher interventions and their impact on the interplay of social interactions, affect, and mathematical thinking.

Endnote

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References

Alston, A., Goldin, G., Jones, J., McCulloch, A., Rossman, C., & Schmeelk, S. (2007). The complexity of affect in an urban mathematics classroom. In T. Lamberg & L. R. Wiest (Eds.), *Proceedings of the 29th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Stateline (Lake Tahoe),* (pp. 326-333), NV: University of Nevada, Reno

Ball, D., & Bass, H. (2003). Making mathematics reasonable in the schools. In J. Kilpatrick, W. Martin & D. Shifter (Eds.), *A Research Companion to Principals and Standards for School Mathematics.* Reston, Virginia: NCTM.

Cobb, P., & Yackel, E. (1998). A constructivist perspective on the culture of the mathematics classroom. In F. Seeger, J. Voight & U. Waschescio (Eds.), *The Culture of the Mathematics Classroom.* London: Cambridge University Press.

Davis, R. B., & Maher, C. A. (1997). How students think: The role of representations. In English, L. (Ed.), *Mathematical Reasoning: Analogies, Metaphors, and Images* (pp. 93-115). Hillsdale, NJ: Lawrence Erlbaum Associates.

DeBellis, V. A., & Goldin, G. A. (2006). Affect and meta-affect in mathematical problem solving: A representational perspective. *Educational Studies in Mathematics,* 6, 131-147.

Epstein, Y., Schorr, R. Y., Goldin, G. A., Warner, L., Arias, C., Sanchez, L., Dunn, M., & Cain, T. R. (2007). Studying the affective/social dimension of an inner-city mathematics class. In T. Lamberg & L. R. Wiest (Eds.), *Proceedings of the 29th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 649-656). Stateline (Lake Tahoe), NV: University of Nevada, Reno.

Evans, J., Morgan, C., & Tsatsaroni, A. (2006). Discursive positioning and emotion in school mathematics practices. *Educational Studies in Mathematics, 63*(2), 209-226.

Goldin, G. A., Epstein, Y., & Schorr, R. Y. (2007). Affective pathways and structures in urban students' mathematics learning. In *Proceedings of the 9th International Conference of the Mathematics Education into the 21st Century Project*. Charlotte, NC.

Lappan, G., Fey, J. T., Fitzgerald, W. M., Friel, S., & Phillips, E. D. (2006). Variables and Patterns Unit, Investigation 3.1. In *Connected Mathematics 2, (*pp. 50-51) Boston: Pearson Prentice Hall.

Lubienski, S. (2007). Research, reform and equity in U.S. Mathematics Education. In Nasir, N. & Cobb, P. (Eds.), *Improving Access to Mathematics: Diversity and Equity in the Classroom*. New York. Teachers College Press.

Martin, D. B. (2000). *Mathematics Success and Failure among African-American Youth.* Mahwah, NJ: Lawrence Erlbaum Associates.

McLeod, D. (1994). Research on affect and mathematics learning. Journal for Research in Mathematics Education, 25, 637-647.

Moschkovich, J. (2002). A situated and sociocultural perspective on bilingual mathematics learners. *Mathematical Thinking and Learning,* 4 (2&3), 189-212.

Schoenfeld A. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In G. A. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning,* (pp. 334-360). NY: MacMillan.

STUDENT TEACHERS' ACQUISITION OF PEDAGOGICAL KNOWLEDGE OF ALGORITHMS

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The research results presented in this paper are only a small part of an action research performed with the main aim of improving student teachers' understanding of mathematics. The re-teaching of mathematics was integrated with the teaching of pedagogy by asking student teachers (STs) to perform children's activities which have the potential to develop conceptual understanding of the subject. This paper presents some results concerning STs' difficulties in acquiring conceptual understanding and pedagogical knowledge of alternative and standard algorithms for operations with natural numbers.

SOME RELATED LITERATURE

According to Pimm (1995), there are four ways of performing calculations: (i) with the aid of concrete materials, (ii) mentally, (iii) with written symbols and (iv) with the aid of calculators. Each way presents both strengths and weaknesses and the more or less suitability of some of these ways depends on the numbers involved in the problem. Three types of written calculations for the four operations with natural numbers are described in the literature: standard, alternative and students' invented algorithms (Schiro and Lawson, 2004). Even standard algorithms vary from one culture to another and from one generation to another (Leinhardt, 1988). Mathematics educators have debated which type of written algorithm should be the focus of school curricula, but the debate is not finished and research seems to be inconclusive (e.g., Laing and Meyer, 1982; Kamii and Dominick, 1997; and Schiro and Lawson, 2004).

I remember being taught the "equal addition" algorithm for subtraction. However, as a teacher I taught the "decomposition or trading" algorithm (Schiro and Lawson, 2004, pp. 204, 205). Yet I still use my own invented algorithm when solving my everyday subtraction problems. It involves reasoning that if I add what is left to what is taken away, the result is what I had before. That is, I transform any subtraction into an addition (e.g., if $345 - 158 = X$, then $X + 158 = 345$). In a subtraction such as $345 - 158$, I start by searching for a number that added to 8 results 15. I check that it is 7 by mentally doing: $7 + 8 = 15$. I record the 7 under the 8 (as in the equal addition algorithm) and record a small "carry one" near the digit 5. Then I search for a number that added to 6 (1+5) results in 14 and so on. I never had to memorize any subtraction facts. I did all my subtraction sums well and my teachers never managed to notice that my algorithm was different from the ones they were teaching. It was only later, when I became a teacher, that I noticed that I was using a different method from the one provided by the textbooks.

Teachers can not prevent students from inventing and using different algorithms from the ones they teach. Problems only happen when invented algorithms are faulty (Ashlock, 1982). Both Hart (1993) and Ball and Bass (2004) express the importance of evaluating the validity of students' methods. Hart (1993) points out that these methods "may be useful if a teacher has the time and sees the need to keep track of the child's methods and to help the move to greater sophistication" (p. 21). Teaching alternative and standard algorithms in conceptual ways and monitoring students' invented algorithms are complex tasks that demand great conceptual knowledge from teachers (Ball and Bass, 2004). It is also difficult for teachers to find the time to monitor different invented algorithms in classrooms with 30 to 40 students. With big classes, I prefer to work with several concrete and mental methods, but to focus on a single symbolic algorithm which can be the standard one or simpler versions which Ashlock (1982) calls low-stress algorithms.

Orton (1994) hypothesises that some students resist using a procedure "unless they have in some way made it their own" (p. 36). He thinks that there is a greater possibility of incorporating a procedure which has also been conceptually understood than a procedure which has only been rote memorised. Students can be asked to compare their different ways of working with concrete materials and decide which is the quickest or the more economical method of finding and recording the solution and why. The classroom agreed quickest algorithm can be called the "common way", adopted for whole classroom discussions and translated into a written algorithm. For natural numbers the quickest algorithms coincide with the actions behind the standard symbolic algorithms. Such reflections and comparisons seem to be a good way of helping students make certain standard algorithms "their own". When solving problems students can be asked to try to find the solution by both using their own methods and the "common" method. In this manner students who have more problems in translating from concrete materials to symbols can at least rely on one effective written algorithm and answer the problems; and those who know more than one way can use one way to check the other.

I take the view of Schiro and Lawson (2004) who think that standard algorithms are an important part of students cultural heritage and teachers "do not need to choose between teachers teaching algorithms and children inventing their own algorithms, but that these two activities can complement and enrich each other" (p. 97). Research tends to confirm this view. For example, Resnick and Ford (1981) found that instruction helped a student to connect her conceptual knowledge of place value with the procedural knowledge in a standard subtraction algorithm. The connection in turn helped the student to establish, mostly on her own, further place value connections and invent an alternative subtraction algorithm.

Understanding of algebra algorithms are said to be dependent on the understanding of arithmetic algorithms (English and Halford, 1995). Mathematics is not only beautiful and useful in everyday life but it is also the language of science. Although the more informal and oral mathematics used by Brazilian street sellers (Carraher et al. 1989) is an important tool in everyday life, it is not enough to change their social status. The mastery of school written mathematics can help students to acquire the necessary conditions to progress in mathematics itself and in many other subjects. Whereas success only "in the out-of-school mathematics will just assure the children of continuity in the low-status jobs they are already engaged in" (Abreu et al., 1997, p. 238). It is teacher educators' responsibility to help student teachers to acquire enough conceptual understanding and pedagogical knowledge to teach both alternative and standard algorithms and to evaluate students' invented algorithms. Therefore, one particular research question related to the present study was: "In what ways can primary school STs be helped to acquire a more conceptual understanding and some pedagogical knowledge of the algorithms in the primary school curriculum?".

METHODOLOGY

I carried out an action research at University of Brasília through a mathematics teaching course component in pre-service teacher education (Amato, 2004). The component consists of one semester (80 hours) in which both theory related to the teaching of mathematics and strategies for teaching the content in the primary school curriculum must be discussed. This is the only compulsory component related to mathematics offered to primary school STs at University of Brasília. There were two main action steps and each had the duration of one semester, thus each action step took place with a different cohort of STs. As the third and subsequent action steps were less formal in nature and involved less data collection, not many results will be reported from the latter.

A new teaching programme was designed with the aims of improving STs' conceptual understanding of the content they would be expected to teach in the future. In the action steps of the research, the re-teaching of mathematics was integrated with the teaching of pedagogical content knowledge by asking the STs to perform children's activities which have the potential to develop conceptual understanding for most of the contents in the primary school curriculum. About 90% of the new teaching program became children's activities. The children's activities performed by the STs had four more specific aims in mind: (a) promote STs' familiarity with multiple modes of representation for most concepts and operations in the primary school curriculum; (b) expose STs to several ways of performing operations with concrete materials; (c) help STs to construct relationships among concepts and operations through the use of versatile representations (Amato, 2006); and (d) facilitate STs' transition from concrete to symbolic mathematics. A summary of the main activities in the teaching program can be found in Amato (2004). The sequence of activities performed by STs for alternative and standard algorithms for each operation with natural numbers in the first and second action steps of the research was:

I. Practical work and discussion about different concrete algorithms. STs manipulate concrete materials on a special board called place value board (PVB) (Amato, 2006). There are two versions of the PVB: (i) students' version and (ii) teacher's version used for whole class discussions. At this stage no symbols are used. First I write on the blackboard a simple word problem, breaking the problem into parts that are connected to each line of the PVB. I also display loose straws and bundles of 10 straws on the appropriate places. Then the STs are asked to represent the initial amounts with concrete materials and to manipulate the concrete materials to solve the problem. The STs are also asked to pretend to be children who do not know the sum. They have only to remember that 10 things can not be left for long in a place. The ten things have to be bundled together and displayed on the next place on the left. Finally some STs are asked to show the class how they have solved the problem using the teachers' PVB.

II. Comparing left-handed and right-handed concrete algorithms. STs simultaneously manipulate concrete materials and symbols (number cards) on the PVB with the aim of comparing two specific concrete algorithms for addition and division and decide which was the quickest way of finding and recording the solution and why: (i) starting from the tens (left-handed), or (ii) starting from the units (right-handed).

III. Practical work and discussion about the standard algorithm. STs simultaneously manipulate concrete materials and symbols on the PVB with the aim of internalising the concrete and symbolic actions behind the standard algorithm.

IV. Formalisation of the standard algorithm. Through systematic questions asked by me, STs are asked to look back at their previous actions with concrete materials and symbols in activity III and verbalise their past actions (e.g., What did you do next with the tens blocks?). The objective is to construct the symbolic standard algorithm separated from the concrete materials. Each step in the symbolic algorithm is written by me on the chalkboard after each question is answered by the class.

V. Recording the concrete and symbolic actions behind the standard algorithm. STs are asked to record with pictures and symbols the actions they had previously performed in the third type of activity (III). The recording is done on specially designed sheets called "reports". STs record the initial position of the concrete materials (the sum) on a first picture of the PVB. The next pictures of the PVB are for recording the sequence of actions in the standard algorithms. The reports are a way of organising STs' recording and to save time as they do not have to draw pictures of PVBs as three (addition and multiplication) or four PVBs (subtraction and division) are printed for them on each sheet.

VI. Recording different symbolic algorithms. I use the teachers' PVB and concrete materials to perform the previous concrete algorithms manipulated by the STs in the first and second stage (I and II) and to perform other symbolic algorithms extracted from the literature. I record on the blackboard with symbols each concrete action performed by me on the teachers' PVB. Finally, I provide STs with handouts summarising the symbolic algorithms presented in the class for each operation and ask them to use all algorithms to calculate the result of two new sums as a home assignment.

The last two types of activities (V and VI) are considered teachers' activities because it involves recording standard and alternative algorithms in iconic and symbolic ways.

Four data collection instruments were used to monitor the effects of the strategic actions: (a) researcher's daily diary; (b) middle and end of semester interviews; (c) beginning, middle and end of semester questionnaires; and (d) pre- and post-tests. The questions in the questionnaires and interviews focused on STs' (i) perceptions about their own understanding of mathematics and their attitudes towards mathematics before and after experiencing the activities in the teaching programme, and (ii) evaluation of the activities in the teaching programme. Much information was produced by the data collection instruments but, because of the limitations of space, only some STs' responses related to their activities concerning alternative and standard algorithms for operations with natural numbers will be reported here.

SOME RESULTS

One of the teaching strategies used in the action steps of this research involved the use of activities which could help the STs to notice that there can be different ways of performing an operation. Some STs mentioned that the practical activities and discussions about different algorithms were useful to their understanding of operations and to their learning of pedagogical knowledge:

> Interview 11(3) ST140 ... The work with concrete materials inside the classroom. We did two processes on the PVB. First by starting from the loose ones [units] and then by starting from the big bundles [hundreds]. Then we divided the methods into stages and compared them. At the end we noticed that it was easier to start from the loose ones. Otherwise we would have to add another big bundle later. This practical aspect inside the classroom is very important for working with children.

> Interview 21(6) ST203 ... [she was already a primary school teacher who had done a vocational course at high school] I did not know there were other methods of doing addition and subtraction. I had a student who did subtraction in a different way. Her results were always correct but I never stopped to pay attention to how she did the sums. Seeing all those new methods [of performing an operation] opened my mind.

Different algorithms in the concrete mode

The work with different algorithms using concrete materials or with "concrete algorithms" was considered very successful in all semesters. Some STs were curious about why the process of doing an addition 'from left to right', that is, starting with the tens place, had not been adopted in the past. The comparisons made among the different concrete algorithms used by the class were helpful in the understanding of the standard algorithms:

> Questionnaire Pos-und (1)(b) ST136 I understood that it is possible to solve mathematical problems without being so rigid about using standard sums and that the standard [sums] corresponds to a historical construction.

Similar comparisons were not possible for subtraction and multiplication because of lack of time, but the STs were advised to use similar comparisons for those operations with their future students. In the case of division, they could notice that division behaves in a different way from other operations as it is the only operation which is started from the left side in the standard algorithm. The work with different concrete algorithms was thought to be quite important to help the STs improve their conceptual understanding of operations and to become aware of the validity of different methods of carrying out an operation.

In the first semester the initial addition of natural numbers with concrete materials was performed with two-digit numbers. Later there was a lively discussion with the classroom divided into two views: (i) one group thinking that joining the tens first and then the units (left to right) was a quicker method and (ii) another group thinking that joining the units first and then the tens (right to left) was quicker.

The STs took a long time to reach a conclusion because with two-digit numbers the economy is not great and so it did not seem to be perceived by some STs. So it was decided to repeat the activity in the next lecture with three-digit numbers and with the plane version of Dienes' blocks instead of straws. Working with bigger numbers was thought to be better in making the processes and relationships involved clearer as STs are exposed to more place value ideas and trading actions. Therefore, most activities during the rest of the semester and in other semesters were performed with three-digit numbers. Apart from that, no major changes for the practical activities concerning concrete algorithms were proposed for the second and subsequent semesters.

Different algorithms in the symbolic mode

The work with different algorithms in the symbolic or written mode or "symbolic algorithms", was considered interesting by some STs and in the case of ST234 it had helped to change her understanding of natural numbers: "Interview21(5)(b) To know the existence of other methods of performing sums.

The standard ways were chosen because they were considered more practical". On the other hand, some STs found the alternative symbolic algorithms difficult. ST222 commented in the classroom that he found even the low stress algorithms for addition very confusing. According to Ashlock (1982), low stress algorithms are meant to help children by reducing the intermediate numbers that have to be committed to the memory while adding each column of digits. ST234 interrupted and said that she considered the ideas as an option for the teacher to work with children who are having difficulties with the standard way of adding. Then ST207 commented that he thought they were nothing more than the standard algorithm recorded in a different way. The class concluded that not many alternative symbolic algorithms should be presented to young students as they could cause confusion, but STs should know them for:

- Helping their future students to learn the standard algorithms by using the low stress algorithms as ST207 and ST234 had noticed;
- Using some alternative algorithms as a recreational activity with older children. Some children find it very interesting to know how the Egyptians did multiplication without having to memorise the times tables; and
- Getting more flexibility in thinking about the operations at a more formal or symbolic level. That, in turn, could help them to: (i) understand different algorithms used by their students. They could have a student who studied for some time abroad or a student whose parents or other teacher have taught a different algorithm; and (ii) analyse the validity of their students' invented algorithms. Some teachers cannot cope with anything different from their own reasoning. They simply cross the problem as wrong.

Many STs seemed to have enjoyed the handouts and learning about different symbolic algorithms. However, such work, even in the case of natural numbers, was thought to be difficult for some STs. Therefore, some changes were made during the first action steps of the research. Similar practical activities and handouts had been designed for rational numbers but they were not administered to the STs in any semester. In a single semester, STs had to accommodate the idea of representing alternative and standard algorithms for natural numbers with concrete materials. That was already considered a difficult step for some STs.

The number of STs who complained in the questionnaires and interviews about their difficulties in learning about alternative symbolic algorithms for natural numbers increased from 1 in the first semester to 8 in the second semester. No major changes were made in the activities from the first to the second semester, so the increase of complaints may also have been due to the fact that more data was collected in the second semester. However, the standard symbolic algorithms they had memorised at school appeared to be interfering with the learning of new algorithms:

> Interview 21(6) ST216 ... You ask us to forget the procedures we learned at school, but it is very complicated to do that. The different methods of doing sums are very complicated for me. The sums had to be done in the standard way and that was all. There was no other way of doing them. Suddenly appears lots of methods for doing them. They are very difficult for us who are used to the standard methods.

Even the work with symbolic alternative algorithms for natural numbers was excluded from the programme in the third and subsequent semesters. Only the Egyptian algorithm for multiplication was presented as a recreational activity. Helping the STs to understand different symbolic algorithms was thought to be too difficult for a single semester. Besides the STs suggested that more activities concerning fraction concepts and operations were needed (Amato, 2004).

SOME CONCLUSIONS

In order to deal with students' invented algorithms, teachers must, themselves, be confident and fluent in performing algorithms in all four ways described by Pimm (1995) and in all modes of representation. Ideally, they should also have a good conceptual understanding to be able to discuss with children the reasoning behind different algorithms in symbolic form. Yet in a first course component about teaching mathematics it was thought to be more urgent to help STs to draw out connections between the standard and symbolic ways of operating natural numbers they had in their minds before starting the course and other more informal representations so that different representations could be incorporated in the same schema.

Apart from improving their conceptual understanding, knowing well these connections could provide STs with the confidence they needed to start teaching conceptually with the use of concrete materials and be an important starting point for their future understanding of alternative and invented symbolic algorithms. Indeed the low stress algorithms involve only small variations of the standard algorithms.

On the other hand, certain standard algorithms appear unreasonable. One example is the algorithm of division of fractions that is transformed into multiplication and does not resemble the previous schema for division. In those cases it is not difficult to convince STs to learn and use an alternative algorithm. Besides extending the standard algorithms for operations with natural numbers to the operations with rational numbers (Amato, 2006) was thought to be quite important to STs' acquisition of conceptual knowledge as it involves relating new content to previous learned content and so to the acquisition of meaningful knowledge (Ausubel, 2000). If, however, more teaching time is provided in the future, the STs could benefit not only from learning about alternative symbolic algorithms for natural numbers, but also from learning about the history of algorithms, that is, learning how algorithms changed over time and progressed to the present day notation.

References

Abreu, G., Bishop A, J., & Pompeu, G. J. (1997). What Children and teachers count as mathematics. In T. Nunes & P. Bryant (Eds.), *Learning and Teaching Mathematics: An International Perspective*, Hove: Psychology Press.

Amato, S. A. (2004). Improving student teachers' mathematical knowledge, *Proceedings of the 10th International Congress on Mathematical Education*, Copenhagen, Denmark.

Amato, S. A. (2006). Improving student teachers' understanding of fractions, *Proceedings of the 30th International Conference for the Psychology of Mathematics Education*, (Vol. 2, 41-48), Prague, Czech Republic.

Ashlock, R. B. (1982). *Error Patterns in Computation*, Columbus: Merril.

Ausubel, D. P. (2000). *The Acquisition and Retention of Knowledge: A Cognitive View*, Dordrecht, The Netherlands: Kluwer.

Ball, D., & Bass, H. (2004). Knowing mathematics for teaching. In R. Strasser, G. Brandell, B. Grevholm, & O. Helenius (Eds.), *Educating for the future*, Proceedings of an International Symposium on Mathematics Teacher Education, 159-178, Sweden: The Royal Swedish Academy of Sciences.

Carraher, T. N., Carraher, D., & Schliemann, A. (1989). *Na Vida Dez, na Escola Zero*, São Paulo: Cortez.

English, L. D., & Halford, G. S. (1995). *Mathematics Education Models and Processes*, Mahwah, New Jersey: Laurence Erlbaum.

Hart, K. M. (1993). Confidence in success, *Proceedings of the 17th International Conference for the Psychology of Mathematics Education,* (Vol. 1, 17-31), Tsukuba, Japan.

Kamii, C., & Dominick, A. (1997). To teach or not to teach algorithms, *Journal of Mathematical Behaviour*, 16(1), 51-61.

Laing, R. A., & Meyer, R. A. (1982). Traditional Division Algorithms, *Arithmetic Teacher*, 29 (9), 10-12.

Leinhardt, G. (1988). Situated Knowledge and Expertise in Teaching, in J. Calderhead (Eds.), *Teachers' Professional Learning*, London: Falmer.

Orton, A. (1994). Learning Mathematics: Implications for Teaching, in A. Orton and G. Wain (eds.), *Issues in Teaching Mathematics*, London: Cassel.

Pimm, D. (1995). *Symbols and Meanings in School Mathematics*, London: Routledge.

Resnick, L. B. & Ford, W. W. (1981). *The Psychology of Mathematics for Instruction*, Hillsdale, New Jersey: Lawrence Erlbaum Associates.

Schiro, M. & Lawson, D. (2004). *Oral Storytelling and Teaching Mathematics: Pedagogical and Multicultural Perspectives*, Thousand Oaks, CA: Sage Publications.

METHODS FOR THE GENERALIZATION OF NON-LINEAR PATTERNS USED BY TALENTED PRE-ALGEBRA STUDENTS

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This study focuses on the generalization methods of mathematically talented middle-school students in solving a quadric pattern task. A qualitative analysis of the solutions revealed two main approaches: an expansive recursive approach, either by drawing or by numerical means, and a visual-based approach. The latter was found to be the most efficient in achieving a global functional rule. The results of this study demonstrate the importance and value of challenging talented students with non-linear patterns, as the cognitive demands of such tasks have the potential for providing rich mathematical experiences.

THEORETICAL BACKGROUND

The prominence of generalization in mathematics has been noted by numerous researchers (e.g. Doerfler, 1991; Kruteskii, 1976; Polya, 1957; Skemp, 1986).

Pattern problems have been found to be efficient in developing and revealing the ability to generalize. Several studies have focused on generalizing patterns; they vary in types of patterns –numerical, pictorial or repeating patterns, and differ in population– from children to pre-service school teachers (e.g. Amit & Neria, 2008; Becker & Rivera, 2004; English & Warren, 1998; Ishida, 1997; Rivera, 2007; Stacy, 1989; Zazkis & Lijendak, 2002).

Concerning linear patterns, Stacey (1989) distinguishes between '*near generalization*' tasks, in which finding the next pattern or elements can be achieved by counting, drawing or forming a table, and '*far generalization*' tasks, in which finding a pattern requires an understanding of the general rule. Garcia-Cruz and Martinon (1998) referred to generalization strategies as local generalizations, based on recursive-additive approaches and global generalizations, based on searching for the functional relationship.

Studies that address non-linear patterns (e.g. Ebersbach & Wilkening, 2007; Krebs, 2003) have found additive strategies to be common and there was an evident tendency towards linearity, even when the patterns were clearly non-linear. Moreover, while in linear pattern problems using additive (expansive) strategies can lead to a global generalization (because the difference between each two successive patterns is constant, and more obvious to the solver), in non-linear patterns this approach can prevent them from seeing the global structure; a more productive approach involves using visual approaches (Amit & Neria, 2008; Krebs, 2003; Rivera, 2007; Steele & Johanning, 2004).

In this study, we examined the generalization methods of talented middle-school students when solving quadric pattern problems.

METHODOLOGY

Population

Fifty mathematically talented middle-school students (age 12-14) who participate in "Kidumatica" - an after-school math club in the southern region of Israel.[3]

The students participating in this study were new members in the club had no prior extra-curricular studies, just their school curriculum.

Settings and Tools

The research tool was a questionnaire comprised of six non-routine tasks that included the pattern task discussed here[4] (Fig. 1).

The questionnaire served as a pre-test aimed at investigating the abilities of the club's new participants, prior to any mathematical activities in the club.

The questionnaire was designed according to the cognitive abilities of mathematically talented students described by Kruteskii (1976), one of which is the ability to generalize.

Although the students had sufficient background to meet the challenge, the problem was considered non-routine, requiring students to use their pre-existing knowledge in an unfamiliar way, thereby effectively reconstructing what they know. It provided an opportunity to use different strategies and representations.

The task held potential for the construction of new mathematical ideas and concepts – in this case, the potential for developing generalizations. The students were required to fully document and justify the solution process.

The tasks' 'givens' consisted of a small finite set of figural patterns of a sequence, and included four questions based on previous research on generalization (Stacey, 1989; English & Warren, 1998).

Item a - finding the next pattern, in accordance to the theoretical 'near generalization'. The item served as a "warm up" item that enabled the solvers to examine and investigate the pattern.

Item b - finding the tenth pattern, in accordance with the theoretical term 'far generalization'. A correct answer could be obtained by extending the pattern (using numbers or by drawing) or by finding the functional rule.

Item c - the 'intuitive generalization' (informal generalization), enabling the students to represent the generalization in any form they felt comfortable with. For the

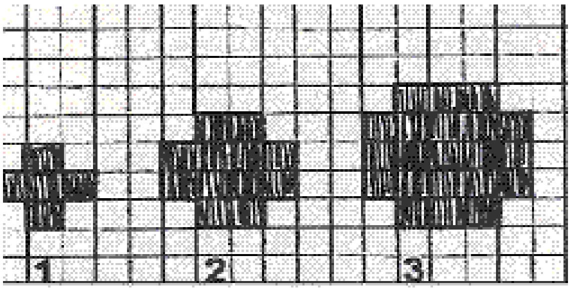
[3] Kidumatica Math club was founded in 1988 in Ben-Gurion University of the Negev. Every year, around 400 students ranging from ages 10-16, from 60 schools, participate in the clubs' activities. The weekly activities increase their creative thinking and mathematical skills, through subjects such as game theory, logic, combinatorics, and algebra. Students are chosen for their high mathematical abilities and their interest in developing these talents. The activities are run by experienced educators, who have been specially trained to instruct gifted students. Since its establishment, the Kidumatica math club has become a prestigious program that draws a multitude of applicants.

[4] Adopted from Zareba (2003).

researchers, this item was an indicator of generalization abilities. It was based on prior research indicating that students find it easier to verbalize generalizations than to write them symbolically (English & Warren, 1998), and on the fact that the study population was comprised of pre-algebra students.

Item d - the 'formal generalization', which contained an explicit requirement to represent a generalization in a formal mode, striving towards algebra. The aim of item D was to investigate how the students symbolize prior to formal studies in algebra.

The following illustration presents the first three patterns in a sequence:



a. How many tiles are needed to make the next pattern?
b. How many tiles are needed to make pattern 10?
c. Suggest a method to calculate the number of tiles needed to make any pattern in this sequence.
d. Suggest a method to calculate the number of tiles needed to make the n^{th} pattern in this sequence

Figure 1. The quadric pattern task.

Data analysis

All students' answers were analysed qualitatively according to their correctness and their generalization strategy.

Based on previous studies, (English & Warren, 1998; Ishida, 1997; Lee, 1996), generalizations were categorized into recursive (local) strategies versus functional ones. Strategies for generalizing were divided into numerical - such as the use of finite differences in a table, drawing and counting or visual strategies (Becker & Rivera, 2004; English & Warren, 1998; Ishida, 1997; Krebs, 2003; Rivera, 2007).

FINDINGS AND INTERPRETATION

Two main strategies were found: additive strategies, either by expanding the pattern by drawings or by numerical means (tables or lists), and visual based approaches (Table 1).

Strategy		Item A The next pattern	Item B The 10^{th} pattern
Additive strategies	Expansion by drawing (drawing and counting)	31	7
	Expansion by numerical means (tables, lists etc.)	9	16
Global strategies	Of the pattern	5	14
	Of the sequence of differences		4
Unclear/ not coherent		5	5
No answer		0	4
Total		50	50

Table 1. Distribution of solving strategies

Expansion by drawing

Of the fifty students who performed this task, 31 began solving it by drawing the next one or two patterns (Fig. 2). As noted by Lowrie and Kay (2001), using visual methods to complete complex or novel problems and in situations where the problem is not immediately understood is efficient in helping the solvers to organize and access relevant knowledge. Once students grasped the initial pattern, most of them turned to other approaches, and only 7 students continued to expand the pattern by drawing to find the tenth pattern. These students did not manage to find a global generalization.

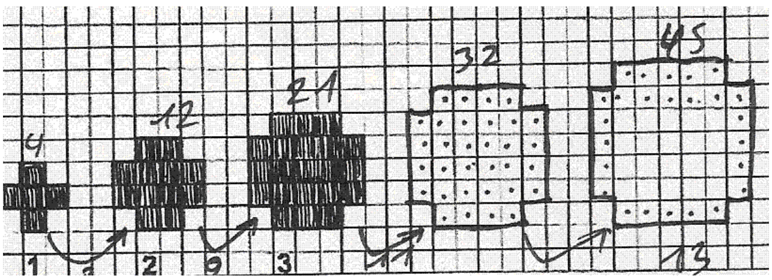


Figure 2. Expansion by drawing.

Expansion by numerical means

Sixteen students used number sequences. They adopted a recursive approach and achieved local generalization, as illustrated in Figure 3. Once the student counted the squares in the givens, she abandoned the pictorial figures and concentrated on the numerical representation. Grasping the regularity, she linked between the number of a pattern (bottom line) and the number of tiles in this pattern (middle line). In the upper line she wrote the difference between the numbers of squares in successive patterns.

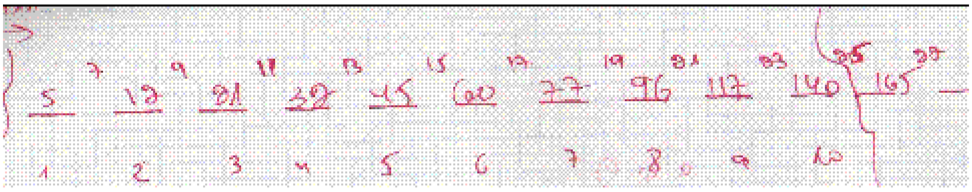


Figure 3. Numerical approach.

Though the students that carried out this approach formed correct lists, these lists and tables had no figurative meaning. Extending the list enabled them to achieve recursive generalizations, such as: "*the difference between patterns 1 and 2 is 7, and between patterns 2 and 3 is 9; between patterns 3 and 4 it's 11 and so on. The difference increases by 2 (from 1 to 2, from 2 to 3 etc.), and then you add the number of squares to the difference between the next and the previous.*"

These results are in line with Swafford and Langrall (2000), who found that although forming tables is useful in helping solvers make sense of a problem, it may also cause distraction from a more global view. This seems to be more prominent when solving non-linear patterns since the constant difference cannot be recognized straight away and the mathematical relationship between the listed numbers is not as obvious as in the case of linear patterns.

In four cases in this study, the numerical representations distracted and misled the students into focusing and generalizing the sequence of differences, which in this case was linear, and more noticeable than the non-linear pattern, a phenomenon described as an "irresistible tendency towards linearity" (De Bock, Van Dooren, Jansens & Verschaffel, 2002).

Global visual-based approach

Visual-based approaches were found to be more productive and led solvers to global generalizations. The fourteen students who generalized globally were those who divided the pattern into parts, whose areas had a *constant relation* to the pattern's place in the sequence.

In this case, what remained constant throughout the generalization process was the manner of division, and not the number of added squares. For example, in Figure 4, the student dismantled the given figure into a central rectangle whose sides are n and $n+2$, so the area is the multiplication of n by $n+2$, and then added two additional rectangles whose sides are 1 and n. He was able to find a global functional relation between the squares of a pattern and the pattern location in the sequence and used an algebraic notation for representing the functional rule: "$n \cdot (n + 2) + n \cdot 2$".

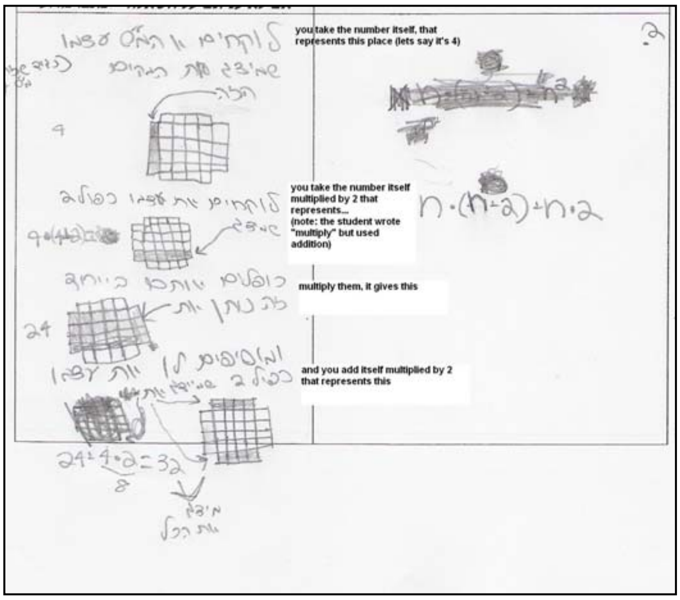


Figure 4: Figurative approach.

The students who were able to detect the variables (pattern number, dimensions) in the figural structure and differentiate them from the constants (shapes) achieved a correct global generalization. These findings are in accordance with former studies (Krebs, 2003; Rivera, 2007) that found that using visual approaches when generalizing non-linear patterns leads to success.

DISCUSSION

This study focused on the solving strategies of a quadric pictorial pattern task of mathematically promising students. The importance of pattern problems lies in their extensive mathematical potential. They not only encourage generalization, they also require students to pool their existing knowledge resources, rebuild and reconstruct them (e.g. Amit & Neria, 2008; English & Warren, 1998; Rivera, 2007).

Most students are familiar with linear or proportional relations, but have difficulties in generalizing less familiar situations, such as non-linear relationships (De Bock et al, 2002). The cognitive demands of the non-linear pattern problems differ from those of linear ones. In linear patterns, a global generalization can be achieved either by visual means or by numerical means, since the difference between each two successive patterns is constant; in non-linear patterns, relying merely on numerical lists may help solvers to achieve local-recursive generalizations, but it may also prevent them from discovering the functional rule.

In this study, only the students who visualized the growth in the pattern achieved a global generalization. In order to generalize productively, they divided the pattern into parts whose areas had a constant relation to the pattern place in the sequence. In this case, what remained constant throughout the generalization process was the manner of division, and not the number of added squares (Amit & Neria, 2008).

Previous studies have found a tendency toward linearity, even when the relationship is clearly non-linear. This phenomenon is explained by the extensive attention paid to linear and proportional relationships in elementary and secondary mathematical education, which may lead to a "fixation" on linear relationships (De Bock et al, 2002). In this study, the tendency for linearity was negligible, and all but four students were not distracted by linearity. Although for most students in this study this was their first experience dealing with non-linear patterns, they recruited existing knowledge (from geometry and number sequences - two seemingly un-related subjects) and applied it in a new situation, revealing flexibility in applying solving strategies.

In solving this task, the students demonstrated several of the characteristics of the mathematically talented – flexibility, persistence in problem solving, and the ability to generalize (Kruteskii, 1976; Wieczerkowski, Cropley, & Prado, 2000).

The results of this study demonstrate the importance of exposing students, particularly mathematically promising ones, to non-linear patterns, since they increase the challenge of generalization, provide novel mathematical experiences, and have the potential to enhance mathematical development.

References

Amit, M. & Neria, D. (2008). "Rising to the challenge": Using generalization in pattern problems to unearth the algebraic skills of talented pre-algebra students. *ZDM, 40*(1), 111-129.

Becker, J. R. & Rivera, F. (2004). An investigation of beginning algebra students' ability to generalize linear patterns. In M. J. Hoins & A. B. Fuglestad (Eds.), *Proceedings of the 28th PME* (Vol. 1, p. 286). Bergen, Norway.

De Bock, D., van Dooren, W., Jannsens, D., & Verschaffel, L. (2002). Improper use of linear reasoning: An in-depth study of the nature and the irresistibility of secondary school students' errors. *Educational Studies in Mathematics, 50*, 311-334.

Doerfler, W. (1991). Forms and means of generalization in mathematics. In A. Bishop (Ed.), *Mathematical knowledge: Its growth through teaching* (pp. 63-85). Mahwah, NJ: Erlbaum.

Ebersbach, M., & Wilkening, F. (2007). Children's intuitive mathematics: The development of knowledge about nonlinear growth. *Child Development, 78*, 296-308.

English, L. D. & Warren, E. A. (1998). Introducing the variable through pattern exploration. *The Mathematics teacher, 91,* 166-170.

Garcia-Cruz, J. A. & Martinon, A. (1998). Levels of generalization in linear patterns. *Proceedings of the 22nd PME* (Vol. 2, pp. 329-336). Stellenbosch, South Africa.

Ishida, J. (1997). The teaching of general solution methods to pattern finding problems through focusing on an evaluation and improvement process. *School Science and Mathematics, 97,* 155-162.

Krebs, A. S. (2003). Middle grade students' algebraic understanding in a reform curriculum. *School Science and Mathematics, 103,* 233-243.

Krutetskii, V. A. (1976). The *psychology of mathematical abilities in schoolchildren*. Chicago: University of Chicago Press.

Lee, L. (1996). An initiation into algebraic culture through generalization activities. In N. Bednarz, C. Kieran & L. Lee (Eds.), *Approaches to Algebra: Perspectives for Research and Teaching* (pp. 87-106). Dordrecht, the Netherlands: Kluwer.

Lowrie, T. & Kay, R. (2001). Relationship between visual and nonvisual solution methods and difficulty in elementary mathematics. *Journal of Educational Research, 94(4)*, 248-255.

Polya, G. (1957). *How to solve it* (2nd ed.). Princeton: Princeton University Press.

Rivera, F. (2007). Visualizing as a mathematical way of knowing: Understanding figural generalization. *Mathematics Teacher, 101*(1), 69-75.

Rivera, F. D. & Becker, J. R. (2005). Figural and numerical modes of generalizing in algebra. *Mathematics Teaching in the Middle School, 11*(4), 198-203.

Skemp, R. R. (1986). *The Psychology of Learning Mathematics*. NY: Penguin Books

Stacey, K. (1989). Finding and using patterns in linear generalizing problems. *Educational Studies in Mathematics*, *20*, 147-164.

Steele, D.F. & Johanning, D. I. (2004). Schematic-theoretic view of problem solving and development of algebraic thinking. *Educational Studies in Mathematics, 57*, 65-90.

Swaford, J. O. & Langrall, C. W. (2000). Grade 6 students' preinstructional use of equations to describe and represent problem situations. *Educational Studies in Mathematics, 31*, 89-112.

Wieczerkowski, W., Cropley, A.J., & Prado, T.M. (2000). Nurturing talents/gifts in mathematics. In K.A. Heller, F.J. Monks, R.J. Sternberg, & R.F. Subotnik, *International Handbook of Giftedness and Talent* (pp. 413-425). Oxford: Elsevier.

Zareba, L. (2003). From research on the process of generalizing and on applying a letter symbol by pupils aged 10 to 14. In *Proceedings of the 55th Conference of the International Commission for the Study and Improvement of Mathematics Education* (pp. 1-6). Plock, Poland.

Zazkis, R. & Liljedahl, P. (2002). Generalization of patterns: The tension between algebraic thinking and algebraic notation. *Educational Studies in Mathematics, 49*, 379-402.

THE RELATIONSHIP BETWEEN RESEARCH AND CLASSROOM IN MATHEMATICS EDUCATION: A VERY COMPLEX AND OF MULTIPLE LOOK PHENOMENON

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This work of inquiry, part of our doctorate research in finalization process, investigates the relationship between research and classroom in Mathematics Education with special attention to documenting processes related to the questions that fellow. What is the impact of Mathematics Education research in the classroom? How does research and researchers relate to the classroom? What do researchers have to say about the Mathematics classroom, and what has it shown them? More specifically, we present this paper a focus discussion of the study object, literature and theoretical background, methodology and data collection, some results, analysis and conclusion. We will show that the relation between research and classroom in Mathematics Education is a very complex and of multiple look phenomenon.

INTRODUCTION AND JUSTIFICATION

It seems to be a consensus among Mathematics educators that, on one hand, the scholastic failure of students in Mathematics and, on the other, the great importance of this discipline in the school curricula and in all the nations of the world have been main reasons to originate the field of mathematical education - a study area that, in a direct or indirect form, has always been involved with the Mathematics classroom. According to Kilpatrick (1992), "*the mathematical education started to be developed as mathematicians and educators have turned their attention to how and what Mathematics is, or might be, taught and learned in school*". However, it has been pointed out that the research and the researchers of this area are not relating themselves, in an efficient and coherent way, to the Mathematics classroom. These concerns have become stronger from the moment that we perceive that the data set disclosed in some research about the reality of the Mathematics classroom indicates that there is a mismatch between academic literature and the Mathematics classroom. That the research and the researchers have not related, in an efficient way, to the Mathematics classroom. Therefore, a systematic study on the relation between research and classroom in Mathematics Education is necessary, in order to point out more effective ways to change the Mathematics classroom and contribute towards a qualitative change in the relations between research and researchers and the Mathematics classroom.

LITERATURE AND THEORETICAL BACKGROUND

Theoretically, we have been working on this subject mostly with studies regarding to the theme research and practice and handbooks of Mathematics Education.

As an autonomous field of knowledge, Mathematics Education is recent and it is still being discussed, with frequency, what is Mathematics Education? What is the research in Mathematics Education? The expression "Mathematics Education" is still strange for many Mathematics teachers in Brazil and perhaps around the world. A historical synthesis of the research in Mathematics Education was published by Kilpatrick (1992) and a study of the Mathematics Education, as a field of academic study, was edited by Sierpinska & Kilpatrick (1998): "*Mathematics Education as a research domain: a search for identity* ", that argues, in great depth, questions of the type: Is the Mathematics Education a science? Is it a discipline? In what way? What is its role inside the other domains of research and academic discipline? What is its specificity? In it, the Mathematics Education researcher will find a broad range of possible answers to these questions, a variety of analyses of the direction of the research in Mathematics Education in different countries and a set of visions for the future of Mathematics Education. More recent publications like the *Second International Handbook of Mathematics Education* (Bishop, A. J. et al., 2003), the *Handbook of International Research in Mathematics Education* (English, 2003) and the "*Second Handbook of Research on Mathematics Teaching and Learning* (Lester, 2007) has also deepened such debate.

In the specific case of the researchers, there is also a concern over what is and how to do research in this area of knowledge. The objective of this is that the research in Mathematics Education reaches its own identity. Research in this area has been each and every time more molded by the research models in Education and in the Social Sciences.

But, facing all these discussions we, constantly, question ourselves: And the Mathematics classroom, how does it stand? How the research and the researchers have been communicating/relating to the Mathematics classroom? How have they been speaking of it? How have they been looking at it? How have they been facing its dilemmas? How have they gotten there? What have been the results of such relations for the Mathematics classroom itself? In what have the research and the researchers contributed to change the Mathematics classroom? What have been their concerns, discourses and actions about the Mathematics classroom? How can they make more effective changes in the Mathematics classroom? And what the latter has to say to the researchers?

These concerns became stronger when we came to realize that there is a misalignment between academic literature and the Mathematics classroom. That the research and the researchers have not been relating, in an efficient way, to the Mathematics classroom. Being, necessary a study on the relation between research practice and classroom practice.

For example, in our master's degree research (Andrade, 1998) and in Mathematics Education courses (from 1998 to 2007) that we have presented to Mathematics teachers in Brazil, specifically in the area of Problem Solving, we have verified that the academic literature on Problem Solving does not match what the teachers know and practice in the Mathematics classroom. While in the research in Mathematics Education, Problem Solving is conceived as a teaching methodology, in school practice it is not even perceived as content application, but simply as technique application (recipes, drills...). In content such as fraction, for instance, teachers teach -separately and without any connection to what has been previously given- all the operation rules before teaching problems with fractions. This attitude is in accordance with the 'banking' concept of education, which is criticized by Paulo Freire (1987).

Teachers do not even believe they can do otherwise. Only one out of seven teachers, with whom we worked with during our master's research in Rio Claro (SP), Brazil - a city with a tradition in Mathematics Education research-, showed some approximation/awareness between her theory/practice and the literature on Problem Solving. Another teacher was aware of the current trends in Problem Solving, but did not use them. She alleged that she could not apply in class what she had recently learned in college, consequently continuing in traditional teaching.

Recently, Regarding with a better approach between research and classroom has been emerging preoccupations in publications as "Lessons learned from research" (Sowder 2002), "Teachers engaged in research: inquiry into Mathematics classrooms" (Mewborn, 2006) and in events as ICME 10 (2004), especially in the sessions ST1 (Survey Team 1): The relation between research and practice in Mathematics Education and DG2 (Discussion Group 2) - The relationship between research and practice in Mathematics Education.

METHODOLOGY AND DATA COLLECTION

Regarding the methodology of research in Mathematics Education we understand that some researchers seem to be linked to a unidirectional paradigm of research of the type research $\rightarrow$ methodology $\rightarrow$ problem. It seems to be a concern to fit the problem of research in one determined methodology, not realizing that it is the problem that, in a multidirectional process of the type research/theoretical referential /world visions $\Leftrightarrow$ problem $\Leftrightarrow$ research/ theoretical referential/world visions $\Leftrightarrow$ methodology determines the methodology to be used in the development of the research. It is necessary that we endeavor to select strategies that fit each research problem instead of labeling it and casting it under a peculiar methodological denomination. In this sense, we stress out that the researcher, respecting the compatibilization of processes and the epistemological foundation, can work with some methodological resources to make his research. Problematization and methods are inseparable. When one formulates a research problem, one also invents a peculiar way to search, to produce and to propose alternative answers.

It doesn't matter the method we use to arrive at the knowledge; what in fact makes a difference is the interrogations that can be formulated within one way or another of conceiving the relations between subject, method, knowing and power. It is the looks that we place on things that create the problems of the world. The statements do more than a representation of the world; they produce the world. To Foucault (2004a, 2004b), they are the *visible* elements – non-discursive formation - and the *enunciable* elements - discursive formation - that will make the world what it seems to be to us. We should problematize all the certainties, all the declarations of principles. It is necessary a look that goes beyond what others already have looked at, a restless look, a look that surprises, disarms, disturbs and introduces the disturbances in the interior of the debate, in the plan of discourses.

Specifically, this study, the research methodology has mainly been based on discourse analysis and studies from the perspective of Michel Foucault (1996, 1999, 2004a, 2004b) that this way we seek to explain the fragile and strong points of the relationship between research practice and the classroom practice, type a topographical and geological summary.

We take under consideration here that what in fact makes a difference in the methodology is the questioning that can be formulated within another way of conceiving the relations between subject, method, knowing and power. The method consists then of understanding that the things are not more than practical objectifications of specific practices, whose determination must be exposed to light, since consciousness does not conceive them. And, in this context, the movement of the relation research/classroom is perceived as practice that systematically forms the objects that are spoken of and the ideas and theories are taken as the keys of a toolbox. We have also thought simultaneously with Foucault and, among others, Jacques Derrida, for example. We have found fertile convergences between Derrida's deconstruction (1974) and Foucault's splitting analytics that disturbs what was previously considered at a standstill; fragments what was considered amalgamated; shows the heterogeneity of what was imagined consistent with itself. Together, theses theories take on a provocative and irresistible energy (St. Pierre, 2004). This way, our research methodology would also be a deconstruction one, to keep things in process, to disrupt to keep the system in play, to set up procedures to continuously demystify the realities we create, to fight the tendency for our categories to congeal.

The survey of data/facts and their analysis include speeches of 71 Mathematics Education researchers (44 Brazilians and 27 from other countries), P01 to P71 - obtained through opened and discursive research questionnaire; speeches of teachers of Mathematics - selected of our Master Degree research and speeches of the works presented in the sessions ST1: The relation between research and practice in Mathematics Education and DG2: The relationship between research and practice in Mathematics Education, ICME 10 (2004).

SOME RESULTS, ANALYSIS AND CONCLUSION

Based on the set of the gathered data/facts, as described in the methodology mentioned above, we single out the following partial result: that it seems to exist, in the set of the discourses of a good many researchers, a certain defense of research and projects of the collaborative type, action-research, participative or similar, in the belief that such research and projects would have a better impact in the classroom than others. The declarations below, extracted of our data collection, from a Brazilian researcher (P24) and one from abroad (P49), are examples in this direction.

P24: The research is still very distant from the classroom. One of the reasons is that the school teachers do not understand the texts and the academic language do not identify themselves with the contexts being presented. During all this time of production in the area, the research has been about the teachers and for the school teachers. I believe that, only when there is a radical change and the research starts being produced with the teachers is that these will begin to produce the desired effect.

P24: In this sense, there are some innovative experiences that have been disclosing how much the teacher searches for processes of formation that mean something for him or her. The problem is that they rarely find them. In the last 10 years, several were the researches produced in the area of Mathematics Education that have been pointing to new alternatives of teacher education. These researches reveal that successful experiences are those carried through with the teachers, from their necessities, angst and search for solutions to the problems they find in their daily school life.

P49: One of the bigger successes I have had in research is working "with" schools and teachers – ie the action-research-type model of research. This is a process where the teachers (and students) feel a commitment to the research and hence become active participants in the change, take ownership of the change/process and real outcomes can be achieved. The less successful model is that where the research is 'done' on classrooms. This research tends to be less valued by the schools/teachers and less likely to have an impact. It does make for good research that is easier to publish and hence improve the career prospects of the researcher! The action research type research is less easy to publish as it does not conform with the general parameters of what constitutes good research in the field and hence more difficult to publish in high quality journals read by maths educators.

The research-action, collaborative and similar approaches as resources to bring research and classroom closer together represent only one of the several points discussed by the researchers, it does not represent the thinking of the whole group.

A speech such as the researcher's P53, problematize such debate questioning if the researchers really are interested in this. P53 says: "*I am not sure most researchers actually do want to do this. They are doing a job of work*".

Researcher P03 states out that we, researchers, could contribute to a change in the classroom if we managed to institute new forms of relation with the knowledge.

P03: The research objects are very "local" or very "broad", they do not reach the classroom directly, in the generic sense. This is not going to change. We, researchers, could contribute if we could institute new forms of relation with knowledge.

Researcher P26 places as the main point that it is necessary to go back to the classrooms and look closely at the student and discover who he or she is and then think how to give him or her support.

P26: It is necessary to go there, to the classrooms, and look at the student a lot. We have to disclose him, this student connected to the contradictions, vicissitudes, assets and benefits of the modern society.

Skovsmose (2004), in document presented at the ICME 10 (2004), defends, among other points, that it is necessary that the research in Mathematics Education be focused in classrooms of the non model type, in classrooms at a poverty-stricken neighborhood, in classrooms of the 4th world. He questions the fact that a certain model classroom seems to dominate the field of research in Mathematics Education, that in many cases it seems to be selective regarding which practice to address. To him, the discourse in Mathematics Education has been dominated by the prototype of the model classroom.

He suggests we defy the hegemony of the discourse bred around the model classrooms and, he adds that a non-standard classroom would have an enormous number of students, it would be located in a poverty-stricken neighborhood, it would be infected by violence. To him, research on the non standard Mathematics classroom can focus on many declarations: the violence, poverty, immigration and discrimination in general etc.

Speeches such as the teacher's below, subject 04, taken from our master dissertation (Andrade, 1998), also seems to point to the necessity of there also being research focusing on non-model classrooms.

> Subject 04: Well, the school... it is kind of problematic today. I guess the teachers are with no incentive. We, in a general way, are. Another day, in a meeting, a teacher came and spoke so: look she does activities with newspapers in the Portuguese language class. And she makes the students bring news, because sometimes they do not have time to read. They read the news and later they explain to the class. Every week, one day is reserved to this. Imagine that there was a day, reading a newspaper about drugs, they got to talk, and she found out that in the class, most of them were all druggies, everyone was an addict. And she started to understand the behavior of the class. The adult education classroom is a classroom where I have no problems, but, the others do. Then, when I get to, if I get to, because I do not want to get to, in this case, the regular class, I do not know what to do. Because I never used drugs, I never had the problems that they have in life. What am I going to do with an adult about his or her problems? I said like this: my! Poor girl! What an awful situation! I have nothing to say to them. I won't know how to act with them, how to deal. The school today needs, not teachers, but yes, I am speaking about my school, a center where the students can be recovered, because what there is a lot of here are problems, students with problems. And they form a problematic classroom.

Lester & Wiliam (2004) placed, among other points, the dimension that the research has to reach the makers of educational politics. Sfard (2004), enters in the debate, looking at research and practice as discursive activities. Researches like P01, for example, declare that there is some impact from the research in the classroom, but such impact has been to keep the *status quo*.

> P01: The Mathematics Education and the Education in general is the main strategy of the power structure [State, or Church, or Corporations] to maintain and to consolidate themselves. There is interest in "filtering" those that go through the educational system in order to be able to co-opt those convenient to the power structure. History teaches us this.

P01: There is some impact, as long as it allows the improvement of the strategy mentioned above. The great majority of the research is related to the models in practice [improvement of the same-old-same-old].

The discourses above indicate that the theme the relation between research and classroom in Mathematics Education is a very complex phenomenon and of multiple looks. The current text has been a brief look in search of a representative map of this complexity and multiplicity, in a deconstruction process that teaches us, on one hand, about the possibilities and impossibilities of impact of the research of Mathematics Education in the classroom, but, on the other hand, does not bring a key to the real impact.

For example, when P01 declares that the Mathematics Education and the Education in general are main strategies of the power structure. We here have an impossibility for the real impact. But, there is another declaration of P01 in our data collection that points that we have to think about a Mathematics Education that can necessarily include Ethics. Here, we have, therefore, a possibility for the real impact.

The different speeches of the researchers bring us then a deconstruction on the word impact, regarding the relation between research and classroom. Each speech/statement is transactional. They teach us something about the conditions of the production of making impact of the research in the classroom, but they do not give a key for the real impact. They teach us about the possibility and impossibility of such impact happening or not. They teach us something on essentialisms of being among the conditions of producing the doing, knowing, being, but they do not give a key to the real impact.

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References

Andrade, S. (1998). *Ensino-aprendizagem de Matemática Via Resolução, Exploração, Codificação e Descodificação de Problemas e a Multicontextualidade da Sala de Aula* (Dissertação de Mestrado em Educação Matemática). Rio Claro: IGCE-UNESP.

Bishop, A. J., et al. (Eds.). (2003). *Second International Handbook of Mathematics Education*. Dordrecht: Kluwer Academic Plubishers.

English, L. (Ed.). (2003). *Handbook of International Research in Mathematics Education*. Mahwah: Lawrence Erlbaum Associates.

Derrida, J. (1974). *Of Grammatology* (G. C. Spivak, Trans.). Baltimore: The Johns Hopkins University Press. (Original work published 1967).

Foucault, M. (1996). *A Verdade e as Formas Jurídicas* (R. Machado & E. J. Morais, Trans.). Rio de Janeiro: NAU. (Lecture delivered 1973).

Foucault, M. (1999). *Em Defesa da Sociedade* (M. E. Galvão, Trans.). São Paulo: Martins Fontes. (Original work published 1997).

Foucault, M. (2004a). *A Arqueologia do Saber* (L. F. B. Neves, Trans.). 7. ed. Rio de Janeiro: Forense Universitária.

Foucault, M. (2004b). *A Ordem do Discurso* (L. F. A. Sampaio, Trans.). 11. ed. São Paulo: Loyola. (Original work published 1971 & Lecture delivered 1970).

Freire, P (1987). *Pedagogia do Oprimido*. 17. ed. Rio de Janeiro: Paz e Terra.

ICME 10 (2004). *The 10th International Congress on Mathematical Education.* Copenhagen, Denmark. **(**www.ICME10.dk).

Kilpatrick, J (1992). A History of Research in Mathematics Education. In D. A. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and T* (p. 3-38). New York: Macmillan.

Lester, F. K. & Wiliam, D (2004). On the purpose of Mathematics Education Research: making productive contributions to policy and practice. In *10th International Congress on Mathematical Education* (ICME10). Copenhagen, Denmark. **(**www.ICME10.dk).

Lester, F. K. (Ed.) (2007). *Second Handbook of Research on Mathematics Teaching and Learning*. Greenwich: Information Age Publishing.

Mewborn, D. S. (Ed.). (2006). *Teachers Engaged in Research: Inquiry nto Mathematics Classrooms*. Greenwich: Information Age Publishing.

Sfard, A (2004). On Mathematics Education research and practice, and on what they can do for one another - a conceptual introduction. In *10th International Congress on Mathematical Education* (ICME10). Copenhagen, Denmark. **(**www.ICME10.dk).

Sierpinska, A. & Kilpatrick, J (Eds.), (1998). *Mathematics Education as a Research Domain: A Search for Identity*. Dordrecht: The Netherlands.

Skovsmose, O. (2004). Research, practice and responsibility. In *10th International Congress on Mathematical Education* (ICME10). Copenhagen, Denmark. **(**www.ICME10.dk).

Sowder, J. T. (ED.) (2002). L*essons learned form Rresearch*. Reston: NCTM.

St. Pierre, E. A. (2004). Care of the self: the subject and freedom. In B. M. Baker & K. E. Heyning (Eds.), *Dangerous Coagulations? The Use of Foucault in the Study of Education* (p.325-358). New York: Peter Lang.

DEVELOPING ALGEBRAIC GENERALISATION STRATEGIES

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Student transition from arithmetic to algebraic reasoning is recognised as an important but difficult process. Functions and numeric patterning activities provide opportunities to integrate early algebraic reasoning into mathematics classrooms. This paper examines student use of generalisation strategies when engaged in numeric patterning activities and explores how young students can be supported to use flexible efficient strategies. Results suggest that use of generalisation strategies can be extended through purposely designed tasks and specific teacher actions.

INTRODUCTION

Over the past decade the teaching and learning of algebraic reasoning has been a focus of both national and international research and reform efforts (e.g., Ministry of Education (MoE), 2007; National Council of Teachers of Mathematics (NCTM), 2000). Such attention has arisen primarily in response to the growing recognition of the inadequate algebraic understandings many students develop during their schooling and the role this has in denying them access to prospective educational and employment opportunities (Knuth, Stephens, McNeil, & Alibabi, 2006). In response, some curricula advocate teaching arithmetic and algebra as a unified strand across the curriculum (e.g., NCTM, 2000; MoE, 2007). This approach focuses on using students' informal knowledge and numerical reasoning to build early algebraic thinking. Tasks involving functions and numeric patterning activities offer an opportunity to integrate early algebraic reasoning into the existing mathematics curriculum. The research reported in this paper examines student use of generalisation strategies when participating in numeric patterning activities. The focus of the study is to explore how the students aged from nine to eleven years of age were supported to use flexible efficient generalisation strategies.

Recent research (e.g., Becker & Rivera, 2007; Swafford & Langrall, 2000; Warren, 2005) indicates that young children, in making the transition from numeric to algebraic reasoning, exhibit forms of functional thinking. Functional thinking is described as "representational thinking that focuses on the relationship between two (or more) varying quantities, specifically the kinds of thinking that lead from specific relationships (individual incidences) to generalizations for that relationship across instances" (Smith, 2008, p. 143). The inventing or appropriation of a representational system to represent the generalisation is evidence of algebraic reasoning. Through analysis of tasks that involve functional thinking –henceforth referred to as functional tasks– Lannin, Barker, and Townsend, (2006) illustrated that the strategies students use to generalise numeric situations emerge through different types of reasoning. Their framework outlines a continuum of generalisation strategies that students can use. Less sophisticated use of recursive generalisations involved students identifying the relationship between consecutive values using an additive strategy. More proficient strategies included ‘chunking’ in which the students construct a “recursive pattern by building a unit onto known values of the desired attribute” (p. 6), and ‘whole-object strategies’ in which a portion is used as a unit “to construct a larger unit using multiples of the unit” (p. 6). The most sophisticated strategy identified by Lannin et al. involved students’ use of an explicit generalisation in which a rule is constructed to allow “for immediate calculation of any output value given a particular input value” (p. 6).

Student use of generalisation strategies is influenced by a range of task related factors. For example, students in Lannin et al. (2006) and Swafford and Langrall (2000) studies commonly used recursive strategies when completing patterning tasks with closely related input values and used whole-object strategies when input values were multiples or doubles of previous values. These researchers suggest that setting tasks which require students to consider increasingly large input values is an effective ways to encourage students’ movement towards explicit generalisation strategies. The notion of efficiency is also identified as an important factor influencing students’ choice of generalisation strategies. Lannin and his colleagues showed how students used flexible strategies and explicit rules in order to establish more efficient strategies. Visual images also influence students’ use of explicit generalisations. When students are able to link the rules to a visual representation they are more flexible in their strategy use and accurate in developing explicit rules (Healy & Hoyles, 1999; Warren, 2000). However, developing students’ proficient use of generalisation strategies is complex and difficult. It requires more that the provision of appropriate tasks; it requires considerable time and explicit teacher attention.

The theoretical framework of this study uses the emergent perspective taken by Cobb (1995). The socio-constructivist learning perspective links Piagetian and Vygotskian notions of cognitive development connecting the person, cultural, and social factors. In this paper, construction of algebraic understanding is recognised as both an individual constructive process and the social negotiation of meaning.

METHOD

The findings reported in this paper are a small component of a larger study involving a 3-month classroom teaching experiment (Cobb, 2000). The research was conducted at a New Zealand urban primary school and involved 25 students between 9-11 years old. The students came from predominantly middle socio-economic home environments and represented a range of ethnic backgrounds.

The teaching experiment approach supported a collaborative teacher-researcher partnership. A hypothetical learning trajectory and sequence of learning activities, focused on developing students’ early algebraic understanding, was collaboratively developed. Data were generated and collected through student interviews, participant and video records, and classroom artefacts.

On-going data analysis shaped the study and involved the researcher and teacher in collaborative examination of classroom practices and modification of the instructional sequence and associated learning trajectory. Retrospective data analysis took a grounded approach, identifying categories, codes, patterns, and themes. Both on-going and retrospective data analysis were used to develop the findings of the one classroom case study.

RESULTS AND DISCUSSION

Mathematical tasks were purposely designed to support student development of early algebraic understanding. Following on from task activities focused on exploring the properties of number and associated computations, the students were provided with problems designed to develop algebraic reasoning through functions and patterning activities. These were comprised of linear functional problems and included tasks with geometric contexts. The design of the problems was aimed, with the assistance of teacher scaffolding and modelling, to promote the use of flexible, efficient generalisation strategies. Drawing on the framework provided by Lannin et al. (2006) we were aware of the need for the tasks themselves to promote students to progressively adopt recursive, chunking, whole-object, and explicit strategies.

Recursive strategies

In the initial lesson, many of the students applied additive recursive strategies –listing successive values until the desired output number was reached. For example, during small group work while solving a functional relationship problem[1] a student, Ruby, introduced the recursive pattern into the discussion as follows:

Ruby: Look there's five people here but there's three added on.

Heath: We are plusing three, so on one table there is five, on two tables which makes eight.

Matthew: So then four tables will be fourteen.

Rani: So that is just showing we add another three on.

Sharing of strategies for the same problem appeared to be a useful way to encourage most, but not all, students to consider more effective strategies.

For some students, however, shifting beyond the use of recursive strategies was challenging. Despite Ruby sharing a more efficient chunking strategy for the table problem, Rani continued to promote the use of a recursive strategy:

Rani: You have to keep adding three all the time and if you do it this way twenty-seven won't come here, nine would be twenty-nine and ten would be thirty-two.

---

[1] **Table problem**

At the table 5 people sit like this …… ∘
∘ /_\\∘
∘∘

When another table is joined this many people sit around it… ∘∘∘
∘/\\/∘
∘∘∘

Can you find a pattern? How many people could sit at 3 tables or 5 tables or 10 tables? See if your group can come up with a rule and make sure you can explain why your rule works.

In following lessons, students were frequently observed to use recursion as their initial strategy before seeking more efficient strategies. We related this to the ease with which they could recognise the recursive relationship in the patterning problems. For some students it appeared that the confidence to generate and answer this way provided the space for them to risk trying alternative strategies.

Chunking strategies

To extend student flexibility and efficiency in strategy use the teacher used questioning to prompt students to consider issues of efficiency. In the following example the prompt was implicated in a student developing her recursive strategy into a chunking strategy using the known values:

Teacher: What would be a quicker way than going plus three?

Ruby: [points to model] The first table is five so you could ignore that and just go nine times three…you could just ignore that because you know it is five, so nine times, because that's table one, nine times three then add the five on.

Whole-object strategies

In a lesson early in the sequence, a task containing input values which were multiples led to some students using an erroneous whole-object generalisation strategy. The table problem required that they calculate how many people could sit around ten tables. Pressing further, the teacher asked them to calculate the number of people around 100 tables. Both Heath and Matthew over-counted in their generalisation strategy:

Heath: [points to 10 in the table of data] If it is a hundred we will just plus a zero to that.

Matthew: [points to 10 and 32] You can add a zero to that and a zero to that.

The teacher's observations of students' using whole object strategies that over-counted led to the provision of additional tasks which facilitated further examination of the whole-object strategy. By structuring the input values of the problems she was able to prompt the students to examine and discuss the whole-object generalisation strategy in-depth. For example, one problem[2] involved input values that doubled.

When Gareth's explanation over-counted the values he was challenged:

Gareth: So if four is twenty-one so it is twenty-one plus twenty-one.

Ruby: Instead of just doing twenty-one plus twenty-one, you don't because you wouldn't just build another four separate and there is not going to be another six one so it's not really adding twenty-one...

Teacher: So you're saying you can't just double it because there's not going to be another six one like at the start.

Ruby: So you just do twenty.

---

[2] **House problem**

Jasmine and Cameron are playing "Happy houses". They have to build a house and add onto it.

The first one looks like this….. / \ |_| The second building project looks like this…. / \ / \ |_|_|

How many sticks would you need to build four houses? How many sticks would you need to build eight houses? Can you find a pattern and a rule?

Gareth responded to the reasoned argument by correctly using a whole-object generalisation strategy to find the output value:

Gareth: So it's only twenty because you take a one away at the start, you add on twenty from here… like Ruby said you can't add on six that would mean there would be two of those sides [points to middle stick] so it can't be twenty-one plus twenty-one so it's twenty-one plus twenty.

Explicit strategies

In all the lessons, the teacher explicitly encouraged students to be aware of the range of generalisation strategies and explore and examine more efficient generalisation strategies:

Teacher: Is there another way you can do it without adding? Can you think of an equation or a rule that would help you get from four to twenty-one?

Initially, when many of the students did not use an explicit generalisation strategy to begin solving a problem, this press to consider alternative and effective strategies often led to the development of a final strategy involving an explicit generalisation. For example, Ruby's challenge to find a quicker strategy to solve the house problem facilitated other students to shift from recursive generalisation towards a more efficient strategy:

Ruby: It would be five more because the first one was six but they don't need another wall there [points to the middle stick between the two houses].

Susan: You just add on. Yeah it changes.

Ruby: But the easier way is adding five but what I am thinking is instead each time you could just.

Susan: Plus five.

Ruby: If you are doing four houses instead of going five, plus five, plus five, you can just go four times five then add one.

Susan: Well, that's kind of a problem because this is six.

Ruby: I know, but look times four then add one. You are just timesing that and then adding one so that one [points to first house] is still six.

Heath: So you just keep plusing five.

Ruby: But keep plusing five isn't good because you want a quicker way.

Gareth: You could count it but that would take ages. If you wanted to get it to a hundred or something it would take too long.

In many cases it was observed that the geometric structure or visual image of a problem assisted students to use explicit strategies and construct correct rules. The teacher pressed the students to connect their explicit rules with the geometric problem representation. For example, when Hamish explained an explicit generalisation in response to the table problem the teacher pressed him to connect his contextual explanation to the geometric representation:

Hamish: Thirty-two people sit at the table…you get the ten and times it by three and the two people who are sitting on those ends, one of them stays there and the other one gets moved to the end of the new table.

Teacher: Hamish can you show…the times three part of your model there and the plus two part?

Subsequently, elaboration with reference to a geometric representation became a more frequent and expected way for students to explain and justify the rules they had constructed from their explicit generalisation strategy. The following explanation illustrates how Ruby draws on geometric representation when sharing her group strategy for the house problem:

Ruby: [builds model] The first one is six but then when you add another house it is only five because you don't need another wall…if you wanted to see how many for eight you could just go eight times five and then plus the one. You plus one because you have to understand that is six [points to first house].

Such practice was also observed to be appropriated within small group discussions. For example, students consistently referred to the geometric context of a problem[3] when justifying their explicit generalisation and rule for finding the number of squares across and the total number of squares in a cross-shaped object:

Josie: This is cross one. There is one on each side plus one in the middle. This is cross two, so two here and two here and one in the middle so that makes five. So you double it and then add one to get the number across…

Steve: So when you double it, what are you actually trying to get to by doubling it?

Josie: [covers the vertical row so only the horizontal row is visible] The number of squares in that line there… this little bit here is also three squares wide [points to right horizontal line] and this is three squares wide [points to left horizontal arm] … so to get the bit across here in the middle you do times two plus one.

CONCLUSION

This study sought to explore student use of generalisation strategies and how they could be supported to use flexible, efficient strategies as they engaged in numeric patterning activities. The description of the learning activities presented in this paper, although only a small sample of those used in the teaching experiment, demonstrate that the use of deliberately designed functional tasks and specific teacher actions can successfully extend student use of generalisation strategies.

Similar to the findings of other researchers (e.g., Lannin et al., 2006; Swafford & Langrall, 2000), the students initially employed additive recursive generalisation in order to solve functional relationship problems. The use of functional tasks designed with specifically selected input values resulted in different generalisation strategies being utilised. Multiple or double input values led to student examination of whole number generalisation strategies. Students were pressed to use more efficient explicit generalisation strategies through the extension to large input values. Additionally, the use of specifically designed functional tasks including those with numeric and geometric patterns offered possibilities for students to integrate their visual and numeric schema.

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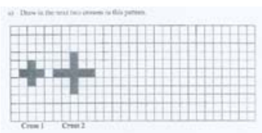
3 

Whilst tasks features invoked a range of strategies, specific teacher actions led to the students' flexible use of a range of strategies. The teachers' pedagogical press included questions and prompts that progressed student reasoning toward the use of more efficient strategies. Requiring that students link their explicit rules to the geometric basis of the functional problem also supported them to develop explicit generalisation strategies based on the geometric structure of the problem. The geometric representation had the advantage of providing a thinking tool that was able to be shared with other students within the explanation and justification processes associated with forming and defending generalisations.

The forward and backward shifts students made between recursive and explicit generalisations strategies were evident in this study. Multiple opportunities for students to create representations involving models, diagrams, and tables of numeric patterning activities were needed. In combination with effective pedagogical support, opportunities to for students to engage with functional relationships problems and connect their actions to appropriate representational systems enabled them, at various levels, to form generalisation of relationships across instances. As such, these patterning problem types should form a significant part of elementary curricula aiming to support students' development of algebraic reasoning.

References

Becker, J. R. & Rivera, F. (2007). Factors affecting seventh graders' cognitive perceptions of patterns involving constructive and deconstructive generalizations. In J. Woo, H. Lew, K. Park, & D. Seo (Eds.), *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 129-136). Seoul: PME.

Cobb, P. (1995). Cultural tools and mathematical learning: A case study. *Journal for Research in Mathematics Education, 26*(4), 362-385.

Cobb, P. (2000). Conducting teaching experiments in collaboration with teachers. In A. Kelly & R. Lesh (Eds.), *Handbook of Research Design in Mathematics and Science Education* (pp. 307-333). Mahwah, NJ: Lawrence Erlbaum.

Healy, L. & Hoyles, C. (1999). Visual and symbolic reasoning in mathematics: Making connections with computers. *Mathematical Thinking and Learning, 1*(1), 59-84.

Knuth, E., Stephens, A., McNeil, N., & Alibabi, M. (2006). Does understanding the equal sign matter? Evidence from solving equations. *Journal for Research in Mathematics Education*, *37*(4), 297-312.

Lannin, J., Barker, D., & Townsend, B. (2006). Algebraic generalisation strategies: Factors influencing student strategy selection. *Mathematics Education Research Journal, 18*(3), 3-28.

Ministry of Education. (2007). *The New Zealand curriculum: Draft for consultation 2006*. Wellington: Learning Media.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: Author.

Smith, E. (2008). Representational thinking as a framework for introducing functions in the elementary curriculum. In J. Kaput, D. Carraher, & M. Blanton (Eds.), *Algebra in the early grades* (pp. 133-160). New York: Lawrence Erlbaum Associates.

Swafford, J. O., Langrall, C. W. (2000). Grade 6 students' preinstructional use of equations to describe and represent problem situations. *Journal for Research in Mathematics Education, 31*(1), 89-112.

Warren, E. (2000). Visualisation and the development of early understanding in algebra. In T. Nakahara & M. Koyama (Eds.), *Proceedings of the 24th conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 273-280). Hiroshima: PME.

Warren, E. (2005). Young children's ability to generalise the pattern rule for growing patterns. In H .L. Chick & J. L. Vincent (Eds.), *Proceedings of the 29th conference of the International group for the Psychology of Mathematics Education* (Vol. 4, pp. 305-312). Melbourne: PME.

INDIRECT ARGUMENTATIONS IN GEOMETRY AND TREATMENT OF CONTRADICTIONS

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In proof by reductio ad absurdum, the impossibility of a mathematical object is drawn from the deduction of a contradiction. The relationship between the statement and the contradiction is logical in nature and it is one of the main obstacles for students. An analysis of indirect argumentations produced by students in geometry enlightens how they sometimes by-pass this obstacle transforming the geometrical figure so that the (false) proposition becomes true and the link between the contradiction and the statement is reconstructed. This analysis reveals some interesting differences in the treatment of the contradiction in argumentations and in proofs, identifying important difficulties in understanding proof by contradiction.

INTRODUCTION

In the last decades, many researchers have investigated proof in mathematics education. Some studies have focused on proof by contradiction and have identified many students' difficulties with this type of proof. Obstacles are found in the formulation and interpretation of the negation (Wu Yu et al., 2003; Antonini, 2001), in the treatment of the false properties generated by the assumptions of the statement negation (Mariotti & Antonini, 2006; Leron, 1985) and in the last step, that is the passage from the contradiction to the conclusion (Antonini & Mariotti, accepted for publication).

On the other side, it seems that *indirect argumentations* – argumentations fitting the scheme "*...if it were not so, it would happen that...*" (Freudenthal, 1973) – are common in students discourses and are spontaneously produced by them also in mathematics (Reid & Dobbin, 1998; Thompson, 1996; Freudenthal, 1973), in particular when they are dealing with open-ended problems (Antonini, 2003).

Therefore, we think it is important and interesting to study indirect argumentations generated by students and to compare them with proofs. A comparative analysis can give elements to identify specific characteristics of proof by contradiction and of cognitive processes leading to its construction, that are far from those we find in indirect argumentation and then could be cause of significant difficulties.

In particular, in this paper we present an exploratory study on the treatment of the contradiction in indirect argumentations in geometry context.

THEORETICAL FRAMEWORK

Studies on proof have often considered relations between argumentation and proof and, in spite of significant differences in their epistemological and didactical approaches, have contributed with many results that are important both for teaching and from theoretical point of view (Pedemonte, 2007, 2002; Knipping, 2004; Garuti et al., 1998; Duval, 1992-93). The work we are presenting here is part of a wider research on argumentation and proof in the theoretical framework of *Cognitive Unity* (Pedemonte, 2002; Garuti et al., 1998). The studies on Cognitive Unity focused on the analogies between argumentation and proof and in particular between the processes leading to their constructions. From didactical point of view an approach to proof based on the students' generation of the conjectures is suggested because of the richness of argumentative processes that open-ended problems can promote. Of course, to implement educational activities, studies on argumentations are needed. In this paper, we investigate indirect argumentations by which students justify the impossibility for a geometrical figure to have some properties.

In proofs by contradiction in geometry, we assume the existence of a geometrical figure with some properties and we aim to prove its non-existence, or, and it is logically the same, that it can not have these properties. Starting from the existence of this (impossible) figure, some deductions are drawn according to a mathematical theory (usually Euclidean geometry) until we reach a proposition contradicting a theorem, an axiom or another proposition previously deduced in the proving process. The achievement of a contradiction, according to a meta-theorem, a logical theorem on the derivation between propositions, assures that the geometrical figure does not exist or that it can not have these properties (for an analysis from a cognitive prospective of the meta-theorem, see Antonini & Mariotti, accepted for publication). The figure, the object of the reasoning, has a temporarily role (as any object in a proof by contradiction): once deduced a contradiction it has accomplished its goal. Briefly speaking, the meta-theorem states that, if a contradiction can be drawn from a statement, this is false and its negation is valid. In other words, when from the existence of a mathematical object we can deduce a contradiction, this object does not exist, it has never existed.

Two concepts are relevant here: the impossibility and the contradiction. As underlined by Toulmin (1958, pp. 30-38) in his famous book, the notion of impossibility is common not only in mathematics but in many fields, as Physics, Physiology, Linguistic, etc., and the criteria of impossibility depend on these fields (are *field-dependent*). In mathematics, contradictoriness is a criterion of impossibility but in other fields different criteria could be used[1]. In these terms, we aim to observe if the contradiction is a criterion of impossibility in students' argumentations in geometry.

METHODOLOGY

The empirical data are part of the main research on argumentation and proof and consisted in recording of clinical interviews and of some regular lessons. The subjects

[1] We are not saying here that deriving a contradiction is the only way to prove an impossibility. If a statement A is proved, of course the impossibility of non-A is stated as well, and sometimes it is also possible to prove an impossibility after exhaustive analysis of cases (see Winicki-Landman, 2007).

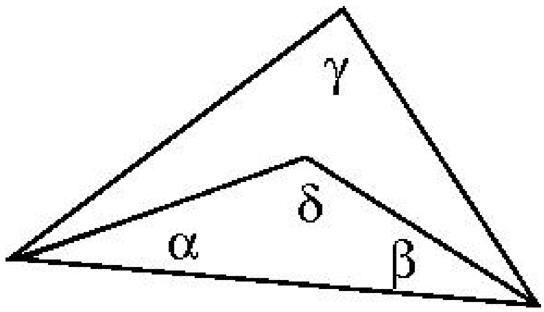
are secondary school students (grades 10-13) and university students. In the interviews, they were asked to express their thinking processes aloud and to work in couple, in order to favour argumentative processes. In this paper we report an analysis of the solution of a geometrical problem consisting in formulating and proving a conjecture; an excerpt of a regular lesson will also present in the discussion.

THE CONTRADICTION AND A NEW GEOMETRICAL FIGURE

We analyse two excerpts. The task was the geometrical open-ended problem: *what can you say about the angle formed by two bisectors of a triangle?* Students dealt with it in the paper-and-pencil environment. In the transcript, the interviewer is indicated with “I” and the students with the first letter of their names (pseudonyms).

Excerpt 1

Elenia and Francesca are university students (second year of the degree in Biology). Named the angles as in the picture, they are evaluating the possibility that the angle δ is right and they have just deduced that if it is so then $\alpha+\beta=90$ and $2\alpha+2\beta=180$. In this brief excerpt, only Elenia speaks.



46 E: … there is something wrong.

47 I: Where?

48 E: In 180.

49 I: Why?

50 E: Because, is not the interior sum of all the three angles?

51 I: Yes, the sum of the interior angles of a triangle…

52 E: is 180 [degrees].

53 I: Yes.

54 E: Right.

55 I: And then?

56 E: And then there is something wrong! They should be $2\alpha+2\beta+\gamma=180$. […]

60 E: …and then it would become $\gamma=0$…

61 I: And then?

62 E: But equal to 0 means that it isn’t a triangle! If not, it would be so [she joins her hands]. Can I arrange the lines in this way? No... […]

85 E: And then there is essentially not the triangle any more.

86 I: And now?

87 E: …that it cannot be 90 [degrees].

88 I: Are you sure?

89 E: Yes.

90 I: Why?

91 E: Because, in fact, if γ=0 it means that… it is as if the triangle essentially closed on itself and then it is not even a triangle any more, it is exactly a line, that is absurd.

The assumption that δ is right leads to the proposition "$2\alpha+2\beta=180$" that contradicts a well known theorem. The consequence is the falshood of the starting assumption and the validity of its negation: the geometrical figure, object of the reasoning, does not exist and the fact that the angle δ is not right is proved. Nevertheless, initially the students look astonished and disoriented. The non-sense of the contradiction induces them not to take it into account to formulate and to argument a conjecture. Therefore, it seems clear that the contradiction is not a criterion for the impossibility of the figure. Subsequently, students give a sense, drawing further conclusions. From γ=0 they identify a new geometrical figure in which the false proposition is true: the triangle becomes a line (in fact, the triangle should become two parallel segments but it does not seem important for our discussion).

The transformation of the figure allows them to give a sense to the false proposition and at the same time to formulate and to argument a conjecture: it is impossible that the angle is right because otherwise *the triangle closes on itself*. The figure does not have a temporary role as in the proof by contradiction, because its status is different from that assumed in a proof. In this argumentation the figure is a dynamic entity: it is initially a triangle; then, in order to have the properties deduced in a mathematical theory, is transformed and "*it is not even a triangle any more*". The impossibility of that triangle is not a consequence of the contradiction but of the transformation process that has changed it.

Excerpt 2

The following is the solution process of Paolo and Riccardo (grade 13). They named K and H respectively the angles that in the previous picture were indicated as 2α and 2β. Also in this excerpt they are involved in the case of the right angle.

63 R: … it cannot be.

64 P: Yes, but it would mean that K+H is ... a square […]

65 R: It surely should be a square, or a parallelogram.

66 P: […] [it] would mean that […] K+H is 180 degrees...

67 R: It would be impossible. Exactly, I would have with these two angles already 180, that surely it is not a triangle. […]

71 R: We can exclude that [the angle] is $\pi/2$ [right] because it would become a quadrilateral.

As in a previous interview, referring to an important theorem, the students deduce that K+H=180, and even here this proposition does not seem sufficient for them to formulate and to prove a statement until a new geometrical figure with this property is identified. The quadrilateral arises during the exploration phase of the solving process but it comes back subsequently as the main actor of the argumentation. The figure was initially a triangle but later the students better identify the figure they are treating and it is a different figure: this seems very convincing for them, more than the deduction of a contradiction (for further details on this protocol, see Antonini & Mariotti, accepted for publication).

DISCUSSION

The protocols enlighten some differences between mathematical proofs and students' argumentations. In the interviews, students produce and justify a conjecture through indirect argumentations: they assume that a geometrical figure has some properties and then they claimed that it does not have. But, differently from what happens in proofs by contradiction, in students' argumentations the contradiction is not a criterion of impossibility; it does not even seem that the contradiction has some links with any statement: initially the students do not manage to assign any sense to it and they consider it as "*something wrong*". Subsequently, the students aim to find a geometrical meaning in the false proposition they have deduced (look at the frequency of the verb "*to mean*" in the protocols: "*if γ=0 it means that…*", "*it would mean that K+H is ... a square*", etc.) through a transformation of the figure (the triangle "*becomes*" a line or a quadrilateral). Now, the false proposition is a (true) property of a new geometrical figure. Only at this point students feel satisfaction and manage to conclude; to assign a geometrical meaning to the false proposition has then relevant consequences to their argumentations. The geometrical (impossible) figure is not rejected because it has a consequent contradiction but it is adjusted in order to be coherent with the (false) proposition and according to the mathematical theory. Elenia says that "*there is essentially not the triangle any more*" not because its existence lead to a contradiction but because it is transformed in something different ("*it is as if the triangle essentially closed on itself and then it is not even a triangle any more* "); in the same way, Riccardo concludes that "*we can exclude that [the angle] is $\pi/2$ [right] because it would become a quadrilateral*". Note the expressions like "*any more*", "*become*", "*closed on itself*" by which students refer explicitly to the dynamic status of the figure and to its transformations. Summarizing, the figure is transformed in order to find a geometrical meaning in the false proposition and to reconstruct a link between this proposition and a statement. Moreover, the transformation of the figure in something different seems to be an accepted criterion for the impossibility.

In this way, the students overcome one of the main obstacles involved in proof by contradiction. In fact, an important aspect is the assumption of false hypotheses and the consequent deductions from them (Mariotti & Antonini, 2006). As revealed by Leron (1985), in a proof by contradiction students are asked to generate a false, impossible world and, instead of a construction of the results of the theorem, deduced a contradiction, this false world has to be rejected. Students can feel confused and dissatisfied for the destruction of the mathematical objects on which the proof was based. In a proof by contradiction, the geometrical figures have a temporarily role, their function is exhausted when a false proposition is deduced; after that, they have to be rejected and it is stated that they have never existed. Differently, in the described argumentation they are modified.

We observe that these argumentative processes can be analysed in the Harel & Sowder's framework (1998). As a matter of fact, these argumentations are examples of *Transformational Proof Scheme.* We briefly recall the characterization:

> "…the transformational proof scheme is characterized by (a) consideration of the generality aspects of the conjecture, (b) application of mental operations that are goal oriented and anticipatory, and (c) transformations of images as part of a deductive process." (Harel & Sowder, 1998, p. 261).

In particular, in the cases we have analysed the generality does not seem a problem, the goal of mental operations was the research of a figure for which the deduced false proposition is meaningful and true, and it seems also that subjects anticipate the results of the transformations. Moreover we have seen the transformations of the figure to be really "*part of a [students'] deductive process*". As in the examples reported by Harel and Sowder, our students treat the mathematical object as *dynamic entity* that can be transformed. It is the false property of a figure that promotes the important form of reasoning called by Martin (1996) *transformational reasoning*, with the goal to overcome the lack of a meaning and to conclude the argumentation.

We have seen here the particular case of argumentation of impossibility, but we recall that the activation of mental dynamics in production and in justification of a conjecture is one of the main aspects of the Cognitive Unity framework (Garuti et al., 1996). We also notice that our study, as in general the results in Cognitive Unity framework, allows significant analysis and explanations of students' difficulties and behaviour even outside the situations of the production of conjectures. The following episode is part of the regular didactical activity in a classroom (grade 10).

> The teacher has to prove the statement "if r is parallel to s, then $\alpha=\beta$" (look at the picture) and he proposes the following proof by contradiction: "*Suppose that $\alpha>\beta$ and let $\delta=\alpha$. For a theorem proved in the previous lesson, t is parallel to r. Then we have two lines, parallel to r and passing through the point P; this is false for a Euclidean axiom. Then $\alpha=\beta$.*"
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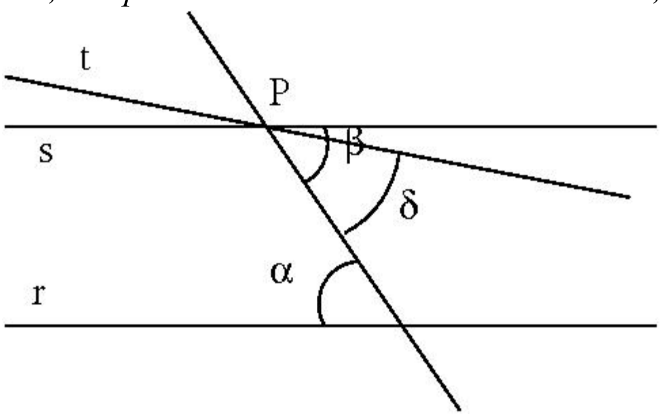
> 
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> Students are astonished and confused: they do not understand and they do not accept this reasoning. A teacher tries to argument in another way: "*Ok. Listen to me. For Euclidean axiom there exist only one parallel line, then, in fact, the line t and the line s are the same! Then the angles β and δ are the same angle; and, because $\delta=\alpha$, then $\beta=\alpha$*". Almost every student understood this argumentation, they accept it and they prefer to the first one.

The teacher proposes a proof by contradiction and then an argumentation like those we have analysed. Differently to the proof, the argumentation is both understood and accepted. In the proof, the equality of the two angles is based on the deductive chain starting from the assumption of their diversity and ending in the negation of an axiom. In the argumentation, the teacher offers a different conclusion. The false proposition becomes true after a modification of the figure according to the axiom: there are not false propositions any more and the link with the statement is reconstructed. In our opinion, the reconstruction of a geometrical meaning and of a link with the angles equality determined the immediate understanding and acceptability of this argumentation.

CONCLUSION

We have described particular justifications of some impossibilities in geometry. Other forms of indirect argumentations are possible. For example, a different process that leads to claim a statement formulated in a positive form, is analysed by Leung & Lopez-Real (2002) who studied the production of proof by contradiction in dynamic geometry environments (e.g. Cabri-Géomètre, Geometer's Sketchpad).

However, further researches are necessary to identify different indirect argumentations and to better understand the processes leading to their constructions. These studies could be significant to enlighten the potentialities of argumentative processes and also the differences between argumentations and proofs that could explain students' difficulties and that have to be carefully considered in teaching.

References

Antonini, S. & Mariotti, M. A. (accepted for publication). Indirect Proof: what is specific of this mode of proving? *Zentralblatt für Didaktik der Mathematik.*

Antonini, S. (2003). Non-examples and proof by contradiction, *Proceedings of the 2003 Joint Meeting of PME and PMENA*, (v. 2, 49-55), Honolulu, Hawai'i, U.S.A.

Antonini, S. (2001). Negation in mathematics: obstacles emerging from an exploratory study, *Proceedings of the 25th PME Conference, Utrecht, The Netherlands*, v.2, 49-56.

Duval, R. (1992-93). Argumenter, démontrer, expliquer: continuité ou rupture cognitive?, *Petite x* 31, 37-61

Freudenthal, H. (1973). *Mathematics as an Educational Task*, Reidel Publishing Company: Dordrecht, Holland.

Garuti, R., Boero, P., & Lemut, E.(1998). Cognitive unity of theorems and difficulties of proof. In *Proceedings of the 22th PME Conference, Stellenbosch, South Africa,* (v.2, 345-352).

Garuti, R., Boero, P., & Mariotti, M.A.(1996). Some dynamic mental processes underlying producing and proving conjectures. In *Proceedings of th 20th PME Conference, Valencia,* (v. 2, 121-128).

Harel, G. & Sowder, L.(1998). Students' Proof Schemes: results from exploratory studies. In A. Schoenfeld , J. Kaput, & E. Dubinsky (Eds.), *Research on Collegiate Mathematics Education*, (v.3, 234-283), M.M.A. and A.M.S.

Knipping, C. (2004). Argumentation structures in classroom proving situations. Paper contributed to working Group 4: Argumentation and Proof. In Mariotti, M. A. (Ed.), *Proceedings of CERME 3*, Bellaria, Italy.

Leron, U. (1985). A direct approach to indirect proofs. *Educational Studies in Mathematics* 16(3), 321-325.

Leung, A. & Lopez-Real, F. (2002). Theorem justification and acquisition in dynamic geometry: a case of proof by contradiction, *International Journal of Computers for Mathematical Learning*, 7, 145-165.

Mariotti, M.A. & Antonini, S. (2006). Reasoning in an absurd world: difficulties with proof by contradiction, *Proceedings of the 30th PME Conference, Prague, Czech Republic,* 2, 65-72.

Martin, A. S. (1996). Beyond inductive and deductive reasoning: the search for a sense of knowing, *Educational Studies in Mathematics*, 30, 197-210.

Pedemonte, B. (2007). How can the relationship between argumentation and proof be analysed? *Educational Studies in Mathematics*, 66(1), 23-41.

Pedemonte, B. (2002). *Etude Didactique et Cognitive des Rapports de l'Argumentation et de la Démonstration dans l'Apprentissage des Mathématiques*, Thèse, Université Joseph Fourier, Grenoble.

Reid, D. & Dobbin, J. (1998). Why is proof by contradiction difficult? *Proceedings of the 22th PME Conference*, (V. 4, 41-48), Stellenbosch, South Africa.

Thompson, D.R. (1996). Learning and teaching indirect proof. *The Mathematics Teacher* 89(6), 474-82.

Toulmin, S. (1958). *The Uses of Argument*, Cambridge University Press.

Winicki-Landman, G. (2007). Making possible the discussion of 'impossible in mathematics'. In Boero, P. (Ed.), *Theorems in School: from History, Epistemology and Cognition to Classroom Practice*, 185-195, Sense Publishers.

Wu Yu, J., Lin, F., & Lee, Y. (2003). Students' understanding of proof by contradiction, *Proceedings of the 2003 Joint Meeting of PME and PMENA,* (Vol. 4, 443-449), Honolulu, Hawai'i, U.S.A.

CONNECTING THEORETICAL FRAMEWORKS: THE TELMA PERSPECTIVE

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In this text, we report on a research project developed within the European research team TELMA (Technology Enhanced Learning in MAthematics) of the Kaleidoscope network of excellence created in 2004. We describe the conceptual and methodological tools we have progressively built for allowing productive research collaboration and overcoming the difficulties resulting from the diversity and heterogeneity of our respective theoretical backgrounds. We also show how these tools have contributed to give us a clearer idea of what is needed in terms of theoretical connection and integration in mathematics education, of what seems accessible today and how.

INTRODUCTION

Research in mathematics education does not obey a unified paradigm. On the contrary, it often appears as a field broken into a multiplicity of local communities that develop more or less independently, generating an overflow of conceptual and methodological tools poorly connected. In spite of the multiplicity of international conferences and groups, in spite of evident common trends, exchanges remain often superficial. Even if anyone understands the necessary sensitivity of the educational domain to social and cultural contexts, this situation conveys the negative image of an immature scientific field and does not encourage at considering the results obtained in it as convincing and valuable. Such a situation appears more and more problematic, increasing the attention paid to issues of comparison and connection between theoretical frames, as illustrated for instance by two recent issues of the Zentralblatt für Didaktik der Mathematik (ZDM 2005 Vol. 37(6), ZDM 2006 Vol. 38(1)), the chapter by Cobb in the second NCTM Handbook of Research on Teaching and Learning Mathematics (Cobb, 2007) or the existence of a working group especially devoted to these issues at the two last conferences of the European Association for Research in Mathematics Education (Bosch, 2006). Research concerning digital technologies does not escape this rule as evidenced for instance by the meta-study (Lagrange & al., 2003) but, due to the normal ambition of artefact designers to develop tools not restricted to one particular local community and able to migrate from one educational context to another one, researchers in that area are perhaps more sensitive to the problems raised by the current fragmentation of the field.

Within the European research team TELMA, we faced the difficulties generated by this situation when exploring possibilities for collaboration between the six different teams involved. In this paper, we report on the TELMA enterprise which began four years ago and led us to develop specific tools for overcoming these difficulties. We first briefly present the TELMA structure then focus on the conceptual and methodological tools that we have developed. After describing these, we try to show how these tools have contributed to give us a clearer idea of what is needed in terms of theoretical connection and integration in mathematics education, of what seems accessible today and how.

TELMA: AIMS, CHARACTERISTICS AND FIRST STEPS

TELMA (Technology Enhanced Learning in Mathematics) is a sub-structure of the Kaleidoscope European Network of Excellence. It includes six European teams from four different countries (England, France, Greece and Italy), and its main aims is to promote networking and integration among such teams for favouring the development of collaborative research and development projects on the teaching and learning of mathematics with digital technologies. The TELMA teams have a long experience in that area but they live in different educational contexts, the digital technologies they have developed are diverse, ranging from half baked microworlds to diagnostic and remedial tools, and the theoretical frameworks they rely on are also quite diverse. A first attempt made for identifying these (ITD, 2004) showed the existence of at least eight main theoretical frameworks: theory of didactical situations, anthropological theory of didactics, activity theory, instrumental approach, theory of semiotic mediation, social semiotics, socio-constructivism and constructionism, not to mention the theoretical approaches referred to in the AIED community and mobilized in the design of digital artefacts (Grandbastien & Labat, 2006).

For facilitating research collaboration, TELMA teams decided first to structure their collaborative work regarding the design and use of digital technologies around two main issues: representations and contexts, and to produce a description of each team according to common categories: main research aims, theoretical frameworks of references, digital tools designed and used… in order to make visible similarities and differences. As mentioned above, the descriptions produced evidenced a striking diversity in terms of theoretical frameworks, language and concepts used, and the difficulty we had to understand up to what point and how these differences affected our respective research and perspectives on the issues at stake. The notion of didactical functionality (see below) was then introduced as a reading key, general enough and based on elements relevant for all the teams, to be used to describe and compare frameworks. It was also decided to ask each team to select some few publications it considered the most appropriate for promoting mutual understanding and to work on these. Soon enough we experienced the limitation of such an enterprise: the reading of selected papers gave us only a rather superficial view of the exact role played by theoretical frames in our respective research projects. Theoretical frames were of course evoked or even discussed but their links with the details of the actual research work were missing or remained fuzzy. The idea of developing a specific methodology: the cross-experimentation methodology, presented in the next part, emerged from the awareness of these limitations.

TELMA CONSTRUCTS

The first construct introduced in TELMA was the notion of didactical functionality. It was seen as a reading key as mentioned above and a means to link theoretical reflection and practice, helping us approach theories in more operational terms, beyond the declarative level dominating in the set of selected papers.

The notion of didactical functionality

The notion of *didactical functionality* (Cerulli et al, 2005) indeed individuates three different dimensions to be taken into account when considering a learning environment integrating one or several digital artefacts, for purpose of design or analysis of use:

- a set of features/characteristics of the considered digital artefact(s);
- one (or a few coordinated) educational goal(s);
- the modalities of use of the artefact(s) in the teaching and learning activity enacted to reach such goal(s).

These three dimensions are not independent of course: although characteristics and features of a digital tool can be identified through an *a priori* inspection, these features only become functionally meaningful when understood in relation to the educational goal for which the artefact is being used in a given context and to the modalities of its use. Nevertheless, identifying and distinguishing these dimensions helped us structure the reflection and analysis, and approach theoretical frameworks in operational terms. For progressing in the understanding of our similarities and differences, we needed then to complement this structure by appropriate descriptors or categories. This was the source of the notion of *key concern* we introduce below.

The notion of key concern

In spite of its limitations, the analysis of selected papers carried out showed that the different teams shared evident common sensitivities (for instance common sensitivity to semiotic and instrumental issues, to the social and situated dimensions of learning processes), but they generally took these into consideration through different constructs and approaches. Retrospectively, the existence of such common sensitivities has nothing strange: even if we live in different educational cultures and have different trajectories, we are partly facing similar challenges and issues. Seeing theoretical frameworks and constructs as tools that we build for understanding and addressing challenges and issues, we thus conjectured that, for comparing and identifying possible productive connections between our respective theoretical frameworks and concepts, a good strategy could be to approach theories and concepts through the main sensitivities and needs they try to respond to. For tracing these common sensitivities and needs, we needed a common language not dependent on some particular theoretical approach. This was the source of the notion of key concern. A set of key concerns was thus attached to each dimension of the notion of didactical functionality, expressing the main sensitivities evidenced by the analysis carried out in the first phase of TELMA work (Artigue & al., 2005).

If we consider for instance, the first dimension of the notion of didactical functionality corresponding to the analysis of the tool for identifying potentially interesting characteristics, we distinguished between different dimensions, questioning the usability of the tool, how the mathematical knowledge of the domain is implemented in the tool and what kind of relationships with mathematical objects this implementation allows, the forms of social and didactic interactions offered by the tool, the distance with institutional and cultural objects. This resulted in a set of 8 different key concerns for this dimension.

The theoretical frame(s) that a team relies on contribute to creating a partial hierarchy between key concerns. We decided to use these hierarchies, once identified, for organizing the comparison and connection between theoretical frameworks that we wanted to achieve, considering that priority had to be given to the cases where the same key concern or set of key concerns was given a high position by two or more different teams. In such cases, we expected to be able to trace how similar or close needs were fulfilled by different theoretical constructions, better understand the functionality of these, and infer from that possible interesting connections.

We had thus a structure and the meta-language of concerns for approaching theoretical connection, but what made these tools productive was the cross-experimentation methodology we developed for supporting the analysis.

The cross-experimentation methodology

The cross-experimentation methodology was supposed to enable comparison among teams highlighting similarities and differences in their research approaches. In order to do this TELMA teams developed a set of simultaneous teaching experiments according to the principles described below.

First of all it was decided that *each team would develop a teaching experiment making use of an IT-based tool developed by another team*. This was expected to induce deeper exchanges between the teams, and to make more visible the influence of theoretical frames through comparison of the vision of didactical functionalities developed by the designers of the digital artefacts and by the teams using these in the cross-experimentation. These simultaneous experiments needed to be gathered together to allow comparisons. For this reason it was decided the *collaborative development of a common set of guidelines expressing questions to be addressed* by each designing and experimenting team in order to frame the process of cross-team communication. This document was meant to draw a framework of common questions providing a methodological tool for comparing the theoretical basis of the individual studies, their methodologies and outcomes. Furthermore, to increase the visibility of theoretical choices and discussions, and also to make the experimental situation more realistic, it was decided that in each team *PHD students and young researchers would be in charge of the experimentation*.

Finally the range of some variables was limited: in order to facilitate the comparison between the different experimental settings, it was agreed to address common mathematical knowledge domains (fractions and introduction to algebra), to carry out the experiments with students between the 5th to 8th grade, and to perform classroom experiments of about the same duration (one month).

These principles were put in practice through an on-line collaborative activity that brought the involved young researchers characterised by the 4 main phases: **1.** Production of a pre-classroom experiment version of the guidelines, containing plans for each experiment and answers to some questions (*a priori* questions); **2.** Implementation of the classroom experiments; **3.** Analysis of the experiments; **4.** Production of the final version of the guidelines containing answers to all of the addressed questions (including the *a posteriori* questions).

Each phase was interlaced with reflection tasks were the involved researchers were requested to review in-itinere the other teams' answers to the questions contained in the guidelines, and to comment on them and ask for clarifications. In this way a constant dialogue could be set up, enabling researchers to bring to light implicit assumptions and to compare the different teams' approaches (Cerulli & al, 2007). In a sense the guidelines may be considered both as a product and as a tool supporting TELMA collaborative work. A product in the sense that the final version contains questions and answers to questions as well as plans, descriptions of the experiments and results. A tool in the sense that the guidelines structured each team's work by:

- providing research questions concerning contexts, representations, and theoretical frameworks;
- establishing the time when to address each question (ex. before, or after the classroom experiment, etc.);
- establishing common concerns to focus on when describing classroom experiments, on the basis of the definition of DF;
- gathering, under the same document, the answers provided by each team to the chosen questions, in a format that could possibly help comparisons.

The guidelines were finally complemented by a final analysis of the cross experiment based on a set of interviews: a senior researcher in each team, who was not directly involved with the experimental work, interviewed the young researchers who carried out the field experiments (Artigue & al., 2007). Interviews followed a specific technique named “interview for explicitation” (Vermesch & Maurel, 1997): young researchers were asked to tell what they had done and how, but they were not directly questioned about the rationale for their actions.

THE LESSONS DRAWN FROM THE TELMA CROSS-EXPERIMENT

As was expected, the cross-experiment methodology, thanks to the perturbation it introduced in the normal functioning of the research teams, contributed to make visible the invisible, explicit the implicit. The space limitations of this research report do not allow us to enter into the necessary details, but we will try to show some important lessons that we drew from this cross-experimentation regarding both the role played by theoretical frames in design and analysis, and the needs and potentials in terms of coordination of theoretical frames. In the oral presentation, we plan to illustrate these results by using the two particular cases which are provided by the TELMA teams of the two co-authors of this research report: the DIDIREM team which experimented a digital artefact: Arilab, designed by the ITD team and the ITD team which experimented a digital artefact: Aplusix, designed by the Metah French team sharing the same didactical culture as DIDIREM.

The cross-experiment confirmed the conjectured relationship between theoretical frames and the key concern hierarchy, and showed the precise effects of this relationship in the design of the experiments, from the selection of the digital artefact to be experimented, the type of tasks proposed to the students, the diversity of semiotic mediations considered and the role given to these, the granularity in the planning of their management, the respective role given to the teacher and the student, to the attention paid to the distance with institutional and cultural habits. Moreover, it was evidenced that this influence was more or less conscious to the researchers. Familiar constructs were often used in a naturalized way and that was also the case regarding values. For that reason, the reflective interviews introduced in the cross-experimentation methodology were especially productive.

Another important result was that, even if important, the role of theoretical frames and concerns in shaping the design was limited. Answers to the guideline questionnaires and interviews evidenced the existing gap between what the theories offered and the decisions to be taken in the design. A lot of design decisions were determined by usual habits and experience and not under the control of theory. The same occurred in the implementation of the experimental design. Moreover, it clearly appeared that, for a given team, the hierarchy of key concerns was dependant on the moment of the experimentation: for instance concerns which played major role in the design of the experiment were less apparent in the analysis of the experiment. Vice versa, during the analysis phase, researchers often realized that they had underestimated specific needs in the design, and this awareness also contributed to move the concern hierarchy. They also faced unexpected events that were not so unexpected when adopting other theoretical perspectives, for instance those offered by other teams.

More generally, regarding connection and integration issues between theoretical frames, we draw from this experience a number of lessons potentially helpful for future research. We list below three of these.

<u>The necessity of distinguishing, when looking at integration, possibilities and needs between design and a posteriori analysis.</u> The economical and coherence needs of design are different of those of a posteriori analysis. Incorporating two many different theoretical frames can make design quite impossible, but in a posteriori analysis introducing new theoretical frames for instance for explaining unexpected events, producing alternative explanations, is easier and can be an effective support towards theoretical integration. For instance, the cross-experiment made clear that the theory of didactic situations and theory of semiotic mediation, which have a crucial role in

design for the DIDIREM and the ITD team respectively, induce to control and anticipate in the design of an experiment is quite different but that each vision has its own coherence and leads the design in a different and potentially productive direction. But we also got the evidence that the theoretical tools of one approach can enrich the a posteriori analysis of the other one.

<u>The fact that the hierarchy of concerns can be exploited for looking at possible theoretical connections in different ways.</u> In TELMA work, similarities in hierarchies were first exploited for establishing connections between theoretical frames and concepts, but contrasted priorities can also been exploited for looking at possible complementarities between theoretical frames.

<u>The fact that progressing in the comparison and connection between theoretical frames needs the development of specific structures and languages making the communication possible.</u> In our case, these structure and languages were provided by the notion of didactical functionality and the language of concerns. They obliged us to approach theories in terms of functionalities and this approach was really productive.

<u>Beyond that, progression needs also the building of some form of collaborative practice supporting the comparison and connection work.</u> Knowledge in this domain as in others cannot only result from readings, explanations and discussions. In our case, the cross-experimentation was asked to play this role, and the results it allowed us to achieve led us to reinvest this methodology in a new and more ambitious European project: the Remath project (Representing Mathematics with Digital Technologies) where the collaboration is extended towards the development of digital artefacts, of a common language for scenarios, and of an integrative platform MathDils. In this project, each team experiments both faliliar and alien digital artefacts in realistic contexts and cross-experiments. Moreover each team experiments both its own ILE and an alien ILE in realistic contexts, and the methodological tools built in TELMA are no longer only used to foster communication per se but also to achieve specific common research goals.

References

Artigue, M., et al. (2005). *Towards a Methodological Tool for Comparing the Use of Learning Theories in Technology Enhanced Learning in Mathematics*. TELMA Deliverable 20-4-1. Kaleidoscope Network of Excellence. http://telma.noe-kaleidoscope.org

Artigue, M., et al. (2007). *Comparison of Theories in Technology Enhanced Learning in Mathematics*. TELMA Deliverable 20-4-2. Kaleidoscope Network of Excellence. http://telma.noe-kaleidoscope.org

Bosch, M. (Ed.). *Proceedings of the IVth Congress of the European Society for Research in Mathematics Education* (CERME 4), Barcelona: Universitat Ramon Llull Editions.

Cerulli, M., Pedemonte, B., & Robotti, E. (2006). An integrated perspective to approach technology in mathematics education. In *Proceedings of CERME 4*, Sant Feliu de Guíxols, Spain (pp. 1389-1399). http:// ermeweb.free.fr/CERME4/CERME4_WG11.pdf

Cobb, P. (2007). Putting philosophy to work: coping with multiple theoretical perspectives. In F. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning*. Information Age Publishing, Inc., Greenwich, Connecticut.

Grandbastien, M. & Labat, J. M. (Eds.) (2006). *Environnements Informatiques pour l'Apprentissage Humain.* Paris: Hermes.

ITD (Ed.) (2004). *Theoretical Frameworks of Reference*. www.itd.cnr.it/telma.

Lagrange, J.B., Artigue, M., Laborde, C., & Trouche, T. (2003). Technology and mathematics education: A multidimensional study of the evolution of research and innovation. In, A. J. Bishop, et al. (Eds.). *Second International Handbook of Mathematics Education*, 239-271. Dordrecht: Kluwer Academic Publishers.

Vermesch, P. & Maurel, M. (Eds.). (1997), *Pratiques de l'Entretien d'Explicitation.* Paris: ESF.

HOW TO CHOOSE THE INDEPENDENT VARIABLE?

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A case study is presented, where the paper and pencil environment and the technological one are combined together and designed to face a subtle mathematical problem: how to choose the dependent Vs independent variables in modelling situations? We show how the combined approach allows to pose the problem in an adequate way for 9th grade students, provided the teacher interventions support suitably their learning processes. The case is analysed through two lenses from the literature: the so called instrumental approach and the notion of semiotic mediation.

INTRODUCTION

The paper presents a case study that illustrates how the combined use of technologies and paper and pencil environments can offer the teacher first the opportunity of focusing subtle but important mathematical problems not so easily accessible in only one environment, and second the tools for a positive mediation with respect to the consequent difficulties met by the students.

The "combined environment" can be thought as a tool that triggers problem posing and supports problem solving activities, provided the teacher suitably designs her/his interventions. The example we discuss here is emblematic of similar cases we met in the teaching experiments we are developing from many years with secondary school students, where the curriculum for the secondary school is "function-based" (Chazan and Yerushalmy, 2003) and developed through the combined use of new technologies (e.g. spreadsheets, DGS or CAS: see Paola, 2006) and paper and pencil environments.

The combined approach philosophy ensues from the following observations. From the one side, the students, who solve problems within technological environments, often develop practices that are significantly different from those induced by paper and pencil environments and this may offer fresh didactical opportunities:

> The curriculum with technology…changes the order and the intensity in which students meet key concepts. This change in order allows students to solve some kinds of problems that students typically might find difficult; it also either restructures points of transition between views or introduces new points of transition (Yerushalmy, 2004, p.3).

From the other side, sometimes they "naturally" use a mixed approach, where paper and pencil environment survives beside the technological one. In such cases it can be useful to exploit the didactical positive interactions of the two, suitably designing their combined use. We have observed that this methodology can be particularly useful in approaching some delicate mathematical problems, where remaining within only one environment (technological or not) may not be so productive. We shall illustrate this point showing how students choose the independent Vs dependent variables for modelling sequences of geometrical figures defined by recursive rules.

THE THEORETICAL FRAME

To properly describe our case study we use two theoretical frames in a complementary way: (1) the notion of *instrumental approach* (see Verillon & Rabardel, 1995); (2) the notion of *semiotic mediation of the teacher* (see Bartolini & Mariotti, 2008, Arzarello & Robutti, 2008).

1. *Instrumental approach.* Teaching-learning mathematics in computer environments introduces a strong instrumental dimension into the processes developed by the students. Verillon and Rabardel (1995) speak of *instrumented actions*, insofar the actions of the subjects are deeply ruled by the instrument's schemes of use (for a description of these phenomena within another theoretical frame, see Yerushalmy, 2004): e.g. to compute the roots of an equation, students can use the suitable function in the calculator modality. Instrumented actions have strong consequences on the cognitive dimensions of didactic phenomena and must be carefully considered. We shall point out how in the combined approach of paper and pencil with a specific software (TI-*n*spire) students instrumented actions contribute to modify their approach to the choice of independent Vs dependent variables in a modelling problem on recursively given sequences of geometrical figures (see below). But their instrumented actions alone are not enough to allow them to completely grasp the situation. Appropriate interventions of the teacher are necessary, as sketched in (2).
2. *Semiotic mediation of the teacher.* According to Vygotsky's conceptualization of ZPD (Vygotsky, 1978, p. 84), teaching consists in a process of enabling students' potential achievements. The teacher must provide the suitable pedagogical mediation for students' appropriation of scientific concepts (Schmittau, 2003). Within such an approach, some researchers (e.g. Bartolini & Mariotti, 2008) picture the teacher as a *semiotic mediator*, who promotes the evolution of signs in the classroom from the personal senses that the students give to them towards the scientific shared sense. We shall describe how the semiotic mediation of the teacher is crucial to support the students towards a deep understanding of the functional relationships among the variables of our problem. As a consequence, they can make an aware choice of the independent variables and draw a graph that suitably represents the situation.

THE CLASSROOM BACKGROUND AND THE TASK

The activity we shall comment concerns students attending the first year of secondary school (9th grade; 14-15 years old) in Italy. They attend a scientific course with 5 classes of mathematics per week, including the use of computers with mathematical software. Since the beginning the students have the habit of working in small collaborative groups. The classroom has been chosen for experimenting a new mathematical software, TI-*n*spire (see: www.ti-nspire.com/tools/nspire/index.html) of Texas Instruments, within an international project, whose aim is to investigate the software effectiveness in mathematics learning. The students have used TI-*n*spire from the second month of school for about 2-3 hours per week. Each student has also the software at home to make her/his homework. As to the curriculum they follow, it is strongly based on the notion of function and on modelling activities through problem solving. While making the activity described below (on March 15, 2007) the students were already able to use the (first and second) finite differences techniques for analysing if and how a function grows; and to distinguish between the polynomial and exponential growing of functions or between linear and quadratic growings. For more information (in Italian) on the curriculum and these activities, see http://www.matematica.it/paola/Corso%20di%20matematica.htm.

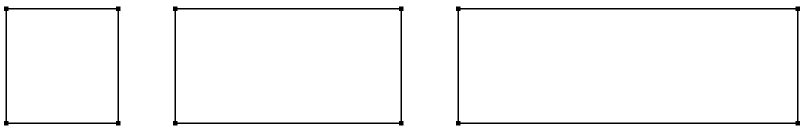
In the activity we analyse, the students, grouped in pairs, must solve a problem taken from Hershkovitz & Kieran (2001), according to the following task sheet (its working methodology is usual in the classroom).

Task

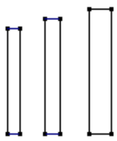
Listen carefully to the reading of the problem by the teacher. For 10 minutes think individually to the problem: do not use paper and pencil or TI-*n*spire. Produce conjectures about the change of the rectangles areas. In the successive 10 minutes discuss your conjecture with your mate; use paper and pencil only; share possible strategies to approach the problem (for validating or exploring) within TI-*n*spire. In the successive 60 minutes you can use TI-nspire to verify your conjectures, to explore the problem and eventually to solve it.

Problem

Consider the following three sequences a), b), c) of rectangles:
a) The height is constant (1 cm); the base of the first rectangle is 1 cm, while the successive rectangles are got by increasing the base 1 cm each time, as suggested by the following figures:



b) The first rectangle has height of 1 cm and base 0.1 cm; the successive ones are got increasing of 0.1 cm both the base and the height each time, as suggested by the following figures:



c) The first rectangle is a square with the side of 0.01 cm; the successive rectangles have the height always of 0.01 cm, while their bases are got each time doubling the base of the previous rectangle, as suggested by the following figures:

What can you say about the type of growing of the rectangles area in each sequence? Justify your answer."

All the pairs have produced a final document within TI-*n*spire and one of them has been videorecorded by two cameras: a fixed one for the computer screen and a second mobile one for recording the two students (L and S) while working. In the next paragraph we shall present and comment some excerpts from this videorecording. L and S are two good level achievers in mathematics.

THE SOLUTION STRATEGIES BY L AND S

In this paragraph we shall comment the strategies elaborated by L and S to solve the three questions. We shall analyse what happened only in the last two phases of their work (with paper and pencil and with TI-*n*spire). It must be observed that the classroom has been divided into two groups: one in one room with L and S and the researcher, who videorecords them but does not intervene; and the other with all the other students, who work in another room. The teacher goes back and forth from one group to the other. Hence there are long periods of time in which L and S work alone.

In phase 2, L and S do not hesitate to agree that the area in a) changes linearly. The study of the sequence b) is not so immediate. L and S build a 2 columns table, where they write the first values of the height and of the base (Table 1). L observes that the areas seem to "grow more and more" (it is the shared expression to indicate a function that increases with the concavity upwards). L wonders if this type of growing can concern all the data and not only the few considered in the table. His conjecture is that it is so provided the base does not exceed 1.

h	B
1	0.1
1.1	0.2
1.2	0.3
1.3	0.4

Table 1

H	B
1.9	1
2	1.1
2.1	1.2

Table 2

Hence he builds a second table (Table 2), which starts with the value 1 in the second column. This method is a typical strategy within paper and pencil environment; using the spreadsheet of TI-*n*spire the strategy would have been different, since students could have easily considered a lot of values and studied them with the first and second differences. At this point L generalises his conjecture saying: "It seems that it grows more and more…even because if one enlarges…it must grow more and more…two sides are always growing…hence it must grow"; and with the pencil traces in the air the "drawing" of an increasing curve with concavity upwards.

Then they pass to the sequence c). Also in this case the two students produce a table like above. At this point the teacher interacts with them and asks them what kind of growing they expect. S makes a gesture, which in their previous discussion had been used to indicate the doubling of the base. L says explicitly: "exponential…there are always powers of 2". Then L calculates some first differences, observes that they reproduce the same values of the function and this confirms his conjecture of an exponential growing. Even if with some perplexity S accepts. Hence the students are ready to pass to the software already with many given answers. One could so expect that in TI-*n*spire they find the confirmation of their (right) conjectures. This regularly happens with the sequence a): the graphic and numeric information they get from the software are coherent each other and confirm their conjecture of a linear growing. More interesting their work for the sequence b). Once they have done the work with the spreadsheet of TI-*n*spire they wish to produce a graphic and must decide what is the independent and what the dependent variable. The second choice is obvious: it is the area. But what about the independent variable? They have some uncertainty:

L: With respect to the variation of what? Of the base?

S: Hmm…

L: Yes, L3 [he refers to the name of the variable in the spreadsheet]

S: However, it is not only the change of the base ...

L: Both are changing…both are changing… with respect to the variation of what otherwise?

S: Yes but both are changing…

After a while, the teacher recalls them that when its second differences are constant the function is quadratic and then asks them: "that this is a second degree growing, could we have foreseen it?". The students remain silent for a while; then there is the following interaction between L and T (the teacher):

L: Hence they both increase [namely height and base]

T: Before you told me that when you have thought individually you thought to the fact that to find the area you multiply the base times the height. Isn't it? You have thought to this formula…

L: Yes, hence the area could be…Then we multiply the starting number…area-one equals b times a number, b times c. The second area equals (b+1) times (c+1), hence…

L gets lost with these computations: the symbols he uses are not so good to clarify why the sequence is quadratic.

T: Of what type is the change of the base?

L: Linear as that of the height.

T: If both the base and the height grow linearly, what happens to the area?

L: The area will grow…two things that grow linearly and are multiplied…ah yes x times x!

Hence they decide that the independent variable may be indifferently either the base or the height and draw the consequent graph with TI-*n*spire: a quadratic function of the area Vs the base.

The work for the sequence c) is very interesting. The students wait for an exponential graph, but when they draw the graph area Vs base a linear function appears! The graph is so unexpected that L suggests not to consider it and eliminates it from the screen of TI-*n*spire. It is the teacher to oblige them to reconsider what has happened.

T: What about the third graph?

L: Ehmm…we do not understand, it seems that it is a linear function […]

T: What were you waiting for?

L: More and more…[i.e. a growing function with the concavity upwards]

T: The area is growing […] why?

L: Because as a base…Because we have put…also the base is changing… it changes with the *same step*.

T: Hence it is correct, isn't?

S & L: Yes, yes, yes […]

T: Be careful! We were waiting an exponential function. Namely the area were increasing exponentially, but with respect to what?

L: Of an x that *went on regularly…*

T: Well, what is this x that changes regularly? […]

L: With a *constant increment*

T: Yes but in what manner…when you have said that the area grows exponentially […] with respect to what you have thought it was increasing?...Not with respect to the base. In fact if the base grows up exponentially it is clear that the area …if the base doubles, the area doubles with respect to what?

L: With respect to what? […]

T: The area of the first rectangle is […]

L: 0.0001

T: The area of the second rectangle measures…

L: Ah, *with respect to the places*.

T: Good, with respect to the places! This problem does not appear in the preceding sequences: why?

L: Because all change with a *constant step…the base*

It is interesting to observe how the students arrive to the linearity of the graph in the dialogue (see italics) and their explanation in the notes: "…the area of the sequence grows exponentially. This appears very clear to us looking at the values of the first and second differences [of the base], which result the same as those of the area". Namely for them it is clear that linearity depends on the choice of the independent variable [the base], which in this case changes proportionally with respect to the areas. So it is clear that they do not feel the necessity of making it explicit in their notes.

CONCLUSIONS

The three questions a), b), c) are essentially solved by the students in paper and pencil environments, but at different levels of understanding. Students are pushed to enter more deeply into the relationships among the variables that model the different situations by the instrumented actions they produce. In fact, they must choose a column of the spreadsheet as independent variable to validate with the software what they are waiting for: the task is obvious in case a); problematic in case b), very difficult in case c). We call this the *problem of the independent variable.* In case b) they acknowledge that the quadratic dependence results because of the increase given to both the height and to the base of the rectangle. The reflection about the structure of the area formula (suggested by the teacher) produces L's understanding of the real nature of the quadratic law ("The area will grow…two things that grow linearly and are multiplied…ah yes x times x!"). The semiotic mediation of the teacher is based on two ingredients: (i) the necessity of passing from the signs of the spreadsheet to those of the graph environment of TI-*n*spire, which requires to explicit the two variables of the graph; (ii) the reflection on the way the multiplicative area formula incorporates twice the linear increment of the sides (bilinearity of the area function). The combined effect of these two ingredients supports the cognitive processes of L. The third case is more complex: none of the variables in the spreadsheet changes linearly with the "place". The place is a hidden variable that has supported all the previous thinking processes of the students in cases a) and b). When passing to the software, they changed the independent variable, without realising it. But while in case a) and b) the hidden variable could in some way be represented through the variables they had in the spreadsheet (case b already posed some difficulties), in case c) this is not any longer possible: it is now necessary to explicit the hidden place-variable, to see what they are waiting for. The problem could not have cropped out so "naturally" in the paper and pencil environment. Students' instrumented actions generate it in cases b) and c) but it is the intervention of the teacher to make the students aware of the problem. Its solution is crucial for developing an algebraic thinking apt to sustain the formal machinery that is necessary for modelling mathematical situations. It requires to shift from the neutral reading of the relationships among the variables of a formula (e.g. Area = base × height) to a functional reading of the same formula (e.g. Area = linear function of the base, provided height is constant, as in a). The epistemological relevance of this shifting was already pointed out by J.L. Lagrange (1879, p.15): "Algebra…is the art of determining the unknowns through functions of the known quantities, or of the quantities that are considered as known". Its didactical relevance has been stressed by many researchers, e.g. see Bergsten (2003, p.8).

Comparing what happened in our classroom with the results in Hershkowitz & Kieran (2001), we find some analogies and some differences. Our experience is more similar to what happened in their Israeli 9th grade classroom, where students "were first invited to suggest hypotheses without using the computerized tool, then to use it to check them" (*ibid.*, p. 99). In that case students could find the closed algebraic formulas for problem c), even if with some difficulties; successively they could draw the three graphs using the graphic calculator. We must observe that the focus of the problem in that experience concerned more the comparison among the relative growth of the rectangles, while in our case the attention is more on the choice of the independent Vs dependent variables. During the discussion with the teacher, the Israeli students were able to match "together representatives from different representations: the algebraic, the numerical, the graphic, and the phenomenon itself" and "the evidence provided by the different representations of the software was accepted even if, for some students, it was unexpected" (*ibid.*, p. 100). In our case the students concentrated more on the finite difference techniques and got a meaningful model of the situation; however their successive instrumented actions with the software disorientated them because of some unexpected answers, particularly in case c). In our case the software acted also as a source of problems and it has been necessary a further strong mediation of the teacher. In fact, the independent variable problem is a subtle question that has been grasped by the students because of the instrumented actions fostered by the software and of the semiotic mediation of the teacher. The two have produced a meaningful reflection on this issue and avoided that "computerized tools reduce students' need for high level algebraic activity" (*ibid.,* p.106): the instrumented actions made the question accessible to the students; the teacher fostered their thinking processes by asking them the right questions at the right moment. The use of software in this example has been complementary and not substitutive to that of paper and pencil environment. Using both has allowed to get two goals. The first one concern students learning: the dialectic between what they have foreseen in the paper and pencil environment and what they are seeing within the TI-*n*spire environment poses the problem of the independent variable and gives fuel for solving it. The second concerns the researcher in mathematics education: *combining* both environments in the teaching experiment has allowed to face the issue of the use of technologies in mathematics teaching-learning according to a fresh perspective. Our point is that the curriculum with technology "changes the order and the intensity in which students meet key concepts" not only in the "substitutive" sense that it makes "natural" different approaches to the same problem, making it easier. It changes things also in a "integrated" sense: in fact, for many reasons the paper and pencil environment continues to live in our students thinking models even if they massively use technological environments. It can be useful to *combine* didactically the two in order to pose and solve mathematical problems that could be posed and solved with more difficulty remaining only within one single environment.

References

Arzarello, F. & Robutti, O. (2008). Framing the embodied mind approach within a multimodal paradigm. In Lyn English, M. Bartolini Bussi, G. Jones, R. Lesh, & D. Tirosh (Eds.), *Handbook of International Research in Mathematics Education* (LEA, USA), 2nd revised edition.

Bartolini, M.G. & Mariotti, M.A. (2008). Semiotic mediation in the mathematics classroom. In Lyn English, M. Bartolini Bussi, G. Jones, R. Lesh e D. Tirosh (Ed.), *Handbook of International Research in Mathematics Education* (LEA, USA), 2nd revised edition.

Bergsten, C. (2003). *A Classification of Algebraic Tasks*. Presentation at the seminar New trends in mathematics education research: an international perspective. Bologna, February 27, 2003.

Chazan, D. & Yerushalmy, M. (2003). On appreciating the cognitive complexity of school algebra: Research on algebra learning and directions of curricular change. In Kilpatrick, J., Schifter, D., & G. Martin (Eds.) *A Research Companion to the Principles and Standards for School Mathematics*. pp. 123-135 Reston: NCTM.

Hershkowitz, R & Kieran, C. (2001) Algorithmic and meaningful ways of joining together representatives within the same mathematical activity: an experience with graphing calculators. In *Proceedings of the 25th conference of the international group for the Psychology of Mathematics Education*, (v.1, pp. 96 – 107), Utrecht.

Lagrange, J.L. (1879). *Œuvres*, tome VIII, Traité de la résolution des équations numériques de tous les degrés. Paris: Gauthier-Villars.

Paola, D. (2006). Sensing mathematics in the classroom through the use of new technologies. In *Changes in Society: A Challenge for Mathematics Education*, Proceedings CIEAEM 58, Srnì, pp. 30 – 35.

Schmittau, J. (2003). Cultural-Historical Theory and Mathematics Education, in A. Kozulin, B. Gindis, V. S. Ageyev and S. M. Miller, (eds.), *Vygotsky's Educational Theory in Cultural Context*. (pp. 225-245), Cambridge: Cambridge University Press,

Yerushalmy, M. (2004). Does technology transform the content of algebra curricula? An analysis of critical transitions for learning and teaching. *TSG9 – Teaching and Learning of Algebra*, ICME 10, July 2004 (http://construct.haifa.ac.il/~michalyr/Yerushalmy.pdf).

Verillon, P. & Rabardel, P. (1995). Cognition and artifacts: A contribution to the study of thought in relation to instrumented activity. *Europ. Journ. of Psych. of Education*, **10** (1), pp. 77-101.

Vygotsky, L. S. (1978). *Mind in society*, Cambridge, MA: Harvard University Press.

STUDENTS' VERBAL DESCRIPTIONS THAT SUPPORT VISUAL AND ANALYTIC THINKING IN CALCULUS

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This study adds momentum to the ongoing discussion clarifying differences between visualization and analysis in mathematical thinking. By virtue of a new instrument for understanding the thinking of calculus students, we present data from its first use with 195 Advanced Placement calculus students from five high schools. Our results indicate that the new framework predicts individuals' preferences for visual or analytic thinking and, moreover, advances an alternative model involving more than this simple duality. As a result of interviews with students, we report that successful students use a combination of visualization and analysis, and that verbal-descriptive thinking is the linchpin sustaining the use of visual and analytic thinking.

OBJECTIVES

The value of calculus lies in its potential to reduce complex problems to simple rules and procedures. However, as mathematics educators have seen, many students in calculus classrooms are either unsuccessful or appear to have resigned themselves to learning strategies in order to cope without understanding; they often lack an understanding of the conceptual foundations of calculus and its practical value. One means of effecting innovation involves curricular change. With potential significance for such change, this study focuses on how students understand the derivative function with a goal of enriching learning environments in calculus classrooms.

By virtue of a new instrument for understanding the thinking of calculus students, we present data from its first administration with 195 Advanced Placement calculus students. From the perspective of developmental research, this study completed one research cycle in preparation for future examination and classroom trials by researchers and teachers. Although this new instrument classifies elements of visual and analytic mathematical thinking, more than this simple duality appeared to be involved.

Thus, in addition to the development of a new framework, we interviewed students to whom the instrument had been administered as we sought to understand the complexity of visualization and analysis as (internal) cognitive processes and to explore their roles in students' understanding. We found that a significant number of students resort to verbal descriptions as internal processes to support their analytic or visual processing. We argue that such description of mathematical objects is one of the most pervasive and useful modes of internal processing, supporting visual and analytic processing. We refer to this internal processing as verbal-description and introduce a new model, to illustrate critical intersections among visualization, analysis, and verbal-description, as internal processing by individuals, to understand external representations of mathematical objects – graphs, equations, tables, and words.

BACKGROUND

A primary theme we develop is that verbal-description is both a mode of internal processing and a means of representation of a concept; for example, in a case described below, an individual uses analysis and verbal-description, as internal modes of processing, to create a function represented by a graph and by words. We contend that such verbal-descriptions are critical to understanding for many individuals and occupy a special place as cognitive support for tables, graphs, and equations. To develop this theme, we visit three domains – one to describe what we mean by visualization, one to describe our Krutetskiian perspective for cognitive processing, and a third to propose written or verbal expressions as a way to know and understand mathematics.

Calculus is a topic that includes graphs –in addition to arrangements of symbols, tables, and other diagrams– and it is appropriate to explore learners' thought processes that relate to visual processing. The term visualization has been used in different ways in the past two decades of mathematics education research, and thus it is necessary to define how it is used in this study. Following Presmeg (2006), when a learner creates, or considers, a mathematical object, a visual image in the learner's mind guides this creation. By visual image, we do not mean merely a mental picture; instead our depictions are informed by Piaget's (1977) distinctions among visual images based on actions taken on the image leading to the creation of new cognitive structures. We follow Zazkis, Dubinsky, and Dautermann (1996) in the contention that analysis is the manipulation of these cognitive structures, with or without the use of symbols; we too do not see analysis as incompatible with visualization and insist that neither could exist without the other. Mathematical visualization then includes processes of creating or changing visual mental images, a characterization that includes the construction and interpretation of graphs.

Our work is framed by Krutetskii's (1976) classifications of learners as analytic or geometric (visual) in which visual learners are characterized as those who prefer to use visual methods when there is a choice; below, we provide descriptions of these elements for our work. We agree with Aspinwall, Shaw, and Presmeg (1997) that it is not useful to classify individuals in categories since mathematical problem solving is situation-specific and the approach used by an individual may vary according to the situation. Accordingly, we generally refer to types of processing rather than types of individuals.

Students' attempts to express their thinking in words without mathematical symbols of the derivative and integral in calculus enrich their understanding of connections among graphic, algebraic, and numeric representations (Aspinwall & Miller, 1997; 2001). Aspinwall and his associate investigated written and verbal mathematical expressions as a fourth representation and demonstrated that when provided structured writing prompts as a way of learning, students developed a positive reliance on writing for conceptual understanding and continued its use, independent of instructor solicitation, in other mathematics classes.

We found evidence for Zazkis et al.'s (1996) model that described an interchange between analysis and visualization by students in an abstract algebra course. However, the model was insufficient for all calculus students in our study as we also observed a third component that we describe as neither visualization nor analysis.

METHODS

Our work is supported by the view that posing and analysing rich tasks for students provide windows into their thinking with ramifications for curriculum and instruction. For this study, we required a valid and reliable instrument for capturing the manifold nature of students' understanding of calculus to determine the relative presence and value of the visual and analytic elements of their thinking. Because no adequate instrument was available, one had to be developed as a component of this study, and we developed and field tested the Mathematical Processing Instrument for Calculus (MPIC). Presmeg's (1985) Mathematical Processing Instrument (MPI) was the catalyst and model for the MPIC.

The MPIC classifies the processing of students according to their preferences for visual and analytic thinking in calculus. Validity and reliability are critical to any research study that employs an instrument of measurement, and the nature of students' cognitive activity makes it difficult to measure; thus careful attention was paid to techniques to make the instrument valid and reliable. For example, to insure such credibility, individual items on the MPIC are based on the standards advocated by the National Council of Teachers of Mathematics ([NCTM], 2000), an international organization with standards related to the teaching and learning of mathematics. Moreover, pilot tests of the MPIC were conducted with research mathematicians, mathematics teachers, and mathematics education professors. The Chronbach alpha correlation coefficient is a calculation resulting from a formula that is based on two or more parts of the instrument. The coefficient can take a value between 0 and 1, and a higher coefficient indicates a more credible instrument. Our field testing with the MPIC yielded a Chronbach alpha coefficient of .862, indicating that the instrument is trustworthy.

This study generated two sets of data. First, we used the newly-created MPIC to develop a quantitative understanding of the, necessarily internal, visual and analytic cognitive processes of 195 Advanced Placement (AP) students in eleven classrooms in five North Florida high schools. The instrument provides a score that reveals the extent to which students employ visualization or analysis to determine their answers. Second, to investigate students' thinking, we developed case studies by means of task-based interviews with students who, having been classified by the MPIC as either analytic or visual, described their thinking in greater detail. We asked students to draw a derivative graph when presented with the graph of a function. We describe one of the tasks in this paper and, as parts of case studies, excerpts of interviews with two students, one whose scores on the MPIC were analytic, and one whose MPIC scores were visual.

It is useful here to describe the *Elements of Visualization, Analysis, and Verbal-Description* that guided our explorations of students' thinking. When students acted on the external visual object, in this case the graph of a function, we considered this an example of Visualization, Analysis, or Verbal-Description, based on our meaning for these Elements, as follows.

Elements of Visualization

Visual solutions are dynamic and image-based. Students using such solutions can operate on their images without feeling the necessity of another thinking process. They are able to visualize the changing slopes of tangent lines to the function and accordingly are able to construct an entire derivative graph with no need to consider individual parts such as critical points or intervals. These individuals are able to determine the shape of derivative graphs based on their estimates of slopes.

Elements of Analysis

Analytic solutions are generally equations-based. An analytic solution to a task presented graphically typically may involve translation to an equation, computing the derivative of the equation, and then using this new equation to draw the derivative graph. In addition, we observed students whose analytic processes do not necessarily involve precise estimation of equations; these individuals referred to basic groups of functions such as cubic functions or quadratic functions, and their graphs associated with odd or even powers of *x*, respectively. They described a process of using analytic information obtained from tasks presented graphically.

Elements of Verbal-Description

Students using thinking processes that are verbal-descriptive determine critical points and intervals on the graphs, distinguish among different elements in the tasks, determine a hierarchy for these elements, and then combine them to draw the derivative graph. This process enables them to assemble descriptions of evidence they use to create their graphs.

The individuals in our study tended to use some combination of visual and analytic strategies, just as Zazkis and her associates reported in their 1996 study. However, we observed a cohesive third component that

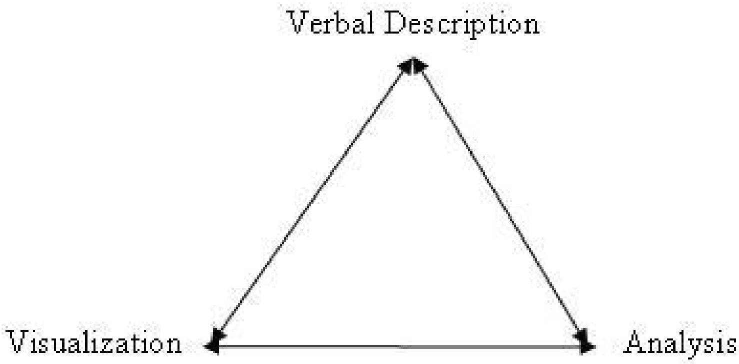

Figure 1: Triangle of Mental Processes.

supported visualization and analysis; we demonstrate with excerpts from our interviews below that these students are using a verbal-descriptive mode of thinking. We propose a model that unites the elements of visualization and analysis with students' verbal expressions. In our model (Figure 1), the processes are depicted as the vertices of the triangle, and our examples describe how individuals progress from one "vertex" to another in making decisions. We created transcripts of the interviews with Al, whose thinking required all three processes in the model, and Bill, for whom only two processes in the model were needed. Our cases describe how Al and Bill used the processes in the triangle to create their sketches of the derivative graph of the function in Figure 2. The purpose of the interviews was to understand better how they created their sketches.

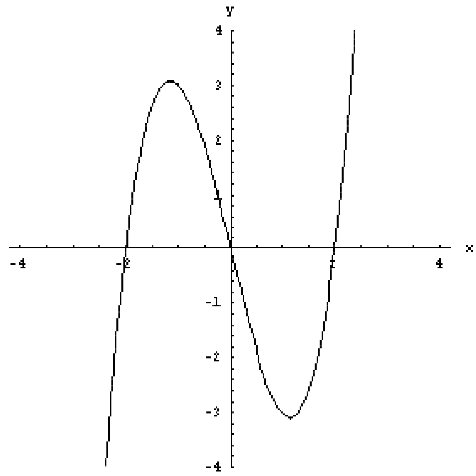

Figure 2: The Task.

RESULTS

The Case of Al

Al's results on the MPIC reveal that his responses were visual. He demonstrated his preference for visual processing for the graph in figure 2 as the following excerpt suggests. When we asked how he drew his graph, his descriptions were image-based as he described the changing slopes of the graph.

Al: The slope is] pretty big negative number around here [points to the interval between -1 and 1], here [points to the interval between $-\infty$ and -1] it [slope] is positive so you know it [x intercept of the derivative graph] is going to be somewhere here. To the left of horizontal tangent line, to the left of -1 so it will be positive and to the right of 1, that's also positive.

Interviewer: What do you mean when you say positive or negative?

Al: Slopes.

At this point it seems Al had constructed an image of the derivative graph, as his MPIC results predicted. The transcript above demonstrates that Al supported his visualization with verbal description revealed in his reference above to critical intervals for the changing slopes for the graph in Figure 2. Thus, he was shifting between two of the vertices of the model – visualization and verbal-description. But he then resorted to analytic processes to support his (verbal-description-supported) visualization. We considered this a shift from visualization to analysis, and when we continued to probe, he described the elements of his analysis.

Al: It [points to the original graph] looks like cubic so the derivative would be a parabola. It would look something like this [draws the derivative graph shown in Figure 3].

Interviewer: How did you know that it was cubic?

Al: Well you don't but it's about cubic the shape of it. The derivative of a cubic is a parabola. That should also look somewhat like a parabola.

The descriptions by Al support the Zazkis' model. As Zazkis and her associates found in their study, we too observed that visual and analytic processes are mutually dependent in mathematical problem solving; that is, students translated between them as they solved graphical tasks. When we asked how he drew his graph in Figure 3, Al's descriptions were dynamic and image-based as he described the changing slopes of the graph. And the interview revealed analytic and verbal descriptive support for his visual images, implying the dichotomy between visual and analytic processes may be an inadequate classification for describing all students' learning. All three processes of the model in Figure 1, including the verbal-descriptive as a linking component, were necessary for Al, and it may be an essential element in the internal processing of others. Consider now the case of Bill.

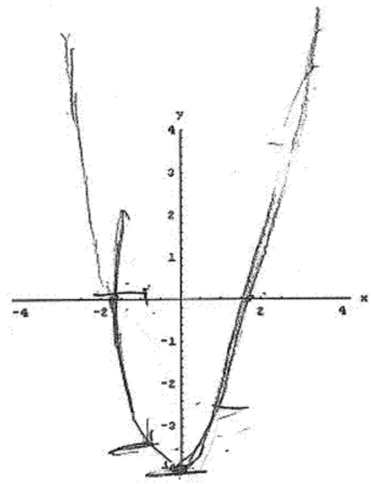

Figure 3: Al's Sketch.

The Case of Bill

Bill's results on the MPIC reveal that his solutions were analytic, and he demonstrated his preference for analytic processing in the interviews. But as he explained his thinking, his descriptions contained elements of verbal-descriptions as well. He first tried unsuccessfully to estimate a possible equation of the graph in Figure 2:

Bill: [Pauses] I am trying to think of an equation, what equation makes these minima and maxima?

We considered this to be attempts at analysis. He then shifted from analysis to verbal-description as he surrendered his attempts at translating (for analysis) and turned his attention to critical points and intervals on the graph.

Bill: Right here [points] is where the derivative is going to be equal to 0, which means that here and here [points] is where it is going to cross the x axis. And here [points] the derivative is going to be negative, and here it is going to be positive and positive.

These determinations of critical points and intervals on the graphs, with his explanations for their meaning, are elements of verbal-description that he used to sketch the graph. When we asked how he had determined the minimum value of his derivative graph, he shifted back to analysis.

Bill: Because the slope where $x = 0$ is, roughly guessing, if you take -1 and $+2$, the slope appears to be about -2. [draws box shown in Figure 4].

We considered Bill's drawing of the box in figure 4 an act of analysis as he was analytically determining the slope between -1 and 0. He shifted again to verbal-description as he described how he distinguished and assembled elements of these descriptions to create his graph in Figure 5:

Bill: Kind of piece by piece. I know first, these two points are right here on the graph. They have to be, that x value for the derivative has to be 0. So, whatever the graph looks like, it is going to go through those two points [on the x axis]. Then I found the vertex by estimating the slope here [points to the origin]. So, it goes through these two points. But it's not going to look like this [draws a V on his paper, on the side].

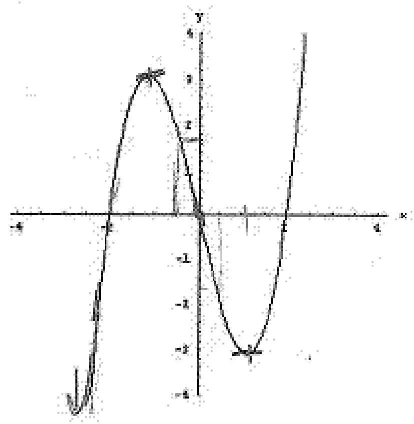

Figure 4: Bill's First Sketches.

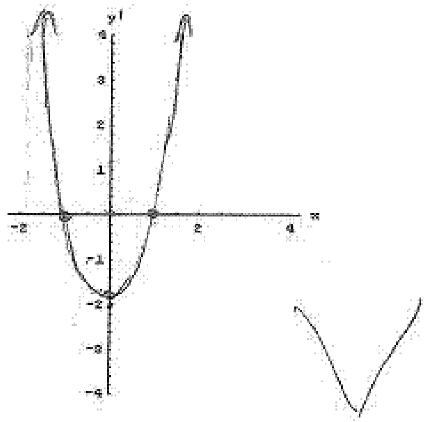

Figure 5. Bill's Sketches.

As the interview suggests, Bill translated between Analysis and Verbal-Description. He examined points and intervals "piece by piece" as he gathered evidence to draw his graph. His descriptions of slopes are dramatically different than the changing and dynamic images of slopes of which Al spoke. The box he drew near the origin suggests elements of analysis that he used as he translated to analysis and used this analysis to support his verbal descriptions. He made little reference to a mode of thinking that we considered visualization. Therefore, only analysis and verbal-description, as two processes of the model in Figure 1, were necessary for Bill, who preferred analytic solutions on the MPIC.

SIGNIFICANCE

We defined visualization, analysis, and verbal-description, and our definitions were theoretical. We suggest that the definitions, along with our data, provide useful windows into the thinking of students. Furthermore, we think the data are reasonably consistent with our model (figure 1). We have concluded that students invoke words as an amalgam to support their visual and analytic understanding of mathematical equations, graphs, and tables. Further, for some, verbal-description is possibly used in lieu of accessible visual images or symbolic mathematical expressions. Our research with the data base of students who have been tested with the Mathematical Processing Instrument for Calculus continues. Future study will help us determine the use and degree of interactions among the three elements of analysis, visualization, and verbal-description.

To the extent that these data make sense for our model of visual, analytic, and verbal-descriptive thinking in students' understanding of elementary calculus, we think the model they suggest may be useful for learning and instruction of mathematics in other areas. Successful students, if success is measured by conceptual understanding, use a combination of strategies. The element of verbal-description, described in this study as a third mode of internal processing, is a critical link between visualization and analysis, and may sometimes be used in lieu of these modes. An increased focus on how students understand and know calculus has the potential to enrich classroom instruction and conceptual learning.

References

Aspinwall, L. & Miller, D. (1997). Students' positive reliance on writing as a process to learn first semester calculus. *Journal of Instructional Psychology, 24*, 253-261. Research funded by College of Graduate Studies, Faculty Research and Creative Activities Committee, Middle Tennessee State University.

Aspinwall, L. & Miller, D. (2001). Diagnosing conflict factors in calculus through students' writings: One teacher's reflections. *Journal of Mathematical Behavior, 20*, 89-107.

Aspinwall, L., Shaw, K., & Presmeg, N. (1997). Uncontrollable mental imagery: Graphical connections between a function and its derivative. *Educational Studies in Mathematics, 33*, 301-317.

Krutetskii, V. A. (1976). *The Psychology of Mathematical Abilities in Schoolchildren*. J. Kilpatrick & I. Wirszup (Eds.), Chicago: The University of Chicago Press

National Council of Teachers of Mathematics (2000). *Principles and Standards for School Mathematics.* Reston, VA: The Council.

Piaget, J. (1977). Mental images. In H.Gruber & J. Voneche (Eds.), *The Essential Piaget* (pp. 652-684). New York: Basic Books.

Presmeg, N. C. (1985). *The Role of Visually Mediated Processes in High School Mathematics: A Classroom Investigation.* Unpublished Ph.D. dissertation, University of Cambridge.

Presmeg, N. C. (2006). Research on visualization in learning and teaching mathematics: Emergence from psychology. In A. Gutierrez & P. Boero (Eds.), *Handbook of Research on the Psychology of Mathematics Education: Past, Present and Future* (pp. 205-235). Rotterdam, The Netherlands: Sense Publishers.

Zazkis, R., Dubinsky, E., & Dautermann, J. (1996). Coordinating visual and analytic strategies: A study of students' understanding of the group D_4. *Journal for Research in Mathematics Education, 27*, 453-457.

CLASSROOM ENVIRONMENT FIT IN MATHEMATICS ACROSS THE TRANSITION FROM PRIMARY TO SECONDARY SCHOOL

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In the present paper we investigate students' perceptions of the actual and the preferred classroom environment in mathematics across the transition from primary to secondary school. The analysis of 220 students' responses to a questionnaire suggests that there is a developmental mismatch between the actual and the preferred classroom environment across the transition. More specifically, our findings indicate that students perceive fewer actual opportunities to participate in learning and carry out investigations after than before the transition; they also express a preference for more interactive teaching and independence after than before the transition. The level of congruence between students' actual and preferred perceptions declines after the transition regarding personalization/participation and investigation.

BACKGROUND AND AIMS OF STUDY

The period surrounding the transition from primary to secondary school has been found to result in a decline in students' motivation in mathematics (see e.g., Athanasiou & Philippou, 2006, MacCallum, 2004). This decline in motivation in mathematics was found to be related to certain dimensions of the school and classroom culture (e.g. Eccles et al., 1993, Urdan & Midgley, 2003). It has been suggested that during this transition there are inappropriate changes in a cluster of classroom organizational, instructional and climate variables. The two types of schools were characterized as very different organizations with respect to "ethos" as well as to practices, and that this discrepancy influences students' motivation and performance (Midgley et. al., 1995).

The dimensions of the school culture that were found to affect motivation during this systemic transition include the perceived classroom goal structure (Urdan & Midgley, 2003), teacher's sense of efficacy and his/her ability to discipline and control students (Midgley et al., 1989), teacher-student relations and opportunities for students to participate in decision making (Athanasiou & Philippou, 2006).

A slightly different analysis of the possible environmental influences associated with the transition to middle school draws on the idea of person-environment fit (PEF). PEF theory (Eccles et al., 1993) states that the behaviour of an individual is jointly determined by his/hers characteristics and the properties of the environment in which the person functions. Therefore, within this theoretical framework, it is the fit between the needs of the adolescent and the educational environment that is important, that is the fit between the preferred and the actual classroom environment (Eccles et al., 1993). If it is true that different types of educational environments may be needed to meet the needs of different age groups, then it is also possible that some types of changes in educational environments may be inappropriate or regressive at certain stages of development, such as the early adolescent period, during which students move to secondary school. Exposure to such changes is likely to create a particularly poor person-environment fit, which could account, to a certain extend, for the decline in motivation seen at this developmental period.

Despite the above theoretical considerations, we have located only a few studies that examined the fit between the actual and the preferred classroom environment and all of them focused on a single dimension, namely decision-making (e.g. Midgley & Feldlaufer, 1987). In these studies, students where found to perceive fewer actual decision-making opportunities after than before the transition and that the congruence between students' actual and preferred perceptions declined after the transition.

The purpose of the present longitudinal study is to chart the developmental changes of the fit between the actual and the preferred classroom environment in mathematics during the transition from primary to secondary school, focusing on four classroom dimensions: opportunities provided to students to: a) participate and interact with the teacher, b) investigate, c) make decisions regarding movement and sitting, and d) be treated differently according to their own individual abilities and pace. Since the transition to secondary school in the educational system of Cyprus, where the study is conducted, occurs after Grade 6, the research questions were formulated as follows:

- Is there any mismatch between the actual and the preferred classroom environment in mathematics as perceived by sixth and seventh graders?
- Are there any changes in students' perceptions of the actual and the preferred classroom environment in mathematics across the transition to secondary school?
- Are there any developmental differences in the fit between the actual and the preferred classroom environment in mathematics across the transition to secondary school?

METHODOLOGY

Participants in this study were 220 students (97 boys and 123 girls) who were followed over a period of two consecutive school years, from Grade 6 in elementary to Grade 7 in secondary school. Data were collected from these students in four waves through a self-report questionnaire, which was an adaptation of the Individualized Classroom Environment Questionnaire (Fraser, 1990). The first measurement was taken at elementary school and the other three in each of the three trimesters in secondary school. The exact timing of the measurements was based on the organization of the school year in the specific educational system where the study is conducted and on the Phase Model of Transitions by Ruble (1994).

The Questionnaire included 20 items tapping students' perceptions of the classroom environment in four dimensions: a) personalization/participation (**Pers/Part**) (e.g. "The teacher considers students' feelings in mathematics"), b) investigation (**Inv**) (e.g. "Students carry out investigations to test ideas in mathematics"), c) independence (**Ind**) (e.g. "The teacher decides where students sit in mathematics") and d) differentiation (**Diff**) (e.g. "All students do the same work at the same time in mathematics").

The questionnaire was completed by students in two parallel forms, eliciting the perceived as actual classroom environment and the preferred or expected classroom environment in each of the four dimensions. For instance, the preferred version in a Diff statement was: "I would prefer all the students to do the same work at the same time in mathematics". The statements were presented at a five-point Likert-type format (1=Strongly Disagree, 5=Strongly Agree).

Data processing was carried out using the SPSS software. The statistical procedures used were paired-samples t-test and multivariate analysis of variance (MANOVA). Post-hoc tests (Bonferroni multiple comparison procedure) were performed as follow-up tests to examine whether there are significant differences between the means of each pair. Confirmatory Factor Analysis was undertaken to determine the effectiveness of the translated instrument in the specific environment; all scale items were clustered in the expected factor in all four measurements for the **Ind**, **Inv** and **Diff** dimensions of the scale, whereas the items regarding personalization and participation clustered in a joint factor (**Pers/Part**). The reliability estimate (Cronbach's alpha) for the whole scale was found to be quite high (a = .81).

RESULTS

To examine whether there is any mismatch between the actual and the expected classroom environment, as perceived by students, pairwise t-tests were performed to compare the means in the respective forms of the questionnaire at each of the four waves of measurement in each scale dimension. Table 1 presents the means of the students' perceptions of the actual and the preferred classroom environment.

	Wave 1			Wave 2			Wave 3			Wave 4		
	M	SD	T	M	SD	t	M	SD	t	M	SD	t
Personalization/Participation												
Actual	3.85	.76	-4.69*	3.65	.86	-9.24*	3.79	.81	-4.81*	3.84	.79	-2.38*
Preferred	4.13	.61		4.26	.59		4.06	.64		3.98	.64	
Investigation												
Actual	3.51	.90	-.62	3.29	.88	-4.58*	3.33	.97	-4.44*	3.37	.80	-2.32*
Preferred	3.55	.94		3.63	.98		3.63	.91		3.56	1.0	
Independence												
Actual	2.88	.90	-3.20*	3.25	.80	-3.43*	3.08	.86	-2.32*	3.05	.86	-3.02*
Preferred	3.15	.97		3.52	1.0		3.23	.93		3.31	1.0	
Differentiation												
Actual	2.43	1.0	6.20*	2.04	.84	2.84*	2.13	.88	-.37	2.36	.95	.53
Preferred	1.88	.93		1.85	.89		2.15	.88		2.32	.92	

*p<0.01

Table 1. Means and Standard Deviations of students' perceptions of actual and preferred classroom environment

Table 1 shows that across all four waves, students reported that the actual classroom environment was significantly lower than the environment they prefer on the dimensions of **Pers/Part** and **Ind**. A different pattern is, however, observed with respect to **Diff**, where students reported that the actual classroom environment was significantly higher than the preferred in Waves 1 and 2. Furthermore, with respect to **Inv** there was no difference between the actual and the preferred classroom environment in the pre-transition period (Wave 1), indicating that in the primary school students' expectations are in this respect well met, while in the secondary school (Waves 2, 3 and 4) students perceived the actual classroom environment as being below their expectations on the factor **Inv**.

To look for changes in student perceptions of the actual and the preferred classroom environment, repeated measures multivariate analyses of variance (MANOVA) were performed, including one within-subjects factor, which was the wave of measurement (4 levels). Post hoc tests were performed as follow-up tests to determine whether the means differed significantly from each other.

Table 2 summarizes one-way MANOVAs on the actual and the preferred classroom environment scores.

The analysis indicated that the time of measurement effect was significant for all the factors of the actual classroom environment indicating that students' perceptions change over time. More specifically students' mean perceptions of the actual classroom environment in three dimensions (**Pers/Part**, **Inv**, and **Diff**) were at the lowest value immediately after the transition to secondary school (Wave 2).

According to the post-hoc tests though, only students' perceptions regarding **Diff** were significantly higher at Wave 1 than at Waves 2 and 3, suggesting that the decline was a transition effect.

A different pattern of change was observed for **Ind** with students reporting more actual independence opportunities after the transition than before. The post-hoc tests showed that students' perceptions of actual independence were significantly lower at Wave 1 than at Wave 2, suggesting that the increase was a transition effect.

The MANOVA on the preferred classroom environment dimensions showed a significant time of measurement effect on the three scale dimensions: **Pers/Part**, **Ind** and **Diff,** but not on the **Inv** factor. More specifically, students' perceptions of the preferred **Pers/Part** and **Ind** dimensions of classroom environment appear at peak after the transition (Wave 2). Post-hoc tests showed that students' perceptions of the preferred **Pers/Part** dimension were significantly higher on Wave 2 than at Waves 3 and 4, whereas in the case of the **Ind** dimension, their perceptions were significantly higher at Wave 2 than at Wave 1, suggesting that the increase was a transition effect. Students' perceptions of the preferred **Diff** classroom environment were the lowest after the transition (Wave 2) and increased at Waves 3 and 4; post-hoc tests showed that their perceptions were significantly lower at Waves 1 and 2 rather than at Waves 3 and 4.

Variable	Df	MS	F
	Actual Classroom Environment		
Personalization/Participation	2.460	2.148	5.053*
Investigation	2.891	2.166	3.886*
Independence	2.723	5.524	9.365*
Differentiation	2.813	7.934	14.412*
	Preferred Classroom Environment		
Personalization/Participation	2.854	3.123	11.736*
Investigation	2.849	.452	.699
Independence	2.689	6.235	8.892*
Differentiation	2.918	11.579	18.982*

*p<0.01

Table 2. Effect of Time of Measurement on students' perceptions of actual and preferred classroom environment

To test for differences in the fit between the actual and the preferred classroom environment, we took as fit scores the differences between the respective means in the two formats of the questionnaire, in each scale dimension (actual – preferred classroom environment). A negative value of the fit score indicates that students reported that they did not experience but they would expect the classroom environment mentioned. A positive value indicates that students reported that they actually had experienced the classroom environment but they should not have, whereas a zero value of the fit score indicates that students reported that the classroom environment they actually had coincides with what the have expected to have or that they actually did not and should not have the classroom environment mentioned. The results of the analysis are shown in Table 3.

Variable	df	MS	F
Personalization/Participation	2.636	9.647	15.330*
Investigation	2.851	4.298	3.448**
Independence	2.785	.773	.600
Differentiation	2.654	16.273	12.360*

*p<0.01, **p<0.05

Table 3. Effect of Time of Measurement on students' fit score

The analysis indicated that the time of measurement effect was significant for the three factors of the fit between the actual and the preferred classroom environment that is for **Pers/Part**, **Inv** and **Diff** dimensions. The mean fit scores in each measurement wave regarding **Pers/Part** were: -.27 (SD = .88), -.60 (SD = .97), - .27 (SD = .84), and -.14 (SD = .90), for Waves 1,2,3,4, respectively. Likewise, the fit scores for the **Inv** factor were: -.04 (SD = .98), -.34 (SD = 1.11), -.30, (SD = 1.00), and -.18 (SD = 1.17), for Waves 1,2,3,4, respectively. Clearly, the fit of the actual and the preferred classroom environment on the **Pers/Part** and **Inv** dimensions of the scale had the most negative values immediately after the transition to middle school.

The post-hoc tests showed that the mismatch between students' actual and preferred **Pers/Part** and **Inv** dimensions of classroom environment increases after students enter middle school, since students' mean fit was significantly higher at Wave 2 than at Wave 1, suggesting that the increase was a transition effect. The means of the fit score for **Diff** were: .54 (SD = 1.31), .19 (SD =1.02), -.02 (SD = .88), and .04 (SD = 1.19) for Waves 1,2,3,4, respectively. The post-hoc tests showed that the mismatch between students' actual and preferred **Diff** classroom environment decreases after students enter the middle school, since students' mean fit was significantly higher at Wave 1 than at Waves 2, 3 and 4.

DISCUSSION

The purpose of the present study was to examine the developmental changes of the fit between the actual and the preferred classroom environment in mathematics as perceived by students over the transition from primary to secondary school.

The analysis of the data indicated that there is a mismatch between the actual and the preferred classroom environment. At both the pre- and the post- transition level students' preferences are out-of-synch with their environment regarding two scale dimensions, **Pers/Part** and **Ind**; they would like considerably more opportunities for participation and interaction with the teacher, and more independence than they perceive they actually have. According to PEF theory, when the needs of the individual are congruent with opportunities granted by the environment, favourable affective, cognitive and behavioural outcomes should result for that individual. Conversely, when a discrepancy exists between the needs of the individual and opportunities available in the environment, unfavourable outcomes should result (Midgley & Feldlaufer, 1987). In line with this theory, the lack of fit between students' preferences and the actual environment they encounter daily in class, should predict unfortunate consequences for those students whose needs are not being met.

Longitudinal studies should address this issue, since studies regarding students' decision-making confirmed this prediction, showing that students whose desire for decision-making in mathematics was discrepant with the opportunities available in the classroom were less positive about mathematics and about their potential in mathematics than the students whose desires and opportunities were congruent (Midgley & Feldlaufer, 1987).

A different pattern of findings was observed for the dimensions of **Inv** and **Diff**. For the former, pre-transition students' needs are being met, while post-transition students would expect more opportunities for mathematical investigations. This finding is pretty logical taking into consideration that elementary school classrooms as compared to middle school classrooms are characterised by a greater emphasis on student involvement and investigation in learning mathematics. The findings about **Diff** indicate that the actual classroom environment was significantly higher than the preferred environment; the students were found to prefer less differentiation than they perceive they actually had. If we consider that differentiation in both contexts has to do with selective treatment of students, based on ability and therefore with weaknesses in mathematics and social discrimination, then it seems logical that students do not want the teacher to offer different teaching materials or aids to students with special abilities in mathematics.

The results of the study contribute to our understanding of the fit between the actual and the preferred classroom environment in mathematics during the transition to middle school. It is remarkable that the mismatch between students' actual and preferred **Pers/Part** and **Inv** classroom environment had the most negative value immediately after the transition to middle school, whereas the mismatch between students' actual and preferred differentiation classroom environment decreases after students enter middle school. Given the differences in the school culture between elementary and secondary schools reported in other studies (Athanasiou & Philippou, 2006, Urdan & Midgley, 2003) it is not surprising that elementary school students perceive that in their mathematics classroom the teacher is friendly, caring and helpful and that he/she encourages investigation and participation more than the teachers in middle schools.

Exposure to such changes leads to a particularly poor person-environment fit and this lack of fit could account for some of the declines in motivation seen at this developmental period. Therefore, the environmental changes often associated with the transition to middle school seem especially harmful in that they emphasize lower level cognitive strategies at a time when the ability to use higher level strategies such as investigation is increasing; they emphasize pathetic learning at a time of heightened need for participation and involvement in learning; and they disrupt social networks with the teacher at a time when adolescents are especially concerned with close adult relationships outside of the home.

The findings of the present study highlight the developmental differences in students' perceptions of the fit between the actual and the preferred classroom environment in mathematics. Longitudinal studies addressing this issue, need not examine students only as a whole group. Recent research in the area of students' perceptions of classroom environment adds credence to the view that students' do not all perceive the same environment in the same way, at least not in all dimensions (MacCallum, 2004). Also there is a need to understand not only the effects of what is most prevalent in classrooms but also to determine what the most facilitative environments are, even if they are uncommon, in order to test the effects of these environments on the nature of change in student motivation in mathematics.

Such longitudinal studies can assist in unravelling the complexity of motivational change across the transition from primary to secondary school, by providing information of the dimensions of the classroom culture that influence motivational change. Such studies will be useful for teachers, educators, and policy makers in their planning to make systemic transitions easier for students. These preventive steps can include the identification of the dimensions of the school culture that have a positive or a negative impact on students motivation and the strengthening of the support structures provided to students either by their family or by the school through transition programs.

References

Athanasiou, C. & Philippou, G.N. (2006). Motivation and perceptions of classroom culture in mathematics of students across grades 5 to 7. *Proceedings of* PME30 (Vol. 2, pp. 81-88). Prague, Czech Republic

Eccles, J. S., Wigfield, A., Reuman, D., & MacIver, D. (1993). Changes in self-perceptions and values at early adolescence. *Paper Presented at the Annual Meeting of the American Educational Research Association,* Washington, DC.

Fraser, B. (1990). *Individualised Classroom Environment Questionnaire.* Melbourne: Australian Council of Educational Research.

MacCallum, J. A. (2004). The contexts of individual motivational change. In D. M. McInerney & S. V. Etten (Eds.), *Research on Sociocultural Influences on Motivation and Learning,* Vol.1. Information Age Publishing, Greenwich.

Midgley, C. & Feldlaufer, H. (1987). Students' and teachers' decision-making fit before and after the transition to junior high school. *Journal of Early Adolescence, 2,* 225-241.

Midgley, C., Feldlaufer, H., & Eccles, J. S. (1989). Change in teacher efficacy and student self and task-related beliefs during the transition to junior high school. *Journal of Educational Psychology, 81,* 247-258.

Midgley, C., Anderman, E. M., & Hicks, L. (1995). Differences between elementary and middle school teachers and students: A goal theory approach. *Journal of Early Adolescence, 15,* 90-113.

Ruble, D. N. (1994). A phase model of transitions: cognitive and motivational consequences. *Advances in Experimental Social Psychology, 26,* 163-214.

Urdan, T. & Midgley, C.(2003). Changes in the perceived classroom goal structure and pattern of adaptive learning during early adolescence. *Contemporary Educational Psychology, 28,* 524-551.

VIEWS OF MATHEMATICS EDUCATORS ON THE ROLE OF MATHEMATICS LEARNING IN THE DEVELOPMENT OF DEDUCTIVE REASONING

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This study examines the views of people involved in mathematics education regarding the role of mathematics learning in the development of deductive reasoning that is not restricted to mathematics but can be used in other domains as well. The data source includes 21 individual semi-structured interviews. All interviewees said that developing deductive reasoning is one of the goals of mathematics teaching. However, none of them seemed to think that this is at all possible, but for different reasons. Three distinct views were identified: the reservation view, the intervention view, and the spontaneity view. Each interviewee's view was interrelated with the interviewee's approach to deductive reasoning and its nature in mathematics and outside it.

In a previous PME meeting we presented findings on different approaches to the meaning of deductive reasoning and its nature in mathematics and outside it that were expressed by mathematics educators (Ayalon & Even, 2006). This paper extends this research and examines the views of these mathematics educators on the role of mathematics learning in the development of deductive reasoning.

INTRODUCTION

There are various sorts of thinking and reasoning. Among them are association, creation, induction, plausible inference, and deduction (Johnson-Laird & Byrne, 1991). Deductive reasoning is unique in that it is the process of inferring conclusions from known information (premises) based on formal logic rules, where conclusions are necessarily derived from the given information and there is no need to validate them by experiments. There are several forms of valid deductive argument, for example, *modus ponens* (If p then q; p; therefore q) and *modus tolens* (If p then q; not q; therefore not p). Valid deductive arguments preserve truth, in the sense that if the premises are true, then the conclusion is also true. However, the truth (or falsehood) of a conclusion or premises does not imply that an argument is valid (or invalid). Also, the premises and the conclusion of a valid argument may all be false.

Mathematics and deductive reasoning have twofold connections. On the one hand, deductive reasoning serves mathematics because it is a key to work in mathematics. Rigorous logical proof, which is a unique fundamental characteristic of mathematics, is constructed using deductive reasoning. Although there are some other accepted forms of mathematical validation, deductive proof is considered as the preferred tool in the mathematics community for verifying mathematical statements and showing their universality. And indeed, deductive reasoning is often used as a synonym for mathematical thinking.

But deductive reasoning is not only a servant of mathematics. Since the early days of Greek philosophical and scientific work, deductive reasoning has been considered as a high (and even the highest) form of human reasoning (Luria, 1976). Already Aristotle, who laid down the foundations for this kind of thinking in the 4th century B.C., perceived a person who possesses deductive ability as being able to grasp the universe in more profound and comprehensive ways. Similarly, more than two thousand years later, Luria (1976) viewed deductive ability as necessary for gaining new knowledge. Throughout human scientific development, great scientists, such as Descartes and Popper, emphasized the importance of this kind of reasoning to science. Johnson-Laird & Byrne (1991) emphasized its importance for work in science, technology, and the legal system, and Wu (1996) for facilitating wise decision making related to politics and the economy.

Learning mathematics has been traditionally believed to be an effective tool for teaching deductive reasoning, altering the connections between mathematics and deductive reasoning and, in a way, making mathematics a servant to deductive reasoning. The Greeks, more than two thousand years ago, taught logic by teaching arithmetic and geometry (Nisbett et al., 1987). And today, curriculum guidelines, textbooks and teacher guides in many countries state explicitly that mathematics helps students develop their ability to reason logically, and that one of the goals of mathematics instruction is the development of deductive reasoning. For example,

> Mathematics equips pupils with a uniquely powerful set of tools to understand and change the world. These tools include *logical reasoning*, problem-solving skills, and the ability to think in abstract ways (emphasis added) (Qualifications and Curriculum Authority, 2006).

Do people involved in mathematics education, such as, curriculum developers, teacher educators, teachers, and researchers view mathematics learning as an effective tool for teaching deductive reasoning? If they do, what do they mean by that? If they do not, why do they think so? And - how do people with different approaches to deductive reasoning, and to the usability of deductive reasoning outside mathematics, view the role of learning mathematics in the development of deductive reasoning? This study addresses these questions.

This study continues a previous study (Ayalon & Even, 2006) in which we identified two different approaches to deductive reasoning among the participating mathematics educators. One, which was expected, describes deductive reasoning as an action of inference based on the rules of formal logic. The other approach, which we did not anticipate when starting the study, describes deductive reasoning as a systematic step-by-step manner for solving problems, with no attention to issues of validity, formal logic rules, or necessity – the very essence of deductive reasoning. Moreover, whereas all study participants agreed that deductive reasoning is essential to mathematics, different approaches regarding the usability of deductive reasoning outside mathematics were identified. Those approaching deductive reasoning as a systematic step-by-step manner for solving problems considered the use of deductive reasoning in mathematics to be the same as its use in other domains or in daily life. In contrast, those emphasizing formal logic as the basis of deductive reasoning, distinguished between mathematics and other domains in the usability of deductive reasoning. The latter view is in line with the argumentation literature (Duval, 2002, Krummheuer, 1995; Mariotti, 2006; Toulmin, 1969; Voss & Van Dyke, 2001), where it is frequently claimed that *rationality*, in the sense of "taking the best choice out of a set of options whereby what counts as the best is a matter of negotiation" (Krummheuer, 1995, p. 229), better describes reasoning in real life situations (Toulmin, 1969). Thus, instead of analytical arguments (i.e., based on formal logical deduction), substantial arguments (Toulmin, 1969), which do not have the logical rigidity of formal deductions, are claimed to often be more suitable. Some of the study participants were moderate in their approach, claiming that in non-mathematical situations people apply other “softer” rules of inference in addition to the rigorous formal ones. The other participants were more radical and claimed that people do not, or even cannot, use deductive reasoning in non-mathematical contexts.

METHODOLOGY

Research participants

Twenty-one people participated in the research. Most of them (17) belonged to different sub-communities in the field of mathematics education. This group was chosen to be as heterogeneous as possible in terms of the kinds of involvement they had in mathematics education, in order to increase the potential of diversity in their approaches. The group included mathematics teachers at various levels (from secondary school teachers to research mathematicians who teach undergraduate or graduate university mathematics), curriculum developers, pre- and in-service teacher educators, and researchers in mathematics education. Naturally, some of these participants belonged to several sub-communities (i.e., a curriculum developer who was also a teacher educator, and so on). All the participants had a reputation of being experienced and knowledgeable in their respective fields, all had solid university or college education in mathematics; many also in mathematics education.

The four remaining participants out of the 21 were not connected to mathematics. They were chosen because their deep knowledge in issues related to logic and deductive reasoning. The aim was to enrich the data in order to contribute to the analysis and interpretation of the approaches of the math participants, which are the focus of the study. Two of these participants were logicians in the philosophy department of a leading university; the other two were university researchers in science education who had a long history of studying students' development of logical thinking.

Data collection and analysis

The data sources were individual semi-structured interviews that lasted between one to two hours. They focused on different issues related to the role of learning mathematics in developing general deductive reasoning (that is not restricted to mathematics, but can be used in different fields and situations). For example, What does the concept of deductive reasoning mean to you? To what extent do you think deductive reasoning is significant in our lives? Where? Why? How do you perceive the connections between learning mathematics and the development of general deductive reasoning? How, if at all possible, can deductive reasoning be improved through learning mathematics? Probing during the interviews aimed at elaboration and explanation of general statements, continuously asking the interviewees to give specific examples from their own experiences.

Data analysis was based on the Grounded Theory method (Strauss & Corbin, 1990). Thus, no prior assumptions were made regarding the interviewees' opinions and approaches, or regarding possible differences or similarities among different sub groups. The interviews were transcribed and read carefully several times in their entirety, in no specific order. We then used open coding to generate initial categories. For example, the significance of deductive reasoning, the role and use of deductive reasoning in daily life, the likelihood of developing deductive reasoning. The initial categories were constantly compared with new data from the interviews and refined. We identified relationships and hierarchies among the categories, and created core categories which are "the central phenomenon around which all the other categories are related" (Strauss & Corbin, 1990, p. 116). We used the core categories as a source for theoretical constructs. One of the categories that was developed through this process and is discussed in this paper is the view regarding the claim that learning mathematics can develop deductive reasoning.

VIEWS ON THE ROLE OF MATHEMATICS LEARNING IN THE DEVELOPMENT OF DEDUCTIVE REASONING

All interviewees said that developing deductive reasoning is one of the goals of mathematics teaching. However, this statement had different meanings for different interviewees. Three distinct views regarding the role mathematics learning could play in the development of deductive reasoning were identified. Each interviewee's view appeared to be consistent throughout the interview, and was interrelated with the interviewee's approach to deductive reasoning and its nature in mathematics and outside it, as identified in Ayalon and Even (2006).

View 1: Intervention

Most of the interviewees, 13 of them (several of the following: school teachers, teacher educators, curriculum developers, mathematicians, and researchers in mathematics education, and all four non-mathematics participants), held the *intervention* view. They claimed that there should be a deliberate intervention in the process of teaching mathematics in order to achieve significant improvement of deductive reasoning. As found in Ayalon and Even (2006), all these interviewees approached deductive reasoning as an action of inference based on formal logic rules, and claimed that different kinds of factors affect reasoning outside mathematics.

Thus, they argued, in non-mathematical situations people apply "softer" rules of inference, in addition to the rigorous formal ones that are used in mathematics.

When addressing the question whether mathematics learning can develop deductive reasoning, they responded positively, but replaced the term deductive reasoning with terms related to argumentative habits of mind and skills, such as, justifying, articulating claims, and evaluating arguments. These interviewees referred to mathematics as a suitable subject matter to teach argumentative habits of mind and skills that students could use also in discourses other than mathematics, where, these interviewees said, the connections between claims and their supporting reasons are not necessarily deductive. For example, an interviewee was asked whether she thinks learning mathematics could assist in promoting deductive reasoning that can be implemented in domains other than mathematics. In her reply she distinguished between the nature of reasoning in mathematics and in other domains, but also pointed out that some aspects are common to both:

> Look, developing the ability to think logically, to think deductively, has been always presented as a central goal of mathematics education. But first we have to define exactly what it means to promote such abilities... When I think about this goal, I think that mathematics learning can contribute to the improvement of habits such as providing reasons for one's views and evaluating others' views. Now, in mathematics the arguments may have to be more deductive. But the habits of giving reasons and examining arguments could still be used in informal discussions such as conversation in family dinners or in listening to and reading the news. The important thing in these kinds of conversation is not the deductiveness of arguments, but the fact that these arguments should be clearly articulated, with sound reasons, not in an ambiguous way. So I believe that mathematics education should direct itself towards the enhancement of these skills (interviewee no. 10).

These interviewees viewed the development of argumentative habits as something that requires an explicit attention during the process of instruction. They thought that mathematics is indeed a suitable subject matter for enhancing argumentative skills. However, many of these interviewees claimed that talk in classrooms is commonly non-argumentative in its nature, and thus does not support the development of argumentative habits of mind and skills, such as, justifying claims, listening to others, following others' reasoning, determining whether what was presented made sense and why, voicing disagreement and providing reasons for it. These interviewees pointed out the crucial role that the teacher plays in creating an atmosphere that encourages practices and norms of argumentation. For example,

> In order to develop habits of argumentation among the children, the mathematics teacher should invite students to participate in discussions, to justify their thinking. She should teach them to listen to their friends and to attempt to understand their explanations and to intervene when they think that these explanations are wrong (interviewee no. 9).

View 2: Reservation

Four interviewees (three researchers in mathematics education, one of which is also a mathematician, another is also a curriculum developer, and one curriculum developer who is also a teacher educator) held the *reservation* approach. They claimed that learning mathematics might have an influence on students' deductive reasoning, and that its development is a valued goal of mathematics education. However, they found it hard to point out how this goal might be achieved, and questioned the possibility of actually achieving it, as the following excerpt exemplifies:

> Well, it is a difficult question. I think it is a worthy goal of mathematics to promote deductive reasoning. Even if we do not use deductive arguments outside mathematics it is important that we all be familiar with the logical rules so we can value our ideas against logic standards… It also looks like something that can be done. However, I don't know of any method to achieve this goal, and I doubt if there exist any at all (interviewee no. 2).

As found in Ayalon and Even (2006), like the intervention group, all these interviewees approached deductive reasoning as an action of inference based on formal logic rules. However, unlike the intervention group's interviewees, these interviewees claimed that outside mathematical context, people do not, or even cannot, use deductive reasoning. Similarly, as exemplified above, when addressing the question whether learning mathematics develops deductive reasoning that can be used in non-mathematical situations, these interviewees revealed hesitation and deep reservation.

View 3: Spontaneity

Four interviewees (two school teachers and two curriculum developers, one of which is also a teacher educator) held the *spontaneity* view. They regarded learning mathematics as spontaneously improves students' deductive reasoning, and claimed that there is no need for deliberate intervention to achieve such an improvement. Yet, as found in Ayalon & Even (2006), these interviewees approached deductive reasoning as a systematic step-by-step manner for solving problems both in mathematics and in other domains, with no attention to the logical validity of the inference. Thus, when discussing the development of deductive reasoning, these interviewees talked about developing systematic habits of mind and ascribed this development to the systematic structure of mathematics and to the methodical, step-by-step manner of solving mathematical problems. According to them, doing mathematics provides experiences in working systematically, and consequently encourages the spontaneous formation of students' systematic habits of mind. For example, an interviewee was asked whether learning mathematics could improve deductive reasoning. She replied:

> I think that mathematics improves deductive reasoning, and that it is one of mathematics' main goals… It teaches them [the students] to be methodical, that everything requires a method and that there is order, and these are very important values… For example, in solving a problem of personal relationships in work or in the family, and also in making plans in the economical field of expenses and incomes, of getting out of debt, buying an apartment – how to get organized. It helps a lot, organizes your thinking, and helps in building a systematic procedure of how to reach the sum of money you need, how to save (interviewee no. 11).

Later she emphasized that this learning to be organized and work systematically happens spontaneously, simply as a result of experiencing the learning of mathematics:

> When you learn mathematics and practice it you internalize systematic principles: to read a problem, to extract from it the relevant data, to examine them one by one to see how they relate to the question posed in the problem, to plan a method for solving it, to progress step by step in an organized way. The student internalizes this systematic approach to problem solving. By the mere mathematical practice the student develops this attitude. And these principles would become usable when the student faces problems also in non-mathematical contexts. It might be that the nature of the data and the problem will be different, but the ability to organize, to systematically examine the situation, the possible ways of solutions – all these could be actualized. And that happens mostly thanks to learning mathematics (interviewee no. 11).

CONCLUSION

In contrast with common statements in curriculum guidelines, and their own, that one of the goals of learning mathematics is the development of deductive reasoning that is not restricted to mathematics but can be used in other domains as well, the study participants did not view mathematics learning as an adequate means for developing deductive reasoning (logic based inference), but for different reasons. Three views were identified among the interviewees. These views were interrelated with the interviewees' approaches to deductive reasoning and its use and usability in mathematics and outside it. One group of interviewees (the reservation group) approached deductive reasoning as an action of inference based on formal logic rules that is useful and usable in mathematical contexts only. They doubted straightforwardly the possibility of achieving the goal of developing deductive reasoning that is not restricted to mathematics. Another group (the intervention group) approached deductive reasoning as an action of inference based on formal logic rules that is used outside mathematics alongside "softer" rules of inference. They claimed that there is a need for explicit intervention during the process of instruction in order to achieve the goal of developing deductive reasoning, but modified the goal from developing deductive reasoning to developing argumentative habits and skills, in ways that do not necessarily comply with rigorous deduction. Still, another group (the spontaneity group) approached deductive reasoning as a systematic step-by-step manner for solving problems both in mathematics and in other domains, with no attention to the logical validity of the inference. They claimed that learning mathematics develops systematic habits of mind spontaneously.

The findings of this study show that the goal of using mathematics learning to develop deductive reasoning has different meanings for different mathematics education practitioners.

These different interpretations shape *what* these people try to develop when they teach mathematics, design teaching and learning materials, or prepare teachers to teach mathematics (e.g., systematic habits of mind, argumentation habits and skills, the use of formal logic to make and evaluate inferences). The different interpretations shape also *whether* and *how* these people try to do that (e.g., provide experiences in mathematics problem solving, encourage justification and evaluation of arguments, provide experiences in the use of deduction).

Our study raises several issues for future research. E.g., What would happen when a teacher with a systematic approach teach a logic-based curriculum? Or vice versa? Do specific sub-groups of mathematics educators tend to approach the development of deductive reasoning in different ways? Do different countries do?

References

Ayalon, M. & Even R. (2006). Deductive reasoning: Different conceptions and approaches. In J. Novotna (Ed.), *Proc. 30th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol 2, pp. 89-96). Prague, Czech Republic.

Duval, R. (2002). Proof understanding in mathematics. In *Proceedings of 2002 International Conference on Mathematics: Understanding Proving and Proving to Understand* (pp. 23-44). National Taiwan Normal University. Retrieved from: http://www.math.ntnu.edu.tw/~cyc/_private/mathedu/me1/me1_2002_1/duval.doc

Johnson-Laird, P. N. & Byrne, R. M. J. (1991). *Deduction*. Hillsdale, NJ: Erlbaum.

Krummheuer, G. (1995). The ethnography of argumentation. In P. Cobb, & H. Bauersfeld (Eds.), *The Emergence of Mathematical Meaning: Interaction in Classroom Cultures* (pp. 229-269). Hillsdale, NJ: Erlbaum.

Luria, A. R. (1976). *Cognitive Development: Its Cultural and Social Foundations*. Cambridge, MA: Harvard University Press.

Mariotti, M. A. (2006). Proof and proving in mathematics education. In A. Gutierrez and P. Boero (Eds.), *Handbook of Research on the Psychology of Mathematics Education* (pp. 173-203). Rotterdam: Sense.

Nisbett, R. E., Fong, G. T., Lehman, D. R., & Cheng, P. W. (1987). Teaching reasoning. *Science, 238,* 625-631.

Qualifications and Curriculum Authority (2006). *National Curriculum Online*. Retrieved from http://www.qca.org.uk

Strauss, A. & Corbin, J. (1990). *Basics of Qualitative Research: Grounded Theory Procedures and Techniques.* Thousand Oaks, CA: Sage.

Toulmin, S. E. (1969). *The Uses of Arguments.* Cambridge, UK: Cambridge University Press.

Voss J. F. & Van Dyke, J. A. (2001). Argumentation in psychology: Background comments. *Discourse Processes, 32*, 89-111.

Wu, H. (1996). The role of Euclidean geometry in high school. *Journal of Mathematical Behavior, 15,* 221-237.

BRIDGING ARITHMETIC AND ALGEBRA: EVOLUTION OF A TEACHING SEQUENCE[1]

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The aim of this paper is to describe the evolution of a teaching learning sequence for grade 6 students beginning algebra learning over a period of two years that included multiple trials. The teaching learning sequence was designed to enable the students to make a transition to algebra from arithmetic by connecting their prior knowledge of arithmetic and operations and exploiting the structure of arithmetic expressions. In the process, the study aimed to identify the concepts, rules and procedures which facilitate the connection between arithmetic and algebra and enable the transition. The repeated trials allowed us to see the potential of the two concepts 'term' and 'equality' identified during the study and the nature of tasks that help in making the connection between the two domains.

INTRODUCTION

Researchers in algebra education have suggested a variety of approaches for introducing algebra. One set of approaches introduces symbolic algebra to students in the lower secondary grades through generalized arithmetic, emphasizing structure of arithmetic expressions and replacing the number by the letter to represent generalized rules and properties of operations in arithmetic (e.g. Thompson and Thompson, 1987; Liebenberg et al., 1999; Livneh and Linchevski, 2003). Much of this research has focused on building a sense of the structure of arithmetic and algebraic expressions among students. Earlier exploratory students (e.g. Chaiklin and Lesgold, 1984; Linchevski and Livneh, 1999) had shown that the lack of the understanding of structure was a major factor in not understanding the manipulation of algebraic expressions. Although the teaching studies just mentioned identified important elements of a beginning algebra curriculum, they have not yielded a well elaborated model of teaching and learning of algebra using arithmetic as the base. Some of these studies suggested the need to focus away from computation to be an important criterion for transiting to algebra from arithmetic (e.g. Liebenberg et al. 1999). Elsewhere, we have reported aspects of a teaching approach that aimed to develop a structural understanding of arithmetic expressions (Subramaniam and Banerjee, 2004, Banerjee and Subramaniam, 2005). In this paper, we describe the evolution of the teaching approach as part of a design experiment, highlighting the changes and the decisions made and the reasons for these decisions.

THE RESEARCH STUDY

In a two year long study involving a design experiment methodology (Cobb et al., 2003), we developed a teaching approach to learning algebra using students' prior

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knowledge of arithmetic and operations. The approach aimed to build a strong structure and procedure sense of arithmetic and algebraic expressions by giving visual and conceptual support. In the process, we wanted to identify the nature of concepts, rules and procedures which would facilitate building the connection between the two domains. The study started with only a conjecture about the possibility of using the structure of arithmetic for teaching algebra and the many assumptions had to be progressively tested in order to build the sequence. The teaching learning sequence co-evolved with the developing understanding of the researchers about the phenomena under study as well as with the growing understanding of the students as evidenced from their performance and reasoning on various tasks. After each trial, the strengths and limitations of the concepts, ideas and tasks were identified leading to suitable modification of the sequence in the next trial of teaching.

The study was conducted with grade 6 students from nearby English and vernacular medium schools during vacation periods in summer and mid-year. Each trial had two to three student groups, with each group receiving 11-15 days of teaching, 1.5 hours per day. The teaching sequence, which included concepts and task that went well beyond those introduced in the school, was developed over five trials between 2003-2005 with the first two trials being exploratory in nature and considered pilot trials (PST-I and PST-II) and the last three trials forming the main study (MST-I, MST-II, MST-III). Different groups of students attended the pilot trials whereas the same students who attended MST-I were invited for MST-II and III. The data was collected through students' performance in pre and post tests, interviews, teachers' daily logs and video recordings of classroom discussions.

THE TEACHING CYCLE

The evolution of the teaching approach was similar to the 'mathematics teaching cycle' and the '*hypothetical learning trajectory*' described by Simon (1995). The approach was developed keeping in mind the insights from the literature, using arithmetic as a 'template' to build the new algebraic symbolism. The main focus of the sequence was to move the students away from a sequential, procedural understanding of expressions to a relational, structural understanding, which is important for algebra. Besides learning to parse expressions correctly, developing understanding of structure of expressions requires students to turn the processes of computation into 'objects' (Sfard, 1991) or flexible 'procepts' (Tall et al., 2000). This would allow them to think mentally about operations, suspend computations, anticipate the outcome of actions and attend to the relations within components of the expressions as well as between two expressions. The sequence tried to achieve this gradually by creating appropriate learning tasks, and by identifying concepts, rules and procedures, together with visual and verbal support which could consolidate the reification of the processes of arithmetic.

A teaching sequence was constructed for the first trial which aimed at identifying instructional material as well as testing their efficacy, sequencing and identifying pre-requisite concepts or skills needed for developing structure sense among students.

Tasks were chosen, adapted and modified from the existing literature for the trials. Students' intuitive as well as formal ideas about operations, symbols and procedures were given due importance in the classroom, allowing the students to articulate their reasoning, so as to be able to build on them. During the enactment of the teaching sequence in the classroom, the students were engaged in making sense of the tasks and the responses expected (e.g. that they have to explain their solution, that they have to understand the explanations given by others) and the teachers were engaged in observing and making sense of the students' responses and actions. This not only led to changes in the subsequent trials but also small immediate changes, with regard to examples and explanations in the same trial.

In the following paragraphs, we give an account of the processes that led to the evolution of the teaching approach and the rationale for emphasizing certain concepts/ ideas and choosing and changing some of the tasks.

THE PILOT TRIALS

The first two trials (PST-I and PST-II) explored how students' knowledge of arithmetic could be harnessed as a preparation for symbolic algebra. We began the trials with the understanding that procedure and structure sense are two separate pieces of knowledge and building the structure sense is enough to make the transition to algebra. But as we tried out the instructional sequence in the first trial, we found that building of the structure sense itself required adequate procedural understanding. This led us to include tasks which strengthened students' procedural knowledge, like working with brackets and later integer operations as well.

One of the goals of the first trial was to move the students away from a computational understanding of expressions towards a relational understanding. This was the first step towards attending to the structure of the expressions and appreciating the duality: that the expression stands for a number which is the value of the expression and that all the expressions for a number 'express' different information about the number, in the form of a relationship among two or more numbers. For example, students learnt that the expression 5 + 8 stands for the number 13 and conveys the information that it is '8 more than 5'. Many other phrases like 'more than', 'sum', 'difference between', 'less than', 'product of', 'times' and 'quotient' were introduced.

Rules of evaluating simple expressions, like 13 – 5 + 8 and 6 + 2 × 4, were explained to them in the traditional fashion by explicating the precedence rules (giving precedence to '×' operation and computing from left to right) and strengthened using the meaning of the expressions. For example, 9 – 3 + 4 is four more than the difference between nine and three whereas 9 – (3 + 4) is difference between nine and the sum of three and four, suggesting the difference in the way the computation is be carried out.

Another goal of the first trial was to develop among students an ability to judge equality of expressions without computation. However first, their understanding of the '=' sign needed to be broadened. They then compared expressions which could be seen as related such as 27+32 and 29+30. Although students were able to view expressions relationally, we saw overgeneralizations from the addition context to expressions with a negative sign. Such situations required students to keep track of the transformations on the number for which they did not have the resources, like the use of brackets. For expressions with brackets, simple bracket opening rules were introduced with the use of phrases like 'adding/ subtracting a sum or difference'. For example, 12 – (6 + 4) and 12 – 6 – 4 are equal and one can subtract the sum of 6 and 4 from 12 or subtract them one by one as in the expression 12 – 6 – 4. Other tasks included finding the value of an expression given the value of a related expression (find 228+149 if 227+148=375). The students were expected to explain their answers verbally. Attempts by the teacher to help students with symbolic justifications were not very successful. As students worked on these tasks, the concept of 'term' was introduced as a component of an expression (e.g. in 12 + 4 – 3 the terms are +12, +4, –3), and the students soon learnt by verification that the value of an expression remains the same on rearranging the terms. This concept not only helped the students parse an expression correctly but also allowed them to see relationships between the terms and with the expression as a whole, leading to the important idea of 'equal expressions'. Thus, 'terms' and 'equality' were the two key concepts identified during the first trial.

The second trial sought to build this sequence by extending it to include algebraic expressions. It had a two group design: students who were taught algebra together with the approach to arithmetic expressions as outlined above and a group who were taught algebra without any arithmetic beyond the instruction in school. The first group of students who worked on both arithmetic and algebra were taught the concept of term immediately after dealing with the procedures of evaluating arithmetic expressions. Terms were categorized into 'simple term' (e.g. +3, –4) and 'product term' (e.g. $+3\times4$, $+2\times y$). But the use of terms was restricted to tasks of comparison of expressions. In contrast to the group which had been exposed to only algebra, this group of students performed better in both procedures of evaluating expressions and using the surface structure of the expressions to identify and generate equal expressions, where terms and numbers were rearranged, in both arithmetic and algebra. However, the appreciation of surface structure did not allow abstraction of procedures to manipulate algebraic expressions, which needed a deeper understanding of rules and properties of operations. On retrospect, we realized that the procedures used with arithmetic expressions for evaluation and with algebraic expressions for simplification (by collecting like terms) were disparate, not allowing for transfer between the two, many students making the conjoining error ($3+5\times x=8\times x$) due to non-appreciation of the constraints on operation. Also, students were introduced to bracket opening rules by embedding them in story situations which could lead to two ways of representing and solving. This proved to be quite cumbersome and did not succeed in explicating the structure of the expressions.

At the end of these two trials, it was evident that although strengthening the understanding of arithmetic was helpful in making sense of algebra and rules of transformation of algebraic symbols, there was a need to make the sequence more coherent and bridge the gaps between procedure and structure and between arithmetic and algebra, so that the understanding developed in the context of arithmetic could be fruitfully used in the context of algebra (see Subramaniam and Banerjee, 2004). There was also a need to pay attention to negative numbers and bracket opening rules.

MAIN STUDY TRIALS

The three main study trials (MST-I, II and III) were carried out with two fresh groups of students. The students came soon after appearing for their grade 5 exams for MST-I (Summer, 2004), were in the middle of grade 6 during MST-II (mid-year vacations, 2004) and finished grade 6 during MST-III (Summer, 2005). These trials were aimed at achieving better coherence in the teaching learning sequence. In all the three trials, the concept of 'term' was introduced in the beginning and was used for both procedural and structural tasks in an increasingly integrated manner.

Students were introduced to the idea of 'terms' of an expression immediately after developing an understanding of expressions in MST-I. Terms were made visually salient by putting them in boxes (e.g. terms of 19 – 7 + 4 are $\boxed{+19}$ $\boxed{-7}$ $\boxed{+4}$) and were used to decide the precedence rule to be applied for evaluating arithmetic expressions and to identify like terms in the context of algebraic expressions. They were subsequently used to compare expressions, identify and generate equal expressions as earlier. Students again failed to make the connection between the simplification procedures of arithmetic and algebraic expressions due to the persisting disparity as in PST-II. Some efforts to make the connection explicit included evaluating algebraic expressions for a given value of the letter (e.g. $5+4\times x$, $x=2$) and finding easy ways of evaluating expressions like 28-17-8+17, emphasizing non-sequential computation. These efforts were not entirely successful partly due to the rigidity of the rules of evaluation. Rules for transformation of expressions with brackets (+ and '–' to the left of the bracket) were connected to the idea of equal expressions verified through computations. Area of a rectangle model was used for distributive property. Number line and letter-number line were used to give meaning to the integers and the letter. The letter-number line served the dual purpose of understanding expressions like x-1 as denoting a number by means of a relation (a number which is one less than x) and the process of decrementing 'x'. It could further be used in tasks like the journey on the letter-number line and finding the distance between two points on the letter-number line, both of which required the students to create a representation and manipulate it.

As we have noted, a strong connection between the procedural and the structural components of the expressions was still lacking in MST-I. The students also could not use their knowledge of rules of transformation in contexts where algebra was being used as a tool for justification (like, think-of-a-number game). It was clear that simply the presence of structural notions and explicating the surface structure is not sufficient to make the connection between procedure and structure and between arithmetic and algebra. The structural notions had to be used differently in such a manner that one could reflect on possibilities and constraints on operations, enhancing flexibility and anticipation with respect to results of carrying out operations.

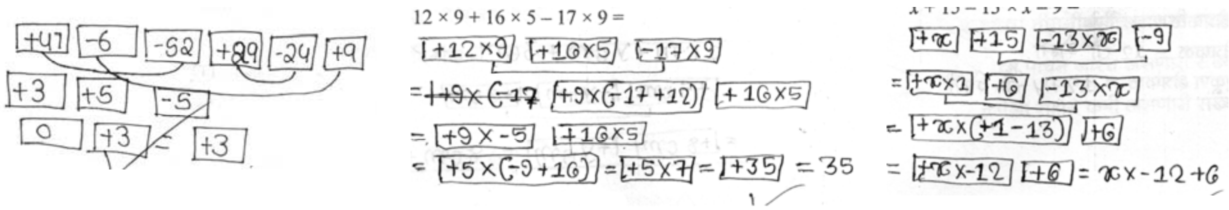

Figure 1. Illustration of students' solutions while evaluating/ simplifying expressions.

In the second main study trial (MST-II), terms and equality were made more central to the teaching sequence and the approach was made radically structural. Terms were now classified as simple and complex terms (examples of the latter are product term, bracket term). The procedures for combining terms for evaluating expressions were introduced as a structural reinterpretation of the precedence rules. The rules of evaluation were made flexible by including the idea of combining terms in any order, thus subsuming integer addition. Positive terms increased the value of the arithmetic expression, while negative terms decreased the value. A product term needed to be converted into a simple term before combining with other simple terms. Two product terms with a common factor could be combined using the distributive property. This paved the way for integrating the transformation rules of arithmetic and algebraic expressions (where this flexibility and non-sequential computation is essential) as well as complement procedure sense with structure sense. Figure 1 illustrates the flexible ways in which students evaluated/ simplified expressions as they learnt this approach. The complementary nature of procedure and structure was strengthened by the tasks of finding easy ways of evaluating expressions and generating expressions equal to a given expression (both arithmetic and algebra) using various transformations, requiring abilities to mentally operate in forward and reverse direction. Even the bracket opening rules were reformulated using 'terms' and 'equality' in conjunction with ideas of 'inverse' (taking care of the integer subtraction) and 'multiple'. This evolved sequence was called the 'terms approach' and gave the students the vocabulary and visual and conceptual support to reason about the syntactic based transformations. The two structural concepts of 'term' and 'equality' and the reformulation of the rules of transformation enabled the students to consider the arithmetic processes as potential processes which could be suspended for a while and combined with other terms based on structural relations. Further, generating equal expressions separated the denotation from the meaning of the expressions, the transformations keeping the value same but changing the surface structure of the expressions.

The last trial (MST-III) aimed to consolidate the teaching learning sequence focusing on students' verbalization and articulation of various concepts and rules and their use in different contexts. Evaluation of expressions with brackets (e.g. 23-(4+5×3)) together with understanding general principles of keeping the value of an expression invariant were the focus of this trial. Also a fair amount of time was spent on tasks which embedded the use of algebra in contexts requiring generalizing and proving/ justifying (Think-of-a-number game, pattern generalization from growing patterns). Building on our earlier observations of students' inability to use their knowledge of syntactic transformations in such contexts, students were engaged in verbalization of explanations of the answers before introducing symbolic justifications. These activities led to fruitful discussions about semantic and syntactic aspects of algebra: meaning of letters, correct representation and proper use of brackets and generalization from particular instances ('seeing the general in the particular') and goal directed manipulation of expressions. The study ended after this trial with indications that the transfer to 'reasoning *with* expressions' in context is not trivial but 'reasoning *about* expressions' in the course of working with syntax based transformations can play a part in predisposing students to think about situations with the help of expressions.

Clinical interviews were conducted with a subset of students after MST-II (14) and III (17) to probe the robustness of their understanding. The interviews revealed students' ability to appropriately articulate the reason for the incorrectness of the solution of an expression like 22-7+9 = 22-16 by pointing out the need for a bracket around 7 and 9 for the above solution to be correct or that -7+9=+2. Probing specific abilities of students with respect to simplification of algebraic expressions (e.g. $5\times a+6-2\times a+9$) at the end of MST-III, almost all students were able to convincingly explain the procedure of simplification by drawing on their knowledge of evaluating arithmetic expressions. They stated the rules for combining terms to explain why expressions like $3+5\times x$ cannot be simplified further. Also, eleven of the students understood that each step in the simplification process yields equivalent expressions. The remaining six students needed to calculate the simplified and the original expression to arrive at this conclusion, generalizing their understanding from evaluating arithmetic expressions with similar structure.

CONCLUSIONS

The design experiment led to the development of a teaching learning sequence with the potential to bridge the gap between arithmetic and symbolic algebra for students beginning algebra learning. Through a long term engagement with the process and our own reflections on the assumptions and the tasks, the study helped us understand the nature of arithmetic and the tasks required to make the transition possible. The transition is not a trivial affair and the connection is not spontaneously seen by the students. Using arithmetic as a template, and enhancing both computational as well as non-computational reflective understanding of operations and their properties by the use of two structural concepts 'term' and 'equality' enabled the students to develop a new symbolic system of algebra and simple operations on them. The 'radicalized' structural treatment created meaning for the symbols in the context of syntactic transformation and allowed us to convert the processes of addition, subtraction, multiplication into 'objects' which could feed into the development of the algebraic symbols.

References

Banerjee, R. & Subramaniam, K. (2005). Developing procedure and structure sense of arithmetic expressions. In H. L. Chick & J. L. Vincent (Eds.), *Proc. 29th Conf. of* *the Int. Group for the Psychology of Mathematics Education,* (Vol. 2, pp.121-128), Melbourne, Australia: PME.

Chaiklin, S. & Lesgold, S. (1984). *Prealgebra Students' Knowledge of Algebraic Tasks with Arithmetic Expressions.* Paper presented at the annual meeting of the American Research Association.

Cobb, P., Confrey, J., diSessa, A., Lehrer, R., & Schuble, L. (2003). Design experiments in educational research. *Educational Researcher*, *32*, 9-13.

Liebenberg, R. Linchevski, L. Sasman, M. C., & Olivier, A. (1999). Focusing on the structural aspects of numerical expressions. In J. Kuiper (Ed.), *Proc. 7th Annual Conference of the Southern African Association for Research in Mathematics and Science Education* (SAARMSE) (pp. 249-256). Harare, Zimbabwe.

Linchevski, L. & Livneh, D. (1999) Structure sense: The relationship between algebraic and numerical contexts. *Educational Studies in Mathematics*, *40*, 173-196.

Livneh, D. & Linchevski, L. (2003). *Teaching Arithmetic for Algebraic Purposes: Does it Work?* Paper presented at CERME 3, Bellaria, Italy.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, *22*, 1-36.

Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education*, *26*(2), 114-145.

Subramaniam, K. & Banerjee, R. (2004). Teaching arithmetic and algberaic expressions. In M. J. Hoines & A. B. Fuglestad (eds), *Proc. 28th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 3, pp. 121-128). Bergen, Norway: PME.

Tall, D., Thomas, M., Davis, Gray, E., & Simpson, A. (2000). What is the object of encapsulation of a process? *Journal of Mathematical Behavior*, *18*(2), pp. 223-241.

Thompson, P. W. & Thompson, A. G. (1987). Computer presentations of structure in algebra. In J. Bergeron, N. Herscovics, & C. Kieran (Eds.), *Proc. 11th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol 1. 248-254). Montreal, Canada.

HYBRID DISCOURSE IN MATHEMATICIANS' TALK: THE CASE OF THE HYPER BAGEL

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One issue that has arisen in research on the nature and role of mathematical discourse in thinking about Mathematics is to better understand the relationship between everyday discourse and mathematical discourse. Little, if any, of this work has, however, examined how mathematicians talk about mathematics. In this paper, I use ideas from discursive psychology to analyse an example of mathematicians' talk, taken from a live radio broadcast about the Poincaré conjecture. My analysis highlights some of the discursive resources the mathematicians draw on in their thinking. Moreover, my analysis suggests that these mathematicians' talk is an example of a hybrid discourse incorporating both the mathematical and the everyday.

INTRODUCTION

There has been a good deal of research examining different aspects of the nature of mathematical discourse and its role in the teaching and learning of mathematics (for a summary, see Barwell, 2008). This work is fairly diverse in scope, ranging from concerns with understanding how classroom interaction shapes students' learning, to specific concerns about multilingual or second language learners, to the nature of formal academic mathematical discourse. For this paper, however, I want to highlight one issue in particular. This issue concerns the problem of defining mathematical discourse and its relation with 'everyday' discourse. While it seems reasonable to think of mathematical discourse as the way language, gestures, symbols and other resources are used in doing mathematics, it is not really possible to demarcate clear boundaries with everyday (i.e. supposedly non-mathematical) discourse (Moschkovich, 2003, 2007). As Moschkovich (2007) implies, it seems questionable to assume that mathematical discourse consists only of formal practices, such as formal definitions. She hints at how more informal, everyday discourse has a role in how students make sense of mathematics. Indeed it is likely that informal, everyday discourse has a role in how professional mathematicians think about mathematics. It would, therefore, be useful to examine examples of mathematicians talking about mathematics in a variety of settings to explore if and how they draw on the everyday. Research on mathematical discourse has, however, generally analysed mathematics *classroom* discourse, usually spoken, occasionally written (e.g. Pimm, 1987; Kieran *et al*., 2002). Fewer studies have looked at the nature of mathematical discourse produced by professional mathematicians and most, if not all, of this work looks at formal *written* mathematics (e.g. O'Halloran, 2005). There appear to be few examples of analyses of mathematicians' *talk*. In this paper, then, I report some initial analysis of an example of spontaneously produced mathematicians' talk from a live radio broadcast. This analysis is framed by discursive psychology, which is outlined in the next section.

DISCURSIVE PSYCHOLOGY

Discursive psychology (e.g. Edwards, 1997; Edwards & Potter, 1992) has been described as offering an anti-cognitivist, anti-realist, anti-structuralist account of the relationship between discourse and cognitive processes, such as thinking, meaning or remembering [1]. This perspective involves a shift in theoretical and analytic focus from 'what happens in the mind' to how 'what happens in the mind' is discursively accomplished. This shift is based on the principle that 'what happens in the mind' is played out through talk or other forms of interaction. What participants in interaction think or know or believe are, as Edwards (1997) puts it, topics of discourse. From this perspective, for example, the cross-examination of a witness in a courtroom, although ostensibly a process of establishing a set of pre-existing facts and rememberings, can instead be understood as a process of *constructing* these facts. A key principle, then, is that versions of reality are situated – the accounts given by a witness are shaped by the court's proceedings and as such are jointly produced. In the case of mathematics, mathematical thinking can also be seen as situated, with particular versions of mathematical reality being produced and accounted for in different circumstances (Barwell, 2007). Furthermore, these processes are seen as primarily social, rather than cognitive. That is, the organization of different versions is a response to the unfolding social situation, which is itself reflexively produced through the interaction.

The inevitable variation that arises from the situated discursive construction of cognitive processes provides a way into analysis (Wetherell & Potter, 1992). By looking for variation, the researcher can begin to explore, for example, how participants' accounts of experiences vary in response to potential challenges, questions or counter-stories. Analysis of variation in turn leads to the identification of discursive resources: different ways of talking and organising discourse which can be deployed to, for example, head off challenges or accusations. The idea of discursive resources provides a link between the broader patterns of language use into which human beings are socialised and the here-and-now of particular moments of interaction. The emphasis, therefore, is on how language is used and on the effects that different kinds of language use have, rather than on listing and categorising different resources.

How then, do mathematicians present their ideas to non-mathematicians through talk? How do they use the everyday as a discursive resource to talk about mathematics? What are some of the other resources they use? How are these resources used? What effects do they have?

DATA: MATHEMATICAL RADIO BROADCASTS

The data for this study come from a set of radio broadcasts from the UK. The data consists of five programs, of which, in this paper, I will discuss one. The programs come from a regular series broadcast on BBC radio called *In Our Time*. The series, hosted by Melvyn Bragg, according to its website [2], concerns 'the history of ideas'. A wide range of topics has been addressed, in the fields of religion, philosophy, history, culture and science. The format of the program is that three experts on the week's topic (usually academics) are invited. Bragg introduces the topic and the speakers. He then chairs a live, unscripted discussion lasting 45 minutes. Although unscripted, the discussion is fairly tightly structured, usually chronologically, by Bragg. Thus the early part of the program might summarise the origins of an idea or the career of a key actor. The development of these ideas is then traced through time so that the last few minutes of the program generally concern issues of contemporary relevance. Bragg prompts the contributors to talk about particular aspects of the topic. There is often a reasonable degree of interaction between all the participants, particularly as the program unfolds. Nevertheless, the format invites fairly extended contributions from participants from time to time. For this study, I have selected five broadcasts with a mathematical theme: they are about symmetry, negative numbers, pi, prime numbers and the Poincaré conjecture. These broadcasts have been transcribed.

The analysis reported in this paper is of the broadcast about the Poincaré conjecture. The episode features Ian Stewart (Warwick University), Marcus du Sautoy (Oxford University) and June Barrow-Green (Open University). This episode was selected for initial analysis because of the particularly advanced nature of the mathematics of the topic. The program begins with some biographical material about Poincaré followed by some summarising of his significance and contribution to mathematics. The origins of topology are introduced and the middle part of the program involves some elaboration on what topology is about. This material leads into an account of what the conjecture itself is about. Finally, the last segment of the program gives a sketch of a recent possible proof of the conjecture.

The analysis reported in this paper is presented in two parts. In the first part, an overview is given of some of the discursive resources apparent in the program, along with necessarily brief illustrations. In the second part, I present a more extended extract from the data to illustrate these resources and to enable some examination of how they are used and to what effect.

ANALYSIS

First, then, I will highlight three particular discursive resources that arose from noticing variations of different kinds. The purpose here is not to provide an exhaustive list of all the discursive resources apparent in the program, but rather to highlight some that seem to be more significant in constructing a version of mathematical reality.

1. *Narrative-like forms*: the presentation of ideas in chronological sequence and involving some semblance of a situation, of actors and of a situation that needs to be resolved. One of the functions of narrative is to implicitly provide a sense of motive for human action (Bruner, 1996). At one point, for example, du Sautoy

describes the Konnigsburg bridge problem and Euler's solution of it, in the form of a vignette about the people of Konnigsburg amusing themselves by trying to cross all seven bridges once only and wondering if it could ever be done. In du Sautoy's account, it would appear that Euler's motive for solving the problem was to put the people of Konnigsburg "out of their misery."

2. *Agents*: specific or general active 'doers' of mathematics or other activities arising in the participants' accounts and explanations. As Pimm (1987) has argued, the use of different pronouns in mathematics serves to include or exclude different participants and revolves around the relationship between participants and mathematics. Agents used in this program include 'you', 'we', 'mathematicians', Poincaré, Euler and 'I'. 'You' arises frequently as a form of generalisation, as in: "if you try to write down mathematical formulas for how those bodies move you're not really going to get anywhere." It is unlikely that many listeners would be inclined to tackle this problem, so the 'you' must be heard as a generalised agent. 'I' also occurs frequently, often as a way to include the listener in a way that 'you' does not. Du Sautoy, for example, is explaining what topology is and says: "a rugby ball in (.) topologically would be the same as a **foot**ball because I can sort of squash it to one from one to the other," which invites the listener to imagine du Sautoy or themselves deforming the ball. This use of 'I' presents mathematics in a more human-centred way than much written mathematics.

3. *Everyday discourse*: interpreted broadly to include vocabulary, expressions, analogies and references to popular culture. As discussed in the introduction, the relationship between the everyday and the mathematical is not easy to untangle. Everyday vocabulary used in the broadcast includes pushing, pulling, squashing and gluing. Expressions used include 'discovered to their horror' and 'ditch the formulas'. Analogies used include references to bagels, pretzels, coffee cups and teapots, as well as accounts of lasoos tightening around footballs. References to popular culture include Homer Simpson and the map of the London Underground.

In the rest of this section, I will refer to and discuss the extract transcribed below (see [3] for transcript conventions). The extract consists of the whole of a single turn made by Ian Stewart, which I present in 6 sections, the rationale for which is explained afterwards. Stewart's contribution follows a lengthy exchange between Bragg and du Sautoy in which the latter summarises the Poincaré conjecture. Ian Stewart is then invited by Bragg to 'develop that a little further'.

IS: 1 [uh: (.) it's (.) yeah uh:m (.) I think it's important to **rea**lize that we live in a three dimensional space=a par**ti**cular three dimensional space=we actually live in a fairly small **bit** of it=we don't explore **hu**ge amounts of this space uh:m we have a rather naïve view that (.) the model of space we have in our heads=euclidean three dimensional space is really **a:**ll there is=

2 I mean marcus is talking here about uh:m (.) ways of **ben**ding three dimensional **space** and uh: I'm sitting here thinking now I understand this stuff but that's a pretty strange thing to want to do uh:m **how** can you bend it where can it **go** uh:m (.)

3 now the way mathematicians bend three dimensional space is actually they don't **ben**d it what they just do is they slice it into bits and then they tell you how conceptually you glue the pieces together again so the **wa:y** that this (.) pr**oo**f that surfaces a:re spheres or bagels or two handled bagels or bagels with seventeen holes or som:e specific number of holes and that's all there is=the way that that works is let's say you **ch**op the surface into triangles (.) uh:m you (.) work out how all the triangles fit together edge to edge (.) and then you do a massive mathematical simplification of this sort of huge jigsaw puzzle a:nd you end up discovering that you can (.) simplify the structure **dow**n until you can actually count how many holes there are and that really is the only thing that's going on (.)

4 you can do this in three dimensions (.) you can chop space into (.) let's say (.) I mean one of the=one of the **sim**plest three dimensional **cur**ved spaces to=well curved in a sense to understand it=is=is called the flat torus (.) uh:m it's=it's sort of hyper bagel (.) you just take a **cu**be (.) and you have a rule which says if you go off the cube you immediately come in again on the corresponding position on the other si:de=it's like these video games where something goes off the edge of the television screen=it comes back goes out on the right=it comes back on the left as if the screen wraps around=you can wrap the faces of a cube ^around^ (.)

5 and it's **that** representa:tion that suddenly opens up a **hu**ge pile of different (.) wei:rd (.) fascinating three dimensional shapes

6 from the **in**side it just sort of looks like all these lumps you know if you looked around in a limited **re**gion (.) it would look as if you were just in our ordinary three dimensional space (.) but as marcus says if=if=if=if you look fa:r enough (.) you might discover you're looking at the back of your head

The three sets of discursive resources summarised above can be seen in use in Stewart's contribution. The turn as a whole is structured in a narrative-like way. He begins by outlining a situation (section 1): we live in a three dimensional space. Next he 'troubles' this situation (section 2): bending space is a strange thing to want to do. Most of the turn is taken up with an account of how mathematicians deal with this trouble, which appears to consist of two separate parts. A description of slicing up surfaces and counting the holes (section 3) is followed by a description of the flat torus as an example of a curved space (section 4). The end of the turn explains the consequence or resolution of these mathematical actions (section 5: they open up "a huge pile of different…shapes") and a corollary (section 6: "you might discover you're looking at the back of your head") – not unlike the moral of a story. One effect of this structure is to provide a degree of coherence. The situation and trouble serve to situate and motivate the mathematical ideas that follow.

The various sections involve several different agents. The opening situation begins with a broadly inclusive statement: "we live in a three dimensional space". The inclusiveness is brought about, in part, by the use of 'we'. Gradually, however, this initial wide claim is qualified, first with 'a particular bit' and then with 'a fairly small bit' and finally with 'a rather naïve view that [that] is really all there is." Stewart starts with a general, apparently uncontroversial idea, highlights its limits and then highlights our limitations for having had such a simple idea in the first place!

Furthermore, these limitations are all presented as facts - a rhetorical device that 'creates' truths. It does not seem unreasonable to say that we live in 'a fairly small bit' of space. Here, however, space is used ambiguously, implying 'the universe' as much as 'topological' space. 'A fairly small bit' might be an everyday version of localness (as in 'locally flat') or neighbourhoods, but can also be heard more literally to mean 'where I live.'

The 'trouble' is presented from Stewart's own perspective – 'I' – "I'm sitting here thinking now I understand this stuff but that's a pretty strange thing to want to do." The troubling of the mathematical ideas is presented as Stewart's thoughts, particularly about bending space in four dimensions. It is unlikely that Stewart does find the ideas particularly strange: he's spent his life thinking about them! His voicing of these 'thoughts', however, act as proxy for the possible thoughts of listeners: they introduce an outside perspective. His subsequent expansion of this 'strange thing' as "**how** can you bend it where can it **go"** actually appears to be equally strange: he does not question the *possibility* of curving space, which, despite being rather a complex idea, is taken for granted. Stewart's concern is rather with how one goes about bending space. His 'thoughts', however, allow the questions to be asked and positioned as coming from the same outside perspective as the initial observation of strangeness.

The move to mathematical action involves another change of agent and a contrasting move or counter-story: "the way mathematicians bend three dimensional space is actually they don't **ben**d it what they just do is they slice it into bits and then they tell you how conceptually you glue the pieces together again." The agent in this section is mathematicians, signalling that we are now in the realm of mathematics rather than our world of space and our naïve views. And mathematicians have their own way of doing these things (slicing, not bending), although Stewart expresses them in everyday terms, with words like 'slice', 'bits' and 'glue'. These kinds of words avoid the mathematicians appearing too obscure, despite them doing things their own way. This account also minimises the mathematical work involved, glossing over it as 'they tell you conceptually.' This minimising of mathematical work arises again during these sections, with rather vague expressions like 'massive mathematical simplification,' 'you end up discovering' and 'simplify the structure down'.

The consequence or resolution (section 5) does not appear to have an obvious agent, "it's **that** representation that suddenly opens up a **hu**ge pile of different (.) weird (.) fascinating three dimensional shapes," although the 'representation' has a degree of agency and could be seen as standing for mathematics more generally. Again, though, the rather mathematical trait of giving agency to a representation is softened by everyday language like 'a huge pile' and 'weird'.

DISCUSSION: A HYBRID DISCOURSE

Throughout Stewart's turn, there is a constant movement between more 'everyday' and more 'mathematical' discourse, such as the interweaving of references to tori and bagels. This last example seems to be fairly clear-cut: bagels do not seem like mathematical discourse and 'torus' is a fairly technical mathematical term. In many cases, however, the distinction is less clear. At first glance, words like 'chop' and 'glue' appear to be from everyday discourse. I suspect, however, that such words, and even bagels, arise rather frequently in more informal, particularly spoken, mathematical situations. Similarly, the account of a representation of the flat torus incorporates the everyday *and* the mathematical: "you have a rule which says if you go off the cube you immediately come in again on the corresponding position on the other si:de." This account involves much that can be seen as everyday: 'if you go off' 'you immediately come in again' and the account is a kind of mini-narrative. At the same time, it is informally mathematical: it includes a rule and an if/then condition and some fairly precise description ("corresponding position"). Similarly, the variation in agents in Stewart's turn incorporates more everyday discourse ("I'm sitting here thinking") and more mathematical discourse ("it's **that** representa:tion that suddenly opens up…").

I have been using words like 'movement between' and 'interweaving' to describe how the everyday and the mathematical are related. These words imply mutually exclusive discourses, however, when often both are present in a word like 'slice'. Perhaps a better description is to say that the mathematicians' talk is *hybrid* (e.g. Pennycook, 2005), a fusion of the everyday and the mathematical (for another topological example, see Barton, 2008, pp. 60-61). This idea is particularly well illustrated by the form 'hyper bagel', which incorporates a mathematical modifier with an everyday object. I conjecture that this kind of hybridity is actually widespread in the world of professional mathematics. In the above transcript, it seems to be a crucial resource in the participants' attempts to relate some sense of advanced topology to a general audience and perhaps to try to avoid the impression that mathematics is abstract, abstruse and irrelevant. The participants' talk, moreover, constructs a version of mathematics and of mathematical thinking. These versions involve active agents, narrative-like structure and are very much part of the everyday world.

Endnotes

1. The three antis are from a response given by Margaret Wetherell as part of a UK Linguistic Ethnography Forum colloquium at the annual meeting of the British Association for Applied Linguistics, Bristol, 15-17 September 2005.

2. See www.bbc.co.uk/radio4/history/inourtime/ where you can access an archive of all editions broadcast to date.

3. Transcription conventions used are as follows: bold indicates emphasis; colons (:) indicates phoneme extension within a word (one colon for every approx. 0.1 sec); (.) is a pause < 1.5 secs; ^ ^ encloses whispered or very quiet speech; = shows latching (no gap between words). I am grateful to Jennifer Bene for her careful transcribing.

References

Barton, B. (2008). *The Language of Mathematics: Telling Mathematical Tales*. New York: Springer.

Barwell, R. (2007). The discursive construction of mathematical thinking: the role of researchers' descriptions. In Woo, J-H., Lew, H-C., Park, K-S, & Seo, D-Y. (Eds.) *Proceedings of PME31*, vol. 2, pp 49-56.

Barwell, R. (2008). Discourse, mathematics and mathematics education. In Martin-Jones, M., de Mejia, A-M, & Hornberger, N. (Eds.). *Encyclopedia of Language and Education (2nd edn.)*, *Volume 3: Discourse and Education*, pp. 317-328. New York: Springer.

Bruner, J. (1996). *The Culture of Education.* Cambridge, MA.: Harvard University Press.

Edwards, D. (1997). *Discourse and Cognition.* London: Sage.

Edwards, D. & Potter, J. (1992). *Discursive Psychology*. London: Sage.

Kieran, C., Forman, E., & Sfard, A. (Eds.) (2002). *Learning Discourse: Discursive Approaches to Research in Mathematics Education*. Dordrecht: Kluwer.

Moschkovich, J. (2003). What counts as mathematical discourse? In Pateman, N. A., Dougherty, B. J., & Zilliox, J. (Eds.) *Proceedings of the 2003 Joint Meeting of PME and PMENA*, vol. 3, pp. 325-340.

Mosckovich, J. (2007). Examining mathematical discourse practices. *For the Learning of Mathematics* 27(1) 24-30.

O'Halloran, K. L. (2005). *Mathematical Discourse: Language, Symbolism and Visual Images*. London: Continuum.

Pennycook, A. (2005). Global Englishes, Rip Slyme, and performativity. *Journal of Sociolinguistics* 7(4) 513-533.

Pimm, D. (1987). *Speaking Mathematically: Communication in the Mathematics Classroom*. London: Routledge.

Wetherell, M. & Potter, J. (1992) *Mapping the Language of Racism: Discourse and the Legitimation of Exploitation.* London: Harvester Wheatsheaf.

RESEARCH EMPOWERING THE RESEARCHED: REFLECTIONS ON SUPPORTING INDIGENOUS TEACHER AIDES TO TUTOR MATHEMATICS

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This paper looks at the implications of decolonising methodologies on mathematics education research with Indigenous communities. It uses a study of remote Indigenous assistants being supported to become effective mathematics tutors of at-risk Indigenous students to draw implications for the application of the Empowering Outcomes research model for remote Indigenous research sites. It discusses the results of the study in terms of benefit and empowerment, and draws conclusions with respect to research designs that benefit the researched.

In 2001/2, authors Baturo, Cooper and Warren began to work in remote Australian Indigenous communities supporting teachers to enhance the mathematics learning of Indigenous students. As such, they joined the army of mostly non-Indigenous researchers who have made Indigenous people the most researched group in countries like Australia and who, generally, have brought little or no benefit to these Indigenous people or their communities. In 2003/4, aware of their limitations and realising that non-Indigenous research of Indigenous issues can be part of the ongoing oppression of Indigenous people, these authors sought out Indigenous researchers, of whom authors Matthews and Underwood are presently part, with whom to collaborate in setting up a research group that came to be called Deadly Maths. In this, the initial members of Deadly Maths were strongly influenced by L. Smith (1999) who cogently argued that non-Indigenous research of Indigenous people has been "implicated in the worst excesses of colonialism" (p. 1), continued constructing Indigenous peoples as the problem, and "frequently failed to improve the conditions of the researched" (p. 176).

Deadly Maths group

Deadly Maths was set up to undertake Indigenous mathematics-education research with the primary focus on benefitting the researched, a focus that cannot be violated even to maintain so-called excellence in scientific design. Informing this research were the two imperatives for education espoused by Indigenous people across the 16 Queensland Indigenous communities that Deadly Maths members visit − namely, that students in Western schooling learn to be: (1) "solid" (strong) and "deadly" (smart) Indigenous people who have pride in their heritage; and (2) successful people in terms of enhanced employment and life chances. The Deadly Maths projects are embedded in decolonising methodologies (L. Smith, 1999) and attempt to incorporate her seven cultural positions, namely: (a) to have respect for people, (b) to present yourself to the people face-to-face, (c) to "look, listen … speak" (p. 120), (d) to share and host people and be generous, (e) to be cautious, (f) to not trample over people's dignity, and (g) to not flaunt your knowledge. Deadly Maths projects also recognise G. Smith's (1992) *Empowering Outcomes* model that addresses the sorts of questions that Indigenous people want to know in ways that empower these people; and *Mentoring,* and *Power Sharing* models through having Indigenous researchers on all projects and collaborating with Indigenous Community members as equal partners.

This paper

This paper reflects on these decolonising methodologies with respect to a project (funded by Australian Research Council Linkage grant LP0562352) to support Indigenous teacher assistants (ITAs) to tutor more effectively Indigenous students at-risk with respect to mathematics. It discusses the study's design in terms of the Empowering Outcomes model for remote Indigenous research sites, discusses the results of the study in terms of benefit and empowerment, and draws conclusions with respect to research designs that benefit the researched.

DECOLONISING METHODOLOGIES AND REMOTE COMMUNITIES

The education of Australian remote Indigenous students is inherently unjust (Warren, Cooper & Baturo, 2007) with the lowest retention and performance rates in Australia's school system particularly in mathematics (Queensland Studies Authority, 2004; 2005; 2006). This is due to social factors (Fitzgerald, 2001) such as racism, poverty, remoteness, unemployment and welfare dependence, and education systemic issues (Matthews, Watego, Cooper & Baturo, 2005) such as culturally disempowering forms of teaching, curriculum and assessment, particularly the use of Standard English which is, at best, a second language for Indigenous people. Due to remoteness, there is a scarcity of resources and services, and schools are generally staffed by inexperienced non-Indigenous teachers with little Indigenous education knowledge who, in turn, are supported by ITAs with little training in what and how to teach. As well, the relationship between the non-Indigenous teachers and the ITAs has, in general, led to the further disempowerment of the ITAs within the school. For remote Indigenous communities, the continued low educational performance of their children is a major issue and one for which there are only a few examples of success (e.g., Sarra, 2003) which have not been sustainable when key people have left.

Empowering outcomes research with Indigenous people is post-positivist and qualitative in nature and requires persistent face-to-face contact (L Smith, 1999). This is fiscally and physically challenging in remote communities because of their isolation − even with air travel, and their limited and irregular facilities and services. Air travel can take two days, requiring connections to be made between regional and local airlines (that often fail to meet their timetables). Shops have limited opening hours (food may have to be carried by researchers on the plane), accommodation is restricted to a few highly in-demand school cottages, and email is very slow. However, face-to-face contact is crucial to building collaborative relationships in which there can be "two-way" sharing of ideas and joint researcher-TA activity to improve Indigenous students' mathematics performance.

As argued by L Smith (1999), cultural sensitivity and humility are important for acceptance and success in Indigenous communities. Western expectations that research visits take priority over other daily events and that timetables and structures developed in Brisbane will be adhered to in the remote community is not respecting local culture. Furthermore, it is important not to apply non-Indigenous perspectives to Indigenous student attendance and behaviour in classrooms as low attendance and confronting behaviour is a product of cultural resistance of Western schooling (Matthews et al., 2005) as well as students meeting their family and cultural obligations within the community (looking after siblings and attending funerals). As Partington (1998) argued:

> As a consequence of the treatment they experience in the classroom - even from the first day of school – many Indigenous students become alienated and start on the path that ends only when they drop out of school. They do not become alienated voluntarily but as a consequence of the way they are treated (p. 19).

Australia's schooling system remains largely Eurocentric in structure and curriculum. Indigenous teachers and ITAs "are not given input into the strategic plans of the school" (Matthews, Howard, & Perry, 2003, p. 11) and "are denied access to facilitators and services that other teachers take for granted" (MCEETYA, 2000, p. 16). The way in which Indigenous students learn, their languages, cultures and values, are not respected within this environment. In particular, mathematics is not contextualised into Indigenous culture which is perceived as primitive and simplistic (Matthews et al., 2005). Where no real attempts have been made to reverse this, Indigenous teachers and ITAs cannot mediate between Community and school often resulting in half of the students being absent each day (Fitzgerald, 2001). This can severely affect most types of "scientific" research; for example, it makes pre-post testing and persistent observation problematic.

Empowering Outcomes research is not so troubled with the perceived uncertainties of Indigenous Community life; with its focus on community benefit, it takes the existing situation as the starting point for collaborative activity. Thus, the involvement of Indigenous researchers and community members in looking at problems they feel are important enables research to advance (although, as will be discussed later in this paper, sometimes not with the "scientific" structure wanted by many editors). For Deadly Maths, decolonising research approaches were particularly important in reducing difficulties in interviewing Indigenous students who were reluctant to answer direct questions. As Barnes (2000) noted, Indigenous students "may find it difficult to respond to questions or display knowledge in the presence of adults or other persons in authority. This may be misinterpreted as ignorance or resistance" (p. 9).

Once a respectful relationship is built, students feel able to talk freely, particularly to Indigenous researchers.

TEACHER ASSISTANT STUDY

One focus of Deadly Maths projects which has attracted strong Indigenous support has been collaborative work with untrained ITAs to improve their (and the researchers') abilities to tutor at-risk Indigenous students in mathematics (Baturo, Cooper & Doyle, 2007). The study discussed in this paper occurred in two remote Queensland sites (referred to as Junction and Kanoona) where the Professional Learning (PL) activities were undertaken and involved four local Indigenous communities (Junction, Ooting − Site 1; Kanoona, Beachall at Site 2). Altogether, 10 TAs were involved in this project − 7 females; 3 males; 8 Indigenous, 2 non-Indigenous.

At each PL site, a non-Indigenous and an Indigenous researcher worked with four Tas to provide PL with respect to addition and subtraction (meaning, mental computation strategies, and algorithmic procedures). All participants were provided with 5 booklets of addition and subtraction tutoring materials which they were asked to trial in the weeks following the PL sessions. Data were gathered by: (a) observations (video-taped) of the PL sessions, (c) interviews (audio-taped) with the TAs before, during and after their tutoring trials of the addition and subtraction materials, and (d) TAs' records of their tutoring trials with Indigenous students.

Professional learning sessions and tutoring trials

Junction/Ooting's PL sessions were held in a regional city which has one plane flight a day. Kanoonga required two days of flying, but by driving a hired four-wheel drive vehicle from the regional city, the travel time each-way was reduced to one day. By travelling Sunday and Friday, four days were available for the PL sessions for both groups. As the next visit by researchers was in two months, the tutoring trials with the booklets meant to follow the PL sessions had to be undertaken by the TAs with only phone support by the Indigenous researcher.

The PL sessions incorporated 10 PL principles (Baturo, & Cooper, 2004; Baturo, Cooper, & Doyle, 2007) which encompassed mathematics and pedagogy, professional development, and social principles. The sessions thus focused on activities that would develop structural rather than procedural knowledge; they were designed to provide the TAs with the same level of material as preservice teachers.

The *content* focus of the PL/booklets was on: (a) building conceptual meaning for the two operations within both set and length models using games and activities to connect different representations (materials-diagrams, language, symbols) using the approaches of Payne and Rathmell (1977) and Duval (1999); (b) covering three strategies for computation, namely, separation (adding/subtracting in place-value positions), sequencing (adding/subtracting parts of second number to the first) and compensation (changing both numbers to make computation easy yet maintain equivalence); and (c) developing abilities to both interpret and construct real-world problems using the part-part-total concept and forward, backward and comparison stories.

The *pedagogic* focus was on: (a) concept and strategy development; (b) relating operations to the everyday world of the student (contextualisation); and (c) making out-of-school knowledge legitimate within school; getting answers was de-emphasised. The TAs were encouraged to have high expectation of themselves and their students. The pedagogy was based on a mixture of social constructivism and the holistic interactive Indigenous learning approaches espoused by Grant (1997). The extent the above was attempted can be seen in two aspects of the teaching (Cooper & Baturo, 2008): (a) local issues were used for context, including an Indigenous card game learnt from assistant A1 which was used to teach computation; and (b) compensation was connected to identity by showing that equivalent computations involve adding/subtracting numbers equivalent to zero.

Trialling each of the main tutoring ideas with students directly after the ideas were covered in the PL sessions was part of the plan, but this was not possible. For the Junction-Ooting TAs, the PL sessions were in a regional office and no students were available. For the Kanoonga-Beachall TAs, one-and-a-half days of the PL sessions available had to be cancelled for a funeral; all remaining time was assigned to covering the main ideas in the materials. This meant that the TAs had no experience at all of using the booklets with students when they returned to their classrooms.

Findings from TA interviews

Interviews were conducted by the Indigenous researcher over the phone with nine of the twelve TAs, three from Junction-Ooting (classified as J1, J2, J3) and six from Kanoonga-Beachall (K1, K2, K3, K4, K5, K6) (the other TAs had left their Communities). These interviews were transcribed and combined with observations of the PL sessions. These data were analysed in terms of commonalities in the TAs' responses. This section describes the central ideas from these data on perceptions of the PL and trials.

<u>Empowerment/confidence</u>. All the nine TAs stated that they felt empowered through the learning experiences in the PL sessions and tutoring trials, and felt confident to teach mathematics. In particular, they remarked on being able to return to their classrooms with knowledge of what to do. In one case, the TA said she took over the teaching of an area of mathematics. Examples of responses were:

J1 … now I have different ways of actually, like in case they are not getting in one way of teaching it I actually have something to go back on…I can change it to simplify it

J2 Yes I am [more confident]. I have the Deadly Maths by myself and I do that by myself while the teacher is in the classroom

J3 … the different ways you have are better than the ways we sort of put it together … and the kids are sort of getting the work.

K2 The other teachers have seen me doing and showing the teacher I am with, when it does come to maths I take over and do a bit of the games with them with the maths sheets

K4 … how to start from scratch basically to give the kids more of an understanding … she [the teacher] will say 'can you do this?' and I can say 'oh how about we do this'

Knowledge/Relationships. All TAs felt that they had gained important knowledge that they could share with the students in their community. They liked the PL sessions because, as J2 stated, *you get to see other people from other places and know who they are*. In particular, they liked the sharing and they appreciated their differences:

J1 … everybody has got their say in this didn't work or that didn't work, or I did it this way … then everybody has their own thing. If you came to our school it would be just me and [J2] and we don't really see what the other people have been doing and what their outcomes are too. If you have the group then it is better. Everybody then knows what's going on. [J4] is out there but she's not Indigenous and we are different to her. We probably teach in a different way to her.

However, they also felt that the experience would be improved with more attention to applying this knowledge in tutoring students, particularly their students. As K5 stated, *it is really hard to try and do it with someone else's kids, at least our kids know us*.

All TAs felt that good relationships had been built with researchers and were very comfortable speaking with the Indigenous researcher in particular. However, K2 felt that there could be more face-to-face and phone contact.

PL sessions/Tutoring trials. The TAs valued the support provided by the researchers and that it had been provided in community; they felt that the structure of PL sessions was appropriate for discussing ideas but not for applying ideas to tutoring. The TAs liked the resource booklets − *they were very useful ... like ready to go ... and when you read it you add more information for ... the better way to teach the kids* (J1); *they* [students] *all liked it and they would run to it when they had free time* (J4). This led to requests for more games and activities in future booklets. The TAs also liked the contextualisation − *it just surprised me of what you can use and how to use it ... you can just use the things in your own school and around the classroom* (J5). However, not everything was appreciated − J1 and J2 felt that the separate session on theory given at Junction-Ooting (and not at Kanoonga-Beachall) needed to be integrated into the activities.

School support/Student improvement during trials. School support depended on the assistants' school and teacher, and lack of support from some schools/teachers inhibited the tutoring trials. Some assistants were not able to tutor the mathematics (addition and subtraction) from the PL sessions in their classrooms, while others were supported or partially supported. Assistant A1 was assigned to Prep Year so her tutoring did not cover operations. This variation in support can be seen in the three responses below:

J2 …. like every time we come back and I told her that we were going to try this and see if it works with the kids and stuff like that and she was OK, you know, she would just say 'if you think it is going to work, then do it'.

K5 … I haven't really put it [tutoring program] into practice … only when we came back in the beginning, the first two weeks after the training, we were able to do things with kids.

K6 No [could not tutor the program] … it's just a bit hard when you have got teachers that have their own agenda.

Only four (J2, J4, K1 and K4) of the nine assistants interviewed were able to trial the tutoring of the addition and subtraction material. However, their results were encouraging; students' mathematics outcomes appeared to improve. J2 described her success with the number board, *they didn't really realise how easy it* [computation] *was like if you started at a number and wanted to add 10 more you could just jump down 10 and 1 over and 2 back*. J4 referred to her pre and post testing which she said showed *that they become very good at it*. K1 declared, *... they are moving up levels in their maths*, while K4 felt she had success in co-teaching with her teacher.

Relationships with their teachers were particularly sore points with Indigenous TAs. To attract teachers to the remote communities, the state Education Department has a point system that means that a remote community teacher can transfer to any school after two years and most do. As J1 said when asked how about her Community, *the only thing that changes in* [Junction] *is the teachers*.

DISCUSSION AND CONCLUSIONS

The study described in this paper is a typical Deadly Maths project using L. Smith's (1999) decolonising approaches, namely, and G. Smith's (1992) Empowering Outcomes research model. The focus is on benefiting the TAs and, in the long run and of paramount importance, benefiting the students. For this reason, the researchers visit the Communities, set up relationships with the TAs that promulgate sharing of ideas (e.g., the card game that is the focus of Booklet 4 – Cooper & Baturo, 2008), and limit intrusive data gathering (e.g., no pre-post tests on TAs' mathematics performance were given as this might inhibit building relationships: another disempowering process).

The feedback from the TAs is that the study worked at their level; they felt: (a) empowered in the classroom, (b) supported as Indigenous educators using contextualised mathematics (Matthews et al., 2005), and (c) happy to continue working with us. However, trialling with students was ambivalent; many assistants were not given the time to trial the ideas even though this aspect of the study had been negotiated with the schools beforehand, but those that could, appeared to have success. The structure of the materials was also acceptable.

Deadly Maths will continue the work with the TAs at the sites in 2008 with multiplication, division and fraction material. As a consequence of this study, the work will focus more on tutoring students in classrooms and on including the TAs as co-managers and co-researchers in the trials (the Sharing Power research model).

> J1 No I think just the group gathering first until I can get it implemented into working with the staff, teachers; then I could ask you to come and see how or if I am doing it right.

In conclusion, three implications are evident. First, the TAs have shown that they are solid and deadly and have the potential to be the major sustainable provider of quality mathematics education to Communities into the future. Second, such quality beneficial outcomes require compromise with regard to data gathering and a focus on relationship before data; there has to be a way that such research can be accepted in prestigious journals. The development of ways to publish findings from decolonising projects is one aim of Deadly Maths. Third, L Smith's (1999) arguments apply equally to research on all disempowered peoples in Australian society. For these peoples, it is the opinion of the Deadly Maths researchers that the only acceptable research methodologies are those whose prime purpose is to make the disempowered active participants and beneficiaries of the research outcomes.

References

Barnes, A.L. (2000). Learning preferences of some Aboriginal and Torres Strait Islander students in the veterinary program. *Australian J. Indigenous Education, 28*(1), 8-16.

Baturo, A. R., Cooper, T. J., & Doyle, K. (2007). Authority and esteem effects of enhancing remote Indigenous teacher-assistants' mathematics-education knowledge and skills. In J. Woo, H. Lew, K. Park, & D. Seo (Eds.), *Proceedings of the 31st Annual Conference of the International Group for the Psychology of Mathematics Education*. Seoul, South Korea: Seoul National University.

Cooper T. J. & Baturo, A. R. (2007). *Deadly Maths Tutor Program: Addition and Subtraction, Trial Packages 1, 2, 3 4, and 5*. Resource materials, School of Mathematics, Science and Technology Ed, QUT, Brisbane, 4059, Australia (www.deadlymaths.oz-teachernet.edu.au).

Duval, R. (1999). Representations, vision and visualization: Cognitive functions in mathematical thinking. Basic issues for learning. In F. Hitt & M. Santos (Eds.), *Proceedings of the 21st Conference of the North American Chapter of the International Group for the Psychology of Mathematics Educati*on (Vol 1, pp. 3-26).

Fitzgerald, A. (2001). *Cape York justice study*. Brisbane, Qld: Queensland Government.

Grant, E. (1997). *Holistic Approach to Teaching and Learning*. Paper presented at the Aboriginal Studies in the 90s: Sharing our Stories. 7th Annual ASA Conference, Sydney.

Matthews, S., Howard, P., & Perry, B. (2003). Working together to enhance Australian Aboriginal students' mathematics learning. In L. Bragg, C. Campbell, G. Herbert, & J. Mousley (Eds.), *Proceedings of MERGA* (26, pp. 9-28) Sydney: MERGA.

Matthews, C., Watego, L., Cooper, T.J., & Baturo, A. R. (2005). Does mathematics education in Australia devalue Indigenous culture? Indigenous perspectives and non-Indigenous reflections. In P. Clarkson, A. Downton, D. Gronn, M. Horne, A. McDonough, R. Pierce, & A. Roche (Eds.), *Proceedings of MERGA 28* (Vol 2, pp. 513-520). Melbourne: Vic: MERGA

Ministerial Council on Education, Employment, Training and Youth Affairs (MCEETYA). (2002). *A Taskforce on Indigenous Education* (www.mceetya.edu.au/aboutmc.htm)

Queensland Studies Authority. (2004). *Overview of statewide student performance in Aspects of Literacy and Numeracy: Report to the Minister for Education and Minister for the Arts*. Brisbane, Qld: Queensland Studies Authority.

Partington, G. (1998). No simple solutions exist: Perspectives on education as the key to change. In G. Partington (Ed.), *Perspectives on Aboriginal and Torres Strait Islander education* (pp. 3-26). Katoomba, NSW: Social Science Press.

Payne, J., & Rathmell, E. (1977). Numeration. In J. Payne (Ed.), *Mathematics Learning in Early Childhood*. (1977 NCTM Yearbook). Reston, VA: NCTM.

Sarra, C. (2003). Young and black and deadly: Strategies for improving outcomes for Indigenous students. In Australian College of Educators (Eds.), *Practitioner Perspectives* (Quality Teaching Series). Deakin, Vic: Australian College of Educators.

Smith, G. H. (1992). Research issues related to Maori education. In Research Unit for Maori Education (Ed.), *The Isue of Research and Maori*. Auckland, NZ: University Auckland.

Smith, L. T. (1999). *Decolonising Methodologies: Research and Indigenous Peoples*. Dunedin: University of Otago Press.

Warren, E., Baturo, A., & Cooper, T. (in press). Power and authority in school and community: Interactions between non-Indigenous teachers and Indigenous assistants in remote Australian schools. In J. Zajda (Ed.), *Handbook of Social Justice*. Springer, NY.

INSTRUCTIONAL ANALOGIES AND STUDENT LEARNING: THE CONCEPT OF FUNCTION

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This paper examines the effectiveness of analogies in the teaching and learning of the function concept. Our findings indicate that instructional analogies cannot support students' understanding of the functions unless students are given epistemologically appropriate analogues and illustrated the structural relations between the analogues and the targeted concepts. Provision of analogies to emphasize procedures, algebraic or otherwise, may confine students to a limited way of thinking about the concept.

INTRODUCTION

Educators emphasize the importance of analogies in teaching and learning science and mathematics (Podolefsky & Finkelstein, 2006; Shulman, 1986). Simply defined, analogy refers to descriptions which tell how two things are similar to each other. It entails using a familiar system – the so called source or base – as a foundation for drawing inferences about an unfamiliar system – referred to targeted concept (Spellman & Holyoak, 1996). Rattermann (1997) states that "*A good analogy conveys large amount of information with very little explanation, it inspires scientific discovery, and it provides new information about an unfamiliar domain*" (p. 247). Shulman (1986) suggests that teachers need an expertise in using most appropriate analogies to align the logic of scientific notions to the students' comprehension. They need to have rich repertoire of analogies to transform subject-matter into forms that could be grasped by the students of different ability and social background (ibid).

The role of analogy in teaching and learning has been extensively researched in science education (see, for example, Heywood, 2002; Podolefsky & Finkelstein, 2006; Reiner et al, 2000). In recent years, instructional analogies have received attention from mathematics educators (Alexander et al, 1997; Fast, 1996; Kathy et al, 1999; Richland et al, 2004). Fast (1996) indicated that provision of analogies can cause conceptual changes in students' understanding and help them revise and reconstruct their knowledge of probability. Kathy et al (1999) researched the effectiveness of analogies in teaching and learning the fraction concept. They used seven concrete analogues (e.g., pizza, ice-cream bars, and licorice straps) and evaluated them with respect to their '*ecological validity*' – how realistic for pupils was the sharing context engendered by the analogues – and their '*ease of partitioning*' – how easy were the analogues to physically partition into quotients. The results indicated that both '*ecological validity*' and the '*ease of partitioning*' were crucial features of the source analogues that were greatly affecting the students' ability to draw inferences about the fraction concept.

A review of available literature suggests that analogies could promote students' understanding of mathematics. Thinking by analogy means transferring structural information from a familiar system to an unfamiliar system. Educators place a great emphasis on the transfer of knowledge from an analogue to a targeted concept, suggesting that it is this phenomenon that has significant implication for teaching and learning mathematics (English, 1997). The present study contributes to a growing body of research in the filed by examining an experienced teacher's analogy-based teaching approach and relating it to his students' understanding of the function concept.

BACKGROUND: THE CONCEPT OF FUNCTION AND THE TURKISH CONTEXT

The contemporary literature suggests that the concept of function can be construed in two fundamental ways: as a process and as an object. The former entails an ability to interpret a function as a process transforming inputs to outputs (Dubinsky & Harel, 1992). Those who possess a process conception would manipulate a function in various ways, for instance they could reverse a function process or combine it with other processes (ibid). The constant reflection upon a function process may lead to its eventual encapsulation as an object (Breidenbach et al, 1992), and this level of understanding enables one to use a function in further processes, such as manipulating a function as a single entity in the process of derivative or integral. No matter how one conceives a function, the concept has two fundamental properties: the univalence and the arbitrariness conditions. The univalence states that every element of the domain must be assigned to a unique element in the co-domain (Malik, 1980). The arbitrariness suggests that a function could do transformation in an arbitrary manner (ibid); thus it rules out attributing a mechanical rule, algebraic or otherwise, to the concept.

The participant teacher, Burak[1], introduced the idea of function through a definition – "*A function is a relation that matches every element of the domain to a unique element in the co-domain*". He illustrated the definition through various examples in the set-diagrams and ordered pairs with a particular emphasis on the univalence condition. Burak continued to use above definition as he moved into algebraic and graphical context; yet his teaching encouraged an understanding of the function as a computational (algebraic or arithmetical) process and mostly engaged the students with the rules, procedures and the factual knowledge associated with the visual properties of algebraic and graphical representations. Consequently, in the coming sections we shall use the above definition as a benchmark and refer to its dynamic (process-like) nature as we comment upon analogy-based teaching instances derived from Burak's lessons. The essence and the properties of the sub-concepts of the function (e.g., inverse function, constant function) will be illustrated through data presentations. We shall do this to facilitate an evaluation of the extent to which

[1] Teacher's and students' names are altered.

Burak's analogy-based teaching could have prompted his students' understanding of the functions as a process transforming elements from domain to co-domain.

RESEARCH METHOD AND THE DATA ANALYSIS

The research employed a qualitative case study using classroom observations and semi-structured interviews as the main sources of data. The research sample included an experienced mathematics teacher - Burak who had 24 year teaching career - and his 9th grade students (age 15). The teacher was observed 14/19 lessons teaching all aspects of the functions. The lessons were tape recorded and annotated field notes were taken. During the sampling process Burak had stated his belief that the use of analogies could promote students' understanding of the functions thus when observing his lessons we gave a particular attention to analogy-based teaching instances and took narrative summary of the context where the analogies were used and how they were used. After the course on functions Burak's students (27 students) were given a questionnaire which encouraged them to provide reasons (including instructional analogies) for their answers. The questionnaire aimed to assess students' conceptual (process-conception) and procedural understanding of the function concept, its properties and sub-notions. Based on their achievements in the questionnaire three students were selected for the interviews. A semi-structured interview was conducted and the aspects of clinical interview (Gingsburg, 1981) were considered to delve into the students' thinking. If a student revealed analogical thinking we probed him/her to see whether he/she was able to use analogies to draw inferences about the situation at hand.

Overall, literature about epistemology of the functions (Dubinsky & Harel, 1992) and the role of analogies in teaching and learning science and mathematics (Kathy et al, 1999; Podolefsky & Finkelstein, 2006) provided a conceptual base for the data analysis. Content analysis (Philips & Hardy, 2002) was conducted to discern meaning in the teacher's written and spoken expressions. Lessons were fully transcribed and considered line by line whilst annotated field notes were used as supplementary sources. The first phase of data analysis included detecting analogy-based teaching instances and identifying source analogue and the targeted concepts, for instance 'fixed minded person - constant function' and 'clothes worn under certain weather conditions - piecewise function'. The subsequent phases embraced in-depth examinations of spotted cases in accord with two criteria: 'purpose of use – whether analogies were offered to explain function-related ideas or to emphasize procedures' and 'content validity - whether the source analogues had the epistemological power to represent function-related ideas; if they had so, how they were used'. As it was the manner in the analysis of the teacher data students' interviews were fully transcribed and considered. Learning instances in which the students indicated analogical thinking were identified and, then, they were scrutinized to determine whether or not the students were able to use analogies to resolve the function problems they were given. Lastly, since the research involved multiple cases, it was necessary to use the strategy of cross-case analysis (Miles & Huberman, 1994) to establish the relationships between the variables. The comparison between the sets of data was made in two ways. We examined analogy-based teaching instances and then looked for the corresponding learning outcomes in the students' data, or we did the reverse.

RESULTS

Provision of analogies was a distinguishing feature of Burak's teaching. He offered analogies as an 'advanced organizer' to prepare his students for the concept of function or as an 'activator' to stimulate their knowledge when they were solving problems in the domain. Table-1 provides a list of analogies that Burak used in his teaching, and it illustrates the contexts where the analogies were used and how they were used.

	Source analogue	**Targeted concept**	**Content validity of the analogues/Purpose of use**
1	Washing machine, camera	Pre-image – Image	Analogies were used to explain how to calculate the images when the pre-images were given, or vice versa.
2	A relation of dance…	Univalence condition	The source analogue had the content validity to illustrate the idea of univalence; yet the teacher did not illustrate the structural relations between the two.
3	Identity numbers, 1 and 0.	Identity function	The analogues were epistemologically inappropriate to illustrate the notion of identity function…
4	A 'fixed-minded' person who rejects all the ideas proposed…	Constant function	The analogue of a 'fixed-minded person' was epistemologically inappropriate to illustrate the concept.
5	Clothes worn under certain weather conditions…	Piecewise function	Analogy emphasized the procedure – selection of the right formulas to operate on each sub-domain…
6	Blanket used for warmth.	Onto function	Analogy was used to emphasize a surface property of the concept… **Analogy:** "... *As we go to sleep we cover ourselves with the blanket... an 'onto function' is like that it covers up every element in the co-domain...*"
7	Football matches in the first premiership…	One-to-one function	The analogue had the epistemological power to illustrate the targeted concept, but the teacher did not illuminate the structural relations between the two.
8	The idea of 'inverse operation', and the examples from every day life…	Inverse function	Analogies emphasized the idea of inverse operation, not the concept of inverse function. The idea of inverse operation included inverting a sequence of operations in a function process…

Table 1. A list of analogies that Burak used in his teaching of the functions

It is seen from this table that there are three basic limitations in Burak's analogy-based teaching approach and these include:

- Provision of analogues which had no content validity,
- Not clarifying the relations between the analogues and the targeted concepts, and
- Using analogies to over-emphasize procedures and the factual knowledge.

Content validity is concerned with the epistemology of an analogue in that the analogue should potentially incite an idea of a function as a process transforming every input to an output. It should have properties and components that match up those of the function concept. Nevertheless, this property was not evident in many of the analogues Burak used. For instance, preparing his students for the idea of identity function Burak gave the explanation: "... *The number one (1) has no effect in the operation of multiplication; likewise zero (0) has no effect in the operation of addition. ... You would think of the identity function like these numbers. ...*". The teacher suggests his students to think of the identity function like identity numbers, 1 and 0. However, neither of these numbers represent a transformation process, they are mathematical entities used in the process of multiplication or addition. On some occasions, although Burak gave his students epistemologically appropriate analogues he did not illustrate the structural relations between the analogues and the targeted concepts. He talked about the analogues very much in tune of their daily meaning (like story telling) and this shifted, apparently, students' attention from function-related ideas to daily events. Consider the following which was given to illustrate the univalence aspect of the function concept (**Episode-1**):

> … Suppose that we are invited to a party, OK. …the party is going on… music is playing…and it is time to have a dance… And there are two groups in the party; the group of guys and the group of ladies…we are going to dance… Yet, we have a rule… … First, all the guys must dance…[1]. Second, every guy can choose only one partner…nobody is allowed to dance with more than one girl, OK [2]. …if there is a beautiful lady…you can queue in front her…this does not break the rule…you can queue in front of her [3]. The function is like the relation of dance…it must satisfy certain conditions.

Notice that Burak's description suggests, implicitly though, all the conditions, [1], [2], and [3], which correspond to the features of univalence condition; yet he does not clarify this in an explicit manner. The structural relations between the two could be best established by converting the source analogue into a mathematical task – sketching two set-diagrams, putting the names of students into the sets (first set: the group of guys, and second set: the group of ladies), and then illustrating all the alternative mappings between the elements of these sets. The teacher does not do this nor does he encourage his students to find out the relations by themselves.

It was common for Burak to use analogies to emphasize procedures and the factual knowledge associated with the algebraic and graphical representations of the functions. Consider the following which was given when they were reversing an algebraic function, $f(x)=x^3+7$, (**Episode-2**):

> … We got up in the morning; we had a breakfast, put on our clothes…and then we locked the door when we left home… When we return back to home…first of all we shall

open the door, and then take off our clothes, and so on… Yes, this is the logic we are going to implement… The last operation here is addition of 7; therefore in the first step we should subtract 7 from the *x*… The operation before the last one is the 3rd power of *x*, so we should take the 3rd root of *x-7* (obtains the inverse function through appropriate manipulations).

An inverse function undoes what a function does and it is in this sense the notion of 'undoing' captures the essence of the inverse function (Even, 1992). The property of 'one-to-one and onto' is the basic criterion that a function must meet to be reversed. However, Burak communicates neither of these ideas through the analogy. He presents the analogy in a way that confines the essence of the concept – the notion of 'undoing' – to the idea of 'inverse operation' – reversing an algebraic function by inverting a sequence of algorithmic operations in a function process.

LEARNING OUTCOME

An analysis of the student data indicated that none of the students revealed analogical thinking in their written responses in the questionnaire. During the interviews two students recalled and used analogies to interpret the function problems they were given. However, both failed at executing the mapping process between the analogues and the targeted concepts. Belgin's response to '*Does the relation m={(4,9), (3,6), (2,7), (1,8), (4,6)} represent a function?*" is typical:

B- Yes, it does. …it (4) can go to many elements… Our teacher told us that a guy cannot dance with more than one lady; yet many guys can queue in front of a beautiful lady…[laughing]…this is like that.

Int- Could you tell me who are the guys, and who are the ladies here?

B- …[Silence]…these [first components] are ladies; is not it? …umm…these [second components] must be guys then…[Silence]…

Int- So, you think…two guys are queuing in front a lady; and this does not bother you.

B- …umm…[Silence]…yes, two guys are waiting… I remember the example…many guys can queue in front of a lady [laughing]; we can think like that. …

This exchange shows that Belgin is unable to transfer structural information from the analogue to the target so that she can conclude that the situation is not a function because it matches an element of the domain to more than one element in the co-domain.

DISCUSSION AND CONCLUSION

The purpose of this paper was to illustrate the influence of analogy-based teaching practices on students' understanding of the function concept. However, the study has limitations. Teaching is a social and cognitive activity offered to help students acquire knowledge (Leinhardt, 1993). Learning is a mental process through which individuals construct their knowledge by interacting with the external stimuli. The mediating process between the two is open to influence of many internal and external factors that may include individuals' cognitive ability, and parental involvement in students' education. The difficulty in controlling all these influences does permit considering analogy-based teaching instances in isolation and relating it to the students' learning.

Having cited these limitations this study produced evidences which have implications for pedagogical considerations and classroom practices. As we have seen there are three basic limitations in Burak's analogy-based teaching approach. The first is concerned with the content validity of the source analogues. Many analogues that Burak offered to his students had no epistemological power to illustrate the function concept, its properties and sub-notions. Fundamentally, the source analogues lacked the ability to represent a function as a process transforming every input to an output. Second, although on several occasions Burak used epistemologically appropriate analogues he did not link them to the targeted concepts nor did he encourage his students to find out the structural relations between the two. Consider again the analogy of '*a relation of a dance*' (see Episode-1). The source analogue permits communicating the idea of univalence: a function could do 'one-to-one' and 'many-to-one' mappings and it should not omit an element in the domain. Nevertheless, Burak does not establish the relations between the aspects the analogue and those of the univalence condition. We can see the negative impacts if this on students' learning. In the interviews, one of Burak's students, Belgin, recalled the same analogy but could not utilize it to interpret the situation she was given. Third, as it is seen in Episode-2 Burak introduces analogies to emphasize procedures (the idea of inverse operation), not the function-related ideas (the notion of undoing). This approach might have intensified the importance of routines for the students and, as a result, confined their understanding of the concept to algorithmic procedures (Bayazit & Gray, 2004).

In conclusion, our evidence suggests that instructional analogies cannot support students' understanding of the function concept unless the content validity is established in them. Provision of appropriate analogues does not help students develop a meaningful learning unless students are explained the structural relations between the two. The efficiency of analogies in teaching and learning the functions depends upon the teacher's and students' expertise at executing the mapping process between the analogues and the function concept. One may use analogies to emphasize procedures associated with the algebraic and graphical functions; yet this might shift students' attention from function-related ideas and, as a result, confine their understanding of the concept to mechanical manipulations.

References

Alexander, P. A., White, C. S., & Daugherty, M. (1997). Analogical reasoning and early mathematical learning. In L. D. English (Ed.), *Mathematical Reasoning: Analogies, Metaphors, and Images* (pp. 117-147). New Jersey: Lawrence Erlbaum Associates.

Bayazit, I. & Gray, E. (2004). Understanding inverse functions: the relationship between teaching practice and student learning. In M. J. Honies & A. B. Fuglestad (Eds.), *Proc. 28th Con. of the Int. Group for the Psychology of Mathematics Education* (Vol. 2, pp. 103-110). Bergen, Norway.

Breidenbach, D., Dubinsky, Ed., Hawks, J., & Nichols, D. (1992). Development of the process conception of function. *Educational Studies in Mathematics*, *23(3), 247-285*.

Dubinsky, Ed. & Harel, G. (1992). The Nature of the Process Conception of Function. In G. Harel & Ed. Dubinsky (Eds.), *The Concept of Function: Aspects of Epistemology and Pedagogy* (pp. 85-107). USA: Mathematical Association of America.

English, L. D. (1997). Analogies, metaphors, and images: vehicles for mathematical reasoning. In L. D. English (Ed.), *Mathematical Reasoning: Analogies, Metaphors, and Images* (pp. 3-18). New Jersey: Lawrence Erlbaum Associates Inc.

Even, R. (1992). The inverse function: prospective teachers' use of 'undoing'. *International Journal of Mathematical Education in Science and Technology, 23(4), 557-562*.

Fast, G. R. (1996). Analogies and reconstruction of mathematical knowledge. *Annual Meeting of the Mid-Western Educational Research Association Conference*. Chicago.

Gingsburg, H. (1981). The clinical interview in psychological research on mathematical thinking: aims, rationales, techniques. *For the Learning of Mathematics, 1(3), 57-64*.

Heywood, D. (2002). The place of analogies in science education. *Cambridge Journal of Education*, *32(2), 233-247*.

Kathy, C., Rod, N., & Tom, C. (1999). Mathematical analogs and the teaching of fractions, *Annual Meeting of the Australian Association for Research in Education and the New Zealand Association for Research in Education*. Melbourne: Australia.

Leinhardt, G. (1993). On teaching. In R. Glaser (Ed.), *Advances in Instructional Psychology* (vol. 4, pp. 1-54). Hillsdale, NJ: Lawrence Erlbaum Associates.

Malik, M. A. (1980). Historical and pedagogical aspects of the definition of function. *International Journal of Mathematical Education in Science and Technology*, 11(4), 489-492.

Miles, M. B. & Huberman, A. M. (1994). *Qualitative Data Analysis* (*An Expanded Sourcebook*). London: Sage Publications.

Phillips, N. & Hardy, C. (2002). *Discourse Analysis: Investigating Processes of Social Construction*. United Kingdom: Sage Publications Inc.

Podolefsky, N. P. & Finkelstein, N. D. (2006). Use of analogy in learning physics. *Physics Education Research*, 2, pp. 101-110.

Rattermann, M. J. (1997). Commentary: Mathematical reasoning and analogy. In L. D. English (Ed.), *Mathematical Reasoning: Analogies, Metaphors, and Images* (pp. 247-264). New Jersey: Lawrence Erlbaum Associates Inc.

Richland, L. E., Holyoak, K. J., & Stigler, J. W. (2004). Analogy use in eight-grade mathematics. *Cognition and Instruction*, *22*(1), 37-60.

Shulman, L. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15,* 4-14.

Spellman, B. A. & Holyoak, K. J. (1996). *Pragmatics in Analogical Mapping*. *Cognitive Psychology*, *31,* 307-346.

COMPUTER ALGEBRA SYSTEMS, SEMIOTIC ACTIVITY AND THE COGNITIVE PARADOX

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I propose that a semiotic perspective provides an illuminating view of mathematical activity. In line with this position I suggest that a Computer Algebra System (CAS) may be viewed as a semiotic tool. Given the capacity of a CAS to transform signs within and between registers, I specifically argue that the use of a CAS may facilitate the learning of mathematics. This argument is based on Duval's (2006) cognitive paradox: how can a learner distinguish the represented object from its semiotic representations when there is no access to the mathematical object apart from its semiotic representations? I illustrate these theoretical arguments with a semiotic analysis of a pair of mathematical learners at first–year university level engaging in mathematical activity whilst using a CAS.

INTRODUCTION

The important role that technological tools may play in the learning of mathematics is well-recognised within the mathematics education world. For example, there is research that focuses on the possibilities and drawbacks of the use of technology (for example, Hershkowitz and Kieran, 2001). Other research (for example, Artigue, 2002) argues that the successful introduction of technology into a mathematics classroom involves the development of a complex relationship between user and tool; the learner has to construct personal schemes which turn the tool into an instrument for learning (instrumental genesis).

In this paper, I argue that a Computer Algebra System (CAS) is a tool for semiotic activity. I show how such a perspective illuminates the process whereby the use of CAS may promote mathematical understandings.

A SEMIOTIC APPROACH

Mathematics as a semiotic system

The idea of adopting a semiotic perspective when looking at the nature of mathematics and mathematical activities has its modern roots in the writings of the fathers of contemporary semiotics, Peirce (1839–1914) and Saussure (1857–1913). During the twentieth and current century, a semiotic view has been developed and applied to mathematics or mathematics education by, for example, Rotman (1993), Radford (2005), Presmeg (2006).

In this paper I use Ernest's (2006) formulation of mathematics as a semiotic system as my broad framework. Ernest (ibid.) argues that mathematics consists of three components: a set of signs which may be written or uttered or encoded electronically, a set of rules for sign production and "a set of relationships between signs and their meanings embodied in an underlying meaning structure" (p. 70). Furthermore, he argues that sign use is socially located: it is part of social and historical practice. Ernest's semiotic view of mathematics links the individual (who constructs her own meaning from the mathematical signs) with the social (the individual's successful use of mathematical signs must be compatible with their use by the community of mathematicians). It also links the subjective with the objective "For signs are intersubjective, and thus provide both a basis for subjective meaning construction, as well as the basis for shared human knowledge, which …is what is taken for objective knowledge" (ibid, p.68).

Clarification of how I use word 'sign' is required: "A sign is a thing which serves to convey knowledge of some other thing, which it is said to stand for or represent" (Peirce, 1998, p.13). An essential aspect of a sign is that it is experienced meaningfully. That is, it must signify to someone something other than itself. For example, a green traffic light is a sign that tells one to go; it is not there to make one think of greenness. In the phrase '*a*=*b*', '=' is a sign which tells us that *a* and *b* are equal; it is not there to make us think of the shape '−' or the combination of shapes '='.

According to Peirce (1998) all signs have three parts: a representamen (signifier) which refers to the form which the sign takes (not necessarily material), an object (a physical thing or an abstract concept) and an interpretant (the idea or meaning of the object). Peirce refers to the interaction between the representamen, the object and the interpretant as 'semiosis'. Significantly the interplay of the sign's components leads to the possibility of infinite semiosis whereby the representamen stands for an object which entails an interpretant and this interpretant in turn becomes the representamen for yet another object and so on. In 'good' learning, semiosis continues until the learner is able to use the mathematical sign in a way that is meaningful to herself and is commensurate with its use by the relevant mathematical community.

Examples of mathematical representamen are symbols, words, graphs. Examples of mathematical objects are the conceptual objects: the function, the rectangle. Examples of interpretants are an idea or interpretation of the function, the rectangle. For example a graph of a parabola (the representamen) is a particular representation of the mathematical object, a quadratic function. Different individuals may construct different interpretants for the quadratic function (for example, the shape of the parabola or the fact that any value in the range other than the vertex is generated by two different values in the domain, and so on).

CAS and the cognitive paradox

I propose that a view of CAS as a tool which facilitates the production and transformation of mathematical signs helps illuminate how it may serve as a tool for learning mathematics. My argument is based on Duval's thesis that "the only way to have access to [mathematical objects] and deal with them is using signs and semiotic representations" (2006, p. 107). Hence we get the cognitive paradox of access to knowledge objects: how can a learner distinguish the represented object from its semiotic representations when there is no access to the mathematical object apart from its semiotic representations? (*ibid,* 2006).

Duval argues that there are four "registers of representation" (2001, p.2) which are relevant for mathematical activity. Briefly these comprise the register of natural language (as used in proofs), the register of numeric, algebraic and symbolic notations, plane or perspective geometrical figures and Cartesian graphs. Representamen (that is, representations) necessarily differ from one register to another. Mathematical comprehension involves the capacity to change from one register to another "because one must never confuse an object and its representation" (2001, p.7). Duval (ibid.) calls the process of transforming the representation (or representamen) of a mathematical object from one register to another, a "conversion". He argues that two representations (or representamen) of the same mathematical object in two different registers do not have the same content – they may denote the same object but different registers make different properties of the object explicit. He also claims that another type of transformation that is intrinsic to mathematical activity is a treatment (also called processing). A treatment is a transformation of a representation (or representamen) that occurs within the same register; for example solving an equation given symbolically within the symbolic register.

My argument is that a CAS is a tool that can transform mathematical signs in accordance with the standard rules and procedures of mathematics. Hence, and in terms of the cognitive paradox, its use *may* facilitate the comprehension of mathematical objects and relationships. For example, a user may be able to move easily from a symbolic to a graphic representation (a conversion). Seeing the same object using different representamen may enable the learner to construct different interpretants for the same object. In this way, the student may notice important properties of the object not previously perceived. Also seeing different objects in the same registers may help the student discriminate between properties of these different objects.

The use of the word 'may' in the above paragraph is important: the use of a CAS does not in itself guarantee that a user is able to move from one mathematical sign to another (for example, she may not know the correct CAS syntax to generate a representamen) nor that the user is able to recognize the same object in different registers (for example, she may not recognize that a graph of the derivative of sin x is a graph of cos x).

Duval's framework has also been used by Winsløw (2002) who argued that the use of a CAS may enable mathematical activity on a conceptual level higher than possible without the tool (the lever potential).

Semiotic activities with a CAS

In order to examine mathematical activity with a CAS, a further refinement of Duval's semiotic registers is required. In particular it will be useful to distinguish the different notation systems (symbolic, algebraic and numeric) from one another. For example, using *Mathematica* one can approximate a limit numerically (numeric register); using paper and pencil, one can determine the limit using laws of algebra (algebraic register); using *Mathematica*, one can evaluate the limit (symbolic register). Of course, I am assuming that the limit exists in all these instances.

Furthermore I distinguish different media (CAS or pencil and paper) from one another. Chandler (2002, p. 232) argues that "signs and codes are always anchored in the material form of a medium – each of which has its own constraints and affordances". (The specifics of the CAS medium differ according to which particular CAS is being used, eg *Mathematica*, Derive, etc. But each CAS uses its own representamen which are different to the representamen used in traditional mathematical notation.) The production of mathematical signs in the CAS medium involves, inter alia, learning a new syntax and being able to distinguish certain mathematical objects or operations (which may look the same in the paper and pencil medium) from each other. Indeed the precise notation of a CAS may be problematic for students and may act as an impediment to the effective use of the CAS (Pierce & Stacey, 2004).

For example, to solve the equation $x^2 - 4 = 0$ using *Mathematica,* the user has to use the double equal sign: Solve [x^2 – 4 = = 0, x]. To define a function, $f(x) = \sin x$, the user needs to use the single equal sign preceded by a colon: f[x_]: = Sin [x]. To define a constant, say area of a circle where r is radius the user enters: *area* = π*r^2.

On the other hand, CAS can enable the production and transformation of mathematical signs. That is, CAS can be used to execute numerical operations, to generate graphs, to define functions and to manipulate symbols in a mathematical way. For example, even if the user does not yet know how to differentiate the Arcsine function, she can enter the command D[ArcSin[x], x] (symbolic register) and get the response $1/\sqrt{1-x^2}$ (a treatment in the symbolic register). The user can use the new sign (within the CAS medium or not) to produce yet another sign, for example, a graph, which she may then use to produce yet another sign and so on. This is reminiscent of the notion of unlimited semiosis. This production of mathematical signs by CAS depends largely on the initial production of signs by the individual using the CAS medium. Besides syntactical concerns, the user may need knowledge of specific properties of the mathematical object. (For example, to plot a graph of the Arcsine function, she needs to know the domain of Arcsine.)

But the crucial point is that the learner, given sufficient knowledge of the CAS syntax and the mathematics, may use the CAS to produce new mathematical signs. These new signs may enable her to recognise the same object in different registers thus enriching or supporting her conception of the mathematical object. That is, different representamen may enable the user to construct different interpretants for the same object. In this way the learner can construct or deepen her knowledge of the mathematical object.

EXAMPLE OF SEMIOTIC ANALYSIS OF STUDENTS ENGAGING IN A CAS-BASED TASK

In this brief example, I hope to demonstrate how a semiotic perspective provides the researcher or teacher with a fresh insight into students' activities with a CAS. In particular I demonstrate how learners use various signs to generate yet new signs. These new signs with their new interpretants eventually enable appropriate mathematical activity.

The Context and Task

Mathematica was introduced several years ago into the first−year Mathematics Major Course at the South African university at which I lecture mathematics. Every two weeks students use this CAS during a tutorial to solve mathematical problems and to consolidate or anticipate new mathematical material.

Near the end of the previous academic year all students in the class were given an assignment. The assignment was designed to introduce students to the concept of the Maclaurin polynomial before the students had been introduced to the concept in regular mathematics lectures. The assignment involved the use of CAS and paper and pencil.

Five pairs of volunteer students were audio−taped (a pair at a time) while working on this mathematical assignment. The CAS keystrokes were recorded by Bulent software. I took field–notes during the recordings which took place at a computer in my office.

In the episode below, Temba and Sipho are working on a particular task from this assignment. Previously in the assignment they had to generate, symbolically and graphically, the quadratic approximation (second order Maclaurin polynomial) p, of $f(x)$ = Cos x given that $p(0) = f(0)$, $p'(0) = f'(0)$ and $p''(0) = f''(0)$. They did this successfully and found that $p(x) = 1 - ½\, x^2$. They now proceed with the following task:

Determine the values of x for which the quadratic approximation $p(x)$ found above is accurate to $f(x)$ within 0.1. [Hint: Graph the functions, $f(x) = \cos x$, $y = p(x)$ and $y = \cos x + 0.1$, $y = \cos x - 0.1$ on a common screen.]

Activities and Analysis

After some discussion about the meaning of the question in the task, Temba and Sipho plot all four graphs on one screen, as per the hint. They use domain $(-4\pi, 4\pi)$. This results in a picture (Figure 1) in which all four graphs are very close together; it is consequently difficult to distinguish one graph from another.

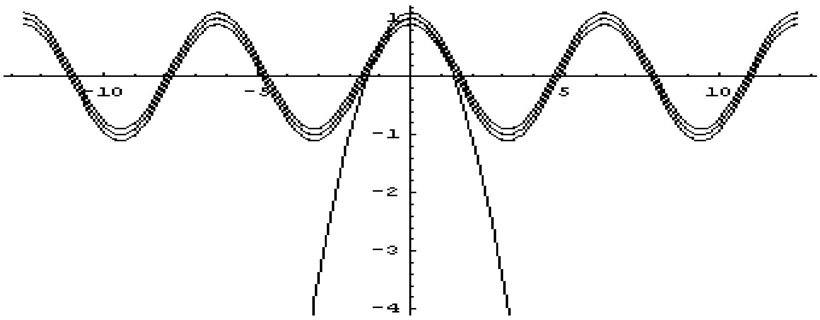

Figure 1.

Despite this Temba and Sipho are able to generate several meaningful signs from these CAS-generated graphs.

1. Temba: No! What's happening there (referring to screen, ie Figure 1).
2. Sipho: I'm not too sure. Okay, oh ya.
3. Temba: Oh ya.
4. Sipho: I can see what is happening. It's shifted in two directions.
5. Temba: Oh. The centre one. The one in the centre. If you can see. That's probably the Cos one, Cos x. And then minus 1 for the bottom one. Minus 0.1, I mean. And plus 0.1.
6. Sipho: They are saying: which values of x… its accurate to within 0.1. Wouldn't that be where they intersect? Do you see what I am saying? Like you have this one.
7. Temba: Um
8. Sipho: You have, you have a Cos graph coming like this. And you have Cos plus 0.1 and you have Cos minus 0.1 (drawing with pencil the four graphs – Figure 2). Then you have this quadratic estimate over here.

:

9. Temba: Okay do you see at this end… I'd say, um. You see where… what will, what will the quadratic do here. Won't it cut the Cos minus 0.1 there. And then not go into these graphs? Right? (Looking at hand–drawing and screen).

:

10. Temba: Like what I am trying to say to you is, we must equate our p(x) to that point there and this point here (darkening points of intersection on Figure 2). So it's in between there… the values where it is accurate.

Analysis: At first Temba & Sipho struggle to interpret the graphical sign (Figure 1) that they have generated with the CAS (lines 1 – 3). But in line 4, Sipho "sees what is happening" (the interpretant) which he partially explains (line 4) using the language register. Temba elaborates further (line 5) by explaining correctly that the centre graph is Cos x, and that the lower graph is Cos x – 0.1.

In line 6, Sipho rhetorically asks: "Wouldn't that be where they intersect?" Presumably this question (a sign in the language register) is consequent upon previous interpretants. Also this question anticipates eventual mathematical activity (finding the points of intersection of two graphs).

Sipho goes on to generate yet another graphical sign (line 8) using paper and pencil. In this drawing (Figure 2), he sketches the four graphs with the points of intersection of p(x) and Cos x – 0.1 highlighted. This new graphical sign both depends on the previous signs (with their interpretants) and looks forward to the generation of future signs. (This hand-drawn graph represents a similar object to that of Figure 1, but with a different domain and scale. Possibly Sipho chooses to hand-draw rather than generate the graphs with CAS due to a lack of confidence with CAS?)

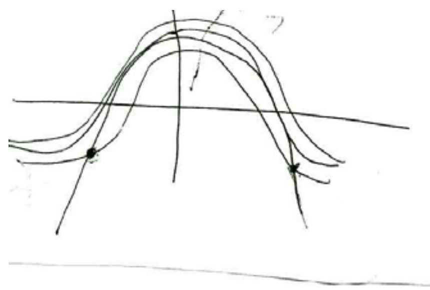

Figure 2.

After further discussion about points of intersection (omitted here), Temba argues correctly that the p(x) graph will only cut the Cos x – 0.1 graph (line 9). A little later (line 10) he is able to transform this sign (interpretant) into an appropriate plan of mathematical activity (equate p(x) to Cos x –0.1).

Due to space constraints, I cannot provide further transcript for analysis. Suffice to say that the students generate further CAS graphs (consisting of $p(x)$ and Cos x–0.1 on a single set of axes). Consequent upon the generation of these signs (with their new interpretants) they use the appropriate command, FindRoot (symbolic register in CAS), to find the points of intersection of $p(x)$ and Cos x – 0.1. They are then able to infer the values of x for which $p(x)$ approximates Cos x to within 0.1.

DISCUSSION

In the above episode we see how the students initially use the CAS as a tool to transform the instructions (signs) in the task (in symbolic and language register) into signs in the graphical register. The students' interpretation (the interpretants) of the CAS-based graphical sign leads them to successfully generate further signs (in language register and paper-and-pencil and CAS graphical registers). The interpretation of these further signs ultimately leads to their finding the relevant points of intersection using the symbolic register in CAS.

This semiotic analysis illustrates my elaboration of Duval's argument: Seeing the same mathematical object (in this case, the values of x for which the quadratic approximation $p(x)$ is accurate to $f(x)$ within 0.1) using different representamen (in graphical and language registers primarily) enables the learner to construct different interpretants (for the same object) and consequently to embark on an appropriate course of mathematical activity.

CONCLUDING SUMMARY

I have argued and demonstrated that CAS is a tool for semiotic activity. Within this framework, I have examined one small episode of students using CAS and paper and pencil. Drawing on Duval's (2006) notion of the cognitive paradox, I have demonstrated how the movement between signs in different registers and media facilitates mathematical activity. Implications for designing CAS–based tasks include the use of activities which exploit CAS's multi–representation affordances.

References

Artigue, M. (2002). Learning mathematics in a CAS environment: The genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *International Journal of Computers for Mathematical Learning*, 7, 245-274.

Chandler, D. (2002). *Semiotics: The Basics*. Oxon: Routledge.

Duval, R. (2001). The cognitive analysis of problems of comprehension in the learning of mathematics. Paper presented at Discussion Group on Semiotics and Mathematics Education at 25th Conf. of the Int. Group for the Psychology of Mathematics Education, Utrecht.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics*, 61(1-2), 103−131.

Ernest, P. (2006). A semiotic perspective of mathematics: The case of number. *Educational Studies in Mathematics*, 61(1-2), 67−101.

Hershkowitz, R. & Kieran, C. (2001). Algorithmic and meaningful ways of joining together representatives within the same mathematical activity: An experience with graphing calculators. In M. van den Heuvel–Panhuizen (ed.) *Proc. 25td Conf. of the Int. Group for the Psychology of Mathematics Education,* (Vol. 1, pp. 96–107). Utrecht, The Netherlands: PME.

Peirce, C.S. (1998). In Peirce Edition Project (Ed.), Volume 2. *The Essential Peirce*. Indiana University Press, Bloomington.

Pierce, R. & Stacey, K. (2004). A framework for monitoring progress and planning teaching towards the effective use of computer algebra systems. *International Journal of Computers for Mathematical Learning*, 9, 59-93.

Presmeg, N. (2006). Semiotics and the "Connections" standard: Significance of semiotics for teachers of mathematics. *Educational Studies in Mathematics*, 61(1-2), 163 − 182.

Radford, L. (2005). On the relevance of semiotics in mathematics Education. Paper presented at Discussion Group on Semiotics and Mathematics Education at 25th Conf. of the Int. Group for the Psychology of Mathematics Education, Utrecht.

Rotman, B. (1993). *Ad Infinitum: The Ghost in Turing's Machine*. Stanford: Stanford University Press.

Winsløw, C. (2002). Semiotic analysis of Dreyfus' potential in first–year calculus. In A.D.Cockburn and E.Nardi (eds.) *Proc. 26th Conf. of the Int. Group for the Psychology of Mathematics Education,* (Vol. 4, pp. 409–416). Norwich, UK: PME.

TEACHERS' AND THEIR STUDENTS' PERCEPTIONS OF THEIR MATHEMATICS CLASSROOM ENVIRONMENTS

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Teacher and student versions of the same instrument were used to compare the perceptions of their classroom environments of a sample of mathematics teachers with those of their students. Overall the data suggest that the teachers had realistic views of the extent to which their classrooms conformed to constructivist principles, but significant differences were found for a one quarter of the items suggesting aspects of the classroom environment that may warrant consideration by teachers. Teachers' descriptions of ideal and typical mathematics lessons provided insights into factors that teachers perceived as constraining their capacities to create the kinds of classroom environments that they wanted and suggest possible reasons for the discrepancies between their's and their students' perceptions.

LITERATURE & BACKGROUND

Classroom environment refers to the overall psychological and social context of the classroom (Fraser, 1991) and is the net result of myriad cognitive, affective and social elements to which teachers and students alike contribute (Shuell, 1996). Considerable research has found that students learn more effectively when their perceptions of their classroom environments are positive (Dorman & Ferguson, 2004) and instruments have been developed to measure particular aspects of classroom environments. Among these is the Constructivist Learning Environment Survey (CLES), used in the study reported here, which was designed to measure the extent to which the classroom environment is consistent with a constructivist view of learning (Taylor, Fraser, & Fisher, 1993).

As a theory of learning, constructivism does not prescribe particular teaching practices, but it is possible to identify principles or beliefs, held by the teacher, that are consistent with a constructivist view of learning and which are necessary for the creation of a constructivist classroom environment (Pirie & Kieren, 1992). For example, Pirie and Kieren (1992) asserted that the teacher must recognise: the differing mathematical understandings that students bring with them; the unpredictability of students' learning; that there is more than one pathway to understanding a given mathematical concept; and that for any topic various levels of understanding exist and that the process of coming to understand is never completed.

Consistent with this, the subscales of the CLES comprise a set of broad aspects of a classroom environment that could be described as constructivist. Each could be manifested in a variety of ways and hence do not relate to specific pedagogical practices. They were described by Taylor et al. (1993, p. 6) as follows:

Autonomy: Extent to which students control their learning and think independently.

Prior Knowledge: Extent to which students' knowledge and experiences are meaningfully integrated into their learning activities.

Negotiation: Extent to which students socially interact for the purpose of negotiating meaning and building consensus.

Student-Centredness: Extent to which students experience learning as a personally problematic experience.

Recent and ongoing curriculum reform efforts in many places are also underpinned by constructivist views of learning. In this environment it is increasingly important that teachers understand the implications of constructivism for their teaching and are able to incorporate aspects like those identified in the subscales of CLES into their mathematics classroom environments. In Australia, where this study was conducted, mathematics is regarded as the discipline that underpins the development of *numeracy*, and it is numeracy, with its inherent emphasis on the application, relevance, and usefulness of mathematics, which is central to the curriculum. For students to appreciate the usefulness of mathematics, its applications need to be included in curricula (National Council of Teachers of Mathematics (NCTM), 2000) and connections made with the lives and interests of students (Wiske, 1998).

Consistent with social constructivist learning theories, the value of student talk has been highlighted in policy documents (e.g., Department of Education Training and Youth Affairs, 2000), curriculum documents (e.g., DoET, 2002) and in research reports and advice for teachers (e.g., Watson, De Geest, & Prestage, 2004). Amit and Fried (2005) linked the control of classroom talk with the control of ideas and the notion of shared authority which they saw as consistent with constructivism, and which are implicit in the Autonomy, Negotiation and Student-Centredness subscales of the CLES.

Differences between mathematics classrooms and those in other school subjects have been attributed to the prevalence of such beliefs as, there is just one way to solve mathematics problems, and that achievement in mathematics is more strongly related to innate ability than is achievement in other subjects (Ryan & Patrick, 2001). These beliefs are at odds with the constructivist views of learning that underpin ongoing reform efforts in mathematics teaching and hence consideration of the extent to which teachers and students perceive their mathematics classrooms to conform to constructivist principles indirectly reflect the prevalence of those beliefs (Beswick, 2005). Comparisons of teachers' and students' perceptions raise questions about the particular features of classroom life to which the two groups are attending in forming their perceptions and have the potential to alert teachers to aspects of their practice that students may be interpreting differently from them.

Associations between students' perceptions of various aspects of their mathematics classroom environments and behaviours likely to be important to their learning include: links between the social aspects of the environment and students' engagement and motivation (Ryan & Patrick, 2001); and between classroom goal structure and teacher discourse, and students' tendency to use avoidance strategies (Turner et al., 2002) or other self-handicapping behaviours (Dorman & Ferguson, 2004). One of few studies to have examined associations between students' and teachers' perceptions of their classroom environments (including mathematics) found that teachers perceived their students' effort and use of strategies less favourably than did the students (Meltzer, Katzir-Cohen, Miller, & Roditi, 2001).

THE STUDY

The following questions provided the focus of the current study.

- What differences are there between students' and teachers' perceptions of the extent to which their mathematics classroom environments can be described as constructivist?
- What factors do teachers believe constrain their capacity to create their preferred mathematics classroom environments?

Instruments

Teacher and student versions of the CLES (Taylor, Fraser, & Fisher, 1993) were used. These differed very slightly with, for example, the word "students" in the teacher version replaced with "I" in the student version. Both instruments comprised 28 items with seven contributing to each of the four subscales described by Taylor et al. (1993). Each item required a response on a 5-point Likert scale indicating the frequency, from *Never* (scored 1) to *Very Often* (scored 5), with which the respondent perceived the event described to occur.

A sample of teachers were asked firstly to describe an ideal mathematics lesson, and then a typical one, in terms of what they and the students would be doing, the physical environment, mathematics content, teaching methods, and resources.

Participants and Procedure

A total of 25 mathematics teachers (i.e., teachers with two or more mathematics classes) in 6 secondary (Grades 7-10) schools in one rural region were asked to complete the CLES (teacher version) for at least two of their mathematics classes. They then administered the CLES (student version) to all students in their classes. A number of teachers chose to complete just one version for two or more classes as they perceived the classroom environments in both/all to be essentially the same. Thirty teacher surveys were thus completed. Several teachers also administered the student version to just one class resulting in a total of 39 classes contributing data. Of these, 34 were also mentioned on the teacher versions. The 25 teachers had from 1-38 years of teaching experience, 17 were male, 8 had studied mathematics to third year university level, including 3 with majors in the subject, and 4 had a Masters degree in education. The eight interviewed teachers were representative of the diversity of survey results and of the teaching experience, gender, and mathematics background of the 25 teachers, but included 3 of the teachers with a Masters degree.

RESULTS

For the majority of classes the teachers scored the Student-Centredness scale lower than the other scales. The average totals for the four teacher version subscales were: Negotiation, 24; Prior Knowledge, 23; Student-Centredness; 15, and Autonomy; 23. The corresponding averages for the 39 classes were Negotiation, 23.7; Prior Knowledge, 22.1; Student-Centredness, 15.0 and Autonomy, 23.2. The average classroom environment, as perceived by both teachers and students, was therefore one in which students negotiated meaning through social interaction, were autonomous in their learning and thinking, and engaged in learning activities that were integrated with their prior experience and existing knowledge. However, students were perceived to be relatively unlikely to experience their learning as personally problematic. That is, teachers were regarded as primarily responsible for deciding on content, setting tasks and deadlines, and providing solution methods.

Independent sample *t*-tests comparing the teachers' responses and the average responses of students to individual items revealed significant differences for seven of the 28 items. Effect sizes were calculated as described by Burns (2000) and in each case were either medium or large. The 34 classes which completed the CLES (student version) and in relation to which their teacher completed a CLES (teacher version) were considered in this analysis. The results are shown in Table 1.

CLES (teacher version) item	Teacher Mean n=34	Class Mean n=34	Mean diff. (teacher -student)	Std Dev.	Sig. (2-tailed)	Effect size
1. In this class students ask each other about their ideas.	3.71	3.10	0.61	0.16	0.000**	0.92
2. In this class I help students to think about what they learned in past lessons.	3.88	3.08	0.80	0.12	0.000*	1.66
3. In this class students think hard about their own ideas.	3.50	3.82	-0.32	0.12	0.010*	0.65
5. In this class students don't ask other students about their ideas.	2.00	2.49	-0.49	0.10	0.000**	1.14
17. In this class students try to make sense of other students' ideas.	3.09	3.43	-0.34	0.14	0.017*	0.61
18. In this class students learn about things that interest them.	2.79	3.30	-0.50	0.13	0.000**	0.94
24. In this class I insist that students complete activities on time.	3.65	4.00	-0.35	0.12	0.005**	0.71

$^*p<0.05$. $^{**}p<0.01$.

Table 1. Significant differences between teacher and class average CLES responses

Differences between the average perception of teachers and those of their classes concerning the frequency with which students asked one another about their ideas were significant (Items 1 and 5, $p=0.000$) with teachers more likely to perceive this as a frequent occurrence. On average, the students were more likely than their teachers to believe that they tried to make sense of their peer's ideas (Item 17, p=0.017) and that they thought hard about their own ideas (Item 3, p=0.010). Teachers were significantly more inclined than their students to believe that they often helped students to think about what had been learned in past lessons (Item 2, p=0.000), but less likely to consider that students often learned about things of interest to them (Item 18, p=0.000). Compared with their students, teachers tended to believe that they enforced deadlines for task completion less often (Item 24, p=0.005).

Interview data were listed and clustered (Miles & Huberman, 1994) according to whether they referred to students, teachers or other factors, and then according to whether they referred to an ideal or existing situation, or were seen as a constraint on achieving the ideal. The factors mentioned more than once, and the numbers of mentions (sometimes more than once by the same teacher) are shown in Table 2.

Ideally students…	No.	Constraints: students	No.
would be motivated/engaged	5	Many students are unmotivated, especially;	
would discuss their work	4	average/lower ability students (4), older	
could be trusted	3	students (2), after lunch (2)	12
would sit and listen	2	Some students are disruptive	3
would come up with problems for themselves	2	Social/peer group more important than results/mathematics	2
would be able to hypothesis		You need to be quite rigid, discipline is poor	2
and plan investigations	2	Students don't like thinking	2
I actually try to…	**No.**	**Constraints: teachers/ teaching**	**No.**
make students think	4	Teachers need to understand the mathematics	3
avoid telling students		Teachers need to know students as learners	3
answers	3	Heterogeneous classes (grades 9 & 10)	3
emphasise why we are doing things – more than utilitarian	2	There is insufficient professional learning for teachers	2
I would like to…	**No.**	**Constraints: Other**	**No.**
make links to real people		Some topics are not suited to practical tasks	6
using and enjoying		Lack of budget, resources, status	4
mathematics	2	Inadequate physical space	3
		Irrelevant curriculum (grades 9 & 10)	2
		Maths is not valued in society	2

Table 2. Interview responses concerning ideal and typical mathematics lessons

There were also single mentions that students ideally would; be challenged and stimulated, help one another, and be prepared to take risks. Individual teachers professed attempting to; use ambiguity and conflict to stimulate students' thinking, listen more and talk less, emphasise the importance of mathematical communication, use positive relationships with students to motivate them, use practical tasks, work with students on problems that he hadn't already found answers to, and keeping students occupied. One teacher said that he would like to include some history of the ideas that they were dealing with in mathematics lessons. In terms of constraints there were single mentions of; the struggle that students have to volunteer their own ideas for investigation, students' difficulties with communication, the need for older lower ability students to have life skills rather than other aspects of the curriculum, and some students' tendency to gossip if allowed to talk. Individual teachers observed; that teachers are very isolated in their classrooms, that teaching as he would like required enormous amounts of preparation time, and that non-traditional teaching was exhausting.

DISCUSSION

The different directions of the differences between the average perceptions of teachers and their classes concerning the frequencies of students asking one another about their ideas (perceived by teachers as more frequent), and of students trying to make sense of each other's ideas and thinking hard about their ideas (perceived by students as more frequent) suggests that teachers are more inclined than their students to judge student interactions as other than genuine efforts to understand. The fact that the most frequently mentioned constraint on achieving their ideal mathematics lesson was students' lack of motivation is consistent with this. Indeed, many of the constraints mentioned related to students and their shortcomings. In all cases the teachers were referring to a subset of their students or to particular classes. In the latter case these were always lower ability or older students (grades 9 and 10). For example one teacher said,

> the [grade 7] class I've got at the moment, because they're quite good, an enjoyable class to teach. They're all sort of proactive and enthusiastic, but by nines and tens particularly the lower levels, I think their social and peer group's are more important to them than probably, for a lot of them, than their, how would you say, their results or anything.

It seems likely that teachers' concern to maintain order, and the effort spent in attempting to motivate some students led to them more readily to associate student talk with off-task behaviour than perhaps was the case.

All but the two interview responses (Ideally students would sit and listen) about the behaviours of students in an ideal lesson were consistent with constructivist ideas, as were all of the things that teachers professed to be trying to do and wanting to do. This suggests that these teachers were aware of and largely in agreement with reform thinking based on constructivist ideas.

The constraints related to teachers and the demands and circumstances of their teaching were all offered by a total of four of the teachers, including all three with a Masters degree, with the majority coming from just two. It is likely that the further study (in education not mathematics) undertaken by these teachers had alerted them to broader issues in mathematics education. The six comments on the difficulty of using practical tasks for some topics were from other teachers, each of whom illustrated their ideal lessons with descriptions of particular topics that they had taught in non-traditional ways. It could be that they lacked knowledge of how to teach other topics in similar ways.

Amit and Fried (2005) contrasted the clear distinction, usually apparent in mathematics classrooms, between the one with authority (the teacher) and those under that authority (students), with the ideal of a community of mathematical thinkers (teacher and students) subject to the authority of the discipline. They connected this with traditional, but still prevalent, ideas about classroom control. At a practical level Watson et al. (2004) described specific ways in which teachers had moved towards a situation approaching Amit and Fried's (2005) ideal, and acknowledged the opportunity that such changes could afford for off-task behaviour, and need for perseverance on the part of the teacher in establishing new work habits in their classrooms. Concerns about students' behaviour, and teachers' conception of their role, particularly in terms of the nature of their authority, could be important obstacles to the creation of constructivist classroom environments. Differences between teachers' and their classes' perceptions concerning who decided when tasks should be completed may also be linked with these concerns.

The finding that teachers were less inclined than their classes to believe that students often learned about things that interested them, may indicate that the emphasis on relevance found in recent curriculum documents has been effective in raising teachers' awareness of the importance of this issue. Alternatively, it could suggest that students are less concerned about this than has been assumed, although other researchers (e.g., Bay, Beem, Reys, Papick, & Barnes, 1999) have reported positive student responses to curricula including "real-world" applications.

CONCLUSION

This study revealed discrepancies between the perceptions of teachers and their classes regarding the extent to which their classroom environments can be described as constructivist. Most of the teachers in this study aspired to create constructivist environments but felt constrained by a range of factors, principally unmotivated students and their concerns for the implications of this for student behaviour. Future research, including studies incorporating the voices of students, is needed to further unpack the particular features that contribute to these perceptions.

References

Amit, M. & Fried, M. N. (2005). Authority and authority relations in mathematics education: A view from an 8th grade classroom. *Educational Studies in Mathematics, 58*(2), 145-168.

Bay, J. M., Beem, J. K., Reys, R. E., Papick, I., & Barnes, D. E. (1999). Student reactions to standards-based mathematics curricula: The interplay between curriculum, teachers and students. *School Science and Mathematics, 99*(4), 182-187.

Beswick, K. (2005). The beliefs/practice connection in broadly defined contexts. *Mathematics Education Research Journal, 17*(2), 39-68.

Burns, R. B. (2000). *Introduction to Research Methods* (4th ed.). French's Forest, NSW: Longman.

Department of Education Tasmania. (2002). *Essential Learning Framework 1*. Hobart: Author.

Department of Education Training and Youth Affairs. (2000). *Numeracy, a Priority for All: Challenges for Australian Schools* (Commonwealth Numeracy Policies for Australian Schools). Canberra: DETYA.

Dorman, J. P. & Ferguson, J. M. (2004). Associations between students' perceptions of mathematics classroom environment and self-handicapping in Australian and Canadian high schools. *McGill Journal of Education, 39*(1), 69-86.

Fraser, B. J. (1991). Two decades of classroom environment research. In B. J. Fraser & H. J. Walberg (Eds.), *Educational environments: Evaluation, antecedents and consequences* (pp. 3-27). Oxford: Pergamon Press.

Meltzer, L., Katzir-Cohen, T., Miller, L., & Roditi, B. (2001). The impact of effort and strategy use on academic performance: Student and teacher perceptions. *Learning Disability Quarterly, 24*(2), 85-98.

Miles, M. B. & Huberman, A. M. (1994). *Qualitative Data Analysis: An Expanded Sourcebook* (2nd ed.). Thousand Oaks, CA: Sage.

NCTM. (2000). *Principles and Standards for School Mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Pirie, S. & Kieren, T. (1992). Creating constructivist environments and constructing creative mathematics. *Educational Studies in Mathematics, 23*(5), 505-528.

Ryan, A. & Patrick, H. (2001). The classroom social environment and changes in adolescents' motivation and engagement during middle school. *American Educational Research Journal, 38*(2), 437-460.

Shuell, T. J. (1996). Teaching and learning in the classroom context. In D. C. Berliner & R. C. Calfee (Eds.), *Handbook of educational psychology* (pp. 726-764). New York: Simon & Schuster Macmillan.

Taylor, P., Fraser, B. J., & Fisher, D. L. (1993, April). *Monitoring the Development of Constructivist Learning Environments.* Paper presented at the annual convention of the National Science Teachers Association, Kansas City.

Turner, J. C., Midgley, C., Meyer, D. K., Gheen, M., Anderman, E. M., Kang, Y., et al. (2002). The classroom environment and students' reports of avoidance strategies in mathematics: A multimethod study. *Journal of Educational Psychology, 94*(1), 88-106.

Watson, A., De Geest, E., & Prestage, S. (2004). *Deep Progress in Mathematics: The Improving Attainment in Mathematics Project*. Oxford: University of Oxford.

Wiske, M. S. (1998). What is teaching for understanding? In M. S. Wiske (Ed.), *Teaching for understanding: Linking research with practice* (pp. 61-86). San Francisco: Jossey-Bass.

CURRICULUM REFORM IN PRIMARY MATHEMATICS EDUCATION: TEACHER DIFFICULTIES AND DILEMMAS

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This paper examines primary classroom teachers' preparedness of implementing a new curriculum model. The new curriculum displays a paradigmatic shift from a behaviourist approach to more of a constructivist one. The development of problem solving skills is particularly emphasised in the new curriculum. Two questionnaires including items on students' different solution strategies to problems are applied to roughly 500 teachers to seek how teachers value and make sense of different strategies. The data reveals that the teachers are not open to different strategies, have difficulties in evaluating students' responses to the open-ended questions and experience serious mathematical difficulties in assessing students' solutions. We discuss issues raised by the findings with regard to the curriculum implementation.

INTRODUCTION

Dissatisfaction with the long-lasting poor conditions of the educational system has compelled the Turkish Ministry of National Education to put the system in the primary level under close scrutiny. Parties concerned with the poor conditions of the system have decided that what need to be done is more than just window dressing. A paradigmatic shift regarding how learning and teaching are viewed and conducted was considered to be necessary. Endeavours in this direction eventually, unsurprisingly, resulted in a massive curricular change at primary level (MEB, 2004).

In Turkey, primary education lasts for eight years. Students are taught by one classroom teacher in the first five years and different teachers who are specialists in their subject areas in the last three years. Compared with the previous one, the new school mathematics curriculum for the first five years in which we are interested in this paper displays a shift from a behaviourist approach to the one with constructivist flavour (Babadogan & Olkun, 2005). It proposes fundamental changes in learning, teaching and assessment. It adopts a student-centred approach where students are active in their learning. More emphasis is placed upon conceptual understanding rather than procedural one. Such macro skills as problem solving, reasoning, communications and use of technology are emphasised (ibid.).

Teachers' roles are redefined and new roles are assigned to them. They are deemed as facilitators rather than sole transmitters. Teachers are expected to conduct activity-based teaching in which students are encouraged to reason, work cooperatively, communicate with others and share their ideas. The new curriculum also proposes changes in terms of how assessment is conducted. Process and performance-based evaluation rather than product evaluation is emphasised. Students' performance evaluation with such tools as portfolios and projects is suggested.

The curricular change works started in 2004. The new curriculum was piloted in the academic year of 2004-2005 and started to be implemented in 2005-2006 nationwide. Classroom teachers were trained only for a week to get to know about the whole new primary school curriculum. As mentioned above, the new curriculum particularly defines and determines new roles for teachers that they were never used to before. With little training, it is not known how well equipped classroom teachers are to handle their new roles. This study takes a step in this direction and aims to shed light on this issue. The question of how we do this is the focus of following two sections.

THE THEORETICAL FRAMEWORK OF THE STUDY

A curriculum with its philosophy behind, at least theoretically, defines and determines roles for students, teachers, school administrators and parents. It does shape how textbooks are written, which technologies and teaching tools are going to be employed, and how teacher education programmes are/should be designed. A curriculum change, therefore, means changes in all these parameters' roles or uses.

The literature provides evidence that change in a curriculum does not necessarily mean a change in the actual classroom practices (e.g. Ball&Cohen, 1996). Cuban (1992) uses the terms 'intended' and 'taught' curriculum to draw attention to this issue and notes that change in the intended curriculum does not easily reflect itself in delivery in classrooms. Papert (2000) also points to difficulties of implementing a new curriculum with the idea(s) behind it. He claims that when ideas go to school they lose their power and are subjected to disempowerment. He notes his appreciation and shares intentions of contemporary movements of school reform but claims that "in practice these would-be reform movements have allowed themselves to be assimilated to School's way of thinking and in the end bolster rather than reform the fundamentals of School mentality they set out to reform" (p. 722).

The reason that the ideas lose their power or meet resistance when they enter the school is perhaps because they enter an institution in which institutional rules are already well-established, organisational patterns are firmly structured, space and time utilisation is well configured, and roles and authority relations are customarily appropriated (Waks, 2003). A new curriculum with a powerful idea behind it means introducing new institutional rules and therefore fundamental changes in all these parameters (Cuban, 1992). Any attempt in this direction would perhaps encounter the resistance of 'the establishment' particularly formed by the school teachers and administrators (Waks, 2003). The resistance against the implementation of a new curriculum does not come solely from within. Such external factors as standard textbooks, achievement tests and university admission requirements can also hinder the implementation of a new curriculum (ibid.).

The related literature suggests that one of the main reasons that new curricula have not a deep influence on school practice is because the influences of teachers on their curriculum had been neglected too often by curriculum researchers and designers (e.g Manouchehri & Goodman, 1998). The lack of research on teacher influence has since forced researchers to examine how teachers cope with the demands of new curricula (e.g. Manouchehri, 1998). A large body of studies have come into being particularly examining teachers' beliefs, practices (e.g. Middleton, 1999), their subject matter and pedagogical content knowledge with reference to new curricula (e.g. Manouchehri, 1998; Ball & Bass, 2003). These studies suggest that teachers' beliefs, experiences, personal theories, level of content and pedagogical content knowledge all have influences on how they teach and implement a curriculum.

Teachers surely have the chief role in the implementation of a new curriculum. With the new curriculum model in Turkey, the big idea is to shift learning and teaching from a behaviourist approach to more of constructivist one. This assigns dramatically new roles and responsibilities to teachers. Development of problem solving skills is one aspect that is particularly emphasised and teachers are expected to create classroom environments in which students' non-standard solutions to open-ended problems are encouraged. In this study, we aim to explore how well-equipped classroom teachers are to take up their new role in this regard through the following two research questions.

- How open are the primary classroom teachers to different solution strategies to mathematical problems?
- How do primary classroom teachers evaluate students' responses to the open-ended questions?

THE CONTEXT AND METHODOLOGY OF THE STUDY

The project that gave rise to this paper set out to investigate the level of preparedness of classroom teachers in coping with the demand of the new curriculum. We aimed to explore this in two phases: (1) to elicit a large group of teachers' preparedness through questionnaires; (2) to follow a small representative sample of the teachers in the classroom settings to see how they get on with the new curriculum. The data we provide in this paper comes from the first phase.

Two questionnaires with open-ended questions were developed to seek whether teachers themselves are actually open to non-standard solutions and value them. Both questionnaires included items regarding mathematical concepts covered in primary school mathematics curriculum. In this paper, due to space limitations, we focus only on one item from each questionnaire. The first item is related to multiplication (item-1) and the second one is concerned with the calculation of the area of a rectangle (item-2). In both items, teachers are presented with students' different solutions to the problems and asked to evaluate these fictional solutions.

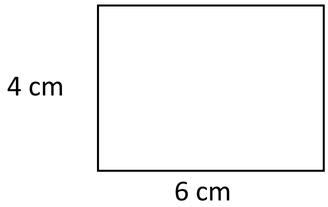
Item-1: $\begin{array}{r} 32 \\ \times\ 25 \\ \hline \end{array}$ Below students' three different responses to this multiplication are presented. All three students have reached the same result. Please evaluate each response and explain which one or ones you would accept as an answer and why? (adopted from Ball & Bass, 2003).

A
$$\begin{array}{r} 32 \\ \times\ 25 \\ \hline 160 \\ +\ 64 \\ \hline 800 \end{array}$$

B
$$\begin{array}{r} 32 \\ \times\ 25 \\ \hline 10 \\ 150 \\ 40 \\ +\ 600 \\ \hline 800 \end{array}$$

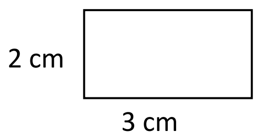
C
$$\begin{array}{r} 32 \\ \times\ 25 \\ \hline 50 \\ 150 \\ +\ 600 \\ \hline 800 \end{array}$$

Item-2: Fourth and fifth grade students are presented with following problem:



What can be the dimensions of a rectangle with exactly half the area of this rectangle? Please explain your answer.

The responses of two students to this problem are presented below. How would you grade these responses **over a range from 0 to 10 and please explain why**? (adopted from Hansen et al., 2005).

	Student's Answer and Explanation	**Score**	**Reason**
Student K	To find out half area of the rectangle, I do this: $\frac{6+4}{2}=5$. Then each dimension can be 5 cm.		
Student L	"I would have the half of each dimension: $6 \div 2 = 3$ and $4 \div 2 = 2$. Then I would come up with a rectangle with a one side being 3 cm and the other 2cm". And draws the following figure: 		

The first questionnaire that included item-1 was applied to approximately 300 classroom teachers and we had 216 returns. The second questionnaire that included item-2 was applied to approximately 200 teachers and we had 177 returns. Of them, 148 teachers responded to item-2. The participating teachers differed in terms of the years of teaching experiences ranging from 2 to 35 years. Those teachers taking the first questionnaires were working in 104 different schools in a large province and those taking the second one were working in 10 different schools in three different provinces in Turkey.

DATA ANALYSIS AND RESULTS

This section presents data analysis and results concurrently. The analysis and related results of each item are provided respectively.

With regard to item-1, a frequency analysis is first carried out to determine the number of teachers accepting any of solutions of A, B, C; both of any A, B, C; or all of them. Table 1 shows that a vast majority of teachers (67%) states that they would accept only solution A, 15% accept both A and B, and only 17% accept all A, B, and C as an answer to the multiplication problem.

	A	**A&B**	**A&B&C**	**No answer**	**Total**
Number	145	33	36	2	216
Percentage	67%	15%	17%	1%	100%

Table 1: Teachers' responses (frequencies–percentages) to item-1

A further analysis is conducted on those teachers who cited accepting only solution A. The aim was to find out why they would accept only solution A. This analysis consisted of repeated readings of participants' reasons for accepting solution A. The analysis eventually generated five categories which encompass the teachers' reasoning for their choices (Table 2):

Categories	**Explanations for categories**
Rule	Teachers cite algorithmic rule of multiplication
Practical	Teachers cite responses like "Solution A is easy, practical and take little time"
B and C being difficult	Teachers' finding these solutions difficult and complex to understand or/and teach
Accept A but listen to B and C	Teachers cite to be open to both B and C solutions but would accept only A as an answer
Not categorised	No reasoning or statements like "I accept only solution A"

Table 2: Analysis of teachers' reasoning for accepting only solution A in item-1

Establishment of the categories is carried out by two researchers simultaneously and 100% agreement was reached for every teacher's response to a category. Frequencies of these categories are presented in Table 3 below. Note that some responses fall under more than one category and hence the total percentage exceeds 100%.

	Those teachers who accept only solution A (145)				
	Rule	Practical	Accept A but listen to B and C	B and C being difficult	Not categorised
Number	79	39	16	17	20
Percentage	54%	27%	11%	12%	14%

Table 3: Responses of teachers who only chose solution A in item-1

Of those teachers who state to accept only solution A, 54% cites rule and 39% cites practicality in explaining their reasons (Table 3). Only 11% of the teachers indicates to listen to B&C solutions too and 12% finds these two solutions difficult.

With regard to item-2, it was applied to 144 teachers to see how teachers view different solutions to open-ended questions, how they evaluate erroneous student answers, whether they are aware of, and able to propose any remediation to, common student misconceptions. A frequency analysis is first conducted to determine how teachers grade student K and L's responses over a range from 0 to 10 (Table 4).

Scores	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Student K	76 51%	13 9%	7 5%	7 5%	5 3%	13 9%	4 3%	3 2%	4 3%	1 1%	15 10%
Student L	35 24%	4 3%	6 4%	6 4%	6 4%	18 12%	3 2%	3 2%	1 1%	0	65 44%

Table 4: Teachers' responses (frequencies–percentages) to item-2

The data reveals that teachers graded students' wrong responses for different reasons over a range from 0 to 10. For solution K, 51% of the teachers gave a score of 0, 10% gave a score of 10 and the rest ranged between. For solution L, 44% of the teachers unexpectedly gave a grade of exact 10 to the wrong response of the student L, 24% gave a grade of 0 and the rest ranged from 1 to 8. Those teachers who knew that the responses were wrong but gave grades from 1 to 5 provided various reasons including "because at least students attempted to solve the problem", "as an encoura-gement or award", and "because the student knows at least how to calculate the area".

DISCUSSION

The results, overall, have shown that classroom teachers are not open to different solution strategies to mathematical problems (Table 1), have difficulties in evaluating students' responses to the open-ended questions, and experience serious difficulties in assessing whether student solutions to open-ended problems are mathematically correct or not (Table 4). Further examination of teachers' reason reveals that they value 'routine', 'rule' and 'practical' aspects of mathematical solutions (Table 3).

We interpret these findings as signalling three potential difficulties in the implementation of the new curriculum. The first one is related to the classroom teachers' difficulties in mathematics. This is particularly evident in the teachers' evaluation of student L's wrong response to item-2 to which 44% of teachers gave a grade of exact 10 (see Table 4). Such a high percentage was unexpected to us and we do not, on the basis of our data, tend to over-generalise this trend to the whole primary teacher population in Turkey. Yet this is an important proportion and points to a possible source of challenge, that is lack of mathematical content knowledge, in implementing the new curriculum. These findings, in fact, before anything else, raise concern with regard to the competency of the teachers' teaching mathematics to the students let alone the implementation of the reformed curriculum.

The second challenge, as our data indicate, is related to the issues of assessment of students' non-standard solutions to open-ended questions. The new curriculum puts a heavy emphasise on the use of open-ended questions for both formative and summative assessments. Yet asking and expecting teachers to employ open-ended questions is one thing but using such questions during instruction and in exams is quite another. Open-ended questions mean variations and unexpected responses in students' solution strategies to the questions that sometimes raise challenges for teachers to make sense. The teachers' responses to item-1 clearly show that teacher tend to privilege rule-based and practical solutions and have difficulties in making sense of different (but correct) solutions (see Table 3). Further to this, on what bases responses to open-ended questions would be evaluated especially if students make an effort to answer? Responses to the item-2 show great variations even in grading of those who found the solutions wrong. For instance, 51% of teachers graded solution of student K in item-2 with 0 and expressed that because it was wrong. Yet student K's solution also received the grades ranging from 1 to 5 from those teachers who, while stating the inaccuracy of the solution, noted that "the student at least tried". This variation in our view is important as the grades send signals to the students what is valued (mathematical accuracy or making effort). To some extent a certain level of variation might be understandable for subjective judgements yet this indicates lack of assessment criterion which teachers draw on in the evaluation of students' work.

The third one is related to overall teachers' already formed personal theories, views, orientations and beliefs with regard to mathematics, its learning and teaching. This is particularly evident in the teachers' reasons for choosing solution A in item-1 in that some appear to hold the view that solutions to mathematics problems should take little time, be practical and employ procedural rules. Of these teachers, for instance, one cites to accept only solution A "because there is only one way to the truth (*right conclusion*)" and another one cites not to accept B and C as correct answers and if his students "attempt to do the multiplication like in B and C, he would interfere at the very beginning not to do so". This stance, in fact, is sharply in conflict with what the new curriculum sets out to achieve, which encourages teachers to "create classroom environments in which students can bring different solutions to the posed problems so that students learn to value different solution strategies in the process of problem solving" (MEB, 2004, p. 11). Our findings, however, suggest that most teachers themselves are not appreciative of and do not value non-standard solution strategies.

We do not see classroom teachers' mathematical difficulties and problems arising from these with regard to assessment and not being open to non-standard approaches in learning and teaching just being peculiar to Turkey. There is evidence that teacher difficulties in mathematics and other aspects especially at primary level is a reality all around the world (e.g. see Ma, 1999; Manouchehri, 1998). This might be understand-able given the fact that these teachers are not specialised in mathematics and responsible for teaching different subjects. Yet we, as mathematics educators, need to take these difficulties seriously and to search ways of improving in-service primary teachers' mathematical content knowledge in a wide scale, probably nationwide. This certainly requires serious consideration about not only the content of such in-service courses but also methods of implementing them. Achieving this collaboration at an international level could be a possibility and perhaps a necessity. Without attending to teachers' mathematical difficulties in the first place, in the words of Papert (2000), curricula changes with its big idea behind (in Turkish case this being constructivism) would meet resistance from the teacher and run the risk of giving into the school way of thinking, losing its power and hence being disempowered.

References

Babadogan, C. & Olkun, S. (2005). Program Development Models and Reform in Turkish Primary School Mathematics Curriculum. *International Journal for Mathematics Teaching and Learning* [Online]. *http://www.cimt.plymouth.ac.uk/journal/default.htm*

Ball, D.L. & Bass, H. (2003). Toward a practice-based theory of mathematical knowledge for teaching. In E. Simmt & B. Davis (eds.), *Proceedings of the 2002 Annual Meeting of the Canadian Mathematics Education Study Group* CMESG, Edmonton, AB, pp. 3–14.

Ball, D.L.&Cohen, D.K. (1996). Reform by the book: What is-or might be-the role of curriculum materials in teacher learning and instructional reform? *Educational Researcher*, 25(9), 6-8.

Cuban, L. (1992). Curriculum stability and change. In P. Jackson (ed.), *Handbook of Research on Curriculum*, Ch. 8. New York: MacMillan.

Hansen, A., Drews, D., Dudgeon, J., Lawton, F.& Surtees, L. (2005). Children's errors in mathematics: understanding common misconceptions in primary schools, Exeter: Learning Matters.

Ma, L. (1999). *Knowing and teaching elementary mathematics: teachers' understanding of fundamental mathematics in China and the United States*. N.J.: Lawrence Erlbaum Asc.

Manouchehri, A. (1998). Mathematics curriculum reform and teachers: what are the dilemmas? *Journal of Teacher Education,* 49(4), 276-286.

Manouchehri, A. & Goodman, T. (1998). Mathematics curriculum reform and teachers: understanding the connections. *The journal of Educational Research*, 92 (1), 27-41.

MEB. (2004). *Talim ve Terbiye Kurulu Baskanligi, Ilkögretim Matematik Dersi (1-5. Siniflar)Ögretim Programi*. Ankara: MEB Basimevi.

Middleton, J. A. (1999). Curricular Influences on the Motivational Beliefs and Practice of Two Middle School Mathematics Teachers: A Follow-Up Study. *Journal for Research in Mathematics Education*, 30 (3), 349-58.

Papert, S. (2000). What's the big idea? Toward a pedagogy of idea power. *IBM Sytems Journal*, 39 (3&4), 720-728.

Waks, L. J. (2003). How globalization can cause fundamental curriculum change: An American perspective. *Journal of Educational Change,* 4 (4), 383-418.

PERSISTENT IMAGES AND TEACHER BELIEFS ABOUT VISUALISATION: THE TANGENT AT AN INFLECTION POINT

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The role of visualisation in the mathematical reasoning teachers present to students may be influenced by several factors (e.g. mathematical, epistemological and pedagogical). Our study explores these potential influences through engaging teachers with tasks that invite them to: reflect on/solve a mathematical problem; examine flawed (fictional) student solutions; and, describe, in writing, feedback to students. Teachers are also interviewed. We discuss responses to one Task (which involved recognising a line as a tangent to a curve at an inflection point) of 91 teachers in order to explore the influence on the teachers' feedback to students of: (i) persistent images of the tangent line; (ii) beliefs about the sufficiency of a visual argument; and (iii) beliefs about the role of visual arguments in student learning.

INTRODUCTION

In the last twenty years or so the debate about the potential contribution of visual representations to mathematical proof has intensified (e.g. Mancosu et al, 2005), not least because developments in IT have expanded this potential so greatly. Central to this debate is 'whether, or to what extent, visual representation can be used, not only as evidence or inspiration for a mathematical statement, but also in its justification' (Hannah & Sidoli, 2007, p73). Recent works (e.g. Giaquinto, 2007) argue that visual means are much more than a mere aid to understanding and can be resources for discovery and justification, even proof. Whether visual representations need to be treated as adjuncts to proofs, as an integral part of proof or as proofs themselves remains a point of contention.

Within mathematics education the body of work on the important role of visualisation has also been increasing and has been focusing on issues as diverse as: curriculum development with an emphasis on visualisation (and often on related IT); mathematicians' perceptions/use of visualisation; students' seeming reluctance to engage (and difficulty) with visualisation; gender differences; links with embodied cognition; etc. – see Presmeg (2006) for a substantial review. Overall we still seem to be rather far from a consensus on the many roles visualisation can play in mathematical learning and teaching. So, while many works clearly recognise these roles, several (e.g. Aspinwall et al, 1997) also recommend caution with regard to 'the 'panacea' view that mental imagery only benefits the learning process' (p315). One of the aims of the study we report in this paper is to explore the role of visualisation with particular reference to the reasoning and feedback that teachers present to students. To do so we first introduce the study briefly and then discuss some of our data.

THE STUDY AND THE TANGENT TASK

The data we draw on in this paper originate in an ongoing study in which we invite teachers to engage with mathematically/pedagogically specific situations which have the following characteristics: they are hypothetical but likely to occur in practice and grounded on learning and teaching issues that previous research and experience have highlighted as seminal. The structure of the tasks we ask teachers to engage with is as follows – see a more elaborate description of the theoretical origins of this type of task in (Biza et al, 2007): reflecting upon the learning objectives within a mathematical problem (and solving it); interpreting flawed (fictional) student solution(s); and, describing, in writing, feedback to the student(s).

In what follows we focus on one of the tasks (Fig. 1) we have used in the course of the study. The Task was one of the questions in a written examination taken by candidates for a Masters in Mathematics Education programme. Ninety-one candidates (of a total 105) were mathematics graduates with teaching experience ranging from a few to many years. Most had attended in-service training of about 80 hours. The first level of analysis of the scripts consisted of entering in a spreadsheet summary descriptions of the teachers' responses with regard to the following: perceptions of the *aims* of the mathematical exercise in the Task; mathematical *correctness*; *interpretation/evaluation* of the two student responses included in the Task; *feedback* to the two students. Adjacent to these columns there was a column for commenting on the means the teacher used (verbal, algebraic, graphical) to convey their commentary and feedback to the students across the script. The discussion we present in this paper is largely based on themes that emerged from the comments recorded in this column. In addition to the scripts we also collected data through interviewing a selection of the participating teachers: their individual interview schedules were based on the first level analysis briefly described above. Interviews lasted approximately 45-60 minutes.

The mathematical problem within the Task in Fig. 1 aims to investigate students' understanding of the tangent line at a point of a function graph and its relationship with the derivative of the function at this point, particularly with regard to two issues that previous research (Biza et al, 2006; Castela, 1995; Vinner, 1991; Tall, 1987) has identified as critical:

- students often believe that having one common point is a necessary and sufficient condition for tangency; and,
- students often see a tangent as a line that keeps the entire curve in the same semi-plane.

The studies mentioned above attribute these beliefs partly to students' earlier experience with tangents in the context of the circle, and some conic sections. For example, the tangent at a point of a circle has only one common point with the circle and keeps the entire circle in the same semi-plane.

Since the line in the problem is a tangent of the curve at the inflection point *A* the problem provides an opportunity to investigate the two beliefs about tangency mentioned above – similarly to the way Tsamir et al (2006) explore teachers' images of derivative through asking them to evaluate the correctness of suggested solutions. Under the influence of the first belief Student A carries out the first step of a correct solution (finding the common point(s) between the line and the curve), accepts the line tangent to the curve and stops. The student thus misses the second, and crucial, step: calculating the derivative at the common point(s) and establishing whether the given line has slope equal to the value of the derivative at this/these point(s). Under the influence of both beliefs, and grounding their claim on the graphical representation of the situation, Student B rejects the line as tangent to the curve.

Year 12 students, specialising in mathematics, were given the following exercise:

'Examine whether the line with equation $y = 2$ is tangent to the graph of function f, where $f(x) = 3x^3 + 2$.'

Two students responded as follows:

Student A

'I will find the common points between the line and the graph solving the system:

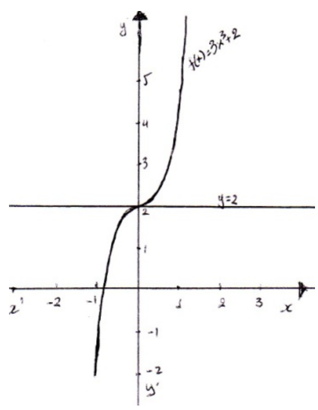
$$\begin{cases} y = 3x^3 + 2 \\ y = 2 \end{cases} \Leftrightarrow \begin{cases} 3x^3 + 2 = 2 \\ y = 2 \end{cases} \Leftrightarrow \begin{cases} 3x^3 = 0 \\ y = 2 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = 2 \end{cases}$$

The common point is $A(0, 2)$.

The line is tangent of the graph at point A because they have only one common point (which is A).'

Student B

'The line is not tangent to the graph because, even though they have one common point, the line cuts across the graph, as we can see in the figure.'



a. In your view what is the aim of the above exercise?
b. How do you interpret the choices made by each of the students in their responses above?
c. What feedback would you give to each of the students above with regard to their response to the exercise?

Figure 1. The Task.

With regard to the Greek curricular context, in which the study is carried out, the Year 12 students (age 17/18) mentioned in the Task have encountered the tangent to the circle in Year 10 in Euclidean Geometry and the tangent lines of conics in Analytic Geometry in Year 11. In Year 12, they have been introduced to the tangent line to a function graph as a line with a slope equal to the derivative of the corresponding function at the point of tangency. Although in Years 11 and 12 the tangent is introduced as the limiting position of secant lines, this definition is rarely used in problems and applications.

One of the themes that emerged from the comments recorded in the spreadsheet with regard to the means the teachers used (verbal, algebraic, graphical) to convey their commentary and feedback to the students concerned the *beliefs (epistemological and pedagogical) of the teachers about the role of visualisation.* For example, with regard to the teachers' evaluation/interpretation of Student B's solution and feedback to Student B we explored questions such as: does the teacher turn the student away from the graphical approach (which may have led the student to an incorrect claim) and towards an algebraic solution in order to help the student change their mind about whether the line is a tangent or not? Does the teacher compare and contrast the algebraic solution to Student B's solution or do they proceed directly to the presentation of an algebraic solution? What types of examples/counterexamples, if any, do they employ in this process? What is the teacher's position towards Student B's grounding their claim on the graph? Etc.

In the course of this part of our analysis we noticed several influences on the teachers' responses: for example almost all teachers distinguished between (and often juxtaposed) Student A's *algebraic* approach and Student B's *graphical* approach. In almost all cases, in both scripts and interviews, the teachers included in their comments an evaluative statement regarding the sufficiency/acceptability of one or both approaches. And often they referred explicitly to their beliefs about, for example, the sufficiency/acceptability of the graphical approach; or about the role visual thinking may play in their students' learning. The teachers' responses also appeared significantly influenced by the mathematical context of the problem within the Task; namely, by their own perceptions of tangents and their own views as to whether the line in the Task must be accepted as a tangent or not.

At this point of our analysis we were somewhat surprised by the fact that 43 out of the 91 teachers supported Student B's claim that the line in the Task is not a tangent line – explicitly (25/91) or implicitly (18/91). In what follows we present examples from our analysis of the data from the 25 teachers who explicitly supported Student B's claim in order to examine the interplay between the teachers' *mathematical views* on whether the line is a tangent or not, *beliefs about the sufficiency/acceptability of the visual argumen*t used by Student B and *beliefs about the role of visual thinking in their students' mathematical learning*. We note that our examples originate in the scripts only, and not in the interviews, due to limitations of space.

PERCEPTIONS OF TANGENTS AND BELIEFS ABOUT VISUALISATION

Mathematical views on whether the line is a tangent or not. Of the twenty-five teachers who explicitly accepted Student B's claim, ten rejected the line as a tangent without stating an argument (phrasing their responses as if this was obvious). The other fifteen stated that the line intersects with the curve without being its tangent either because point *A* is an intersection point but not a tangency point; or because it 'cuts across' the graph as student B argued. Three of these fifteen based the rejection on the fact that the line does not keep the entire curve in the same semi-plane. For example, *Teacher 101* claimed that 'it is not sufficient that the tangent line has only one common point, but it must keep the graph on the same side' and offered the graph in Figure 2. Taking a *local* perspective on Student B's '*cutting across*' argument the teacher also offers Figure 3 and says: 'the tangent could cut across the curve … the line is a tangent at x_0 [in Figure 3] although it cuts across the curve [at another point, *our addition*]'.

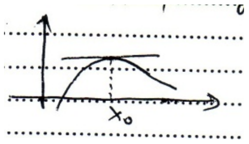


Figure 2.

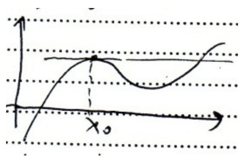


Figure 3.

Beliefs about the sufficiency/acceptability of the visual argument used by Student B. Of the twenty-five teachers, ten did not dispute this visual argument. Of these ten, eight made no reference at all to an algebraic argument. One teacher (*Teacher 81)* made some reference to both the 'algebraic and graphical methods' implying that she accepted the validity of both. She wrote that 'the aim of the exercise is that the students examine whether the line is tangent to the graph either graphically (if they can) or algebraically with the derivative' and later on observed that 'the exercise does not specify which way should be used to solve it'. (We return to this teacher's hint at the superiority of the graphical solution – 'if they can' – later in this section). Another one of these ten teachers (*Teacher 6*) set out with some reference to an algebraic argument that involved accepting the line; on the way, as she proceeded to a consideration of Student B's solution, she deleted the algebraic argument and concluded her response with agreeing with Student B.

The other fifteen teachers, while basing their inference on the graph and supporting Student B's claim, stated the need for supporting and verifying the claim algebraically (11 explicitly and 4 implicitly). These teachers, although they hinted at the algebraic solution for the justification of the answer, did not employ it in the argument they offered the students. As a result they did not confront the inconsistency of their statements. For example, *Teacher 97* wrote:

> To the first student I would say that it is not sufficient that the line and the curve have one common point but that the line must not split the graph as well. The derivative of the curve at *A*(0,2) must be equal to the derivative of *y*=2 at *A*(0,2). To the second student I

> would say that his conclusion is correct and I would encourage him to give a better justified answer.

'Better justification' of a correct answer seems to sum up the views of these teachers. 'Better' is meant as:

- More general, feasible, useful:

> It is not acceptable to answer through graphical representations, because in a more complex case this approach is not feasible, *Teacher 1*

> […] even if a graphical understanding of functions is particularly useful, [Student B] should not forget that it is not always possible to use graphical representations and that he should learn to solve problems also algebraically, *Teacher 80*

- Offering a 'more rounded view of the problems', *Teacher 69*

- Accurate, because, for example, 'however helpful the graph may be, it is never totally accurate when done by hand', *Teacher 53*

- The graph not necessarily constituting a valid complete proof:

> If the exercise is asking for a proof it is better that the graph is accompanied also algebraically by the criterion for finding tangents through the first derivative and monotonicity, *Teacher 64*

In the above examples the teachers, while appreciating their students' employment of visualisation to reach a conclusion, are keen to stress that ultimately students are expected to demonstrate their capacity to complete the task algebraically. It is therefore possible that these teachers' embrace of the visual approach evident in Student B's solution is driven more by their belief in the gradual enculturation (Sierpinska, 1994) of the students into formal mathematical practice and their belief in the assistance that visualisation can provide towards reaching a conclusion (rather than a belief in the completeness of a graph-based argument). We cite below some evidence of these teachers' support for the employment of visualisation by their students.

Beliefs about the role of visual thinking in students' learning. A substantial number of the twenty-five teachers (nine) declared overtly their view of the graphical approach employed by Student B as evidence of 'conceptual' understanding. For example, *Teacher 68* applauded Student B and would say to him that he has a 'rounded way of thinking', 'his idea to investigate the problem graphically is very good', 'understands to a good degree what mathematical thinking is about' and 'carry on this way'.

Five teachers saw students' employment of the algebraic method as 'instinctively' driven by the conditioning students are subjected to in 'traditional' mathematics teaching. For example, *Teacher 4* wrote: 'Student A used the method mechanically' and 'Student B has a complete understanding of the problem'.

Eight teachers declared that the graphical solution is more 'quick and ready', 'clever' and 'not wasting any time', even 'natural' and 'real'. For example, *Teacher 48* claimed that ‘through following a formal procedure we do not always reach correct results if we do not try at the same time to offer a ‘natural’ interpretation of the result’ and would ‘encourage Student A to always try to find the real dimension of the problem (e.g. through drawing the graph)’.

Several teachers, while expressing their appreciation for the graphical approach, stressed that students are not always at ease with it (see also *Teacher 81*’s comment earlier) and are often reluctant to use visualisation.

Overall many of the twenty-five teachers described the pedagogical role of graphical approaches as supporting students’ use of their mathematical *intuition* and *imagination*. For example, *Teacher 69* stated that ‘the aim of the exercise is to encourage students to combine their knowledge in mathematics with imagination in order to reach a result’ and *Teacher 64* that ‘Student B’s answer is better, purely intuitive based, that is, only on the graph’.

It is perhaps interesting to see how some of these twenty-five teachers responded to Student A, with particular regard to that student’s exclusive use of the algebraic method. Many attempted to balance their feedback to the students with regard to the approach they encouraged students to employ:

> It has been observed that many students have difficulty with algebraic manipulation while they are rather facilitated with visualisation, while for others the opposite applies. The teacher must encourage students to work in both ways, *Teacher 53*

In this spirit, and as we saw above, many teachers encourage Student B to work more algebraically. Analogously they encourage Student A to work also graphically:

> To Student A I would explain the mistake he has made and I would suggest one or two directions so that he tries to solve the problem again. I would also tell him to make the geometrical interpretation as this would help him. To Student B I would say that his answer is correct but that he would also need to justify it also algebraically. That is to make a synthesis of the algebraic and the geometric frame, *Teacher 85*

CONCLUDING REMARK

The Task in Fig. 1 invited the teachers to offer feedback to two students one of which had used (incompletely) the algebraic method for deciding whether the line is a tangent and the other had used (incorrectly) a graphical representation of the problem. In this occasion the graph contained information that conjures up images that may lead to the rejection of the line as a tangent. About half of the teachers in our study appeared to get ‘carried away’ by this information – or, in Aspinwall et al’s (1997) term, by these ‘uncontrollable’ images – and agreed with Student B’s incorrect claim that the line is not a tangent. In the evidence we presented above the teachers appeared to get ‘carried away’ not simply by the images that they hold about tangents, conjured up by the graph in Student B’s response, but also by a compelling tendency to support what they described as the more ‘conceptual’, ‘imaginative’ etc. approach of Student B. To them the mathematical problem in the Task offered an opportunity to convey their appreciation for the employment of visualisation.

However lack of awareness of the problems that certain imagery may cause, in this case the graphical representation of tangency at an inflection point, stands in the way of fulfilling the potential within the employment of visualisation.

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References

Aspinwall, L., Shaw, K. L., & Presmeg, N. C. (1997). Uncontrollable mental imagery: Graphical connections between a function and its derivative. *Educational Studies in Mathematics, 33*, 301-317.

Biza, I., Christou, C., & Zachariades, T. (2006). Students' thinking about the tangent line. In J. Novotna, H. Moraova, M. Kratka & N. Stehlikova (Eds.), *Proceedings of the 30th PME International Conference*, *2*, 177-184. Prague, Czech Republic.

Biza, I., Nardi, E. & Zachariades, T. (2007). Using tasks to explore teacher knowledge in situation-specific contexts. *Journal of Mathematics Teacher Education, 10,* 301-309.

Castela, C. (1995) Apprendre avec et contre ses connaissances antérieures: Un example concret, celui de la tangente. *Recherche en Didactique des Mathématiques, 15*(1), 7-47.

Giaquinto, M. (2007). *Visual thinking in mathematics.* New York: Oxford University Press

Hanna, G. & Sidoli, N. (2007). Visualisation and proof: A brief survey of philosophical perspectives. *ZDM Mathematics Education*, *39*, 73-78.

Mancosu, P., Jorgensen, K. F., & Pedersen, S. A. (Eds.). (2005). *Visualization, Explanation and Reasoning Styles in Mathematics*. Dordrecht, NL: Springer.

Presmeg, N. C. (2006). Research on visualization in learning and teaching mathematics: Emergence from psychology. In A.Gutierrez & P. Boero (Eds.), *Handbook of Research on the Psychology of Mathematics Education* (pp. 205–235). Dordrecht, NL: Sense Publishers.

Sierpinska, A. (1994). *Understanding in Mathematics*. London: The Falmer Press.

Tall, D. (1987). Constructing the concept image of a tangent. In J. C. Bergeron, N. Herscovics & C. Kieran (Eds.), *Proceedings of the 11th PME International Conference, 3,* 69-75. Montréal, Canada.

Tsamir, P., Rasslan, S. & Dreyfus, T. (2006). Prospective teachers' reactions to right-or-wrong tasks: The case of derivatives of absolute value functions. *Journal of Mathematical Behavior, 25,* 240-251.

Vinner, S. (1991). The role of definitions in the teaching and learning of Mathematics. In D. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 65-81). Dordrecht, NL: Kluwer.

A TEACHER'S USE OF GESTURE AND DISCOURSE AS COMMUNICATIVE STRATEGIES IN THE PRESENTATION OF A MATHEMATICAL TASK

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This study reports on an experienced sixth-grade teacher's communicative strategies when introducing a mathematical task involving different semiotic representations. The analysis has revealed the following communicative strategies: focusing the pupils' attention, posing questions (speech), pointing and sliding (gestures). Working on the task in groups, two girls have difficulties in coordinating a two-dimensional Cartesian diagram. The interplay between the teacher's gesture and speech is a mediating device in her explanations to the girls. The gestures make the connection between the semiotic representations, figure and diagram.

INTRODUCTION

Gestures and discourses are central tools in the teaching and learning of mathematics. Both modalities are also fundamental tools to interpret communicative strategies used by teachers in the classroom. In a series of studies related to the use of mathematical signs (Bjuland, Cestari & Borgersen, 2007; Bjuland, Cestari & Borgersen, in press), used by two groups of sixth-grade pupils working on a task which involves moving between different semiotic representations, we have identified their gestures and discourse strategies. The aim of the present paper is to focus on an experienced teacher's communicative strategies while presenting this particular task in her sixth-grade lesson. These strategies involve two communicative modalities: verbal and gestural, considered as mediating devices and important resources for understanding the task. We will particularly analyse one extract of the teacher-pupils dialogue in the beginning of a lesson, and one extract when two girls, working on the task in a small group, need help from their teacher.

This study addresses the following research question: What kinds of communicative strategies does an experienced teacher use in her dialogues with pupils, introducing a task that involves moving between different semiotic representations?

THEORETICAL BACKGROUND

Gestures can be defined as "movements of the arms and hands … closely synchronized with the flow of speech" (McNeill, 1992, p.11). McNeil has classified gestures in four major categories: *iconic*, *metaphoric*, *deictic*, and *beat*. In our work we are mostly concerned with deictic gestures which are defined as "pointing movements, which are prototypically performed with the pointing finger" (McNeill, op. cit., p. 80). Edwards (2005) reported that almost all the gestures produced during the mathematical discourse of prospective female school teachers consisted of deictic pointing to different aspects of their problem solving while they were working in pairs, trying to solve some mathematical problems related to fractions. We have in an earlier work defined *pointing* and *sliding* as deictic gestures (Bjuland et al., in press). These gestures are identified when pupils' point to different semiotic elements in a task (pointing) and if they move their fingers/hands continuously within or between two semiotic representations (sliding). We have also distinguished between *linear sliding* (pupils move their fingers along a line, for instance along the figurative representation of the task or along one of the axes of the diagram) and *circular sliding* (circular movement of the hand, for instance between two semiotic representations).

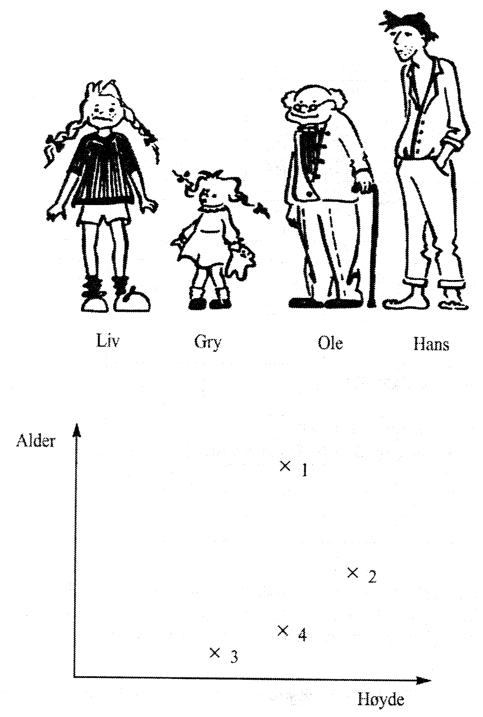
According to Roth (2001), teachers employ many gestural resources crucial for understanding a concept. It is therefore important that pupils attend to both their teachers' speech and their gestures in order to access important information presented in a lesson. In Bjuland, Borgersen, and Cestari (paper submitted), we have revealed how the multimodal components speech, gesture, and written inscriptions develop synchronically. These major components of the objectification process (Radford, 2003), have stimulated the pupils to come up with a solution. In this paper we illustrate how the teacher makes use of these multimodal components in her dialogues with her pupils in full class and in a small-group dialogue with two girls.

METHOD AND CONTEXTUAL BACKGROUND

A four year developmental research project, Learning Communities in Mathematics[1] (LCM), started in Spring 2004 at University of Agder (UiA) in Norway, is "rooted in a philosophy of learning through *inquiry* with the aim of forming *inquiry communities* between teachers in schools and didacticians in the university" (Jaworski, Fuglestad, Bjuland, Breiteig, Goodchild, & Grevholm, 2007, p. 11). The experienced teacher in our study has been a member of the project and transposed ideas from workshops at UiA to her own sixth-grade classroom. We have chosen two extracts of teacher-pupils dialogues from a 19-minute video clip from a particular lesson in order to focus on her communicative strategies.

In order to analyse the teacher's dialogues with her pupils, we have adopted the dialogical approach to communication and cognition (Marková & Foppa, 1990; Cestari, 1997; Linell, 1998). This approach allows us to identify interactional processes, which are the teacher's questioning and her use of particular words to draw the pupils' attention. Using video to collect data, we have also been able to identify the teacher's gestures. These gestures are situated within a theoretical framework that considers cognition as an embodied phenomenon (Edwards, 2005). In our analysis, we focus on the teacher's gestures and speech embodied and situated during the teacher-pupils dialogues.

The pupils were working in groups of two and three on the following task: <u>Write down which person corresponds to each of the points in the diagram</u> (the Norwegian words *alder* and *høyde* mean age and height respectively. This mathematical task has been carefully analysed in Bjuland et al. (in press). We observe that the task challenges the pupils to move between the three semiotic representations, figure, diagram and written text.



Liv corresponds to point ....

Gry corresponds to point ....

Ole corresponds to point ....

Hans corresponds to point ....

THE TEACHER'S PROPOSITION OF THE TASK IN THE CLASSROOM

The aim of analysing the following dialogue is to identify the teacher's communicative strategies while presenting the task for her pupils by using an overhead projector. The pupils are focusing on the screen, siting in a semi-circle. The teacher (Tea) is sitting close to them.

15 Tea: Look, this is about Liv and Gry and Ole and Hans. Do you notice anything about Liv and Gry and Ole and Hans? Can you see any differences between them? Kari what do you see?

16 Kari: They have different heights.

17 Tea: That's right. Mm. Can you see some more differences? Sofie?

18 Sofie: Different age.

19 Tea: Yes, that's clear that they are different ages. Yes, then you know that these four persons have been out for a walk and, and then we're going to try to find out where the different persons are (*The teacher goes from her chair towards the screen*). Who is number one? (*Pointing at point 1 followed by a circular sliding up to the picture*). Who is number two? (*The circular sliding from the picture ends in pointing at point 2*). Who is number three? (*From point 2 with a decreasing circular sliding without reaching the picture, pointing at point 3*) and who is number four? (*From point 3, a decreasing sliding before pointing at point 4*). Hm! How can we find out this?

20 Pupil Number one

21 Tea Don't say it loud yet, don't say it loud. Now I have thought that you should go in groups. And you should try to find out who are the different persons. We should only read the task, very carefully we read it.

The teacher uses the verb *see* many times, inviting her pupils to be attentive to the visual image of the task (15). She also uses a singular *you* (Norwegian singular *du*), indicating that each individual pupil has to focus on the screen. The word *see* is used in three consecutive, open questions. The third question stimulates the pupils to *see* any differences in the figure, indicating a first approach to identifying the two dimensions height and age (15). This question is posed to Kari, inviting her to tell what she *sees* from the figure. The visual image perceived by Kari elicits her response, introducing the variable height in the dialogue (16). Then the teacher asks another open question, inviting Sofie to *see* more differences (17), when the variable age emerges in the conversation (18). The communicative pattern between the teacher and the two pupils shows the well known IRE sequence (initiative – response – evaluation) identified by Sinclair and Coulthard (1975) and used by Mehan (1979) in analysing classroom discourse.

The teacher is contextualising the task by suggesting that the persons have been out for a walk (19). Locating the task in a concrete life situation, could be helpful for the pupils. The contextualisation is also introduced based on a singular *you* perspective, indicating that each individual pupil has to be concerned with the task. She is then focusing on the transition from the picture to the Cartesian coordinate diagram by making a connection between people and the labelling of points by talking and making gestures simultanously. While standing at the screen, the teacher makes four consecutive *pointings* to the diagram with a gradually decreasing *circular sliding* between the diagram and the picture. In this way she indicates the relationship between these two semiotic representations, figure and diagram. The teacher has now introduced the task, and her open question shows that she is ready to let the pupils explore the task in groups.

The pupils's utterance (20) suggests some interference, indicating that he/she is eager to start working on the problem, but the teacher does not want to have any class discussion at this moment (21). This indicates that the pupils should get the opportunity to spend some time in order to explore the task in groups in order to find out the answers themselves. The strategy of reading the task carefully is emphasised.

Analysis has revealed that the teacher clearly focuses on the figurative representation of the four persons in her presentation of the task. She is also concerned with the transition between the representations figure and diagram. The variables height and age are introduced in the dialogue related to the persons in the picture. However, these variables are not related to the two axes in the diagram, indicating that she does not put much emphasis on this representation in her presentation. Nor does she focus on the points in the diagram, for instance by posing a question like, what does a point represent in a diagram? The teacher does not read the task together with her pupils, showing that the third representation, the written text, is left out in her presentation. However, the teacher clearly emphasises that the pupils must read the task very carefully.

THE TEACHER IN DIALOGUE WITH THE TWO GIRLS

The dialogue below shows that two girls have difficulties in capturing the connection between the two variables height and age. They need help from their teacher twice. The sequence of turns illustrates some aspects of the first dialogue between them.

96 Pupil 5: We didn't understand it (*Teacher stands behind the two girls, Pupil 4 and Pupil 5)*.

97 Tea: Didn't you understand it? (*The task*)

98 Pupil 5: No. (*Erasing her written solution*)

99 Tea: No. Mm. But what have you looked at?

100 Pupil 4: We have looked at the height (*Moving her pencil around without any specific pointing or sliding*) [because Hans is highest there]

101 Pupil 5: [It tells that height there and age there]

102 Tea: Have you looked at the age?

103 Pupil 4: that Gry, she is youngest.

.

.

113 Pupil 4: But I didn't understand what these labels meant.

114 Tea: No. These are which persons they are (*Linear sliding along the picture*). One of those persons is number three (*Linear sliding along the picture followed by pointing at point 3 in the diagram*). One of those persons is number four (*Linear sliding along the picture followed by pointing at point 4 in the diagram*). One of those persons is number two (*Linear sliding along the picture followed by pointing at point 2 in the diagram*) and so on, aren't they?

115 Pupil 4: Okay, but those then?

116 Tea: Yes the points one, two, three, four. Those are four different points.

117 Pupil 4: Should we write the name to those points? (*Moving her pencil between the diagram and the written text*)

118 Tea: Yes, you should only write one, two, three or four on these, according as where you find that those are the different [persons]

119 Pupil 4: [okay]

After Pupil 5 expresses that they have some difficulties (96), the teacher repeats this pupil's utterance as a yes-no question (97), making it explicit that the pupils need some help (98). The teacher's open question (99) invites the pupils to express what they have done so far. Pupil 4 focuses on the one-dimensional perspective height (100), while Pupil 5 seems to be concerned with the coordination of the variables height and age (101). The teacher continues to ask for the dimension, age, probably challenging Pupil 4 since she has only focused on the variable height (102). Pupil 4 focuses on the *extreme location,* Gry (103), who is the youngest person. From the teacher's yes-no question (97), the open question (99), and the specific question (102), we have implicitly observed that Pupil 4 has difficulties in making any connection between age and height, for instance by locating Gry at point 3 in the diagram.

In the continuation of the dialogue, Pupil 4 expresses what she does not understand (113), and the teacher shifts from posing questions to making an explanation. The teacher's verbal explanation is supplemented with gestures. She makes two *linear slidings* along the picture followed by one pointing at point 3 in the diagram. This gestural sequence (linear sliding, pointing) is repeated two more times (114), with the important function of making connection between the two semiotic representations figure and diagram.

Pupil 4 expresses that she has understood the explanation given by the teacher (115). However, the dialogue between Pupil 4 and the teacher (115-119) illustrates that this pupil needs some help to understand the connection between the two semiotic representations diagram and written text (116), (118).

DISCUSSION AND CONCLUSION

The analysis of the sequence of the teacher-pupil dialogue from the whole-class has revealed some of the teacher's strategies when presenting the task. She focuses on seeing (the four different people, and possible differences between them), on posing open questions bringing the variable height and age into the discourse and on contextualising the task, without using any gestural resources. However, when the teacher focuses on the transition from the two different semiotic representations, figure and diagram, she uses both speech and gestures by making a connection between people and the labelling of points. More specifically, the communicative strategy of *questioning* is used simultaneously with *pointing* to the diagram followed by a gradually decreasing *circular sliding* between the diagram and the picture. This sequence (questioning, pointing, circular sliding) has been repeated four times, illustrating how the teacher's gestures and speech function as mediating devices, helping the pupils to acquire a preliminary understanding of the task.

Roth (2001) emphasises that in a conjunction with a teacher's body position, his or her gestures can orient pupils to aspects of a visual representation being pointed to and highlighted. It is therefore important that pupils have access to both the teachers' gestural resources and their speech. Following Roth (op. cit.), the teacher's gestures make the pupils aware of the translation between the two semiotic representations, from the concrete picture to the more abstract Cartesian diagram.

According to McNeill's classification scheme (1992), the teacher uses *deictic* gestures (pointing and circular sliding) in her presentation. This is also the case in the dialogue between the girls and the teacher, when she uses, three times, the gestural sequence *linear sliding* along the picture followed by *pointing* at a specific point in the diagram. In the dialogue with the two girls, the interplay between the teacher's gesture and discourse is a mediating device in her explanations to the girls. These gestures also reinforce the verbal explanations, stimulating the pupils to make connections between the semiotic representations and helping them to be aware of the two dimensions, age and height.

According to Brekke (1995), most pupils are familiar with one-dimensional graphs. It is therefore probable that the pupils identify the visual differences on the figure and the fact that the persons have different heights. However, it is far more complex for them to realise that a graph could also show the connection between two variables. The communicative strategies reveal that the teacher focuses on the figure of the four persons in her presentation of the task. The two other representations, diagram and written text, are left for the pupils, and they are told to read the task very carefully themselves. The pupils are therefore challenged to cope with this difficulty of coordination in their groups. By posing different questions (yes-no, open, specific), the teacher observes that the girls have problems in understanding the task. When the girls express their difficulties, which are related to the coordination of the two dimensions and the transition between the second and third representation, the teacher makes explanations, combining gestural resources like pointing and circular sliding to make connections between figure and diagram. We could wonder if the difficulties, identified in the girls' solution process, are related to the teacher's avoidance of focusing on the diagram and the written text in her presentation.

From this microanalysis we have observed that gesture and discourse are natural mediating devices when this teacher introduces a new task, involving the representation of figures in a Cartesian table. We have identified the following main communicative strategies used by this experienced teacher: Focusing the attention of pupils, posing questions, using gestures of pointing and sliding. The teacher's approach of using gestures and verbal explanations when moving between semiotic representations in a task stimulates the pupils to go on working on the task in groups. These modalities are also used in her dialogue with the two girls, starting by posing questions, then making verbal explanations in combination with gestures. In future studies we would ask what kind of communicative strategies novice teachers use when introducing a mathematical task of the same nature.

Endnote

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References

Bjuland, R., Cestari, M. L., & Borgersen, H. E. (2007). En lærers bruk av problemløsningsverksted: Fra LCM til klasserommet. In B. Jaworski, A. B. Fuglestad, R. Bjuland, T. Breiteig, S. Goodchild, & B. Grevholm, (Eds.), *Læringsfellesskap i Matematikk* (pp. 151-160). Bergen, Norway: Casper Forlag.

Bjuland, R., Cestari, M. L., & Borgersen, H. E. (in press). Pupils' mathematical reasoning expressed through gesture and discourse: A case study from a sixth-grade lesson. Paper presented and accepted for publication in proceedings for CERME5, Larnaca, Cyprus, 2007.

Bjuland, R., Borgersen, H. E., & Cestari, M. L. (paper submitted). The interplay between gesture and discourse as mediating devices in collaborative mathematical reasoning. A multimodal approach. (Submitted to *Mathematical Thinking and Learning*)

Brekke, G. (1995). *Introduksjon til diagnostisk undervisning i Mtematikk.* Oslo: Nasjonalt læremiddelsenter.

Cestari, M. L. (1997). *Communication in Mathematics Classroom. A Dialogical Approach.* Doctoral Dissertation. Oslo: University of Oslo, Norway.

Edwards, L. (2005). Gesture and mathematical talk: Remembering and problem solving. Paper presented at American Educational Research Association Annual Meeting, Montreal, Canada.

Jaworski, B., Fuglestad, A. B., Bjuland, R., Breiteig, T., Goodchild, S., & Grevholm, B. (2007). *Læringsfellesskap i Matematikk*. Bergen, Norway: Casper Forlag.

Linell, P. (1998). *Approaching Dialogue. Talk, Interaction and Contexts in Dialogical Perspectives*. Amsterdam: John Benjamins.

Markovà, I. & Foppa, K. (1990). *The dynamics of dialogue.* Hemel Hempstead: Harvester Wheatsheaf.

McNeill, D. (1992). *Hand and Mind: What Gestures Reveal about Thought.* Chicago: Chicago University Press.

Mehan, H. (1979). *Learning Lessons: Social Organization of the Classroom*. Cambridge, Mass.: Harvard University Press.

Radford, L. (2003). Gestures, speech and the sprouting of signs. *Mathematical Thinking and Learning*, *5*(1), 37-70.

Roth, W. M. (2001). Gestures: Their role in teaching and learning. *Review of Educational Research*, *71*(3), 365-392.

Sinclair, J. McH. & Coulhard, M. C. (1975). *Towards and Analysis of Discourse: The English Used by Teachers and Pupils.* London: Oxford University Press.

DEVELOPMENT OF FUTURE MATHEMATICS TEACHERS DURING TEACHER EDUCATION - RESULTS OF A QUASI-LONGITUDINAL STUDY

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Germany took part in a 6-country study on the efficacy of mathematics teacher education. Three cohorts of future teachers (beginning, midterm and final) were tested on their professional knowledge and their beliefs. The paper presents results of the German sub-sample. If interpreted in a quasi-longitudinal way, the data strongly support the main hypotheses which led the study: Future teachers' knowledge develops significantly during teacher education. The differences between the cohorts reflect the emphases of the teachers' programmes and are specifically large in those fields emphasized in the professional programmes.

INTRODUCTION

In Germany as well as in many other countries systematic knowledge about how teachers perform at the end of their education is almost non-existent (Sikula, Buttery & Guyton, 1996; Wilson, Floden & Ferrini-Mundy, 2001; Blömeke, 2004; Cochran-Smith & Zeichner, 2005). Even in the field that is covered by most of the existing studies – the education of mathematics teachers – research deficits have to be stated: the research is often short term, of a non-cumulative nature, and conducted within the own training institution (Krainer, Goffree & Berger, 1999; Adler et al., 2005a). Only recently more empirical studies on mathematics teacher education have been developed (cf. Chick et al., 2006, Baker & Chick, 2006, Adler et al., 2005b). The important relation of the qualifications of practising mathematics teachers and their teaching or the achievements of their students was recently explored in several empirical studies (Ball & Bass, 2003; Hill, 2007; Ferrini-Mundy et al., 2006; Schmidt et al., 2006; Brunner et al., 2006).

The project "Mathematics Teaching in the 21st Century (*MT21*)" aims to shed light on the important field of mathematics teacher education from a comparative perspective. In an attempt to fill existing research gaps, the knowledge and beliefs of future lower secondary teachers are investigated. In Germany, lower-secondary teachers are prepared in two different routes. In the first, future primary and lower-secondary teachers (grades 1 through 10) are trained; whilst the second graduates lower- and upper-secondary teachers (grades 5 through 13). Both routes include two phases. Phase 1 is situated at universities, lasts for 3.5 or 4.5 years respectively and involves study in mathematics, mathematics pedagogy and general pedagogy theoretically oriented towards mathematics as a discipline. Phase 2 follows and takes place in separate state institutions, partly at school. This second phase lasts for 1.5 or 2 years respectively and involves study in mathematics pedagogy and general pedagogy under a more practical perspective.

MAIN FOCUS "PROFESSIONAL COMPETENCIES"

MT21 mainly refers to the concept of "professional competencies", which is defined referring to Weinert (1999) as core professional tasks that teachers must be able to master (Bromme, 1992). Instruction and assessment are evaluated in MT21 as central teacher tasks. To accomplish these two tasks teachers need:

- cognitive abilities and skills in terms of *professional knowledge* as well as.
- professional convictions and conception of values in terms of *beliefs*.

The professional knowledge that future mathematics teachers need can be divided into three general facets well-known in the literature: content knowledge, pedagogical content knowledge and general pedagogical knowledge (Shulman, 1985; Blömeke, 2002; Baumert & Kunter, 2006; for an elaboration of the theoretical approach of Shulman see Leikin & Levev-Waynberg, 2007). In MT21 the content is mathematics.

Beliefs are defined as "psychologically held understandings, premises, or propositions about the world that are felt to be true" (Richardson, 1996, p. 103). If beliefs are operationalised closely to the content a teacher has to teach, the correlation of student performance with teacher beliefs was found to be high (Bromme, 2005). Beliefs yield an orientational and action-guiding function (Grigutsch, Raatz & Törner, 1998). With this a bridge is built between knowledge and action (Peterson et al., 1989; Leder, Pehkonen & Törner, 2002). For further information on the development of beliefs during teacher education see the international *MT21* project report (Schmidt et al., 2007) and the German report (Blömeke, Kaiser & Lehmann, 2008).

SAMPLING

The sampling of countries in *MT21* followed a careful selection of criteria based on existing international comparisons. The countries had to cover the two main kinds of teacher education (one-phase programmes with content, pedagogical content and general pedagogy taught simultaneously; two-phase programmes with content taught first, followed by pedagogical content and general pedagogy). The spectrum of student performance as shown in TIMSS and PISA had to be covered as well. In addition, the participating countries had to represent socio-cultural contexts that had proved significant in studies of former international comparative educational research. Finally, the countries had to show at least a middle grade degree of industrialisation in order to avoid serious bias through socio-economic differences. According to these criteria, Bulgaria and Germany, South Korea and Taiwan, Mexico and the US were sampled.

Within countries *MT21* sampled at the institutional level. The goal was to obtain a reasonably representative sample of each country including the variation found across all teacher education institutions in the country. In Germany four regions were selected to take part in *MT21*. The sample cuts across important structural characteristics of German teacher education, e.g. differences in the structure and content of the taught curricula, the connections to school practice. In these four regions all teacher-education institutions were sampled. Within the institutions the goal was to take a complete census.

The overall sample size was 849 with 368 students in the first cohort (beginning), 195 in the second (midterm students) and 286 students in the third cohort (final). Since the response rate differed between the three cohorts the results were weighted according to the total number of future teachers prepared in each cohort and in each teacher education route at each of the sampled institutions in order to make the sample as representative as possible for the four regions.

TEST DESIGN

MT21 sought to measure what individuals learn in their teacher education programmes, which was done by testing the future teachers' knowledge. In order to consider the situation-specificity of teaching and to avoid the measurement of so-called "idle knowledge" a special item format was used in addition to traditional items, namely, teaching situations which can only be handled by using and linking several knowledge dimensions. The test was given in a paper-and-pencil format. Two versions of the test were used; each knowledge area shared common items between the forms but most of the questions were unique to one version. All scales were developed and multi-piloted under the consideration of the curricula from the participating countries and expert assessments.

For the scaling of the test sections on mathematics and mathematics pedagogy the two-dimensional *Random Coefficients Multinomial Logit* model implemented in Conquest was used. Item difficulties were estimated by Maximum Likelihood procedures based on a 65% probability to solve a problem. Weighted Likelihood Estimators were used to estimate individual abilities. The general pedagogical scale represents classical sum scores. The distribution of the three scales was transformed so that each had a mean of 100 and a standard deviation of 20.

Since all mathematics related items – those in the mathematics part as well as those in the mathematics pedagogy part – measure some kind of content, five scales were developed that cover number, algebra, functions, geometry and statistics. For the scaling we used in this case the five-dimensional *Random Coefficients Multinomial Logit* model. Again the distribution was transformed so that the mean of the scale was 100 and the standard deviation 20. Regarding general pedagogical knowledge the test covers three sub-domains: lesson planning, assessment and socio-economic differences in student achievement. These scales represent sum-scores. They were transformed onto a mean of 50 and a standard deviation of 10.

RESULTS

The basic hypothesis of *MT21* was:

H1: Future teachers' professional knowledge increases significantly during teacher education, that is, in each of the three areas mathematics, mathematics pedagogy and general pedagogy.

The data strongly support this hypothesis (see table 1). The differences between the results of cohort 1 and 3 are highly significant. On average cohort 3 students scored 16.5 points higher in mathematics, 10.7 points higher in mathematics pedagogy and 3.2 points higher in general pedagogy than cohort 1 students. The effect sizes indicate that the differences are of practical relevance in two of the three cases. In mathematics and mathematics pedagogy they mount up to more than a half standard deviation. Only in general pedagogy the difference between the cohorts is of minor relevance.

	Cohort 1 (*n* = 368)			**Cohort 3 (*n* = 286)**			**Comparison Cohort 1 – Cohort 3**	
Knowledge Dimension	M	SE	SD	M	SE	SD	Difference $M_{K1, MK3}$	$d_{K1,K3}$
Mathematics	93.7	0.9	18.3	110.2	0.6	21.6	+ 16.5*	0.8
Mathematics Pedagogy	95.1	1.0	18.0	105.8	0.5	18.4	+ 10.7*	0.6
General Pedagogy	98.3	1.0	17.3	101.5	0.5	18.1	+ 3.2*	0.2

Note. Means (M), Standard Errors (SE), Standard Deviations (SD) und Effect Sizes (Cohen's *d*) represent pooled sampling estimates. Significant differences ($p < .05$) between the two cohorts are marked (*).

Table 1. Future Teachers' professional knowledge at the beginning and at the end of teacher education

In addition to this basic hypothesis, *MT21* was led by more sophisticated hypotheses which try to capture the relation between opportunities to learn in teacher education and outcomes in more detail. One of them was:

H2: Future teachers' professional knowledge depends significantly on their opportunities to learn. They gain more knowledge in areas strongly emphasized in teacher education.

Since the differences in opportunities to learn between the two teacher-education routes are large in Germany, H2 has to be investigated separately for future teachers for primary and lower-secondary schools and future teachers for lower and upper-secondary schools; otherwise possible effects might be washed out.

In mathematics the curriculum for future primary and lower-secondary teachers strongly emphasizes aspects of number whereas the curriculum for future lower and upper-secondary teachers additionally emphasizes algebra and functions besides topics from advanced mathematics. Bearing this in mind, it becomes obvious that the data support H2 as well (see table 2). Future primary and lower secondary teachers have their largest achievement gains in number. The effect size is twice as high as in the other areas. Number is an area in which future lower and upper-secondary teachers score very high as well (see table 3). In addition, the latter group shows the expected large achievement gains in functions and especially in algebra. In algebra the effect size is three times larger than in geometry and statistics.

	Cohort 1 (*n* = 294)			Cohort 3 (*n* = 133)			Comparison Cohort 1 – Cohort 3	
Content Area	**M**	**SE**	**SD**	**M**	**SE**	**SD**	**Diff.** $\mathbf{M_{K1,K3}}$	$\boldsymbol{d_{K1,K3}}$
Number	101.6	1.0	18.8	113.8	0.8	19.8	**+ 12.2***	**0.6**
Algebra	92.4	0.9	17.0	96.9	0.8	19.9	**+ 4.5***	**0.3**
Functions	96.8	0.9	17.9	101.6	0.8	20.1	**+ 4.8***	**0.3**
Geometry	102.8	0.9	17.6	105.9	0.8	19.3	**+ 3.1***	**0.2**
Statistics	99.8	1.0	19.4	105.9	1.0	22.2	**+ 6.1***	**0.3**
Lesson Plan	48.3	0.5	9.1	52.3	0.4	10.1	**+ 4.0***	**0.4**
Assessment	49.8	0.4	7.4	52.2	0.3	8.1	**+ 2.7***	**0.3**
SES	48.5	0.4	7.5	51.3	0.4	10.0	**+ 2.4***	**0.3**

Table 2. Knowledge of future teachers at primary and lower secondary schools in different content areas at the beginning and at the end of teacher education

	Cohort 1 (*n* = 74)			Cohort 3 (*n* = 153)			Comparison Cohort 1 – Cohort 3	
Content Area	**M**	**SE**	**SD**	**M**	**SE**	**SD**	**Diff.** $\mathbf{M_{K1}}$, $\mathbf{M_{K3}}$	$\boldsymbol{d_{K1,K3}}$
Number	115.0	2.5	16.0	127.1	0.6	18.9	**+ 12.1***	**0.7**
Algebra	94.4	2.8	16.8	111.6	0.7	18.3	**+ 17.2***	**1.0**
Functions	109.3	2.3	16.4	121.3	0.8	20.7	**+ 12.0***	**0.6**
Geometry	108.4	3.2	18.3	113.5	0.6	17.6	**+ 4.9***	**0.3**
Statistics	110.2	3.3	20.9	116.1	0.8	21.3	**+ 5.9***	**0.3**
Lesson Plan	49.5	1.6	9.0	50.9	0.4	9.9	+1.4	0.2
Assessment	52.5	1.4	10.2	46.7	0.3	7.1	**- 5.8***	**0.7**
SES	46.1	0.9	7.2	52.5	0.3	8.9	**+ 6.4***	**0.8**

Table 3: Knowledge of future teachers at lower and upper secondary schools in different content areas at the beginning and at the end of teacher education

In an indirect way H2 is also supported by the fact that future primary and lower secondary teachers of cohort 1 showed relatively even performance in number, geometry and statistics and similarly for lower and upper-secondary teachers in functions, geometry and statistics. In cohort 3 the two groups scored very differently though in these areas. Another support of H2 can be derived from the data that were used for analysing H1. The differences between cohort 1 and 3 were largest in mathematics, of middle size in mathematics pedagogy and smallest in general pedagogy. This reflects precisely differences in learning opportunities. Overall, the amount of future teachers' education in Germany is highest in mathematics and lowest in general pedagogy.

In contrast to these convincing results, the results in the sub-domains of general pedagogy are more irregular and due to space constraints we only name a few. For example in the curriculum of general pedagogy lesson planning is emphasized. The knowledge of future primary and lower secondary teachers corresponds with this and therefore our hypothesis is supported at least partly by the data. However, the difference between cohort 1 and cohort 3 in the group of future lower and upper secondary teachers in lesson plan is not statistically significant and in assessment the scores of the third cohort have even decreased, which does not fit with H1.

DISCUSSION

The MT21 data strongly support the hypothesis that teacher education matters. Interpreting our cohort data in a quasi-longitudinal way makes it obvious that future teachers' professional knowledge significantly increases during teacher education - and this in all areas: mathematics as well as mathematics pedagogy and general pedagogy. It is especially noteworthy that the effect sizes are rather high given the fact that we tested beginners when they had already taken one year university seminars and final cohort students half a year before they actually completed their exams. This means that the teacher education effect probably is considerably underestimated.

Our additional hypothesis that opportunities to learn significantly influence the development of professional knowledge is supported by the data as well. Both subgroups of future lower secondary teachers show by far the largest gains in test scores in those areas which are highly emphasized in teacher education. In areas which are not emphasized the knowledge of future teachers does not increase approximately even if they were at a similar level at the beginning of their studies.

The only result that contradicts our hypotheses occurs in the test part about general pedagogy within the subgroup of future lower and upper secondary teachers. Regarding this a particularity of German teacher education has to be considered: Traditionally, general pedagogy is of low importance in the curriculum of the lower and upper secondary teachers, it is often not a compulsory subject and/or the requirements to pass the examinations are quite low.

It has to be mentioned that – as always in quasi-longitudinal studies - the study design shows weaknesses: We cannot rule out the possibility that the effects documented go back to differences in cohort composition. We controlled for important background characteristics like route and institutional affiliation but it has to be considered that teacher education in Germany lasts for five to seven years. It might be that the first cohort entered teacher education with different knowledge than the previous cohorts or that their experiences during teacher education differed.

A subsequent internationally comparative study, conducted by IEA under the name "Teacher Education and Development Study: Learning to Teach Mathematics (TEDS-M)" will allow implementing the respective future research. TEDS-M will take place in 20 countries – including Germany. It involves representative samples which allow testing the hypotheses outlined in this paper and thus has the potential to contribute to an evidence-based reform of teacher education.

References

Adler, J., Ball, D. L., Krainer, K., Lin, F. L., & Novotna, J. (2005a). Reflections on an emerging field: Researching mathematics teacher education. *Educational Studies in Mathematics, 60*, 359–381.

Adler, J., Davis, Z., Kazima, M., Parker, D., & Webb, L. (2005b) 'Working with students' mathematical productions: elaborating a key element of mathematical knowledge for teaching'. In H.L. Chick & J. L. Vincent (Eds.), *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education* (vol. 2, pp.1-8). Melbourne: Melbourne University, Department of Science and Mathematics Education.

Baker, M. & Chick, H. L. (2006). Pedagogical content knowledge for teaching primary mathematics: A case study of two teachers. In P. Grootenboer, R. Zevenbergen, & M. Chinnappan (Eds.), *Identities, Cultures and Learning Spaces.* Proceedings of the 29th annual conference of the Mathematics Education Research Group of Australasia (pp. 60-67).

Ball, D.L. & Bass, H. (2003). Toward a practice-based theory of mathematical knowledge for teaching. In B. Davis & E. Simmt (Eds.), *Proceedings of the 2002 annual meeting of the Canadian Mathematics Education Study Group* (pp. 3–14). Edmonton: CMESG/GCEDM.

Baumert, J. & Kunter, M. (2006). Stichwort: professionelle kompetenz von lehrkräften. *Zeitschrift für Erziehungswissenschaft, 9*, 469–520.

Blömeke, S. (2002). *Universität und Lehrerausbildung*. Bad Heilbrunn: Klinkhardt.

Blömeke, S. (2004). Empirische befunde zur wirksamkeit der lehrerbildung. In S. Blömeke, P. Reinhold, G. Tulodziecki, & J. Wildt (Eds.), *Handbuch Lehrerbildung* (pp. 59–91). Bad Heilbrunn/Braunschweig: Klinkhardt/Westermann.

Blömeke, S., Kaiser, G., & Lehmann, R. (2008). *Professionelle Kompetenz Angehender Lehrerinnen und Lehrer. Wissen, Überzeugungen und Lerngelegenheiten Deutscher Mathematikstudierender und -Referendare - Erste Ergebnisse zur Wirksamkeit der Lehrerausbildung.* Münster: Waxmann.

Bromme, R. (1992). *Der Lehrer als Experte. Zur Psychologie des professionellen Wissens*. Bern: Huber.

Bromme, R. (2005). Thinking and Knowing About Knowledge: A Plea for and Critical Remarks on Psychological Research Programs on Epistemological Beliefs. In M.H.G. Hoffmann, J. Lenhard, & F. Seeger (Eds.), *Activity and Sign – Grounding Mathematics Education,* (pp. 191–201). New York: Springer.

Brunner, M., Kunter, M., Krauss, S., Baumert, J., Blum, W., Dubberke, T. et al. (2006). Welche zusammenhänge bestehen zwischen dem fachspezifischen professionswissen von mathematiklehrkräften und ihrer ausbildung sowie beruflichen fortbildung? *Zeitschrift für Erziehungswissenschaft, 9,* 521–544.

Chick, H. L., Baker, M., Pham, T., & Cheng, H. (2006). Aspects of teachers' pedagogical content knowledge for decimals. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Proceedings of the 30th Annual Conference of the International Group for the Psychology of Mathematics Education* (vol. 2, pp. 297-304). Prague: PME.

Cochran-Smith, M., & Zeichner, K.M. (Eds.) (2005). *Studying Teacher Education. The Report of the AERA Panel on Research and Teacher Education*. Mahwah, N.J.: Lawrence Erlbaum.

Ferrini-Mundy, J., Schmidt, W.H., Bates, P., Joyner, T., Leroi, G., & Wigent, Ch. (2006). *Knowing Mathematics: What We Can Learn from Teachers* (PROM/SE Research Report Nr. 2). East Lansing: Michigan State University.

Grigutsch, S., Raatz, U., & Törner, G. (1998). Einstellungen gegenüber Mathematik bei Mathematiklehrern. *Journal für Mathematik-Didaktik*, *19*, 3–45.

Hill, H.C. (2007). Mathematical knowledge of middle school teachers. implications for the no child left behind policy initiative. *Educational Evaluation and Policy Analysis, 29*, 95–114.

Krainer, K., Goffree, F., & Berger, P. (1999). European Research in Mathematics Education I.III – From a Study of Teaching Practices to Issues in Teacher Education. Osnabrück: Forschungsinstitut für Mathematikdidaktik.

Leder, G.C., Pehkonen, E., & Törner, G. (Eds.) (2002). *Beliefs: A Hidden Variable in Mathematics Education?* Dordrecht: Kluwer.

Leikin, R. & Levev-Waynberg, A. (2007). Exploring mathematics teacher knowledge to explain the gap between theory-based recommendations and school practice in the use of connecting tasks. *Educational Studies in Mathematics*, 66(3), 349-371.

Peterson, P., Fennema, E., Carpenter, T.P., & Loef, M. (1989). Teachers' pedagogical content beliefs in mathematics. *Cognition and Instruction, 6*, 1–40.

Richardson, V. (1996). The role of attitudes and beliefs in learning to teach. In J. Sikula, T. Buttery, & E. Guyton (Eds), *Handbook of Research on Teacher Education* (2. ed., pp. 102–119). New York: Macmillan.

Schmidt, W.H., Tatto, M.T., Bankov, K., Blömeke, S., Cedillo, T., Cogan, L., Han, S.I., Houang, R., Hsieh, F.J., Paine, L., Santillan, M.N., & Schwille, J. (2007). *The Preparation Gap: Teacher Education for Middle School Mathematics in Six Countries - Mathematics Teaching in the 21st Century (MT21)*. East Lansing, MI: MSU [last access 12-12-2007, under http://usteds.msu.edu/related_research.asp].

Schmidt, W.H., Ferrini-Mundy, J., Houang, R.T., & Cogan, L.S. (2006). *Curriculum insights from PROM/SE*. East Lansing: Michigan State University.

Shulman, L.S. (1985). Paradigms and research programs in the study of teaching: A contemporary perspective. In M.C. Wittrock (Ed.), *Handbook of Research on Teaching* (3rd ed., pp. 3–36). New York: Macmillan.

Weinert, F.E. (1999). *Konzepte der Kompetenz. Gutachten zum OECD-Projekt "Definition and Selection of Competencies: Theoretical and Conceptual Foundations (DeSeCo)",* Neuchatel, Schweiz: Bundesamt für Statistik.

DESIGNING OPEN-ENDED PROBLEMS TO CHALLENGE PRESERVICE TEACHERS' VIEWS ON MATHEMATICS AND PEDAGOGY

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This study examines preservice elementary teachers' reported experiences of posing open-ended mathematics problems. Responses of 33 students in a mathematics teacher education course were analysed for the strategies participants used, what they learned and the challenges encountered from an opportunity to collect digital images and pose open-ended problems related to those images. Results indicate that preservice teachers reported a shift in the ways they viewed mathematics and how it might be taught. The school curriculum both constrained and provided possibilities for preservice teachers in noticing mathematics beyond the textbook and mathematics classroom. This study adds to our understanding of teaching as a learning practice and the art of posing mathematical problems as a significant aspect of that practice.

INTRODUCTION

Selecting, adapting and/or extending mathematics problems are a significant pedagogical practice for teachers. Mathematical "tasks convey messages about what mathematics is and what doing mathematics entails" (National Council of Mathematics, 1991, p. 24). They can provide a context for student learning about mathematical concepts and skills as well as mathematical inquiry. Tasks can also help students frame ideas about what it means to do mathematics. As Schoenfeld (1989) argues, students develop beliefs about the discipline of mathematics from their experiences with classroom mathematics activities. What counts as a good mathematics task has varied interpretations. Henningsen and Stein (1997) refer to worthwhile tasks as high-level tasks having the potential for high cognitive demand by students. Sullivan and Lilburn (2002) define good questions for mathematics teaching as having three features: 1) requiring more than recalling a fact or skill; 2) educative for both students and teachers; and 3) having possibly several acceptable answers. Whereas Gutstein (2006) argues that good tasks include those that are culturally relevant, that is, those that are related to students' lives, offer the possibility of teaching for social justice, and "rely more on students' own meaning making rather than with outside sources like the teacher or answer sheet" (p. 103). These definitions share an openness that offers students opportunities to explore mathematics in meaningful ways. They recognize that what makes a task 'good' does not necessarily reside in the task itself but rather in the relationship between the task and the student (or the teacher).

Learning to develop, adapt, select and pose good tasks is neither simple nor trivial. Teachers who have had few opportunities to experience posing their own mathematics problems or even asking "what if?" questions when they themselves were students may find it challenging to now select and pose more open-ended mathematics problems as teachers. Moreover this challenge may be amplified for those elementary teachers who come to teaching without a strong background in mathematics. How do teachers learn the practice of selecting and posing good mathematics tasks?

The field of mathematics education does not currently have a well-developed knowledge base on particular ways in which teachers learn to pose non-routine or open-ended mathematics tasks. We do know that it is extremely difficult for teachers to maintain with students the high cognitive demand of potentially high-level tasks (Henningsen & Stein, 1997). We also know that changes in problem posing strategies are possible and that preservice teachers can move from posing traditional single step problems to more open-ended cognitively complex problems (Crespo, 2003; Sinclair & Crespo, 2006). One factor that seems to support this change is opportunities for teachers and preservice teachers to explore new kinds of problems in varied contexts.

Our study adds to this research and examines preservice teachers' experiences with opportunities to pose new kinds of problems: those that are open-ended and grounded in images of real-life activities. In this study we explored elementary preservice teachers' perspectives on posing open-ended tasks inspired by a set of digital images that preservice teachers collected for the specific purpose of investigating mathematics with students. We offer an example of the kinds of images preservice teachers collected and the kinds of related problems they posed (see Nicol & Bragg, forthcoming for a more detailed analysis of these posed problems). We focus in this paper more specifically on preservice teachers' strategies for developing problems in the context of a mathematics teacher education course, what they report they learned, and the challenges encountered from an opportunity to collect images and pose open-ended problems related to those images.

THEORETICAL CONSIDERATIONS

For many preservice teachers' their prior experiences with mathematics has situated their knowledge and beliefs of mathematics as procedural, rule-bound, and closed. Supporting beginning teachers as they examine their underlying knowledge, beliefs and practices about teaching and learning mathematics is challenging. Lampert (2001) and others propose that teaching and learning can be understood as *learning practices*. A focus on teaching as *learning practices* draws attention to the activities teachers attend to in the activity or practice of teaching mathematics. Crespo (2003) and Nicol (1999) suggest that learning to pose mathematical problems, listen to and interpret student responses, and respond to students are central learning practices for teaching. Crespo (2003) describes a context in which preservice teachers developed their problem-posing practices through penpal letter writing activities where preservice teachers and Grade 4 students exchanged mathematical problems. Preservice teachers in Crespo's study were not provided with explicit direction on defining or creating open-ended problems, leaving us to wonder how preservice teachers might respond with more explicit instruction.

Situated theories of learning and cognitive apprenticeships suggest that what is learned is intricately tied to the context in which it is learned (Lave, 1996; Lave & Wenger, 1991). From this perspective knowledge is inseparable from the activity, context, and culture in which it is developed and used. Heckman and Weissglass (1994) contend that "a key and vital factor in acquiring knowledge through cognitive apprenticeships is situating the learning experience in an environment that is real to the student" (p. 30). This requires learning to pose mathematically and pedagogically interesting problems that connect to students' lives. Where do preservice teachers see mathematics? How do they see or notice mathematics in their own or their students' lives? Inspired by Richard Phillip's Problem Pictures CD-ROM (http://www.problempictures.co.uk/index.htm) that offers hundreds of digital photos as a source of mathematics problems together with Sullivan and Lilburn's (2002) description and examples of open-ended mathematics problems our study examines the experiences of preservice teachers who developed open-ended mathematics problems around their personal collection of digital photographs. How did preservice teachers approach this task, what did they learn and find challenging, and to what extent does this problem-posing task offer insight into learning practices of teaching?

CONTEXT AND DATA COLLECTION METHODS

The Problem Pictures task was posed to elementary preservice teachers as a course assignment in a 13-week mathematics teacher education course taught by Author A. The task involved preservice teachers in collecting their own photos with digital cameras, selecting four photos from their collection, analysing the photos, and then posing 3 to 4 open-ended mathematics problems associated with each photo. Preservice teachers were encouraged to collect photo images that they thought would be engaging to students and would offer opportunities to explore interesting mathematics related to the elementary school curriculum. They collected images over a 2-week period and collated and submitted their pictures and problems in a PowerPoint file. A range of contexts were chosen by preservice teachers as places to pose problems. Figure 1 is representative of the kinds of photos and problems developed by participating preservice teachers.

Problem: You are a giant spider and this is your web. If you catch one or two flies in every "hole" in your web, how many flies might you catch for supper? BOO!

Figure 1. Problem picture photo and question designed by preservice teacher.

Participants for the study were enrolled in a 3 hour per week, 13 week mathematics education course as part of a two-year post-baccalaureate teacher education program in a large Canadian university. Students enrolled in this course were also members of the Diversity cohort–a programme option for students entering the teacher education problem with interests on issues of diversity, social justice, and equity. Thirty-three of the 40 students volunteered to participate in the study. Participants' backgrounds included Asian-Canadian (14), First Nations (4), and Caucasian (15). Participants were in their first year and first term of the teacher education program.

Data collected included researcher field notes, a written response survey completed by students upon completion of the course and copies of students' work in the form of the Problem Pictures assignment (as described above). For this paper we draw upon researcher field notes and students' written survey responses. The survey was administered through SurveyMonkey (an online survey program) and was developed to learn more about preservice teachers' experiences with the Problem Pictures task. It involved 15 open response questions asking students to share their thoughts about how they approached the assignment, what they learned and did not from it and what they found useful and challenging. Four questions were selected for analysis in this paper. These questions specifically examined participants' approach to creating open-ended problems based on original photos, the challenges the preservice teachers faced in this assignment and the impact of this task on their future as an educator.

A qualitative computer program, Nvivo, was employed to collate and analyse the data gathered from the online survey and field notes. A preliminary phase of analysis consisted of reviewing the students' responses and implementing a coding scheme. The responses were coded by the researchers independently according to the common themes that emerged and then cross-checked for commonality and consistency. The themes were categorised and reviewed again for emerging sub-themes. The data from the interviews are presented in a narrative form, and the interpretation presented in the discussion. These data are seen as broadly representative of the general views of the participating preservice teachers. Field note excerpts supplemented these data from the researcher's perspective.

RESULTS

Preservice teachers' reported varied responses in their strategies for approaching the Problem Pictures assignment; however most (85%) stated that they began by looking around them (indoors and outdoors) for mathematical contexts. These preservice teachers indicated that they began by seeking out mathematically-centred photos and developing their questions based on these images. Heather (all names used are pseudonyms) detailed this process:

> Firstly, I took my digital camera and snapped photographs of what I thought could turn into a mathematical question. Capturing photographs proved to be harder than I thought,

> because I did not just randomly snapped pictures. I would stare at a potential scene for a few good seconds, thinking if I could come up with at least two "good" diverse questions, and if I could not, I would just move on and started to walk elsewhere.

The formulation of open-ended questions is challenging for experienced teachers who do not have the added restriction of matching the questions to an original photo as outlined in the assignment criteria. It appears that the preservice teachers' limited knowledge of the mathematical curriculum was an added barrier and made the task more difficult than anticipated.

Some preservice teachers chose to stage their photos based on personal interests, as illustrated by Ava's comment:

> I first approached this assignment by taking photos around UBC that I felt contained possibilities for good questions. However, I felt uninspired. I then took some photos around my neighbourhood (playground, streetscape, etc.) but still did not feel great about what I was coming up with. After turning ideas over in my head, I decided to set up a Scrabble board with math words. After that the questions wouldn't stop coming! I decided to make a list of things I really enjoyed doing; baking, basketball, and playing with my nephew inspired me for the remaining pictures. The questions came easily once I felt a connection and excitement with the photos. I used the IRPs [curriculum documents] as a guide and tried to cover a variety of the Prescribed Learning Outcomes with the questions.

Preservice teachers sought images that depicted school mathematics (particularly the topics of space and shape) more than images that were connected to their own lives, interests and passions. That preservice teachers did not readily see their own passions and interests as a resource to collect photos and develop problems indicates the disconnect many felt with mathematics and their personal lives. An informal analysis of their photos and problem contexts confirms this claim.

As some preservice teachers (48%) became more familiar with the provincial curriculum documents and the nature of the Problem Pictures assignment their approach to collecting photos changed. Rather than selecting and taking photos that inspired possible open-ended questions, these students began searching for photos that would match questions they had already posed. Sophia's comment is indicative of this change:

> At first, I just took pictures of things that I thought I could formulate questions around. But once I refered [sic] to the IRPs [curriculum] it seemed like that wouldn't necessarily work. Instead I ended up looking at the IRPs [curriculum] and then generating questions and ideas for potential photos. So in the end, I really thought of the questions first, and then went out and took the pictures that I had in mind for those questions.

As these preservice teachers began to focus on the mathematical needs of their students they created Problem Pictures based on the requirements of the curriculum documents. Thus for some collecting photos became an act of finding images to illustrate the problem rather than finding images that could ground or situate the problem.

Mathematical and Pedagogical Possibilities

Participants were asked to comment on what they learned through the Problem Pictures task. Most students (95%) reported that collecting images and exploring open-ended problems with the photos helped extend or challenge their previous views about the nature of mathematics and how it might be taught. Connie's observation was representative of this realisation, "I learned that math is really all around me, and that it is useful to me in everyday life, not just in school for homework from textbooks and tests from teachers." Preservice teachers further stated that creating open-ended tasks challenged their previously-held views about the nature of mathematics. This can be seen in Yvette's comment:

> It taught me that there is such thing as "doing math without ONE correct answer". Prior to taking this course, I held the WORST fears and anxieties toward math and teaching math, most likely due to my poor math performances in 11th and 12th grade. Looking back, it certainly would have been nice to have these types of questions to do back then to build confidence.

Although many preservice teachers in the course expressed some mathematics anxiety, it is interesting to note that even those who stated they had enjoyed mathematics and was successful with it as a school student reported their experience with the Problem Pictures task changed their ideas about mathematics. Isabel stated:

> I love math, but I think it was because I could get the right answers most of the time because I was good at the repetitiveness of close-ended questions. However, these open-ended questions… I actually enjoyed them more… they provoked more enthusiasm and excitement in math.

For preservice teachers, such as Isabel, multiple possible solutions to questions provided a less apprehensive lens and a more exciting context through which to view and experience mathematics.

Creating open-ended questions, reported some preservice teachers (25%), gave them a sense of empowerment through being able to create mathematical problems beyond the textbook and classroom walls. This process provided them with a sense of ownership of their questions. Heather stated, "Because these questions are thought of by me, I would feel much more comfortable explaining and teaching the concepts of these questions because I made them up as compared to teaching through the textbooks." On the one hand, this expression of autonomy is impressive for a beginning teacher. However, discarding textbook problems developed by experienced teachers and researchers that could be a source of possible problems is worrisome. It is important for preservice teachers to not only develop but also consider and critically evaluate well-constructed open-ended questions designed by experts in the field (see Sullivan & Lilburn, 2002).

DISCUSSION AND CONCLUSION

Our results indicate that preservice teachers designing open-ended questions based on photographs prompted a shift in their understanding of pedagogical approaches and ways in which they viewed mathematics. However, at least one-third of the participants stated that developing open-ended questions was complex. One participant stated, “I debated with my classmates for hours trying to figure out whether or not my question was open-ended or not.” This statement supported in-class observations of the preservice teachers’ struggle with the idea of open-endedness. As Ilana wrote, “…it was hard to distinguish between a very good close-ended question and an open-ended one.” It was observed that many students initially viewed an open-ended question as having multiple, complicated steps to achieving one correct answer.

A second challenge for preservice teachers involved their limited experience with the mandated school mathematics curriculum. Unfamiliarity with the curriculum made it difficult for preservice teachers to pose mathematically appropriate and suitable problems for particular grade levels. This unfamiliarity may explain some students’ reported struggle to find the mathematics within a particular photo scene. However, students’ understanding of mathematics also framed what they were able to attend to or notice that was mathematical within a photo. That some preservice teachers turned to the elementary school curriculum for a list of mathematical topics and concepts (e.g., number, patterns, relations, space and shape) that could be used to analyze a photograph speaks to the structure the curriculum offered them for noticing school mathematics outside the textbook. Thus the curriculum both constrained and provided possibilities for preservice teachers to explore and create open-ended problems.

Further analysis that includes an examination of preservice teachers’ collected problems and photos can help determine the extent to which the preservice teachers were able to develop mathematically interesting problems. Sinclair and Crespo (2006) found that when preservice teachers were intent on posing problems for their students they created mathematically less interesting problems than if they were to create problems for themselves. However, Crespo (2003) previously found that having an authentic audience, such as preservice teachers posing problems to young children, was one supporting factor that helped preservice teachers move from posing procedural computation-type problems to more open-ended problems. Our study results indicate that preservice teachers’ overwhelmingly reported the mathematical and pedagogical benefits of creating open-ended problems related to photos. As Del best stated, “…now I carry my digital camera around and have noticed more math in real life.” Nonetheless, we wonder if the kinds of problems preservice teachers create in this context are mathematically interesting (Sinclair & Crespo, 2006) and if preservice teachers consider themselves as the authentic audience or their future students. These questions are important if we are to further our understanding of teaching as a learning practice and the art of creating and posing mathematical problems as a significant aspect of that practice.

References

Crespo, S. (2003). Learning to pose mathematics problems: Exploring changes in preservice teachers’ practices. *Educational Studies in Mathematics,* 52, 243-270.

Gutstein, E. (2006). *Reading and Writing the World with Mathematics: Toward a Pedagogy for Social Justice*. New York: Routledge.

Heckman, P. & Weissglass, J. (1994). Contextualized mathematics instruction: Moving beyond recent proposals. *For the Learning of Mathematics,* 14(1) 29-33.

Henningsen, M. & Stein, M. (1987). Mathematical tasks and student cognition: Classroom-based factors that support and inhibit high-level mathematical thinking and reasoning. *Journal for Research in Mathematics Education, 28*, 524-549.

Lampert, M. (2001). *Teaching Problems and the Problems of Teaching*. New Haven: Yale University Press.

Lave, J. (1996). Teaching, as learning, in practice. *Mind, Culture and Activity* 3(3), 149-164.

Lave, J. & Wenger, E. (1991). *Situated Learning: Legitimate Peripheral Participation*. New York: Cambridge University Press.

National Council of Teachers of Mathematics [NCTM] (1991). *Professional Standards for Teaching Mathematics*. Reston VA: NCTM.

Nicol, C. (1999). Learning to teach mathematics: Questioning, listening, and responding. *Educational Studies in Mathematics,* 37, 45-66.

Nicol, C. & Bragg, L. A. (forthcoming). How preservice teachers see mathematics through problem pictures.

Schoenfeld, A. (1989). Teaching mathematical thinking and problem solving. In L. Resnick & L. Klopfer (Eds.), *Toward the Thinking Curriculum: Current Cognitive Research* pp. 83-103. Alexandria, VA: ASCD.

Sinclair, N. & Crespo, S. (2006). What makes a good problem: An aesthetic lens. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education,* 5, 129-136. Prague: PME.

Sullivan, P. & Lilburn, P. (2002). *Good Questions for Math Teaching: Why Ask them and what to Ask*. Sausalito, CA: Math Solutions Publications.

TOWARDS A LANGUAGE OF DESCRIPTION FOR CHANGING PEDAGOGY

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In this paper I develop a set of codes for describing teacher moves as they respond to learner contributions. The codes are used to analyse four teachers' changing practices, as they begin to work towards engaging with learners' ideas. The analysis shows that the teachers did shift towards more "reform-type" moves. At the same time they maintained the more "traditional" type moves, thus developing hybrid pedagogies.

It has long been understood that teacher-learner interaction and classroom discourse is an important influence on mathematics learning in classrooms. Research on mathematics classroom discourse suggests that it is usually relatively constrained and does not allow for the development and deepening of learners' mathematical thinking. A key aim of mathematics reforms worldwide has been to shift the nature of classroom discourse so that it can better support conceptual mathematical reasoning among learners. Changing patterns of discourse can be a difficult and demanding task for teachers and recent research has begun to describe different ways in which teachers both manage and struggle with this challenge (Gamoran Sherin, 2002; Nathan & Knuth, 2003). Much of this research has been done as case studies with individual teachers. In this paper, I develop a set of codes to describe teacher responsiveness to learner contributions across teachers. These codes serve as the beginnings of a language of description for teaching and show how teaching practices shift as teachers begin to develop strategies for enabling participation and engaging with learners' thinking in their classrooms.

RESEARCH PARTICIPANTS AND CONTEXTS

The subjects in this study were four secondary school mathematics teachers (one grade 10 or 11 class for each teacher), in four differently resourced schools in Johannesburg, South Africa. Two schools are in poor socio-economic areas, are under-resourced, and serve exclusively black learners. One school is in a lower-middle class area, with adequate resources and with a racially diverse learner profile. The fourth is a private, extremely well resourced school, serving very wealthy learners who are predominantly white and all boys. At the time of the study, each teacher had between 7 and 15 years of mathematics teaching experience in secondary schools. Each teacher also has relatively strong mathematical knowledge and pedagogical content knowledge (established through task-based interviews and through classroom observation). All of the teachers were enrolled on an in-service post-graduate degree programme at a university in Johannesburg.

The teachers volunteered for the study and formed a purposive sample in that they were well informed about new curriculum developments in South Africa. Since much teacher development around the new curriculum is at a generic pedagogical level rather than subject related, many teachers' understandings of the mathematical implications of the curriculum are relatively superficial (Chisholm et al., 2000). I therefore thought it important to work with teachers who were reasonably well informed mathematically. While my sample of teachers is unique in this way, my initial classroom observations and interviews with the teachers suggested a range of teaching styles and approaches. Table 1 gives details of the schools and classes in which the research was conducted.

Teacher	Grade	SES	Class size	Tracking/Level of class
Mr. Nkomo	11	low	28	Untracked: Weak knowledge
Mr. Daniels	11	mid	35	Tracked: Knowledge at Grade level
Mr. Peters	10	very low	45	Untracked: Extremely weak
Ms. King	10	high	27	Tracked: Strong knowledge

Table 1. Description of research classes

The teachers were observed and videotaped for two weeks each. The first week was in February or March, towards the beginning of the school year and each teacher taught according to his/her own teaching plan in his/her usual ways. The second week was in April or May. In preparation for this week, the teachers worked together to plan a number of tasks that would support mathematical reasoning among their learners and talked with each other about possible teaching approaches. The teachers had been exposed to ideas about interaction in reform-oriented mathematics classrooms and they spent some time thinking about different ways in which they might respond to learner contributions. They all used a combination of group work and whole class discussion, where the whole class discussion was based on report backs from the group work. The teachers each used about a week of lessons on the set of tasks. All of the lessons were videotaped, transcribed and coded on the transcripts while watching the videotapes. The analysis in this paper focuses on the whole class sessions.

CLASSROOM DISCOURSE

I developed a set of codes to describe the teachers' shifts in responsiveness to learner contributions across the two weeks, drawing on the work of Mehan (1979) and others on the Initiation-Response-Feedback/Evaluation (IRF/E) exchange structure of classroom discourse. In this structure, the teacher makes an *initiation* move, a learner *responds*, the teacher provides *feedback* or *evaluates* and then moves on to a new *initiation*. Often, the feedback/evaluation and subsequent initiation moves are combined into one turn, and sometimes the feedback/evaluation is absent or implicit. This gives rise to an extended sequence of initiation-response pairs, where the repeated initiation works to achieve the response the teacher is looking for. When this response is achieved, the teacher positively evaluates the response and the extended sequence ends.

Many classroom researchers see the IRE structure as a constraining form of classroom interaction. Although this structure requires a learner contribution every other turn (the response move), and therefore apparently gives learners time to talk, much research has shown that because teachers tend to ask questions to which they already know the answers (Edwards & Mercer, 1987) and to 'funnel' learners' responses toward the answers that they want (Bauersfeld, 1980), space for genuine learner contributions is limited. Therefore, many reform proponents suggest that a complete shift of the IRE structure is necessary to achieve the goals of student engagement and inquiry. They argue that classroom discussions should become more like conversations, with the teacher being a participant in similar ways to learners (Davis, 1997; Nystrand & Gamoran, 1991).

However, aside from the enormous challenges involved in creating such conversations (Brodie, 2007), such suggestions ignore the particular functions of the teacher in the classroom, which include evaluating learners. Some argue that the IRE is a particular form of classroom discourse that can be used for a range of functions, both positive and negative (Wells, 1999). In trying to understand different possibilities for interaction, it is important to try to understand the benefits that the IRE can afford, which depend on the nature of the elicitation and evaluation moves.

TEACHER MOVES

Teacher initiations and evaluations are often fused in form, although not in function. I developed a set of codes to describe the function of teacher utterances as they initiate and evaluate. My unit of coding is the teacher move, which is constituted by all or part of a teacher turn that can be described with one code. Thus codes help to determine moves, it is not possible to identify moves prior to coding, and codes distinguish between different moves in one turn of teacher talk.

The key category in my data is *follow up*, which describes all teacher moves that respond to learner contributions. My *follow up* code is closely related to, although broader than, Nystrand and Gamoran's (1991) notion of "uptake". *Follow up* includes some teacher moves that "uptake" might not include. A teacher move is coded as *follow up* when the teacher picks up on a contribution made by a learner, either immediately preceding or some time earlier. This could be in a form of a request for clarification or elaboration; the teacher can ask a question or challenge the learner. Usually there is explicit reference to the idea, but there does not have to be. Usually the idea is in the public space, but it does not have to be; for example when a teacher asks a learner to share an idea that she has seen previously in the learner's work. Repeating a contribution counts as *follow up* if it functions to solicit more discussion in relation to the learner's contribution. An initial coding of my data showed that there were several ways in which the *follow up* move was used by teachers each of which functioned differently. Table 2 lists the subcategories of the *follow up* move. Examples of these can be found in Brodie (2004).

Insert	The teacher adds something in response to the learner's contribution. She can elaborate on it, correct it, answer a question, suggest something, make a link etc.
Elicit	While following up on a contribution, the teacher tries to get something from the learner. She elicits something else to work on learner's idea. Elicit moves can sometimes narrow the contributions in the same way as funneling.
Press	The teacher pushes or probes the learner for more on their idea, to clarify, justify or explain more clearly. The teacher does this by asking the learner to explain more, by asking why the learner thinks s/he is correct, or by asking a specific question that relates to the learner's idea and pushes for something more.
Maintain	The teacher maintains the contribution in the public realm for further consideration. She can repeat the idea, ask others for comment, or merely indicate that the learner should continue talking.
Confirm	The teacher confirms that s/he has heard the learner correctly. There should be some evidence that the teacher is not sure what s/he has heard from the learner, otherwise it could be press.

Table 2. Subcategories of *Follow up*

These categories are informed by the literature as follows: *Insert* is motivated by a similar rationale to that of Lobato et al (2005), that teachers cannot avoid "telling" and this should be recognised as an appropriate part of their practice. *Elicit* is closest to Edwards and Mercer's (1987) "repeated questions imply wrong answers" or Bauersfeld's (1980) "funneling", which can constrain as much as enable learner thinking. It is likely that Nystrand and Gamoran (1991) would not have included such moves in their notion of uptake because they may not represent a serious consideration of learner ideas. I chose to include *elicit* moves under *follow up*, because for my purposes it is illuminating to distinguish between different kinds of *follow up* rather than to exclude a range of moves that teachers might intend as a *follow up* from this category. *Press* is similar to Wood's "focusing" and to Boaler and Brodie's (2004) fourth question category "probing". *Elicit* and *press* moves can sometimes seem similar to each other, they are distinguished in similar ways to how Wood (1994) distinguishes focusing from funneling – a press move orients towards the learners' thinking, rather than towards a solution. *Maintain* is similar to "social scaffolding" (Nathan & Knuth, 2003), and supports the process of learners' articulating their contributions. It is also similar to revoicing (O'Connor & Michaels, 1996) and often involves a repetition or rephrasing of the learner's contribution which keeps the idea in the public realm for further consideration. Although *confirm* exists as a category, it occurred seldom in my data and did not have major consequences for the interaction, so it is not discussed in detail.

These moves can be arranged on a continuum of less to more intervention as follows: *confirm* and *maintain* make no intervention in the learner contribution, merely keep it in the public space; *press* tries to get the learner to intervene in her own contribution; *elicit* intervenes more directly, where the teacher has something in mind that she steers the learner towards; and *insert* is the most direct intervention, where the teacher makes her own mathematical contribution. *Maintain* and *press* moves can be seen to be more "reform-oriented" moves. *Maintain* moves suggest a more neutral teacher or a "facilitator", where the teacher holds back on her/his own ideas in favour of enabling the conversation to continue or supporting a learner to work harder to articulate, clarify and deepen her/his engagement with the ideas. *Elicit* and *insert* moves might be considered to be more traditional, in that the teacher brings in more of her own knowledge of the discipline, asking questions that suggest particular answers (*elicit*) or explaining concepts or making her/his own points in the conversation (*insert*). Part of the argument of this paper is that even as teachers shift their pedagogy towards reform-oriented moves (maintain and press), they maintain elements of previous pedagogy (elicit and insert).

DISTRIBUTION OF TEACHER MOVES

For each teacher, I coded all of the whole-class teaching during the two weeks. Table 3 gives the distribution of follow up moves for each teacher during each week.

Teacher	Week 1	Week 2
Mr. Daniels	177; 71%	174; 61%
Mr. Nkomo	319; 44%	238; 70%
Mr. Peters	929; 82%	432; 68%
Ms. King	405; 79%	209; 52%

Table 3. *Follow up* moves: Week 1 and 2 (numbers and percents)

The most striking result in table 3 is the predominance of *follow up* moves in all the classrooms. For each teacher during both weeks, except for Mr. Nkomo in week 1, *follow up* moves accounted for the majority of moves. A significant result is that the percentages of *follow up* moves decreased from week 1 to week 2 for all the teachers except Mr. Nkomo. This is somewhat surprising, given that the teachers were trying to engage their learners' meanings and reasoning more in week 2 than in week 1. However, as discussed above, the way in which *follow up* was defined included some moves that could be considered as more constraining of learner thinking, particularly the *elicit* move.

Table 4 shows the shifts in the subcategories of *follow up* from week 1 to week 2. There was a substantial increase in *press* moves for all teachers and an increase in *maintain* moves, although these increased substantially more for the two Grade 11 teachers, Mr. Daniels and Mr. Nkomo, than for the two Grade 10 teachers, Mr. Peters and Ms. King. The increase in *press* and *maintain* moves was accompanied by a decrease in *elicit* moves. This suggests that the teachers decreased the amount of narrowing or constraining of the tasks and their questions. The picture for *insert* is slightly more complex, with the two Grade 11 teachers, Mr. Daniels and Mr. Nkomo, decreasing and the two Grade 10 teachers, Mr. Peters and Ms. King increasing. Across all four teachers, the shift in *insert* moves is less than the shifts in the other moves.

		Insert	**Elicit**	**Press**	**Maintain**	**Confirm**
Mr Daniels	Week 1	38	28	6	21	6
	Week 2	24	5	20	42	9
Mr Nkomo	Week 1	30	24	4	38	5
	Week 2	18	10	13	50	9
Mr Peters	Week 1	19	49	4	27	2
	Week 2	24	23	20	30	4
Ms King	Week 1	29	34	2	31	4
	Week 2	31	21	7	39	2

Table 4. Subcategories of follow up (percents)

The above distributions provide a first level description of the teachers' changing practices. They show that Mr. Nkomo shifted in *following up* on learner contributions, with a substantial increase in week 2. Within this increase in *follow up*, he also increased the percentage of *press* and *maintain* moves while decreasing *elicit* and *insert* moves. Mr Daniels decreased his *follow up* moves but also increased his *press* and *maintain* moves while substantially decreasing his *elicit* and *insert* moves. Mr. Peters and Ms. King decreased their *follow up* moves while increasing their *press* and *maintain* moves, slightly increasing their *insert* moves and decreasing their *elicit* moves (however, these remain substantially higher than both Mr. Daniels' and Mr. Nkomo's). This description suggests that the two Grade 11 teachers shifted in somewhat similar ways, while the two Grade 10 teachers shifted in similar ways. This can be somewhat accounted for by the fact that they used similar tasks in week 2. Another interesting point to note is that Mr. Peters worked with the lowest SES learners in the sample, whose mathematical knowledge was weakest and Ms. King worked with the highest SES learners whose mathematical knowledge was strongest. This suggests that at this level of description, when teachers work together, shifting pedagogies can be seen as similar across SES and mathematical knowledge of learners.

The above analysis suggests a number of important points about the teachers' changing pedagogies. First, they all did shift to the more reform-type moves: *press* and *maintain*. However, even with the shift to *press* and *maintain* moves in week 2, the teachers continued to use all the moves. This resonates with Boaler and Brodie's (2004) finding that while teachers using a reform curriculum asked more conceptual and probing questions, they still continued to use a substantial amount of questions requiring recall or procedures. So the analysis suggests that teachers develop hybrid pedagogies (Cuban, 1993) and illuminates some of the ways in which they do so.

CONCLUSIONS

In this paper I have presented a set of codes for describing teacher discourse in mathematics classrooms. The codes build on existing analyses of discourse and take these further by further elaborating a notion of *follow up*, which is key to the teaching work of seriously engaging with learners' contributions. These codes, together with their conceptual bases provide the beginnings of a language of description for teaching as teachers engage with learners' thinking. The codes enable comparisons across teachers as well as a means of understanding each teacher's practices on its own terms. They also enable descriptions of teachers' practices that cut across vague descriptions such as traditional and reform and thus enable us to describe, analyse and understand shifting practices in contexts of curriculum and pedagogical change. The codes provide a broad level of description, which can be deepened with more qualitative analyses, and they provide a framework for such analyses.

Using this language of description, I presented an analysis of the ways in which four mathematics teachers engaged with learners' contributions, and how this engagement shifted when the teachers explicitly attempted to engage learners' mathematical thinking. The teacher moves that are most usually associated with deepening learner engagement are the teacher *press* (Boaler & Brodie, 2004; Kazemi & Stipek, 2001) and *maintain* moves. In my data, there was a significant increase in these moves from week 1 to week 2, suggesting that the teachers were attempting to engage learners' thinking. There was also an increase in *maintain* moves, which shows that the teachers were supporting learner talk.

Research on teacher learning has shown that learning to support mathematical reasoning and engagement is a difficult task for teachers. It is clear that a range of interventions is necessary and appropriate to support teachers in this task. While a language of description is essentially a research tool, it can also be a tool for teachers and educators. Being able to describe what you are doing is a first step in understanding and improving on it. While the language that I have provided is certainly not the only possibility, I suggest that it can and will give teachers a way in which they can talk about and reflect on their own practices of responding to and interacting with learners' contributions.

References

Bauersfeld, H. (1980). Hidden dimensions in the so-called reality of a mathematics classroom. *Educational Studies in Mathematics, 11*, 23-41.

Boaler, J. & Brodie, K. (2004). The importance, nature and impact of teacher questions. In D. E. McDougall & J. A. Ross (Eds.), *Proceedings of the 26th annual meeting of the*

North American chapter of the International Group for the Psychology of Mathematics Education (Vol. 2, pp. 773-781). Toronto: Ontario Institute of Studies in Education/University of Toronto.

Brodie, K. (2004). Working with learner contributions: Coding teacher responses. In D. E. McDougall & J. A. Ross (Eds.), *Proceedings of the 26th Annual meeting of the North American chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 689-697). Toronto: Ontario Institute of Studies in Education/University of Toronto.

Brodie, K. (2007). Teaching with conversations: Beginnings and endings. *For the Learning of Mathematics, 27*(1), 17-23.

Chisholm, L., Volmink, J., Ndhlovu, T., Potenza, E., Mahomed, H., Muller, J., et al. (2000). *A South African curriculum for the twenty first century. Report of the review committee on Curriculum 2005.* Pretoria: Department of Education.

Cuban, L. (1993). *How Teachers Taught: Constancy and Change in American Classrooms*. New York: Teachers' College Press.

Davis, B. (1997). Listening for differences: An evolving conception of mathematics teaching. *Journal for Research in Mathematics Education, 28*(3), 355-376.

Edwards, D. & Mercer, N. (1987). *Common Knowledge: The Growth of Understanding in the Classroom.* London: Routledge.

Gamoran Sherin, M. (2002). A balancing act: developing a discourse community in a mathematics classroom. *Journal of mathematics teacher education, 5*(205-233).

Kazemi, E. & Stipek, D. (2001). Promoting conceptual thinking in four upper-elementary mathematics classrooms. *Elementary School Journal, 102*, 59-80.

Lobato, J., Clarke, D., & Ellis, A. B. (2005). Initiating and eliciting in teaching: a reformulation of telling. *Journal for Research in Mathematics Education, 36*(2), 101-136.

Mehan, H. (1979). *Learning Lessons: Social Organisation in the Classroom*. Cambridge, MA: Harvard University Press.

Nathan, M. J. & Knuth, E. J. (2003). A study of whole classroom mathematical discourse and teacher change. *Cognition and Instruction, 21*(2), 175-207.

Nystrand, M. & Gamoran, A. (1991). Student engagement: When recitation becomes conversation. In H. C. Waxman & H. J. Walberg (Eds.), *Effective Teaching: Current Research* (pp. 257-276). Berkeley, CA: McCutchan Publishing Corporation.

O'Connor, M. C. & Michaels, S. (1996). Shifting participant frameworks: Orchestrating thinking practices in group discussion. In D. Hicks (Ed.), *Discourse, Learning and Schooling.* (pp. 63-103). New York: Cambridge University Press.

Wells, G. (1999). *Dialogic Iquiry: Toward a Sociocultural Practice and Theory of Education*. Cambridge: Cambridge University Press.

Wood, T. (1994). Patterns of interaction and the culture of mathematics classrooms. In S. Lerman (Ed.), *Cultural Perspectives on the Mathematics Classroom* (pp. 149-168). Dordrecht: Kluwer.

EXPLORING THE NEED FOR A PROFESSIONAL VISION TOWARDS CURRICULA

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In this paper, we describe a novel research approach to exploring the character of teachers' implementations of a Standards-based curriculum. Specifically, we propose a theoretical reconceptualisation of the intended curriculum and then use this construct to explore teachers' enactments of lessons. We conclude by discussing the need for a professional vision towards curricula shared by teachers, curriculum developers, and researchers in mathematics education.

INTRODUCTION

The United States has spent nearly 93 million dollars over the past two decades developing *Standards*-based curricula (NRC, 2004) that incorporate the recommendations of the National Council of Teachers in Mathematics' *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989) and *Principles and Standards for School Mathematics* (NCTM, 2000), only to have these curricula bombarded with attacks on their effectiveness. Furthermore, developers of *Standards*-based curricula have been asked to meet unprecedented research standards (Reys, 2001). These events highlight some of the difficulties faced by researchers and *Standards*-based curriculum developers as they strive to enact the reforms described in the NCTM's *Standards* documents.

One benefit of the current controversies surrounding mathematics education in the United States is that they have increased both attention to mathematics education and opportunities for research on mathematics teaching, learning and curricula (Reys, 2001). For example, after an extensive review of evaluation studies of 13 curricula, researchers concluded that existing research was insufficient to determine the effectiveness of the curricula (NRC, 2004). One response to this finding was a call for research that analyses "the quality, extent, and means of curricular implementation" (NRC, 2004, p. 4). Along the same lines, and in reaction to national policy changes, Schoenfeld (2006) argues:

> Indeed, one can imagine curricular materials that, when used in the way intended by the designers, result in significant increases in student performance, but, when used by teachers not invested or trained in the curriculum, result in significant decreases in student performance. Hence, data gathering, coding, and analysis must try to indicate the character of the implementation and its fidelity to intended practice (p. 17).

In this paper, we describe a novel research approach to exploring the character of teachers' implementations of an elementary *Standards*-based curriculum, *Math Trailblazers*. Specifically, we propose a theoretical reconceptualization of the *intended curriculum*. We then identify aspects of the written curriculum that appear to play a role in teachers' enactments of the intended curriculum. We conclude with a discussion of how teachers' use of curricula indicates the need for the development of a professional vision towards curricula shared by teachers, curriculum developers, and researchers.

RECONCEIVING THE INTENDED CURRICULUM

Curriculum theorists define the *intended curriculum* as the learning objectives, determined at a national or sub-national level, expected to be taught within an educational system (McKnight, 1979). Models of the ways in which these learning objectives play a role in student outcomes often view these objectives as directly influencing curricula, and consequently, classroom lessons (Valverde et al., 2002). This definition and perspective of the intended curriculum is problematic in the United States for two reasons. First, we have neither a national curriculum nor do we have national learning objectives. Educational learning objectives are determined at either state or local district levels with little consistency from one set of objectives to the next (Reys, et al., 2006). Second, textbooks in the United States are produced, almost exclusively, by private publishing companies who in order to exist must meet the requirements of many states and therefore multiple sets of state objectives, recommendations, or standards. Thus, curricula in the United States do not address an intended curriculum, but rather many intended curricula. Efforts to move towards unified learning objectives, however, have been put forth in the National Council of Teachers of Mathematics *Standards* documents (NCTM, 1989; NCTM, 2000). The National Science Foundation furthered these efforts by funding or co-funding the development of "*Standards*-based" curricula. Researchers, however, have raised questions about how standards are interpreted through curricula and, more generally, what it means to be *Standards*-based (Ferrini-Mundy, 2004). Thus, it is unclear even in the context of *Standards*-based curricula, how the curricula address the intended curriculum. It is also difficult, if not impossible, to say the extent to which an implementation represents "fidelity to intended practice."

The word *intended* is derived from the Latin word *intendere*, which means to aim towards. The position taken in this paper is that specific learning objectives guide curriculum authors' decisions when developing curricula, whether or not national or sub-national learning objectives exist. Moreover, learning objectives do not live in isolation. Authors' selection of learning objectives are influenced by their interpretations of others' learning objectives and their ideas about how students come to know mathematics; that is, the authors' epistemological stance. Moreover, the authors' interpretations of others' learning objectives and their epistemological stance influence the selection, sequencing and more importantly, the conceptualization of learning objectives; that is, the ways in which authors design activities that address or embody learning objectives. Thus, written curricula, within the United States, can and should be viewed as addressing the authors' *intended curriculum*. In the case of current U.S. reforms, authors of *Standards*-based curricula attempted to address not only novel learning objectives but also learning objectives of a different form. Specifically, learning objectives have historically been interpreted as specific skill and content knowledge goals. In contrast, the learning objectives of recent reforms in the United States and described within the NCTM's *Standards* documents focused on content and on the ways in which students' might engage in that content during classroom activities. In other words, these learning objectives focus on students' *opportunities to learn* mathematics, as described by Hiebert (2003):

> "Providing an opportunity to learn what is intended means providing the conditions in which students are likely to *engage* in tasks that involve the relevant content" (p.10, emphasis in original).

It is for these reasons that, in the context of curricula in the United States, we have reconceptualized the notion of an intended curriculum as existing at a curricular level; and in the context of *Standards*-based curricula, construed the *authors' intended curriculum* as consisting of the intended *opportunities to learn* mathematics. Viewed in this way, questions concerning the implementation of *Standards*-based curricula, and more importantly, the character of implementations, become questions about the extent to which the intended opportunities arose as teachers enacted the lessons. Such questions have yet to be addressed within American curricular studies.

EXAMINING TEACHERS' ENACTMENTS OF LESSONS

The work reported in this paper was part of a larger NSF-funded study, the Whole Number Study (grant # ESI-035-2345). This study was aimed at understanding teachers' use of whole number lessons, students learning of whole number concepts, and developing research-based recommendations to inform revisions to the *Standards*-based, comprehensive, elementary curriculum, *Math Trailblazers*. In this paper, we focus on our explorations of teachers' use of whole number lessons as illustrated by two teachers' enactments of the grade 2 lesson, *Base-Ten Subtraction*.

To begin, we analysed the curriculum in terms of the authors' intended opportunities to learn. We then derived from a combination of resources -which included the curriculum, *Standards* documents cited in the curriculum, authors' philosophical statements embedded in curriculum resources, and discussions with the authors- a series of codes for the authors' intended opportunities to learn (see Table 1).

Data, in the form of videotaped lessons, was transcribed and then segmented. A segment, in this context, refers to a portion of transcript that begins and ends with a shift in activity. Pairs of researchers independently coded each segmented transcript by selecting the most appropriate opportunity to learn codes, rating the extent to which these opportunities arose and providing rationales for each rating. The researchers then compared the codes, discussed discrepancies, and produced a rating of the observation's degree of alignment to the intended curriculum.

A. Opportunities to Reason About Mathematics
A1. Opportunities to reason to solve problems; Opportunities to reason about mathematical concepts**.**
A2. Opportunities to use or apply concepts, strategies, or operations, reasoning about how to refine strategies so that they become more efficient.
A3. Opportunities to select from multiple tools, representations, or strategies.
A4. Opportunities to compare and make connections across tools, representation, or strategies
A5. Opportunities to validate strategies or solutions; reason from errors; inquire into the reasonableness of a solution.
B. Opportunities to Communication About Mathematics
B1. Opportunities to communicate mathematical ideas or ways of reasoning
B2. Opportunities to interpret another student's way of reasoning about tools, representations, strategies, or operations.
B3. Opportunities clarify reasoning or explanations; provide supporting rationale.
B4. Opportunities to characterize mathematical operations.

Table 1. Opportunities to learn codes

RESULTS

For this paper, we will focus on the lesson, *Base-Ten Subtraction*. We begin by exploring the written lesson materials for *Base-Ten Subtraction*. We then consider two implementations of *Base-Ten Subtraction* to illustrate differences in the extent to which opportunities to learn arose during the teachers' enactments of the lesson.

The written *Base-Ten Subtraction* lesson materials emphasize creating situations in which students can build on previously developed subtraction strategies and representations of number to explore subtraction with base-ten pieces. The lesson is part of a unit where students work on subtraction with regrouping and precedes a lesson that introduces a subtraction algorithm. In terms of opportunities to learn, the lesson focuses on providing opportunities for students to select and compare strategies, and to explore and discuss representations and strategies.

The curricular materials make these aims apparent both in background information about expectations for working with students to develop their understandings of computational operations, as well as in the lesson text itself. For instance, the written lesson materials do not specify a procedure that students' are expected to use to solve subtraction problems in this lesson, and there are numerous statements that indicate students should share their solution strategies. For example, "pairs share their solutions" would seem to imply that students should compare different ways of solving the problem. "If students represent 52 with the fewest number of pieces" would seem to imply that there is an expectation for variety in students' strategies.

The two implementations of *Base-Ten Subtraction* were videotaped in the Spring of 2004. Both sites were public, urban elementary schools, with moderate-income students. Teacher A's school is a mixed ethnicity school, typical of large urban areas. Teacher B's school is predominately white, with a small percentage of minority students. Videotaped observations of both teachers' implementations of *Base-Ten Subtraction* indicate that the teachers followed the written lesson. In particular, we documented that both teachers followed the lesson steps for the set-up, main activities, and conclusion of the lesson. Thus, for the remainder our discussion we focus on the degree of alignment between the enacted lesson and the intended opportunities to learn, where "enacted lesson" refers to the classroom lesson that results from interactions between the teacher and the students (and vice-versa) as they engage with the lesson materials (Reys & Roseman, 2004).

We rated Teacher A's enactment of *Base-Ten Subtraction* as having a high degree of alignment to the intended curriculum, indicating that the observed opportunities to learn aligned with the authors' intended opportunities to learn. In contrast, we rated Teachers B's enactment as having a low degree of alignment to the intended curriculum, indicating that the observed opportunities to learn did not align with those the authors' intended. Thus, Teacher A and Teacher B represent different ends of a continuum—a trend we observed in the larger data set. To illustrate differences in the teachers' enactments, we will focus students' opportunities to select from multiple tools, representations, or strategies (see A3) during the two enactments.

In Teacher A's enactment of *Base-Ten Subtraction*, she asks students to solve problems in ways that make sense to them. Notice below, how the teacher invites students to solve problems, in the first case, as they chose, and in the second case, as they chose but with a particular tool-the base-ten pieces.

Transcript Segment 1-Teacher A

> Teacher A: Okay. Now let's try to solve this problem by your favourite way of solving subtraction.

Transcript Segment 2-Teacher A

> Teacher A: Now, please use Base Ten Pieces to solve...
>
> Student 1: YAY!
>
> Teacher A: …fifty-two minus fourteen.

In both of these segments, immediately following the teachers' statements, students began working on the problem. As seen in the segments, Teacher A does not specify what strategies or representations students should use. Instead, students select how they will solve the problems. In general, the opportunities students had during this enactment aligned well with the authors' intended opportunities to learn.

Teacher B's enactment does not provide evidence of opportunities for students to select representations or strategies. Teacher B, for example, started the task 78-42 by remarking "So they want me to take away forty-two." She proceeded by specifying how students should operate on the base ten pieces, as demonstrated below.

Transcript Segment 3-Teacher B

> Teacher B: Why do I always start with the bits? Do you know? In math, can I start over here on the tens side? I can't, it's the opposite of what? What did we say it's the opposite of?
>
> Student 2: Reading.
>
> Teacher B: Reading. It's the opposite of reading and reading we work left to right, in math we work...
>
> Students: Right to left.
>
> Teacher B: Right to left. So we always have to start with the ones.

In this segment, Teacher B describes a procedure for the students to use when solving subtraction problems, as opposed to providing an opportunity for students to select strategies for solving the problem as intended by the authors. The teacher prompts students to begin with the ones place and work "right to left" while solving the problem. This structuring of students activities by the teacher is not suggested anywhere in the materials.

The lack of opportunities for students to select strategies had an impact on the potential for other opportunities to learn. Since students used the teacher-specified procedure, there were not multiple strategies to discuss or compare. They did not have to evaluate each other's thinking or respond to strategies different from their own. Therefore, as a result of the instructional decision to model an approach for solving base-ten subtraction problems, Teacher B's enactment lacked many of the intended opportunities to learn for *Base-Ten Subtraction*.

Considering the Written Instructional Materials

Although both teachers implemented the literal steps of the written lesson, there seems to be something in their professional and personal experiences that afforded them different interpretations of the written lesson. Teacher B appears to have interpreted the written lesson as one in which the primary goal was to introduce students to a procedure for solving subtraction problems with base-ten pieces. In contrast, Teacher A appears to have interpreted the written lesson as one in which the primary goal is to develop and explore students' subtraction strategies. One must question why the teachers' interpretations differed this way. One explanation is that Teacher B held particular views about students' learning of subtraction and therefore, read the lesson in a way that aligns with these views. Another explanation is that as Teacher B read the lesson, she focused on a subtraction strategy illustrated in the margin of the written lesson and concluded that she should model the strategy. Since the written lesson materials indicate that students should explore and discuss subtraction strategies, one can argue that Teacher B should have recognized that many strategies should arise during the lesson. The lesson materials, however, do not explicitly state that teachers should not model a strategy; information that Teacher B may have been unable to infer. If the later explanation is valid, then the curricular materials included information that supported Teacher B's interpretation, while at the same time leaving the teacher to infer information that contradicted her interpretation.

TOWARDS A SHARED PERSPECTIVE OF CURRICULA

In Miriam Gamoran Sherin's (2001) introduction to the notion of a professional vision towards classroom events, she described the following scenario:

> Imagine that you are standing at the site of an archeological dig. On your left you see a large rock with a dent in the middle. Next to it you see a pile of smaller stones. Aside from this, all you see is sand. An archeologist soon appears at the site. What looked like just a rock to you, he recognizes as the base of a column; the small stones, a set of architectural fragments. And where you saw only sand, he begins to visualize the structure that stood here years before (p. 75).

After which, Sherin introduces Goodwin's notion of a professional vision as: "socially organized ways of seeing and understandings events that are answerable to the distinctive interests of a particular social group" (Goodwin, 1994; cited in Sherin, 2001). Sherin then makes the case that there is a need for a professional vision towards classroom events shared by researchers and teachers. In this section of our paper, we extend Sherin's notion of a professional vision. In particular, we demonstrate why reform curricula may depend on the development of a *professional vision towards curricula* shared by authors, teachers and researchers.

As researchers familiar with the research basis cited in *Math Trailblazers* and with the *Math Trailblazers* teacher resources, we were able to approach the lessons within the curricula from a common foundation. This common foundation allowed us to interact with the curriculum authors to develop a shared perspective of the authors' intended curriculum at a lesson level and provided common ground for the communication of research findings. Considered in the context of Sherin's (2001) discussion, one can argue that the researchers and authors developed "socially organized ways of seeing and interpreting" the curriculum - that is, we developed a shared *professional vision towards the curriculum*.

Our work on teachers' implementation of whole number lessons, however, indicates that teachers differ in terms of the extent to which their lesson enactments align with the intentions of the authors. While some of the teachers in our study, teachers such as Teacher A, seem to hold a perspective of the curriculum that they share with the authors, other teachers appear to interpret the materials in ways that do not align with the authors' intentions. These findings indicate that while some teachers' ways of seeing and interpreting curricula may align with those held by authors of *Standards*-based curricula, we cannot assume that these ways of seeing and interpreting curricula are shared. In other words, we cannot assume that teachers, curriculum authors and researchers share a professional vision towards curricula. We can argue, however, that enactments such as Teacher B's enactment of *Base-Ten Subtraction* indicate that current reform efforts in the United States are likely to suffer until a shared professional vision of curricula develops. In particular, teachers are likely to continue to implement lessons in ways that fail to align with the intentions of authors. Furthermore, curriculum authors and researchers, in an effort to improve curricula, are likely to struggle with the form and content of information to provide to teachers. Specifically, it seems likely that a common language will develop without shared meanings – as is common to reforms.

References

Ferrini-Mundy, J. (2004). SIRG Brief 3: What does it means to be Standards-based? Issues in Conceptualizing, Measuring, and Studying Alignment with Standards. In Lester, F. K. & Ferrini-Mundy, J. (Eds.) *Proceedings of the NCTM Research Catalyst Conference* (pp. 25 – 32). Reston, VA: National Council for Teachers of Mathematics.

Hiebert, J. (2003). What research says about the NCTM Standards. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A Research Companion to Principles and Standards for School Mathematics* (pp. 1–23). Reston, VA: National Council for Teachers of Mathematics.

McKnight, C. C. (1979). Model for the Second International Mathematics Study. *SIMS Bulletin 4*. (pp. 6 – 39).

National Council of Teachers of Mathematics. (1989). *Curriculum and Estandards for School Mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics. (2000). *The Principles and Standards for School Mathematics*. Reston, VA: Author.

National Research Council. (2004). *On Evaluating Curricular Effectiveness: Judging the Quality of K-12 Mathematics Evaluations*. Washington, DC: Mathematical Science Education Board, Center for Education.

Reys, B. J., Dingman, S., Olson, T., Sutter, A., Teuscher, D., & Chval, K. (2006). Analysis of K-8 number and operation grade-level learning expectations. In B. J. Reys (Ed.), *The Intended Mathematics Curriculum as Represented in State-Level Curriculum Standards: Consensus or Confusion?*, (pp. 15–58). Charlotte, NC: Information Age Publishing.

Reys, B. & Roseman, J. (2004). Report of Working Group 3: Instructional Materials and Curriculum. In F. K. Lester & J. Ferrini-Mundy (Eds.) *Proceedings of the NCTM Research Catalyst Conference* (pp. 133 –140). Reston, VA: National Council for Teachers of Mathematics.

Reys, R. E. (2001). Curricular controversy in the math wars: A battle without winners. *Kappan Professional Journal*. Retrieved at URL:http://pdkintl.org/kappan/k0111rey.htm on January 2, 2008.

Schoenfeld, A. H. (2006). What doesn't work: The challenge and failure of the what works clearinghouse to conduct meaningful reviews of studies of mathematics curricula. *Educational Researcher*, 35(2), 13-21, March 2006.

Sherin, M. G. (2001). Developing a professional vision of classroom events. In T. Wood, B. S. Nelson, & J. Warfield (Eds.), *Beyond Classical Pedagogy* (pp. 75 - 93) Mahway, New Jersey: Lawrence Earlbaum Associates, Pulishers.

Valverde, G. A., Bianchi, L.J., Wolfe, R.G., Schmidt, W. H., & Houang, R. T. (2002). *According to the Book. Using TIMSS to Investigate the Translation of Policy Into Practice Through the World of Textbooks*. Dordrecht: Kluwer Academic Publishers.

8TH GRADE STUDENTS REPRESENTATIONS OF LINEAR EQUATIONS BASED ON CUPS AND TILES MODELS

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This study examines 8th grade students' use of a representational metaphor (cups and tiles) for writing and solving equations in one unknown. We base our analysis within a framework of referential relationship of meanings (Kaput, 1991). Our data consist of videotaped classroom lessons, student interviews and teacher interviews. Our results indicate that addition and (implied) multiplication operations only are the most meaningful and relevant operation types in dealing with these representational models. Only one student was able to construct a "family of meanings" in sense making of and in connecting the algebraic expressions and the representational metaphor when negative quantities were involved. We conclude that quantitative unit coordination and conservation are necessary constructs for overcoming the cognitive dissonance (between the two representations) experienced by students and teacher.

BACKGROUND

Physical objects, also often referred to as manipulatives, can serve as essential representational models in the course of experiential learning (NCTM, 2000). However, research has shown that the use of physical objects can be an obstacle to mathematical progress in some cases. Howden (1986) showed that even though students were successful mathematically at the concrete level, that was not always the case in the abstract level. Uttal et al. states that "part of the difficulty that children encounter when using manipulatives stems from the need to interpret the manipulative as a representation of something else." (1997, p .38) We believe that a reference to any kind of physical object brings with itself the necessity to think about the object under consideration as some sort of quantity possessing a name, a value, and a measurement unit (Schwartz 1988, Thompson 1993). Attending to the quantitative nature of manipulatives may be an asset for students' success in relating the manipulatives to their written symbolic referents. The physical object itself can not be a representation of a written symbol without "meanings" projected into these "concrete objects." (Olive & Vomvoridi, 2006). A successful mapping of the "concrete" to the "abstract" depends on the manipulative itself and a "family of meanings" attached to these objects.

THEORETICAL FRAMEWORK

Our study draws on Kaput's Referential Relationship Representational Model (1991, p.60). In this model, physical observables (e.g., algebraic expressions, their representations, etc.) are denoted by letters A, and B, respectively (Figure 1). The upper part of this model claims that meanings are projected onto the mental levels in the realms of physical observables A and B, respectively; whereas the bottom rectangle serves for connecting the two realms. The double arrows in both configurations imply a continuum of forward- and backward- shifts. The notations Cog $_A$ and Cog $_B$ represent the cognitive levels one must take into account in order to describe the corresponding representational acts on the physical observables A and B (p. 60). The shared referential meaning is a result of the cognitive operation connecting Cog $_A$ and Cog $_B$.

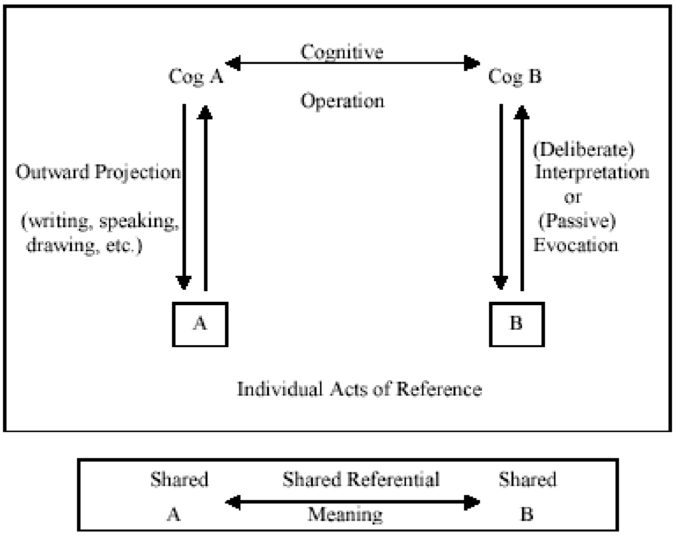


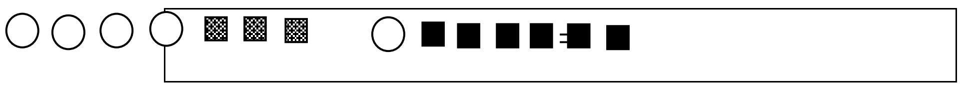
Figure 1. Kaput's referential relationship (1991, p. 60).

In this paper we explore the writing and solving of equations in one unknown using a representational metaphor of cups (that hold an unknown number of tiles) and tiles. Through our analysis of the classroom discussions, students' explanations and responses to interview tasks, along with interviews with the classroom teacher, we have come to realize that addition and (implied) multiplication operations only are the most meaningful and relevant operations when using drawn representations of cups and tiles--there is no way to represent subtraction.

CONTEXT AND METHODOLOGY

This study took place in an 8th-grade classroom in a rural middle school in the southeastern United States. The 24 students were between 13 and 14 years old and had been placed in the algebra class based on their success in 7th-grade mathematics. The students were racially, socially and economically diverse, with an approximately equal distribution of gender. All eight class lessons on a unit that focused on writing and solving algebraic equations from word problems were videotaped using two cameras, one focused on the teacher and the other on the students. Four students were interviewed twice in pairs (a pair of girls and a pair of boys) during the three weeks of the study. The classroom teacher was also interviewed twice during the three weeks. All interviews were videotaped. Excerpts from the classroom videotapes were used during both student and teacher interviews to initiate discussion of the learning that was taking place in the classroom. Excerpts from the videotapes of student interviews were also used in the teacher interviews. The second author conducted all of the interviews.

The data for this study came from two class lessons, two student interviews and two teacher interviews. With the Cups & Tiles representational model, each occurrence of the unknown in the linear equation is represented by a small circle (a cup), and the known quantities are represented by small squares (tiles). Each tile corresponds to one unit. Positive quantities are drawn in black and negative quantities (cups or tiles) in red. The unspecified rule is that the same number of tiles is hidden in each cup. The problem for the students is to solve the equation by determining how many tiles are in each cup. For example, "4 cups – 3 tiles = 1 cup + 6 tiles" would be represented as follows:



Students were instructed to solve this pictorial equation by first adding 3 black tiles to each side of the equation (so as to make zero pairs with the 3 red tiles on the left side) and then to remove 1 cup from each side. These actions would result in 3 cups being equal to 9 black tiles – thus, there must be 3 black tiles in each cup. Cups and Tiles representation gave rise to student difficulties that can be explained in terms of unit identification and coordination, as well as the need for a notational system in agreement with what the Cups and Tiles Representational metaphor models—namely, the *combining* of physical quantities.

RESULTS

The analysis of cups and tiles starts when the teacher (Ms. Jennings[1]) invites students to draw pictures of simple equations, that is to say, equations involving positive integers and plus sign only. Ms. Jennings starts the class lesson on 11/01/04 with the following introduction:

Protocol I: Introducing cups and tiles to students (From classroom video on 11/01/04)

Ms. Jennings: We're gonna draw pictures of equations… I want you to write this equation on your paper: [She writes 2c+1=7 on the white board] Now, before you tell me that I already know what the answer is, and that's fine if you already know the answer, the point is, you need some kind of a strategy to solve equations, so that you can solve much harder ones. So if you already know the answer, please don't tell me, I trust that you do know the answer. We're gonna draw the picture of what this looks like. We're gonna use what we call cups and tiles. You know, like cups [she shows the class the plastic white cup on her desk] and what goes in the cups. The cup is what you don't know and the tiles are what you count; you know, like the algebra tiles we've done on the overhead. Those are tiles, just like that... OK, so, what am I gonna have on this side [meaning the left hand side of the equation 2c+1=7] if I am drawing it?

Gary: Oh, I got it!

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[1] All names are pseudonyms.

[Ms. Jennings then draws two cups on the left, and then another student jumps in]

Cliff: Plus one tile! [meaning that we draw one tile on the left hand side]

[Ms. Jennings then draws seven tiles on the right hand side to balance the equation.]

Ms. Jennings: Now, remember. I want my equation to balance. Everything on the left is everything on the right. You've got to trust that it is. Everything on the left is everything on the right, just in a different form.

This warm-up example is a straightforward one for introducing the cups and tiles metaphor in that it deals with positive quantities only. Moreover, it is a very simple equation with a positive integer solution [some students already knew that the answer would be 3] and this simplicity could help students connect what they already know about solving equations to the situation represented with cups and tiles. There is, however something more that needs to be emphasized about the cups holding tiles, i.e., that all the cups must hold the same number of tiles, as the following protocol indicates:

Protocol II: Solving 2c+1=7 with Cups and Tiles (From classroom video on 11/01/04)

Ms. Jennings: What do I have on both sides that are the same [meaning tiles]?

Several students together: They are all tiles!

Ms. Jennings: Tiles on both sides? How many tiles on the left?

[Different answers come from various students, e.g., 1, 6, 3]

Ms. Jennings: How many tiles on the right?

Several students together: 7

Ms. Jennings: OK, well, I am gonna take one tile off both sides. If I take one off both sides, are they still equal?

Several students together: Yes

Ms. Jennings: Am I doing the same thing to both sides? [students affirm] OK this is gone [crosses one tile off left hand side] and this is gone [crosses one tile off the right hand side] I took them both off… Now my equation looks a little bit different, but it's still equal; I got 2 cups [meaning on the left] and six tiles [meaning on the right]. How many tiles do you think will fit in each cup?

Several students: Three

Ms. Jennings: And how do you know that?

Brent: 3 times 2… 6 [inaudible] and three times 2c is 6 [this is interesting because Brent includes 2c, not just 2 in his multiplication]

Ms. Jennings: Well, that's true, but how will I know three tiles will go in each cup?

Brent: Just put them there, in two lines of three.

Ms. Jennings: In two lines of three. O. K. I can put all this [circling the three top tiles in figure 2] in one [draws the top arrow from the top three tiles to the first cup] and all this [draws an oval around the bottom three tiles in figure 2] in the other [draws an arrow from the bottom three tiles to the second cup - see figure 2]

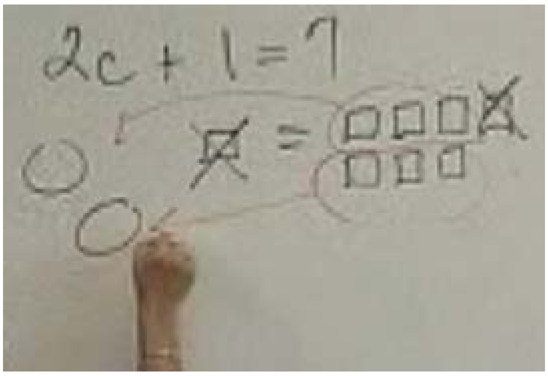

Figure 2. Ms. Jennings using drawings of cups and tiles to solve an equation.

Cliff: Six divide by two

Ms. Jennings: Six divide by two? I don't have six?

Cliff: Six tiles divided by two cups.

Ms. Jennings: Oh, six tiles divided by those two?

Cliff: Yes.

Ms. Jennings: OK. So that means my c is equal to...

Cathy: 3

[Ms. Jennings then writes c=3 on the white board underneath her drawing and asks her students to substitute this value back in the original equation to check that it works.]

Any representation is prone to yield some sense of dealing with different units. In fact, the cups and tiles representation necessitates a unit coordination task (Authors, 2008). There are two different units to be coordinated: First, each cup is to be filled with the same number of tiles; therefore, there is the same number of tiles per cup. In fact, in the example above (2c+1=7) c must stand for the intensive unit number of tiles per cup. There seems to arise a paradoxical situation here, as to what then the number 2 in front of c stands for. If it were just a scalar number, then the multiplication of 2 by c would not give a different unit, therefore, the unit for 2c would still be number of tiles per cup. However, 1, which is added to 2c, itself represents a "tile"; and has a different unit, number of tiles, therefore, we cannot add it to 2c (unit conservation, see Authors, 2008). We need to redefine "2" as "2 cups" in order to resolve the contradiction. Hence there is an implicit unit, number of cups, hidden in "2". In this way, 2c really means (2 cups) times (c tiles per cup) and the resulting unit will be just "tiles" as desired. The foregoing analysis is not obvious to most people and is certainly not made explicit in the teacher or student materials. Cliff's comment: "Six tiles divided by two cups" in the above protocol, however, does suggest that the representation generates an intensive unit as the answer for c.

To continue with our analysis of the cups and tiles data, we jump to a problem presented later on in the lesson, where Ms. Jennings draws 3 black circles and 2 red squares on the left, and 1 black circle and 3 red squares on the right. [Note: black tiles or cups represent positive quantities whereas red tiles or cups represent negative quantities]. There is an equality sign in the middle that separates the shapes (see Figure 3). The task is then to write an equation for this representation.

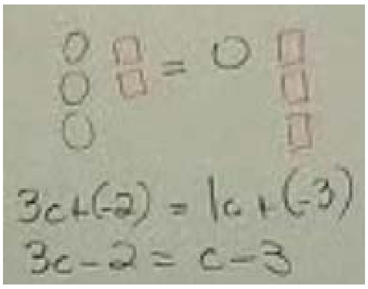

Figure 3. Showing 3 black cups and 2 red tiles on the left and 1 black cup and 3 red tiles on the right of the equal sign. The algebraic equations were added later.

The discussion between Gary and Ms. Jennings is interesting:

Protocol III: Converting cups and tiles picture to an equation (From 11/01/04)

Gary: I think I got it. Aren't reds negatives?

Ms. Jennings: That's a really good guess.

Gary: I think they are... Would it be "three c plus negative two equals one c plus negative three"? (Ms. Jennings writes 3c + (-2) = 1c + (-3) on the board-see Figure 3 above.)

Gary's interpretation of the figurative equation is compatible with the cups and tiles picture. First of all, he prefers 3c+(-2) instead of 3c-2, which, in our opinion shows that this student believes that the only operation that's going on, if any, must be addition because he understands that the contents of the cups and the tiles are to be combined. Finally, Gary's suggestion 1c+(-3) is interesting in that he emphasizes the "1" in front of c, indicating an implied multiplication. In our opinion, this is consistent with his unit coordination that there must be a counting number that he is using to count the cups, and c left alone, without the number "1" in front of it would not be enough to give the full explanation of what is going on with the pictures. That "1" therefore, could represent the unit "number of cups," and c could have the unit of "number of tiles per each cup." These are our conjectures at this point in the lesson.

Protocol IV: Teacher's reaction to Gary's proposed equation (from 11/01/04)

Ms. Jennings: Gary, I am liking this except one thing. It's a little "symbolly" to me. Could we simplify this any? I think it has some symbols in it we don't need.

Gary: Umm, can you put "minus 2" instead of "plus negative 2?"

Ms. Jennings: I like that a whole lot better.

Gary: OK. Three c minus two equals one c minus three. (Ms. Jennings writes 3c-2=c-3 on the board-see Figure 3 above.)

Ms. Jennings: Can I make it c instead of 1c?

Gary: Err, yeah.

Ms. Jennings: Simple! It's all about making it simple.

The teacher's intervention is potentially significant because the resulting problem solving procedure is no longer modelled by the cups and tiles. When all the cups and tiles, whether red or black, are present in the drawn representation, it would be hard to suggest a subtraction operation (see Figure 3 above). Therefore, if the student wants to follow the simplified version 3c-2=c-3, s/he is no longer using the initial model represented by the black-red cups and tiles combination: The problem lost its original appearance and became something else. In fact, two days later, after watching this classroom video, Ms. Jennings admitted that the original form of the equation suggested by Gary would work better.

The classroom lesson continued with the drawn representation, and both equation forms written on the whiteboard, as in Figure 3, above. Ms. Jennings asked the students what to do next. Interview student, Ben, suggested removing one black cup from each side and 2 red tiles from each side, which Ms. Jennings did by crossing out one cup and two tiles on each side of the drawn equation on the whiteboard. The simplified equation (3c-2=c-3) was further simplified to 2c=-1 but another student, Cathy had some doubts about this, as the following protocol demonstrates:

Protocol V: Cathy's doubts about 2c=-1 (from classroom video on 11/01/04)

Cathy: I don't understand how you're adding the 2c with the negative 1.

Ms. Jennings: How many c's are left?

Cathy: Two

Ms. Jennings: How many tiles are left?

Cathy: One

Ms. Jennings: Well, that's where they came from...

Cathy: Okay. I guess.

Ms. Jennings: Cathy, why does that not make sense? I am just looking at what's left. Red is negative...

Cathy: I know that.

Ms. Jennings: Okay...

Cathy: But they are two different things like. How can they equal each other though?

Cathy's, as well as other students' dissatisfaction with Ms. Jennings' explanation and their inability to see how 2 black cups could equal a red tile can be explained by the missing concept of unitizing variables. On the one hand, Ms. Jennings and participating students agreed that they could add or subtract only like terms. On the other hand, the equation 2c=-1, admits that there is something missing with this configuration that troubles students' understanding. We claim that for all the numbers and the letters in the original equation: 3c+(-2) = 1c+(-3), and hence naturally, in the last equation: 2c = -1, there is a need to designate a quantitative unit for each variable (number of cups and number of tiles per cup) in order to conserve units. In Protocol V above, the student appears to think of "2c" as 2 cups, and of "-1" as a red tile. The equality of these different representational objects does not make sense because the initial construction of units is missing. The "2" really has a unit, "number of cups." The "c" then must correspond to the intensive unit “number of tiles per cup,” rather than “number of tiles” (quantitative unit coordination). Emphasizing all these is necessary here. It naturally follows, then, that the product “2c” really has a unit, and that unit must be just “number of tiles”. It then would make perfect sense since we have -1 on the right hand side, and that corresponds to a red tile, having the same unit, namely “number of tiles” (quantitative unit conservation).

CONCLUSIONS

Our main conclusion is about the existence of a disconnect between mental operations and physical operations (Kaput, 1991, p. 57) that was the case for both the classroom teacher and many students. The difficulty that Cathy expressed in making sense of the result “2c=-1” in Protocol V suggests a cognitive dissonance in the referential relationship between Cog $_A$ and Cog $_B$ in Figure 1. Here Cog $_B$ is Cathy’s interpretation of “2 cups and one red tile” as physical quantities and Cog $_A$ is Cathy’s interpretation of “2c=-1” as a statement of equality between those two quantities. Gary, on the other hand, appeared to be successful both in inducing a meaningful algebraic notation of his own (Cog $_A$) and in making sense of the physical objects with reference to quantitative unit coordination and conservation constructs (Cog $_B$). Gary’s family of meanings is a system consisting of black and red cups and tiles as quantities existing on their own (elements), black and red cups and tiles as quantities existing in relation to each other (relationships among elements), the addition and the implied multiplication operations (operations that describe how the elements interact), and quantitative unit coordination and conservation constructs (patterns or rules that apply to the preceding relationships and operations). We conclude that quantitative unit coordination and conservation are necessary constructs for overcoming the cognitive disconnect between the two representations A and B.

References

Howden, H. (1986). The role of manipulatives in learning mathematics. *Insights into Open Education, 19,* 1-11.

Kaput, J. (1991). Notations and representations as mediators of constructive processes. In E. von Glaserfeld (Ed.), *Radical Constructivism in Mathematics Education* (pp. 53-74). The Netherlands: Kluwer Academic Publishers.

NCTM (2000). *Principles and Standards for School Mathematics.* Reston, VA.

Olive, J. & Vomvoridi, E. (2006). Making sense of instruction on fractions when a student lacks necessary fractional schemes: The case of Tim. *Journal of Mathematical Behavior, 25,* 18-45.

Schwartz, J.L. (1988). Intensive quantity and referent transforming arithmetic operations. In J. Hiebert & M. Behr (Eds.), *Research Agenda for Mathematics Education: Number Concepts and Operations in the Middle Grades* (pp. 41-52). Reston, VA: NCTM.

Thompson, P. W. (1993). Quantitative reasoning, complexity, and additive structures. *Educational Studies in Mathematics*, *25,* 165-208.

Uttal, D. H., Scudder, K. V., & DeLoache, J. S. (1997). Manipulatives as symbols: A new perspective on the use of concrete objects to teach mathematics. *Journal of Applied Developmental Psychology, 18,* 37-54.

USING VIDEO-CASE AND ON-LINE DISCUSSION TO LEARN TO "NOTICE" MATHEMATICS TEACHING

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This research report presents part of the findings of a research project whose goal was to characterize how prospective secondary mathematics teachers learn to notice mathematics teaching through the analysis of video-cases and the participation in online discussions. In this context, we understand to learn to "notice" when prospective secondary mathematics teacher link empirical evidence to theoretical information as a process of identifying relevant aspects in mathematics teaching and interpreting them. The findings show that the specific structural aspects of a web-learning environment might explain some relationships between the different topics in on-line discussions and the characteristics of learning to notice mathematics teaching.

In this research project we assume that "notice" teaching mathematics can be learned (Lin, 2005; Mousley & Sullivan, 1996; Van Es & Sherin, 2002; Sullivan & Mousley, 1996;) understood as linking the events in a mathematics lesson with theoretical ideas originating in the didactics of mathematics as a process of identifying and interpreting different aspects of a mathematics lesson (Morris, 2006; Lin, 2005). This process of interpretation is generated by relating the particular to the general, and thus forms a starting-point for the development of professional knowledge in prospective teachers.

Nowadays, recently developed technologies can be used to support interaction among prospective teachers. Online discussions make it possible to extend the boundaries of the class and to provide opportunities for written interactions with peers and expanded discussion spaces by allowing students to reflect and to develop skills that facilitate learning from practice (Derry et al., 2000). In these social interaction spaces, questions are generated on the cognitive effects of interactions that involve explanation and justification, in particular the question of how the different modes of participation operate to mediate meanings about mathematics teaching (Llinares & Oliveros, in preparation). Here, the activity of analysing a video-clip and partipating in virtual debates are therefore semiotically directed and enables us to analyse the "products generated" by the prospective teachers as particular examples of knowledge-building; on the other hand, as the prospective secondary mathematics teachers (PSMT) are able to integrate what they consider to be relevant information in the analysis of mathematics teaching, we can observe how they construct this knowledge (Wells, 2002).

From these two viewpoints (analysing the mathematics teaching throught video-clips and the participation in virtual debates), we designed several virtual learning environments for prospective secondary mathematics teachers during their final year of their mathematics degree. The goal of this research was to characterize how prospective secondary mathematics teachers conceptualize mathematics teaching through analysing video-cases and participating in online discussions.

DESIGN OF INTERACTIVE LEARNING ENVIRONMENTS

For the last four years, we have been carrying out a research project using a design experiment approach (Cobb et al., 2003) about how prospective mathematics teachers endow mathematics teaching and learning with meaning through analysis of video-cases of mathematics lessons and through participating in online discussions (Llinares & Valls, 2007).

The multimedia learning environments we designed included the following: a video-clips of part of a mathematics lesson, a virtual debate, theoretical informative documents relating to the teaching of mathematics and documents containing information on the actual classroom context, which included the teacher's lesson plan, previous activities and classroom organisation. The PSMT were expected to exchange views with their colleagues on the analysis and interpretation of the videoed episode, and to come to an agreement on the text of a written report which was to be prepared in groups of four or five and handed in as a final assignment.

The documents with theoretical information described critical classroom features that promote mathematical understanding (tasks, tools, norms, structuring and applying knowledge, reflection and articulation), and one characterization of mathematical competence as a multidimensional construct (conceptual understanding; development of skills; communication; posing, representing and solving problems; positive attitudes; mathematical confidence in oneself) (Fennema & Romberg, 1999).

The following two questions were offered to guide PSMT in their analysis of the video-clip (Pea, 2006):

> Q1. What features of mathematical competence are improved by Sara's (the teacher) interaction with her pupils?
>
> Q2. What aspects of teaching (the mathematical task proposed, methodology, management of the teaching process …) influence the development of different features of mathematical competence in this situation?

The video-clip shows the interaction between a teacher (Sara) and a group of pupils (14-15 years of age) while attempting to solve a problem consisting in drawing graphs to show the relationship between the quantities of water poured into jars of different shapes and with different surface levels. The teacher's role consisted in helping them in the process of drawing the graph corresponding to each of the vessels and establishing the significance of the differences between the graphs in order to lend meaning to the underlying concept of slope of a linear function.

Data used in this paper come from one of these learning environments which was operative during 2005-2006. The participants were 23 PSMT. We analysed the 109 PSMT' postings in the online-discussion in the 17 days during which the web-learning environment was activated. These PSMT had experience in face-to-face and e-learning activities before participating in this learning environment. The e-learning activities formed 40% of this subject of mathematics education. That is to say, the course has the structure of b-learning.

ANALYSIS

During the debate some of the contributions were organised as conversational chains trough dialogical interactions. A conversational chain is a set of interactions all relating to the same topic. The characterisation of these chains enabled us to identify the topics and the ways in which the PSMT interacted. The PSMT' contributions to the debate were analysed on three dimensions: participation, interaction and cognition (Schrire, 2006).

On the *participation* dimension we paid attention to who contributed and when. In the present paper we shall present the global participation features of the group and we shall not take the time factor into consideration.

Ways of interacting were analysed by considering 6 categories: *Supplies information* (SI), *Clarifies* (Cl), *Agrees* (A), *Agrees and amplifies* (A+A), *Disagrees* (D) and *Disagrees and amplifies* (D+A).

As regards the analysis of cognition we have established four distinct levels considering the content of participations, based on the sources used, the way in which the ideas were expressed in the contribution and the way in which were interrelated. We also considerated whether relationships were established between ideas from a general point of view or whether the student examined the actual mathematical content shown in the video-clip. The four levels (L) were as follows:

- *L1. Descriptive:* The PSMT responds by describing in a "natural" manner what he/she has seen, but does not make use of the theoretical ideas which might be relevant to the analysis of the situation.
- *L2. Rhetorical:* The PSMT uses the theoretical ideas contained in the documents in order to construct a response, but without establishing relationships between the ideas and the situation. The discourse may be said to lack cohesion.
- *L3. Identification and initial instrumental use of the information provided:* The PSMT identifies one or more relevant aspects of the situation and links them to one or more of the theoretical ideas, thus generating an interpretation of the situation.
- *L4. Theorising and conceptualising: relational integration:* The discourse generated shows a process of integration of different ideas to explain different aspects of mathematics teaching.

PROCEDURE

We considered the different theoretical bases of level of knowledge building, ways of participating and perspective-taking, in order to make a first draft of the category system reflecting the different analytical dimensions. The category system was revised after the researchers became familiar with the postings of PSMT in the online discussion. Next, we independently codified the different participations considering ways of participating, level of knowledge building and the perspective-taking stages. Finally, the discrepancies were discussed until a unitary evaluation was reached. We present here the findings of the first phase of analysis.

RESULTS

We identified two conversational chains whose content is related to each of the questions posed at the start of the debate.

C*hain 1* (C1): This chain deals with the meaning of the idea of mathematical competence as the interrelation between: (a) conceptual comprehension, (b) development of procedural skills, (c) communicating, explaining and arguing mathematically, (d) the capacity to formulate, represent and solve problems (strategic thought), (e) the development of positive attitudes towards mathematics, and (f) achieving mathematical confidence in oneself.

Chain 2 (C2): This chain deals with the teacher's handling of the situation in order to develop the pupils' mathematical competence. Following the initial response to the starter-question, five new areas of debate were opened up: (i) the rigorous use of language and the role the teacher should play, (ii) group work, (iii) characteristics of teacher-pupil interaction, (iv) ways in which the teacher can encourage pupils' participation, and (v) the appropriateness of the mathematical task proposed.

The number of contributions and the centre of interest of the discourse were different in the two chains. Of the 109 contributions to the debate, 16 referred to the Chain 1 and 93 referred to the Chain 2.

Our analysis of the ways in which students participated revealed that although there was less participation in Chain 1, the type of interaction was similar to that in Chain 2. 75% of the contributions in both conversational chains corresponded to a reply to someone else or a clarification of something expressed previously in an attempt to make oneself understood (see Table 1). The PSMT were in greater disagreement, however, on which aspects of the lesson seemed to promote the development of mathematical competence (33.31% in Chain 2), than in their indication of evidence of mathematical competence in the pupils (25% in Chain 1).

The cognitive levels reached in both chains also showed differences (Table 2). In Chain 1, 75% of the contributions (12 out of 16) were considered to be at level 3 (L3). Of all the contributions at this level, 9 referred explicitly to the mathematical content of the videoed lesson. The PSMT concentrated on assessing the potential of the mathematical activity generated by the problem proposed by the teacher in the video-clips.

	SI	Interaction with others					Others	Total
		CL	A	A+A	D	D+A		
Chain 1	4	4 (25%)	3 (18.75%)	1 (6.25%)	0	4 (25%)	0	16 (100%)
	(25%)	75%						
Chain 2	19	15 (16.13%)	14 (15.1%)	8 (8.6%)	14 (15.1%)	17 (18.21%)	6	93
	(20.43%)	75%					(6.37%)	(100%)
Total	23	19	17	9	14	21	6	109

Table 1. Modes of participation in each chain

In contrast, only 39.78% (37 out of 93) of the contributions in Chain 2 were considered to be at level 3, and of those only 6 alluded directly to the mathematical content; the majority concentrated on more general aspects such as the rigorous use of mathematical language in class, features of group work, the role of the interaction between the pupils and the teacher, the nature of the task, the way in which the teacher handled the relationship between achieving the objective of the lesson and dealing with the pupils' answers, and a description of the context exemplified in the video-clips. None of the contributions was identified as being at level 4 (L4).

	L1	L2	L3	L4	Others	Total
Chain 1	3 (18.75%)	1 (6.25%)	12 (9[1]) (75%)	0	0	16 (9) (100%)
Chain 2	33 (35.48%)	19 (1) (20.43%)	37 (6) (39.78%)	0	4 (4.31%)	93 (7) (100%)
Total	36	20	49	0	4	109

Table 2.Cognitive levels in each chain

DISCUSSION

The two questions given to initiate the debate referred to learning and teaching, or more specifically to the dimensions of mathematical competence which can be enlarged by teacher-pupil interaction, and the role of the teacher in the enlargement of those dimensions. These two aspects are interrelated: What do we want the pupils to learn? How should we modify the instruction process to achieve this end?

The two chains, however, showed different characteristics. In the first place, the first chain concentrated on a single point of interest, while in the second the discourse was more destructured and there were several different points of interest. Secondly, the students participated and disagreed with each other on the characteristics of the teaching process considerably more in Chain 2 than in Chain 1. And thirdly, the contributions to

[1] The number in brackets indicates those contributions which referred explicitly to specific mathematical topic in the video-clip.

Chain 1 referring to evidence of the dimensions of the pupils' mathematical competence were of much better quality. These differences may be explained by the nature of the topics under discussion and by the type of information available to the student teachers. These two points will form the central issue of our discussion.

The online discussion topics

The questions to be debated were of different types. The first (Q1) was more conceptual and required answers based on the characterisation of the idea on mathematical competence given in the documents provided. The second question (Q2) however, could be seen as referring to the social factors involved in the teaching process and could be answered more subjectively, with answers based on personal educational experience and beliefs, or simple descriptions of what is "seen" in the video-clip. To answer question 1, therefore, the PSMT had to use the ideas contained in the documents (level 3 contributions), to a greater extent than in their answers to the second question (Q2).

When the discourse was centred on the idea of mathematical competence as revealed in the behaviour of the pupils while they were deciding how to draw the graphs and while they were interpreting the difference between the finished graphs (the idea of slope) (Q1), the online conversation was of a higher quality but the PSMT' involvement and degree of disagreement was lower. On the other hand, when the discourse was centred on the teacher's handling of the situation there was more disagreement and a poorer quality of discourse. These differences reveal how the topic under discussion determines the way in which PSMT discuss it. Furthermore, the topic of conversation also seems to determine the way in which relevant aspects of the situation are identified and linked to some theoretical idea to generate an interpretation. If we take these two aspects of the online discussion together, it seems to be clear that the PSMTs in this experiment became more easily involved in social factors related to teaching than in cognitive aspects of learning, though this greater degree of involvement was only maintained at a superficial level.

The interpretation process. The difference between aspects of teaching and the identification of mathematical competence.

The documents containing theoretical information with which the PSMTs were provided referred both to the characterisation of mathematical competence and to certain aspects of the teaching process. The information provided mentioned neither the specific mathematical content of the videoed lesson nor the specific aspects of teaching involved. The information was handled in different ways by different PSMTs, which indicated how they related it to the empirical evidence observed in the video-clip. They found it more difficult to relate the characteristics of mathematical competence to the behaviour of the pupils than to do the same with aspects of the teaching process, which they found relatively easy.

For instance, one of the documents on teaching stated that "in order to create a classroom atmosphere conducive to investigation and mutual respect, the teacher should encourage the generation of arguments by asking the pupils to clarify and justify their ideas.” Recommendations of this type helped the PSMTs to identify in the video-clip some features of the teacher’s performance which could be associated with this characterisation. This fact may have caused the PSMTs to focus their discourse more on matters relating to the teaching process (Q2). In such cases the PSMTs could simply describe what they saw and identify it with a characteristic given in the document. They could then disagree on the degree to which they thought that what the teacher was doing was, for example, relevant to the encouragement of argument-generation among the pupils. But when the information was of a more general nature, such as “one dimension of mathematical competence is conceptual comprehension of mathematical topics, by which we mean the way in which secondary-school pupils are able to link different mathematical ideas together and explain their meanings”, the PSMTs were obliged to focus their attention on the mathematical cognitive processes of the pupils while they were interacting to solve the problem, and then to interpret what they did as manifestations of mathematical competence. It appears that this kind of activity required a much greater effort on the part of the PSMTs.

This difference between interpreting the characteristic of the teaching process (Q2) and identifying manifestations of the pupils’ mathematical competence (Q1) could explain the nature of the debate generated (differences in the quality of the discourse, and differences in modes of participation). These results are similar to those obtained by Lin (2005) via videoed case studies shown to student teachers, where the students tended to focus their attention on the teaching process and had difficulty in “noticing” the development of the pupils’ mathematical competence.

Our results, however, like those of Sullivan & Mousley (1996), show that the use of videoed material is a powerful tool in relating theory to practice and in enabling PSMT to develop a high cognitive capacity in their analysis of teaching. At the same time, our research reveals that the design of the learning environments may facilitate to a greater or lesser degree the construction of knowledge about the teaching of mathematics. The fact that contributions to the debate refer separately to learning and teaching without explicitly interrelating them reveals that the PSMTs approached the analysis and interpretation of teaching through an analytical thought-process which examined each aspect of the situation in turn, instead of looking at it globally and holistically. This fact might be explained by the actual structure of the online debate and the presentation of two initial questions, though the students were never asked to answer them separately. The directions in which the debate developed could have been corrected by a chairperson (tutor), who could have suggested to the students that they try to establish more connections between different ideas and thus construct cognitive knowledge of a higher quality.

Endnote

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References

Cobb, P. Confrey, J., diSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in *Educational Research,* 32(1), 9-13.

Derry, Sh.J., Gance, S., Gance, L.L., & Schaleger, M. (2000). Toward assessment of knowledge-building practices in technology-Mediated work group interactions. In Lajoie, E. (ed.) *Computers as Cgnitive tools. No More Walls. Volume 2*. (pp. 29-68). Mahwah, NJ: Lawrence Erlbaum Associates.

Fennema, E. & Romberg, T. (eds.) (1999). Mathematics classrooms that promote understanding. Mahwah, NJ: Lawrence Erlbaum Associates, Pbs.

Lin, P. (2005). Using Research-Based Video-Cases to Help Pre-service Primary Teachers Conceptualize a Contemporary View of Mathematics Teaching, *International Journal of Science and Mathematics Education*, 3, 351-377

Llinares, S. & Olivero, F. (in preparation). Interactions, discourse and communities in prospective mathematics teachers' learning. The role of new information and communication tools. In T. Wood (Series Ed.) & K. Krainer (Vol. Ed), *International Handbook of Mathematics Teacher Education: Vol. 3. Participants in Mathematics Teacher Education: Individuals, Teams, and Networks*. Rotterdam: Sense Publishers.

Llinares, S. & Valls, J. (2007). The building of preservice primary teachers' knowledge of mathematics teaching: interaction and online video case studies. *Instructional Science*, DOI: 10.1007/s11251-007-9043-4.

Morris, A. (2006). Assessing pre-service teachers' skills for analyzing teaching. *Journal of Mathematics Teacher Education*, 9, 471-505.

Mousley, J. & Sullivan, P. (1996). *Learning about teaching. An interactive tutorial program to facilitate the study of the teaching*. ASMT: Adelaida; CSMSEE: Deakin University; MTLC; ACU.

Pea, R. (2006). Video-as-data and digital video manipulation techniques for transforming learning science research, education and other cultural practices. In J. Weiss, J. Nolan, Hunsinger, J. & P. Trifonas (Eds.) *The International Handbook of Virtual Learning Environments,* Vol. 2. (pp. 1321-1394). Netherland: Springer.

Schrire, S. (2006). Knowledge building in asynchronous discussion groups: Going beyond quantitative analysis. *Computers and Education*, 46, 49-70.

Sullivan, P. & Mousley, J. (1996). Learning about teaching: the potential of specific mathematics teaching examples, presented on interactive multimedia. En: L. Puig y A. Gutiérrez (Eds.). *Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, 283-290, Universidad de Valencia.

Van Es, E. & Sherin, M.G. (2002). Learning to notice: scaffolding new teachers' interpretations of classroom interactions. *Journal of Technology and Teacher Education*, 10(4), 571-596.

Wells, G. (2002*). Dialogic Inquiry. Towards a Sociocultural Practice and Theory of Education.* Cambridge: Cambridge University Press.

REVISITING UNIVERSITY STUDENTS' KNOWLEDGE THAT INVOLVES BASIC DIFFERENTIAL EQUATION QUESTIONS

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We document and analyze the extent to which university students utilize diverse representations and mathematical processes to interpret and respond to a set of questions that involves fundamental concepts associated with the study of differential equations. Results indicate that the students' idea to solve a differential equation is reduced to the application of proper solution methods to a certain types of forms or equation differential expressions. They failed to remember the solutions methods and lacked clear comprehension of the concept of solution. Results suggest that instructional activities should promote the students' use of several representation systems in which they can reflect on the various aspects associated with the concept itself, the solution methods, the procedures, and the corresponding meaning and connections among those representations.

INTRODUCTION

Traditionally the teaching of Differential Equations has been undertaken with an algorithmic focus in the sense that some guidelines have usually been taught so that these are then classified as certain types and solved, sometimes using certain techniques that lead to the solution either explicitly or implicitly. However, what do students deem to be relevant when using these techniques or algorithms to solve problems? Will the students remember them and will they be able to use them when needed? Do they recognize that that it is not possible to give an explicit or implicit expression for the solutions for most differential equations? How do they behave when faced with a differential equation which cannot be solved by using the methods they have studied?

In this study we document and analyze students' types of behaviour when attempting to solve some differential equation problems which are presented from a different perspective than they normally appear during the process of instruction. We describe the processes associated when solving these tasks, as well as the strategies for solving problems that students use when carrying out the activities set.

Our research responds four main questions. Do students use knowledge gathered during their previous studies (meaning of the derivative, function concept, graphics representations, etc.) to answer questions on differential equations that do not necessarily require methods belonging to this field? What use do they make of the various systems of representation? What influence does the wording of the question have on students' mode of approaching it? And, finally, what types of strategies and representations do students use when faced with contextualized problems?

CONCEPTUAL FRAMEWORK

The learning or development of mathematical knowledge is a process that demands continual reflection on the part of students to help them represent and examine mathematical concepts from different points of view and lead them to construct a network of relations and meanings associated with this concept (Camacho et al, in press). Development of this process of construction depends directly on the systems of representation used and the coordination between these (Duval, 1993). On the other hand, the learning of a mathematical concept is directly related to the activities undertaken to solve problems (Santos, 2007). Problem solving, then, should form a major part of teaching. In this context, the student formulates questions, puts forward conjectures, seeks different ways of validating them and communicates his or her answers or results in a suitable language. Thurston (1994, quoted in Camacho et. al, in press) stated that the comprehension of the concept of derivative involves thinking of diverse ways to define, operate, represent, and to interpret its meaning:

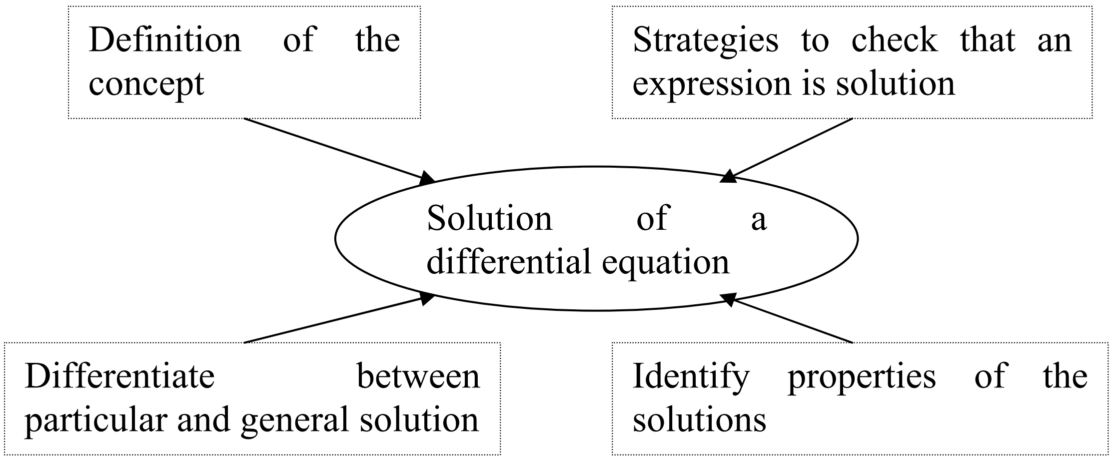
- Infinitesimal: the ratio of the infinitesimal change in the value of a function to the infinitesimal change in a function.
- Symbolic: the derivative of x^n is nx^{n-1}, the derivative of $Sin(x)$ is $Cos(x)$, the derivative of $f \circ g$ is $f' \circ g * g'$, etc.
- Logical: $f'(x) = d$ if and only if for every ε there is a δ such that when $0 < |\Delta x| < \delta$, $\left| \frac{f(x+\Delta x) - f(x)}{\Delta x} - d \right| < \varepsilon$
- Geometric: the derivative is the slope of a line tangent to the graph of the function, if the graph has a tangent.
- Rate: the instantaneous speed of $f(t)$ when t is time.
- Approximation: The derivative of a function is the best linear approximation to the function near a point.
- Microscopic: The derivative of a function is the limit of what you get by looking at it under a microscope of higher and higher power.

Thus comprehending the concept of derivative or those that involve the study of differential equations requires or demands that students relate and transit, in terms of meaning, through the ideas and representations associated with each way of thinking about those concepts.

Based on these premises, we can see that the understanding of a mathematical concept passes through various stages or phases, among which there is the phase where the student understands the definition of the concept itself, the phase where this concept is used algorithmically, and the phase where the concept is recognized as an instrument to solve problems. Along the route taken for constructing mathematical knowledge it is important to identify the previous knowledge and forms of thinking students use when attempting to understand mathematical ideas and solve problems.

The concept of solving a differential equation and the direction field associated with it are some of the meanings that are closely connected with the concept of

Differential Equation. The graphic nature of the direction field (geometric meaning in the sense attributed by Thurston) and the traditionally algebraic focus from which differential equations are taught (symbolic meaning in the sense attributed by Thurston) suggest that we need to analyze the balance or complementariness of the relations between the various systems of representation. The understanding of the concept of the solution of an ODE develops as the definition of the concept joins up with other elements, among which we can find those that can be seen in the following diagram. Also, the concept of direction field associated with an ODE includes two related phases that are different from the cognitive point of view: interpretation and representation.



METHODOLOGY

A total of 21 students took part in this study, ten of whom were studying for a Mathematics degree and eleven were studying Physics (University of La Laguna). The main difference between the two groups is the instruction they receive: the Mathematics students receive a more theoretical approach while the Physics students a more practical instruction. This difference is due in part to the nature of the subjects the students take. The Mathematics students were studying a fifth semester subject devoted solely to Differential Equations, while Physics students cover the material in their second semester and the subject they take covers Differential Equations as well as other Calculus concepts.

Analysis of the strategies used by students to solve the activities set is made based on a questionnaire designed specifically for this end. The questionnaire is made up of eleven problems which can be solved using several methods or for which something more than the application of rules, formulas or algorithms is required (Santos, 2007). Selection of these problems was made taking into account the results given in the literature review and some were chosen from textbooks used in Differential Equations courses. Other tasks were specifically designed in order to respond to the main questions of our study. The questionnaire includes activities where students need to use properly algebraic and graphic systems of representation, both separately and together, and it was necessary for students to use their knowledge of solving of problems set in a real context. Tasks were classified into four types:

Type 1: These questions require knowledge of the concept of solution. This type of question is used to check whether an algebraic expression is a particular or general solution to a differential equation (Q3, Q4 and Q11) and to analyze some general properties of the solutions in function of the terms of this expression (Q5). Two examples are:

> **Q3**: Say whether the following statements are true or false and give reasons for your answer: a) The function $y = e^{\int e^{t^2} dt}$ is a solution for the differential equation $\frac{dy}{dx} = 4e^{t^2} y$. b) The function $y = f(x)$ which allow $-x^3 + 3y - y^3 = C$ are solutions for the differential equation $\frac{dy}{dx} = \frac{x^2}{1-y^2}$.
>
> **Q5**: Say whether the following statement is true or false and give reasons for your answer: "Take the first order differential equation $y'(x) = f(x, y)$. If the function $f(x, y)$ is defined as $\mathbf{R}^2$, solutions for the differential equation will also be defined as $\mathbf{R}^2$".

Type 2: Solution of this type of question can be achieved through use of logical reasoning (Q1) or using simple algebraic methods (Q2). This type of question implies graphic representation of elemental functions, but do not involve either the construction or interpretation of the direction field or the interpretation of data from or towards a mathematical context. An example:

> **Q1**: Represent graphically some solutions for the following equations a) $\frac{dy}{dx} = 0$; $x \in [0,2]$ b) $\frac{dy}{dx} = \cos x$.

Type 3: Questions where solving requires representation and/or interpretation of the direction field of a differential equation (Q6, Q8 and Q10). An example:

> **Q10**: Draw the direction field for the differential equations $\frac{dy}{dx} = 1$ and, based on this, solve the following initial value problem $\begin{cases} \frac{dy}{dx} = 1 \\ y(-2) = 4 \end{cases}$.

Type 4: Activities where it is necessary to interpret information supplied in algebraic or graphic terms, in a real context, or vice versa (Q7 and Q9). An example:

Q7: we know that the population of a city grows constantly over time, substantiating the differential equation $\frac{dP}{dt} = K$, $K > 0$. If the population has doubled in 3 years, and in 5 years it has reached a total of 40,000 inhabitants, how many people lived in the city at the beginning of the five-year period?

DISCUSSION AND RESULTS

We mainly focus here on three of the questions taken from the questionnaire in order to analyze the processes of solution followed by the students when the tasks are framed in different types of contexts. To this end, we analyze the answers from students for Questions Q1a (Type 2), Q7 (Type 4) and Q10 (Type 3) from the questionnaire. We represent the Mathematics students as MSj (j=1 … 10) and the Physics students as PSj (j=1 … 12). We eliminated from our analysis the student PS10 as this student did not manage to answer any of the questions in the questionnaire.

Those students who solved tasks Q1a and Q7 but not Q10 (MS3, PS7, PS10, and PS 12) share the common characteristic of having shown that they know some algebraic methods for solving differential equations but that they fail to represent any of the direction fields asked for in the questionnaire and also fail to make mathematical interpretations. Another characteristic that can be underlined regarding these four students is that, while they indeed attempted to solve both questions Q1a and question Q7, they did so without using the same form of reasoning. While they all solved the equation of the problem Q7, taking it as one of separate variables, only MS3 used this solution strategy for Q1a. Moreover, of these four students, only PS12 correctly solved the differential equation. The other three students omitted the integration constant when applying the method of separate variables, an error they did not make when solving the equations in problems Q1 and Q2.

Student PS4, who is the only one who deals with the question Q10 and not Q7, answers hardly any of the questions set in the questionnaire; she only answers the problems Q1 and Q10 (Figures 1 and 2). In spite of this, she is able to find a particular solution to each of the differential equations set. So, in the question Q10, although she was asked to solve the problem based on the direction field, this student expressed the solution for the initial value problem using logical reasoning in order to find that it would be a linear function, and guesswork in order to find the constant that was missing. This is an example of how intuitive the ordinary differential equation is.

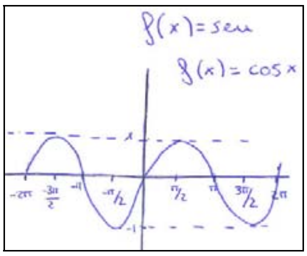


Figure 1. Answer from PS4 to Q1.

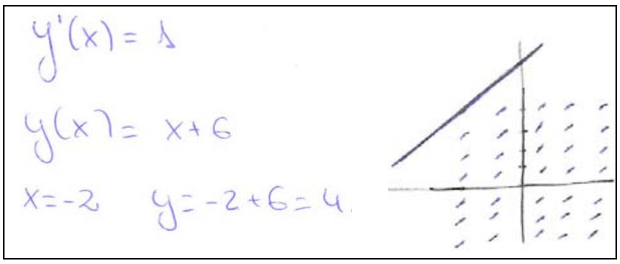


Figure 2: Answer from PS4 to Q10.

We now analyze the answers from those students who attempted to answer the three questions that we are studying. Given the characteristics of the questions text, Q10 induces use of the direction field associated with a differential equation in order to find a solution, which clearly distinguishes it from Q1a and Q7. We might think, then, that this problem is going to be solved by students in a way different from that used in the other two questions, due precisely to the use of the graphic representation system. However, we find that of the 10 students under study only two, MS4 and PS9, set about this task using the direction field. The other students depend on an algebraic solution of the equation. Regarding the strategies used by students to solve the problems Q1a, Q7 and Q10, we find the following types of behaviour. Students PS2 and PS11 demonstrate throughout the questionnaire that they know some methods for solving differential equations; however, they do not take those methods into account when solving task Q7. PS11 uses a rule of three to solve it, while PS2 does not attempt to solve the equation on finding that he cannot translate the data in the problem into mathematical language (Figure 3). This is the only differential equation that this student does not solve, which shows us that the student fails to relate the equation that appears in the text of Q7 with those that the student has solved in the rest of the activities.

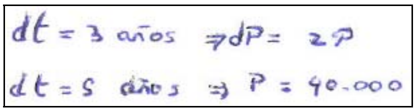


Figure 3. Answer from PS2 to Q7.

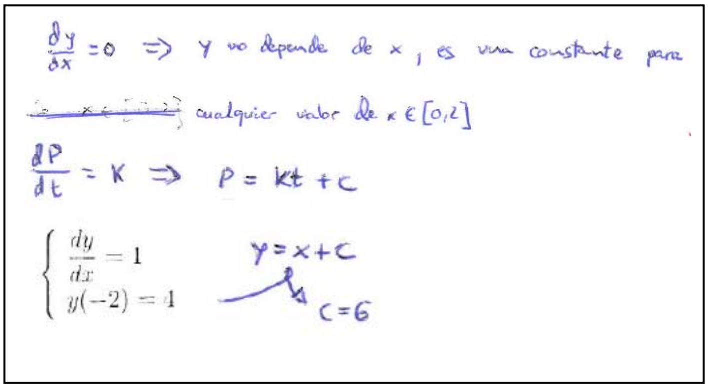


Figure 4. Solution from PS6 to various differential equations.

Student MS1 and PS6, on the other hand, find solutions to the equations in the three problems by either using knowledge they have acquired prior to their study of Differential Equations or directly, without our being able to appreciate use of specific methods for the solving differential equations.

Four students (MS7, MS10, MS9 and PS3), once they have solved the equation $y'(x)=0$ using concepts and procedures they had learnt prior to their study of Differential Equations or by expressing the solution of the equation algebraically, without explicitly showing the use of any specific method of solution, then use the method for solving separate variable equations when they solve tasks Q7 and Q10. The context of the question does not influence them when they make certain mistakes. In all their answers they avoid the integration constant (MS9 and PS3). However, MS10 only makes this mistake in the contextualized problem Q7, while student MS7 finds herself in difficulties by not being able to interpret the information on population supplied in the problem text in mathematical terms, which means she is unable to solve Q7 correctly.

Finally, MS4 and PS9 use different strategies when solving these three activities, limiting themselves to the stipulations of the question texts. PS9, meanwhile, makes some modification when solving the equations in Q1a and Q7, using definite integrals for the latter but using indefinite integrals for the rest of the differential equations solved (Figure 5).

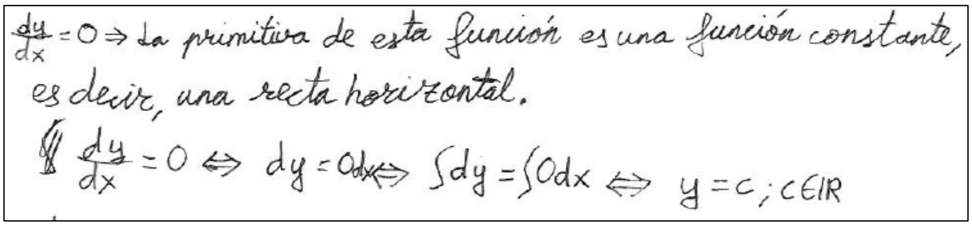


Figure 5. Answer from PS9 to Q1a.

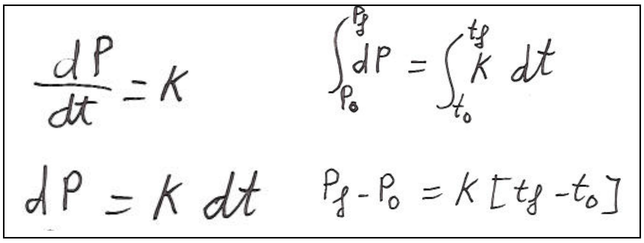


Figure 6. Part of answer from PS9 to Q7.

CLOSING REMARKS

Students' answers to the various questions set show once more that they prefer to use the algebraic rather than the graphic and verbal register. This might be a result of the instruction they have received where algebraic aspects have been predominant, graphic studies have only been superficially covered and where there is no incentive to find a possible verbal solution (González-Martín, Camacho, 2004). To this should be added the students' deficiencies when undertaking activities associated with the solving of problems, such as analysis of the problem, decision making and the evaluation of the solution (Santos, 2007).

Many of the students conceive of the concept of Differential Equation as an isolated mathematical entity unconnected to other notions they know. For the students, solving a differential equation is merely a matter of finding an implicit or explicit algebraic expression of the solution, so that for them the relevant information supplied to them in a differential equation is that this information can lead them to apply some method in order to give them the solution. Also, in general, it has been shown that after a certain amount of time the student cannot remember the methods and that they do not have an understanding of the concept of solution that allows them to solve problems (for example, Q7) without using the algorithmic methods studied.

We believe that introducing concepts based on others already known can allow the student to make connections between the different themes or questions studied. Also, this will permit a broader vision of the concept of Differential Equation, and not limit this to the use of certain "tricks" which are easily forgettable and unfruitful. Accepting the system of graphic representation as legitimate in the process of solution can broaden understanding of the concept. Teaching based on the solving of problems wherein students are shown the need to take into account and work with different registers of representation will motivate them to tackle questions related to mathematics in general, and to differential equations in particular, in a more open-minded and complete form, greatly increasing their chances of success when it comes to solving problems.

Endnote

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References

Camacho, M., Depool, R., & Santos, M. (in press) *Students' Use of DERIVE Software in Comprehending and Making Sense Of Definite Integral and Area Concepts*. Research in Collegiate Mathematics Education.

Duval, R. (1993). *Registres de Représentation Sémiotique et Fonctionnement Cognitif de la Penseé*, Annales de Didactique et de Sciences Cognitives, 5, IREM de Strasbourg 37-65.

González-Martín, A. S. & Camacho, M. (2004). *Legitimization of the Graphic Register in Problem Solving at the Undergraduate Level. The case of the improper integral.* Proceedings of PME28 (2), 479-486.

Santos, M. (2007). *La Resolución de Problemas Matemáticos. Fundamentos Cognitivos*, Asociación Nacional de Profesores de Matemáticas, Editorial Trillas.

Thurston, W. P. (1994). *On Proof and Progress in Mathematics*. Bulletin of the American Mathematical Society, 30(2), 161-177.

CONCEPTIONS AND MISCONCEPTIONS OF ELEMENTARY PRESERVICE TEACHERS IN PROPORTIONAL REASONING

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Michael Gilbert — University of Hawai'i

Keith Adolphson — Eastern Washington University

As Elementary Preservice Teachers (EPSTs) worked through a proportional reasoning problem that was posed in three different ways, we wondered: What range of mathematical approaches might EPSTs use? What connections between representations might they make, and how might their conceptual understanding be affected by the different versions of the problem? In this paper, after describing the three versions and how we used them in class, we'll profile some of the different ways of mathematical thinking that EPSTs demonstrated. Of particular interest were some of the difficulties that our subjects had in reasoning about the task, and the attendant misconceptions which were revealed. This research points to a need for university programs to ensure a firm understanding of proportional reasoning for EPSTs.

INTRODUCTION

The purpose of this paper is to report on research aimed at discerning the conceptions held by Elementary Preservice Teachers (EPSTs) in a proportional reasoning context. Just as proportional reasoning is of paramount importance in the K-12 school curriculum (Lesh, Post, & Behr, 1988; Behr, Harel, Post, & Lesh, 1992), so too is it critical that those in university teacher training programs have a deep understanding of this concept. As teacher educators, we believe that deep mathematical understanding includes the higher level skills of comparing, contrasting, and making connections between different computational approaches.

Thus, we were motivated to pose the following central research question concerning our students (the EPSTs): What conceptions of proportional reasoning do students exhibit when asked to explore a task that is posed in three different ways? We wanted to investigate the range of mathematical approaches that students might use, what connections they might make between the parallel problems, and how their conceptual understanding might be affected. We called the set of all three versions of the task *Ratio Triplets*. The essential feature of the activity involved determining which of two packages of ice cream was the better buy: A 64-ounce container selling for $6.79 or a 48-ounce container selling for $4.69.

While *Ratio Triplets* centers on a fairly common type of proportional reasoning task, we found that using different versions of the task afforded a good opportunity for our students to consider the meaning of ratios in multiple ways. In this paper, after considering the theoretical background relevant to the current research, we'll describe the three versions of the task and how we used them in class. Then, we'll profile some results of the different mathematical thinking that students demonstrated by using examples from each of the three versions, and finally we'll offer a brief discussion of these results.

THEORETICAL BACKGROUND

While relatively few studies have been aimed specifically at the proportional understanding of preservice teachers, there is a substantial body of prior research that exists on children's' understanding of ratio and proportion (see Behr, et. al., 1992, for a good overview). Several themes emerge from this corpus of research that are germane to the current study, since the *Ratio Triplets* task fundamentally involves a comparison of two rates. One theme is the importance of unit recognition, which is included with partitioning and equivalence as part of the "basic thinking tools for understanding rational numbers" (Behr, Lesh, Post, & Silver, 1983, p. 109). As Lamon (1993) notes, of particular importance to reasoning proportionally is "the ability to construct a reference unit or a unit whole, and then to reinterpret a situation in terms of that unit" (p. 133). Lamon also describes how "the process of norming can achieve yet another level of sophistication" (p. 137), whereby an independent unit may be selected as a basis for comparison. For example, while two natural units of measurement present themselves in the basic *Ratio Triplets* task scenario as dollars or ounces, a pint (16 ounces) could be chosen in the norming process. A second theme is the numbers that are used in the task and how these numbers may influence the comparison strategies employed by students. Two common strategies are comparing "within" the same measurement contexts (such as the amount of ice cream in one container to the amount in the other), or "between" contexts (such as the amount of ice cream to the cost of that amount). For many children, a bias seems to exist for finding "between" relationships, although "there is some evidence that on the certain tasks the "within" relationship becomes more popular" (Hart, 1988, p. 203). Research has shown that children are more successful when the "between" ratios are integers, while any presence of non-integer ratios presents more challenges (e.g. Friedlander, Fitzgerald, & Lappan, 1984). *Ratio Triplets* involves non-integer ratios. Finally, the basic theme of additive versus multiplicative structures is of key importance in discerning proportional reasoners. As Resnick and Singer (1993) note, "the early preference for additive solutions to proportional problems is a robust finding, replicated in several studies" (p. 123). Even with the EPSTs in our study, we found a similar reliance on additive strategies.

METHODOLOGY

Knowing how the NCTM Standards (2000) echoes calls from the literature to stress only not only the importance of using multiple representations in problem solving, but also the importance of communication, we designed *Ratio Triplets* to incorporate three versions of the same basic problem. This design choice allowed us to examine EPSTs' understanding of ratio and proportion in a way that also promoted discourse about different ways of thinking mathematically.

Version A is shown in Figure 1, and it suggests a dollars-per-ounce strategy for Mark as well as an ounces-per-dollar strategy for Alisha. The correct result for each calculation is also provided.

Mark and Alisha were sent to buy ice cream for a class party. Their favourite flavours came in a 64-ounce package for \$6.79 and a 48-ounce package for \$4.69.

► To find which is the better buy, Mark divided like this:

$$\frac{6.79}{64} = .10609375 \qquad \frac{4.69}{48} = .097708\overline{3}$$

Explain how these ratios can tell Mark which ice cream is the better buy.

► Alisha claimed she could use different ratios to solve this problem.

She divided like this:

$$\frac{64}{6.79} \approx 9.42562592 \qquad \frac{48}{4.69} \approx 10.2345418$$

Is Alisha correct? Explain your answer.

Figure 1: Version A of the task behind *Ratio Triplets.*

We wondered if students would recognize the meaning behind those calculations, and further, would they understand the interpretations of the results? Version B was identical to Version A in all but one respect: The outcome of each calculation was not provided. That is, the decimals were missing as well as the equality and approximation symbols. Since calculators were available, we wondered if students would just do a unit-rate conversion (effectively mirroring the computation results in Version A), or would they try something different? Version C had the same initial situation description, but omitted any reference to what strategies Mark or Alisha might have used. Instead, it invited any strategy for Mark and any *different* strategy for Alisha. The focus on all versions was in getting the students to provide explanations and justify their thinking.

Our subjects were 75 EPSTs in the same university in the Northwest USA who were spread out amongst three sections of a course (taught by the authors) that provided both content and methods of mathematics for teaching. The mathematics background for the subjects was largely restricted to their precollege education, as common or varied as that may have been. In the classroom setting, we broke students up into small groups: Each member within a group got the same version, with different versions going to different groups. Our instructions were for students to solve the problem individually and to write down their approaches. Then, within their groups, they were to explain their thinking to each other. We asked them to attend to the different approaches their partners might have used. Later, we regrouped the students so that there were all three versions within each group, and again asked students to share their thinking with each other. Finally, we had a whole-class discussion about their reactions and responses to *Ratio Triplets*. The small-group and whole-class discussions were videotaped, and the transcriptions of these discussions along with the written responses then analyzed to determine the different types of thinking represented.

RESULTS

Across all versions, we found a surprising diversity of explanations given by EPSTs, some which were *reasonable* and many which were *questionable*. To illustrate the types of thinking offered, sample results will first be presented in terms of what the subjects had to say about Mark (the first strategy in all versions), and then about Alisha (the second strategy).

Mark's Strategy

We first distinguished responses about Mark's strategies according to whether they were *reasonable* or *questionable* according to the types of themes that emerged from the literature on children's proportional reasoning. For example, responses that invoked a unit rate, or demonstrated a "between" or "within" comparison, or relied on multiplicative structures generally were coded as *reasonable*, with finer levels of analysis being used to further stratify responses according to types of thinking. Also, the response did need to lend support for the inference that the 48-ounce container was the better buy. Responses that were *questionable* typically exhibited some level of confusion, lack of clarity, or simply erroneous thinking. Also, support for the inference of the 64-ounce container as the better buy usually accompanied a *questionable* response. Of the 75 total subjects, 81.3% (n = 61) gave *reasonable* responses for Mark, which was encouraging. The exact breakdown of the numbers of responses by version is given in Table 1.

	Version		
	A	B	C
Reasonable	21	24	16
Questionable	3	4	7

Table 1: EPSTs responses for Mark's strategy

As an example of the kind of *reasonable* response given in Version A, consider AJ who wrote that:

> AJ: Well, the [6.79/64] tells Mark he would spend around 11 cents per ounce and the [4.69/48] would tell him he would be spending around 10 cents per ounce, letting Mark know the [48 oz.] is a better deal.

We can see that AJ recognizes a unit rate, and can properly decipher the decimals provided in Version A as being related to fractions of a dollar. He understands Mark is finding a cost-per-amount, and wants to associate the lower cost with the better buy. As in Version A, in Version B most of the *reasonable* responses also reflected a somewhat standard line of expected thinking, to the effect that Mark was looking for price per ounce. In Version C, where the subject needed to determine a strategy for Mark, some EPSTs looked for a common multiple for the ounces of ice cream as an attempt at a norming process:

> FB: What you want to do is start by making these into rates and finding a commonality between the 2 so that would be finding the same size packaging and comparing the prices after that. By comparing the same units it gives you a common ground to start comparing. [Has written: (6.79/64 x 48/48 = 325.92/3072)(4.69/48 x 64/64 = 300.16/3072)]

Whereas the *reasonable* responses showed some conventional ways of thinking, we were surprised at the *questionable* responses for Mark's strategies. We wondered, for example, in Versions A & B where the ratios were already set up, where did those subjects get confused? One EPST wrote about Version A that:

> GC: The bigger number [.10609375] would explain how much you get for the amount you paid and therefore you would get a better deal for the money from the [64-ounce package].

His response shows a lack of understanding of the unit rate in the sense that the decimal has been mischaracterized as ounces and not dollars. We suspect that there also may be some primal linguistic association for the phrase "better buy" with larger numbers. On Version B, even if a decimal was obtained (since calculators were provided) there still occurred problems of interpretation:

> BT: If you change the decimals to a percent you could get 10.6% and 9.77% this could allow Mark to determine which is the better deal. 64 ounces for $6.79 is the better choice.

Here BT seems to just be doing division blindly, performing a computation without an understanding of how to interpret the result. Note how GC and BT both point to the 64-ounce container as the better buy, with dramatically different interpretations of the provided structure in Version A. On Version C, we saw evidence of a classic additive structure as opposed to multiplicative thinking:

> DC: The better deal is 64 ounces for 6.79. This is because there is 2 dollar & 10 cent difference between the two prices, but the more expensive one is 16 ounces more than the cheaper, so it turns out to be a better deal.

By finding the differences in prices and comparing that with the differences in amounts, the notion of proportion is bypassed for DC.

Alisha's Strategy

The same essential techniques used in coding responses for Mark's strategy was also used in coding responses for Alisha's strategy. Two additional emphases were on Version C, where subjects needed to come up with a strategy for Alisha that really was different from that of Mark, and on Versions A & B where the question was asked whether or not Alisha was correct in her strategy. In contrast to results for

Marks' strategy, however, we found that the majority of EPSTs actually had *questionable* responses for Alisha. Of the 75 total subjects, only 37.3% (n = 28) gave reasonable responses for Alisha, which was surprising to us. The exact breakdown of the numbers of responses by version is given in Table 2.

As in the *reasonable* responses to Mark's strategy, so too did this category of responses for Alisha fall across the expected themes for proportional reasoning. For example, HP offers the follows thinking on Version B:

> HP: [Has written: (64/6.79~ 9.42562592)(48/4.69~ 10.23454158)] Alisha is also correct because she can look at the two results and see that the bottom ice cream is the better buy because it gets more ounces per dollar than the top package.

He correctly discerns the amount-per-cost involving the unit of one dollar, and compares the ounces delivered in terms of each dollar spent. In this case, of course, the better buy is associated with the higher number.

	Version		
	A	B	C
Reasonable	7	12	9
Questionable	17	16	14

Table 2: EPSTs responses for Alisha's strategy

More *questionable* were the many responses that said, in effect, that Alisha was not correct in attempting a different approach, and that the only way to find a better buy was to consider cost-per-amount (as in Mark's strategy in Versions A and B). For example, here are some sample responses from Version A:

> AJ: No, you want to find the cost per ounce to see what is the best deal.
>
> ST: No, because Alisha got a completely different price than Mark did.
>
> BJ: No, this is incorrect. This equation does not tell them anything about which one is better buy.
>
> LO: No. She is finding the ounce per price which doesn't really matter.
>
> JK: Alisha is incorrect because she divided the ounces by the price. The answer she got is irrelevant to the problem. Mark's answer shows how much the ice cream costs per ounce. To find that, you divide the total cost by the number of ounces. Alisha did it backwards.

Notice how AJ is still thinking in terms of cost-per-amount, and ST misinterprets the computational result as being a price (per amount). BJ can't see the possible relevance of Alisha's ratios to finding a better buy, and even LO (who correctly interprets the ratio as amount-per-cost) can't relate the results to making an inference for which is the better buy. More strident were some of the *questionable* responses for Version B,

> MG: Alisha is not correct because by dividing her way you get ounces/cost and not cost/ounce. This will not tell you the better buy. It tells you for every penny spent you get so many ounces.

JL: Alisha is incorrect because dividing this way doesn't determine the price per ounce. It is incorrect to divide the ounce by price, it is correct to divide the price by ounce.

Especially for MG, who seems to know that Alisha is aiming for "ounces/cost", the implication is that MG cannot make sense of how this might help in inferring the better buy. And the flavour that emerges from JL is that Alisha is simply wrong in her strategy. Again, as in Mark's *questionable* responses, we saw more additive structures emerging for Alisha in Version C:

HG: She could have added the difference between the two to see the cheaper one of the two, to see if she was really getting her money's worth

TC: [Has written: (64+48 = 112) (6.79+4.69= 11.48) (11.48/112)]

Neither of these two responses is particularly clear, although they do seem to appeal to addition or subtraction as a key factor in making some sort of comparison.

DISCUSSION

As we sought to engage our EPSTs in a discussion about mathematics for teaching, we gave Ratio Triplets as a way to examine what they knew about proportional reasoning and also to promote conversations about different ways of thinking. We knew a priori that proportional reasoning is often difficult for students, "especially for those who do not understand what is actually meant by a particular proportional situation or why a given solution strategy works" (Weinberg, 2002, p. 138). We did see evidence of range of *reasonable* explanations for Mark's or Alisha's strategy, such as finding common multiples of ounces and comparing costs on that basis, or determining how many ounces might be obtained for a fixed cost. There was an impressive array of differing strategies that prompted fruitful discussion in class about representing mathematical situations in a variety of ways, and especially about the importance of communicating one's own thinking and understanding that of others.

However, it was surprising to us that 18.7% of our EPSTs either gave *questionable* responses for Mark's strategies (on Versions A and B) or could not come up with a proportional strategy for Mark (in Version C). Particularly on Versions A and B, it was disturbing to find some these young adults unable to interpret the initial ratio set-ups that were offered. Moreover, the 62.7% of our EPSTs who gave *questionable* responses for Alisha's strategies often showed a very limited understanding of proportional reasoning.

In their comments about schoolchildren, Resnick & Singer (1993) point out how "we know that ratio and proportion are difficult concepts for children to learn. They constitute one of the stumbling blocks of the middle school curriculum, and there is a good possibility that many people never come to understand them" (p. 107). As our exploratory research suggests, even those preparing to be teachers may not be entering their university training with a sufficiently robust conceptual understanding in proportional reasoning.

While more research is necessary to further unpack the dimensions of thinking exhibited by EPSTs in a proportional reasoning context, this research takes important first steps toward that process. Of particular importance was the structure of our *Ratio Triplets* task, since the lively class discussions that ensued when debating each others' explanations across the three versions helped foster a better conceptual sense of the actual mathematics while also modeling the kinds of pedagogical practices we'd like to see carried into the classrooms where these preservice teachers eventually will serve.

References

Behr, M., Harel, G., Post, T., & Lesh, R. (1992). Rational number, ratio, and proportion. In D. Grouws (Ed.), *Handbook of Rresearch on Mathematics Teaching and Learning* (pp. 296-333). New York: Macmillan.

Behr, M., Lesh, R., Post, T., & Silver, E. (1983). Rational-number concepts. In R. Lesh & M. Landau (Eds.), *Acquisition of Mathematical Concepts and Processes,* (pp. 91-128). New York: Academic Press.

Friedlander, A., Fitzgerald, W., & Lappan, G. (1984). The growth of similarity concepts at the sixth grade level. In J. Moser (Ed.), *Proceedings of the Sixth Annual Meeting of PME-NA* (pp. 127-132). Madison: University of Wisconsin.

Hart, K. (1988). Ratio and proportion. In J. Hiebert & M. Behr (Eds.), *Number Concepts and Operations in the Middle Grades* (pp. 198- 219). Reston, VA: National Council of Teachers of Mathematics.

Lamon, S. (1993b). Ratio and proportion: Children's cognitive and metacognitive processes. In T. Carpenter, E. Fennema, & T. Romberg (Eds.), *Rational Numbers: An Integration of Research* (pp. 157-195). Hillsdale, New Jersey: Lawrence Erlbaum Associates.

Lesh, R., Post, T., & Behr, M. (1988). Proportional Reasoning. In J. Hiebert & M. Behr (Eds.), *Number Concepts and Operations in the Middle Grades* (pp. 93- 118). Reston, VA: National Council of Teachers of Mathematics.

Marshall, S. (1993). Assessment of rational number understanding: A schema-based approach. In T. Carpenter, E. Fennema, & T. Romberg (Eds.), *Rational Numbers: An Integration of Research* (pp. 261-288). Hillsdale, New Jersey: Lawrence Erlbaum Associates.

National Council of Teachers of Mathematics. (2000). *Principles and Standards for School Mathematics*. Reston, VA: NCTM.

Resnick, L. & Singer, J. (1993). Protoquantitative origins of ratio reasoning. In T. Carpenter, E. Fennema, & T. Romberg (Eds.), *Rational Numbers: An Integration of Research* (pp. 107-130). Hillsdale, New Jersey: Lawrence Erlbaum Associates.

Weinberg, S. (2002). Proportional easoning: one problem, many solutions! In B. Litwiller & G. Bright (Eds.), *Making Sense of Fractions, Ratios, and Proportions: 2002 Yearbook.* Reston, VA: NCTM.

DESCRIPTION OF A PROCEDURE TO IDENTIFY STRATEGIES: THE CASE OF THE TILES PROBLEM

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In this paper we present a procedure to describe strategies in problems which can be solved using inductive reasoning. This procedure is based on some aspects of the analysis of the specific subject matter, concretely on the elements, the representation systems and the transformations involved. We show an example of how we used this procedure for the tiles problem. Finally we present some results and conclusions.

The researchers related to inductive reasoning process are usually developed in problem solving context (Cañadas, Deulofeu, Figueiras, Reid, & Yevdokimov, 2007; Christou & Papageorgiou, 2007; Küchemann & Hoyles, 2005; Stacey, 1989). These investigation pay attention to the cognitive process as well as to the general strategies that students used to solve the problems proposed.

In this paper, we present part of a wider investigation (Cañadas, 2007), which is focused on the inductive reasoning process and on the specific strategies developed by students to solve problems which involved a specific mathematical subject matter. One of the methodological contributions of this research consists on a procedure to describe strategies in problem solving. We use this procedure to identify and to describe strategies of students in problems that involved linear and quadratic sequences.

This paper consists of four main parts. First, we present some theoretical and methodological aspects of our research, which are important to introduce a procedure to identify and to describe inductive strategies, which conforms the second part. Third, we show the application of such procedure for the tiles problems. Finally, we present some results and conclusions related to this problem.

THEORETICAL FRAMEWORK

Inductive reasoning

Inductive reasoning is a process that produces scientific knowledge through the discovery of general rules starting from the observation of particular cases (Neubert & Binko, 1992). Following this idea, we took as starting point the Polya´s proposal about induction (1967)[1]. We consider working on particular cases and generalization as two states in the process of inductive reasoning (Cañadas, 2007). One of our research objectives was to produce a systematic procedure for exploring the inductive reasoning of students in the context of problem solving.

[1] Pólya talks about *induction* in the same sense as we refer to *inductive reasoning*. This conception is different from mathematical induction or complete induction, which refers to a formal method of proof, based more on deductive reasoning.

Inductive strategies in problem solving

Problem solving is considered a highly formative activity in mathematics education. It promotes different kinds of reasoning (Rico, Castro, Castro, Coriat y Segovia, 1997), specifically inductive reasoning. Induction is a heuristic and its aim is to provide regularity and coherence to data obtained through observation (Pólya, 1967).

Strategies are the “ways of performing on mathematical tasks, which are executed in concepts and relationships representations”[2] (Rico, 1997, p. 31). We use the expression *inductive strategies* to refer to the strategies used in problems which can be solved through inductive reasoning as heuristic.

Representation systems play an important role in problem solving because they allow expressing the reasoning performed. In our research, we focused on external representation used by students in problem solving. We analyzed the way that students performed to solve written problems through the external representations.

Mathematic subject matter

Given that we choose linear and quadratic sequences as the specific subject matter, we needed to describe it to select adequate problems to propose to the students and to obtain criteria to describe students´ work on those problems. We based this subject matter description on some ideas of the subject matter analysis (Gómez, 2007). Through some aspects of this analysis, we obtained useful information about linear and quadratic sequences to elaborate a procedure to describe inductive strategies. Particularly, we focused on the elements of the sequences, the representation systems and the transformations.

The elements of sequences are the particular and general terms, and the limit. Since our interest was inductive reasoning[3], we selected particular and general terms to work on.

Since sequences are a particular kind of functions, we took into account four representation systems for functions, following Janvier (1987): Graphic, numeric, verbal and algebraic. On the one hand, particular terms can be expressed numerically, graphically or verbally. On the other hand, general terms can be expressed algebraically or verbally.

We considered three sorts of transformations:

- Transformations among different representations of the same element: *synonymous transformations* (Janvier, Girardon, & Morand, 1993).
- Transformations among the same element inside the same sort of representation systems: *syntactic transformations* (Kaput, 1992).
- Transformations among different elements expressed in different representation systems.

[2] My personal translation.

[3] We consider that inductive reasoning is the process that begins with particular cases and produces a generalization from these cases. Pólya (1967) adds the idea of validation based on new particular cases to this kind of reasoning.

PROCEDURE TO IDENTIFY INDUCTIVE STRATEGIES

We elaborated a procedure to identify strategies based on representation systems. Each strategy is constituted by a sequence of transformations. To identify a strategy in a specific problem response, we start from particular terms expressed in the statement of the problem and we detect the kinds of transformations performed.

Element		Particular Term		
		Numeric	Graphic	Verbal
Particular Term	Numeric	TSN	T3	T5
	Graphic	T1	TSG	T6
	Verbal	T2	T4	TSV

Table 1. Transformations involving particular terms

In Tables 1, 2 and 3, we collect how we refer to all the possible kinds of transformations. Tables 1 and 2 contain transformations from the term in the first column to the term in the second column. For example, in Table 1, T6 refers to a transformation from a particular term represented graphically to a verbal representation of such term.

Element		General Term	
		Algebraic	Verbal
General Term	Algebraic	TSA	T8
	Verbal	T7	TSV

Table 2. Transformations involving general term

In Table 3, C refers to a transformation from particular term to general term and CB to a transformation in the inverse sense.

Element		General Term			
		Algebraic		Verbal	
Particular Term	Numeric	C1	C1B	C4	C4B
	Graphic	C2	C2B	C5	C5B
	Verbal	C3	C3B	C6	C6B

Table 3. Transformations involving general and particular terms

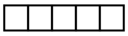
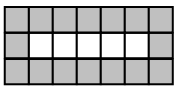
METHODOLOGICAL FRAMEWORK

We asked 359 Spanish students to work on a written questionnaire. Students belonged to years 9 and 10 in four different schools.

The questionnaire had six problems which involved linear and quadratic sequences that could be solved using inductive reasoning as a heuristic. Given that our interest was inductive reasoning, we considered problems that contained information about particular cases. One of these problems was the "tiles problem".

The tiles problem[4]

In the following lines, we present the tiles problem as it was presented in the questionnaire:

> Imagine some white squares tiles and some grey square tiles. They are all the same size. We make a row of white tiles:
>
> 
>
> We surround the white tiles by a single layer of grey tiles.
>
> 
>
> - How many grey tiles do you need to surround a row of 1320 white tiles?
>
> - Justify your answer.

DATA ANALYSIS

One example

In what folows, we will use the described procedure to identify the inductive strategy observed in one student´s response to the tiles problem. Figure 3 shows the student´s solution

1320
x 2
2640
2640 + 6 = 2646 tiles

We need the double number of white tiles, plus 3 tiles at each of both ends.

Figure 3. One solution to the tiles problem.

We observe that, first; s/he makes a transformation from graphic system of the particular term of the statement to numeric system (T1, see Table 1). After that, the student makes a transformation in this representation system (TSN, see Table 1). Finally, s/he gets the generalization verbally (C4, see Table 3). So, s/he used the following inductive strategy: T1-TSN-C4.

[4] We present the English version of the problem we posed in our research.

Inductive strategies in the tiles problem

We applied the described procedure to identify inductive strategies for each student's response to the tiles problem. Table 4 shows such strategies, the number of students who used each of them, the elements involved, and whether they produced the generalization or not.

Inductive Strategies	Freq	Elements	Generaliz	Partial Freq
No transformations	52			52
T1	10	Particular terms	No	247
T1-T5	6			
T1-TSN	151			
T1-TSN-T5	54			
TSG-T1	2			
TSG-T1-TSN	14			
TSG-T1-TSN-T5	3			
TSG-T6	1			
T6	2			
T6-T2-TSN	4			
T1-TSN-C1-TSA	1	Particular and general terms	Yes	60
T1-C4	9			
T1-TSN-C4	36			
TSG-C1-C1B-T5	1			
TSG-T1-C4	1			
TSG-T1-TSN-C4	7			
TSG-C4-C4B-TSN	2			
T6-C3-C3B-TSN	1			
C5-C4B-TSN	2			
Total				359

Table 4. Inductive strategies in the tiles problem

RESULTS

We identified 19 different inductive strategies in this problem, whether they generalized or not. There were 247 students that remained working on particular terms (C does not appear in the sequence of inductive strategies). On the other hand, there were 60 students who obtained the expression of the general term.

T1-TSN, T1-TSN-T5 y T1-TSN-C4 were the strategies used by most students. Observing the information in Tables 1, 2 and 3, we deduce that these students performed a transformation from the graphical representation to the numeric system (T1) and, after that, a syntactic transformation in the numerical representation (TSN).

Through the different strategies identified, we observe that students used the four possible representation systems: numeric, verbal, graphic and algebraic. Although the tendency to use the numeric representation is clear, there were 31 students who started their responses in the graphic representation (TSG). The verbal representation usually appeared at the end of the response (T5, T6, C4 or C5 at the end of the inductive strategies).

We now describe strategies of students who did not generalize and strategies of students who did, separately.

Students who did not generalize

Six of the students who answered to the problem started working on the verbal representation, as shows the transformation T6 in their strategies (T6 and T6-T2-TSN). There were 20 students who started with the graphical representation (TSG as the first term of the sequence that represent the strategy: TSG-T1, TSG-T1-TSN, TSG-T1-TSN-T5 y TSG-T6.

In general, the numeric system was the most frequent representation used by students who did not achieve the generalization (247 students).

The verbal representation was performed by 70 students, as we deduce from the frequencies of strategies that include T5 and T6. 63 of these students used this kind of representation at the end of their response, when they tried to justify their answers using particular terms.

Students who generalize

Of the 60 students who achieved the generalization, just two generalized directly from the statement (as reveals strategy C5-C4B-TSN). These students reached the generalization without any previous transformation among particular terms.

There were 55 students that generalized and had previously worked on particular term in the numeric representation (T1 precedes C1 or C4) and three students worked on particular term in the graphic representation before generalizing (TSG-C1-C1B-T5, TSG-C4-C4B-TSN). Eight of the students that generalized, combined graphical and numeric representation before expressing the general term for the sequence.

The generalization was expressed algebraically by three students. The respective strategies are T1-TSN-C1-TSA, TSG-C1-C1B-T5 and T6-C3-C3B. The remaining 57 students that get the generalization used the verbal representation to express it.

As part of the strategies of students who generalized, we paid attention to how they used the generalization. On the one hand, two of the three students who expressed the general terms algebraically, used the generalization to calculate the particular term required by the problem (students who use the strategies TSG-C1-C1B-T5 and T6-

C3-C3B). The third one got the generalization in the last transformation, so s/he did not use the general term for the first task proposed in the problem. This student used the general expression as a way to justify her/his response. On the other hand, four of the 57 students that generalized verbally, used such expression to calculate the particular term that the problem asks for (students who used inductive strategies TSG-C4-C4B-TSN and C5-C4B-TSN).

CONCLUSIONS

The procedure presented in this paper allowed us to identify and describe strategies used by students in the tiles problem. The information obtained through the procedure allowed us to get conclusions related to the work on particular cases and the generalization, as part of the inductive reasoning process. Moreover, we got data about the representation systems used related to these states of inductive reasoning. In this paper we have shown some of these results for the tiles problem.

In the tiles problem, we get some conclusions related to the inductive strategies and to the inductive reasoning process. First of all, we highlight that students denote a preference for the numeric system, although the four possible representation systems are employed by different students. Another general conclusion is that most of the students remain working on particular cases.

Students show a tendency to use verbal representation at the end of their responses. This fact reveals us that they use this system in the justification of their responses. The verbal representation is also the most frequent way of expressing generalization. This is surprising if we consider that students used to express the generalization algebraically in their classrooms. The majority of the students that generalize verbally tend to do so when they try to justify their answers, and not as a way to calculate new particular terms of the sequence. Probably it could be interesting for teaching to consider this way of expressing generalization before working on it algebraically.

The generalization, both algebraic and verbal, is occasionally used to calculate the particular term required in the problem.

The procedure to identify inductive strategies can be useful for other mathematical subject matters and maybe for other cognitive processes. In the case of other mathematical subject matters, we could consider an analogous procedure based on specific elements, representation systems and transformation of such subject matter.

References

Cañadas, M. C. (2007). Descripción y caracterización del razonamiento inductivo utilizado por estudiantes de educación secundaria al resolver tareas relacionadas con sucesiones lineales y cuadráticas. Granada: Departamento de Didáctica de la Matemática de la Universidad de Granada. Available at http://cumbia.ath.cx:591/pna/Archivos/CannadasM07-2850.PDF.

Cañadas, M. C., Deulofeu, J., Figueiras, L., Reid, D., & Yevdokimov, O. (2007). The conjectuting process: Perspectives in theory and implications in practice. *Journal of Teaching and Learning, 5*(1), 55-72.

Christou, C. & Papageorgiou, E. (2007). A framework of mathematics inductive reasoning. *Learning and Instruction, 17*(1), 55-66.

Gómez, P. (2007). *Desarrollo del Conocimiento Didáctico en un Plan de Formación Inicial de P de Matemáticas de Secundaria*. Granada: Departamento de Didáctica de la Matemática de la Universidad de Granada.

Janvier, C. (1987). *Problems of Representation in the Teaching and Learning of Mathematics*. Mahwah: LEA.

Janvier, C., Girardon, C., & Morand, J. (1993). Mathematical symbols and representations. In P. Wilson (Ed.), *Research Ideas for the Classroom: High school Mathematics* (pp. 79-102). Reston, VA: National Council of Teachers of Mathematics.

Kaput, J. (1992). Technology and mathematics education. In D. A. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 515-556). New York: Macmillan.

Küchemann, D. & Hoyles, C. (2005). Pupils awareness of structure on two number/algebra questions. In M. Bosch (Ed.), *Proceedings of European Research in Mathematics Education IV* (pp. 438-447). Barcelona: FUNDEMI IQS-Universitat Ramón Llull.

Neubert, G. A., & Binko, J. B. (1992). *Inductive reasoning in the secondary classroom*. Washington, DC: National Education Association.

Pólya, G. (1967). *Le Découverte des Mathématiques*. París: DUNOD.

Rico, L. (1997). Consideraciones sobre el currículo de matemáticas para educación secundaria. In L. Rico (Coord.), *La Educación Matemática en la Enseñanza Secundaria* (pp. 15-38). Barcelona: Horsori.

Rico, L., Castro, E., Castro, E., Coriat, M., & Segovia, I. (1997). Investigación, diseño y desarrollo curricular. In L. Rico (Ed.), *Bases Teóricas del Currículo de Matemáticas en Educación Secundaria,* (pp. 265-318). Madrid: Síntesis.

Stacey, K. (1989). Finding and using patterns in linear generalising problems. *Educational Studies in Mathematics, 20*(2), 147-164.

RELATIONSHIPS BETWEEN CHILDREN'S EXTERNAL REPRESENTATIONS OF NUMBER

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Previous studies of children's use of notation have pointed to different types of notational strategies (Alvarado, 2002; Brizuela, 2004; Cayton, 2007; Scheuer et. al., 2000; Seron & Fayol, 1994). In a recent study (Cayton & Brizuela, 2007) we found that first grade children were producing a great number of unconventional responses when writing large numbers. This study follows those same children into grade two to see how the children perform after another year of experience with writing larger numbers. We also examine the relationship between the children's written numbers and another type of external representation through valued tokens.

RATIONALE AND PAST RESEARCH

Studies of children's numerical understanding over the last decade suggest that there are identifiable progressions in how children develop number concepts (Cobb, 1997; Fuson, 1997). According to Fuson (1997), children construct meanings for numbers through the various interactions that they have with these numbers both in and out of school. Elementary school mathematics classrooms encourage or facilitate the development of various number concepts through the language that is used by the teachers and students, the type of materials that are used, the problems that are solved, and the class activities. These components act in concert with one another to support children's construction of meanings for numbers. One important and interesting aspect of numbers is that they can be represented in many different formats: written numerals, oral numbers, arrays of dots, tallies and more. What are the relationships, for children, among these representations of number?

In relation to written numbers, for instance, Bialystok and Codd (2000) ask, "what do children believe that written representations of quantity mean?" (p. 117). To illustrate the interconnections between written representations of number and other external number representations, studies have found that transcoding zeros within numbers proves particularly problematic amongst young children. For example, several recent studies (e.g., Cayton, 2007; Cayton & Brizuela, 2007; Scheuer et. al., 2000; Seron & Fayol, 1994) highlight that children's errors point to problems in both implementing the representational actions as well as in appropriating the number system itself.

In the study presented here, we wished to follow up to two previous studies. In Cayton and Brizuela (2007), presented at PME 31, we found that at the end of first grade, students were still producing a great number of unconventional responses in three different systems of external representations of number: oral, written, and a third system where numbers are represented by tokens of various colors, each a power of ten. In Cayton (2007), we found that two seemingly-similar strategies for written numbers (Full Literal Transcoding [FLT] and Compacted Notation [CN], described below) produced by kindergarten and first grade children were, in fact, associated with different strategies in token-building: children using CN tended to perform similarly to children writing conventional numbers, with a large number of children building conventional token arrangements; while children using FLT were more likely to form unconventional token arrangements. We wondered if we would find this same pattern with older children, who have received more instruction and have more experience with written numbers and larger numbers in general.

METHOD

Participants

Twenty-six second grade students (students need to be seven years of age by the time they begin second grade) were interviewed individually. The school is located in an urban suburb of the United Stated of America. The school is ethnically, racially, and socio-economically diverse. In addition, the school provides a two-way bilingual education to children. All children in the second grade were invited to participate. Only children whose parents consented to their participation were included in the study.

Materials and Procedures

Interviews were carried out as clinical interviews (Piaget, 1965). During the course of the interviews, children were presented with the numbers detailed in Table 1. Our goal was to be able to explore children's oral, written, and nonverbal representations of number. Our proposal was to access children's oral representation through their oral naming of numbers; their written representation through their writing of numbers; and their nonverbal representations through their construction, through tokens, of the "value" of the different numbers.

Each of the interviews has three tasks: oral, written, and tokens. Each one of these tasks has both a production and interpretation mode: when numbers are presented by the interviewer in tokens, they can be interpreted through writing or through naming orally; when numbers are presented by the interviewer in writing, they can be interpreted through construction of tokens or through naming orally; when numbers are presented by the interviewer orally, they can be interpreted through construction of tokens or through writing.

Oral task: In this part of the task, children were asked to read from a piece of paper or from a token composition the numbers in Table 1.

Written task: In this part of the task, every child was asked to write at least two numbers from each series in Table 1 after being presented the number orally or through tokens.

Tokens task: This part of the task was designed for the purpose of understanding the consistencies/inconsistencies in the child's understanding of our number system without the use of notation. Children were presented with tokens of different colors.

Tokens were chosen based on the work of Nunes Carraher (1985) performing similar tasks in the understanding of place value in young children and illiterate adults. The child was told that red tokens are worth 1 point, blue tokens are worth 10 points, white tokens are worth 100 points, brown tokens are worth 1,000 points, and maroon tokens are worth 10,000 points. The child was asked to compose a number from Table 1 with the tokens after being presented the number orally or in writing.

Series	Number Type		
Series 1	Three digit – without 0	127	143
Series 2	Three digit – internal 0	101	207
Series 3	Three digit – final 0	300	760
Series 4	Four digit – without 0	1127	3143
Series 5	Four digit – X0XX	3064	2053
Series 6	Four digit – XX0X	2101	3504
Series 7	Four digit – XXX0	1300	3760
Series 8	Five digit–without 0	21127	13143
Series 9	Five digit—XX0XX	43064	52053
Series 10	Five digit—XXX0X	22101	33504

Table 1. Numbers presented to children in the three different tasks (orally, through tokens, or in writing). Numbers were designed based on the work of Alvarado and Ferreiro (2002), Power and Dal Martello (1990), and Seron and Fayol (1994)

Children were randomly assigned to one of six task orders: (a) two possible conditions with an *oral introduction*; (b) two possible conditions with a *written introduction*; (c) two possible conditions with a *token introduction*. See Cayton and Brizuela (2007) for more details on the tasks and conditions.

ANALYSIS

All the interviews were videotaped. Transcripts of the interviews were reviewed along with any notes made during or after the interview, the written work of the children, and the physical manipulations of the children during the tokens tasks as documented in the videos. These pieces constituted the data for the study. This paper only looks at the analysis of two of the study tasks: written and token.

Data was arranged into categories for different types of strategies. In the written task, responses were classified by the strategy used to produce each written numeral. There were eight categories coded:

A) *Idiosyncratic* - No discernable strategy used for writing the number. For example, in one instance, the number 153 was transcribed as 4033.

B) *Missing Digits* - The written numeral is missing digits from the original number, either replaced by zero or deleted entirely. For example, the number 1127 could be represented as 1027 or 127.

C) *Digit Transposition* - Two or more digits of the number are transposed with one another. All of the original digits are still contained in the number. For example, the number 1127 could be represented as 1217.

D) *Full Literal Transcoding (FLT)* – Child writes out number literally, for example, 100701 or 10071 for one hundred seventy-one. This category is taken from Seron and Fayol (1994). This is similar to the Scheuer et al (2000) category of logogramic notation except that Scheuer would only allow for 100701 to be considered in this category. I allow for both types of literal transcoding (100701 and 10071) as FLT since for children who are conventionally writing 2-digit numbers, "71" has become the literal writing of seventy-one.

E) *Compacted Notation (CN)* - Child writes extra zeros in numbers but fewer than the FLT notation, for example, 1071 for one hundred and seventy-one. This category is taken from Scheuer et al (2000).

F) *Error due to incorrect use of comma* - This category only pertains to numbers over 999. The child uses a comma in notation and leaves out zeros. For example, one thousand seventy-one would be written 1,71.

G) *Lexical Error* - Child replaces one digit of the number with a different digit. For example, 137 for 127.

H) *Conventional Response* - Number conventionally represented.

In the tokens task, children were classified by the strategy used to compose the point value of the tokens. Composition number strategies were coded separately for each type of number. The same eight categories were used for each type of number:

A) *No Response* - Child answers "I don't know" and will not provide a guess.

B) *Incorrect Understanding* - Child fails to understand the multiplicative nature of the tokens. While in previous studies with Kindergarten and first graders, the most common example of this was counting every token as one point regardless of its color; this was rarely seen with second graders. The most common example of incorrect understanding in second grade was lining up the tokens in a literal order. For example, 1300 would become one one-point token, one thousand-point token, three one-point tokens, and one-hundred-point token to make "one- thousand - three - hundred."

C) *Counting by ones* - Child does show some understanding of token value, but can only add up the points by counting by ones (i.e., pointing to a 10-point token and counting 1 to 10), always leading to an error when dealing with large numbers.

D) *Incorrect token value* - One value of token was replaced entirely with a token of a different value. For example, the number 127 could be composed with tokens totaling 1027.

E) *Value Missing* - One value of token is missing entirely, with the other values all having the correct total. For example, the number 1127 could be composed to total 1027 due to missing one hundred-value token.

F) *Non-canonical, incorrect total* - Responses where there were more than nine tokens of any single value were classified as non-canonical. In this category, responses were both non-canonical and the total value was incorrect. For example, in composing the number 127, the response could have twenty ten-point tokens and seven one-point tokens, totaling 207.

G) *Non-canonical, correct total* - Responses were non-canonical, but the total value of the tokens was correct. For example, in composing the number 127, the response could have twelve ten-point tokens and seven one-point tokens, totaling the stimulus value of 127.

H) *Canonical and correct* - Number was represented canonically (no more than 9 tokens of any given value) and with the correct total value.

RESULTS

A total of 494 written numbers and 508 token constructions were produced by the twenty-six children. While a majority of written numbers were conventionally written by the second graders (375 of 494, 75.9%), the number of conventionally written numbers dropped dramatically as the length of the number increased, from 145 of 150 (96.7%) of 3-digit numbers to 141 of 200 (70.5%) of 4-digit numbers, and finally 89 of 144 (61.8%) of 5-digit numbers (see Table 2).

While this may seem logical, given that children have more practice with smaller numbers, we could argue that once children have appropriated the rules of the number system, numbers of any length should be equally accessible. Further, the difference in percentage of responses correct between four- and five-digit numbers indicates that this is not due to misunderstanding the word “thousand” or other vocabulary as both of these use all of the same terminology.

Category of Response	Three-digit	Four-digit	Five-digit	**Total**
Idiosyncratic (Idio.)	2	7	2	**11**
Missing Digits (MD)	0	2	16	**18**
Digit Transposition (DT)	1	1	2	**4**
Full Literal Transcoding (FLT)	0	24	17	**41**
Compacted Notation (CN)	2	13	14	**29**
Error due to Comma (Com.)	N/A	2	2	**4**
Lexical Error (LE)	0	10	2	**12**
Conventional Response (Conv.)	145	141	89	**375**
Total	**150**	**200**	**144**	**494**

Table 2. Written number responses by children, n=494

The tokens task produced similar results, with only one three-digit number of 156 total responses not produced conventionally (in fact, the one response was a refusal, not an incorrect representation). With 4-digit numbers the conventional rate dropped to 179 of 207 (86.5%) and 121 of 145 (83.4%) 5-digit numbers (see Table 3).

Category of Response	Three-digit	Four-digit	Five-digit	**Total**
No Response	1	1	4	**6**
Incorrect Understanding	0	10	10	**20**
Counting by Ones	0	3	1	**4**
Incorrect Token Value	0	8	4	**12**
Value Missing	0	2	2	**4**
Non-canonical, Incorrect Total	0	3	3	**6**
Non-canonical, Correct Total	0	1	0	**1**
Conventional Response	155	179	121	**455**
Total	**156**	**207**	**145**	**508**

Table 3. Token compositions by children, n=508

We next compared the tokens results with the written number strategies to investigate whether specific strategies in each representational system were associated to one another. The entirety of our cross-tabular comparisons will be discussed at length in following papers, but for now, we focus our attention to the two notational strategies, FLT and CN, in which we found disparities in previous studies (see Table 4).

Table 4 shows FLT and CN responses in relationship to unconventional (all categories except for the “Conventional Response” category) and conventional token constructions. The results and differences were striking, while 92.6% of CN responses were associated to conventional token constructions, only 62.5% of FLT responses were associated with conventional token constructions [χ^2 (1, N = 67) = 7.71, p<.01].

	FLT	CN	**Total**
Unconventional	15 (37.5%)	2 (7.4%)	**17 (25.4%)**
Conventional	25 (62.5%)	25 (92.6%)	**50 (74.6%)**
Total	**40 (59.7%)**	**27 (40.3%)**	**67 (100%)**

Table 4: Unconventional and conventional token compositions amongst FLT and CN written number responses, n=67

DISCUSSION

While these results do replicate our previous findings with younger children in which we found that FLT and CN were associated with different strategies in token-building: children using CN tended to perform similarly to children writing conventional numbers, with a large number of children building conventional token arrangements; while children using FLT were more likely to form unconventional token arrangements; we still consider it to be quite puzzling: why is it that notations for numbers of the type CN tend to be related to more conventional number constructions (through tokens) than FLT, seeing that both of these strategies are incorrect, never taught, and both formed by the over-use of zero in numerical notation? In the case of FLT, one could call it the numerical equivalent of sounding out the spelling of a word, which children are taught to do in written language. If this is the case, it still does not explain the prevalence of CN. Why do children choose to eliminate some but not all zeros in a number? Moreover, why do children who are performing this way appear to be closer to a conventional understanding of base-ten number construction? Seron and Fayol (1994) posit that perhaps children are absorbing some rules of notation, such as the "overwriting" of zeros, but are not yet grasping the entire concept of the system.

This leads us to the next obvious question, which is: are numerical notation strategies shaped by understandings of number construction or does the notation influence what children understand about numbers (i.e. their understanding of the base-ten construction of numbers)? The finding that numbers represented by conventional token constructions were much more variable in notational strategies than the reverse seems to point in the direction of the notation as a reflection of the child's constructional understanding. That being said, we still have much to learn about how children understand the unique and global system of numerical notation. The data presented in this paper show how much we may be able to learn about how children understand various aspects of the number system by looking at their notational strategies.

The data here also indicate that by the end of the second grade, children are still having difficulty in representing numbers. This should be a great cause for concern as this is the same age in which children begin learning place-value algorithms for arithmetic, and yet, they are demonstrating an incomplete understanding of the use of place-value and of written numbers and what they represent.

References

Bialystok, E. & Codd, J. (2000). Representing quantity beyond whole numbers: some, none, and part. *Canadian Journal of Experimental Psychology, 54*(2), 117-128.

Cayton, G. A. (2007). Young children's numerical representations across various modes. *Unpublished Qualifying Paper*. Medford, MA: Tufts University.

Cayton, G. A. & Brizuela, B. M. (2007). First graders' strategies for numerical notation, number reading and the number concept. In J.-H. Woo, H.-C. Lew, K.-S. Park, & D.-Y.

Seo (Eds.), *Proceedings of the 31st Annual Meeting of the International Group for the Psychology of Mathematics Education* (vol. 2, pp. 81-88). Seoul, Korea: Seoul National University.

Cobb, P. (2000). From representations to symbolizing: Introductory comments on semiotics and mathematical learning. In P. Cobb, P., E. Yackel, & K. McClain (Eds.), *Symbolizing and Communicating in Mathematics Classrooms. Perspectives on Discourse, Tools, and Instructional Design* (pp. 17-36). Mahwah, NJ: Lawrence Erlbaum Associates.

Cobb, P., Gravemeijer, K., Yackel, E., McClain, K., & Whitemack, J. (1997). *Mathematizing and Symbolizing: The Emergence of Chains of Signification in One First-grade Classroom*. Mahwah, NJ: Lawrence Erlbaum.

Fuson, K. C., Wearne, D., Hiebert, J. C., Murray, H. G., Human, P. G., Olivier, A. I., et al. (1997). Children's conceptual structure for multidigit numbers and methods of multidigit addition and subtraction. *Journal for Research in Mathematics Education, 28*(2), 130-162.

Nunes, T. (1985, July 22-26). *The Decimal system: Understanding and Notation.* Paper presented at the Ninth International Conference for the Psychology of Mathematics Education, Noordwijkerhout.

Piaget, J. (1965). *The Child's Conception of the World* (J. Tomlinson & A. Tomlinson, Trans.). Totowa, NJ: Littlefield, Adams, & Co.

Scheuer, N., Sinclair, A., Merlo de Rivas, S., & Tieche-Christinat, C. (2000). Cuando ciento setenta y uno se escribe 10071: niños de 5 a 8 años produciendo numerales [When a hundred and seventy one is written 10071: Five to eight-year-olds' production of written numerals]. *Infancia y Aprendizaje, 90*, 31-50.

Seron, X. & Fayol, M. (1994). Number transcoding in children: A functional analysis. *British Journal of Developmental Psychology, 12*, 281-300.

A CASE STUDY OF ELEMENTARY BEGINNING MATHEMATICS TEACHERS' EFFICACY DEVELOPMENT

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The purpose of this study aimed to explore the developmental process of elementary beginning mathematics teachers' efficacy. According to findings, the developmental process and transformative trend of beginning teacher efficacy showed a dynamic feature and moved back and force among the five gradations concluded from the data. Besides, the five developmental gradations were influenced by two-dimensional interactions of internal and external factors, which were found in a previous study.

INTRODUCTION

Teacher efficacy is "their belief in their ability to have a positive effect in student learning" (Ashton, 1985 p.142). Bandura's (1981) indicated that teachers' efficacy expectations will influence their thoughts and feelings, their selection of activities, and the amount of effort they spend, as well as the degree of their persistence while facing obstacles. Actually within last 30 years, this concept has developed continuously relevant to Bandura's (1977) theory of self-efficacy and Rotter's (1966) locus of control and currently gained much attention (Pajares, 1992), which reveal the significance of teachers' beliefs in their own capabilities in relation to the effects of student learning and achievement. Several studies further reported, "Teacher efficacy has been identified as a variable accounting for individual differences in teaching effectiveness" (Gibson & Dembo, 1984, p. 569) and had a strong relationship with student learning and achievement (Allinder, 1995). Moreover, Bandura (1981) mentioned that knowledge and action are two prerequisites, but not limited in, to be successful. However, he argued that one might have one kind of knowledge and understand what should be done but having an inappropriate action. This result is caused by the adjustment of one's self-referent thought, which is exactly the fountain of efficacy (Bandura, 1981), on the relationship between knowledge and action. In addition, teacher efficacy is different from teachers' effectiveness. The latter tends to examine how well a teacher performs in the classroom. This performance is defined as their external behaviors which are usually composed of three dimensions, i.e. cognition, affection, and skill. Nevertheless, teacher efficacy is an internal belief above and significantly controlling a teacher's external behavior and performance (Chang, 2003).

According to the findings of Chang and Wu (2006), beginning mathematics teachers with mathematics and science (M & S) background had a significantly higher increase in their efficacy ratings (both personal teaching efficacy and teaching outcome expectancy) than those who were not both at the beginning and the end of the first year. Further, two categories of factors were found influencing beginning mathematics teacher efficacy: teacher's professional performance (*internal factor*) and assistance obtained from peer interactions and the administration level (*external factor*) (Chang & Wu, 2006). Under the condition of having low teacher efficacy and inadequate readiness in teaching, what would be the developmental process of their efficacy during their first year of teaching? How would their efficacy development be influenced by the internal and external factors mentioned above? Referring to Bandura's (1978) reciprocal determinism, an individual's mental function is determined by a continuous interaction process of three elements, i.e. behavior, cognition, and environment. Thus, this interaction should be analyzed qualitatively under a process that is combining the three elements with a realistic circumstance. Accordingly, a qualitative research should be conducted by entering classrooms of the beginning mathematics teachers for answering the questions proposed above.

PURPOSE AND METHOD

The purpose of this qualitative case study was to explore the development of elementary beginning mathematics teachers' efficacy. It also intended to compare the efficacy development of the two groups of beginning teachers (with and without M & S background, coded as "M" and "N"). Interviews, observations, and both researchers' and beginning teachers' weekly reflection notes (see Table 1) were utilized, according to the research model of teachers' thought proposed by Clark and Peterson (1996), in analyzing the change of beginning teacher efficacy. "Mathematics Teaching Efficacy Beliefs Instruments" (Chinese version, Chang, 2003) were administered for purposefully selecting the research participants from two groups of elementary beginning mathematics teachers. Six teachers totally participated in this study, where each group has one teacher belonging to each level of teacher efficacy—high, medium, and low (numbered as "1, 2, and 3" respectively). Teacher specialty, student learning, and teacher-student interaction were the main topics in the interview and observation protocols, associated with the internal and external influential factors of teacher efficacy. The analysis in context strategy was employed for reaching the objectives.

Source	Coding Example	Note
Interview/Observation	941018 IN/OB M1-1	941018 is the date
Researchers' reflection note	9410-2 Note N1-1	9410-2 shows the week of the month
Teachers' reflection note	9410-4 M1 Note	

Table 1. Coding System

FINDING AND DISCUSSION

Dynamic Developmental Process and Transformative Trend of Teacher Efficacy

Both Fessler (1985) and Katz (1972) considered the developmental process of teachers' profession as a stage model, even though they proposed different kinds of approaches in demarcating and defining the developmental stages. These stages could be divided based on different concepts such as cycles, periods, phases, or topics concerned. In this study, the data was first analyzed in a processing and recursive approach. According to the internal belief, need, feeling, and instructional behaviors of the six beginning mathematics teachers, five developmental gradations of their teacher efficacy were then generalized, as well as the transformative trends. The findings showed that all six teachers experienced the first and the second gradations during the first year of teaching. Two medium-efficacy teachers (M2 & N2) reached the third gradation at the end of the year. Both high-efficacy teachers (M1 & N1) exhibited the characteristics of the forth phase by quickly passing through the first three gradations. The only one, M1, who possessed M & S background even entered the highest level, the fifth gradation.

- *First Gradation-Disorientation:* A teacher fails to notice and realize the changes and special situations occurred within the classroom, and usually feels lost and does not know what to do; the only thing could be handled is the teaching tasks but having lots of difficulties.
- *Second Gradation-Concerning External Teaching Environment and its Change:* A teacher fails to concern and aware her/his own professional development and growth; passively reacting to the instructional problem occurred and then usually failing to solve it effectively. The feeling of frustrations, recognizing the huge gap between her/his own expectation and the realistic situation in the classroom, was another obstacle of being mature.
- *Third Gradation-Self-Attentiveness and Self-Adaptation:* A teacher begins to concern her/his personal capabilities of classroom management and think reflectively to advance self-adaptation. Concerning for the sufficiency of instructional activities provided, how well-prepared for delivering the content, and then conducting both self-examination and self-adaptation.
- *Forth Gradation-Concerning Instruction Itself and Looking for External Resources:* A teacher focuses on her/his instructional performance, i.e. the content or the curriculum selected, the choice of teaching methods and its outcome, and the instructional limitations. Besides, looking for external resources and assistances actively for promoting the quality of teaching.
- *Fifth Gradation-oncerning Learning and Outcome Expectancy:* A teacher cares mostly about the influence of her/his teaching performance on students' learning (i.e. satisfaction of students' needs and mental/physical development) for adjusting the instruction correspondingly.

Based on the results of data analyses, it was found that beginning mathematics teachers who had the same level of teacher efficacy tended to exhibit substantial similarities in their developmental processes and transformative trends. Consequently, the dynamic developmental process and transformative trend were presented in three parts.

A. High-efficacy teachers' fast development

Beginning mathematics teachers with high level of teacher efficacy usually had adequate confidences in their personal subject-matter knowledge and teaching capability at the beginning of the first year. For example, “It took me about one month for adaptation”, said M1 (941024 IN M1-10). He controlled both teaching and learning situations in such a short period and applied multiple teaching strategies to provide various leaning experiences. This effective teaching performance led to a good beginning that his students “had more positive learning attitudes and better academic outcomes” (941024 OB M1-7). Thus, both beginning teachers with high efficacy never exhibited the characteristics of the first gradation, crossed the second gradation after one month, and then entered the third gradation. Further, M1 showed how he concerned about the preparation of all instructional activities. He also used multiple assessment tools for examining students’ learning outcomes, such as “homework of searching stories of mathematicians through the Internet to obtain more background knowledge and be prepared for class discussions” (9410-4 Note M1-2). However, M1 had a short stay at the third gradation. As his self-attentiveness and self-adaptation continued, he became gradually focusing on the instructional activity and content he provided. Because of his M & S background (*internal factor*) and more teaching experiences gained, M1 started to seek out for extra assistances and resources. Consequently, this characteristic of actively looking for external resources (*external factor*) made him entering the fourth gradation at the end of first semester.

At the middle of the second semester, M1 became aware of the influence of his personal teaching performance to his students’ learning based on what he performed at the third gradation. Especially in the group discussion activities, he would actively asked how well students could comprehend about the discussion topic and instructional content, and found out certain students’ learning conditions in particular. This kind of formative assessments gave him significant information for further adaptation and improvement in his personal teaching ability. Said in his last interview,

> …I became realizing that those students who had low academic achievement or who had mental or developmental problems deserved to learn differently. So, I was always thinking what kind(s) of instructional methods would be better for them. My good teaching performance did not guarantee students’ better understandings. So I had to take care of all students, those students who had special needs in particular (950524 IN M1-7).

This clearly showed that M1 had already entered the fifth gradation at the end of the first year. He concerned all students’ learning conditions and tried to satisfy all students’ needs through efforts of adjusting his teaching activities. He also knew how to utilize the integration of instruction and assessment to reach the ultimate goal of enhancing students’ learning outcomes.

Another high-efficacy teacher N1 had fairly similar developmental process and transformative trend, even if the final status of their efficacy development was not quite the same. Although without M & S background, her great interest, past excellent academic achievement, and abundant tutor experiences in mathematics resulted in her high level of confidence in teaching mathematics. Same as M1, she soon entered the third gradation. Her good classroom management skills and various teaching strategies helped her to control the entire learning environment. We also found that the teacher-student(s) interactions happened frequently in her classroom. Students expressed their own opinions or raised questions energetically. However, the use of this cooperative learning model also took her about "3 to 4 weeks to adapt the whole situation of both teaching and learning in the classroom" (9412-3 Note N-2).

In comparison, N1's characteristics of the third gradation were more obvious which made her staying longer. Starting from the beginning of the second semester, she "discussed with other teachers for improvement (*external factor*)" (950308 IN N1-10). Besides, she actively "looked for supplemental materials or exercises" (9502-3 N1 Note) in order to design further instructional activities. She also wanted her students to learn various kinds of problem-solving skills; on the contrary, she "wanted her students to share, listen, and/or obtain various opinions through discussions" (950308 IN N1-2). This thought and action was the main motivity of her professional development, and also kept remaining her the significance of external resources.

Moreover, N1 liked to use instant reinforcements to give students positive feedbacks, especially for those who had special needs. It seemed that she had some features of the fifth gradation. However, her viewpoint and practice on students' assessment contrarily made her teaching quality going backwards. She said that "paper-pencil examination is the main assessment tool" (950508 IN N1-11). She also thought students would learn more effectively under the pressure of examinations, which was the best way to evaluate the learning outcome. Consequently, more observations were essential for ascertaining other evidences to diagnose whether she reached the fifth gradation or not.

B. Low-efficacy teachers' slow-moving and vital need of the external factor

According to the classroom observations, two low-efficacy teachers (M3 and N3) could hardly handle the teaching tasks at the first semester. They usually were found lost in the instructional activities and did not know what to do. They used only the textbook and its teaching guidebook as the instructional content, while employing its exercise book and traditional tests for the evaluation of students' learning outcomes. Besides, they were not conscious of the change occurred in the classroom since the lecture was the only teaching method associated with a rigorous rule for classroom management. For instance, "students put their hands on the back during the whole class period…and no students-teacher interaction at all or no opportunity provided for practice" (941108 OB N3-9). Accordingly, they were both at the first gradation.

M3 admitted that he was struggling at the first semester and felt powerless in teaching. However, this status changed after being lost in one semester. He could explore certain strategies to solve the problem passively, which led him to the second gradation at the middle of the second semester. Additionally, he started to look for ways of improvement. But under the limitation of no out-of-school training (i.e. in-service education) allowed, he felt a little frustrated. Fortunately, there were good peer interactions within his school. Active cares and continuous assistances from experienced teachers (*external factor*) and his professional background (M & S, *internal factor*) produced a positive and reciprocal interaction, which pushed him into the third gradation. He also began to think reflectively of whether the instructional activities provided were adequate and proper, and then reviewed his classroom management rules. One big movement was the use of reading; this strategy was employed to "help students to better understand the assay question or the actual meaning of the mathematical question" (95005-4 M3 Note). However, he stayed at the third gradation till the end of this year.

With regard to the development of N3, she was afraid of mathematics and never achieved well all the time. She "felt terrified" (9501 Note N3-2) while teaching since she was "fearful of teaching the content in a wrong way or guiding students to an incorrect direction" (9412-4 N3 Note). This caused her a serious problem that she could not normally deal with the mathematical teaching problems occurred. Under the circumstance of neither sufficient mathematics education background (*internal factor*) nor assistance from peers and the administration level (*external factor*), it seemed that the efficacy development of N3 was helpless, which kept her at the first gradation till the middle of the second semester. Besides, as described in her weekly reflection journal, she felt that "my loading (except teaching) was so huge that made me so busy and having a tremendous amount of pressure" (9505-2 N3 Note). Instead of giving her more opportunities to learn how to survive in the classroom, this extra work (i.e. administrative tasks, competitions, teaching demonstrations) actually became a negative and heavy burden of her teacher efficacy development in this beginning year. In this poor circumstance, "it was hard for her not to be lost in the classroom" (9505-4 Note N3-3). Fortunately, she finally nerved herself to ask one kindly experienced teacher how to teach mathematics, which was an irregularly scheduled interaction, at the last two months. Within this period (last two months of the year), we also found that "she started to concern changes occurred in the classroom but passively" (950523 OB N3-2), as well as taking the suggestions of that experienced teacher for her classroom management problems. Accordingly, she stepped into the second gradation at the end of the first year eventually.

C. Medium-efficacy teachers' highly similar development

The efficacy development process of both medium-efficacy teachers (M2 and N2) was respectably similar, and both of them reached the third gradation at the end of the first year. They both had the feeling of helplessness while first entering the teaching position, where N2 had more obvious features of this phase and lasted longer. N2 mentioned that "the only thing I mastered was music, neither mathematics itself nor how to teach mathematics!" (9412-4 N3 Note).

This phenomenon lasted till the beginning of the second semester. She obtained active assistances from two experienced teachers in her school (*external factor*), where we found that "she could discuss her instructional problems while being asked (by those two teachers) to raise those problems" (950314 OB N2-14). This positive peer interaction led to a major change of her efficacy development. Not only making her entering the second gradation but also she exhibited some features of the third gradation at the middle of the second semester. She started to "employ group discussion activities" (9504-4 Note N2-4) instead of the lecture all the time even though she was still afraid that the learning order in the classroom might be worse. However, this attempt, in fact, did not bring her further troubles while teaching mathematics. On the contrary, it inspired her to look for external resources more actively for her professional development.

As regards M2 who had M & S background (*internal factor*), he showed the characteristics of the second gradation after teaching two months. The struggle between theories and practices forced him to change his instructional strategies. For example, his pre-service training experience remained him that he could not just lecture in the mathematics course even if he worried about the distraction of the discussion activities. Therefore, he asked students to discuss in small groups and presented their answers (or ways of solving mathematical problems) to other group members or the entire class. "I thought I had to use multiple ways to teaching. As you saw, I was still trying to control the order and getting their attentions" (941229 IN M2), said M2. Compared to N2, M2 did not gain enough external assistances and resources from his colleagues. Nevertheless, as described previously, his mathematics education background stimulated him to rethink what he still needed for improving his teaching practice and even more energetically seeking out for help. Because of the contact with our research team (i.e. members included elementary and secondary experienced teachers and professors), he realized that he might attain external assistances from us. He began eager for asking questions while possible and demanding extra instructional materials or teaching aids. This external and unexpected support accidentally became the inspirational source of his efficacy development. He said, "you all (i.e. the research team members) were truly helpful for me" (9505-1 M2 Note). It also helped him to step into the third gradation and forced him to reflect that "I should actively interact with other teachers, within and out of my school, more frequently!" (9505-4 M2 Note).

CONCLUSION

In summary, under the influences of two-dimensional interactions of internal and external factors, beginning mathematics teachers who had various gradations of teacher efficacy showed different developmental processes and transformative trends. In fact, this efficacy development was a dynamic process and exhibited special movement patterns. For instance, after quickly entering the third gradation, high-efficacy teachers kept moving forward to the forth and the fifth gradation. In the meantime, they repeatedly went backwards and then sequentially moved in these three gradations, while facing new difficulties or challenges; same as those who had medium or low efficacy. Consequently, we believed that, concluding from the data, the development and transformation of elementary beginning mathematics teachers not only matched the characteristics of the stage model mentioned above but also showed a dynamic feature and moved back and force among the five gradations. Echoing to both Vygotsky's (1978) social constructivism and the concept of 'zone of proximal development' and Bandura's (1978) reciprocal determinism, an individual's learning and mental development is dynamic and will be motivated through the acquisition of more knowledge and/or higher levels of thinking. This power will definitely help one to pursue higher achievement or advanced development. Therefore, our next task should be finding out more specific strategies in assisting all elementary in-service mathematics teachers to enhance their teacher efficacy immediately and effectively.

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References

Allinder, R.M. (1995). An examination of the relationship between teacher efficacy and curriculum-based measurement and student achievement. *Remedial and Special Education*, *16* (4), 247-254.

Ashton, P. (1985). Motivation and the teacher's sense of efficacy. In Ames, C. & Ames, R. (Eds.), *Research on tivation in Education* (Vol. 2): The classroom meliu (pp. 141-174). Orlando, FL: Academic Press.

Bandura, A. (1977). Self-efficacy: toward a unifying theory of behavioral change. *Psychological Review*, *84*(2), 191-215.

Bandura, A. (1978). The self system in reciprocal determinism. *American Psychologist*, *33*, 344-358.

Bandura, A. (1981). Self-referent thought: A developmental analysis of self-efficacy. In J. H. Flavelll & L. Ross (Eds.). *Social Cognitive Development Frontiers and Possible Futures*. Melbourne, Australia: Cambridge.

Chang, Y. L. (2003). *Knowledge Assessment and Self-efficacy Ratings in Teacher Preparation*. Paper presented at the annual conference of Northwest Association of Teacher Educators (NWATE), Spokane, WA, U.S.A.

Chang Y. L. & Wu, S. C. (2006). Teacher efficacy and elementary teacher education. *Academic Exchange Quarterly*, *10* (3), 75-79.

Clark, C. M. & Peterson, P. L. (1986). Teachers' thought processes. In M. C. Wittrock, *Second Handbook of Research on teaching*, pp. 255-296. New York: Macmillan.

Fessler, R. (1985). A model for teacher professional growth and development. In P. J. Burke & R. G. Heideman, *Career-long teacher education*. Springfield, Illinois: Thomas.

Gibson, S. & Dembo, M. H. (1984). Teacher efficacy: A construct validation. *Journal of Educational Psychology*, *76* (4), 569-582.

Katz, L. G. (1972). The developmental stages of preschool teachers. *Elementary School Journal*, *73* (1), 50-54.

Pajares, M. F. (1992). Teachers' beliefs and educational research: Cleaning up a messy construct. *Review of Educational Research*, *62*, 307-332.

Rotter, J. B (1966). Generalized expectancies for internal versus external control of reinforcement. *Psychological Monographs*, *80*, 128.

Vygotsky, L. (1978). *Mind in Society: The Development of Higher Psychological Processes.* Cambridge, MA: Harvard University Press.

MATHEMATICAL KNOWLEDGE FOR TEACHING AND THE UNFOLDING OF TASKS IN MATHEMATICS LESSONS: INTEGRATING TWO LINES OF RESEARCH

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This paper reports on an exploratory study investigating whether a particular type of teacher knowledge, namely teachers' mathematical knowledge for teaching *(MKT), matters for teachers' selection, presentation, and enactment of tasks. To this end, I explored the unfolding of tasks in a series of lessons given by a high- and a low-MKT teacher. The analysis of nine videotaped lessons from each teacher and the dissection of the curriculum materials these teachers employed in their lessons revealed notable differences in the unfolding of tasks in these teachers' lessons. The results of interview data with each instructor in regard to their thinking and reasoning on selected mathematical items suggest that the aforementioned differences should not be considered unrelated to teachers' own mathematical knowledge.*

INTRODUCTION

Engaging students in cognitively demanding tasks is critical to the quality of student learning (Hiebert & Wearne, 1993; Stein & Lane, 1996). However, cognitively challenging tasks are not self-enacting; what determines student learning is rather how these tasks are introduced and worked on during instruction. Teachers' critical role in introducing and enacting these tasks with their students has been documented in several studies (e.g., Boaler, 2002; Henningsen & Stein, 1997). Capturing this role, the NCTM *Principles and Standards for School Mathematics* (2000) also suggests that

> [w]orthwhile tasks alone are not sufficient for effective teaching. Teachers must … decide what aspects of a task to highlight, how to organize and orchestrate the work of the students, what questions to ask to challenge those with varied levels of expertise, and how to support students without taking over the process of thinking for them (p. 19).

A range of factors has been proposed to account for how teachers capitalize on tasks to support student learning. These factors include teachers' beliefs and expectations, students' instructional habits and dispositions, the established classroom norms and practices, and a wide array of contextual factors, such as time constraints and pressure to cover the curriculum (Doyle, 1988; Watson & Mason, 2007). While acknowledging the importance of these factors, this study examines the role of teacher knowledge in the unfolding of tasks. Although teacher knowledge has been considered a plausible contributor to how teachers introduce tasks in their lessons and work on them with their students (cf., Stein, Remillard, & Smith, 2007), the pertinent empirical evidence is relatively scarce.

THEORETICAL FOUNDATIONS

To explore the role of teacher knowledge in the unfolding of tasks, this study built on work in two research areas that have yielded useful ideas for the teaching of mathematics, but have moved in parallel paths. First, drawing on the work of the QUASAR (*Quantitative Understanding: Amplifying Student Achievement and Reasoning*) project, the study utilized the *Mathematical Tasks Framework* (MTF), which helped identify three phases during which teachers' decisions and actions appear to affect the cognitive level at which the content is experienced in mathematics classes. This framework (Figure 1) suggests that teachers can influence student learning by the tasks they select, by the way in which they present these tasks, and by the manner in which they work on these tasks with their students.

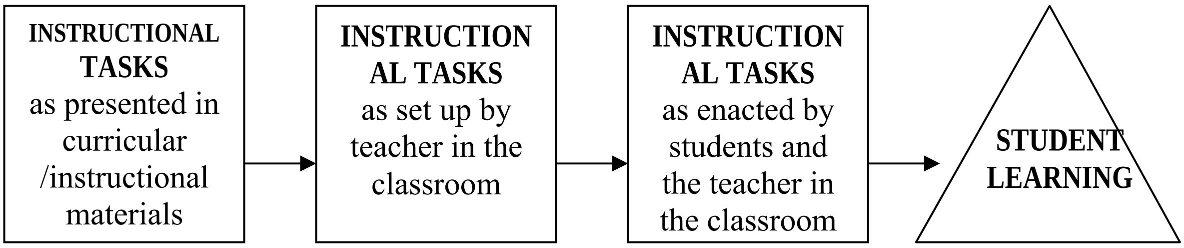


Figure 1. The MTF (reproduced from Henningsen & Stein, 1997, p. 528).

To capture the cognitive level of tasks, the QUASAR researchers have proposed the *Task Analysis Guide* (TAG) that classifies tasks into the following four categories (see Stein, Smith, Henningsen, & Silver, 2000, p. 16 for more elaboration):

- *Memorization (ME)*: These tasks require reproducing previously learned rules or facts.
- *Procedures without connections (PWOC)*: These tasks are algorithmic and focus on producing correct answers rather than developing mathematical understanding.
- *Procedures with connections (PWC)*: These tasks focus students' attention on the use of procedures for the purpose of developing meaning and understanding.
- *Doing mathematics (DM)*: These tasks require complex and non-algorithmic thinking.

The first two categories correspond to tasks of lower level demands, and the latter two to tasks of higher cognitive demand. A fifth category (i.e., *unsystematic exploration, UE)* was also proposed by Stein and Lane (1996) to refer to tasks that could potentially engage students in higher level thinking but during their enactment students engage in unsystematic explorations and hence fail to develop understanding.

The second area of research that informed this study is the work of Ball and associates (Ball, Hill, & Bass, 2005) on *mathematical knowledge for teaching* (MKT). The focus on MKT to explore the inquiry of this study rests on both theoretical and empirical considerations. From a theoretical perspective, MKT captures a unique type of professional knowledge needed for the various aspects of teaching mathematics. From an empirical standpoint, recent studies showed that MKT is positively associated with the quality of instruction (Hill et al., accepted) and with student learning (Hill, Rowan, & Ball, 2005).

Building on these two research areas, this study investigates whether *teachers with dissimilar levels of MKT differ in their selection, presentation, and enactment of tasks, in terms of the cognitive demands of these tasks*. Being exploratory, this study did not endeavor to yield conclusive evidence, but rather to make a first step in examining whether MKT appears to matter for the unfolding of tasks in mathematics lessons.

METHODS

This study utilized a multiple-case approach (Yin, 2006), focusing on a purposive sample (Patton, 2002) of two cases differing in terms of MKT. In particular, a series of eighteen lessons, nine taught by a high-MKT teacher (Karen) and nine by a low-MKT teacher (Lisa) was analyzed, with respect to the three phases of task unfolding (i.e., selection, presentation, and task enactment). Karen and Lisa (both pseudonyms) were part of a non-random, large sample of 640 teachers who completed the paper-and-pencil *Learning Mathematics for Teaching* test measuring their MKT; they were also among 10 teachers who consented to have a series of their lessons videotaped three times in each of three cycles (for more information see Blunk, 2007). Karen scored in the 93rd percentile of the large sample, whereas Lisa was in the 35th percentile. Both were seasoned elementary teachers and taught fifth-grade mathematics for most of the lessons under investigation. Three different types of data were utilized in this study:

(a) Videotaped lessons: Nine videotaped lessons for each of the two teachers were analyzed. The first step in analyzing the lesson data was to turn the visual and auditory images into quantitative and qualitative data. This step required a detailed coding protocol to be developed, including guidelines for parsing the lessons into meaningful chunks whose beginning and end points could be clearly identified; the protocol also detailed how to decide the level of cognitive demand of the content during task presentation and enactment. The unit of analysis of the videotaped lessons was the task, defined as "a segment of classroom work devoted to the development/learning of a mathematical idea" (Stein, et al., 2000, p.7). The cognitive demands of the tasks in these lessons were analyzed by the author of this study using the *Task Analysis Guide* (TAG). A random sample of two lessons per teacher was coded by two independent coders, yielding substantial inter-rater reliability for both task presentation and enactment (Cohen's kappa, κ>.76).

(b) Curriculum documents: The TAG was used to code the cognitive demands of the available curriculum tasks employed in the lessons. The two independent coders coded approximately 15% of the tasks used by each teacher, yielding a perfect inter-rater reliability for the cognitive demands of the curriculum tasks (κ= 1.0).

(c) Interviews: The two teachers were asked to reflect on how they solved 18 of the MKT test items. These interviews were analyzed to identify patterns in the two teachers' thinking and reasoning approaches to these items.

FINDINGS

Thirty nine tasks were identified in Karen's lessons and 30 in Lisa's lessons. Of these tasks, there were available data for *all* phases of task unfolding for only 21 tasks for Karen and 15 for Lisa (i.e., the curriculum materials for the remaining tasks were not available, and thus the phase of task selection could not be captured). Thus, in what follows, I first consider the tasks for which data were available for all phases of task unfolding; I then consider all the 69 tasks (for the phases of task presentation and enactment).

All phases of task unfolding

In 18 of Karen's 21 tasks, Karen consistently maintained the cognitive demand of the tasks at the level of the curriculum materials during presentation and enactment; six of these cases pertained to high-level tasks and 12 to low-level tasks. Only in two cases was Karen observed to *not* maintain the demand of intellectually challenging tasks, and these lapses happened during task enactment. She also elevated the cognitive demand of one task during its presentation, but failed to sustain the challenge during task enactment. Eight of the 15 tasks considered for Lisa were presented as cognitively demanding in her curriculum. Of those tasks Lisa maintained the intellectual challenge during presentation and enactment in only three cases.

In the remaining five cases, the challenge declined during task presentation (four tasks) and enactment (one task). Lisa was not observed to elevate the cognitive demand of any task during presentation or enactment; she was consistent, though, at maintaining the cognitive demand of 7 tasks at the low level of her curriculum materials.

Presentation and enactment phases

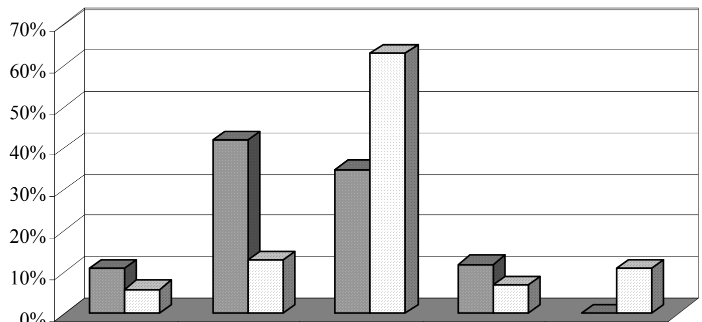
Table 1 provides information about the cognitive demands of all the tasks presented and enacted in the two teachers' lessons. About 40% of the tasks in Karen's lessons were presented as intellectually challenging (i.e., *DM* and *PWC*), compared to only about 16% of the tasks in Lisa's lessons. This finding should not be divorced from Lisa's failure to maintain the cognitive demand of the tasks during task presentation reported above. In fact, Lisa set up most of the tasks (i.e., about 84%) at a lower cognitive level, by emphasizing the procedures involved or by asking students to recall and apply rules and algorithms. Table 1 also shows that Karen was relatively successful in maintaining the cognitive challenge of the presented tasks during their enactment: about 40% of the tasks in her lessons were presented as high level and about 30% were enacted at this cognitive level. In contrast, only about 13% of the tasks in Lisa's lessons were enacted at a higher cognitive level.

Cognitive level of tasks*	Tasks as they were presented				Tasks as they were enacted			
	Karen		*Lisa*		*Karen*		*Lisa*	
	f	%	f	%	f	%	f	%
DM	3	7.70	1	3.33	2	5.26	1	3.33
PWC	12	30.77	4	13.33	9	23.68	3	10.00
PWOC	13	33.33	18	60.00	16	42.11	18	60.00
ME	11	28.21	7	23.34	11	28.95	7	23.34
UE	0	0	0	0.00	0	0.00	1	0.03
Total	39	100.00	30	100.00	38**	100.00	30	100.00

* See the *Task Analysis Guide* (TAG); **No data on the enactment of one of the tasks were available.

Table 1. The cognitive level of tasks in Karen's and Lisa's lessons

If one considers only the *number* of tasks in these teachers' lessons, one could argue that the students in both teachers' classes mainly experienced intellectually undemanding tasks (i.e., notice the number of tasks enacted as *PWOC* and *ME*). However, if one considers the *duration* of the different types of tasks in the two teachers' lessons, a different picture emerges, as depicted in Figure 2. This figure was developed by first calculating the percentage of time allotted to each category of tasks within each lesson, and then by averaging these percents over all nine lessons for each teacher. Figure 2 shows that, on average, the instructional time in Karen's lessons was about evenly distributed between demanding tasks (*DM, PWC*) and less challenging tasks (*PWOC, ME*). In contrast, most of the time in Lisa's lessons was allotted to less demanding tasks or unsystematic explorations (81%). Figure 2 also shows that whereas in Karen's lessons *PWC* tasks were dominant, in Lisa's lessons the dominant tasks were *PWOC*.



	DM	PWC	PWOC	ME	UE
Karen	11%	42%	35%	12%	0%
Lisa	6%	13%	63%	7%	11%

Figure 2. Instructional time allotted to different types of tasks in the two teachers' lessons.

Illustrative episodes from the two teachers' lessons

I present one episode from each teacher's lessons, indicative of the patterns discussed above. The episode from Karen's lessons showcases her emphasis on helping students see meaning in the mathematical procedures at hand. The episode from Lisa's lessons exemplifies her focus on remembering and applying rules and algorithms.

The episode considered for Karen is from a lesson on subtracting integers. In the previous lessons, students used two different types of representations (number lines and plastic tile stoppers shaped as pluses and minuses) and worked on adding integers; during this activity, they figured out the sum of two opposite numbers (e.g., +3 + (-3)). The lesson under consideration started with students' reviewing a couple of examples on adding integers. Karen then projected five "pluses" tiles on the overhead projector and urged students to apply the two different representations they were using in previous lessons to figure out the difference "5 - (-4)," which she presented as "subtracting minus four from the five 'pluses' tiles." After some time for exploration, Karen asked one of the students, Michael, to share his work. Michael suggested removing four of the "pluses" tiles, and argued that the difference was one. Another student observed that Michael took away "positive four instead of negative four." Karen elicited other students' ideas. To scaffold their thinking, she recommended that they "recall an important idea discussed in previous lessons," to which a student responded by referring to the sum of a negative and a positive one being zero. Building on this idea, Karen helped students see that they could add as many pairs of opposite numbers to the number with which they started, without changing the value of this number. After that, Michael modeled the subtraction on the overhead. He first added four pairs of "pluses" and "minuses" tiles to the already existing five "pluses". He then took away all the "minuses" and concluded that the difference of 5- (-4) is nine. Karen pressed for an explanation: "How come? We started with five and now we have nine." At this point, a student mentioned that his older brother told him, that "a negative times a negative makes a positive," to which Karen replied that it might be good for students to avoid such rules, and focus on making meaning, because "in six weeks you might forget [these rules] and then you would be all confused again." Trying to help her students see the underlying meaning of the subtraction of integers, she then asked them to consider several other carefully sequenced problems on this operation, insisting quite adamantly that they use both their representations to show how they were solving these problems.

The episode for Lisa is from a lesson on finding the area of triangles. The lesson started with Lisa's presentation of "the snake farm" task, a cognitively demanding problem that asked students to figure out the area of a given triangle. Lisa then remarked that to solve this task, "we have to know how to find the square area of the triangle." Having said that, she handed out a worksheet that presented a rectangle divided into two triangles (by one of its diagonals), and clarified that the area of a triangle is half the area of a rectangle. She then noted: "So, here's a formula you can remember. One half of a rectangle equals triangle." Following that, she directed students' attention to the word "of": "Whenever we see '*of*,' what do we do?" A student replied that "we divide," to which Lisa disagreed. Another student suggested multiplication. Satisfied with this answer, Lisa went on to write the formula that gives the area of a triangle as "½ x area rectangle" and then "½ x base x height." She then asked students to figure out the area of several triangles of given height and base lengths. Finally, students were given time to solve the "snake farm" task, using the formula introduced in this lesson.

Insights from the interviews

At this point, the critical reader might attribute the differences in the unfolding of tasks in Karen's and Lisa's lessons delineated above to differences in the two teachers' beliefs about mathematics and its teaching. Although this argument cannot be dismissed, the data from the interviews during which the two teachers solved a number of MKT items and explained their thinking and reasoning suggest that these teachers' instructional approaches were not unrelated to the manner in which they themselves understood the content and solved pertinent mathematical problems. Due to space limitations, I consider the teachers' responses to only one of the MKT items. The item under consideration presented a mathematical situation that essentially involved determining the fractional part of another fraction (i.e., 1/5 of ½). Although both teachers eventually solved the item correctly, their solving approaches and their reasoning in solving this item differed remarkably. To reason through the item, Karen drew a picture. She first divided this representation into two halves, and then divided one of these halves into five equal pieces. She concluded by pointing out: "It was very easy. I drew a picture before I did the problem." Lisa, on the other hand, did not use a representation to solve this item; nor did she find the answer to the item right away. Initially, she identified 2/5 as the correct answer. Then, on reconsidering her answer, she murmured: "this would be half *of* something." The word "of" seemed to have triggered her mental schema of multiplication: "Oh, I know what it is." She paused for awhile, and then said: "Because he taught one fifth *of* a half, which is multiplication, [the answer] would be one tenth."

DISCUSSION

This exploratory study makes a first step in investigating the role that a distinctive type of teacher knowledge, namely their MKT, appears to play in task selection, presentation, and enactment. The study showed differences in the unfolding of tasks in the lessons of the teachers under consideration. Karen, the high MKT-teacher, largely maintained the cognitive demand of curriculum tasks at their intended level during task presentation and enactment. She also helped her students see meaning of the procedures at hand, by often urging them to use multiple representations; she was also adamant in eliciting students' thinking and explanations. Lisa, the low-MKT teacher, on the other hand, often proceduralized even the intellectually demanding tasks she was using and placed more emphasis on students' remembering and applying rules and formulas. The study interview data suggest that the patterns observed in the two teachers' presentation and task enactment are not unrelated to how these teachers appear to have understood the content and reasoned through pertinent problems. As was originally intended, this study did not provide conclusive evidence on whether MKT matters for the unfolding of tasks in teachers' lessons. However, the findings reported suggest that MKT should be seriously considered in explorations that aim to capture what informs teachers' decisions and actions during the different phases of task unfolding. It is expected that these findings will also catalyze further thought and research on the effect of teacher knowledge on teachers' instructional approaches.

References

Ball, D. L., Hill, H. C., & Bass, H. (2005). Knowing mathematics for teaching: Who knows mathematics well enough to teach third grade, and how can we decide? *American Educator*, 14-17, 20-22, 43-46.

Blunk, M. (2007). The QMI: Results from validation and scale-building. Paper presented at the *Annual Meeting of the American Educational Research Association,* Illinois, Chicago.

Boaler, J. (2002). *Experiencing school mathematics: Traditional and reform approaches to teaching and their impact on student learning.* Mahwah, NJ: Lawrence Erlbaum Associates.

Doyle, W. (1988). Work in mathematics classes: The context of students' thinking during instruction. *Educational Psychologist, 23* (2), 167-180.

Henningsen, M., & Stein, M. K. (1997). Mathematical tasks and student cognition: Classroom-based factors that support and inhibit high-level mathematical thinking and reasoning. *Journal for Research in Mathematics Education, 28* (5), 524-549.

Hiebert, J., Wearne, D. (1993). Instructional tasks, classroom discourse, and students' learning in second-grade arithmetic. *American Educational Research Journal, 30* (2), 393-425.

Hill, H., Blunk, M., Charalambous, C., Lewis, J., Phelps, G., Sleep, L., et al. (accepted). Mathematical Knowledge for Teaching and the Mathematical Quality of Instruction: An exploratory study. *Cognition and Instruction*.

Hill, H., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal, 42* (2), 371-406.

NCTM. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

Patton, M. (2002). *Qualitative evaluation and research methods,* (3rd ed.). Thousand Oaks, CA: Sage.

Stein, M. K., & Lane, S. (1996). Instructional tasks and the development of student capacity to think and reason: An analysis of the relationship between teaching and learning in a reform mathematics project. *Educational Research and Evaluation, 2* (1), 50-80.

Stein, M. K., Remillard, J., Smith, M. S. (2007). How curriculum influences student learning. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 319-369). Charlotte, NC: Information Age.

Stein, M. K., Smith, M. S., Henningsen, M. A., & Silver, E. A. (2000). *Implementing Standards-based mathematics instruction: A casebook for professional development*. NY: Teachers College Press.

Yin, R. K. (2006). Case study methods. In J. L. Green, G. Camilli & P. B. Elmore (Eds.), *Handbook of complementary methods in education research* (pp. 111-122). Mahwah, NJ: Lawrence.

Watson, A., & Mason, J. (2007). Taken-as-shared: A review of common assumptions about mathematical tasks in teacher education. *Journal of Mathematical Teacher Education, 10,* 205-215.

YOUNG CHILDREN RECOLLECT THEIR MATHEMATICAL THINKING

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Children were interviewed about their mathematical thinking and asked to reflect on their learning as part of a larger study exploring teacher behaviours that challenge children to probe their mathematical understandings. Fifty-three interviews were conducted in 4 schools with 5- to 7-year-old Australian children. The subjects were involved in close conversation with their teachers during the mathematics lesson. Video-stimulated recall was used with a conversational interview to prompt children's recollections and reflections. Findings indicate that young children in the first years of schooling are able to recall events in their mathematics lessons to reconstruct their thinking and reflect on their mathematical learning.

BACKGROUND

The theory of social constructivism underpins this research. Cobb, Wood, Yackel and McNeal (1992) argued that the construction of knowledge occurs within a social and cultural context where discourse is a vital component in establishing an effective learning context. The focus of this research is the meaning constructed between the teachers and children in classrooms.

There has been a long history of interviewing young children to describe their mathematical thinking (e.g., Donaldson, 1978; Gelman & Gallistel, 1978; Hughes, 1996). These interviews often involved children performing mathematical tasks to demonstrate their thinking or development. Task-based interviews have also been used to assess and plot the growth of the mathematical thinking of children over time (Clarke & Cheeseman, 2000). However there appears to be little research that reports young children's reflections on their thinking in post-lesson interviews.

Franke and Carey (1997) conducted interviews to research first-grade children's views about what it means to do mathematics in problem solving classrooms. They found that young children were in fact able to reflect on classroom events.

McDonough (2002) reported procedures that prompted eight to nine year-old children to articulate their beliefs about mathematics. Children found it a difficult to talk abstractly about learning, however, they "held beliefs about mathematics, learning and helping factors and could articulate beliefs when prompted" (p. 270).

To capture some of the complexities of classrooms settings and to collect rich data, the approach termed complementary accounts methodology was used for this study (Clarke, 2001). While the methodology used for the large study differed from that of Clarke, similar fundamental techniques were used: videotaping the whole mathematics lesson, audio taping participants' reconstructions of classroom events, and an analysis of the multiple data sets.

METHOD

In total, 53 children were interviewed on the day their mathematics lesson was conducted. The children were aged five to seven years from four classes, each in a different Australian school. The selection of students was based on classroom observation notes of the researcher.

The interviews were audio taped for transcription and analysis. A video of the lesson was used as a stimulus to recall sections of the lesson directly involving each child. Children were asked to recount events where they were in conversation with the teacher, to say what they were thinking at the time, and to reflect on what they had learned in the mathematics lesson. The interview was conversational in style. While there was an interview script, it was adapted in order to elicit responses from each child. The scripted questions were:

- I am interested in the times when teachers talk to children in maths lessons. I noticed that your teacher had a talk with you in that maths lesson. Can you remember that? Can you tell me what happened?
- I think that we got that on video. Would you like to see it?
- What were you thinking about? (Maybe just watch it at first.)
- Can you say what was happening?
- What did you learn in maths today? Was there anything else?

These questions are modelled on those used by Clarke (2001, pp. 13-32) however these questions have been simplified for young children.

Video-stimulus recall

There appears to be scant literature describing the use of stimulated recall using video with young children to investigate their perspective of mathematics lessons. There are reports of Year 8 children, using video-stimulated interviews to reconstruct the learner's perspective (e.g., Williams, 2003) and reports of teachers video-stimulated recall of the events in their classrooms (e.g., Ainley & Luntley, 2005) but there seems to be no use of this methodology in mathematics education with young children.

Because little was known about how young children would respond to video-stimulated interviews, some piloting was undertaken. It became clear that the best way to prompt recall was to play a little of the beginning of an incident of interest to set the scene for the child then to pause the video and to ask, "Do you remember that, what was happening there?" If a child had no recollection of the event, the entire video episode involving them in conversation with the teacher was played and used as a stimulus to describe their thinking or reflect on their learning. In general, the video was used as a starting point only.

DATA CODING AND ANALYSIS

Interviews were digitally recorded. Seventeen interviews were transcribed in full. The remaining 36 interviews were coded directly from the audio files. Data were considered in terms of the children's recall of an incident or task, description of events, explanation of their thinking, and description of their learning. Categories of response emerged from the data. Descriptors of response were listed in increasing levels of sophistication, with 1 being the least and 5 or 6 as the most sophisticated responses as follows:

Recall of the incident/task

1. no recall
2. child could recall the event only after of the entire video excerpt was replayed
3. recall with the video paused just before the event of interest or with the video playing in the background with no audible sound
4. recall spontaneously with little or no assistance of the video extract

Description of events

1. no description of interaction with teacher
2. describe actions
3. describe outcomes only, e.g., a work sample
4. describe the event from their perspective
5. describe reasoning and/or justify their thinking

Explanation of their thinking

1. no explanation
2. "account for" the videotape e.g., make up a "story"
3. explicit description of thinking
4. explain/reconstruct thinking, reasoning, justifying, evaluating thinking

Description of learning

1. unable to specify learning
2. learned "nothing"
3. learned a behaviour not mathematics e.g., "to share"
4. remembered factual information e.g., number facts
5. learned how to do something e.g., "to count by 6s"
6. described learning at a conceptual level, expressed as a mathematical principle or an insight, e.g., "I can count by 1s, 2s, 3s, 4s, 5s, 6s, 7s, 10s, and 100s and 1000s …once I can count by ten I can count by all the rest. Like 10, 20, 30, 40, 50, and it always has a zero on the end."

In general, the highest level of descriptor was coded when evidenced anywhere in the interview. Codes were then entered into a statistical analysis program (SPSS) to produced descriptive statistics.

Reliability of coding

To improve internal reliability, interviews were re-coded. An independent person, skilled at listening to young children describe their mathematical coded a 20% sample of the audio data. All points of difference were discussed and an agreed understanding of the data was reached. Based on the combined critical analysis, further interviews were transcribed in full (17 in total) and category descriptions were refined. The entire data set was coded again applying the new protocols without any reference to the previous coding. The results of this second coding form the data reported here.

RESULTS

Recall of events

Using videotape of events involving each child in the mathematics lesson of the day to stimulate the recall and an account of the episode from the view of the child was largely successful. This is evident from Table 1, which summarises the categories of responses of children's recall of events, where only 2% of children were unable to recall the events of the lesson. Some children needed to watch the entire replay of the videotape where they were in conversation with the teacher in order to talk about it (23%). Many children, having watched the video of the lesson leading up to the event, could recount their version of what had unfolded after the videotape was paused (30%). In addition almost half of those interviewed could recall a conversation with the teacher before the video was replayed.

Category of response	Frequency as a percent (n = 53)
1. No recall	2
2. Recall with video replay of the event	23
3. Recall with video paused or with no audible sound	30
4. Recall spontaneously	45

Table 1. Categories of Response of Children's Recall of an Event

Description of event

An analysis of the children's descriptions of events revealed an interesting three-way split of responses (see Table 2). Some children described only what they did (23%). The following example illustrates this category of response. James could be seen on the video interlocking blocks but saying nothing:

> Interviewer: So what was happening here?
>
> James: My brain was counting and I wasn't. [James, J2.3:25]

Other children offered a description from their point of view (36%). For example, Ali explained his counting of 5 groups of 5 teddies saying, "It goes 10, 20, 30, 40, 50.

You have to count the ears" [Ali, G1, 7:30]. It is hardly surprising that 36% of children who could remember the event described it from their point of view. In fact what was interesting was that such a large proportion described the event with some reconstruction of their reasoning at the time (28%). This was perhaps the most interesting group of responses. For example, Jessica was explaining how to weigh a dog, Joey, who would not stand on bathroom scales:

Interviewer: Can you tell me about your good idea for maths today please?

Jessica: I thought of holding Joey on the scales. I would know how much Joey weighed. So I hopped on the scales with him and I holded him. And then we took away 19 [from 28] because I was 19 and he was 9 and so that was 9 kilograms and that's what he weighed [Jessica, J3, 0:35].

Category of response	Frequency as a percent (n = 53)
1. No description of interaction with teacher	4
2. Describe actions	23
3. Describe outcomes only, e.g., a work sample	8
4. Describe the event from their perspective	36
5. Describe reasoning and/or justify thinking	28
6. Missing	2

Table 2. Children's Descriptions of Events

Explaining thinking

Table 3 shows the number of children who could explicitly describe their mathematical thinking was high (85%).

Expecting children to be able to communicate their thinking has been an element of Australian mathematics curriculum definition for years (Australian Education Council, 1991; Board of Studies). Certainly based on classroom observational data from the classrooms of the children interviewed here it is a clear expectation of their teachers that they explain their reasoning.

It should be said that these children had been learning mathematics in the classrooms of "highly effective" teachers of mathematics (McDonough, 2003) for 8 months. Perhaps this would account for their readiness to describe their mathematical thinking. Whether children in other classrooms can explain their thinking with this frequency is a question that might be explored by further research.

An example of the type of response that shows a child reconstructing and evaluating his thinking is when Tom offered a thinking strategy for his classmates who could not count by 4. His idea was to use a count by 2.

Interviewer: Now Mrs A says that's a really complicated way to work it out I can't really hear what you were saying. She was looking at a page that had

8 legs and 4 things on each leg. How were you trying to work that one out?

Tom: Oh a different way. You know, when there's 8 legs and I was thinking if people didn't know how to count by 4, I was splitting 4 in half to make two on each side. Then I did 2 X 8 equals 16 then I have to count by 2s up to 32 what it equals. I have to count by 2s 16 times [Tom, G1, 1:00].

A few children could not explain their thinking and another few gave an explanation of their thinking as if telling a story. In examining the knowledge that experienced mathematics teachers access to operate effectively, Ainley and Luntley (2005, p.78) made a distinction that may be pertinent here. Teachers were shown episodes of videotapes of their classrooms and in these interviews some teachers gave an "account for" rather than an "account of" their actions. The children who made up a story to suit the occasion may be doing the same thing or perhaps there is a different mechanism at work. No definitive statements could be made based on the evidence collected here all that can be said is that 3 (6%) children made up a fiction to match the video.

Category of response	Frequency as a percent (n = 53)
1. None	6
2. "Account for" or gave an invented story	6
3. Explicit description of thinking	43
4. Reconstructs thinking, justifies, reasons, evaluates	42
5. Missing	4

Table 3. Children's Explanation of Their Thinking

Specify learning

Only 15% of children did not know what they learned in the mathematics lesson (see Table 4). The category of "nothing" proved unreliable because it became clear that young children translated "What did you learn today?" into "What new things did you learn today?" and these two questions are quite different. Therefore this category will not be discussed. Some children talked about behavioural learning, for example, "to share." Or they referred to non-mathematical things, for example the learning context, "talking about tools and building" [Michael, Jk2]. Totalling the first 3 categories of Table 5 shows that 30% of the children did not specify mathematical learning.

The three categories of most interest were those that made distinctions between learning factual information (15%), learning how to do something (23%) and learning at a conceptual level (21%).

About one third of the children who remembered facts talked in terms of numbers. For example, Annie who had been talking about measuring with a piece of string when asked what she learned said, "I learned that 9 + 11= 20." While it is not possible to be certain from these data, it raises a question as to what these young children think constitutes mathematics learning. Is learning mathematics equated to remembering numbers? Certainly the children interviewed for this research described their learning in detail. For example, Tom talked about his learning saying,

> Tom: I think I might have leant some new times tables.
>
> Interviewer: In which times table?
>
> Tom: I think some were in the, I think some were like 9 X 6. I didn't know that but then I knew it because I just counted by 6 nine times [G1: 6:36].

Some children learned how to do something, for example Jordan, who "learned how to count by nines." Another substantial proportion of the children (21%) reflected on their learning at a conceptual level. For example, Tahani reflected on a lesson where the teacher intended to introduce multiplicative thinking, saying she learned "about groups, to make groups and to count them altogether and I learned to count by 6s."

Category of response	Frequency as a percent (n = 53)
1. Unable to specify learning	15
2. Nothing "new"	9
3. Learned behaviour/ not mathematics	6
4. Remembered factual information	15
5. Learned how to do something	23
6. Specified a conceptual level of understanding	21
7. Missing	11

Table 4. Children's Learning

DISCUSSION AND IMPLICATIONS

Children could recall at least part of their conversations with the teacher during the day's mathematics lesson. These interactions appear to have some lasting effects. This is an important finding because I believe that interactions that challenge children to think about their mathematical understandings are a critical factor in their learning. Therefore knowing that many young children spontaneously remember these conversations and can reconstruct their thinking is an important finding.

The sophistication of their descriptions of events in the classroom were fairly evenly split between recounts of actions, descriptions of the event from the child's perspective and a description that involved some recount of their reasoning. It was surprising and impressive that such a large proportion of five- to seven-year-old children (42%) could reconstruct their thinking and justify it.

It is assumed that the experiences offered to children in mathematics classrooms contribute to their learning. These data indicate that 59% of children could talk about their learning as a result of the lesson--some at a factual level, some at a procedural level, and some at a conceptual level. Further research might investigate factors that influence different levels of understanding reported by young children.

It is also important for researchers to know that video-stimulated recall can be successfully used with five- to seven-year-old children.

References

Ainley, J. & Luntley, M. (2005). The role of attention in classroom practice: Developing a methodology. In P. Clarkson, A. Downton, D. Gronn, M. Horne A. McDonough, R. Pierce, & A. Roche (Eds.), *Building Connections: Research, Theory and Practice* (pp. 73-80). Melbourne: Mathematics Education Research Group of Australasia.

Australian Education Council. (1991). *A National Statement on Mathematics for Australian Schools.* Carlton: Victoria: Curriculum Corporation.

Clarke, D. & Cheeseman, J. (2000). Some insights from the first year of the Early Numeracy Research Project. In *ACER Numeracy Conference.* Melbourne: Australian Council for Educational Research.

Clarke, D. J. (2001). Negotiating meanings - An introduction. In D. J. Clarke (Ed.), *Perspectives on Practice and Meaning in Mathematics and Science Classrooms* (pp. 1-11). Dordrecht: Kluwer Academic Publishers.

Donaldson, M. (1978). *Children's Minds.* London: Fontana.

Cobb, P., Wood, T., Yackel, E., & McNeal, B. (1992). Characteristics of classroom mathematics traditions: An interaction analysis. *American Education Research Journal, 29*(3), 573-604.

Franke, M. L., & Carey, D. A. (1997). Young children's perceptions of mathematics in problem solving environments. *Journal for Research in Mathematics Education*, *28*(1), 8-25.

Gelman, R. & Gallistel, C. R. (1978). *The Child's Understanding of Number*. Cambridge, MA and London: Harvard University Press.

Hughes, M. (1986). *Children and Number.* Oxford: Basil Blackwell Ltd.

McDonough, A. (2002). *Naive Yet Knowing: Young Learners Portray Beliefs About Mathematics and Learning*. Unpublished PhD thesis, Australian Catholic University, Melbourne.

McDonough A.,& Clarke D. M. (2003). Describing the practice of effective teachers of mathematics in the early years. In B. J. N. A. Pateman & J. Z. Dougherty (Eds.), *Proceedings of the 27th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 261-268). Honolulu, HI: College of Education, University of Hawaii.

ELEMENTARY STUDENTS' CONCEPTIONS OF STEEPNESS

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In this study, we interviewed Boston-area students in Grades 2 through 7 to explore their informal knowledge of slope. We are interested in both what these students know as well as what their conceptual difficulties are as they develop an understanding of steepness. Specifically, this study investigates the question, "Which dimensions do students attend to and neglect when describing steepness?" We found that students are able to identify the steeper of two ramps or lines quite accurately; however, they have difficulties accurately describing how different dimensions of the incline contribute to steepness. These results inform teachers and curriculum developers of preconceptions and conceptual difficulties students have before taking algebra in middle school.

INTRODUCTION AND REVIEW OF RESEARCH LITERATURE

Understanding mathematics can lead to personal and professional success. The National Council of Teachers of Mathematics (NCTM, 2000) advocates that all students should study algebra. Since 2000, many states have aligned their graduation requirements with NCTM guidelines, creating a national expectation for students to pass a test covering material learned in an Algebra 1 course ("No Child Left Behind Act of 2001", 2002).

Algebra is a gatekeeper for academic success, and the algebraic topic of linearity is a gatekeeper for other algebraic concepts such as quadratic and exponential relationships (Yerushalmy, 1997). Algebra 1 students in the US, Israel, and Korea performed most poorly on linearity test questions asking for slope of a line on the coordinate plane (Greenes, Chang, & Ben-Chaim, 2007). Other studies found these conceptual difficulties: steepness and height are different, steepness is constant along an incline, slope is a ratio of differences (Cates, 2001; Lobato & Siebert, 2002).

Children who have had the opportunity to experiment with steepness may understand it to a much better extent than we see in schools. Therefore, we believe we can better prepare young children for the study of slope in middle school. The complexity of attainable cognitive tasks develops with age (Frye & Zelazo, 1998). Relational complexity (RC) theory defines complexity as the *arity* of the relation— the number of independent dimensions or variables represented concurrently. Unary relations are defined by a single attribute. If steepness is a single variable, then identifying the steeper ramp is a binary relationship. A more formal understanding of slope involves three variables: horizontal distance, vertical distance and the numerical value of slope. ternary task, attainable at a median age of 5 years (Halford, Wilson, & Phillips, 1998a). Comparing the steepness of two ramps or lines by comparing their horizontal and vertical measures involves a quaternary relation, normally attainable at an age of 11 years (Wood, 1988).

Based on cognitive complexity theory alone, as children enter school, they should be developmentally ready to accurately work with the ternary concept of slope of a ramp or line. By the end of elementary school, they should be able to compare slopes in situations where none of the measurements are held constant. Between grades 1 and 5, students can explore situations where at least one of the dimensions (hypotenuse, vertical or horizontal distance) is constant. Experience with different contexts of steepness can increase students' abilities to understand slope (Halford, Wilson, & Phillips, 1998b).

Most children have had experiences with steepness through building ramps, sliding down slides, or riding a bike up hills. Yet, research of teaching practices in the elementary schools says that pre-existing knowledge is often ignored when slope is introduced, preventing students from making connections between slope and prior knowledge (Fuson, Kalchman, & Bransford, 2005).

The first step in incorporating the mathematics of steepness into elementary schools includes creating activities and explorations which prepare the students for their middle school study of linearity. First, we need to establish what students in different grades know about steepness without instruction.

METHODOLOGY

Semi-structured clinical interview is our primary method of data collection. We have designed the interview protocol and handouts (See http://web.mit.edu/dianasc/www) to guide the interviewer through the required setup of manipulatives and questions to ask. Since the questions build one on the other and have an internal conceptual order, the protocol is quite prescriptive. However, the interviewers were encouraged to ask further questions to elucidate students' thinking.

The interview protocol consisted of five sections: an introduction and four categories of tasks. The *Concrete*, *Imagine*, *Picture*, and *Lines* sections asked the student to identify which of the two ramps presented was steeper and why. Some of the tasks required the student to construct ramps; others also required the student to draw a picture of the ramps. The student's choice of the steeper ramp was coded as *correct* or *incorrect*. For each task, we asked students to explain how they knew that the chosen ramp was steeper in two different ways. We coded their explanations in two ways: *explanation accuracy* and *conceptual category*. The codes for explanation accuracy were *correct* or *incorrect* based on whether the student used a plausible explanation to support his or her answer. The conceptual category codes included: *vertical*, *horizontal*, *hypotenuse*, *incline*, *area/space under ramp*, *speed*, *combinations*, and *other vocabulary*. We collected information from students through several sources: oral descriptions, drawings of the ramps, physical constructions, and worksheets.

We interviewed eight students attending schools in the Boston area. All of the coded interviews showed fragile understanding of steepness and provided over 250 instances of explanations of steepness. There was no evidence of correlation between the number of explanations and students' grade level (squared Pearson correlation coefficient = 0.06). The number of explanations per task also did not correlate with the student's grade (squared Pearson correlation coefficient = 0.09). There was no evidence of significant differences between the task and explanation accuracy scoring schemes (Chi Square statistic = .85, 1 df, p > .35). Using these methods of triangulation, we show that we obtained equivalent data from students across grade levels and that our two coding strategies showed similar results.

ANALYSIS

We found a number of surprising results from our data. We found no evidence of correlation between grade level of the student and his or her accuracy on the tasks. Task accuracy ranged from 71 to 88% and the squared Pearson Correlation Coefficient is less than 0.01.

Accuracy of explanations ranged from 45 to 90% and was also independent of the student's grade (the squared Pearson Correlation Coefficient was 0.04). We can conclude that regardless of age within the Grade 2-7 range, students are relatively accurate in determining which ramp is steeper, but have difficulties providing accurate explanations.

Accuracy of explanations did differ drastically from one task category to another. As we discussed earlier, the *Imagine* task was the most challenging for students. This is not surprising, as the questions in this category required the students to determine what information they would need in order to be able to know which ramp was steeper.

For example in the first *Imagine* scenario, we asked students whether a 20-inch board or a 10-inch board made a steeper ramp. All but one of the students claimed the 10-inch board was steeper. The correct answer is that they would need to know at least one more measure: angle, vertical height, or horizontal distance. Our second *Imagine* scenario asked if students were able to determine which ramp was steeper: one held up by 13 videos or one held up by 12 videos. The correct answer is that they would need to know at least one more measure: angle, ramp / hypotenuse length, or horizontal distance. All of our interview subjects believed the ramp with 13 videos was steeper.

Our list of the conceptual categories of students' explanations is: Incline, Vertical, Horizontal, Hypotenuse, Combinations, Area/Space under Ramp, Speed, and Other Vocabulary.

The category *Incline* includes instances where the students used synonyms or antonyms of "steep" to explain their reasoning. Sample words are: level, flat, tilt, slant, angle, diagonal, steep, pointing up. This category was the most accurately used category, with an accuracy level of 94%.

Explanations in the *Vertical* category included references to the number of videos in the tower or its height. *Vertical* was the most frequently used category which included 40.3% (104/258) of the responses. Approximately 78% of the times when students related steepness to vertical height, they were correct.

Students used *Horizontal* distance in their explanations very infrequently (6%) and inaccurately (53% correct). The difference between the accuracy of the *Vertical* and *Horizontal* explanations was significant as shown by the Chi Square test ($\chi^2 = 4.19$, df =1, $p < 0.041$), showing that students naturally form a more accurate understanding of how the vertical distance affects slope than how the horizontal distance affects it.

Some students focused on the length of the ramp, categorized as *Hypotenuse*. The two boards were the same length, but we created different hypotenuse lengths by sliding the board in and up, creating an overhang. In mathematical drawings and graphs, lines are assumed to extend infinitely. Arrows are often drawn on the end indicating that the lines go on forever. Therefore, basing the slope on the line's length is conceptually inaccurate. In fact, the slope of a line is constant regardless of the segment length. Only 11% of student explanations used the hypotenuse, and these were only 59% accurate, showing no significant difference in accuracy from the explanations using Horizontal distance ($\chi^2= 0.11$, d.f.=1, $p>0.7$).

Combinations of categories lead to a more formal understanding of steepness, namely slope. Slope is a combination (or a ratio) of the vertical and horizontal distances between any two points on a line. Every student, except the 2nd grader, correctly used a combination of categories in at least 10% of explanations.

There are several valid combinations of categories that determine steepness; using incline by itself as an explanation is sufficient, so we analyze the correct non-incline combinations. The most frequently used correct non-incline combination was vertical and hypotenuse (32%), which could be explained by the way we constructed the ramps: the only physical objects in our set-up were the board and the tapes potentially emphasizing the hypotenuse and vertical measures, respectively. It is possible that our manipulatives de-emphasized the horizontal measure.

21% of the correct combinations described by the students included the vertical and horizontal distances. Although none of the students used them in a ratio, this was the closest that they came to formalizing their conceptions of steepness into the idea of slope. Only 7% of the combinations were between the Horizontal and Hypotenuse measures. Explanations including combinations of measurements were 77% accurate, which is not significantly different from the accuracy of the Vertical explanations ($\chi^2=0.01$, df=1, $p>0.90$).

Many of the examples of combinations being used incorrectly happened in the *Imagine* part of our interview. In these questions, the students were given insufficient information and were asked to determine which of the ramps was steeper. The student would need to understand what pieces of information were missing and use them to argue their conclusion.

Four of the eight students used the *Area underneath the ramp* to explain at least one of the scenarios, with explanation accuracy 45%. The area underneath two lines is not a determining factor of steepness or slope, but in our scenarios with finite board lengths, such an explanation could be used correctly. The danger is that this reasoning cannot be extended to more general situations.

An object's speed depends also on the time it spends accelerating down the ramp. Therefore the object's final speed depends on the steepness and the length of the ramp. Using *Speed* alone to justify steepness of a ramp is incomplete, and this misconception is problematic for infinitely long lines. Only 33% of the responses coded under "Speed" were correct; this confirms our idea that the use of speed in relation to steepness can confuse students.

We tried to limit the amount of responses we coded in the *Other Vocabulary* category. One example is: "if someone were to be driving a car over it or skating over it … they would actually like land right here on the tape." This response does not fit under any other category.

Our data show that there are a number of dimensions that elementary and middle school students use to justify their reasoning about slope. The paths of reasoning are displayed in the concept map below.

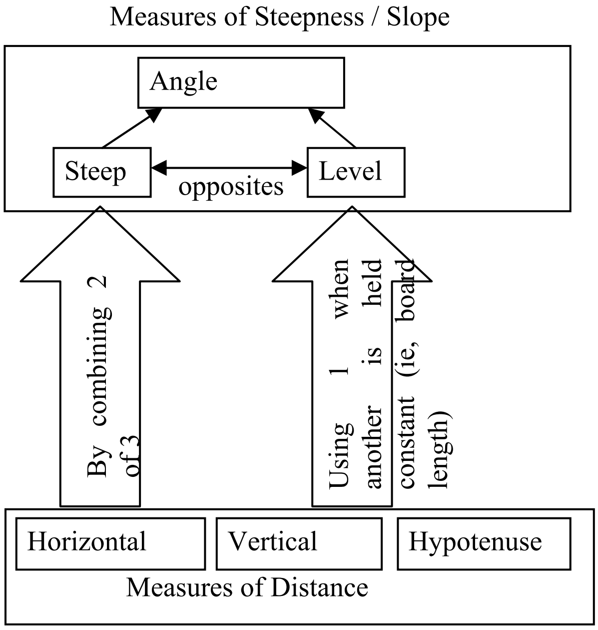

Figure 1. Concept map of the Relationships between Conceptual Categories.

Starting with the top of the concept map, we see the central mathematical idea in our research, steepness or slope. The steepness of a line is a holistic measure of the incline of the line. Slope is a mathematically defined measure of steepness: the ratio of differences of the y-coordinates (Δy) and x-coordinates (Δx) of two points on the line. The angle that a steeper line forms with the horizontal measures closer to 90 degrees, and has a slope value closer to 1. A line that is less steep will be 'flatter'; its angle with the horizontal will be closer to 0 degrees and its slope will be closer to 0. Students often supported their answers by discussing a bigger angle, more tilt, more diagonal up, etc. They were not looking at the numerical value of slope, but they were relying on their intuitive ideas of steepness.

The bottom of the concept map shows the measures of distance that can be involved in the calculation of slope or angle: horizontal, vertical, and hypotenuse distances. At least two of the three of these variables must be given in order to make mathematical conclusions about steepness. If one of the variables is held constant between two scenarios, only one other variable is needed in order to draw conclusions about the relative steepness of the two ramps. For example, if the hypotenuse is held constant (as in our questions using two boards of the same length) then the height of two ramps alone determines which of the ramps is steeper.

CONCLUSION AND FUTURE RESEARCH

The results from this study address the dimensions that students attend to and neglect when describing steepness. We showed that students most frequently refer to the vertical height of the ramp when explaining their conclusions about steepness. They also use the incline of the ramp in their justifications, as well as the hypotenuse length. To a lesser extent, they use the horizontal distance, as well as the predicted speed with which an object would roll down the ramp. Another explanation of interest is the concept of area under the ramp as an indicator of steepness. In addition, students also naturally combine some of these dimensions. Some of these combinations are redundant, while others can be used as basis for defining the mathematical concept of slope as a ratio.

All of the children had a strong intuitive understanding of steepness in familiar contexts and fragile understanding in less familiar contexts. According to RC theory, all of the students should have been capable of working with the ternary tasks that we presented them in this interview. It is possible that the students who did not successfully identify the steeper ramps in the *Concrete* section had less familiarity with the ramps in general. The only tasks that could be classified as quaternary were the *Imagine* questions where none of the dimensions were held constant. It is not surprising that students had much more difficulties completing the *Imagine* tasks.

Even when the students were able to correctly identify the steeper ramp, many used only one dimension (ie, Vertical) to describe its steepness, instead of using a combination of two features (ie, Vertical and Horizontal). Identifying two features to determine steepness is a much more complex cognitive task. When students identified a correct combination of two features, 40% of the time they used the angle of the ramp as one of the features, which is redundant.

This study had weaknesses based on our physical setup of the scenarios. Vertical height was created using a three-dimensional stack of videos and the hypotenuse was represented by the board. None of the students used grid marks on the interviewing table to describe horizontal distance. In the future, we could use a product which has equally salient horizontal and vertical dimensions.

A stronger connection needs to be made between students' experiences with ramps, understanding of the components that define steepness, and their understanding of slope. Our goal is to prepare students for the study of algebra and we must confirm a connection between our suggested experiences and their success. This study has generated many more questions than it answered. However, it has been of tremendous value to us in elucidating some of the preconceptions and misconceptions that the students bring to our classes.

References

Cates, B. (2001). CBL activities and students' conceptual knowledge of functions [Electronic Version]. Retrieved October 1, 2007 from http://archives.math.utk.edu/ICTCM/VOL14/C023/paper.html.

Frye, D. & Zelazo, P. D. (1998). Complexity: From formal analysis to final action. *Behavioral and Brain Sciences, 21*, 836-837.

Fuson, K. C., Kalchman, M., & Bransford, J. D. (2005). Mathematical understanding: An introduction. In M. S. Donovan & J. D. Bransford (Eds.), *How Students Learn Mathematics in the Classroom* (pp. 217-256). Washington, D.C.: The National Academic Press.

Greenes, C., Chang, K. Y., & Ben-Chaim, D. (2007). *International Survey of Secondary School Students' Understanding of Key Concepts of Linearity.* Paper presented at the 31th Annual Meeting of the International Group for the Psychology of Mathematics Education, Seoul, Korea.

Halford, G. S. & Andrews, G. (2004). The development of deductive reasoning: How important is complexity? *Thinking and Reasoning, 10*(2), 123–145.

Halford, G. S., Wilson, W. H., & Phillips, S. (1997). Abstraction: Nature, Costs, and Benefits. *International Journal of Educational Research 27*(1), 21-25.

Halford, G. S., Wilson, W. H., & Phillips, S. (1998a). Processing capacity defined by relational complexity: Implications for comparative, developmental, and cognitive psychology. *Behavioral and Brain Sciences, 21*, 803–865.

Halford, G. S., Wilson, W. H., & Phillips, S. (1998b). Relational complexity metric is effective when assessments are based on actual cognitive processes. *Behavioral and Brain Sciences, 21*, 848-864.

Lobato, J., & Siebert, D. (2002). Quantitative reasoning in a reconceived view of transfer. *Journal of Mathematical Behavior, 21*, 87-116.

NCTM. (2000). *Principles and Standards for School Mathematics*. Reston, VA: National Council of Teachers of Mathematics.

No Child Left Behind Act of 2001, (2002).

Wood, D. (1988). *How children think and learn*. Cambridge, MA: Basil Blackwell, Ltd.

Yerushalmy, M. (1997). Designing representations: Reasoning about functions of two variables. *Journal for Research in Mathematics Education, 28*(4), 431-466.

AN STUDY ON LEFT BEHIND STUDENTS FOR ENHANCING THEIR COMPETENCE OF GEOMETRY ARGUMENTATION

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More than one third of Taiwan junior high students do nothing in a 2-steps geometry proof question after 5 weeks of geometry formal proof lessons. Our previous study show that they are weak in the crucial competence named hypothetical bridging. In this study, we develop the step-by-step unrolled reasoning strategy to help these so-called left behind students. The results show that this strategy can help most of left behind students to do 3-steps of familiar computational question.

INTRODUCTION

The learning and teaching of geometry argumentation in Taiwan

The learning content concerning geometry argumentation in Taiwan is considerably abundant in the elementary and junior high school. The geometry lessons mainly focuses on finding the invariant properties of kinds of geometric figures and apply these properties to solve or prove problems. Include that to find out the measure of an angle or segment, to judge the relationship of one pair of lines or shapes, or to prove a statement or proposition.

The formal deductive approach of argumentation in geometry is introduced in the second semester of grade 8 after teaching the congruence conditions of triangles. In the beginning, students learn how to apply one property to show that a geometry proposition is correct, that is, to infer the wanted conclusion by one acceptable property under the given condition. If two or more properties are necessary in a proof problem the textbook divides the problem into a sequence of single-step proof tasks. In the first semester of grade 9 the students learn how to construct a deductive proof with two or more steps. In particular, they learn how to chain single steps into a proof.

The teaching style in Taiwan junior high school is basically lecturing. Most of the teachers teach geometry lessons by exposition to about 30 students in one classroom. And the geometry proof task is basically treated as writing the reason of a given proposition by applying learnt properties.

In Taiwan, the elementary and junior high education is compulsory, the national curriculum ask all level of students to learn geometry argumentation, including formal proof. Moreover, there is only one version of items in the Junior High Basic Competency Test for entrance into senior high school. In such kind of learning environment, even the low level students have to learn formal geometry proof.

The performance of left behind students in geometry proof

In December 2002, the National Science Council (NSC) conducted a nation-wide survey to investigate Taiwanese junior high students’ competences of mathematical argumentation. The survey asked the grade 9 students, while they had just learnt formal proof in geometry lessons, to construct a proof in a 2-steps unfamiliar question (as Fig1). Students' proof was analysed and evaluated by the project team members including math educators, mathematicians and school teachers. The results show that there is 24.6% of them can construct acceptable proof, 35% of them are able to recognize some crucial elements to prove but missing some deductive process or the concluding step, and 37.4% of them do not have any response in this question (Lin, Cheng and linfl team, 2003). These no response students are named 'left behind' students. As we know now from the results of national wide survey, they learnt nothing in formal proof.

A is the center of a circle and AB is a radius.
C is a point on the circumference where the perpendicular bisector of AB crosses the circle.

Write your answer in a way that would get you the best mark you can.

Triangle ABC is always equilateral.

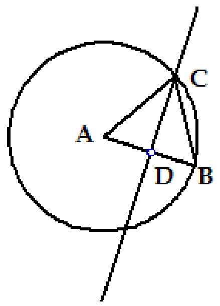

Fig1. The 2-steps unfamiliar question in the national-wide survey.

In our previous study (Cheng, Y. H. and Lin, F. L., 2007), we develop the 'reading and colouring' strategy to help our grade 9 students to enhance their geometry proof performance. The results show that this strategy enhances the quality distribution of multi-steps geometry proof. In reading and colouring class, 60.6% of the students are coded acceptable in the post test. It is quite better than traditional class (30.3%) and than the national survey results (24.6%). Nevertheless, we also find out that this strategy is less-effective to lower 40% of students. Although the reading and colouring strategy can help them to do more trial reasoning, but no one construct acceptable proof in the post test. That is, no matter the traditional or improved reading and colouring strategy can not help these left behind students in geometry proof tasks.

LEARNING DIFFICULTY OF LEFT BEHIND STUDENTS

The process of constructing a multi-steps proof

It is clear that the mental processes of constructing a geometry proof depend on students' individual competence and on the requirements of the concrete proof task. As described in models of the proving process (Boero, 1999), or in cognitive research like conceptual understanding (Vinner, 1991), or constraints in the scientific thinking process of students (Reiss & Heinze, 2004) are influencing the process of constructing a proof.

Healy & Hoyles (1998) propose that the process of constructing a valid proof involves two central mental processes:(1) to sort out what is given, which properties are already known or can be assumed and what is to be deduced, and (2) to organize the necessary transformation to infer the second set of properties from the first into a coherent and complete sequence. Duval (2002) propose a two level cognitive features of constructing proof in a multi-steps question. The first level is to process one step of deduction according to the status of premise, conclusion, and theorems to be used. The second level is to change intermediary conclusion into premise successively for the next step of deduction and to organize these deductive steps into a proof.

A standard geometry proof question in junior high geometry lessons and tests is of the form 'Given X, show that Y' with a figure which the figural meaning of X and Y are embedded in (fig(X,Y). When a student face to a proof question, the information include X, Y, fig(X,Y), and the status (Duval, 2002) of X (as the premise) and Y (as the conclusion). The proof process is to construct a sequence of argumentation from X to Y with supportive reasons. This process can be seen as a transformation process from initial information to new information with reasoning operators such as induction, deduction, visual judgment… (Tabachneck & Simon, 1996). So, we may say that to prove is to ***bridge*** the given condition to wanted conclusion by acceptable mathematical properties.

In a single step proof question, the student might retrieve a property 'IF P then Q' which condition P contain the premise X and result Q contained in Y and finish the proof. We may say this kind of bridging is ***simple bridging***.

The proof process in a multi-steps proof question is much more complex. Since there is no one property can be applied to bridge X and Y. The student has to construct an intermediary condition (IC) firstly for the next reasoning. The IC might be reasoned a step forwardly from X. It is an intermediary conclusion (Duval, 2002) inferred from X as a new premise to bridge Y. Or, it might be reasoned a step backwardly from Y. It is an intermediary premise reasoned from Y as the wanted conclusion to bridge X. So, the first step in a multi-step proof may be a goalless inferring from X and concluding many reasonable intermediary conclusions. The next step is to go on the bridging process to Y by selecting a new premise from the intermediary conclusions. Or, it may be a backward reasoning from Y and finding many reasonable intermediary premises and the next step is to set up a new conclusion from the intermediary premises and going on the bridging process from X. No matter this kind of reasoning is constructed by forward or backward reasoning, it is essentially a process of conjecturing and selecting/testing. We may say this kind of reasoning process is ***hypothetical bridging***.

In summary, constructing an acceptable geometry proof can be seen as a bridging process from given conditions to a wanted conclusion with inferring rules controlled by a coordination process. This includes (1) to understand the given information and the status of these information, (2) to recognize the crucial elements which associate to the necessary properties for deduction, (3) especially in multi-steps proof, to construct intermediary condition for the next step of deduction by hypothetical bridging, and (4) to coordinate the whole process and organize the discourse into an acceptable sequence.

Based on the theoretical analysis presented before we hypothesize that the difficulty of typical proof tasks in junior high school can be determined to a large extend by the distinction into single-step and multi-steps proof.

The limitation of left behind students in constructing geometry proof

It is not easy to analyse the difficulties of left behind students in constructing geometry proof because they write down nothing. We find out some cognitive characteristics of them in our reading and colouring teaching experiment.

As we mention above, even the reading and colouring strategy is more effective than traditional teaching, there are still 40% of students can not construct an acceptable proof after learning it. The post analysis based on the performance of hypothetical bridging in the pretest show that the competence of hypothetical bridging is a crucial element in learning geometry proof (Cheng, Y. H. and Lin, F. L., 2007). In this sense, the performance of students shows that all the acceptable proof constructed by the students who are able to reason with hypothetical bridging. No one of non-hypothetical bridging students can do it. This result shows that if the students' understanding of geometry proof is only restricted in the first level (Duval, 2002) of proving, that is applying one theorem to bridge the premise and conclusion, then they are not able to learn to construct an acceptable proof.

STUDY DESIGN

The aim of the study

In this study, we develop a teaching strategy to help left behind students to develop the competence of hypothetical bridging in geometry argumentation tasks.

The step-by-step unrolled reasoning strategy

We design the teaching strategy based on the principle of continuity of learning. That is, it takes into account the cognitive characteristics of left behind students. Bell(1993) proposed some principles for designing diagnostic teaching for adapting students' misconception. These principles focus on the consideration of cognitive status, such as the task should be related to students' experience and easily to promote the misconception, and operative tool for adaptation, such as immediate feedback of correctness and intensive activities for consolidating new correct concepts. These principles show that designing the learning strategy for enhancing left behind students should focus on students' cognitive status: they are not able to construct intermediary condition(s) in multi-steps argumentation.

Boero (1999) describes an expert model of completing a proof task. This model distinguishes different phases of constructing a proof. The phases are (1) the production of a conjecture. (2) The precise formulation of the statement. (3) the exploration of the conjecture, the identification of mathematical arguments for its validation, and the generation of a rough proof idea. (4) the selection and combination of coherent arguments in a deductive chain, (5) the organization of these arguments according to mathematical standards, and sometimes (6) the proposal of a formal proof. This expert model indicates that the final proof as solution of a proof task gives only an incomplete representation of activities performed during the proving process. Since the left behind students are weak in hypothetical bridging. We should not start the learning activities in the traditional form of “given X, show that Y”. In the sense of Boero, some kind of conjecturing activities based on open-ended reasoning might be better.

After exercising a “thought experiment”(Gravemeijer, 2002) between each possible strategies reported on the literatures (eg. Antonini, 2000; Hoyles et al, 1995; Douek et al, 1999; Reiss, 2005; quoted from Lin, F. L., 2005), we develop the ‘step-by-step unrolled reasoning strategy’ for our left behind students. We give the students a ‘covered’ argumentation task, unroll the first condition to the students, ask students to infer what should be true under such given condition. And then we unroll the second condition, ask students to infer what should be true under such given conditions and conclusion from the first step of inferring, and so on. Moreover, any kind of helpful materials are allowed such as coloured pens, ruler and compass, note of geometry properties and so on in order to reduce the difficulty in learning the heavy subject.

The samples

A questionnaire with four items are developed and tested as pre-test in 5 classes of grade 9 students after they learnt the chapter of formal multi-steps geometry proof. One of the items are single step and three are multi-steps. The students’ performance in these items is coded into three types: hypothetical bridging, simple bridging, and no response. We identify a student is left behind if there is not no response in all items and without hypothetical bridging performance in any one item. After the analysis of performance in the pretest, 40 students are identified as left behind and 25 of them agree to join our experiment. We regroup these students in to an extra class after the regular lessons. We lose some of them because of the self or family reasons and finally 11 of them left. In this paper, we only report the results come from these 11 students.

The process

The whole experiment divided into 4 sections according to the learning content. They are (1) triangle, (2) quadrangle, (3) congruent triangles, and (4) parallel lines. In every section, at first we review the geometry properties of this section which taught in regular lessons. The second step is a learning activity of a step-by-step unrolled reasoning task in one question. It spends about 100 minutes in every section and the experiment last 6 weeks.

A post test is conducted after the 6 weeks of experiment. There are 3 multi-steps questions. One item asks the students to construct the formal proof (as Fig.1) and the other two asks students to find out the measure of unknown angles (as Fig.2). The performance of students is coded into acceptable, incomplete, improper, intuitive response, and no response (Lin, Cheng and linfl team, 2003). We code in this way in order to know the effectiveness of step-by-step unrolled reasoning strategy in constructing geometry argumentation.

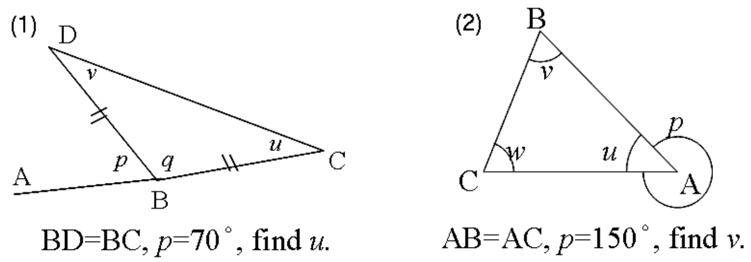

Fig. 2. The computational questions in the post test.

After 4 weeks of post test, we conduct a delay post test with three items of multi-steps of finding out the measure of pointed angles (as Fig.3). The performance of students is mainly coded in the number of correct conclusions from inference in order to know the retaining effectiveness of hypothetical bridging competence under the step-by-step unrolled reasoning strategy.

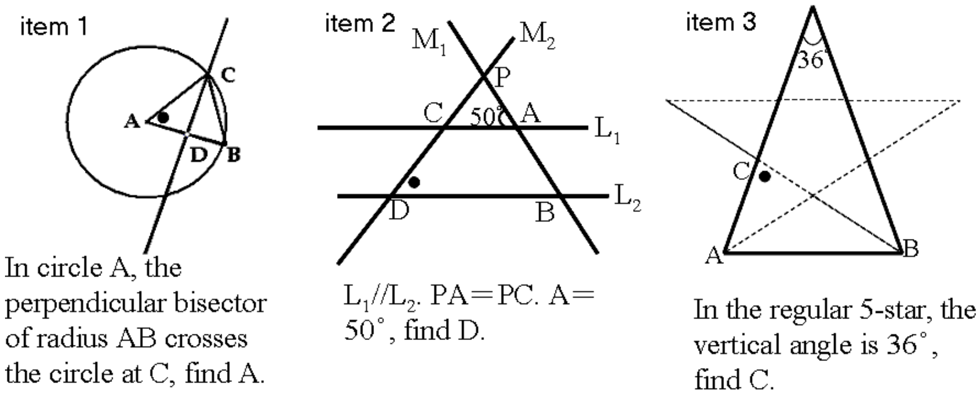

Fig. 3. The computational questions in the delay post test.

RESULTS

The step-by-step unrolled reasoning strategy can help 9/11 students to do 3-steps question which without complex property or knowledge

The performance of samples in the post test shows in Table 1. It shows that the effectiveness of step-by-step unrolled reasoning strategy is not consistence in the three items. In the first question, the formal proof one, only 1 student constructs an incomplete proof. We do not category the others' performance into the coding system because the students seem to do some goalless reasoning, they just write down as many sentences as possible. It seems that the step-by-step unrolled reasoning strategy is not helpful to enhance the performance of constructing formal proof. The performances in the two 3-steps computational questions are different. In item (1), all the properties and knowledge necessary are familiar to the students and 9/11 of students do it correctly. Nevertheless, item (2) is posed in an unfamiliar situation. The external circum-angle and its measure are not familiar to the left behind students and 7/11 of them then do nothing or 'create' a property to do this question. From the results showed in Table 1 we may say that the step-by-step unrolled reasoning strategy can help 9/11 students to do 3-steps question which without complex property or knowledge. Moreover, this strategy seems to be not helpful to formal proof task. It tells us to pay more attention to the different features between computation and proof task in geometry argumentation tasks.

Item of	acceptable	incomplete	improper	not hypothetical bridging
2 steps proof		1		
3 steps computation (1)	9		2	
3 steps computation (2)	4			7

Table 1. The performance of the samples in the post test

The step-by-step unrolled reasoning strategy can help 9/11 students to develop the competence of hypothetical bridging

The performance of samples in the delay post test shows in Table 2. It shows that only 2/11 of samples can not construct intermediary condition for the next step of reasoning in item 1. We may say that The step-by-step unrolled reasoning strategy can help 9/11 students to develop the competence of hypothetical bridging. The number of correct answers shows that the item 1, which modified from the formal proof question, is more difficult. The interview shows that many students can not do this question because '*the conditions tell me nothing about the angles, how can it be possible to find the measure of angle?*'. This response shows an interesting conception about mathematics question of our left behind students. It also needs to pay more attention to know this level of students.

number of correct conclusions from students' inference	0	1	2	3	4	correct answer in this item
in item 1	2	3	1	3		3
in item 2	0	1	8			7
in item 3	1		2		6	6

Table 2. The performance of the samples in the delay post test

DISCUSSION

We conduct an initial study on left behind students to help them to develop higher competence in geometry argumentation. Although the size of samples is small and the results seem not so consistent, it is obvious that the step-by-step unrolled reasoning strategy is effective in developing the competence of hypothetical bridging. In Taiwan, the test format in the Junior High Basic Competency Test, the only one test for entrance into senior high school, is single-choice. All argumentation tasks have to transfer into the form of 'choosing the right answer'. Most of the formal proof tasks then have to transfer into the form of 'finding pointed measure'. Our experiments shows that the step-by-step unrolled reasoning strategy can help most of left behind students to do 3-steps computational question which without complex property or knowledge. So, although is not effective in formal proof task, it is valuable in our junior high education for passing the examinations.

References

Bell A. (1993). *Some Experiments in Diagnostic Teaching Educational Studies of Mathematics*. 24, 115-137

Boero, P. (1999). Argumentation and mathematical proof: A complex, productive, unavoidable relationship in mathematics and mathematics education. *International Newsletter on the Teaching and Learning of Mathematical Proof 7/8.*

Cheng, Y.H. & Lin, F.L (2007) The effectiveness and limitation of Reading and coloring strategy in learning geometry proof. R*esearch Report in PME 31*. Seoul, Korea.

Duval, R. (2002). Proof understanding in MATHEMATICS: What ways for STUDENTS? In *Proceedings of 2002 International Conference on Mathematics: Understanding Proving and Proving to Understand,* pp. 61-77.

Gravemeijer, K. (2001). Developmental Research, a Course in Elementary Data Analysis as an Example. Common Sense in Mathematics Education. In *Proceedings of 2001 The Netherland and Taiwan Conference on Mathematics Education*, 19-23 November 2001.Taipei, Taiwan.

Healy, L. & Hoyles, C. (1998). Justifying and proving in school mathematics. Summary of the results from a survey of the proof conceptions of students in the UK. R*esearch Report Mathematical Sciences*, Institute of Education, University of London.

Lin, F. L., Cheng, Y. H., & linfl team (2003): The Competence of Geometric Argument in Taiwan Adolescents. *International Conference on Science & Mathematics Learning*. 2003.12.16-18.

Lin, F. L. (2005). Modeling Students' Learning on Mathematical proof and Refutation. *PME 29 plenary speech*.

Reiss, K. & Heinze, A. (2004). Knowledge Aquisition in Students' Argumentation and Proof Processes. In G. Törner, R. Bruder, A. Peter-Koop, N. Neill, H.-G. Weigand, & B. Wollring (Hrsg.), *Developements in Mathematic Education in Germany. Selected Papers from the Annual Conference on Didactics of Mathematics, Ludwigsburg 2001* (S. 107-115). Göttingen: Universitätsbilbiothek.

Tabachneck, H. & Simon, H.A. (1996). Alternative representations of instructional material. In D. Peterson (Ed.), *Forms of representation*. UK: Intellect Books Ltd.

Vinner, S. (1991). The role of definitions in teaching and learning of mathematics. In D. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 65-81), Dordrecht: Kluwer.

SAMPLE SPACE: AN INVESTIGATIVE LENS

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This study continues research in probability education by altering a well-known problem, and examining students' responses from novel perspectives. More specifically, students are asked to compare the likelihood of sequences for five flips of a coin. Alternative set descriptions of the sample space—all of which are based upon subjects' verbal descriptions of the sample space—show that normatively incorrect responses to the task are not devoid of correct probabilistic reasoning. The study further demonstrates that alternative set descriptions of the sample space can act as an investigative lens for research on the comparative likelihood task, and probability education in general.

Jones, Langrall, and Mooney's (2007) recent synthesis of probability education literature in the *Second Handbook of Research on Mathematics Teaching and Learning* (Lester, 2007) states: "With respect to probability content, the big ideas that have emerged…are the nature of chance and randomness, sample space, [and] probability measurement (classical, frequentist, and subjective)" (p. 915). The objective of this article is to explore the union of the three big ideas, and demonstrate that they are inextricably linked. Students' verbal descriptions of events are taken into consideration during the analysis of written responses through alternative set descriptions of the sample space. In doing so, alternative set descriptions of the sample space will be suggested as a possible theoretical framework for research in probability education. A task often found in probability education literature—the comparative likelihood task—will act as the medium of exploration.

THE COMPARATIVE LIKELIHOOD TASK

While the Comparative Likelihood Task, hereafter referred to as CLT, can take on many forms, the framework is often essentially the same. Sequences are produced from some type of binomial experiment conducted a certain number of times, and the chances of either of the outcomes occurring are the same (e.g., flips of a coin, or the birth of boys or girls to a family). Two or more sequences are presented in a multiple-choice format and students are asked to determine which of the sequences are less (or more) likely to occur.

According to Tversky and Kahneman (1974), "[p]eople rely on a limited number of heuristic principles which reduce the complex tasks of assessing probabilities and predicting values to simpler judgmental operations" (p. 1124). Application of the *representativeness heuristic*—"in which probabilities are evaluated by the degree to which A is representative of B, that is, by the degree to which A resembles B" (p. 1124)—leads to a number of errors, or biases. The representativeness bias known as *misconceptions of chance* is when "people expect that the essential characteristics of the process will be represented, not only globally in the entire sequence, but also locally in each of its parts" (p. 1125). For example, Tversky and Kahneman (1974) established that subjects found the sequence of coin flips HTHTTH more likely than HHHTTT, because the latter sequence did not "appear" random, and HTHTTH more likely than HHHHTH, because HTHTTH was a representative ratio of heads to tails. The caveat: Normatively, each of the sequences is equally likely to occur.

Researchers in mathematics education have also worked with the CLT. For example, Shaughnessy (1977) found the sequence BGGBGB was considered more likely than the sequences: BBBGGG and BBBBGB. With the new "supply a reason" element brought to the task, Shaughnessy was able to determine that subjects found BBBGGG was not representative of randomness, and BBBBGB was not a representative ratio of boys to girls.

A number of other researchers in mathematics education (e.g., Cox & Mouw, 1992; Batanero & Serrano, 1999; Falk, 1981; Green, 1983; Konold, Pollatsek, Well, Lohmeier, & Lipson, 1993; Rubel, 2006) have worked with variations of the CLT. For example, Falk (1981) determined that randomness was perceived according to frequent switches, and thus short runs. As research has continued on the CLT, researchers in mathematics education have found that students' responses for one sequence being less likely than another stem from two reasons—the ratio of heads to tails, and the perceived randomness of the sequences—each of which stem from the representativeness heuristic.

THEORETICAL FRAMEWORK

One possible explanation of students' incorrect responses to the CLT is that "subjects hold multiple frameworks about probability, and subtle differences in situations activate different perspectives [which] can be employed almost simultaneously in the same situation" (Konold, Pollatsek, Well, Hendrickson, & Lipson, 1991, p. 360). In recognition of this point, this study contends that the traditional sample space is not a sufficient theoretical framework for analysis of the responses to the CLT. Given that, "experimenter and subject will conceptualize different sample spaces or different frames which may provide the impetus for misinterpretation of the data" (Keren, 1984, p. 122), the notion of researchers considering the subjects' different sample spaces provides a new perspective to responses from the CLT; and, furthermore, shows that incongruous answers to the task are a product of the lens with which they are being investigated.

"Identification of the sample space is extremely important since different sample spaces (of the same problem) may lead to different solutions" (Keren, 1984, p. 122). Moreover, events, or subsets of the sample space, can have verbal descriptions and set descriptions. A verbal description of "flipping at least two tails" corresponds to the set description of {{HTT}, {THT}, {TTH}, {TTT}}, and a "a run of two" corresponds to {{HHT}, {THH}, {TTH}, {HTT}}. Thus, responses to the CLT may be analysed against a variety of set descriptions of the sample space; and the particular sample space used for analysis can be based upon verbal clues provided by the student, because "in order to evaluate subjects' responses it is necessary to know what sample space they are employing" (Keren, 1984, p. 123).

METHODOLOGY: TASK AND PARTICIPANTS

Participants in this study were thirty-eight prospective elementary teachers enrolled in a "Methods for Teaching Elementary Mathematics" course, which is a core course in the teacher certification program. The task was presented prior to the introduction of probability to the course.

Students were presented with the following task: *Which of the sequences is (are) least likely to result from flipping a fair coin five times: (A) H H T T H (B) H H H T T (C) T H H H T (D) H T H T H (E) T H H T H (F) All sequences are equally likely to occur. Provide reasoning for your response.* While the wording is similar to the Konold et al. (1993) wording of the task, this new iteration maintains the ratio of heads to tails in all sequences in an attempt to control for ratio responses to the task.

RESULTS

Of the 38 people who completed the task, 27 stated that all sequences were equally likely to occur; however, 5 chose B as least likely and 6 chose D least likely. Sample Response Justifications for B (i.e., HHHTT) and D (i.e., HTHTH) are presented.

Response justifications for HTHTH:

John: D is least likely to occur because the chances of having the coin land on the opposite side each time to create a pattern of HTHTH are very slim, the longer the pattern the less likely it will be. Also, to get 3 H's in a row [sequence B] is probably next least likely.

Kate: I believe there is a 50/50 chance that the first flip will be a heads or a tails. Therefore, I believe that D is least likely to occur b/c the odds of flip a coin from heads to tails is fairly slim.

Jack: With D, an alternating sequence could occur but not necessarily in this order, H + T are more likely to occur at a more random interval.

Hurley: Although there is a 50% chance of getting a H or a T. It is very unlikely that you can get a sequence of alternating sides randomly. The probability of this sequence happening would be the least likely.

Claire: 1st choice: (F) All have the same likelihood of occurring is what I think. It's random. 2nd choice: (D) The chances of a nice tidy pattern like these seems unlikely.

Sawyer: (D) is least likely to occur because with a 50/50 chance it is unlikely that the results will be alternating H/T with each coin flip. It is more likely that the results would be random.

Response justifications for HHHTT:

Boone: (B) because getting three in a row of one type is less probable than the other options of alternating or only two in a row.

Libby: (B) b/c what are the chances to get three H's in a row, and two T's after that?

Charlie: because it is very unlikely that there will be the 3H in a row and then 2T.

Nikki: I thought that it wasn't likely to be on one side for three flips and then the other for the rest.

Shannon: 5 times, and chances are most likely to be 3H 2T or 2H 3Ts. For the sequences they will more likely to be scrambled, because that's fact. I've tried few times, scrambled.

ANALYSIS OF RESULTS

"Obviously, it is possible to consider more than one different set of possible outcomes for an experiment" (Peck, 1970, p. 115). As such, analysis of "incorrect" responses from those who chose HHHTT and HTHTH least likely will be analysed via three alternative set descriptions of the sample space: switches, longest run, and switches and longest run.

Switches sample space

Based upon the verbal descriptions of: John, Kate, Hurley, Claire, and Sawyer a more appropriate, or natural, set description for comparison, corresponding to their verbal descriptions, would be a sample space partitioned according to switches (shown in Table 1), and not to the normative set description (i.e., thirty-two equally likely outcomes) of the sample space.

0 Switches	1Switch	2Switches	3Switches	4Switches
HHHHH	***HHHTT***	HHTHH	HHTHT	***HTHTH***
TTTTT	TTTHH	TTHTT	TTHTH	THTHT
	TTHHH	***HHTTH***	HTHHT	
	HHTTT	TTHHT	THTTH	
	HHHHT	HTTHH	***THHTH***	
	TTTTH	THHTT	HTTHT	
	THHHH	HHHTH	THTHH	
	HTTTT	TTTHT	HTHTT	
		HTHHH		
		THTTT		
		THHHT		
		HTTTH		
P(0S)=2/32	P(1S)=8/32	P(2S)=12/32	P(3S)=8/32	P(4S)=2/32

Table 1. Switches sample space (S denotes switch)

John's verbal response that "the chances of having the coin land on the opposite side each time to create a pattern of HTHTH are very slim," analysed via the switches set partition of the sample space, is correct because the probability of having four switches in five flips of a coin is 2/32 (i.e., P(4S)=2/32). In fact, of all the options presented in the task (emboldened in Table 1) HTHTH is the least likely sequence to occur. Thus, while an incorrect response is derived from the responses being compared to the normative set description, a correct response coupled with insightful probabilistic reasoning is found when compared to a more appropriate set description based on the verbal description presented.

Longest run sample space

Based upon the responses of Boone, and Libby their verbal descriptions of the sample space also do not correspond to the normative set description of the sample space. The verbal descriptions presented imply a more appropriate set description would be a sample space organized according to the length of run, and is shown in Table 2.

LR5	LR4	LR3	LR2	LR1
HHHHH	HHHHT	***HHHTT***	HHTHT	***HTHTH***
TTTTT	TTTTH	TTTHH	TTHTH	THTHT
	THHHH	TTHHH	HTHHT	
	HTTTT	HHTTT	THTTH	
		HHHTH	***THHTH***	
		TTTHT	HTTHT	
		HTHHH	THTHH	
		THTTT	HTHTT	
		THHHT	HHTHH	
		HTTTH	TTHTT	
			HHTTH	
			TTHHT	
			HTTHH	
			THHTT	
P(LR5)=2/32	P(LR4)=4/32	P(LR3)=10/32	P(LR2)=14/32	P(LR1)=2/32

Table 2. Longest Run sample space (LR denotes longest run)

When the response that HHHTT is least likely, because a run of length three is less likely, is compared to the longest runs partition of the sample space, the response is correct in stating that longer runs are less likely (i.e., $P(LR5)<P(LR4)<P(LR3)$). As such, HHHTT would be considered less likely (but not least) among the sequences presented. Boone's response that "three in a row of one type is less probable than the other options of alternating [i.e., HTHTH] or only two in a row [i.e., HHTTH, THHTH]," is also correct when compared against the longest runs sample space. Again, correct responses coupled with insightful probabilistic reasoning are derived when responses are compared to set descriptions that more closely correspond to the verbal descriptions presented. However, when responses are compared to the longest runs sample space HTHTH is in fact least likely, and HHHTT and THHHT are equally likely. The latter point is considered with another set equivalent to verbal descriptions of the sample space.

Switches and Longest run sample space

According to Konold et al. (1991) the multiple frameworks held by a subject activate different perspectives, which can be employed almost simultaneously in the same situation. Unfortunately, when pitted against the normative set description of the sample space this subtlety is ignored. However, when compared to a set description equivalent, based upon verbal descriptions, responses can be shown to possess very subtle innate probabilistic reasoning, which considers more than one factor at a time. For example, Libby, Charlie, and Nikki's verbal descriptions of the sample space do not correspond to the normative set description of the sample space, nor do they correspond to the switches set description, nor to the longest runs set description of the sample space. These verbal descriptions correspond to a more nuanced set description of the sample space. Based upon, for example, the response "because it is very unlikely that there will be the 3H in a row and then 2T," a set partitioned into switches and the longest runs would be a more apt set description for the verbal descriptions of Libby, Charlie, and Nikki, as seen in Table 3.

0 Switch & Longest Run 5	1 Switch & Longest Run 3	1 Switch & Longest Run 4	2 Switch & Longest Run 2	2 Switch & Longest Run 3	3 Switch & Longest Run 2	4 Switch & Longest Run 1
HHHHH	***HHHTT***	HHHHT	HHTHH	HHHTH	HHTHT	***HTHTH***
TTTTT	TTTHH	TTTTH	TTHTT	TTTHT	TTHTH	THTHT
	TTHHH	THHHH	***HHTTH***	HTHHH	HTHHT	
	HHTTT	HTTTT	TTHHT	THTTT	THTTH	
			HTTHH	***THHHT***	***THHTH***	
			THHTT	HTTTH	HTTHT	
					THTHH	
					HTHTT	
P(0S&LR5) = 2/32	P(1S&LR3) = 4/32	P(1S&LR4) = 4/32	P(2S&LR2) = 6/32	P(2S&LR3) = 6/32	P(3S&LR2) = 8/32	P(4S&LR1) = 2/32

Table 3. Switches and Longest Run sample space

When the response that HHHTT is least likely is compared to the switches and longest runs partition of the sample space, the sequence HHHTT is in fact the second least likely sequence of the five to occur. Further, when HHHTT is compared to THHHT, but switches and runs are taken into consideration (based on the verbal descriptions of events) the subjects are correct in saying that HHHTT is less likely than THHHT, even though they both possess a run of length three. Moreover, Nikki's response that "I thought that it wasn't likely to be on one side for three flips and then the other for the rest" is correct when compared to the switches and longest runs set description corresponding to the verbal descriptions provided by the subjects. Once again, subtle, perhaps innate, probabilistic reasoning is exposed through alternative set descriptions of the sample space.

DISCUSSION

Certain research found in probability education literature has, at its core, a comparison between observed data and the normatively correct answer. Moreover, there exists a tone of sacrosanctity to the normative solution, despite the existence of inherent issues (e.g., "the classical definition is essentially circular, since the idea of 'equally likely' is the same as that of 'with equal probability' which has not been defined" (Lipschutz & Schiller, 1998, p. 88)). Further, responses to the CLT are considered correct by stating "that" the sequences are all equally likely to occur, and research has not been concerned with the issue as to "why" the sequences are equally likely to occur. One possible reason for the "that" and "why" distinction having yet to be addressed, is because research in probability education often focuses on incorrect responses to the CLT. However, if the focus on research is to remain on the incorrect responses, the framework for analysis of the incorrect responses should not be "stuck" on the set representation of the normative sample space. In other words, the framework for analysis of responses must evolve beyond the normative solution.

Much like in an experiment involving light, different combinations of light act as a way to investigate colour. A red, blue and green flashlight will provide a number of different perspectives when the lights are singularly, and in combination, shone on an object. In a similar fashion, alternative sample space consideration can shed new light on the CLT. By shining the normative light alone, all one can determine is whether a subject is correct or incorrect, and there exists little explanatory power. However, by combining the lenses of the different partitions of the sample space for analysis of the CLT, it is revealed that when students are determining which of the sequences are least likely to occur, subjects are, in fact, reasoning probabilistically. Different combinations of lenses, along with different numbers of lenses, will allow for multiple perspectives and proper investigation to show how subjects are reasoning probabilistically. Perhaps then the call put forth, that "we need to know more about how students do learn to reason probabilistically" (Maher, Speiser, Friel, & Konold, 1998, p.82) can be addressed.

CONCLUSION

Jones, Langrall, Thornton, & Mogill (1997) state: "research evidence with regard to sample space is conflicting and highlights the need to study […] thinking in this construct more comprehensively, and *within a probability context*" (p. 105, this author's italics). In line with this point of view, this study would be remiss to conclude that individuals who chose sequences HHHTT and HTHTH as least likely are wrong. Instead of saying the individuals were wrong, it would be more appropriate to say that the individuals are wrong, when comparing their responses to the normative solution. In other words, while the CLT has a caveat that, normatively, each of the sequences is equally likely to occur, an ensuing caveat is that subjects' responses to the CLT are incorrect only when compared to the one particular sample space for which the caveat applies. Alternatively stated: "A simple comparison between the normatively correct answer and the observed data has little explanatory power" (Keren, 1984, p.127).

References

Batanero, C. & Serrano, L. (1999). The meaning of randomness for secondary school students. *Journal for Research in Mathematics Education, 30(5),* 558-567.

Bennett, D. J. (1998). *Randomness.* Cambridge, MA: Harvard University Press.

Cox, C. & Mouw, J. T. (1992). Disruption of the representativeness heuristic: Can we be perturbed into using correct probabilistic reasoning? *Educational Studies in Mathematics, 23(2),* 163-178.

Falk, R. (1981). The perception of randomness. In *Proceedings of the Fifth Conference of the International Group for the Psychology of Mathematics Education* (pp. 222-229). Grenoble, France: University of Grenoble.

Green, D. R. (1983). A survey of probability concepts in 3000 pupils aged 11-16 years. In D. R. Grey, P. Holmes, V. Barnett, & G. M. Constable (Eds.), *Proceedings of the First International Conference on Teaching Statistics* (pp. 766-783). Sheffield, UK: Teaching Statistics Trust.

Jones, G. A., Langrall, C. W., & Mooney, E. S. (2007). Research in probability: Responding to classroom realties. In F. K. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning,* (pp. 909-955). New York: Macmillan.

Jones, G. A., Langrall, C. W., Thornton, C. A., & Mogill, A. T. (1997). A framework for assessing and nurturing young children's thinking in instruction. *Educational Studies in Mathematics, 32*, 101-125.

Keren, G. (1984). On the importance of identifying the correct 'problem space'. C*ognition, 16*, 121-128.

Konold, C., Pollatsek, A., Well, A., Hendrickson, J., & Lipson, A. (1991). The origin of inconsistencies in probabilistic reasoning of novices. In D. Vere-Jones (Ed.), *Proceedings of the Third International Conference on Teaching Statistics* (Vol. 1, pp. 357-362). Voorburg, The Netherlands: International Statistical Institute.

Konold, C., Pollatsek, A., Well, A., Lohmeier, J., & Lipson, A. (1993). Inconsistencies in students' reasoning about probability. *Journal for Research in Mathematics Education, 24(5)*, 392-414.

Lipschutz, S., & Schiller, J. (1998). *Introduction to Probability and Statistics*. New York: McGraw-Hill.

Maher, C. A., Speiser, R., Friel, S., & Konold, C. (1998). Learning to reason probabilistically. In *Proceedings of the 20th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. (Vol. 10, pp. 82-88). Raleigh: NC.

Peck, L. C. (1970). *Basic Mathematics for Management and Economics.* Glenview, Illinois: Scott, Foresman and Company.

Rubel, L. H. (2006). Students' probabilistic thinking revealed: The case of coin tosses. In G. F. Burrill & P. C. Elliott (Eds.), *Thinking and Reasoning with Data and Chance: Sixty-eighth yearbook* (pp. 49-60). Reston, VA: National Council of Teachers of Mathematics.

Shaughnessy, J. M. (1977). Misconceptions of probability: An experiment with a small-group, activity-based, model building approach to introductory probability at the college level. *Educational Studies in Mathematics, 8,* 285-316.

Tversky, A. & Kahneman, D. (1974). Judgment under uncertainty: Heuristics and biases. *Science, 185,* 1124-1131.

ISSUES ASSOCIATED WITH USING EXAMPLES IN TEACHING STATISTICS

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Real-world examples, including media items that involve data interpretation, are advocated as a motivator for learning statistics. Moreover, students' ability to interpret real-world data is regarded as a significant outcome of statistics education. The issues associated with using such examples have, however, received little attention. This paper draws on cases from the literature and our own research to raise issues associated with the use of real-world examples in the teaching of statistical concepts. These reveal the importance of content knowledge and pedagogical content knowledge at every step of the process of using an example, but also highlight the need for more systematic study of the issues.

INTRODUCTION

The use of examples in teaching is a well-established practice. In statistics, teachers often utilise a particular data set to illustrate how to calculate measures such as the mean or investigate concepts such as correlation. Many educators have advocated the use of real-world examples as a motivator for learning and suggest that the newspaper, for instance, is a good source of examples for teaching quantitative thinking, especially statistics. Many curricula around the world emphasise interpreting real-world data and situations, so children's school experiences should include these (associated issues are discussed in Watson, 2006). But how trivial is the task of choosing *good* examples? Real-world examples are not inherently good: their real-world status does not always mean useful for teaching or valid statistically. In fact, and ironically, some examples from the media that have significant statistical shortcomings are especially valuable for teaching. Finally, even well-chosen examples are not necessarily easy to implement effectively in the classroom.

In this paper we put forward the case that choosing an appropriate real-world example, identifying what opportunities it offers for teaching, and capitalising on these opportunities when planning for and implementing the example in the classroom, requires complex skills. It certainly involves a deep interplay between content knowledge (CK) and pedagogical content knowledge (PCK). We will begin by reviewing the definitions of *example* and *affordance*, before hypothesising about where and how CK and PCK come into play when recognising affordances and using examples in the teaching of statistics. As there appear to be no studies that have examined the teacher's role in detail, we will do a simple meta-analysis of situations from the literature and our own research to highlight some of the practical complexities associated with teachers choosing and using examples.

BACKGROUND

Examples and affordances

For clarity we define *example* as "a specific instantiation of a general principle, chosen in order to illustrate or explore that principle" (Chick, 2007, p. 5). While there is an extensive literature on examples, with a good overview presented by Bills, Mason, Watson, and Zaslavsky (2006), these authors also point out that there has been very little research done on teachers' *choice* of examples. Indeed, this literature, and the related literature on model-eliciting activities where students develop a mathematical model to address a real-world situation (e.g., Lesh, Amit, & Schorr, 1997), often focuses on the example/task itself rather than how teachers make use of it (see Doerr & English, 2006, for an exception). The role of the teacher, however, can be significant; the cases of two teachers who used a probability game in quite different ways (see Chick, 2007) illustrate this point. If teachers are being encouraged to use contemporary real-world data sets as a stimulus for learning, then their capacity to recognise the usefulness of an example and then decide how to exploit that example in the classroom is critical.

To illustrate the issues simply, consider a birthday party, where the ages of people attending are 5, 6, 8, 9, 9, 9, 10, 13, 37, 71. This situation can be used to illustrate a number of important statistical principles (e.g., measures of central tendency, range, variation, the idea of outliers). The significant issue for teaching is whether or not the teacher can see the possibilities an example offers, or, alternatively, construct an appropriate example to illustrate a desired concept. The term *affordances* (Gibson, 1977) is useful here. It refers to the perceived uses that someone can determine for an object. The data set above, for instance, has affordances for teaching about how to evaluate the median when there is an even number of data points. In Gibson's definition, there is emphasis on the *user's* perceptions, and it is likely that there are affordances that one person will see and another will not. This is particularly significant in teaching. Chick (2007) suggests that it might be useful, especially in education, to refer to *potential affordances*, to highlight that an example may have some inherent use that may or may not be noticed, depending on the expertise of the user. For instance, some teachers may not identify that the data set above affords the opportunity to talk about *when* the median is a good measure of central tendency.

Of course, the data set of birthday party ages above only becomes an example when a teacher puts it to use in the classroom. There must be an activity involving some interaction with the data, and the teacher has to know what that interaction is intended to illustrate. It is at this point—where the example is employed and discussed in the classroom—that it becomes a *didactic object* (Thompson, 2002, p. 198), with the potential to demonstrate the desired general principle or concept.

Teaching with examples

There seem to be three key stages of decision-making for the teacher in using examples to teach statistical principles (or principles in general). The first involves *choosing* the example (or, possibly, constructing one), which requires CK and attention to the potential affordances, as well as PCK to determine if the example is appropriate for the audience. The second stage involves *planning* to use the example in the classroom, where PCK is needed to turn the example into an effective didactic object that illustrates the concept and CK ensures that concepts are addressed correctly. The final stage involves *implementing* the plan in class, and calls on both CK and PCK to keep the lesson on course and deal with issues that arise.

Each stage demands considerable teacher expertise. At the first stage-choosing an example-the teacher might be in one of two situations: the required topic/principle may be known and what is required is an example to illustrate it, or else the teacher might notice a real-world situation and recognise affordances for some topic applicable to a lesson in the future. In the first case, CK is required to identify the attributes of an example that will illustrate the concept. If the principle is correlation, for instance, then data involving two potentially related variables is required, so the teacher must construct or find a data set with the required attributes. In the second case, CK is required to recognise what principles the found example affords. Just finding a newspaper article is not sufficient for teaching effectively; the teacher must be able to identify the affordances, and determine its suitability or adaptability for the class. Ball (2000), talking about mathematics teaching more generally, discussed how tasks (or examples) must be examined by the teacher to determine what they afford students. She pointed out that a significant part of a teacher's work is to decide how to make tasks easier or harder-which is an aspect of PCK-or use them to illuminate certain ideas (see also Chick, 2007).

This work continues in the second stage, where CK and PCK is required when deciding how to introduce the example and use it effectively in the classroom. In particular, the decisions that the teacher makes here will affect how evident the key statistical concepts appear—or, in other words, will govern the power of the example as a didactic object. This stage is not, however, the final step; the third stage of actual teaching brings its own challenges. What a teacher plans to do and then actually does may differ because of things that occur in the actual process of teaching. Students' responses and questions may demand unanticipated additional explanation from the teacher, which again requires teacher expertise in both CK and PCK. There is also the potential for mismatches between intention and implementation: a teacher may, for instance, plan to follow the call of Watson and Chick (2004) to model the kinds of questions that students ought to consider when examining data, but may do so in only a limited way or using poor questions. There needs to be a detailed examination of what occurs at each of these stages, and what CK and PCK are required.

USING EXAMPLES TO TEACH STATISTICS

In order to conduct a preliminary exploration of the issues associated with the use of examples in teaching statistics, we will examine a number of cases arising from published studies and our own research. The cases have been chosen to illustrate the issues that arise at different stages in the example-using process, as discussed above.

They involve statistics teaching at the primary and lower secondary level, where we know teachers may not always have the content expertise of their more specialist counterparts. One reason for looking at a variety of cases is because no one appears to have examined, in a systematic way, the role of the teacher in travelling the path from choosing an example that can highlight a principle, planning how to use the example, and then actually implementing that example in the classroom. We hope these cases begin to shed light on areas that require closer investigation.

Choosing examples: Recognising affordances in a given example (Stage 1)

In our first case we investigate whether it is easy to recognise affordances inherent in an example and identify its potential for teaching. A small study we conducted with primary pre-service teachers (PSTs) examined whether they were able to identify the possible affordances of a rich real-world example (see Chick & Pierce, 2008). The PSTs were provided with an internet resource containing data about water storage levels in their locality over time. The researchers felt the data supported a wide range of statistical affordances, including graph and table reading, graph and table interpretation, hypothesising about explanations for the data, and extrapolation.

The PSTs were asked to imagine that they were teaching an integrated unit of work on "the environment" with a Grade 6 class, and to plan a lesson to teach students some aspect/s of statistics using the water data information. In order to help the PSTs move towards the planning stage, and to investigate their capacity to see affordances, we first asked them to identify statistics topics that *could* be addressed using the water data resource. Whereas all PSTs mentioned some aspect of interpreting or producing graphs as possible affordances, in a few cases the suggestions were vague or inappropriate. Furthermore, although well over half of the PSTs listed some terms associated with measures of central tendency, only one student explicitly identified a relevant situation where these concepts were applicable. The PSTs were also asked to pose questions that Grade 6 students could consider with the water data. Nearly all the PSTs suggested questions involving straightforward data reading and/or interpretation. Less than half, however, identified questions requiring higher levels of reasoning such as extrapolating the data or considering implications.

This study showed that PSTs could recognise the straightforward affordances present, but struggled to identify the possibilities for higher order thinking. There is implicit evidence that the PSTs may have had insufficient CK to interrogate the data well enough to understand it themselves and ascertain what they could do with it, although we did not explicitly examine their CK. Alternatively, the simple suggestions might reflect a tendency to underestimate the capacity of school students (a facet of PCK).

Choosing examples: Recognising affordances in self-chosen examples (Stage 1)

Whereas Chick and Pierce (2008) supplied PSTs with an example and had them identify affordances, Watson and Moritz (2002) asked their primary and secondary PSTs to choose their own example for teaching, as part of a subject assignment. The example had to be selected from the media with the view to allowing the PSTs to link quantitative literacy to daily life. Most of the PSTs chose examples that they perceived would be interesting and accessible to their students. When topics were chosen according to their potential for integration across the curriculum the questions posed by the PSTs often had a less mathematical focus. Watson and Moritz also noted that not all of the affordances in the media article were identified or brought to the fore clearly in the PSTs' suggested questions for students. This mirrors the results and issues of the Chick and Pierce study, despite the fact that this time the PSTs were choosing their own example instead of having it imposed on them.

Choosing examples: Creating an example with particular affordances (Stage 1)

In the previous cases the examples already existed but their affordances had to be identified. In the next case, the general principles were known in advance and the teacher (helped by the researchers) had to construct an example with the desired affordances. In an eight-week study of primary school students' thinking about distribution (Petrosino, Lehrer, & Schauble, 2003), the researchers and teacher knew what mathematical ideas they wanted to convey (ideas of centre and spread) and, with this knowledge, set out to construct an example that would suit their purpose. When the Grade 4 teacher asked the class to find the length of a pencil and the height of a flag pole the purpose was to harness a deep affordance of the example—namely how to deal with variations in measurements—and use this to introduce young students to the characteristics of distributions as a basis for comparing two sets of results. These ideas were later used in the engaging example of comparing the heights reached by two different-shaped model rockets.

Here the teacher's purpose went beyond the immediate problem of finding the length of a pencil or height of a flagpole; rather, these specific examples were used to illustrate more general principles, such as variation in measurement and how statistics can help deal with this, via central measures and variability measures. The successful introduction of these principles depended on the choice of example. The objects involved had contrasting size and needed different measuring techniques (ruler and "height-o-meter"). This steered students to focus on the possible levels of accuracy, and raised the need for a method of finding length from a number of measurements, while accounting for variation. The careful choice of the two examples, with different units and measuring devices, readily revealed the general principles.

Planning to use examples (Stage 2)

In both the Chick and Pierce (2008) and Watson and Moritz (2002) studies the PSTs were asked to prepare a lesson plan based on the real-world data example supplied or found. Chick and Pierce found that in most of the lesson plans based on the water storage data the key concepts were not clearly evident in the written plan. Nearly half of the PSTs did not capitalise on the given data at all but used the "rainfall data" situation as the impetus to have the class collect and graph their own rainfall data, without connecting the outcomes to the supplied data. Only three of the 13 lessons used the resource in a sustained and effective way to exemplify key statistical concepts. Watson and Moritz found similar results, also noting a failure to specify explicit mathematical content. They commented, however, that the variation in the quality of the lesson plans was no more than for other comparable tasks completed by the PSTs in their education program, suggesting that the difficulty of capitalising on the affordances in an example (assuming teachers have recognised them), and then planning to use them to good effect in the classroom is not confined to statistics. We acknowledge, however, that a written lesson plan may not reveal all that a teacher implicitly has in mind to bring out in the actual lesson. Nevertheless, it is of concern that so few deep statistical ideas were clearly articulated in the plans.

As a simple contrast, consider the case of Claire, a teacher involved in an investigation into teachers' PCK (see Chick, 2007, for part of this data). In one of her statistics lessons (not reported elsewhere to date), her plan included having "interpreting line graphs" as an explicit learning objective. She used a textbook as the source of her example-a graph showing the accumulating ticket sales for a fairground over the course of the day-along with some graph interpretation questions for students. In addition, however, Claire added her own "story writing" question at the end, which required students to describe their day as if they were the ticket seller. This plan allowed students to focus both deeply and broadly on interpreting the graph, in a more holistic way than the textbook's narrower questions. Claire's CK and PCK were rich enough to plan to use this example effectively.

Implementing an example in the classroom (Stage 3)

The final stage of example use is the actual classroom implementation. In the study by Watson and Moritz (2002) reported above the PSTs were meant to trial their selected examples and accompanying questions in the classroom. In many cases, however, new examples, more closely related to the school curriculum at the time of teaching, replaced those the PSTs had initially chosen, planned for, and discussed in their assignment. In this case the PSTs' PCK was sufficient to prompt a change in example to suit the current needs of their class. As the actual lessons were not observed in this case, the insights into implementing examples here are limited.

Claire, the teacher who conducted a lesson on graph interpretation, had the capacity to deal with issues that arose unexpectedly. During the lesson one student pointed out that the graph was always increasing (as it was an accumulating total), but Claire asked the students if they could think of a situation that might cause the graph to decrease. She also used spontaneously constructed, open-ended questions that helped students to focus on both the "big picture" and the more specific detail of the data. This reflected her CK, in identifying what concepts to focus on, and also PCK, in being able to target questions at her students' Zone of Proximal Development.

This task of responding, on-the-spot, to students' questions and answers in class is not trivial. During one lesson in the study of Petrosino *et al.* (2003) the class discussion of measuring spread led to the notion of a *spread number* to capture variation. Some students were unconvinced and suggested using the range instead. At this point the teacher and one of the researchers combined efforts to quickly compose a parallel bi-modal example, to illustrate the value of the spread number instead of the range as a measure of variability. It is not clear that the teacher could have constructed this spontaneously without the assistance of the researchers. This admits the possibility that teachers will not always have the CK or PCK required for impromptu responses such as this.

DISCUSSION AND CONCLUSION

The preliminary discussion and the cases examined above highlight a number of critical issues associated with using examples in teaching statistics. These merit more systematic study; in particular it would be illuminating to examine, for a number of teachers, the whole path from choice of example to classroom implementation. Our overview has not examined the extent of student learning that occurred based on the use of the example; this should also be a component of a larger study.

In conclusion, we highlight some of the questions that we have started to address. First, are the posited stages as critical as we think? We note that in some circumstances there will be overlap of the choosing and planning stages, because a teacher may be planning a lesson on a topic and will realise the need for a instructional example. That said, however, planning must still occur *after* the example has been chosen, in order to decide how to present/use the example effectively in the classroom.

Second, what kind of PCK and CK is needed at each stage? Because of space constraints we have not been explicit about the different facets of CK and PCK that are involved in using examples, although some of the cases discussed indicate the importance of knowledge such as (a) being able to assess cognitive demand, (b) awareness of students' current thinking, (c) curriculum knowledge, and (d) teaching strategies that make the concepts more evident.

Third, what are the problems associated with using examples? We might hypothesise that appropriate levels of CK and PCK are sufficient for effective example use. On the other hand, there may be difficulties inherent in the example that reduce its effectiveness (e.g., too much data, or data that are too complicated). It would also be useful to determine if there are pedagogical decisions that reduce the effectiveness of an example for conveying a concept.

Fourth, how do we empower teachers to use examples more effectively? Doerr and English (2006), in the field of model-eliciting tasks, provide an interesting suggestion that examples themselves—perhaps with some of the planning decisions already made—can develop teachers' CK and PCK. This, in turn, might increase teachers' capacity to use examples they have found or constructed for themselves.

Finally, although we have used the domain of statistics, we suspect that most of these issues apply across mathematics. The critical aspect here, and in certain areas of mathematics, is the existence of real-world examples in the "public" domain that are potentially "good for teaching". Clearly we require teachers with the capacity to identify the affordances in these examples and put them to effective use.

References

Ball, D. L. (2000). Bridging practices: Intertwining content and pedagogy in teaching and learning to teach. *Journal of Teacher Education, 51,* 241-247.

Bills, L., Mason, J., Watson, A., & Zaslavsky, O. (2006). RF02: Exemplification: The use of examples in teaching and learning mathematics. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Proceeding 30th Conference of the Interntational Group for the Psychology of Mathematics Education* (Vol. 1, pp. 125-154). Prague: PME.

Chick, H. L. (2007). Teaching and learning by example. In J. Watson & K. Beswick (Eds.), *Mathematics: Essential Research, Essential Practice. Proceeding 30th Conference. of the Mathematics Education Research Group of Australasia*, (pp. 3-21). Sydney: MERGA.

Chick, H. L. & Pierce, R. U. (2008). *Teaching Statistics at the Primary School Level: Beliefs, Affordances and Pedagogical Content Knowledge.* Paper to be presented at the joint ICMI/IASE conference on Statistics Education and School Mathematics, Monterrey, Mexico, June 2008.

Doerr, H. M. & English, L. D. (2006). Middle grade teachers' learning through students' engagement with modeling tasks. *Journal of Mathematics Teacher Education, 9*, 5-32.

Gibson, J. J. (1977). The theory of affordances. In R. Shaw & J. Bransford, (Eds.), *Perceiving, acting and knowing: Toward an ecological psychology* (pp. 67-82). Hillsdale, NJ: Lawrence Erlbaum.

Lesh, R., Amit, M., & Schorr, R. Y. (1997). Using "real-life" problems to prompt students to construct conceptual models for statistical reasoning. In I. Gal & J. B. Garfield (Eds.). *The Assessment Challenge in Statistics Education* (pp. 65-83). Amsterdam: IOS Press and The International Statistical Institute.

Petrosino, A. J., Lehrer, R., & Schauble, L. (2003). Structuring error and experimental variation as distribution in the fourth grade. *Mathematical Thinking and Learning, 5,* 131-156.

Thompson, P. W. (2002). Didactic objects and didactic models in radical constructivism. In K. Gravemeijer, R. Lehrer, B. van Oers, & L. Verschaffel (Eds.), *Symbolizing, Modeling, and Tool Use in Mathematics Education* (pp. 191-212). Dordrecht, The Netherlands: Kluwer.

Watson, J. M. (2006). *Statistical Literacy at School: Growth and Goals*. Mahwah, NJ: Lawrence Erlbaum.

Watson, J. M. & Chick, H. L. (2004). What is Uunusual? The case of a media graph. In M. Johnsen Høines & A. B. Fuglestad (Eds.) *Proceeding 28th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 207-214). Bergen, Norway: PME.

Watson, J. M., & Moritz, J. B. (2002). Quantitative literacy for pre-service teachers via the Internet. *Mathematics Teacher Education and Development, 4,* 42-55.

LANGUAGE AND MATHEMATICS: IMMIGRANT PARENTS' PARTICIPATION IN SCHOOL

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This research addresses the impact of language policies on parental engagement in their children's mathematics education. Our work is situated in primarily Mexican American working-class communities in the Southwest of the U.S. We focus on the implications of a restrictive language policy on immigrant Spanish speaking parents' interactions with their children (and their schools) around mathematics. In particular we raise issues about 1) potential loss of connection and even conflict between parents and children and 2) the role that language plays in the mathematics classroom placements of some of these children.

INTRODUCTION

One of the characteristics of reform-based mathematics education is its emphasis on communication. Students are expected to communicate their thinking about mathematics in writing and orally. Several researchers have written about communication and discussion-rich mathematics instruction in classrooms where students' home language is different from the language of instruction (Khisty, 2006; Moschkovich, 2002; Setati, 2005). Less has been written, however, about the implications of this emphasis on communication on the interactions between parents and children, in particular when the parents' language is different from the language of instruction. Setati (2005) writes about the political role of language and points out, "we must go beyond the cognitive and pedagogic aspects [of language] and consider the political aspects of language use in multilingual mathematics classrooms" (p. 451). In this report we build on this concept of language as political by looking at the impact of language policies on parental engagement in their children's mathematics education, in particular in the case of immigrant parents whose home language is different from the language in their children's schooling.

CONTEXT

For over ten years we have been working on issues related to mathematics education and parental engagement. Our work is situated in Latino (mostly of Mexican origin) working-class communities in the Southwest of the U.S. In our research we have addressed several themes that parents bring up in relation to their children's mathematics education, such as differences in the teaching and learning of mathematics in Mexico and in the U.S., valorization of knowledge, issues of language, and definitions of parental involvement (Civil, Planas, & Quintos, 2005). Our work draws on several bodies of research including research on parental involvement that critically examines issues of power and perceptions of parents, in particular minoritized and working-class parents (Horvat, Weininger, & Lareau, 2003; Pérez, Drake, & Calabrese Barton, 2005); and research on parents and mathematics, particularly that which takes into account culture, ethnicity, race, and context (Abreu & Cline, 2005; Jackson & Remillard, 2005).

Throughout this decade of working with parents we have witnessed the passing of a law that severely restricts bilingual education in the state in which our work takes place. The law allows teachers to use a minimal amount of the child's native language for clarification, but all children are to be taught in English and English Language Learners (ELLs) receive additional English language instruction. The passing of this law has to be seen within the larger political context in which issues related to immigration have taken a prominent role. The context of immigration, the role played by language, and the general living conditions of minoritized groups are realities that need to be taken into account when addressing these children's mathematics education.

The schools where our research takes place use reform-based materials that are quite demanding not only in terms of the mathematics (with topics that parents did not study in their own schooling such as data analysis and problem solving strategies), but also in terms of the language. Problems tend to be contextualized and often require a good command of English. In this report we focus on the implications of a restrictive language policy on immigrant Spanish-speaking parents' engagement in their children's mathematics education.

METHOD

Our research spans over two projects specifically aimed at parents (mostly mothers of Mexican origin) and mathematics education. We use mathematics workshops, courses in mathematics for parents, and mathematical "get-togethers," as settings to engage with parents not only in explorations of reform-based mathematics but also in conversations about mathematics education. Our sources of data include video and transcripts of many of these sessions; individual interviews as well as focus group interviews (audio or video taped); classroom visits in which parents and researchers observe a mathematics lesson and follow-up debriefing (video taped). For this paper we draw on two sets of data. One set is from the 16 sessions (1.5 hours per session) from the last year of the first project with a group of 14 mothers and 1 father. All the mothers in this group had been part of our parental engagement project for at least one year prior to these sessions. Thus, we had established a rapport with them and they had actually indicated that they wanted to continue with the workshops and the dialogues. The second set is from our current project and it consists of interviews and focus groups with a total of 15 parents from three different schools.

Our methodological approach is grounded on phenomenology (Van Manen, 1990), which relies heavily on participants' contributions to the experience and then strives to triangulate the data through multiple experiences and sources of data. The lived experience of each parent is considered significant. All interviews, focus groups, and workshop sessions were transcribed and analyzed using Glaser and Strauss (1967) constant comparative method. This process leads to the development of themes that inform our overarching research goal, which is to document Latino parents' perceptions about the teaching and learning of mathematics. As Van Manen (1990) writes, "themes describe an aspect of the structure of lived experience" (p. 87). A recurrent theme in our analysis is the effect of language policy on parents' engagement in their children's mathematics education. We address this theme first through two short cases (vignettes) and then through a more general discussion of interactions between parents and children.

THE CASES OF TWO MOTHERS

In this section we present two brief cases that illustrate different aspects of language policy. The first case highlights parents' concerns about communication with their children about mathematics because of language issues; the second case addresses the role of language in placement in mathematics classes.

The Case of Verónica

Verónica is a mother who had attended college in Mexico and had some teaching experience in that country. She has lived in the U.S. for several years and in fact her son had started school in the U.S. She has some understanding of English, but she identifies herself as primarily Spanish speaking. Her oldest son was placed in an English-only classroom in second grade by school recommendation. School personnel told Verónica that her son was getting confused in the bilingual classroom and not making progress. This placement affected her ability to participate in her son's schooling:

> I liked it while they were in a bilingual program, I could be involved… When he was in kindergarten … I even brought work home to take for the teacher the next day. In first grade it was the same thing, I went with him and because the teacher spoke Spanish, she gave me things to grade and other jobs like that. My son saw me there, I could listen to him, I watched him. By being there watching, I realize many things. And then when he went to second grade into English-only and with a teacher that only spoke English, then I didn't go, I didn't go.

Although Verónica stopped going to her son's classroom, she continued to support her two sons by attending school meetings, which were usually in English (though some translation was provided). About these meetings, she said, "I attend so that they [her sons] see that I am interested, but not because I think that I'm going to come back with something or that I'm going to understand." At the time of our study, her oldest son was in middle school (11 years old). She told us that she felt confident about her knowledge of mathematics to help him with his homework, however,

> When I sit with him to go over what he's doing, it's like he feels lazy about translating the problem for me. And when it's hard to translate he tells me that he'll just go early to school or will ask someone else, and that's something I don't like…. He is not sure that I am understanding the problem because it's written in English, I don't know how to read it and he doesn't know how to translate well, because he speaks Spanish, he reads

> Spanish, but because we have words and questions that we say differently, he thinks that I studied differently. … He's not sure of me because I don't speak English and he's not sure I am able to help him; "Son, it's mathematics." "Yes, mom, but…" I don't know if it's laziness or maybe he just doesn't find the words. He knows Spanish but when kids learn Spanish here, their vocabulary is not as developed and he doesn't translate like he should so that I'm able to help him.

The situation is particularly upsetting for Verónica because she feels she knows the content and could help her child but her child does not trust her knowledge. Underlying this is the issue of academic language. Because her son had been schooled in English since 2nd grade, he did not have a command of academic Spanish, thus making it harder for him to speak about mathematics in Spanish. This is something that we have documented in interviews with children ages 10 -12 who speak Spanish at home but have been schooled in English. These children have to be able to explain and translate the problems to their parent; this is a process that involves proficiency in the mathematics register in two languages (Moschkovich, 2002). Despite these obstacles, Verónica was determined to support her children. For example, she used the school's after-school tutoring to make sure that they could receive the support (in English) that they needed for the homework.

Verónica's case highlights several issues that we see reflected in other parents. A language policy that basically makes English the language of schooling has limited parents' participation in the schools. We wonder about the equity implications of these language policies at a time in which current educational policy asks for increased parental involvement and mathematics teaching and learning are particularly language rich.

The Case of Emilia

Our work takes place in elementary and middle schools. At the middle school level, what we see happening is a school within a school in which ELLs are kept apart from non-ELLs for many of the core subjects. Almost ten years ago, Valdés (2001) described a similar situation and pointed out that through this "two schools in one" ELLs had very few opportunities to interact with students whose primary language was English. This same situation was echoed recently by Emilia, a mother in our project. Emilia arrived almost three years ago to the U.S. We first interviewed her and her oldest son (Alberto, 11 years of age) shortly after their arrival. At that time, they both talked about how the mathematics he was seeing at his current school, he had already studied it in Mexico and that his main problem was with learning the language ("and here they teach me things that they taught me there; it's just that here it's hard because of the English") (See Civil (2006) for more on this case). This was not a surprise to us since our interviews with immigrant parents consistently document a feeling among these parents that the level of mathematics education in Mexico is higher than what their children are studying in their current school. What caught our attention was Emilia's comment about her child learning things that he already knew:

> That is, for them it's perfect what they are teaching them because in this way it's going to help them grasp it, to get to the level, because for them, with the lack in English that they have, and if to that we were to add, … If they give them all the information, like a lot, very dense, too much teaching during this period, to tell you the truth, it would disorient them more. Right now, what he is learning, what I see is that it's things that he had already seen, but if he gets stuck, it's because of the language, but he doesn't get stuck because of lack of knowledge

How aware are immigrant parents of the process by which their children are placed in mathematics classes? We are concerned about the thinking behind these placements. Emilia seemed to think that this was good for her son because he would not be overwhelmed with having to learn both language and content. This was two years ago. More recently, we interviewed Emilia again. She appeared satisfied with her children's progress in mathematics, although as the interview went on she pointed out what we mentioned earlier, that her children seemed to be interacting mostly with other Spanish-speaking students. She also brought up a concern for how little homework her children seemed to be doing and noticed that her two sons who are supposedly in different grades would sometimes bring the same homework. The reality is that the push for learning English is such that schedules are made around this priority, at the expense in some cases of the learning of content such as mathematics. As Valdés (2001) points out, "students should not be allowed to fall behind in subject-matter areas (e.g., mathematics, science) while they are learning English" (p. 153). We do wonder about the (in)equity implications of some of these placements.

PARENTS' AND CHILDREN'S INTERACTIONS

Verónica is concerned about the conflict she feels between her oldest son and herself around mathematics. She feels she can help him in terms of her content knowledge but language somehow gets in the way. Other parents have brought up the language issue in being able to help their children with homework in mathematics:

> Candida: Well, I remember that they would give her homework in English and in Spanish, and so I could help her a little more. But when it was all in English, no. Then I couldn't. I felt bad. I would be very frustrated because I couldn't explain it to them, I would have liked to explain it to them and I couldn't. I was frustrated.
>
> Selena: Sometimes I cannot explain it to him because I hardly know English. There are things that he reads to me and he translates them into Spanish; sometimes I understand what he's telling me in English, but others, definitely I don't understand anything.
>
> Lucrecia: it was difficult for us, that the boy did the homework, because we didn't know English and were not able to translate the problem for him.

As we mentioned earlier the nature of the homework in many of the reform-based materials is likely to put demands on parents' knowledge of mathematics as well as of English. We are aware that it is often hard to separate what is due to content and what is due to language, but our focus here is on data that illustrate the role that (English) language plays in these parents' access to their children's tasks. We are not referring just to typical word problems, but tasks in which the instructions are rather complex; tasks in which the students are asked to provide written explanations; tasks in which they are given graphs, tables and other representations that assume a certain knowledge of how to interpret these and that tend to have quite complex sentences.

There is another side to this language policy that we want to address here. So far we have talked about how a limited knowledge of the language of instruction (English in this case) may affect parents' participation in their children's school, the most obvious way being as they try to help them with homework. But what about the effect of this policy on the children? Interactions between children and parents concerning each other's mathematical knowledge can be a way for all those involved to gain knowledge. But these interactions can be made particularly difficult when children need to bridge not only school and home knowledge but also different languages. We have seen that by not being able to continue to develop their knowledge of content in Spanish, they loose the ability to, for example, talk about mathematics in Spanish. This affects the communication with their parents. We wonder about the effects of this reduced communication on the relationships parents-children and on children's academic achievement. As Worthy (2006) writes, "as linguistic connections with their families and roots fade, these children also face a loss of cultural knowledge, family values, personal nurturing, and academic support" (p. 140).

IN CLOSING

The parents in our research were concerned about limitations to their participation and the possibility of a loss of connection with their children due to the "language barrier." Language plays a key role in the learning and teaching of mathematics, particularly in reform-based classrooms. It is also true that language is a key component of one's identity. We are aware that in many countries the language of instruction is indeed different from many students' home language. But in our local context what we want to highlight is the political and educational implications of a change in language policy. It is not about whether we teach mathematics in English or in Spanish, but it is about what messages these language policies give about the valorization of certain forms of knowledge over others. We want to point out, however, that several of these parents were quite resourceful in accessing their own networks (Horvat, Weininger, & Lareau, 2003) to provide the necessary support to their children (seeking the help of neighbors, relatives, or teachers). Our parent workshops, focus groups, and interviews have also served as places to network, not just with school personnel and us but also among themselves. The conversations provide a context for reflection, as is the case of Emilia who in her third interview is starting to question issues related to the actual academic achievement of her children. We wonder what Emilia (and others like her) can do as they become more aware of the situation. As one of the mothers, Esperanza, reminds us, there is a difference between language and voice, "*Se me fue quitando el miedo y aprendí que tu voz cuenta, aunque no hables el* *mismo idioma, cuenta*" [The fear slowly went away and I learned that your voice counts, even if you don't speak the same language, it counts].

Endnote

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References

Abreu, G. de & Cline, T. (2005). Parents' representations of their children's mathematics learning in multiethnic primary schools. *British Educational Research Journal*, *31*, 697-722.

Civil, M. (2006). Working towards equity in mathematics education: A focus on learners, teachers, and parents. In S. Alatorre, J.L. Cortina, M. Sáiz, & A. Méndez (Eds.), *Proceedings of the Twenty Eighth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education,* (Vol. 1, pp. 30-50). Mérida, Mexico: Universidad Pedagógica Nacional,.

Civil, M., Planas, N., & Quintos, B. (2005). Immigrant parents' perspectives on their children's mathematics. *Zentralblatt für Didaktik der Mathematik, 37*(2), 81-89.

Glaser, B. & Strauss, A. (1967). *The Discovery of Grounded Theory: Strategies for Qualitative Research*. Chicago, IL: Aldine

Horvat, E. M., Weininger, E. B., & Lareau, A. (2003). From social ties to social capital: Class differences in the relations between schools and parent networks. *American Educational Research Journal, 40*, 319–351.

Jackson, K. & Remillard, J. (2005). Rethinking parent involvement: African American mothers construct their roles in the mathematics education of their children. *The School Community Journal*, *15*(1), 51-73.

Khisty, L. L. (2006). Language and mathematics: Toward social justice for linguistically diverse students. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education,* (Vol. 3, 433-440). Prague, CR: Charles University.

Moschkovich, J. N. (2002). A situated and sociocultural perspective on bilingual mathematics learners. *Mathematical Thinking and Learning*, *4*(2&3), 189-212.

Pérez, G. C., Drake, C., & Calabrese Barton, A. (2005). The importance of presence: Immigrant parents' school engagement experiences. *American Educational Research Journal*, *42*, 465-498.

Setati, M. (2005). Teaching mathematics in a primary multilingual classroom. *Journal for Research in Mathematics Education*, *36*, 447-466.

Valdés, G. (2001). *Learning and not Learning English: Latino Students in American Schools*. New York, NY: Teachers College Press.

Van Manen, M. (1990). *Researching Lived Experience: Human Science for an Action Sensitive Pedagogy*. London, Ontario: The University of Western Ontario.

Worthy, J. (2006). Como si le falta un brazo: latino immigrant parents and the costs of not knowing english. *Journal of Latinos and Education*, *5*(2), 139-154.

MATHEMATICAL ORALITY IN ASIAN AND WESTERN MATHEMATICS CLASSROOMS

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This paper reports research into the nature and occurrence of spoken mathematics in some well-taught classrooms in Australia, China (both Shanghai and Hong Kong), Japan, Korea and the USA. The methodology of the Learner's Perspective Study (LPS) documented the voicing of mathematical ideas in public discussion and in teacher-student conversations and the relative priority accorded by different teachers to student oral contributions to classroom activity. The analysis distinguished one classroom from another on the basis of public "oral interactivity" (the number of utterances in whole class and teacher-student interactions in each lesson) and "mathematical orality" (the frequency of occurrence of key mathematical terms in each lesson). Classrooms characterized by high public oral interactivity were not necessarily sites of high mathematical orality. Of particular interest are the different learning theories implicit in the instructional approaches employed in each mathematics classroom.

INTRODUCTION

The Learner's Perspective Study (LPS) sought to investigate the practices of well-taught mathematics classrooms internationally. Data generation focused on sequences of ten lessons, documented using three video cameras, and interpreted through the reconstructive accounts of classroom participants obtained in post-lesson video-stimulated interviews (Clarke, 2006a). The post-lesson interviews were designed, in part, to address the challenge of inferring student conceptions from video data (Cobb & Bauersfeld, 1994). The LPS approach of conducting case studies of classroom practices over sequences of at least ten lessons in the classes of several competent eighth grade teachers in each of the participating countries offers an informative complement to the survey-style approach of the two video studies carried out by the Third International Mathematics and Science Study (TIMSS) (Hiebert et al., 2003; Stigler & Hiebert, 1999). The criteria for the identification of the competent teachers studied in the LPS were constructed locally, specific to each country, in order to reflect the priorities and values of the school system in that country. In this paper, we report analyses of a subset of the lessons documented in classrooms in Australia, China (Hong Kong and Shanghai), Japan, Korea, and the USA.

Mok has elsewhere suggested that terms such as "teacher-dominating" can be misleading as a characterisation of the teaching in some Chinese classrooms (Mok, 2006). Clarke has similarly challenged the usefulness of the popular dichotomisation of classrooms as "student-centred" or "teacher-centred" (Clarke, 2006b). Attention is therefore focused on what theoretical framework might support cross-classroom comparisons and provide significant insights into essential differences in practice and the principles on which any such differences are based. The distribution of responsibility for knowledge generation has been suggested as a suitable framework (Clarke & Seah, 2005), but the challenge then became how to operationalise this framework in a form that could be applied to classroom data. In this paper, the oral articulation of mathematical terms and phrases during classroom whole-class and teacher-student discussion is employed as the entry point for our analysis.

STUDYING SPOKEN MATHEMATICS IN THE CLASSROOM

The immediate challenge in our recent work has been to interpret the enactment of the distribution of responsibility for knowledge generation in terms of actual classroom actions undertaken (and "observable") by teacher and students. By focusing on the documentation of spoken mathematical (and pseudo-mathematical) terms, through video recording and post-lesson reconstructive interviews, we have employed spoken mathematical terms as a form of surrogate variable, indicative of the location of the agency for knowledge generation in the various classrooms studied (but also of interest in itself) (see Clarke & Seah, 2005).

This paper reports the first two stages of a layered attempt to progressively focus on the significance of the situated use of mathematical language in the classroom. In our first analytical pass, an utterance is taken to be a continuous spoken turn, which may be both long and complex. We restricted our second-pass analysis to those mathematical terms and phrases that referred to the substantive content of the lesson (usually designated as such in the teacher's lesson plan and post-lesson interview). An utterance may contain more than one mathematical term, and our second analytical pass counted mathematical terms rather than utterances.

Bakhtin's use of "utterance" placed emphasis on situating any word, phrase or proposition in its spoken and social setting (Bakhtin, 1979). We take the orchestrated use of mathematical language by the participants in a mathematics classroom to be a strategic instructional activity by the teacher. In this paper, we invoke theory in two senses: (i) the (researchers') theories by which the actions of the classroom participants might be accommodated and explained, and (ii) the (participants') theories implicit in the classroom practices of the teacher and the students. A particular focus is the role of the spoken word in both. The instructional value of the spoken public rehearsal of mathematical terms and phrases central to a lesson's content could be justified by reference to several theoretical perspectives. Interpretation of this public rehearsal as incremental initiation into mathematics as a discursive practice could be justified by reference to Walkerdine (1988), Lave and Wenger (1991), or Bauersfeld (1995). The instructional techniques employed by the teacher in facilitating this progression could be seen as "scaffolding" (Bruner, 1983) and/or as "acculturation via guided participation" (Cobb, 1994).

The oral articulation of mathematical terms and phrases by students could be accorded value in itself, even where this consisted of no more than the choral repetition of a term initially spoken by the teacher. Teachers and students in some of the classrooms we studied clearly attached value to this type of recitation. In other classrooms, the emphasis was on the students' capacity to produce a mathematically correct term or phrase in response to a very specific request (question/task) by the teacher. In such classrooms, both of these activities accorded very limited agency to the learner and the responsibility for the public generation of mathematical knowledge seemed to reside with the teacher. By contrast, in other classrooms, the instructional approach provided opportunities for students to "brainstorm" or to generate their own verbal (written or spoken) mathematics, with very little (if any) explicit cuing from the teacher (e.g. the classrooms in Tokyo). In each classroom the activity of speaking mathematics was performed a little differently.

Our attempts at unpacking the distribution of responsibility for knowledge generation and its potential as a core precept in instructional theory have been hampered by the sheer scale of the logistics of analysing the transcripts of lessons in a wide variety of classrooms distributed across many countries. However, the results that are recorded in this paper certainly suggest that the teachers in this study differed widely in the opportunities they provided for student spoken articulation of mathematical terms and in the extent to which they devolved agency for public knowledge generation to the students.

The demonstration of such differences (and we would like to argue that these differences are profound and reflect fundamental differences in basic beliefs about effective instruction and the nature of learning) in the practices of classrooms situated in school systems and countries that would all be described as "Asian" suggests that any treatment of educational practice that makes reference to the "Asian classroom" confuses several quite distinct pedagogies. This observation is not to deny cultural similarity in the way in which education is privileged and encountered in communities that might be described as "Confucian-heritage." But, the identification of a one-to-one correspondence between membership of a Confucian-heritage culture and a singular pedagogy leading to high student achievement is clearly mistaken, and we must look elsewhere than only at culture in our attempts to single out those instructional practices that might be associated confidently with the educational outcomes that we value.

THE USE OF MATHEMATICAL TERMS

The earlier analysis conducted by Clarke and Seah (2005) distinguished between primary mathematical terms explicitly identified in the teacher's lesson plans, secondary mathematical terms employed in whole class discussion to explicate the primary terms, and transient terms, many pseudo-mathematical (e.g. "steep"), occurring typically once only in the conversations of students discussing the lesson's content among themselves or with the teacher. The initial tabular method of coding and display (see Clarke & Seah, 2005), though revealing, proved so labor-intensive as to be impractical if implemented on the large-scale required by the extensive LPS data set. New video-coding software *Studiocode* has offered a more efficient approach, combining basic descriptive coding statistics with a capacity to reveal temporal patterns in a highly visual form. In this paper, "utterance" and "mathematical term or phrase" require clear specification (below).

Figure 1 shows the number of utterances occurring in whole class and teacher-student interactions in each of the first five lessons from each of the classrooms studied in Shanghai, Hong Kong, Seoul, Tokyo, Melbourne and San Diego. An utterance is a single, continuous oral communication of any length by an individual or group (choral). Used in this way, the frequency (and origins) of public utterances constitute a construct we have designated as public oral interactivity. This does not take into account either the length of time occupied by an utterance or the number of words used in an utterance (problematic in a multi-lingual study like this one). Figure 1 distinguishes utterances by the teacher (white), individual students (black) and choral responses by the class (e.g. in Seoul) or a group of students (e.g. in San Diego) (grey). Any teacher-elicited, public utterance spoken simultaneously by a group of students (most commonly by a majority of the class) was designated a "choral response." Lesson length varied between 40 and 45 minutes and the number of utterances has been standardized to 45 minutes.

Figure 1 suggests that lessons in Melbourne and San Diego demonstrated a much higher level of public oral interactivity than lessons in Shanghai, Hong Kong, Seoul, or Tokyo. There were also substantial differences in the relative frequency of teacher, student and choral utterances. This paper does not address temporal length or complexity of utterance, which will be investigated in a later analysis. It is worth noting that both teacher and student utterances in Shanghai tended to be of longer duration and greater linguistic complexity than elsewhere.

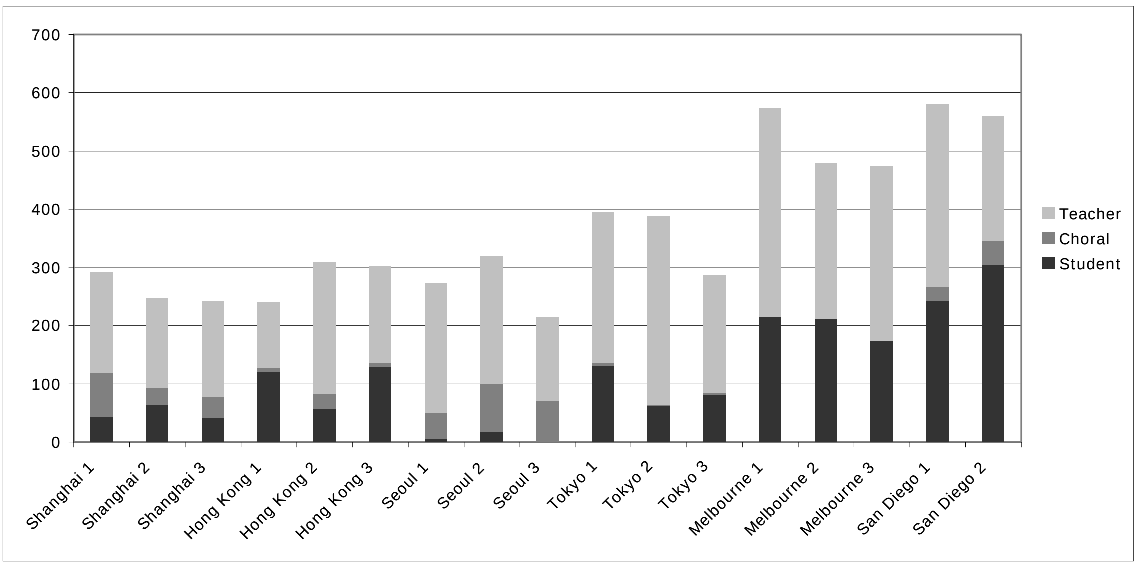

Figure 1. Number of Public Utterances in Whole Class and Teacher-Student Interactions (Public Oral Interactivity).

The classrooms studied can be also distinguished by the relative level of mathematical orality of the classroom (that is, the frequency of spoken mathematical terms or phrases by either teacher or students in whole class discussion or teacher-student interactions) and by the use made of the choral recitation of mathematical terms or phrases by the class. This recitation included both choral response to a teacher question and the reading aloud of text presented on the board or in the textbook. For the purposes of this paper, those mathematical terms were coded that comprised the main focus of the lesson's content. In the terminology of Clarke and Seah (2005), as above, this analysis focused on primary and secondary terms.

Figure 2 shows how the frequency of public statement of mathematical terms varied among the classrooms studied. In classifying the occurrence of spoken mathematical terms, we focused on those terms that represented the main lesson content (e.g. terms such as "equation" or "co-ordinate"). This meant that our analysis did not include utterances that constituted no more than agreement with a teacher's mathematical statement or utterances that only contained numbers or basic operations that were not the main focus of the lesson. In the case of the Korean lessons, the frequent choral responses by students took the form of agreement with a mathematical proposition stated by the teacher. For example, the teacher would use expressions such as, "When we draw the two equations, they meet at just one point, right? Yes or no?" And the class would give the choral response, "Yes." Such student statements did not contain a mathematical term or phrase and were not included in the coding displayed in Figure 2.

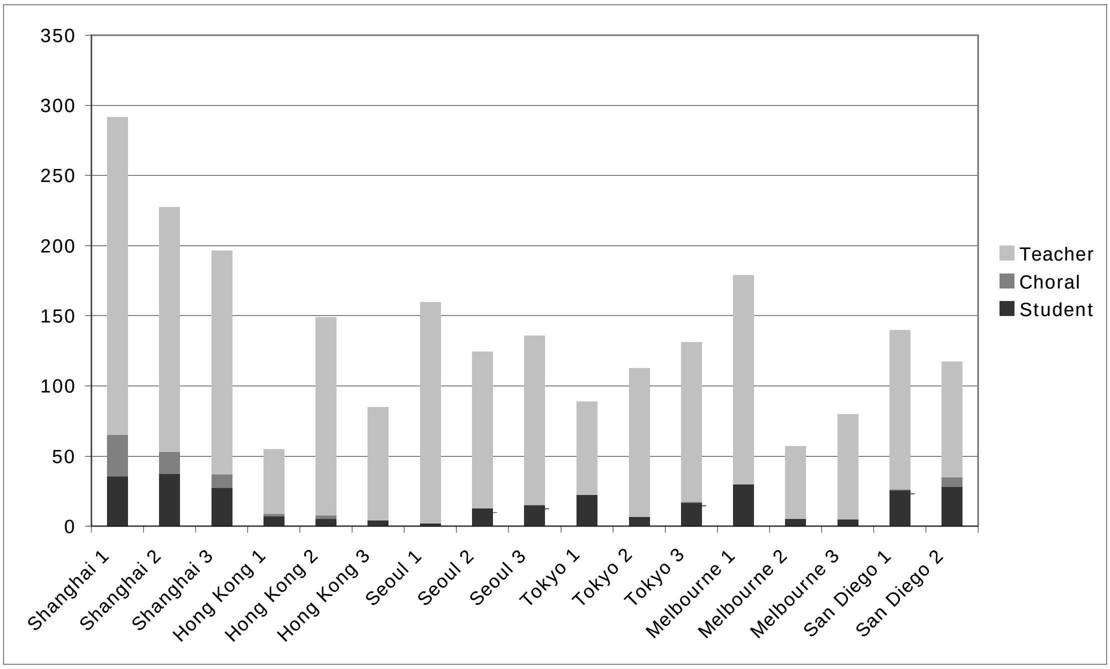

Figure 2. Frequency of Occurrence of Key Mathematical Terms
in Public Utterances (Mathematical Orality).

Similarly, a student utterance that consisted of no more than a number was not coded as use of a key mathematical term. It can be argued that responding "Three" to a question such as "Can anyone tell me the coefficient of x?" represented a significant mathematical utterance, but our concern in this analysis was to document the opportunity provided to students for the oral articulation of the relatively sophisticated mathematical terms that formed the conceptual content of the lesson. Frequencies were again adjusted for the slight variation in lesson length.

The most striking difference between Figures 1 and 2 is the reversal of the order of classrooms according to whether one considers public oral interactivity (Figure 1) or mathematical orality (Figure 2). The highly oral classrooms in San Diego made relatively infrequent use of the mathematical terms that constituted the focus of the lesson's content. By contrast, the less oral classrooms in Shanghai made much more frequent use of key mathematical terms and phrases. Since a single utterance might contain several such terms, and it was terms that were being counted in this analysis, Figure 2 provides a different and possibly more useful picture of the Chinese lessons, where both teacher and student utterances appeared to be longer and more complex than elsewhere. Later analyses will address duration and complexity.

Comparison between those classrooms that might be described as "Asian" is interesting. Key mathematical terms were spoken less frequently in the Seoul classrooms than was the case in the Shanghai classrooms. Even allowing for the relatively low public oral interactivity of the Korean lessons, the Korean students were given proportionally fewer opportunities to make oral use of key mathematical terms in whole class or teacher-student dialogue. In contrast to the teachers in Shanghai and Tokyo, the teachers in the Hong Kong and Seoul classrooms did not appear to attach the same value to the spoken rehearsal of mathematical terms and phrases, whether in individual or choral mode. While the overall level of public oral interactivity in the Tokyo classrooms was similar to those in Seoul, the Japanese classrooms resembled those in Shanghai in the consistently higher frequency of student contribution, but with little use being made of choral response. The value attached to affording student spoken mathematics in some classrooms could suggest adherence by the teacher to a theory of learning that emphasizes the significance of the spoken word in facilitating the internalisation of knowledge. The use of choral response, while consistent with such a belief, could be no more than a classroom management strategy. The Hong Kong classrooms made least use of spoken mathematical terms of all the classrooms studied and student spoken mathematical contribution, whether individual or choral, was extremely low, even though the general public oral interactivity of Hong Kong classrooms 2 and 3 was at least as high as in Shanghai.

CONCLUSIONS

It appears to us that the key constructs Public Oral Interactivity and Mathematical Orality distinguished one classroom from another very effectively. Particularly when the two constructs were juxtaposed (by comparing Figures 1 and 2). The contemporary reform agenda in the USA and Australia has placed a priority on student spoken participation in the classroom and this is reflected in the relatively high public oral interactivity of the San Diego and Melbourne classrooms (Figure

1). By contrast, the "Asian" classrooms, such as those in Shanghai, were markedly less oral. However, this difference conceals striking differences in the frequency of the spoken occurrence of key mathematical terms (Figure 2), from which perspective the Shanghai classrooms can be seen as the most mathematically oral. The relative occurrence of spoken mathematical terms is one level of analysis. We should also distinguish between repetitive oral mimicry and the public (and private) negotiation of meaning (Cobb & Bauersfeld, 1994; Clarke, 2001). However, the frequency of public spoken mathematics does appear to distinguish usefully between classrooms.

Despite the frequently assumed similarities of practice in classrooms characterised as Asian, the Asian classrooms studied displayed significant differences in the level of mathematical orality, particularly with respect to the frequency of spoken mathematical terms and phrases employed by students. A further critical distinguishing characteristic was the form of prompt by which the teacher elicited student spoken mathematics. Students in the Shanghai classrooms had the opportunity to articulate their understanding of key mathematical terms through a structured process of teacher invitation and prompt that built upon the contributions of a sequence of students. The classrooms in Tokyo provided many instances where a student made the first announcement of a mathematical term without specific teacher prompting. These differences in the nature of students' publicly spoken mathematics in classrooms in Seoul, Hong Kong, Shanghai and Tokyo are non-trivial and suggest different instructional theories underlying classroom practice. Any theory of mathematics learning must accommodate, distinguish and explain the learning outcomes of each of these classrooms.

Consideration of the non-Asian classrooms is also interesting. With frequent teacher questioning and eliciting of student prior knowledge, the students in the Melbourne classrooms were given many opportunities to recall and orally rehearse the mathematical terms used in prior lessons. In terms of overall mathematical orality and level of student contribution, Melbourne 1 resembles Shanghai 1 (without the use of choral response). This public orality is potentially augmented by small group discussions, in which students draw upon their mathematical knowledge to complete tasks at hand. Such student-student conversations occurred much more frequently in the Melbourne and San Diego classrooms. For example, in one San Diego lesson (US2-L02), the two focus students made 107 and 97 private utterances, many related to the lesson's mathematics content. These non-public exchanges are not part of the analyses reported in this paper. Levels of mathematical orality in student-student interactions (in which the teacher was not participant) will be examined in a separate analysis. The post-lesson interviews may provide the connection between classroom mathematical orality and student learning outcomes. We suggest that the empirical investigation of mathematical orality (in both public and private domains) and its likely connection to the distribution of the responsibility for knowledge generation are central to the development of any theory of mathematics instruction.

References

Bakhtin, M. (1979). *The aesthetics of Verbal Creation.* Moscow: Bocharov. Cited in Todarov, T. (1984), *Mikhail Bakhtin: The Dialogical Principle.* Translated by Wlad Godzich. Minneapolis: University of Minnesota Press.

Bauersfeld, H. (1994). "Language Games" in the mathematics classroom: Their function and their effects. In P. Cobb & H. Bauersfeld (Eds.), *The Emergence of Mathematical Meaning: Interaction in Classroom Cultures.* Hillsdale, NJ: Lawrence Erlbaum, pp. 271-291.

Bruner, J. (1983). *Children's talk: Learning to use language*. Oxford: OUP.

Clarke, D. J. (2001). Untangling uncertainty, negotiation, and intersubjectivity. In D. Clarke (Ed.), *Perspectives on Practice and Meaning in Mathematics and Science Classrooms*, Dordrecht: Kluwer, 33-52.

Clarke, D. J. (2006a). The LPS Research Design. In D.J. Clarke, C. Keitel, & Y. Shimizu (Eds.), *Mathematics Classrooms in Twelve Countries: The Insider's Perspective*. Rotterdam: Sense Publishers, 15-37.

Clarke, D. J. (2006b). Using International Comparative Research to Contest Prevalent Oppositional Dichotomies. *Zentralblatt für Didaktik der Mathematik, 38*(5), 376-387.

Clarke, D. J. & Seah, L. H. (2005). Studying the distribution of responsibility for the generation of knowledge in mathematics classrooms. In H. L. Chick & J. L. Vincent (Eds.), *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education, Vol. 2, 257-264.*

Cobb, P. (1994). Where is the mind? Constructivist and sociocultural perspectives on mathematical development. *Educational Researcher, 23*(7), 13-20.

Cobb, P. & Bauersfeld, H. (Eds.) (1994). *The Emergence of Mathematical Meaning: Interaction in Cclassroom Cultures.* Hillsdale, NJ: Lawrence Erlbaum.

Hiebert, J., Gallimore, R., Garnier, H., Givvin, K., Hollingsworth, H., Jacobs, J., Chui, A., Wearne, D., Smith, M., Kersting, N., Manaster, A., Tseng, E., Etterbeck, W., Manaster, C., Gonzales, P., & Stigler, J. (2003). *Teaching Mathematics in Seven Countries: Results from the TIMSS 1999 Video Study*. Washington, DC: U.S. Department of Education, National Center for Education Statistics.

Lave, J. & Wenger, E. (1991). *Situated Learning: Legitimate Peripheral Participation*. Cambridge, CUP.

Mok, I. A. C. (2006). Teacher-dominating lessons in Shanghai: The Insiders' Story. Chapter 6 in Clarke, D.J., Keitel, C., & Shimizu, Y. (Eds.), *Mathematics Classrooms in Twelve Countries: The Insider's Perspective*. Rotterdam: Sense Publishers, 87-98.

Stigler, J. & Hiebert, J. (1999). *The Teaching Gap*. New York: Simon & Schuster.

Walkerdine, V. (1988). *The Mastery of Reason: Cognitive Development and the Production of Rationality.* London: Routledge.

HOW DO A PLANE AND A STRAIGHT LINE LOOK LIKE? INCONSISTENCIES BETWEEN FORMAL KNOWLEDGE AND MENTAL IMAGES

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This paper focuses on analysing the ability of pre-service and in-service teachers to visualize straight lines and planes. These concepts are abstract concepts which cannot be understood by the perception alone. The present paper is based on a research which exposed typical misconceptions and a gap between the formal knowledge and the mental picture. The results also demonstrate that awareness and analytical-visual integration are essential in overcoming those difficulties.

INTRODUCTION

This paper focuses on analysing the ability of pre-service and in-service teachers to visualize straight lines and planes. These concepts are abstract concepts which cannot be understood by the perception alone (both are infinite[1] and lack thickness, thus they cannot be concrete objects). This study is a part of broad research dealing with those concepts and with the interrelations between them. The research findings exposed typical difficulties and misconceptions and demonstrated the central role of analytical-visual integration in overcoming them. In the present paper, we focus on two types of difficulties:

1_A. The difficulty to perceive the infinity of a line[2] or a plane. Although the subjects usually say that a line or a plane is infinite, they do not really feel it or "see" it in their minds. They formally acknowledge the fact, but their behaviors reveal that their mental image does not fit this acknowledgement.

1_B. The difficulty to perceive the lack of thickness of a line or a plane (which relates to the notions of infinitesimals and density). Here again, subjects state the facts but do not "sense" or "see" them.

THEORETICAL BACKGROUND

The point, the straight line and the plane are undefined objects. Nevertheless, when we deal with Euclidian geometry, those objects are connected to specific figural images. According to Piaget & Inhelder (1967), geometrical intuition develops not only from perceptions or images, but also from performing some mental operations on them. When dealing with abstract geometric concepts, one cannot help thinking about a concrete image (such as seeing the point as a tiny round surface or the line as a fine thread). But in order to properly understand these concepts, one has to be aware that those are only symbols and not the objects themselves.

[1] "infinite" means here that the straight line and the plane are not bounded (in the mathematical sense)

[2] from here on, "line" will mean "straight line"

Fischbein (1993) sees geometric figures as figural concepts which he defines as mental entities, possessing conceptual and figural properties simultaneously. They have a strong image aspect, but they also have conceptual aspects, as abstract, ideal, logically determinable entities. Even adults, although being aware of the nature of geometrical objects, tend to think in terms of figural models and often draw wrong conclusions about those objects. Vinner (1981) examined 47 prospective teachers and teachers and found that 34% of them believed that geometric concepts are part of the physical world. Without conceptual control, says Fischbein (ibid), it is impossible to fully comprehend geometrical concepts.

THE STUDY

The study utilized both methods of quantitative and qualitative research. The quantitative research enabled the identification of beliefs and difficulties, and the examination of how prevalent they are. The qualitative research served for examination and analysis of participants' perceptions and thinking processes. The research's population mainly included prospective teachers who study in a college of education in Jerusalem and specialize in mathematics, and, in addition, mathematics teachers in elementary or junior high school. The data collection was carried out in several modes: a questionnaire by which the quantitative aspect was examined (341 subjects); Intensive observations of a "base population" (20 subjects) documented by video; Video documentation of another 74 subjects who worked in small groups; A post test which was given to 106 subjects.

In the broad research questionnaire, 3 out of 19 questions involved the direct visual image of a plane or a straight line. Earlier, the students were given a preliminary brief explanation about the tasks and the concepts, in which we made sure that they were aware, at least theoretically, of the infinity and lack of thickness of the plane and the line. After collecting the answers of the questionnaires the students were asked to deal with the same questions again, this time discussing them in pairs, small groups or with the interviewer, and using manipulative visual aids. The visual aids included very thin flat plastic surfaces, with random “cloud” shapes to illustrate planes and straws or thin rods to illustrate lines. Of course, we assured that the students understood that those aids only represented infinite planes or lines with no thickness.

APPROACH AND EMPHASIS IN THE RESEARCH

Individual's capability to utilize visualization and analysis in an integrated manner is referred to in this study as "analytical-visual integration ability". This ability is composed, as suggested here, of three main components:

- **(V→A) the ability to analyse visual information** (visual→analytical).
- **(A→V) the ability to create a visual image corresponding to analytical ideas** (Analytical→visual), namely the ability to create a visual image (physical or in the mind) out of information which is not visual, such as verbal data, abstract relations etc.

- **(AVF) Analytical-Visual Flexibility** which portrays the ability to move freely, in a flexible and efficient way between analytical processing and visual imaging. This flexibility requires, among other things, a clear knowledge of when leaning on a visual example is acceptable and when it is not.

FINDINGS, ANALYSIS AND EXAMPLES

Let us have a close look at one of the questions which directly involves the visual image of a plane or a straight line.

1. Two **different planes** "touch" each other.
What can be **the shape of the intersection**? Circle all the possibilities:

a) one point **b**) two points **c**) a segment **d**) a straight line **e)** a surface

When subjects mark **a**, **b** or **c**, one can suspect that they visualize the plane as being bounded (not infinite). The answer **e** can maybe reveal a mental image of the plane as having thickness or physical existence. Next, these hypotheses shall be supported by examples from the qualitative study. Let's look at the results of this question:

	general success (marked only **d**)	marked **a**	marked **b**	marked **c**	marked at least one of **a, b** or **c**	did not mark **d**	marked **e**
(n=341)	25%	36%	9%	29%	48%	19%	61%

On first glance, it can be seen that the rates of the answers which implicate the misconceptions mentioned hereinabove are significant: about half of the subjects (48%) chose at least one of the possible answers **a**, **b**, or **c**, which imply $\mathbf{1_A}$, and 61% of them chose **e**, which implies 1_B. We shall discuss now these two phenomena separately, while examining them also qualitatively.

1A. The Difficulty to Perceive the Infinity of a Line or a Plane

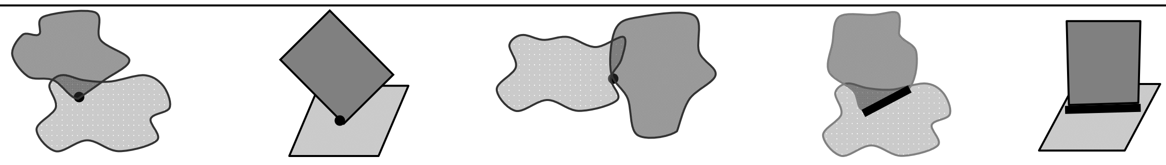

figure **1** figure **2** figure **3** figure **4** figure **5**

As we have seen, about half of the participants chose for the above question, answer **a, b** or **c**, which imply an error from type $\mathbf{1_A}$. The "popular" choice is undoubtedly **a**: 36% of the participants believe that it is possible for two planes to touch each other at one meeting point. From watching the video tapes a clear picture arises regarding the way in which the subjects visualize such a meeting point: they illustrate the planes as depicted in figure no. 1, 2 or 3. Those who believe that the meeting point is in the shape of a segment (29%) visualize it, as depicted in figure 4 or 5. (Fgures 1,3,4 correspond to their illustrations with the visual aids, and figures 2, 5 correspond to their description of what they had visualized).

During the group discussion (after collecting the questionnaires) almost all the subjects corrected these errors (options a-c). The corrections were made usually after one of the subjects in the group had answered correctly, or had reached the correct answer while working, and convinced those who were erroneous. Such discourse took place in most groups. In cases where such discourse did not take place or in the case of personal interviews, the "opposite stand" was presented by the researcher. Following are some typical examples for such a discourse:

Episode 1: Ruth chose option **a** in question **1** and Einat did not choose it.

Einat: it is infinite.

Ruth: so what if it's infinite?

Einat: it can not be one point. It's planes

Ruth: here, a point! (showing with the visual aids something like to figure 1)

Einat: but that can't be a point, it's infinite, it's a plane

Ruth: so what?

Einat: what do you mean 'so what' it's also going down, it continues (using her hands to show the continuation of the plane)

Ruth: where does it continue?

Einat: it continues down and up (illustrating)

Ruth: Yes, down, up, but still…

Einat: it is at the least a line.

Ruth: I don't know… (at last Einat succeeded in convincing Ruth).

In this episode Ruth held her erroneous intuition tightly and it was hard for her even to see the conflict between the analytical explanation and her image ("so what?"). In some other cases, the erroneous subjects acknowledged their errors immediately after the analytical aspect was presented to them. For instance:

Episode 2: Muhamad, on a personal interview (Muhamad = M; Interviewer = I):

1 Muhamad said that one point is possible and illustrated something like figure 3.

2 I: there were some students who claimed that since the plane is infinite, it doesn't end there but rather continues.

3 M: (contemplating for a while) they are right, they are right.

4 I: why are they right?

5 M: It may be like this (illustrates "a clear cut situation of an encounter with a line") and then we have a straight line, or like this (as in figure 3), but then both planes continue and create a line here (illustrates the imaginative intersection line). The shape of the intersection is also a straight line.

6 I: now I shall tell you what others have answered. They said: but it can be like that (illustrates figure 1).

7 M: (unhesitatingly) it is also a straight line, it is not a point, it also creates a straight line (illustrating with his hands).

8 I: why?

9 M: it is infinite, it continues, it continues (illustrating with his hands), it creates only a straight line

10 I: so there's no option for a point?

11 M: not at all.

The instance of Muhamad enables us to trace thinking processes which exemplify a combination of visualization and analysis. At first (1), Muhamad had reached a (wrong) conclusion according to what he saw (the visual aids), namely, a V→A process. From reaction no. 3 it can be inferred that he acknowledged the fact that the plane is infinite, namely his first reaction demonstrated a hidden inconsistency. After the interviewer's question, the hidden inconsistency became conscious and was solved immediately (as a consequence of analytical consideration). Reaction no. 5 demonstrates the process of solving the conflict. Muhamad creates a new mental image of a plane which took into consideration the fact that the plane is infinite. He now visualized the infinite plane. The process here is thus, A→V. In this case the awareness brought the correction of the mental image. In reactions no. 7, 9, 11 and in all the next lessons, there were no episodes which pointed to a loose knowledge in this matter, except one episode in which he corrected himself immediately.

Out of the 14 video taped episodes which relate to option **a** in question 1, eight were very similar to those presented here. In two other episodes the subjects who were erroneous were not convinced, at least until the summarizing class discussion, and in four cases there were no errors to begin with. Similar results were obtained also regarding two other questions which deal with the infinity of the line and the plane. According to this analysis it seems that when the subjects take into consideration consciously the infinity matter, they succeed, some easily, others with effort, to overcome the erroneous intuitive mental image of the plane as having a rectangular bounded shape. In the summarizing discussion, many subjects referred to the process they went through regarding the infinity. Following are some typical citations:

- "We did not pay attention to the fact that the plane continues."
- "I did not think that a plane is actually infinite, so I imagined that they can meet at only one point."
- "I thought that one point is possible since when you take the visual aids and use them, you refer to them and not to what they stand for."

Except questions which deal directly with the infinity of the straight line and the plane, the difficulty regarding this matter was detected also in the subjects' performance regarding other questions. Obviously, there are no traces of it in the quantitative results, yet, in the qualitative analysis all groups had several episodes which point to it. For instance: in tens of episodes subjects put two straws in opposite directions, and treated them as two different straight lines. In the first stages of the work they usually corrected themselves after a remark of a friend or a question posed by the interviewer. In the next stages, when it happened, they corrected it themselves, immediately. Sometimes the lack of awareness to the infinity of the straight line or the plane have yielded correct answers based on the wrong reasons, for instance, Tali wrote in one of her answers that there are infinite planes which are parallel to a given straight line through a given point, but illustrated a situation in which she turned the plastic surface around keeping it all the time on the same plan she chose.

1_B. The Difficulty to Perceive the Lack of Thickness of a Line or a Plane

In the quantitative results we saw that 61% of the subjects chose **e** in question **1**, according to which it is possible for two different planes to touch each other on a plain surface. In this case as well, it happened in most groups, but, not like the infinity case, here, except for few cases, those who were right could not convince those who were wrong. In some cases each one held his own opinion and could not figure out how to convince the other. In other cases those who were wrong convinced those who were correct, in their error. In some of the cases the subjects themselves could not decide! Here too, like in the infinity case, we witness phenomena of inconsistency, in different levels of awareness to it. Following are some typical instances in which the subjects had doubts whether it is possible for two different planes to be "adjacent", namely, to touch each other in all points, yet not to unite.

Episode 3: self discourse within the group discourse

Einat: on the one hand it doesn't have thickness, but on the other hand it is "something"…so I don't know…I tend to believe that it is possible that two different planes touch each other (turning to her friend who said before that in such a case they shall become the same surface), without them being united!

(Einat continues to think it over for a few minutes)…

Einat: I am actually arguing with my self!

I: so what is your answer?

Einat: that it is possible (hesitates) but maybe I'm wrong, because it doesn't have any thickness, but I believe I am right.

Einat allows us here to witness her thoughts while she is having a cognitive conflict. She sees very clearly the contradiction, yet, she cannot solve it, since the image of the plane as "something" which has a physical existence prevents her from understanding the abstract concept, which lacks thickness.

Another instance of an explicit expression of the conflict between knowing that the straight line lacks thickness and the intuitive sense that it has a physical extent is the next citation after the class discussion: "in my mind I can comprehend what a straight line is, but how is a plane created? The plane has to be created as if many straight lines compose it, when you say that if they are close they are united, so how can it be?"

DISCUSSION: OVERCOMING INCONSISTENCY

The phenomenon named inconsistency is well known in the mathematics education literature, and it is described as cases in which subjects hold conflicting beliefs (e.g. Vinner, 1990 Tirosh, 1990). In the instances we examined, different levels of awareness to the inconsistency can be traced: one level includes states in which the subject does not simultaneously examine the contradictory beliefs. This phenomenon is called compartmentalization and it is discussed by Vinner (ibid). This state characterizes most episodes in which the subjects did not correct themselves. The opposite level includes states in which the subjects identify the conflicting components as being inconsistent, treat them as problematic and try to solve the inconsistency. This state is defined as "a cognitive disequilibrium" or a "cognitive conflict". In many episodes in the present study, a gradual change with time, in the awareness level of the subjects, could be traced, which may point to occurrence of learning: from a state in which the inconsistency was a hidden one, and was disclosed only as a consequence of an external interference (a remark or a question by one of the group members or the interviewer), it became more conscious, and the reaction of the subject to it became more independent and quick. An evidence for the learning process can be also found in the post test which was given three months later. In many cases, a stable change in the mental image could be identified. In this matter there was a difference between the two types of difficulty: it seems that most of the subjects indeed visualized the infinite continuation of the straight line and the plane. Only one out of 106 subjects marked this time one of the possibilities **a**, **b** or **c** of question 1. In the explanations which were given after completing the questionnaire, it seemed that the infinity attribute was internalized properly and became intuitive. However, in regard to the lack of thickness, there were still many indications for loose knowledge. 30% of the subjects (in comparison to 61% at the beginning) were still "taken in" and chose the possibility **e** (some of them reconsidered their answer and corrected it while trying to explain).

The awareness of the contradiction and the attempt to solve it are actually analytical considerations which control the primary intuitive answer. We see here, therefore, states in which the combination of the visual-intuitive facet and the analytical facet, yields benefits of overcoming misconceptions.

CONCLUSIONS

The present study had two main goals: one was to examine the difficulty in perception of the concepts "straight line" and "plane", and to expose phenomena of inconsistency between the formal knowledge and the mental image of subjects regarding these concepts. The second goal was to point to the contribution of the analytical visual combination in overcoming the difficulty. As we have seen in the instances presented, what usually led to the solution of the conflict was a discourse which raised awareness and an attempt to process the data analytically. This analytical awareness also enabled a wise use of the visual aids: the analytical thinking served as a watchman that reminded the subject the limitations of such aids and the difference between what is seen and what should be visualized. It seems that the results of the study show an increasing awareness of the subjects to these limitations. Some demonstrated a stable change in the mental image. Others demonstrated an unstable knowledge which still needs "analytical reminder". Thus, the analytical-visual combination is revealed as a supportive factor in the learning process.

References

Fischbein, E. (1993). The theory of figural concepts. *Educational Studies in Mathematics, 24*(2), 139-162.

Piaget, J. & Inhelder, B. (1967). *The Child's Conception of Space*, Routledge & Kegan Paul, London. (first published in France in 1948)

Tirosh D. (1990). Inconsistencies in students' mathematical constructs. *Focus on Learning Problems in Mathematics*, 12(3&4), 111-129.

Vinner, S. (1981). The nature of geometrical objects as conceived by teachers and prospective teachers. In *Proceeding 5th Conference of the International Group for the Psychology of Mathematical Education* (pp. 375-380). Grenoble, Fance: PME

Vinner, S. (1990). Inconsistencies, their causes and function in learning mathematics. *Focus on Learning in Mathematics*, *12*(3&4), 85-98.

LONGTERM-STUDY OF AN INTERVENTION IN THE LEARNING OF PROBLEM-SOLVING IN CONNECTION WITH SELF-REGULATION

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This article provides an overview of a one-year project on teacher training in connection with a material-based teaching concept for the learning of self-regulation in normal maths lessons. The results of the study particularly show increased performance of students in problem-solving during the year of the project. This article focuses on a follow-up-study run with 10 participating classes for one year after the end of the project. The results of this study confirm the stability of the problem-solving skills of the students and especially the sustainability of the teacher training based on this teaching concept.

BACKGROUND

Nationally and internationally, the enhancement of problem-solving competencies in maths lessons is considered important (cf. e.g. NCTM, 2000). The results of the PISA study in 2003 reveal that German students have enough cognitive potential to solve problems yet are not making sufficient use of it for the creation of expert competence (cf. Törner et al., 2007). This result underlines the importance of integrating problem-solving in maths lessons. The education standards for mathematics (cf. Büchter & Leuders, 2005), established in 2003 by the Standing Conference of German Educational und Cultural Ministers (KMK) as a reaction to the results of the international comparative studies, list problem-solving as one of six competencies to be developed by students at the end of intermediate secondary education. This implicitly questions how problem-solving competency can be enhanced in regular maths lessons. Since the eighties, a broad range of ideas has been developed in maths didactics for the enhancement of problem-solving (cf. Törner et al., 2007). Many of these ideas are based on the four-step phase model by Polya as a guideline for problem-solving. The studies on problem-solving, which often concentrate on a small group of students, confirm that problem-solving can be improved by training programs (cf. e.g. Schoenfeld, 1985; Da Ponte, 2007). There are the following research desiderata:

- At present there is a lack of problem-solving concepts empirically tested with a greater number of students (cf. Heinze, 2007),
- In the didactic research little attention was paid to the way teachers are teaching problem-solving in regular maths lessons (cf. Lester & Charles, 1992).

Within the framework of a study sponsored by the German Research Foundation (DFG), a further training course for teachers was run on the basis of a material-supported teaching concept for problem-solving learning in connection with self-regulation in maths lessons of secondary school level I (cf. Collet & Bruder, 2006; Komorek et al., 2007). The study takes up the above-mentioned research desiderata. The field study which was run with 48 teachers (49 school classes) investigates effects on students as well as on teachers through quantitative and qualitative methods.

OVERVIEW OF THE TEACHING CONCEPT AND THE FURTHER TEACHER TRAINING

The material-based teaching concept for the learning of problem solving in connection with self-regulation integrates the four-step model by Polya (1949), ideas regarding the long-term enhancement of problem-solving in maths lessons by Bruder (cf. Bruder, 2003), the process model for self-regulated learning following Schmitz (cf. Schmitz & Wiese, 2006), which is constructed on ideas by Zimmermann (2000) on self-regulation and a theory by De Corte et al. (2000), according to which self-regulated maths learning and problem-solving are connected with characteristics of the person and the environment.

According to Bruder (2003) the following three learning goals are connected with the integration of problem-solving and self-regulation in maths lessons: asking mathematical questions particularly in everyday situations, understanding and learning to adapt heuristic procedures to work on problems and strengthening the willingness to perform and the reflective capability of students. These goals can be reached using the teaching concept for problem-solving in connection with self-regulation (cf. Collet & Bruder, 2006). Problem-solving capabilities are developed in connection with self-regulation in four steps, starting with adaptation and awareness of heuristic approaches with subsequent exercises on the widened context of strategy application.

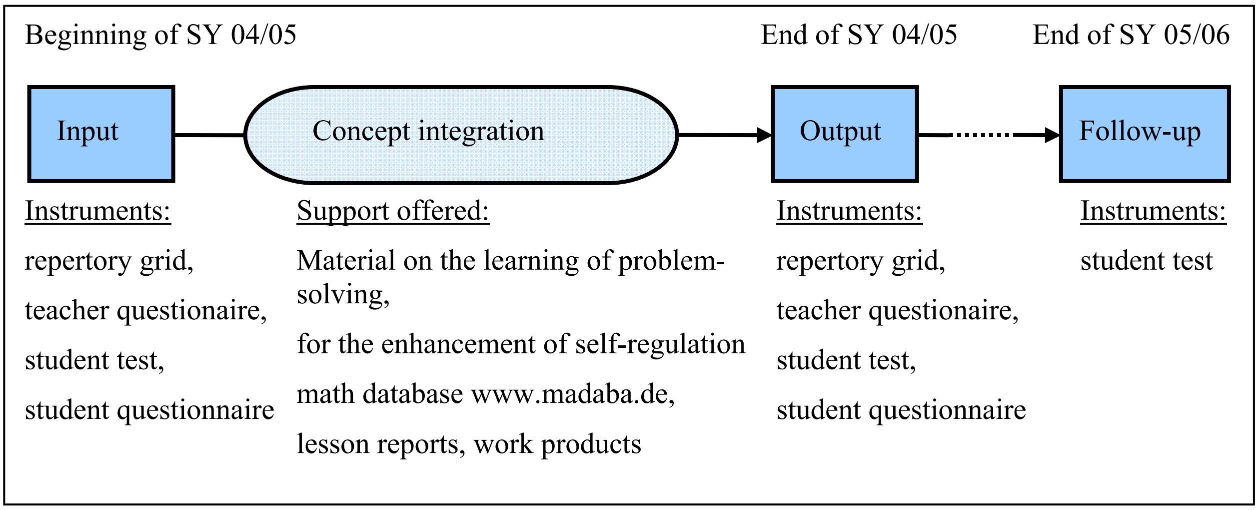

Figure 1. Progress of the further teacher training and instruments used for evaluation.

This teaching concept was first tested in extra-curricular student training programmes with teacher trainees and was positively evaluated (cf. Komorek et al., 2007). A further teacher training on this teaching concept run in the school year 2004/2005 was intended to evaluate the implementation of the teaching concept in regular maths lessons. Fig. 1 shows the progress of the further teacher training and the instruments used for evaluation. The participating teachers underwent training on the teaching concept at the beginning of the school year (SY). During the school year different supporting tools were put at their disposal, e.g. tasks on problem-solving, lesson reports to be kept by the teachers as a monitoring instrument (cf. Collet et al., 2007). The teachers were also asked to submit specific teaching material (a work product) to the project leaders, to document the implementation of the teaching concept (cf. Collet et al., 2007). One year after the further training a follow-up study was run with 10 of the previously participating project classes to check the stability of student performance without further project coaching.

DESIGN OF THE STUDY AND RESULTS OF THE MAIN STUDY

As the superiority of combined teaching of problem-solving and self-regulation could not be taken for granted, three groups were established to integrate respective aspects (problem-solving (PL) or self-regulation (SR) or both in combination (PS)) in the lessons. The main study was carried out in a control group design. The participating 48 teachers were from nine schools with 29 7th classes and 20 8th classes. The teachers were teaching at higher-track schools (Gymnasium) and intermediate-track and lower-track schools. The central question of the main study was:

- What effects has further teacher training with a given teaching concept on the teachers and their students?

The results of the main study in the project year show that the essential contents of the concept were integrated into the lessons by the teachers, e.g. heuristics, elements for internal differentiation and aspects of self-regulation. This is documented in the lesson reports and by the teaching material of the teachers (cf. Collet et al., 2007) as well as in the data provided by the teacher survey (cf. Komorek et al., 2007). The lesson reports reveal that coaching within the school year can have positive effects on the implementation of the further training content in the lessons. Moreover the teachers' knowledge has increased, as revealed in a qualitative Repertory Grid survey (cf. Collet & Bruder, 2006). The students show improved learning efficiency, especially with respect to their problem solving capabilities (cf. Collet & Bruder, 2006). No differences between the three further training groups were verified. Detailed information on the results during the further training year is available in Komorek et al. (2007).

The follow-up study, one year after the project year, asks the question:

- will the effects attained by the teachers' further training concept remain stable for more than one school year without intervention?

DESIGN AND RESULTS OF THE FOLLOW-UP-STUDY

To analyse the time stability of the intervention effects attained with the teaching concept, the problem-solving capability of 10 higher-track school classes of class 7 students who had participated in the project was registered in a follow-up-study with the further teacher training contents PL, PS and SR one year after the end of the intervention with the further training concept, i.e. at the end of class 8. As the performance tests developed for the classes 7 and 8 contain 12 continuous test items (anchor items) it is possible to study the performance of students at three different times (covering almost two years). Table 1 shows the number of the participating students in the different groups, split up according to sex.

Further training content	PL	PS	SR	Total
Male	57	12	20	89
Female	40	20	21	81
Total	97	32	41	170

Table 1. Follow-up-study: Number of students according to groups and sex

The following results refer to 170 students whose problem-solving capability was analysed at three different times. The three periods are defined as follows:

- Pre-test: t1 (beginning of school year 2004/2005)
- Post test: t2 (end of school year 2004/2005)
- Follow-up test: t3 (end of the school year 2005/2006).

The following example shows the test item ("cinema") for problem-solving worked on by the students during these three different test periods.

> Mike proposes a cinema-riddle: "Only a fifth of the seats are taken by adults. 10 more places are taken by boys. Moreover there are 30 girls in the cinema. 20 seats are empty. How many seats has the cinema?"

The students can use different heuristic procedures to solve this problem, e.g. an informative figure, a table of systematic attempts, an equation or combined forward and backward working.

PERFORMANCE DEVELOPMENT AND ANALYSIS OF HEURISTICS OVER THREE MEASURING TIMES

The results of the problem-solving capability of the 170 students attained over the three test periods were as follows: the students improved their performance significantly during the project year. On average they were able to successfully solve one to two more tasks. No differences were observed during the project year regarding the further training content of the teachers, i.e. all groups (PL, PS, SR) developed similarly and showed clear performance development. Significant performance improvements were confirmed also in the follow-up-study. Students whose teachers had been trained in problem-solving and self-regulation (PS) in the project year achieved the best results between the post test period and the follow-up-study.

Heuristic procedures have a central importance in the teaching concept. In order to analyse to what extent heuristic procedures had been used as part of the teaching concept to work on problem-solving tasks the solution methods for 6 of 12 continuous test items were analysed with respect to the heuristics used by the students. One heuristic point was given per used heuristic. The results of this study show that the students in the pre-test are using more or less one heurism. On average it was possible to analyse two to three heurisms in the post test period and in the follow-up-study. The number of the heuristics used by the students in the follow-up-study is stable compared with the post test period. The results show that heuristic approaches thematized in the further teacher training are reflected in the solutions of the students. Frequently used heurisms within certain test items in the post test period were also used in the follow-up-study. The number of the heuristic procedures adopted by the students is almost maintained in the follow-up-study.

RESULTS OF TEST ITEM “CINEMA“

Performance increases over the three test periods become also evident in the test item “cinema“. While only 9% of the students had worked successfully on this test item initially, 19% (post test) and 25% (follow-up test) respectively were able to work successfully on this test item in the post test period. Both in the post test and in the follow-up-study the students mostly applied forward and backwardd working (50 respectively 59 times), an equation (7 respectively 20 times) or an informative figure (4 respectively 8 times) as a heuristic approach. Fig. 2 shows the solution of this test item by a student on the basis of an informative figure.

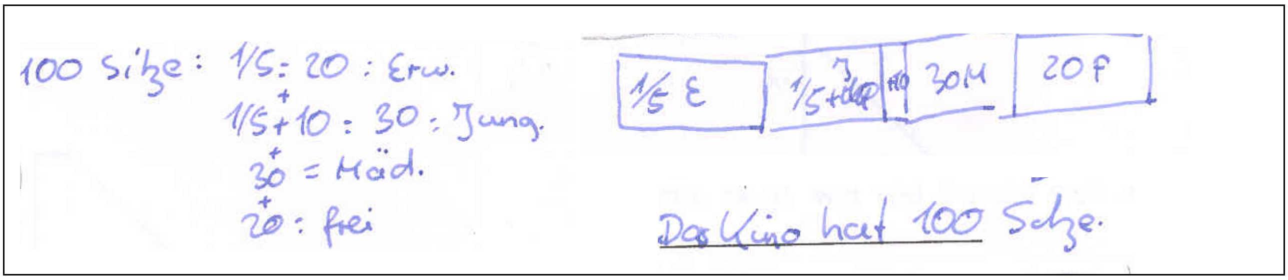

Figure 2. Student solution for the test item “cinema“ - student No. SR_AN09 (E: adults, J: boys, M: girls, F: free seats).

DEVELOPMENT OF PERFORMANCE GROUPS OVER THE THREE TEST PERIODS

In order to analyze the development of performance groups the students were divided into three performance groups: low, medium and good attainers, regarding their problem-solving capabilities at the beginning of the project in the pre-test. The performance groups were established according to the content of the test items. Fig. 3 shows the development of student performance in the three performance groups after division into these groups depending on their performance in the pre-test for the 12 anchor items.

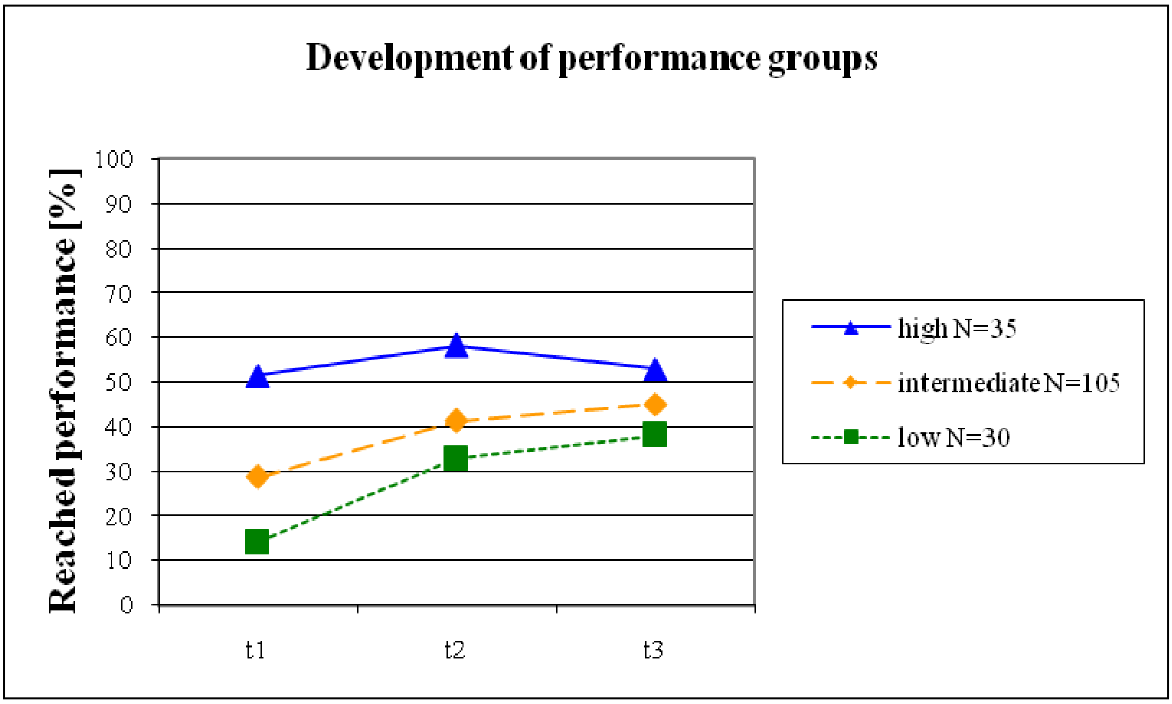

Figure 3. Development of the performance groups depending on pre-test performance.

All performance groups show significant performance increases between pre-test (t1) and post test (t2). The students classified as low attainers in the pre-test reached the level of medium attainers in the pre-test. Low and medium attainers showed significant performance increases between post test and follow-up-study. The development of performance groups between post test and follow-up-study is basically stable regarding the successful processing of tasks.

STABILITY CHECK OF PROBLEM-SOLVING CAPABILITY

A comparison of the results of the student performance at the end of class 8 was intended to find out if the students who had been taught according to the teaching concept in class 7 showed comparable or better attainment at the end of class 8 than students who were taught with the same teaching concept in class 8. All higher-track school students from class 7 whose performance was analyzed over the three test periods (N=170) and all students from class 8 who had participated in the pre- and post test (N=283) were included. Figure 4 illustrates the development of the performance of students from the beginning of class 7 or 8 with regard to the 12 continuous test items. Both groups developed equally between the pre-test (started at the beginning of class 7 or at beginning of class 8) and post test (finished at the end of class 7 or end of class 8) and reached similar performance levels before the beginning of the further training. They reached a comparable level of performance by the end of class 8 (ANOVA).

Figure 4. Performance development of classes for anchor items.

DISCUSSION

The present report shows that on the basis of the teaching concept developed for problem-solving in connection with self-regulation in maths lessons at secondary school level I, teacher competencies for relevant aspects of problem-solving can be enhanced and student competencies for problem-solving developed. The results of the follow-up study run in 10 former project 7th classes in secondary schools confirm the stability of the problem-solving capabilities of the students. This result together with the results of the main study can be evaluated as a success regarding the further teacher training as well as of the teaching concept. Since 2005 the teaching concept has been implemented on the learning platform moodle (www.prolehre.de) for online teacher training following a blended-learning-system.

References

Bruder, R. (2003). *Learning Methods and Techniques How to Solve Problems.* Available at: http://www.math-learning.com/index_e.html

Büchter, A. & Leuders, T. (2005). *Quality Development in Mathematics Education by Focussing on the Outcome: New Answers or New Questions? Introduction.* In. ZDM, 37(4), 263-266.

Collet, C., Bruder, R., & Komorek, E. (2007). Self-monitoring by lesson reports from teachers in problem-solving maths lessons. In J.-H. Woo, H.-C. Lew, K.-S. Park, & D.-Y. Seo (Hrsg.), *Proceedings of the 31th Conference of the International group for the Psychology of Mathematics Education.* 2,169-176.

Collet, C. & Bruder, R. (2006). Evaluation of a teaching concept for the development of problem solving competencies in connection with self-regulation. In: J. Novotná, H.

Moraová, M. Krátká, & N. Stehlíková, (Hrsg.). *Proceedings of the 30th Conference of the International group for the Psychology of Mathematics Education.* 2, 345-352.

Da Ponte, J. P. (2007). *Investigations and Explorations in the Mathematics Classroom.* In. ZDM Mathematics Education, 39, 419-430.

De Corte, E., Verschaffel, L., & Op´t Eynde, P. (2000). Self-regulation. A characteristic and a goal of mathematics education. In M. Boekaerts, P. R. Pintrich, & M. Zeidner (Eds.), *Handbook of Self-Regulation*. 687-725, San Diego: Academic Press,

Heinze, A. (2007). Problemlösen im mathematischen und außermathematischen kontext. Modelle und unterrichtskonzepte aus kognitionstheoretischer perspektive. In. *Journal für Didaktik Mathematik* (JDM), 28(1), 3-30.

Komorek, E., Bruder, R., Collet, C., & Schmitz, B. (2007). Contents and results of an intervention in maths lessons in secondary level I with a teaching concept to support mathematic problem-solving and self-regulative competencies. In. M. Prenzel (Eds.), *Studies on the Educational Quality of Schools. The Final Report on the DFG Priority Programme.* Münster: Waxmann, S. 175-196.

Lester, F. K. & Charles, R. I. (1992). A Framework for Research on Problem-Solving Instruction. In J. P. Ponte, J. F. Matos, J. M. Matos, & D. Fernandes (Eds.), *Mathematical Problem Solving and New Information Technologies.* 1-15 Springer-Verlag: Berlin, Heidelberg,

National Council of Teachers of Mathematics (2000). *Principles and Standards for School Mathematics.* Reston: NCTM.

Polya, G. (1945). *How to Solve it.* Princeton, NJ: Princeton University Press.

Schmitz, B. & Wiese, B. S. (2006). New perspectives for the evaluation of training sessions in self-regulated learning: Time-series analysis of diary data. *Contemporary Educational Psychology,* 31, 64-96.

Schoenfeld, A.H. (1985), *Mathematical Problem Solving.* Orlando, Florida: Academic Press, Inc.

Törner, G., Schoenfeld, A., & Reiss, K. (Hrsg.) (2007). *Problem Solving Around the Word: Summing up the State of the Art.* In ZDM, 39, 5-6. Springer: Berlin, Heidelberg.

Zimmerman, B. J. (2000). Attaining self-regulation: A social cognitive perspective. In M. Boekaerts, P. R. Pintrich, & M. Zeidner (Eds.), *Handbook of Self-regulation.* 13-41San Diego, CA: Academic Press.

EXPANDED TOULMIN DIAGRAMS: A TOOL FOR INVESTIGATING COMPLEX ACTIVITY IN CLASSROOMS

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Krummheuer's adaptation of Toulmin's model of argumentation has been widely used to examine collective argumentation in mathematics classrooms. I propose that an expansion of these diagrams provides a useful tool for examining both the role of the teacher in facilitating argumentation and levels of complexity in argumentation. The use of color or other symbols to denote the contributor of each part of an argument along with diagramming sub-arguments as connected to main arguments in episodes of argumentation allow more specific information to be recorded in each diagram. These expanded diagrams allow different approaches to new and relevant questions about learning, teaching, and classroom interaction by highlighting distinctions between argumentation in different classrooms.

COLLECTIVE ARGUMENTATION IN MATHEMATICS CLASSROOMS

Krummheuer's (1995) description and adaptation of Toulmin's (1958/2003) model of argumentation has been widely adopted in the mathematics education research community for the study of collective argumentation. New questions about the learning and teaching of mathematics through argumentation require an extension of Krummheuer's work through an expansion of these diagrams that captures additional details of the collective argumentation that occurs within whole-class discussions in mathematics classes. These expanded diagrams allow for analysis of classroom interactions in contexts beyond those already examined and with emphases on both the teachers' role and student learning.

Toulmin introduced a structural model of an argument with four main parts (claim, data, warrant, and backing) and two parts that concern the strength and applicability of the warrant (qualifier and rebuttal). Toulmin's use of *argument* is in the classic rhetorical sense of an individual who attempts to convince an audience of the veracity of a claim. The individual and audience operate in a "field" - such as law, science, or mathematics - within which particular backings are accepted as valid. In other words, Toulmin suggested that arguments are valid within particular fields because of the validity of the warrants and backings in that field, even though the basic form or layout of an argument is consistent across many fields (see Figure 1). Toulmin defined each part of an argument as follows: a claim is the statement whose truth is being established; data is evidence presented in support of the claim; a warrant is a bridge between the data and claim, giving reasons that the particular data presented is relevant to the claim; backing, which is usually implicit, is support for the warrant's validity in the particular field in which it is used; a qualifier is indicative of the strength of the warrant (usually a word such as "probably"); and a rebuttal is a description of circumstances under which the warrant would not be valid.

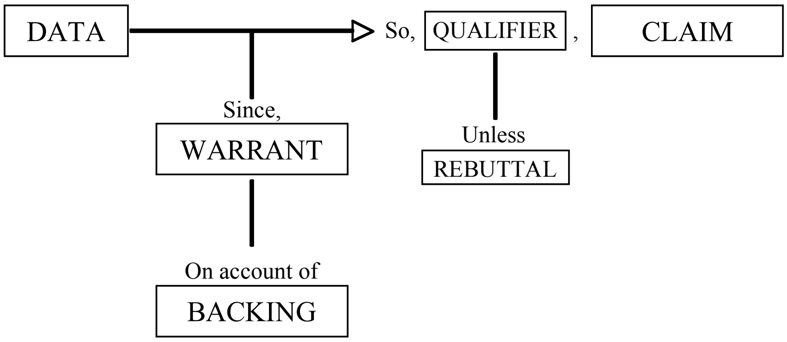

Figure 1. Layout or structure of an argument, adapted from Toulmin (1958/2003).

Krummheuer adapted Toulmin's model of an argument constructed by an individual to one that is appropriate for collective argumentation. Collective argumentation occurs when a group of people (often several students and a teacher) work together to establish the validity of a claim. In his work, Krummheuer focused primarily on what he called "the 'core' of an argument" (p. 243), which was composed of a claim with relevant data and warrant. Krummheuer used the concept of framing to relate Toulmin's field of argument to the context of a mathematics classroom. Within the field of mathematics education, or a particular mathematics education classroom, certain warrants may be accepted as valid (while others are rejected), that is, there are certain backings that are "collectively accepted basic assumptions" (p. 244). On the other hand, individuals have different experiences and different ways of creating meaning. Because of these framings, individuals may accept, infer, or imply different backings for the same warrant in the same argument. Since backings are usually implicit, it is only possible to know what backing an individual intends for a warrant if he or she states it explicitly, and even then, one's understanding of that backing may be different from what the individual intended. Krummheuer argued that the core of an argument is interactively established by participants in the argumentation, and it is the core that is particularly important to thinking about how learning may occur, since the backing is generally implicit and is thus not generally accessible to an observer. On the other hand, Inglis, Mejia-Ramos, and Simpson (2007) argued that qualifiers and rebuttals are important for mathematical argumentation and, as such, should be included in analyses of collective argumentation.

The usefulness of Krummheuer's (1995) adaptation of Toulmin's (1958/2003) model of argumentation for studying student activity within classrooms that are oriented toward problem-solving and class discussions has been well-established. Krummheuer's initial assumption in proposing the analysis of collective argumentation was that collective argumentation contributes to student learning in such classrooms, and, as such, is worth studying. Whitenack and Knipping (2002) used this model to describe opportunities for student learning during collective argumentation in a second grade classroom and hypothesized that the learning they documented through their study may have been attributable to the episodes of argumentation they observed. Yackel (2001) used Toulmin's structure to examine student learning in second grade and college classes, and she suggested that classroom instruction that emphasizes argumentation leads to learning with an emphasis on reasoning.

Several researchers have used Krummheuer's (1995) adaptation of Toulmin's (1958/2003) model of argumentation to examine the work of the teacher in classrooms characterized by an explicit focus on argumentation. Forman, Larreamendy-Joerns, Stein, and Brown (1998) suggested that examining classroom argumentation allows one to focus on both the students' contributions to the class and the teacher's roles within the classroom. They also suggested that examining collective argumentation may give insight into "the impact of educational reform in mathematics" (p. 547). Forman and Ansell (2002) described the main activities of the teacher during collective argumentation as soliciting contributions from students, asking questions to clarify these contributions, and revoicing their contributions. Yackel (2002) has demonstrated the crucial role of the teacher in ensuring that the data and warrants for claims are made explicit in the classroom and in guiding the discussion in order to highlight appropriate mathematical ideas.

Krummheuer's adaptation of Toulmin's model to collective argumentation in mathematics classrooms has been used successfully to examine student learning and to describe the roles and activities of the teacher in classes where one of the main goals of the teacher was to facilitate argumentation (or learning through argumentation). This model, and extensions of this model, may be useful for examining aspects of practice even in classrooms where there is no explicit goal of promoting argumentation, where the emphasis on small group problem solving is less.

In a recent study[1], I used an adaptation of Toulmin's model, based on Krummheuer's work, to examine the practice of three student teachers, Jared[2], Karis, and Lynn. These three student teachers did not have specific goals of facilitating argumentation and used primarily whole class instruction. My goal was to compare how they facilitated or supported argumentation to their conceptions of proof (see Conner, 2007, for more details). As I examined the teachers' roles in facilitating argumentation, it was necessary to distinguish between parts of arguments contributed by the teachers and parts of arguments contributed by their students. In addition to finding that the student teachers' facilitation of argumentation aligned with their conceptions of the purpose and need for proof in mathematics, within this analysis, I observed, as suggested by Whitenack and Knipping (2002), that arguments were often part of longer, more complex episodes of argumentation. I took an episode

[1] This paper is based in part on the author's doctoral dissertation, completed at The Pennsylvania State University under the direction of Rose Mary Zbiek, supported in part by the National Science Foundation under Grant No. ESI0083429 to the University of Maryland with a major subcontract to The Pennsylvania State University and by a research initiation grant from The Pennsylvania State University College of Education Alumni Society. Any opinions, findings, and conclusions or recommendations expressed in this document are those of the author and do not necessarily reflect the views of the National Science Foundation or the College of Education Alumni Society.

[2] All names are pseudonyms to protect the identity of participants.

of argumentation to be a claim together with the data and warrants that supported it and any sub-arguments that were constructed to support the validity of the data or warrants for that claim. Thus, while some episodes contained only one argument, others contained up to nine.

ADDITIONAL INFORMATION FROM MODIFIED DIAGRAMS

Within published studies, arguments are most often presented individually, with data, warrant, and sometimes an inferred backing for each claim. Depending on the study, the argument may be marked as being contributed by an individual or parts of the argument may be attributed to individuals. I propose that more information about collective argumentation, and particularly about roles of the participants in collective argumentation, can be contained in and displayed by a modification of the commonly-used diagram.

Use of color and line style to indicate contributions from students or teacher.

In order to differentiate between the contributors of parts of arguments while still maintaining the form of the diagrams, I used color and line-style to differentiate between parts contributed by the student teachers, parts contributed by the students, and parts interactively contributed by the teachers and students together. Because the study was concerned with the role of the teacher in supporting argumentation, it was not necessary to distinguish precisely which student contributed each part. If this were necessary, using color or line style might become too cumbersome to be efficient, but with only three styles used, it was straightforward to differentiate between the contributions of students and the student teachers, as illustrated in Figure 2.

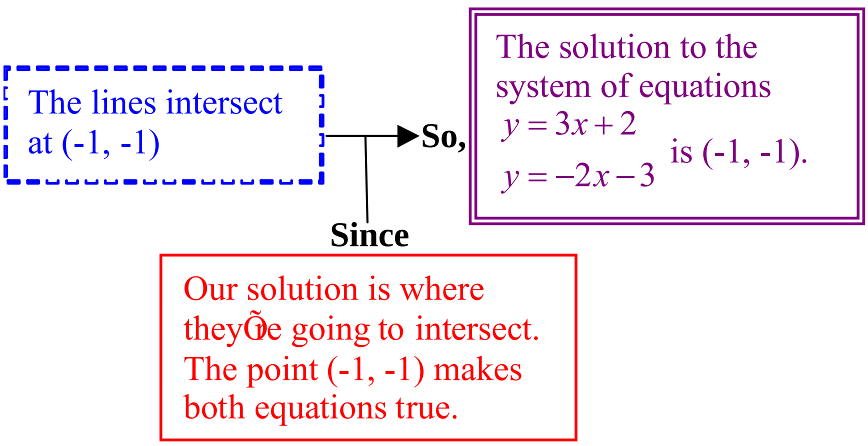

Figure . Example of argument from Jared's algebra class; denotes a student contribution, denotes a teacher contribution, and denotes that both contributed.

While, like Krummheuer, I only diagrammed the core of each argument, employing the concept of framing allowed me to use the warrants that were made explicit along with observations of when warrants were left implicit, to infer the backings the student teachers were using in the collective argumentation in their classes. Because the analysis centered on the student teachers, it was not necessary to infer the backing accepted or implied by the student; it was only necessary to examine the words and actions of the teacher within and across episodes of argumentation, accepting that the students' framing would suggest a myriad of possible backings.

Use of extended diagrams to indicate complexity and depth of argument

To account for the complexity of argumentation within the observed classes, the diagrams were modified to show several levels of sub-arguments while maintaining the styles that signified contributors (see Figure 3). While this was sometimes logistically difficult, patterns in contributions were apparent in this form that could be seen much less readily if each argument was diagrammed separately from the episode in which it occurred. For instance, examining the diagrams of episodes of argumentation led to the hypothesis that in Karis' calculus class, Karis tended to contribute more parts of arguments when the episode of argumentation was longer, that is, when it included more sub-arguments. In addition, these diagrams would allow an analysis of the complexity of argumentation in a classroom, and then an investigation of who contributed to the complex argumentation. In Jared's algebra class, it became clear that arguments dealing with systems of equations tended to be more complex than those in the next unit, involving polynomial expressions, if complexity is measured by number of sub-arguments in an episode of argumentation. Of course, simply counting sub-arguments is not enough to completely characterize complexity. But, an episode of argumentation that contains several sub-arguments, supporting several different parts of the argument, as well as additional sub-sub-arguments, is likely to be more complex than an argument that consists of a claim, a datum, and a warrant.

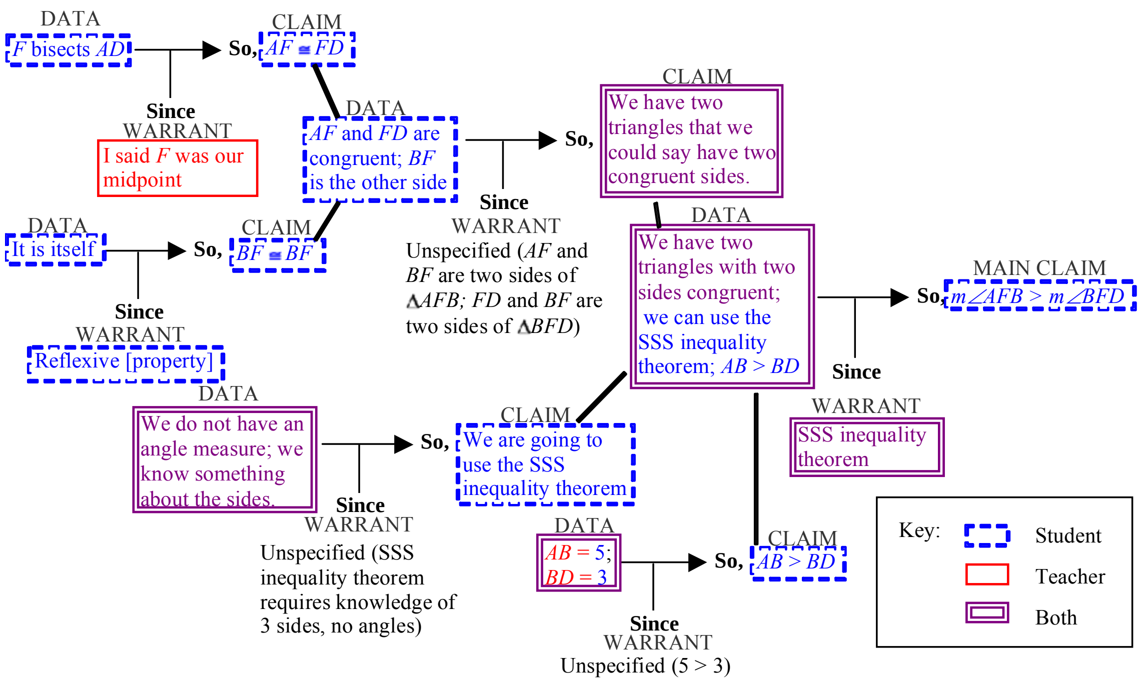

Figure 3. Example of episode of argumentation from Lynn's geometry class; consists of a main argument, three sub-arguments, and two sub-sub-arguments; refers to quadrilateral ABDC in Figure 4.

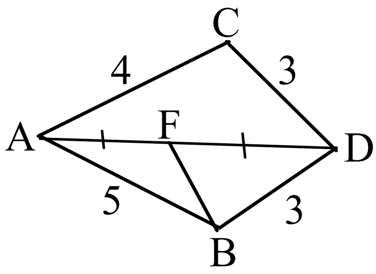

Figure 4. Quadrilateral ABDC, referenced in the argument diagrammed in Figure 3.

When the analysis involves diagrams such as Figure 3, it is important to go back and forth between individual arguments and their place within the structure of the larger episode. For instance, one of the sub-arguments in Figure 3 has a claim (we have two triangles that we could say have two congruent sides) and data (*AF* and *FD* are congruent; *BF* is the other side), but no specified warrant. However, these data are supported by two sub-arguments, each of which has a complete core, and each of which refers to the diagram in Figure 4, at least implicitly. This gives credence to the inferred warrant for the original sub-argument and serves to begin to illustrate the complex nature of the argumentation in this class. To really characterize the complex nature of the argumentation and the utility of these diagrams, it would be necessary to examine multiple diagrams of episodes of argumentation and compare and contrast the various features of them.

Knipping (2003) used a modified diagram to analyse and compare the structure of argumentations in proving situations. These diagrams were similar to the ones described, but did not retain the specifics of the argument or the contributor. Instead, shapes were used to denote parts of the argument. For instance, a rectangle represented the main claim, a circle represented a claim or data, and a square represented a warrant or backing. According to her key, the episode of argumentation diagrammed in Figure 3 would be diagrammed as in Figure 5.

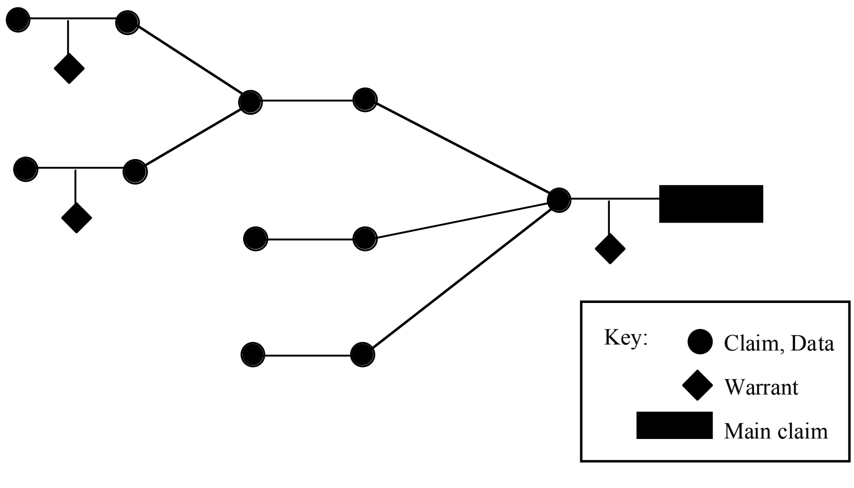

Figure 5: Argument from Figure 3 diagrammed according to Knipping's (2003) scheme to show structure.

I propose that a combination of these modifications would give the most information to the investigator, allowing for an investigation of structure, complexity, and roles of students and teacher in investigating collective argumentation in mathematics classrooms. For instance, a condensed form of argument layout, in which shapes as used by Knipping (2003) were enhanced by color or line style, would allow for an investigation of teacher contributions while maintaining the structural emphasis, allowing for an investigation of, for instance, differences in support for argumentation in classrooms with clearly different argumentation structures. On the other hand, the use of shapes in the background to denote parts of arguments in a diagram such as the one seen in Figure 3 may allow for other pertinent details to be brought to the forefront.

As the study of teaching and learning through collective argumentation begins to be situated in classroom contexts where argumentation is not necessarily an explicit goal of the teacher's instruction, it is important to have tools to distinguish between structures and patterns of argumentation. Investigating the structure of argumentation allows for a characterization of classrooms in which the argumentation is more fruitful (if an analysis of student learning is also carried out). These modified diagrams and the accompanying extensions of analysis allow investigation of diverse and complex questions, including an examination of the teacher's role in argumentation and a search for what components of the teacher's knowledge, beliefs, and experiences may impact the observed patterns, structures, and facilitation of collective argumentation within classrooms.

References

Conner, A. (2007). Student teachers' conceptions of proof and facilitation of argumentation in secondary mathematics classrooms. (Doctoral dissertation, The Pennsylvania State University, 2007). *Dissertations Abstracts International, 68*/05, Nov. 2007 (UMI No. AAT 3266090).

Forman, E. A. & Ansell, E. (2002). Orchestrating the multiple voices and inscriptions of a mathematics classroom. *The Journal of the Learning Sciences, 11*(2&3), 251-274.

Forman, E. A., Larreamendy-Joerns, J., Stein, M. K., & Brown, C. A. (1998). "You're going to want to find out which and prove it": Collective argumentation in a mathematics classroom. *Learning and Instruction, 8*(6), 527-548.

Inglis, M., Mejia-Ramos, J. P., & Simpson, A. (2007). Modelling mathematical argumentation: The importance of qualification. *Educational Studies in Mathematics, 66*, 3-21.

Knipping, C. (2003). *Argumentation Structures in Classroom Proving Situations.* Paper presented at the Third Congress of the European Society for Research in Mathematics Education, Bellaria, Italy.

Krummheuer, G. (1995). The ethnography of argumentation. In P. Cobb & H. Bauersfeld (Eds.), *The Emergence of Mathematical Meaning: Interaction in Classroom Cultures* (pp. 229-269). Hillsdale, NJ: Lawrence Erlbaum Associates.

Toulmin, S. E. (Updated ed.) (2003). *The uses of Argument*. New York: Cambridge University Press. (First published in 1958).

Whitenack, J. W. & Knipping, N. (2002). Argumentation, instructional design theory and students' mathematical learning: a case for coordinating interpretive lenses. *Journal of Mathematical Behavior, 21*, 441-457.

Yackel, E. (2001). *Explanation, Justification, and Argumentation in Mathematics Classrooms.* Paper presented at the 25th Conference of the International Group for the Psychology of Mathematics Education, Utrecht, The Netherlands.

Yackel, E. (2002). What we can learn from analyzing the teacher's role in collective argumentation. *Journal of Mathematical Behavior, 21*, 423-440.

GENERALISING MATHEMATICAL STRUCTURE IN YEARS 3-4: A CASE STUDY OF EQUIVALENCE OF EXPRESSION

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This paper reports on a section of the Early Algebraic Thinking Project (EATP) which focused on Australian Years 3-4 (age 7-9) students' abilities to generalise mathematical structure in relation to equivalence of expressions (with and without unknowns). It focuses on learning activities involving a sequence of representations to show that change resulting from addition-subtraction requires the performance of the opposite change (subtraction-addition respectively) by the same amount in order to return to the original state (e.g., $x = x+p-p$ or $x-q+q$ in algebraic symbols). It shows that children of this age can generalise this mathematical structure and that effective teaching for generalisation uses creative representation-worksheet partnerships.

EATP was a five-year longitudinal project that studied a cohort of students progressively from Years 2 to 6 deriving from 5 inner city middle class state schools in Queensland. The cohort was chosen for their early algebraic thinking, particularly their ability to generalise mathematical structure in patterning, function and equation situations. For EATP, mathematical structure is built around relationship and change (Linchevski, 1995; Scandura, 1971) and is constrained by principles i.e. powerful mathematical ideas where meaning is encoded in the structure between the components not in the form of the components (Ohlsson, 1993). (Note: EATP was funded by Australian Research Council Linkage grant LP0348820.)

An expression is a combination of numbers, operations and/or variables (e.g., 7, 2x5+3, $4x-3$) while an equation is equivalence of expressions (e.g., 13=2x5+3, $4x-3=2x+5$). Expressions are equivalent if the change from one to another is by addition/subtraction of 0 or by multiplication/division by 1. EATP has studied two particular principles associated with equivalence of expressions: the compensation principle, which comes from a relationship view of structure (e.g., 8+5=8+2+5–2=10+3); and the backtracking principle, which comes from a change view of structure (e.g.,?=?+5–5, so ?+5=11 means ?=11-5). EATP has studied how both these principles can be generalised by Year 3-4 students; this paper only focuses on the backtracking principle.

Generalisation and representation. For EATP, early algebra is a way of studying arithmetic that develops number sense, algebraic reasoning and deep understanding of structure (Carraher, Schliemann, Brizuela & Ernest, 2006; Fujii & Stephens 2001; Steffe, 2001). The basis of early algebra (and mathematics in general) is generalisation (Kaput, 1999; Lannin, 2005), for example, generalising from tables of values and patterns to relationships between numbers and pattern rules; and generalising from particular examples in real-world situations to abstract representations, principles and structure. There has been general consensus for some time that mathematical ideas are represented externally and internally (Putnam, Lampert & Petersen, 1990) and that mathematical understanding is the number and strength of the connections in a student's internal network of representations (Hiebert & Carpenter, 1992). It has long been argued that generalising mathematics structures involves determining what is preserved and what is lost between the specific structures which have some isomorphism (Gentner & Markman, 1994; English and Halford, 1995).

EATP's research (Warren, 2006; Cooper & Warren, in press), and that of others (e.g., Carraher et al., 2006; Dougherty & Zilliox, 2003), has shown that young students can generalise to principles. In developing these generalisations, EATP has been influenced by: (i) the reification sequence of Sfard (1991); and (ii) the Mapping Instruction approach of English and Halford (1995). In analysing the act of generalisation, EATP has used: (i) the three generalisation levels of Radford (2003, 2006), factual (gesture driven), contextual (language driven) and symbolic (notation driven); (ii) the two components of Radford, grasping and expressing: (iii) the two generalisation forms of Harel (2001), results (from examples) and process (with justification); and (iv) the quasi-variable notion of Fuji and Stephens (2001). EATP's research suggests that quasi-variable is extendable to generalisation to give a notion of quasi-generalisation, and that the ability to express generalisation in terms of numbers is a step towards full generalisation (Warren, 2006; Cooper & Warren, in press). In designing activities to enable these generalisations, EATP has been influenced by: (i) the four step sequence of Dreyfus (1991), one representation, more than one representation in parallel, linking parallel representations, and integrating representations; (ii) the argument of Duval (1999) that mathematics comprehension results from coordination of at least two representation forms or registers; the multifunctional registers of natural language, and figures/diagrams, and the mono-functional registers of notation systems (symbols) and graphs; and (iii) the contention of Duval that learning involves moving from treatments to conversions to the coordination of registers.

DESIGN OF EATP

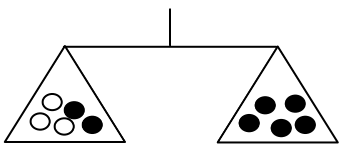
The methodology adopted for EATP was a longitudinal and mixed method using a design research approach, namely, a series of teaching experiments that followed a cohort of students based on the conjecture driven approach of Confrey & Lachance (2000). It was predominantly qualitative and interpretive (Burns, 2000) but with some quantitative analysis of pre-post tests. In each year, the teaching experiments investigated the students' learning in lessons on patterning and functional thinking (using the change perspective), and equivalence and equations (using the relationship perspective). EATP was based on a re-conceptualisation of content and pedagogy for algebra in the elementary school and as such the teaching experiments were exploratory in nature. The representations chosen were intended to be inclusive of all students; however, the necessity to respond to individual student needs was a position acknowledged from the outset. Multiple sources of data were collected and only those findings for which there was triangulation were considered in analysis. Adequate time was spent in the field observing the lessons to substantiate the reliability of the collected data. The instruments used were classroom observations (video and field notes), teacher and student interviews (planned and ad hoc), teacher reflections, yearly and pre-post tests, and artefacts (students' work).

The particular lessons for this paper encompassed teaching the backtracking principle for addition and subtraction as part of the process leading to solving simple addition and subtraction problems for unknowns. They were conducted in a Year 3 (22 students) classroom in a middle class school and a Year 4 (28 students) classroom in a working class school. The Year 3 lesson was conducted following a sequence of lessons introducing the balance rule for addition and subtraction and was designed to be taught with resources including bags containing objects, representing the unknown, a balance beam, and pictures and symbols on worksheets. The Year 4 lesson was undertaken before a similar series of lessons. It involved applying the balance rule to simple addition and subtraction problems with unknowns, and was designed to be taught with a number line and pictures and symbols on worksheets. For both lessons, the worksheets were especially developed to reinforce the backtracking principle. Students were asked to predict and justify in both lessons with no explicit requests to generalise to any number. The lessons used the enquiry approaches of Mapping Instruction (English & Halford, 1995) to discover similarities across different examples and representations.

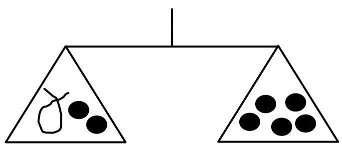
FINDINGS AND DISCUSSION

The data collected was a combination of audio and video transcriptions, pre-post testing, graded worksheets displayed in Excel spreadsheets, field notes and written reflections. This information provided rich descriptions of each teaching experiment that contained relative information between the teaching action and students learning responses, in relation to records of performance and performance change. These descriptions were then analysed for evidence of student learning and generalisation processes followed for that learning.

Year 3 lesson. This lesson focused on addition equations, representing them on a beam balance with objects (for numbers) and cloth bags containing objects (for unknowns), using balance to represent equals (see Figure 1). The representation did not allow for the operation of subtraction to be modelled.



Equation: 3 + 2 = 5



Equation : ? + 2 = 5

Figure 1. Beam balance representations for equations.

Earlier lessons had: (i) connected the beam balance representation with objects to number equations (see Figure 5); (ii) introduced the balance rule (i.e. adding or removing objects from one side of the equation requires the same action with the same number of objects to the other side); and (iii) introduced the notion of the unknown with the cloth bag. The focus lesson discussed how the value of the unknown could be found by using the balance rule, that is, for ?+2=5, determining that the inverse of the operation, subtracting two from both sides, is the balancing action that will give the value of the unknown. This was reinforced by worksheets showing pictures of unknowns and counters in a balance situation, requesting the balancing action and value of the unknown, followed by a final worksheet requesting balancing action and value of unknown, with equations in symbol form. This worksheet contained some questions with large numbers and operations other than addition, and one question with two operations.

Evidence collected through video showed that most students could determine the unknown for the simple equations represented on the balance. This ability was repeated for the picture worksheet. Table 1 shows the number of students who successfully gave the inverse action in the final worksheet. The number of correct responses was high for addition and for subtraction. The number of correct responses reduced markedly for multiplication, division and for two operations, but it should be noted that there was no reference to, or focussed teaching on, these operations prior to the introduction of the worksheet.

Item: What do you do to both sides?	Correct action
? + 11 = 36	22
? – 7 = 6	19
8 + ? = 3	19
? – 30 =54	15
2 x ? = 12	4
? ÷ 3 = 6	5
3 x ? + 4 = 19	1

Table1. Number of correct responses in terms of inverse balancing action (n=22)

Year 4 lesson. This lesson focused on expressions as well as equations. The students first discussed what was required to reach a solution for equations involving addition with unknowns, that is, to determine an action that would leave the unknown on its own. To do this, the lesson focused on the expression that contained the unknown and the operation, and represented the expression in two ways: first by extending the balance representation in Figure 1 to expressions by removing the balance and the objects for the total and using a number line (see Figure 2).

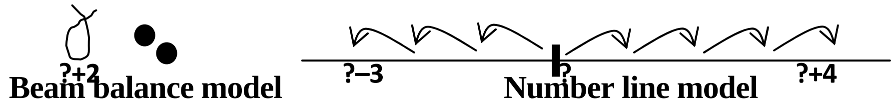


Figure 2. Beam balance and number line representations for different expressions.

The beam balance activity was similar to the Year 3 lessons, except the focus of discussion and worksheets was only on the balancing action, not the unknown's value.

The number line activity was new to Year 4 and required the students to place the unknown anywhere and move right for addition and left for subtraction. After this skill was achieved through discussion and worksheets, the students were challenged to determine the change that would result in returning to the unknown. Discussion focused on generalising the principle that the unknown could be reached by the inverse operation (–4 for ?+4 and +3 for ?–3), as this was equivalent to adding zero. At this point, the learning that had already occurred with regard to functions and identifying their inverses (Warren, 2003, and Warren & Cooper, 2003), reinforced generalisation as did the Mapping Instruction approach of comparing addition and subtraction changes.

A final worksheet was used to ascertain students' understanding of the backtracking principle. It contained items that asked students to draw, for example, ?+6 on the number line and to identify the operation that would result in a return to the unknown. The results were overwhelming; all 28 students were successful for all items except the final two. Twenty-four students correctly answered the first of these items (where the students were requested to draw ?+6 and ? –6 on the same line and give both inverse operations) and 22 correctly answered the second of these items (where the students were requested to draw ?+10 and ?–8 on the same line and give both inverse operations). The number line was a particularly efficacious representation tool for inverse.

However, as a request to write a generalisation was not asked and there were no items that referred to, for example, ?+n, the students were only able to show quasi-generalisation (Fuji & Stephens, 2001) or contextual generalisation (Radford, 2003) at best. Viewing of the video tape showed that some children were able to justify their answers in discussion in a way that indicates process generalisation (Harel, 2001).

Interestingly, the backtracking and balance principles have the opposing actions (the "opposite" operation for inverse and the same operation for balance). After the successful generalising lesson described above which explicitly identified the backtracking principle for expressions with unknowns, some students became confused when this principle was joined with the balance principle to solve for unknowns in later lessons (this is an example of what EATP is calling a "compound" difficulty).

CONCLUSIONS AND IMPLICATIONS

It is difficult to pull conclusions and implications from all the teaching experiments in EATP without a deeper analysis of all the data occurring, including comparisons across generalisations for different principles and structures. However, the two lessons described in this paper indicate the following conclusions.

First, students can learn to understand powerful mathematical structures like the backtracking principle, usually reserved for secondary school, in the early and middle years of elementary school if instruction is appropriate (at least in language and quasi-variable form – Fuji and Stephens, 2001). In EATP, because of separate focus on relationship through equations and change through function machines, there was overlap with regard to the backtracking principle that reinforced inverse in both perspectives. This shows that a teaching focus on structure is a highly effective method for achieving immediate and long term mathematical goals, particularly with respect to portability.

Second, the position that learning is connections between representations (Hiebert & Carpenter, 1992) and conversions between registers and domains (Duval, 1999), was supported. The combination of balance and number line models was particularly powerful. This reinforces the teaching approach of EATP (Warren, 2006) which is based on a socio-constructivist theory of learning, inquiry based discourse and the simultaneous use of multi-representations to build new knowledge. The major representations used in the lesson were effective, particularly in the order that sequences of representations were implemented, from acting out with materials through diagrams to language and symbols. In particular, beam balances, cloth bags and objects and their pictures, integrated with number lines were very effective representations in motivating students, solving problems and building principles and structure.

Third, learning can be enhanced by creative representation-worksheet partnerships. Often teachers restrict worksheets to the symbolic register. EATP has shown that creative use of pictures and directions can allow a worksheet to reinforce understandings (as well as procedures) and to highlight principles.

Fourth, English and Halford's (1995) Mapping Instruction teaching approach to principle generalisation has proved its efficacy in this and many other EATP lessons. It directs us towards comparing activity from different domains (e.g., addition and subtraction) and activity from different representations (e.g., balance and length).

Fifth, although they were developed for older students, some theories regarding development of generalisation have application in early generalisation. This is particularly so for Radford's (2003) theory regarding factual and contextual levels of generalisation, Harel's (2001) theory regarding results and process generalisation, and Fuji and Stephens (2001) notion of quasi variable (which we have adopted as quasi-generalisation). Harel directs us towards justifying as well as identifying generalisation, Radford towards role of gestures (action, movement) and language in early generalisation and Fuji and Stephens towards the acceptability of number-based descriptions of generalisations. As well, Radford's distinction between grasping and expressing generalities was important; these are two aspects often confused by the teacher. In many instances, students' problems with generalisation were with expressing the generalisation, not grasping it. Students often lacked the language with which to discuss generalisation and lessons often became a focus on language development.

Sixth, some activities necessary for building structure affect cognitive load. This is particularly so when large numbers are used to prevent guessing and checking as a strategy for determining answers and to direct students towards the principle. Furthermore, the example in this paper has shown the "compounding" effect of building structure through small steps, with the conflict that occurred between the balance and backtracking principles. It is necessary to build a superstructure into which to place conflicting principles such as backtracking and balance for finding solutions of linear equations.

Finally, although EATP involved creative lesson development and many new activities and outcomes, the students' problems in these lessons as well as in other EATP lessons did not really lie with the new work, but with the basic arithmetic prerequisites. As soon as numbers appeared, students attempted to close on operations and did not attend to pattern and structure to the same extent as in un-numbered situations (similar to findings of Davydov, 1975, supported by Dougherty & Zilliox, 2003). Furthermore, students' abilities to interpret and create real world situations in terms of the actions with materials, diagrams/figures and symbols of early algebra, lagged behind their abilities to process the representations and was a constant difficulty in EATP, a difficulty that increased as the cohort of students moved into middle school years.

References

Burns, R. B. (2000). *Introduction to Rsearch Methods (*4th edn). French's Forest, NSW: Pearson.

Carraher, D., Schliemann, A. D., Brizuela, B. M., & Earnest, D. (2006). Arithmetic and algebra in early mathematics education. *Journal of Research in Mathematics Education, 37*(2), 87-115.

Confrey, J., & Lachance, A. (2000). Transformative teaching experiments through conjecture-driven research design. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of Research Design in Mathematics and Science Education* (pp. 231-265). Mahwah, NJ: Lawrence Erlbaum.

Cooper, T. J. & Warren, E. (in press). The effect of different representations on Years 3 to 5 students' ability to generalise. In *ZDM.*

Davydov, V. V. (1975). The psychological characteristics of the "prenumeral" period of mathematics instruction. In L. P. Steffe (Ed.), *Children's Capacity for Learning Mathematics,* Vol 7 (pp. 109-205). Chicago: University of Chicago.

Dougherty, B. & Zilliox, J. (2003). Voyaging from theory and practice in teaching and learning: A view from Hawaii. In N. Pateman, B. Dougherty, & J. Zilliox (Eds.), *Proceedings of the 27th Conference of the International Group for the Psychology of Mathematics Education,* 1, 31-46. College of Education: University of Hawaii.

Dreyfus T. (1991). Advanced mathematical thinking proceses. In D. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 25-41). Dordtrecht: Kluwer Academic Publishers

Duval, R. (1999). Representations, vision and visualization: Cognitive functions in mathematical thinking. Basic issues for learning. In F. Hitt & M. Santos (Eds.), *Proceedings of the 21st Conference of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol 1, pp. 3-26).

English, L. & Halford, G. (1995). *Mathematics Education: Models and Processes*. Mahwah, NJ: Lawrence Erlbaum

Fujii, T. & Stephens, M. (2001). Fostering understanding of algebraic generalisation through numerical expressions: The role of the quasi-variables. In H. Chick, K. Stacey, J.Vincent &

J.Vincent (Eds.), *The Future of the Teaching and Learning of Algebra. Proceedings of the 12th ICMI study Conference* (Vol 1, pp. 258-64). Melbourne: Australia.

Gentner, D. & Markman, A.B. (1994). Structural alignment in comparison: No difference without similarity. *Psychological Science*, *5*(3), 152-158.

Harel, G. (2001). *The Development of Mathematical Induction as a Proof Scheme: A Model for DNR-Based Induction.* Cited Feb 2007. math.ucsd.edu/harel.

Hiebert, J., & Carpenter, T. P. (1992). Learning and teaching with understanding. In D. Grouws (Ed.), "Handbook for research on mathematics teaching and learning" (pp. 65-97). New York: MacMillan.

Kaput, J. (1999). Teaching and learning a new algebra. In E. Fennema & T. Romnberg (Eds.), *Mathematics Classrooms that Promote Understanding* (pp. 133-155). Mahway, NJ: Lawrence Erlbaum.

Lannin, J. (2005). Generalization and Justification: The Challenge of Introducing Algebraic Reasoning Through Patterning Activities. *Mathematical Thinking and Learning, 7*(3), 231-258.

Linchevski, L. (1995). Algebra with number and arithmetic with letters: A definition of pre-algebra. *Journal of Mathematical Behavior*, *14*, 113-120.

Ohlsson, S. (1993). Abstract schemas. *Educational Psychologist, 28*(1), 51-66.

Putnam, R.T., Lampert, M., & Peterson, P. L. (1990). Alternative perspectives on knowing mathematics in elementary schools. *Review of Research in Education, 16*, 57-149.

Radford, L. (2003). Gestures, Speech, and the Sprouting of Signs: A Semiotic-Cultural Approach to Students' Types of Generalization. *Mathematical Thinking and Learning, 5*(1), 37-70.

Radford, L. (2006). *Algebraic Thinking and the Generalization of Patterns: A Semiotic Perspective.* Paper presented at the 28th annual meeting of the International Group for the Psychology of Mathematics Education (NA), Merida, Mexico.

Scandura, J. M. (1971). *Mathematics: Concrete behavioural foundations*. New York: Harper & Row.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Education Studies in Mathematics, 22*(1), 1-36.

Steffe, L. (2001). What is algebraic about children's numerical reasoning. In H. Chick, K. Stacey, J. Vincent & J. Vincent (Eds.), *The Future of the Teaching and Learning of Algebra. Proceedings of the 12th ICMI study conference* (Vol 2, pp. 556-563). Melbourne: Australia.

Warren, E., (2003). Young children’s understanding of equals: A longitudinal study. In N. Pateman, G. Dougherty, J. Zilliox (Eds.), *Proceedings of the 27th conference of the International Group for the Psychology of Mathematics Education,* 4, 379-387. Hawaii.

Warren, E., (2006). Teacher actions that assist young students write generalizations in words and in symbols. Novotna, J., et al (Eds.), *Proceedings of the 30th Conference of the international group for the psychology of mathematics education,* 5, 377-384. PME: Prague.

Warren E., & Cooper T. J. (2003). Introducing equivalence and inequivalence in Year 2. *Australia Primary Mathematics Classroom*, 8(1), 4-8.

INDIGENOUS VOCATIONAL STUDENTS, CULTURALLY EFFECTIVE COMMUNITIES OF PRACTICE AND MATHEMATICS UNDERSTANDING

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This paper uses Lave and Wenger's (1991) notion of community of practice as a lens to analyse a study in a remote Indigenous Community where Indigenous blocklaying students are being supported to learn the mathematics necessary for certification. The paper shows that the blocklaying students' community of practice is rich in terms of what is shared amongst the members and with whom they interact, involving a sense of service to their community as well as an interest in building. The paper concludes by drawing some implications for teaching mathematics to such students.

As argued in Cooper et al. (2007), Australian remote Indigenous students have the lowest retention and performance rates in Australia's school system (Bortoli & Creswell, 2004; Queensland Studies Authority [QSA], 2004) due to racism, remoteness, English as a second language (ESL), social factors (Fitzgerald, 2001) and systemic issues including non-culturally inclusive forms of teaching, curriculum and assessment (Matthews et al., 2005). Thus, Indigenous unemployment is very high in remote communities leading to a cycle of welfare dependence, disempowerment and the problems identified by Fitzgerald (2001), namely, alcohol and substance abuse, poor mental and physical health, low life expectancy, violence and sexual abuse, and high incarceration rates; this is occurring even when unfilled high-paying skilled jobs in the mining industry are nearby. However, Indigenous Vocational Education and Training (VET) programs within these communities have low retention rates (QSA, 2004) often due to the low education and high anxiety of students with regard to mathematics (Department of Employment, Science and Technology [DEST], 2003; Katitjin, McLoughlin, Hayward, 2000).

The Deadly Maths Group at QUT has entered into a partnership with the Indigenous Lead Centre (a research group set up by the government VET Technical and Further Education [TAFE] Institutes organisation in Queensland) to research and develop effective mathematics programs that assist VET lecturers and trade supervisors, who are untrained in mathematics education. This has emerged from the perceived credibility and success of our work with the Indigenous blocklaying students from the Torres Strait (Cooper et al., 2007) which showed the effectiveness of vocational contexts, structural learning and positive lecturer-student relationships in Indigenous VET mathematics instruction (this research was supported by Australian Research Council grant LP0455667). This paper relooks at this study from a community of practice perspective (Lave & Wenger, 1991; Wenger, 1998) and identifies the particular characteristics and shared practices of the community built within this training program that appeared to relate to the training success.

Communities of practice and student learning.

Lave and Wenger (1991) argue that learning is situated, with the context and culture in which learning takes place inevitably tied up with the type of learning that occurs. They contend that the focus of learning should shift "from the individual as learner to learning as participation in the social world, and from the concept of cognitive process to the more-encompassing view of social practice" (p. 43). This shift is described by Sfard (1998) as a move from an acquisition metaphor, where learning is the accumulation and refinement of information into cognitive structures, to a participation metaphor, where learning is conceived as a process of becoming a member of a certain community and learning activities are never considered separate from the context in which they occur. She argues that the shift involves the permanence of having, giving way to the constant flux of doing.

Hagar (2004) describes the shift of learner from individual acquirer to social participant in terms of product to process. He describes the product view as seeing the mind as a container and knowledge as a type of substance and argues that the stability and replicability of the product view provide foundational certainty for marks and grades. He contends that the product view supports "front end" models of vocational preparation which require students to complete training before qualification and argues that such preliminary training is not sufficient for a lifetime of practice and does not prepare trainees for workplaces. He argues that learning as a process emphasises changes in learners and environments, underlining the impact of social and cultural factors, and best explains vocational education. However, Hagar (2004) goes beyond Sfard (1991) in arguing that the learning metaphors of acquisition and participation are inadequate on their own in understanding the full complexities of vocational learning. He supports the position of Rogoff (1995) that a third metaphor of construction-reconstruction is necessary.

Communities of practice are further elaborated on by Wenger (1998) to include three identifiers – domain, community and practice. Wenger (2007) argues that members of a community of practice are constituted by an "identity defined by a shared domain of interest" (p. 1) where members value each other's skill sets and are committed to learning from each other. Wenger (1998) describes community as a place where members share experiences thus building and maintaining relationships that foster learning and skill building through personal engagement. Wenger (2007) argues these members collectively expand and extend their community's "repertoire of resources" (p. 2) to develop a shared practice (e.g., member knowledge, accounts of the practice problem solving skills).

Communities of practice as an effective approach to learning is strongly supported by Brown, Collins, and Duguid (1989) who explicitly oppose the idea that knowing and doing can be separated; they argue that knowing developed only through doing, learning is a process of enculturation and community; culture, concepts and learning activities are co-dependent:

> The occasions and conditions for use (of a tool) arise directly out of the context of the activities of each community that uses the tool, framed by the way members of that

> community see the world. The community and its viewpoint, quite as much as the tool itself, determine how a tool is used (p. 35).

Brown et al. (1989) argue strongly for authentic mathematics tasks that occur in the discipline field under question and have real-life meaning. They contend that participation in inauthentic tasks causes students to:

> … misconceive entirely what practitioners actually do. As a result, students can easily be introduced to a formalistic, intimidating view of math that encourages a culture of math phobia rather than one of authentic math activity (p. 38).

Communities of practice as effective ways to understand learning, particularly workplace learning, have been critiqued by Guile (2006) who argues that the approach overlooks relationship between training content and workplace practice. Guile argues that although theoretical and everyday are different kinds of knowledge, they are still related to each other: theory allows us to see connections and relations that everyday knowledge would see as separate, and everyday concepts are the foundation for constructing theory. He disagrees with Lave and Wenger's (1991) position that theoretical and everyday practices are equivalent forms of knowledge because it discounts mediation between theory and practice and shifts the focus of research to workplace learning and away from the relation between the vocational curriculum and vocational practice.

BLOCKLAYING STUDY

The methodology adopted for the Blocklaying study was decolonising (L. Smith, 1999) using the *Empowering Outcomes* research model of G. Smith (1992) where research is designed to address the sorts of questions that Indigenous people want to know in ways that empower these people. A qualitative and longitudinal intervention case study was set up with a building and construction lecturer, called Mack, and his blocklaying students at Tropical North Queensland TAFE's Thursday Island campus (see Cooper, Baturo, Ewing, Duus, & Moore, 2007, for a description of this study). Deadly Maths researchers worked collaboratively with Mack to develop approaches and materials that could effectively teach the mathematics needed for TAFE certification. The teaching approach used in the campus was for Mack to be the sole teacher of the students, teaching literacy and numeracy as well as blocklaying. As described in Cooper et al., he was successful with the students, had built strong relationships with them, and emphasized learning to build personal and community capacity as much as to gain certification.

The participants in the study were Mack and the students. Mack was not Indigenous but was a highly qualified master builder with builder-training certification. He had no training in mathematics education; not surprisingly, he saw mathematics teaching in procedural terms. The students were all young (18-26 years old) predominantly-unemployed Torres Strait Island men. Some students came from the outer islands and were selected by their Island's councils and elders to become builders for their communities. Others had just heard about the course. Their mathematics skills were not much more than mid elementary school. The data gathering procedures were observations of classes and professional learning (PL) sessions with Mack (video, audio and field notes), interviews with Mack and the students (audio-taped), and collection of tests results and other examples of students' work. The procedure followed in trialling ideas was four stages: (1) pre-interviews with Mack concerning the focus of the intervention and development of possible materials; (2) pre-interviews with students and PL sessions with Mack (and other TAFE lecturers); (3) trial of the ideas and materials with students and observations of lessons (including some model teaching by researchers); (4) post-interviews with Mack and students, and collection of students' assessments.

The theoretical framework for the study is fully described in Cooper et al. (2007). The first imperative was that mathematics instruction should be situated within a vocational context in line with Baturo and Cooper (2006). This reinforced involvement and ownership which have been identified as the single most important factor of Indigenous success in VET courses (O'Callaghan, 2005). The second imperative was to always take mathematics instruction beyond procedural to structural understanding, at the same time contextualising the instruction by incorporating Indigenous culture and perspectives into pedagogical approaches (Matthews, Watego, Cooper, & Baturo, 2005).

RESULTS

The video and audio tapes of the observations and interviews were transcribed and combined with field notes and records of students' work to give a rich description of the intervention. These data were analysed in terms of three domains of Lave and Wenger (1991), domain, community and practice. The results below are from interviews with ten students and Mack.

Domain

The students saw themselves as blocklayers and felt that they belonged to this domain. However, they expressed another common desire that appeared to be unique to them; they wanted to help people who have helped them, or to give back to their local community. They saw blocklaying as enabling them to provide for their people in ways that, without the course, they would be unable to do. As student P said, *I want to become a contractor. I want to have a chance to give back to people who have helped me*. Similarly, when asked for his motivation for undertaking blocklaying, student E said, *Help the people and help me*. Students shared a love of building and construction and a desire to have more life opportunities. Students A and J put it directly, *I like building*, and *because it's interesting and I've always wanted to do building*, while P focused on opportunities, *to get a good life*.

As mathematics was part of their course, all students had a shared commitment and interest in achieving mathematical competence in relation to their blocklaying skills. Student P said it straightforwardly, *We have to sum all the blocks and pay people's wages. It's important for most parts of* it; while student E described the implications of not knowing mathematics, *If you don't know the maths and you just do it with your eye, a couple of months later you might have to go back and do it all again*.

The domain of shared practice to which the students belonged appeared to be wider to them than just blocklaying. The links that Mack and the TAFE campus had developed with the Torres Strait, and the method of instruction where the students travelled around the Islands undertaking building for the Communities, appeared to lead students to see blocklaying as part of the wider Torres Strait community. Students such as A had entered the course because their local Island Council had nominated them, Student J because he had seen the previous students doing work on his Island, and Student P because a Mate had suggested it. This was reinforced by the students' shared interest in being of service to their community, and by their strong relationship to Mack. As student K said, *yeah, he's all right. He doesn't discourage us if we do something wrong and there's always encouragement from him*. Interestingly, student P saw this as something they were growing into; when he was asked why he felt included, he said, *because everyone's more mature now*.

Community

With Mack, the students formed a strong community based on trust, mutual respect and practical work; as P said, *I reckon it's pretty good how Mack's done this. I forgot to say that they're giving us straight up prac. Usually they explain it to you in theory, but here they show us in the prac*.

This resulted in unexpected ways of demonstrating learning and strong progress in learning; as P said, *we're strong with the prac. We show him that we understand what he's saying by working on the job site. He doesn't expect that sometimes*. The students liked that Mack was a good builder himself and was practically based; as student L said, *you feel stressed sometimes but you practise and you feel better*. They liked the vocational contexts; as A said about building on site, *Sometimes there can be stress. It's hard work. But that's the only way to go. You can't go back to paper and do it again, there's only one chance*. They began to feel comfortable enough to ask for help not only from Mack but also from family and friends, although student K liked this to be one on one, *it is easier to ask for help when it's just one person as opposed to when you're sitting in a whole class*. There was strong communication from experienced to less experienced members of the community, but also back the other way, even to Mack from the students; student K summarised, *We help each other. If I want help I can ask my brother*. Particularly, help was needed for language, and mathematics; as student P said, *we don't really speak English up here very often, we speak broken English ... most of the students are dropouts from 8, 9 and 10, they find the maths hard*. Overall, the strength of the group was relationships, both Mack and the students overcoming language and racial barriers to learning through a shared commitment, and willingness to build a relationship with each other; as Mack explained, *It's not that bad now because I understand their language a lot more. Once I've built relationships with them, ... then they start to relax a bit with me and it's not that hard at all. ... That's the same with all the boys. I have to build a relationship with them before I can get them to do anything.*

Practice

As the year passed, the community built its repertoire of practice and gained a mutually shared set of skills that they were able to draw upon for certain tasks. Interestingly, their views of mathematics were very vocational; for example, when asked how he uses mathematics outside of TAFE, student T said, *Sometimes when I have to do something for my cousins, sometimes they want me to build a barbecue for them*. But, the shared aspect of the knowledge is also strong as students S's and J's discussion of making and levelling mortar shows:

S Usually you use two cement bags and one sand bag full.

J Yeah, we just know what the right mix looks like. Probably just two shovels of sand and two shovels of cement and add some water.

S When we used the big cement truck we had to chuck in ten cement bags and three sand bags. Then fill the water up half way.

J Level? It's too easy. Make sure your bubble's in between the two lines.

S When we first learnt this job, our boss taught us to master the level.

J Plumb all the walls. It took about 1 hour to get it straight.

Again, students' responses showed that their community of practice, the people whom they would go to for blocklaying help, was much wider than blocklayers. It was common for students to ask members of family and extended family as well, if they found something difficult. The many jobs done by the students on different Islands meant that the students' repertoire of shared practice included contributing to local communities. Of course, the Deadly Maths intervention widened the repertoire; as Mack described, *I guess what* [Deadly Maths researcher] *has shown us is that getting the answer is not as important as how you get the answer. So we'll certainly concentrate more on how to get the answer from now on*. But still, no matter the repertoire, it sometimes is not enough to understand; as Mack described, *I didn't think I could teach him. He showed up everyday for ten weeks, and he was the first one to get employed. I don't know how but he has mastered laying blocks.*

DISCUSSION AND CONCLUSION

Thinking of the blocklaying course as a community of practice appears to be a lens that gives rich detail, too rich for this paper to fully investigate, but tantalising in what it appears to say about vocational learning of mathematics. Three conclusions are evident.

First, the <u>domain</u> of the community is not just from a shared interest in building but includes a strong sense of community service. This means that mathematics can be contextualised to the Torres Strait Communities as well as the vocation of blocklaying, and should include respect for the notion that blocklaying is a way of supporting community. This means that the blocklaying community of practice can no longer be contained within the TAFE site; it shows that the classroom is not the privileged locus of learning; as Wenger (2007) states "Schools, classrooms, and training sessions still have a role to play in this vision, but they have to be in the service of the learning that happens in the world" (p. 5). It is interesting to speculate that this is a uniquely indigenous addition to normal non-Indigenous communities of practice.

Second, the blocklaying students' notion of community to which they could turn for support was very wide, much beyond people involved in blocklaying. These multiple interactions formed a living curriculum for the members of the community. As Wenger (2007) stated, "People usually think of apprenticeship as a relationship between a student and a master, but studies of apprenticeship reveal a more complex set of social relationships through which learning takes place" (p. 3). The blocklayers' community of practice drew experience, advice and skilling from family members who had useful knowledge as well as other students, lecturers, and builders. Learning was not just from teacher to student but student to teacher, and student to student. The concept of a living curriculum, appeared to expand beyond the education institution where the actual course was taking place (TAFE) to include family members and situations, work experience groups, members of the island's businesses, and community organisations. This expanding and encompassing ethos, cultivated by the members of the group (students and Mack) complimented the students' needs to be involved in a course that addressed both personal and community needs, implying that mathematics also teaching needs to be seen in both personal and community terms.

Third, the notion of shared practice as applied to the blocklaying community was much expanded, by the involvement of community and by the presence of the Deadly Maths researchers. Relationships and trust and respect had important roles, something that is often missing from mathematics classrooms. Mathematics was also understood in vocational contexts, for example, seeing other uses of mathematics outside of TAFE as building a barbecue for his family, and the showing of knowledge through practice. The Deadly Maths researchers have become integrated into the community of practice by providing additional tools that enable the further learning of the group. They have a shared interest in seeing how and why certain techniques of block laying instruction and mathematics instruction merge to form good teaching practice and improved student understanding, therefore we are active collaborators in the domain of block laying learning. They discuss with students and teacher and other members of the community why certain techniques of learning work and how to improve that learning. As well, they have themselves created a community of practice that sits within and ultimately derives from (or was only made possible through) the blocklayers' community of practice. Thus, they have developed a shared practice with the researched in their work with lecturers and students, interventions and trials, and attempts to refine teaching techniques that will lead to improved learning for blocklaying students.

References

Barnes, A. L. (2000). Learning preferences of some Aboriginal & Torres Strait Islander students in the Veterinary Program. *Australian J Indigenous Education*, *28*(1), 8-16.

Baturo, A. R. & Cooper, T. J. (2006). *Mathematics, Virtual Materials and Indigenous Vocational Training*. Unpublished report, Education Faculty, QUT, Australia, 4059.

Bortoli, L. & Creswell, J. (2004). *Australia's Indigenous Students in PISA 2000: Results from an International Study* (ACER Research Monograph No 59). Camberwell, Vic: ACER.

Brown, J.S. & Collins, A. & Duguid, P. 1989. Situated Cognition and the Culture of Learning. *Educational Researcher, 18*(1), 32-42.

Cooper, T.J., Baturo, A.R., Ewing, B., Duus, E., & Moore, K. (2007). Mathematics education and Torres Straight Islander blocklaying students: The power of vocational context and structural understanding. In Woo, J.H., Lew, H.C., Park, K.S., & Seo, D.Y. (Eds.), *Proceedings 31st Annual Conference of IGPME, 2,* 177-184. Seoul, South Korea: Seoul National University.

DEST. (2003). *National Report to Parliament on Indigenous Education and Training*. Canberra, ACT: Commonwealth of Australia.

Fitzgerald, A. (2001). *Cape York Justice Study Report*. Brisbane, Qld: Department of the Premier and Cabinet.

Guile, D. (2006). Learning across contexts. *Education Philosophy and Theory, 38*(3), 251-268.

Hager, P. (2004). Metaphors of workplace learning: More process, less product. *Fine Print, 27*(3), 7-10.

Katitjin,L., McLoughlin,C., & Hayward,K. (2000). Indigenous Australian adults' perceptions and attitudes to mathematics and on-line learning of mathematics. Proceedings Australian Indigenous Education Conference. *Fremantle, WA: AIEC.*

Lave, J. & Wenger, E. (1991). *Situated Learning: Legitimate Peripheral Participation*. Cambridge University Press: Cambridge.

Sfard, A. (1998). On two metaphors of learning and the dangers of choosing just one. *Educational Researcher, 27*(2), 4-13.

Matthews, C., Watego, L., Cooper, T., & Baturo, A. (2005). Does mathematics education in Australia devalue indigenous culture? Indigenous perspectives and non-indigenous reflections. In *Proceedings 28th conference of MERGA, 2,* 513-520, Melbourne, AU.

O'Callaghan, K. (2005). *Indigenous Vocational Education and Training: At a Glance.* Adelaide, SA: National Centre for Vocational Education Research.

QSA. (2004). Overview of statewide student performance in Aspects of Literacy and Numeracy: Report to the Minister for Education and Minister for the Arts. *Brisbane, Qld: QSA*

Rogoff, B. (1995). Observing sociocultural activity on three planes: Participatory appropriation, guided participation, and apprenticeship. In J. V. Wertsch, P. del Rio, & A. Alvarez (Eds.), *Sociocultural Studies of Mind,* 139-164. New York: Cambridge University Press.

Smith, G. (1992). *The Issue of Research and Maori.* Auckland, NZ: Research unit for Maori education.

Smith, L.T. (1999) *Decolonising Methodologies: Research and Indigenous Peoples.* Dunedin, NZ: University of Otago Press.

Wenger, E. (1998). Communities of practice: Learning as a social system. *Systems Thinker*, June, 1998.

Wenger, E. (2007). Communities of Practice: A Brief Introduction. Retrieved January, 2007 from www.ewenger.com/research.

RATIO-LIKE COMPARISONS AS AN ALTERNATIVE TO EQUAL-PARTITIONING IN SUPPORTING INITIAL LEARNING OF FRACTIONS

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Fourteen clinical interviews of fourth grade students (ages 9 to 11) from an underprivileged social context are analyzed. The interviews included tasks in which students were asked to reason about the relative capacity of cups, specifically of how many of them could be filled with the milk contained in a milk carton. The analysis suggests that these "ratio-like comparisons" could be a viable starting point for supporting students around reasoning quantitatively about unitary fractions; a starting point that, as we discuss, can be an alternative to the "equal-partitioning" (or "equal-sharing") approach that has been traditionally used, and that several authors have judged inadequate for supporting students' development of sophisticated comprehension of fractions.

Mathematics educators have long been concerned about how to introduce the numeric system that expresses quantity as a division of two natural numbers (i.e., a/b), so that students can engage in activities that are readily meaningful to them, and that can serve as a basis for developing sophisticated understandings about the system (e.g., how is it that such numbers can be situated in the number line; Hannula, 2003). The preferred activities have been based on the equal partitioning approach, in which students are oriented to make sense of denominators as numbers that quantify the size of pieces produced by equally partitioning a whole, and of numerators as a number of those pieces (see Figure 1).

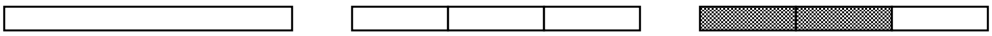

Figure 1. Representations of a whole, thirds, and 2/3 in the equal partitioning approach.

Educators' preferences for this approach have been based on how students can readily and meaningfully engage in equal partitioning and equal sharing activities, even from an early age (cf. Pitkethly & Hunting, 1996). However, several authors have questioned the pedagogical soundness of the approach, both on empirical and conceptual grounds. In the former case, there is evidence that equal partitioning makes it difficult for many students to develop sophisticated comprehension of fractions. For instance, in a broad study that included a survey of 3067 children and 20 interviews, Hannula (2003) identified that the difficulties experienced by many Finish middle-school students in perceiving a fraction as a number on a number line seemed to be related to their reliance on faulty equal-partitioning imagery. This imagery involved interpreting a fraction such as "3/4" as *three out of four*, so that the denominator became construed as something that expressed cardinality (four things), but not size (segments of such a magnitude that each one is one fourth of a whole). Other researchers have previously documented that many students develop similar kinds of interpretations.

In the conceptual arena, Kieren (1980) identified equal partitioning as just one aspect of the rational number construct, and considered that instruction should not be limited to it. Instead, he recommended including situations in which rational numbers are interpreted differently (i.e., as ratios, measures, and operators). Kieren's ideas have been widely accepted (Charalambous & Pitta-Pantazi, 2005; Pitkethly & Hunting, 1996), and have been taken into consideration in the development of important mathematics curricula, such as the Mexican curriculum for elementary schools. Nonetheless, students continue to experience many difficulties in dealing with situations that involve the notion of fractions (cf. Backhoff, Andrade, Sánchez, Peon, & Bouzas, 2006).

Other authors have altogether questioned the convenience of using the equal-partitioning approach to introduce fractions. Freudenthal (1983) labeled this approach as *fraction as fracturer*, and considered it to be "much too restricted not only phenomenologically but also mathematically" (p. 144) as—in principle— it yields only proper fractions. This author regarded fraction as fracturer to be "not only too narrow a start," but also "one sided", and considered it strange "that all attempts at innovation have disregarded this point (p. 147)." He proposed an alternative that consists of approaching fractions *as comparers*, where the big idea is no longer to generate pieces by equally partitioning a whole, and to then identify a certain number of them, but to "put magnitudes into a ratio with each other" (p. 149).

Thompson and Sandanha (2003) also expressed concerns about introducing fractions to students by using an equal-partitioning approach. For these authors:

> The system of conceptual operations comprising a fraction scheme is based on conceiving two quantities as being in reciprocal relationship of relative size: *Amount A is 1/n of the size of amount B means that amount B is n times as large as amount A. Amount A being n times as large as amount B means that B is 1/n as large as amount A* (p. 107; emphasis in the original).

In their view, the equal-partitioning approach leads students to reason about fractions in terms of "additive inclusion—that 1/n of B is one of a collection of pieces—without grounding it in an image of relative size" (p. 108). These authors contend that:

> When students' image of fractions is "so many out of so many," it possesses a sense of inclusion—that the first 'so many' must be included in the other "so many." As a result, they will not accept the idea that we can speak of one quantity's size as being a fraction of another's size when they have nothing in common. They will accept "The number of boys is what fraction of the number of children?," but will be puzzled by "The number of boys is what fraction of the number of girls?" (p. 105).

Freudenthal's (1983) and Thompson and Saldanha's (2003) analyses of fraction coincide in regarding the equal partitioning approach as an inadequate base for supporting students' development of increasingly sophisticated understandings of this concept. These authors also coincide in acknowledging that ratio-like comparisons should be regarded as the essence of initial fraction instruction. A question then arises: What kind of activities would be both compatible with these authors' considerations about the essence of understanding fractions, as well as readily meaningful to novice learners?

Thompson and Saldanha (2003) identified in Steffe's (2002) research the potential nature of such activities. Although the instructional interventions that Steffe reported were grounded in the metaphor of equal partition, he oriented students to think about the size of single fractional pieces (i.e., unitary fractions) not so much as the outcome of equal partition, but in terms of how many iterations (or copies) of it would render something as big as a whole. Steffe's approach is not constrained to orienting students to think about the size quantified by unitary fractions in terms of a quotient of a partitive division, so that 1/3 of a candy bar becomes construed as the amount of candy contained in the pieces that are produced by equally dividing a bar in three (see Figure 1). Instead, his approach seeks to orient students to think about unitary fractions in terms of multiplicands that satisfy a specific iterative criterion, so that 1/3 of a candy bar becomes construed as a piece of candy of such a size that having three of them would render the same amount as what is contained in a whole bar (see Figure 2).

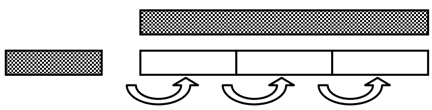

Figure 2. "1/3" as a piece of such a size that three of them would make as much as whole.

The research by Steffe and his colleagues suggests that activities in which unitary fractions are approached more in terms of multiplicands rather than of partitive quotients can be the basis for supporting students' development of relatively sophisticated understandings of fractions (e.g., Olive & Steffe, 2002; Tzur, 1999); understandings that seem compatible with Freudenthal's (1983) and Thompson and Saldanha's (2003) conceptual analyses. However, it must be acknowledged that Steffe and his colleagues reported working with a very small number of student pairs, in non-typical classroom settings that involved intensive use of computers.

It is also worth mentioning that-as is the case with the vast majority of children that participate in mathematics education studies-the students with whom Steffe and colleagues worked most probably belonged to communities where children's enrollment in educational institutions at age four or younger is universal; where parents typically have nine or more years of formal education, and where school-like educational resources (e.g., storybooks, TV shows, toys, educational websites, etc.) are readily available to children. Although such contexts are widespread in the *developed world*, they are alien to millions and millions of elementary-school students in the *developing world*. The question then remains in terms of the instructional viability of activities intended to introduce the notion of unitary fractions as multiplicands in classrooms; particularly in those attended by children who have had limited opportunities for formal education.

DATA COLLECTION AND METHODOLOGY

The data that we analyze in this report comes from 14 clinical interviews of an entire fourth grade classroom located in the outskirts of a middle size town (population 100,000), in southern Mexico. The interviews are part of a larger research project funded by a Mexican government agency that focuses on understanding how fraction instruction could be improved. The purpose of the interviews was to document the kind of mathematical resources developed by fourth grade students whose formal educational trajectories have taken place among impoverished conditions. Following an instructional-design perspective (Gravemeijer, 2004), we were interested in gathering information that could be useful in formulating conjectures about the nature of activities in which all the students in this kind of classrooms could readily engage. We were particularly interested in developing empirically grounded conjectures about the nature of tasks that could help pupils make sense of unitary fractions as multiplicands.

The interviews involved six activities, four of which are discussed in this report. They were conducted in January 2007 on days 90 and 91 of the 200 days included in of the official school calendar. At that point in time, seven of the students were nine years old, six were 10, and one was 11.

Three of the 14 students had not attended preschool. One of the students had repeated second grade and another third grade. The students were the children of socially underprivileged families. To our knowledge, the parents of only one child had had higher education (they were teachers). It is possible that some of the parents had little or no formal education. Ten of the children's families received 140 pesos monthly (about 13 USD) for sending their children to fourth grade, as part of a governmental program intended to prevent "at risk" students from leaving school at an early age because of poverty.

The interviews lasted between 25 and 40 minutes each. They were videotaped. Two researchers were present: one was in charge of presenting the problems to the student and making probing questions; the other was in charge of taking notes and intervening with clarifying questions when she considered it necessary. The interviews were analyzed following the general guidelines recommended by Cobb (1986). Important parts of the interviews were transcribed.

DATA ANALYSIS

Three of the interview activities were aimed at documenting students' understanding of multiplication. One of them was based on a narrative of the number of "tazos" (popular toys that come inside snacks) that several children had. The problem involved having to determine how much was twice ("lo doble"), thrice ("lo triple") and fivefold ("lo quíntuple") of five; for example: "Olga has five tazos and Candelaria has twice as many, how many tazos does Candelaria have?" We decided not to use the Spanish equivalent of the word "times" (veces) from the start so as to facilitate the emergence of interpretations that involved the use of strategies different to repeated addition. However, expressions such as "five times" (cinco veces) were used when students seemed not to understand the meaning of "triple" and/or "fivefold."

All the students were able to readily determine five twice. All the students were also able to determine five thrice, although 8 of them used additive strategies in ways that did not allow them to give an immediate answer (e.g., adding five to 10 or counting five three times). Determining five fivefold became a significantly challenging task for four of the students. They seemed to have trouble keeping a double count (e.g., 5-1, 10-2, 15-3, 20-4, 25-5).

Students were also asked to find out how many cookies would be in a box if it contained 10 packages with 10 cookies in each package. In this case, six students gave an immediate answer, apparently by using the multiplication table (i.e., 10×10). Four students solved the problem by successfully counting 10 times 10 (i.e., 10-1, 20-2, 30-3… 100-10). The remaining four students tried the same strategy but seemed to have trouble keeping track of the double count. By and large, at least eight of the students seemed to have rather primitive notions of multiplication for their grade level.

Another situation involved a candy bar that was physically presented to the students (a rectangle of five by 10 cm). Students were then shown cards with the inscriptions $\frac{1}{2}, \frac{1}{4}$, and $\frac{2}{4}$, and were asked to identify the amount of candy that would correspond to each of them on the bar. In the case of 1/4 and 2/4, they were also asked to explain if it would be more, less, or the same as 1/2. It is worth clarifying that, according to the Mexican curriculum, students should have been familiar with these fractions by the end of third grade and, in the months they had been in fourth grade, they should have already engaged with thirds, fifths and tenths.

All the students were capable of identifying a half of the candy bar, although three of them did not readily relate the "$\frac{1}{2}$" inscription to "one half." Five of the students recognized 1/4 of a candy bar as being less candy than a half; six of them thought it would be more; and three were not sure. Only one student recognized 2/4 of the candy bar as being the same as 1/2. The rest considered it to be more or were not sure. By and large, almost all the students seemed to have inadequate understandings about the meaning of simple conventional fractions.

The main interview tasks involved students reasoning about the volume capacity of cups relative to how many could be filled with the milk in a carton. Students were physically presented with a milk carton like the one shown in Figure 3, but not with the cups. The tasks were intended to orient students to think about the capacity of cups in terms of amounts (multiplicands) that satisfy a certain iterative criterion: that the amount of milk that so many cups of a specific kind could hold would be the same as what the carton contained when being full (e.g. 10 paper cups hold as much milk as a carton). As a consequence, the tasks involved comparing the relative size of magnitudes that were not part of the same thing (i.e., the capacity of a cup was not a part of the capacity of the milk-carton).

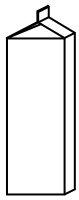

Figure 3. Drawing representing the milk-carton (one litter) used during the interviews.

Students were first told about plastic cups of such a size that the amount of milk in the carton would exactly fill three of them (i.e., servings of 1/3 of the milk in the carton). Students were asked to estimate the place where the milk would be in the carton after serving one, two, and three cups.

Students were then told about glass cups of such a size that the amount of milk in the carton would exactly fill five of them (i.e., servings of 1/5). They were asked to explain if the glass cups could hold more or less milk than the plastic cups (i.e., 1/3 vs. 1/5). Next, they were asked to estimate the place where the milk would be in the carton after serving one, two, three, four, and five cups.

Finally, students were told about paper cups of such a size that the amount of milk in the carton would exactly fill 10 of them (i.e., servings of 1/10). Students were asked if the paper cups could hold more or less milk than the plastic and the glass cups (i.e., 1/10 vs. 1/3 and 1/10 vs. 1/5). Next, they were asked to estimate and mark the place where the milk would be in the carton after serving one and five cups, and to explain if serving five cups would require more, less, or as much as half of the milk in the carton (i.e., 5/10 vs. 1/2).

The cups-capacity tasks appeared to be readily meaningful to all the students, given that the interviewers did not need to give an unusual number of explanations to help the students engage with the problems in a sensible way. All of the students identified the plastic cups as holding more milk than the glass cups (i.e., $1/3 > 1/5$), and the paper cups as holding less than the plastic and glass cups (i.e., $1/10 < 1/3$ and $1/10 < 1/5$). Ten of the students also articulated sensible explanations as to why this were the the case. The following is an example of a comparison between the plastic and the glass cups (1/3 vs. 1/5):

Vicky: Because it's three for the plastic and five for the glass.

Interviewer: And what does that mean?

Vicky: Each one gets a cup, but if you serve five it's going to hold less.

In the case of estimating the place where the milk would be after serving five paper cups, students made marks that indicated where the milk would be after serving one, two, three, four and five cups. Five of the students' estimates coincided with half of the milk carton, eight exceeded half of the milk carton by not much, one felt short, and the remaining exceeded half by a substantial amount.

With respect to the question of whether serving five paper cups would be more, less, or the same as half of the carton (i.e., 5/10 vs. 1/2), the five students whose marks coincided with half readily responded that it would be the same. Although it is possible that their answers were based on the marks they made, and that they had not anticipated that such marks had to coincide with half of the carton, the five students were able to justify their answer mathematically (e.g., "because five and five is 10").

The other nine students clearly based their answers on the mark they had made on the carton, and responded that it would be more or less, depending on where they had made their marks. These students were asked next about how many cups it would be possible to fill with half of the milk carton. All the students responded that it would be five cups. Students were then asked the original question: seven of them now responded that serving five cups would be the same as serving half of the milk carton. The remaining two students continued to base their answers on the original marks they had made. These two students seemed to have trouble reconciling their arithmetical understanding about half of ten being five with imagining pouring milk into the cups.

DISCUSSION

The analysis of the interviews supports the conjecture that ratio-like tasks could be productively used with whole classrooms made up of novice fraction learners, even if these learners are children who have had limited opportunities for formal education in their lives. The cups-capacity tasks appeared to have been readily meaningful to all the students that participated in the interviews, and to have been useful in helping them reason about the capacity of cups in terms of multiplicands that satisfy a certain criterion (i.e., cups holding a volume of milk of such a size that x many of them would amount to the capacity of a milk carton). The tasks also seemed useful in supporting students' reasoning about basic equivalencies (e.g., 1/2 = 5/10). From an instructional perspective, we consider the emergence of this kind of quantitative reasoning among the interviewed students to be particularly relevant, given that it came about regardless of the apparently limited comprehension of multiplication and conventional fractions that most of them seemed to have previously developed.

Our analysis suggests that it is viable to engage novice learners in fraction activities such as the cups-capacity tasks, where the focus is in quantifying *relationships of relative size* by means different to equal partitioning. We thus view it feasible to involve students in fraction learning paths that circumvent the limitations of the equal-partitioning approach, and that can support students' development of relatively sophisticated understandings about rational numbers; understandings that are not typically achieved by pupils, particularly in developing countries like Mexico (cf. Backhoff, Andrade, Sánchez, Peon,

& Bouzas, 2006; Learning for tomorrow's world: First results from PISA 2003, 2004). Specifying the nature of those paths together with the instructional means that would support students' progress along them are important goals of our ongoing research.

Endnote

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References

Backhoff, E., Andrade, E., Sánchez, A., Peón, M., & Bouzas, A. (2006). *El Aprendizaje del Español y las Matemáticas en la Educación Básica en México: Sexto de Primaria y Tercero de Secundaria*. Mexico City: Instituto Nacional para la Evaluación de la Educación.

Charalambous, C. Y. & Pitta-Pantazi, D. (2005). Revisiting a theoretical approach on fractions: Implications for teaching and research. In H. L. Chick & J. L. Vincent (Eds.), *Proceedings of the Twenty Ninth Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 233-240). Melbourne: PME.

Cobb, P. (1986). Clinical interviewing in the context of research programs. In G. Lappan & R. Even (Eds.), *Proceeding of the Eighth Annual Meeting of the International Group of the Psychology of Mathematics Education: Plenary Speeches and Symposium* (pp. 90-110). East Lansing: Michigan State University.

Freudenthal, H. (1983). *Didactical henomenology of Mathematical Structures*. Dordrecht, Holland: Kluwer.

Gravemeijer, K. (2004). Local instruction theories as means of support for teachers in reform mathematics education. *Mathematical Thinking and Learning, 6*, 105-128.

Hannula, M. S. (2003). Locating fraction on a number line. In N. Pateman, B. Dougherty, & J. Zilliox (Eds.), *Proceedings of the 27th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 17-24). Honolulu, USA: PME.

Kieren, T. (1980). The rational number construct--Its elements and mechanisms. In T. Kieren (Ed.), *Recent Research on Number Learning* (pp. 125-149). Columbus, OH: ERIC/SMEAC.

Learning for tomorrow's world: First Results from PISA 2003. (2004). Paris, France: Organization for Economic Co-operation and Development.

Olive, J. & Steffe, L. P. (2002). The construction of an iterative fractional scheme: The case of Joe. *Journal of Mathematical Behavior, 20*, 413-437.

Pitkethly, A. & Hunting, R. (1996). A review of recent research in the area of initial fraction concepts. *Educational Studies in Mathematics, 30*, 5-38.

Steffe, L. P. (2002). A new hypothesis concerning children's fractional knowledge. *Journal of Mathematical Behavior, 20*, 267-307.

Thompson, p. W. & Saldanha, L. A. (2003). Fractions and multiplicative reasoning. In J. Kilpatrick, G. Martin, & D. Schifter (Eds.), *Research companion to the principles and standards for school mathematics* (pp. 95-113). Reston, VA: National Council of Teachers of Mathematics.

Tzur, R. (1999). An integrated study of children's construction of improper fractions and the teacher's role in promoting that learning. *Journal for Research in Mathematics Education, 30*, 390-416.

IMPROVING AWARENESS ABOUT THE MEANING OF THE PRINCIPLE OF MATHEMATICAL INDUCTION

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This work is based on our conviction that it is possible to minimize difficulties students face in learning the Principle of Mathematical Induction by means of clarifying its logical aspects. Based on previous research and theory, we designed a method of fostering students' understanding of the principle. We present results that support the effectiveness of our method with teachers in training who are not specializing in Mathematics.

INTRODUCTION

The Principle of Mathematical Induction (PMI) represents a key topic in the education of teachers in Italy. The approach traditionally used in Italian schools devotes little time to the teaching of a solid understanding of the principle. Most text books do not cover the PMI in depth and only require students to 'blindly' apply it in proving equalities. Students learn to mechanically reproduce the exercises but do not develop a true understanding of the PMI. We propose that it is important and also possible to promote understanding of the PMI, rather than just its application, using non traditional methods. In this paper we present some findings from a study that used a non-traditional approach to teaching the PMI with 44 pre- and in-service middle school (grades 6-8) teachers who were completing a teacher training course. Most of these trainees were not mathematics graduates, but had had some exposure to the PMI during their studies and therefore are a good sample for both examining the 'traces' of their education history and assessing the usefulness of a non-traditional approach to teaching the PMI. In particular, we were interested in promoting comprehension and correcting previously learned misconceptions.

THEORETICAL FRAMEWORK

Previous research has highlighted difficulties that students encounter learning the PMI due to certain misconceptions about it. For example, Ron and Dreyfus (2004) argue that three aspects of knowledge are required to foster a meaningful understanding of a proof by mathematical induction (MI) are essentially three: (1) understanding the structure of proofs by MI; (2) understanding the induction basis; and (3) understanding the induction step. Based on our experience teaching the PMI, we believe that the third aspect, the induction step, is the most important in fostering an understanding of it. Ernest (1984) observes that a typical misconception among students is the idea that in MI "you assume what you have to prove and then prove it" (p.181). Fishbein and Engel (1989) also stress that many students are "inclined to consider the absolute truth value of the inductive hypothesis in the realm of the induction step" (p.276). Both Ernest (2004) and Fishbein and Engel (1989) argue that the source of this misconception is in students' lack of understanding of the meaning of proofs of implication statements. They suggest that a proper approach to teaching the PMI must include logical implication and its methods of proofs. We (Malara, 2002) agree with Avital and Libeskind (1978) who suggest that a way to overcome students' bewilderment in front of the 'jump' from induction basis to induction step is to approach MI by means of '*naïve induction*', which consists of showing the passage from k to k+1 for particular values of k "not by simple computation but by finding a structure of transition which is the same for the passage from each value of k to the next" (p.431).

Another conceptual difficulty experienced by students that is highlighted by research is that many students look at the PMI as something which is neither self evident nor a generalization of previous experience. Ernest (1984) suggests that a way to overcome this problem is to refer to the well ordering of natural numbers, that is: if a number has a property and "if it is passed along the ordered sequence from any natural number to its successors, then the property will hold for all numbers, since they all occur in the sequence" (p.183). Harel (2001) also refers to this way of introducing the PMI, calling it quasi-induction, but he observes that there is a conceptual gap between the PMI and quasi-induction (namely quasi-induction has to do with steps of local inference, while PMI has to do with steps of global inference) which students are not always able to grasp.

In addition, Ron and Dreyfus (2004) highlight the usefulness of using analogies with students when teaching the PMI for two reasons: (1) analogies illustrate the relationship between the method of induction and the ordering of natural numbers and (2) they are tools for fostering understanding of the use of MI in proofs.

RESEARCH HYPOTHESIS AND PURPOSES

We propose that an effective approach to teaching the PMI requires a combination of different points described above. In particular, we propose that the essential steps in a constructive path toward PMI should include: (1) a thorough analysis of the concept of logical implication; (2) an introduction of PMI through the naïve approach, drawing parallels between PMI and the ordering of natural numbers, and the use of reference metaphors; and (3) a presentation of examples of fallacious induction to stress the importance of the inductive basis. Our hypothesis is that a path in which all of these aspects are considered leads to real understanding of the meaning of the principle and therefore its more conscientious use in proofs. Furthermore, a real understanding of the principle does not necessary mean being able to apply it, since many proofs through MI require being able to use and interpret algebraic language.

The purpose of our research is to test the usefulness of this proposed path in instilling a deeper understanding of the PMI. We do this by monitoring trainees during a range of activities and ending with a final exam designed to assess students' true understanding of the PMI. In this paper we present the experience of one trainee, which supports the effectiveness of this approach.

METHOD

The path we propose can be divided into six main phases: (1) An initial diagnostic test; (2) Activities which lead students from conditional propositions in ordinary language to logical implications; (3) Numerical explorations of situations aimed at producing conjectures to be proved in a subsequent phase; (4) An introduction to the method of proofs by MI and to the statement of the principle; (5) Analysis of the statement of PMI and production of proofs; (6) A final test (given 3 weeks after the last lesson). Because of space limitations, we focus on one central phase in the path, because it contains the aspects we propose are essential to a meaningful approach to teaching PMI. The following proof (table 1), which was a starting point in the construction of a lesson, was proposed by a trainee, R., during the numerical exploration phase.

R. intended to prove the conjecture she produced on the sum of the powers of 2: $2^0+2^1+2^2+2^3+\ldots+2^n=2^{n+1}-1$. After having observed that proving this equality is the same as proving $2^0+2^0+2^1+2^2+2^3+\ldots+2^n=2^{n+1}$, R. proceeded in this way: $2^0+2^0+2^1+2^2+2^3+\ldots+2^n= 2\cdot 2^0+2^1+2^2+2^3+\ldots+2^n=$ $=2^1+2^1+2^2+2^3+\ldots+2^n= 2\cdot 2^1+2^2+2^3+\ldots+2^n=$ $=2^2+2^2+2^3+\ldots+2^n=2\cdot 2^2+2^3+\ldots+2^n=\ldots=2^n+2^n=2\cdot 2^n=2^{n+1}$.[1]

Table 1

We showed to trainees R.'s proof and we observed with them that: the individual steps of her proof constitute 'micro-proofs' of the individual implications *P(0)→P(1), P(1)→P(2)…*; the dots testify that she made a generalization. Table 2 illustrates the formal aspects we used in this discussion. We discussed the following points with the trainees: (1) the structure of natural numbers is such that every number n could be obtained from the previous *(n-1)* adding 1; (2) Every sum S_n is obtained by the previous sum adding the n^{th} power of 2, 2^n; (3) The terms of the successions have in common the property of strictly depending on the terms which precede them.

These observations allowed the trainees to agree on the fact that every proposition could be derived recursively from its prior. Starting with this intuition, we highlighted the common structure of R.'s proofs of the 'particular implications' and guided trainees to observe that this structure can be followed every time it is necessary to prove a proposition P(k+1) starting from the previous proposition P(k). Trainees became aware that the complete proof of the statement is based on a chain of implications, such as the ones highlighted in R.'s proof, that can be 'summarized' as "$P(k)\rightarrow P(k+1)\ \forall k\in N$". Together we constructed the proof of this general implication, as a generalization of the step-by-step micro-proofs. Because of the

[1] R.'s proof represents what Harel (2001) defines as quasi-induction.

previous activities on logical implication, trainees were aware that an implication could also be valid when the two components are not valid. It was easy for them therefore gradually to become aware that proving "$P(k) \rightarrow P(k+1)\ \forall k \in N$" means proving that "$P(n)$ is valid $\forall n \in N$", *only if* the first proposition of the chain, $P(0)$, is valid.

$P(n)$: $2^0+2^0+2^1+2^2+2^3+\ldots+2^n=2^{n+1}$ $(n \geq 0)$	
$\underline{2^0+2^0}+2^1+2^2+2^3+2^4+\ldots+2^n=$	*P(0)*: $2^0+2^0=2\cdot2^0=2^{0+1}$
$=\underline{2\cdot2^0}+2^1+2^2+2^3+2^4+\ldots+2^n=$	$P(0) \rightarrow P(1)$
$=\underline{2^1+2^1}+2^2+2^3+2^4+\ldots+2^n=$	$(2^0+2^0)+2^1=2^1+2^1=2\cdot2^1=2^{1+1}$
$=\underline{2\cdot2^1}+2^2+2^3+2^4+\ldots+2^n=$	$P(1) \rightarrow P(2)$
$=\underline{2^2+2^2}+2^3+2^4+\ldots+2^n=$	$(2^0+2^0+2^1)+2^2=2^2+2^2=2\cdot2^2=2^{2+1}$
$=\underline{2\cdot2^2}+2^3+2^4+\ldots+2^n=$	…
	P(k): $2^0+2^0+2^1+2^2+\ldots+2^k=2^{k+1}$
…	$P(k) \rightarrow P(k+1)$
	$(2^0+2^0+2^1+\ldots+2^k)+2^{k+1}=2^{k+1}+2^{k+1}$
	$= 2\cdot2^{k+1}=2^{k+2}$
$=\underline{2^n+2^n}=$	…
$=\underline{2\cdot2^n}=$	
$=2^{n+1}$	

Table 2

ANALYSIS OF TRAINEES' WORK DURING THE PATH: THE CASE OF L

During the activities we proposed them, trainees also worked individually. We collected their protocols in order to analyze the evolution of their acquisition of meaning of the PMI. In particular, we compared the answers they gave in the initial and final tests in order to highlight their effective acquisition of awareness of the meaning and use of PMI. The final test consisted in four questions, two following Fishbein and Engel's questionnaire (1989), the other two concerning the proof of two statements. The purpose was to verify: (1) whether trainees really understood the meaning of the inductive step and the importance of the inductive basis as an integral part of the proofs by MI; (2) whether trainees were able to single out what the key-passages to perform proofs by MI concerning new conjectures are. The results of the questionnaires were really satisfactory because almost all trainees produced correct proofs and, more importantly, many of them demonstrated having acquired an effective comprehension of the sense of the principle. In this paragraph we focus on the analysis of the evolution of another trainee, L., because we observed a remarkable difference between the problematical nature of her initial situation and the level of awareness and the abilities she displays in her answers on the final test. We present two excerpts from her protocols: the first one is taken from the initial test and the second concerns an answer she gave in the final test.

Initial test: The excerpt refers to the proof of the inequality $2^n>3n+1$ *(where* $n\geq4$*)*. L. writes: *1)* $2^4>3\cdot4+1$ *16>13 ok* *2)* $2^k>3k+1$ $k>4$ *It is true.* *Proof:* $2^{k+1}>3(k+1)+1$ $2\cdot2^k>3k+3+1$ $2\cdot2^k>3k+1+3$ *→ 2P(k)>P(k)+3, which is always true because the hypothesis is true (∀k≥4)… but it something I can see at a glance!*

First of all see L.'s erroneous use of the specific symbology; instead of referring to *P(k)* as to the proposition which represents the statement to be proved, she deals with it as representing each of the expressions at the two sides of the inequality. Also to be considered are the logical aspects involved in the use of the principle; i.e., L. directly considers the inequality to be proved, trying to justify it on the basis of the hypothesis, but her arguments rely only on 'evidence'. L.'s difficulties have to be ascribed to a lack of knowledge about logical implication, which is also documented in other answers.

The second excerpt we present refers to a part of the answer L. gave to the following question (final test):

"During a class activity on PMI, Luigi speaks to his mathematics teacher in order to remove a doubt: *We have just proved a theorem, represented by the proposition P(n), by MI, but this method is not clear…I am not sure that the theorem is really true because, in order to prove P(n+1), we had to hypothesise that P(n) is true, but we do not know if P(n) is really true until we prove it!* If you were his teacher, how would you answer to Luigi?"

After correctly enunciating the principle, L. commented:

"It is necessary for Luigi to understand that in the inductive step we do not prove either P(n) or P(n+1), we only prove that the validity of P(n) implies the validity of P(n+1), that is, we prove the implication P(n)→P(n+1) ".

Because of space limitations, we do not report the correct proofs L. produced. This excerpt, however, demonstrates the level of comprehension she attained during the laboratory activities.

CONCLUSIONS

Our observations of the laboratory activities and analysis of trainees' protocols allow us to take some conclusions on the validity of our research hypothesis. L. represents a prototype of an individual for whom a traditional way of teaching left only few confused ideas on the proving method by MI. The different approach L. adopted and her ability both to understand the problem pointed out by Luigi and to respond in a synthetic and precise way to his doubts, represents evidence of the effectiveness of the choices we made in our approach to teaching the PMI. L. is just one example from a large group of trainees who developed a deeper understanding of the PMI in a similar way. The positive outcomes on the final tests testify to the validity of our research hypothesis regarding the aspects fundamental to a productive introduction to the use of PMI as a 'proving tool'. As a future development of our research, in order to test further the effects of this approach, we plan to test the same method in secondary school, with students learning the PMI for the first time. In particular, our aim is to highlight the role played by the teacher in the management of the lessons.

References

Avital, S.& Libeskind, S. (1978). Mathematical Induction in the classroom: Didactical and Mathematical issues. *Educational Studies in Mathematics* 9(4), 429-438.

Ernest, P. (1984). Mathematical induction: A pedagogical discussion. *Educational Studies in Mathematics* 15(2), 173-189.

Fishbein, E. & Engel, I. (1989). Psychological difficulties in understanding the principle of mathematical induction. In Vergnaud, G. Rogalski, J., & Artigue, M. (Eds.), *Proceedings of the 13th PME Conference*. Vol.1 (pp.276-282). Paris.

Harel, G. (2001). The Development of Mathematical Induction as a Proof Scheme: A Model for DNR-Based Instruction. In Campbell, S. & Al. (Eds.). *Learning and Teaching Number Theory* (pp.185-212). New Jersey, Ablex Publishing.

Malara, N.A. (2002). La dimostrazione in ambito aritmetico, quale spazio nella scuola secondaria? In Malara, N.A. (Ed.), *Educazione Matematica e Sviluppo Sociale: Esperienze nel Mondo e Prospettive* (pp.129-166). Soveria Mannelli: Rubettino.

Ron, G., and Dreyfus, T. (2004). The use of models in teaching proof by Mathematical Induction. In M. J. Hoines & A. B. Fuglestad (Eds.), *Proceedings of the 28th PME Conference.* Vol. 4 (pp.113-120). Bergen (Norway).

A STRUCTURAL MODEL FOR FRACTION UNDERSTANDING RELATED TO REPRESENTATIONS AND PROBLEM SOLVING

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The main purpose of this study is twofold, to confirm a model for the structure of fraction addition understanding related to multiple representations flexibility and problem solving ability and to investigate its stability across pupils of two different grades in primary school. Confirmatory factor analyses (CFA) affirmed the existence of seven first-order factors indicating the differential effect of task modes of representation, representation functions and required cognitive processes, two second-order factors representing multiple representations flexibility and problem solving ability and a third-order factor that corresponds to the fraction addition understanding. Results provided evidence for the invariance of this structure across Grades 5 and 6 of primary schools in Cyprus.

INTRODUCTION

From an epistemological point of view there is a basic difference between mathematics and other domains of scientific knowledge as the only way to access mathematical objects and deal with them is by using signs and semiotic representations. Given that a representation cannot describe fully a mathematical construct and that each representation has different advantages, using multiple representations for the same mathematical situation is at the core of mathematical understanding (Duval, 2006).

Nowadays the centrality of different types of external representations in teaching and learning mathematics seems to become widely acknowledged by the mathematics education community (e.g. Elia & Gagatsis, 2006). Furthermore, the NCTM's Principles and Standards for School Mathematics (2000) document includes a new process standard that addresses representations and stresses the importance of the use of multiple representations in mathematical learning. Recognizing the same concept in multiple systems of representations, the ability to manipulate the concept within these representations as well as the ability to convert flexibly the concept from one system of representation to another are necessary for the acquisition of the concept (Lesh, Post, & Behr, 1987) and allow students to see rich relationships (Even, 1998). Moving a step forward, Hitt (1998) identified different levels in the construction of a concept, which are strongly linked with its semiotic representations. The particular levels are as follow: 1) incoherent mixture of different representations of the concept, 2) identification of different representations of a concept, 3) conversion with preservation of meaning from one system of representation to another, 4) coherent articulation between two systems of representations, 5) coherent articulation between two systems of representations in the solution of a problem. However, other researchers (e.g. Presmeg & Nenduradu, 2005) doubt the theoretical assumption that students who can move fluently amongst various representations of the same concept have necessarily attained conceptual knowledge of the relationships involved.

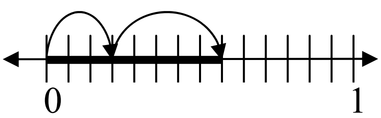
In this study which constitutes a part of the medium research project MED19, funded by University of Cyprus, we incorporated a synthesis of the ideas articulated in previous studies on learning with multiple representations to capture pupils' processes in multiple representations tasks. This may enable us, firstly to gain a more comprehensive picture of fraction addition understanding related to multiple representations flexibility and problem solving ability; secondly, to understand pupils' multiple representations flexibility in a more coherent way; and thirdly, to find out more meaningful similarities in Grade 5 and 6 pupils' representational thinking and problem solving ability. In particular, two hypotheses were tested: a) multiple representations flexibility and problem solving ability influence fraction addition understanding and b) there are similarities between 5th and 6th graders in regard with the structure of their fraction addition understanding.

METHOD

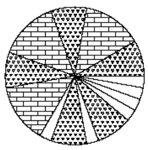
The study was conducted among 829 pupils aged 10 to 12 belonging to 41 classes of different primary schools in Cyprus (414 in Grade 5, 415 in Grade 6). The test that was constructed in order to examine the hypothesis of this study included:

1. Recognition tasks in which the pupils are asked to identify similar (RELa, RECa, RERa, RELb, RERb) and dissimilar (RELc, RERc, RECc) fraction addition in number line, rectangular and circular area diagrams. An example is:

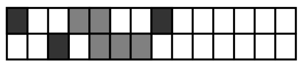
Circle the diagram or the diagrams whose shaded part corresponds to the equation 3/14 + 5/14.



(RELa)

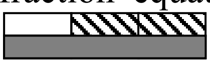


(RECa)



(RERa)

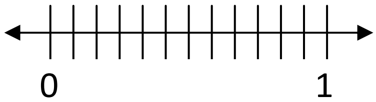
2. Conversion tasks having the diagrammatic and the symbolic representation as the initial and the target representation, respectively. Similar fraction additions are presented in number line (COLSs) and circular area diagram (COCSs), whereas dissimilar fraction additions are presented in number line (COLSd) and rectangular area diagram (CORSd). An example is:

Write the fraction equation that corresponds to the shaded part of the following diagram:  Equation: .... (CORSd)

3. Symbolic treatment tasks of similar (TRSa) and dissimilar (TRSb, TRSc) fraction addition. An example is: 1/6 + 4/12 = ….. (TRSb)
4. Conversion tasks having the symbolic and the diagrammatic representation as the initial and the target representation, respectively. Pupils are asked to present the similar fraction addition in circular area diagram (COSCs) and in number line

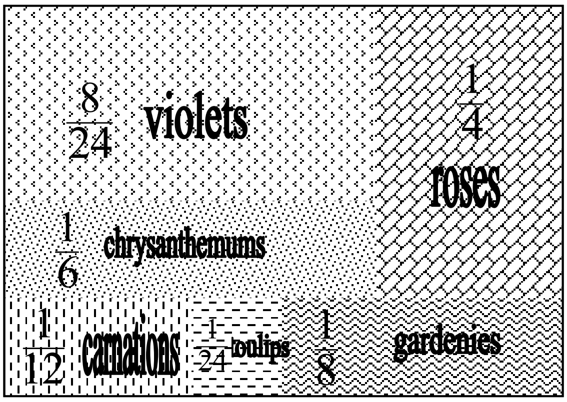
(COSLs), whereas they are asked to present the dissimilar fraction additions in rectangular area diagram (COSRd). An example is:

Present the following equation on the diagram:

1/12 + 7/12=….  (COfSLs)

5. Diagrammatic addition problem in which the unknown quantity is the summands (PD).

Each kind of flower is planted in a part of the rectangular garden as it appears in the diagram below:

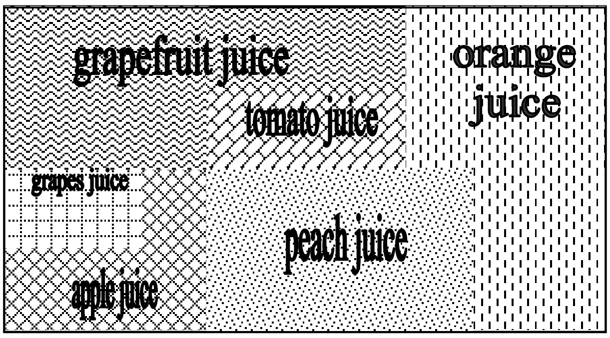


Which three kinds of flowers are planted in the 3/4 of the garden?

6. Verbal problem that is accompanied by auxiliary diagrammatic representation and the unknown quantity is the summands (PVD).

A juice factory produces the following kinds of natural juice:

- 1/4 of the production is grapefruit juice.
- 5/18 of the production is orange juice.
- 3/36 of the production is tomato juice.
- 2/9 of the production is peach juice.
- 1/18 of the production is grapes juice.
- 4/36 of the production is apple juice.



Which four kinds of juice make up 1/2 of the production?

7. Verbal problem whose solution requires not only fraction addition but also the knowledge of the ratio meaning of fraction (PV).

Clowns:	1/2 hour	Animals:	1 hour	Jugglers:	2/1 hour
Dancers:	1/3 hours	Acrobats:	1/6 hour		

Write as a fraction, what part of the total duration of the performance corresponds to the jugglers' program (Evapmib, 2007).

8. Justification task that is presented verbally and is related to similar or dissimilar fraction addition (JV).

In the addition of two fractions whose numerator is smaller than the denominator, the sum may be bigger than the unit. Do you agree with this view? Explain.

It should be noted, that not any diagrammatic representation treatment tasks are included in the test since the students' ability to manipulate diagrammatic representations is examined through conversion tasks in which the target representation is a diagram.

RESULTS

In order to explore the structure of the various fraction addition understanding dimensions a third-order CFA model for the total sample was designed and verified. Bentler's (1995) EQS programme was used for the analysis. The tenability of a model can be determined by using the following measures of goodness-of-fit: x^2, CFI (Comparative Fit Index) and RMSEA (Root Mean Square Error of Approximation). The following values of the three indices are needed to hold true for supporting an adequate fit of the model: x^2/df < 2, CFI > .9, RMSEA < .06. The a priori model hypothesized that the variables of all the measurements would be explained by a specific number of factors and each item would have a nonzero loading on the factor it was supposed to measure. The model was tested under the constraint that the error variances of some pair of scores associated with the same factor would have to be equal.

Figure 1 presents the results of the elaborated model, which fits the data reasonably well (x^2/df=1.911, CFI=0.968, RMSEA=0.033). The third-order model which is considered appropriate for interpreting fraction addition understanding, involves seven first-order factors. The first-order factors F1 to F5 regressed on a second-order factor that stands for the multiple representations flexibility. The first-order factor F1 refers to the similar fraction addition recognition tasks, while the first-order factor F2 to the dissimilar fraction addition recognition tasks in a variety of diagrammatic representations. The first-order factor F3 consists of the similar and dissimilar fraction addition treatment tasks. Conversion tasks in which the initial and the target representation is similar and dissimilar fraction equation and diagrammatic representation, respectively, constitute the first-order factor F4, while the first-order factor F5 refers to the similar and dissimilar fraction addition conversion tasks from a diagrammatic to a symbolic representation.

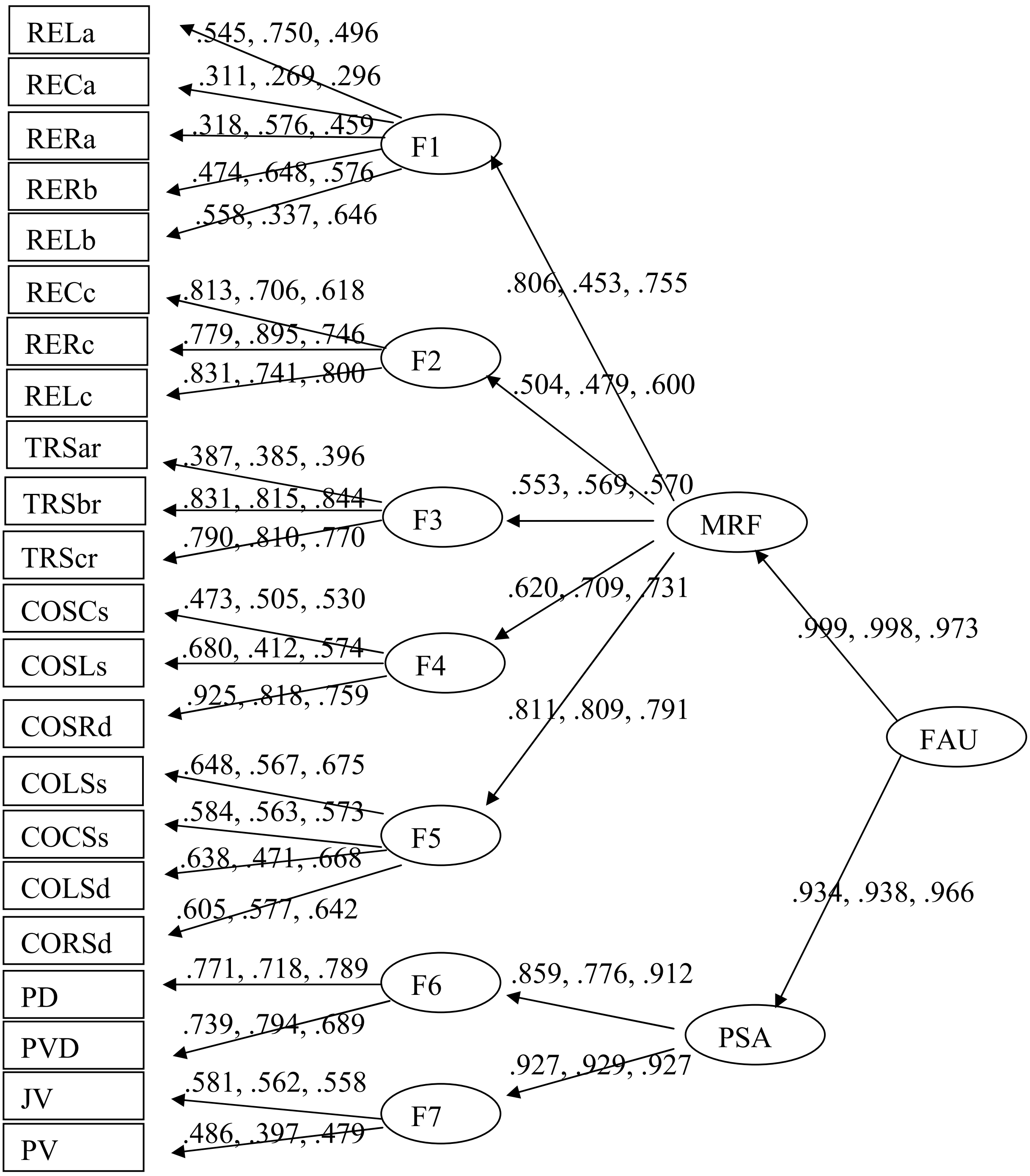


Figure 1. The CFA model of the fraction addition understanding.

Note: 1. The first, second and third coefficients of each factor stand for the application of the model in the whole sample, 5th and 6th graders, respectively. 2. Errors of the variables are omitted. 3. MRF=multiple representation flexibility, PSA= problem solving ability, FAU=fraction addition understanding

The majority of tasks which involve number line have higher loadings than the other tasks, suggesting that the number line model is more strongly related than the circular and rectangular diagrams to multiple representations flexibility. Furthermore, dissimilar fraction tasks loadings are higher than the respective similar fraction addition loadings, indicating that in order to be solved extra mental processes are required since the fraction equivalence understanding is involved, as well. The specific knowledge is also needed to solve similar fraction addition recognition tasks which the number of subdivision is double that of the denominator (e.g. RERa). As a result, higher loadings are observed in these tasks relative to other similar fraction addition tasks. Moreover, the factors loadings indicate that conversion from a diagrammatic to a symbolic representation is more closely associated with multiple representations flexibility than the other first-order factors are. Nevertheless, the first-order factor F1 to F4 loadings strength reveal that the flexibility in multiple representations of similar and dissimilar fraction addition constitute a multifaceted construct in which relations between: a) modes of representation (symbolic, diagrammatic), b) functions (recognition, treatment, conversion) that representations fulfill and c) relative concepts (similar and dissimilar fractions, equivalence) arose.

The other two first-order factor F6 and F5 regressed on a second-order factor that represents problem solving ability. The first-order factor F6 consists of problems having a diagram as an autonomous or an auxiliary representation. Both of them have a common mathematical structure since they have the summands as the unknown quantity. On the other hand, the verbal problem whose solution requires the knowledge of the ratio meaning of fraction and the justification task formed the first-order factor F7, since in order to be solved different cognitive processes are needed. The two second-order factors that correspond to the multiple representations flexibility and to the problem solving ability regressed on a third-order factor that stands for the fraction addition concept understanding. Their loadings values are almost the same revealing that pupils' fraction addition understanding is predicted from both multiple representations flexibility and problem solving ability.

To test for possible similarities between the two age groups' fraction addition understanding, the proposed three-order factor CFA model is validated for each grade separately. The fit indices of the models tested were acceptable and the same structure holds for both the 5th (x^2/df=1.535, CFI=0.954, RMSEA=0.036) and the 6th (x^2/df=1.865, CFI=0.940, RMSEA=0.046) graders. However, some factor loadings are stronger in the group of the 6th graders, revealing that the strength of the relations between these abilities increases across the ages.

CONCLUSIONS

The main purpose of this study is twofold, to test whether multiple representations flexibility and problem solving ability have an effect on fraction addition understanding and to investigate its factorial structure within the framework of a CFA, across pupils of two different grades in primary schools. The results provided a strong case for the important role of the multiple representations flexibility and problem solving ability in 5th and 6th graders fraction addition understanding. Specifically, CFA showed that two second-order factors are needed to account for the flexibility in multiple representations and the problem solving ability. Both of these second-order factors are highly associated with a third-order factor representing the fraction addition understanding.

CFA also show that five first-order factors are required to account for the second- order factor that stands for the flexibility in multiple representations and two first-order factors are needed to explain the second-order factor that represents the problem solving ability. Thus, the results indicate the differential effect of both problem modes of representation and required cognitive processes on problem solving ability. Furthermore, the findings provided evidence to Duval's (2006) view that changing modes of representation is the threshold of mathematical comprehension for learners at each stage of curriculum since the conversion from a diagrammatic to a symbolic representation dimension is more strongly related to multiple representations flexibility than the other dimensions are. Nevertheless, the factors loadings of the proposed three-order model suggest that the flexibility in multiple representations constitute a multifaceted construct in which representations, functions of representations and relative concepts are involved.

It is worth mentioning that the high factor loadings in tasks involving number line reveal the specific model's importance in fraction addition and the different cognitive processes which are activated in order to handle it relative to other diagrammatic representations. In fact the number line is a geometrical model, which involves a continuous interchange between a geometrical and an arithmetic representation. Operations on real number are represented as operations on segments on the line (e.g. Michaelidou, Gagatsis, & Pitta- Pantazi, 2004). That is, the number line has been acknowledged as a suitable representational tool for assessing the extent to which students have developed the measure interpretation of fractions and for reaching fractions additive operations (e.g. Keijzer & Terwel, 2003). Furthermore, the strength of factor loadings in dissimilar fraction addition tasks confirm that different mental processes are required so as to be solved relative to similar fraction addition since the knowledge of fraction equivalence is also needed. The results underline also the high association of the fraction equivalence with fraction addition understanding. Besides, as Smith (2002) points out in order to develop fully the measure personality of fractions pupils need to master the equivalence of fractions.

Concerning the age, it is to be stressed that the structure of the processes underlying the fraction addition understanding is the same across Grade 5 and 6. Even though some factors loadings are higher in the group of 6^{th} graders, indicating that overall cognitive development and learning take place, the results provided evidence for the stability of this structure during primary school years represented here. However, it seems that there is still a need for further investigation into the subject. Taking into account the problems pupils face during the transitions from one educational level to another, it is interesting in future to examine possible differences of the proposed fraction understanding structure as these pupils move to secondary school.

References

Bentler, M. P. (1995). *EQS Structural Equations Program Manual*. Encino, CA: Multivariate Software Inc.

Duval, R. (2006). A cognitive analysis of problems of comprehension in learning of mathematics. *Educational Studies in Mathematics*, *61*, 103- 131.

Elia, I. & Gagatsis, A. (2006). The effects of different modes of representations on mathematical problem solving: Two experimental programs. In J. Novotna, H. Moraova, M. Kratka, & N. Stehlikova (Eds.), *Proceeding of the 30th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 25- 32). Prague, Czech Republic: PME.

Even, R. (1998). Factors involved in linking representations of functions. *The Journal of Mathematical Behavior*, *17*(1), 105-121.

Evapmib (2007). *Une Base se Questions d' Evaluations en Mathématiques*. Retrieved March 25, 2007, from http://ctug48.univ-fcomte.fr/evapmib/siteEvapmib/accueil.php

Hitt, F. (1998). Difficulties in the articulation of different representations linked to the concept of function. *The Journal of Mathematical Behavior, 17*(1), 123-134.

Keijzer, R. & Terwel, J. (2003). Learning for mathematical insight: a longitudinal comparative study on modeling. *Learning and Instruction*, *13*, 285-304.

Lesh, R., Post, T. & Behr, M. (1987). Representations and translations among representations in mathematics learning and problem solving. In C. Janvier (Ed.), *Problems of epresentation in the Teaching and Learning of Mathematics*, (pp. 33-40). Hillsdale, N.J.: Lawrence Erlbaum Associates.

Michaelidou, N., Gagatsis, A., & Pitta- Pantazi, D. (2004). The number line as a representation of decimal numbers: A research with sixth grade students. In M. J. Hoines & A. B. Fuglestad (Eds.), *Proc. of the 28th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 3, pp. 305 – 312). Bergen, Norway: PME.

National Council of Teachers of Mathematics (2000). *Principles and Standards for School Mathematics.* Reston, Va: NCTM.

Presmeg, N., & Nenduradu, R. (2005). An investigation of a preservice teacher's use of representations in solving algebraic problems involving exponential relationships. In H.L. Chick & J.L. Vincent (Eds.), *Proceeding 29th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 105 - 112). Melbourne, Australia: PME.

Smith, J.P. (2002). The development of students' knowledge of fractions and ratios. In B. Litwiller & G. Bright (Eds.). *Making Sense of Fractions, Ratios and Proportions* (pp.3 - 17). Reston, Va: NCTM

SUBVERTING THE TASK: WHY SOME PROOFS ARE VALUED OVER OTHERS IN SCHOOL MATHEMATICS

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This qualitative study on high school mathematics teachers found that some teachers perceive that a major purpose of proof in school mathematics is to enable students to develop both mathematical and transferable thinking skills and that valid proofs that deviate from the intended or expected proof may subvert this purpose by denying the development of student thinking. The validity of a proof is not the only factor determining what counts as a proof in school mathematics, and in fact may be outweighed by the consideration peculiar to school mathematics of how well that proof is perceived to support the development of student thinking. The results of this study have implications for teacher training.

INTRODUCTION

The two-column proof format that dominates American high school geometry classrooms was developed in response to an educational reform around the turn of the twentieth century (Herbst, 2002b). It arose as "a viable way for instruction to meet the demand that every student should be able to do proofs" (p. 285). It has come at the cost of "dissociating the doing of proofs from the construction of knowledge" (p. 307). While proof remains a method of investigating mathematics, proofs are sometimes reduced to little more than logical exercises used to verify trivial statements. This difference between proof and proofs has profound implications in mathematics classrooms.

Herbst (2002a, 2003) reported that teachers must negotiate between competing demands when teaching non-procedural tasks (including mathematical proof) in their classrooms. Typically, high school students are asked to write proofs because they are short and easy rather than because they advance the students' mathematical knowledge. If the stated purpose of a task is to produce a proof, the teacher is expected to support her students in meeting that goal but at the same time, the teacher is expected to help her students advance their understanding of the mathematics involved in the task. Herbst describes the proof task as "an opportunity offered by the teacher for students to produce *the* proof" (2002a, p. 197); while at the same time recognizing that the mathematical goal within the task "is a statement for which the teacher holds students accountable to find *a* proof" (p. 197). Producing *the* (the intended) proof, not only establishes the statement in question but might also provide an opportunity for the student to demonstrate that he or she has some facility with a particular form of proof, with the definitions involved, or with other related concepts. Finding *a* (possibly an alternate) proof still establishes the statement but might not provide the same kinds of additional opportunities for students. This can become a tension between mathematical writing, and mathematical investigation and may place competing or conflicting demands on teachers.

How teachers negotiate these demands depends, in part, on their conceptions of mathematical proof including its purposes. Knuth (2002) investigated the question: What constitutes proof in school mathematics? He found that teachers viewed proof as a deductive or convincing argument that conclusively establishes the validity of a statement. One of the reasons they articulated for teaching proof in high school mathematics was that they believed that proofs help students to develop logical or critical thinking skills that are useful beyond the mathematics classroom. Most did not believe proof to be central to the high school mathematics curriculum and some questioned whether it was an appropriate topic of study for all high school students. He described three levels of argumentation discussed by his participants: formal proofs, less formal proofs, and informal proofs. The formal proofs rigidly adhered to "prescribed formats and/or the use of particular language" (p. 71). The less formal proofs were considered to be valid proofs but lacked the formal structure and language and so were deemed to be less rigorous than the formal proofs. The informal proofs were heuristically based arguments or explanations and not considered by the teachers to be valid proofs.

This paper extends the work of Herbst (2002a, 2003) and Knuth (2002) in exploring the relationship between teachers' perceptions of proof and how they evaluate valid proofs. If a student fails to produce *the* proof but still produces *a* proof, it may be valued differently by teachers who hold different beliefs regarding the purpose of proof in school mathematics. The question this paper will address is: When evaluating valid arguments, what do high school teachers believe counts as a proof in school mathematics?

METHODS

This report is part of a larger study of high school mathematics teachers' perceptions of the purposes of proof in which seventeen high school mathematics teachers were recruited and interviewed three times for approximately one hour each time. The first interview was semi-structured and focused on participants' personal and mathematical histories, and their pedagogical conceptions of mathematical proof. The second interview was task-based (Goldin, 1999) and participants were asked to evaluate a series of fifteen researcher-generated mathematical arguments and to discuss whether each argument conformed to what a proof should be, and in what contexts the argument was sufficient. The third interview was semi-structured and focused on the participants' perceptions of the mathematical purposes of proof. All interviews were audiotaped and transcribed. Both internal and external coding schemes were employed in the analysis of the data.

The participants' responses to the fifteen task items and their discussions during the semi-structured interviews were used to illuminate their perceptions regarding the purposes of proof in school mathematics. Taken together, the interviews sought to find out what the participants believed were the most significant purposes of proof and in what contexts they believed that proofs conforming to different structures and formats were acceptable, allowed, preferred, and why. The views of one of the seventeen participants (Tracy) will be discussed in detail below. We have selected Tracy because her views brought into focus certain key issues that were articulated by other participants. Tracy was approximately 30 years of age and had majored in both mathematics and physics as an undergraduate and held a master's degree in mathematics education. She had been teaching for seven years in an urban high school in the northeast and had taught all levels of high school mathematics.

RESULTS

The results of this study indicate that teachers' conceptions of proof include factors that are peculiar to school mathematics and that these conceptions play a part in determining what counts as proof in school mathematics. Some teachers indicated that a proof requires explicit stepwise reasoning and some teachers indicated that the main purpose of proof in school mathematics is not to verify conjectures but to foster student thinking. Valid proofs that did not include stepwise justifications and proofs that were perceived to curtail the development of student thinking were sometimes deemed by these participants to be unacceptable in high school mathematics.

Explicit Reasoning Required

The analysis of the data suggests that some teachers perceive that validity of argument is but one concern when determining what counts as a proof in school mathematics. Explicit reasoning was considered by some teachers to be vitally important when evaluating proofs and arguments that did not justify each step were sometimes deemed valid yet still unacceptable. For example, Tracy believed that proofs must contain explicit, stepwise reasoning. In a proof, she said, you can't just make a statement, "you have to give a reason for it." A proof, she believed, is a process that documents a thinking pathway. "It's not just about the answer…It involves [a] thought process." When proving a quadrilateral is a rhombus, "you have to show [all four sides have the same length] and say 'A rhombus is something that has all four sides the same." For Tracy, to prove a quadrilateral is a rhombus, it is not sufficient to show that all the sides are congruent; one has to state the definition of a rhombus (quadrilateral with all sides congruent) and has to show that the definition is satisfied.

Because algebraic derivations are typically written without reasons cited for each step, some of the participants perceived algebraic derivations to be somewhat different than proofs. When the participants were shown an algebraic derivation of the quadratic formula, six of the seventeen participants indicated either that it was not a proof or that they were not sure if it was a proof or not, yet none said it was invalid. In particular, because in the statements and reasons do not proceed in lockstep, Tracy believed that an algebraic derivation was something other than a proof. "I don't see that as a real proof." Later, she said, "I would want a little bit more reasoning for it…That's all really very good but you would want to state a little bit more." When asked if she would accept this argument as written from a student she said she would but only for partial credit. "I'd say four out of five…and just say be more specific on what you did." If, however, the student had been asked for a derivation of the quadratic formula rather than for a proof, the argument would be acceptable because, "Then you don't need words." She believed that proofs need stepwise justifications in order to be counted as proofs in school mathematics. Arguments, such as typical algebraic derivations, that fail to justify each step might be valid but were not considered "real" proofs.

Thinking Skills

Sixteen of the seventeen teachers listed the development of student thinking as a purpose for teaching proof in high school mathematics. The majority discussed both mathematical thinking skills (e.g. gaining deeper insight into mathematical content, and learning to think mathematically) and transferable thinking skills (e.g. learning to think logically, critically, or sceptically). In the first case, proof is a tool to help students learn to understand and appreciate mathematics. In the second case, proof is taught to help students develop their minds so that they can be put to other purposes.

The use of proof as a tool to develop students' mathematical thinking skills was articulated in at least three distinct ways: (1) proofs may provide students with a deeper understanding of the mathematics that they have already learned in middle school; (2) proofs may help to solidify mathematical knowledge by helping them to remember facts; and (3) proofs may help students to think mathematically. Tracy discussed proof in terms of helping students learn to think mathematically. Tracy said, "I honestly think that's the most important part of proofs. It's not the 'Can you do a geometry proof?' [it's] 'Do you understand the rules of geometry?'" For Tracy, proofs offered students an opportunity to demonstrate an understanding of mathematical methods.

According to the majority of the participants of this study, a significant purpose for teaching proof in high school has little to do with mathematics, rather it teaches thinking skills that prepare the mind for future activities not necessarily related to mathematics. One participant said about teaching proof, "We're doing everything in the abstract so it can transfer over to any realm of endeavour in the world." Tracy believed that proofs, while useful in teaching students to think mathematically, are not very useful for learning mathematical content. She said, proofs by themselves "don't give [students] an understanding of geometry, but I think all kids can benefit from doing geometry proofs because it develops thinking skills." She said, "Some kids are never going to use the thoughts of geometry proofs but if they develop the ability to think, then the proof itself was helpful." Further, "If you never use it again, at least you developed your ability to think," she said. Learning how to think was of primary importance to her. She believed that the thinking skills developed in high school could have life-long benefits to students. "I don't think the point of having the kids do proof is having kids do proof. I think it's developing a line of thinking that will help you later in life because…very few of our kids are going to continue on in [math] and do proofs." In her view, proof writing does not necessarily develop students' understandings of mathematical content, but the skills that can be developed from writing proofs are valuable in everyday living.

Specific Proof Tasks

The participants were all asked if certain researcher-generated proofs could be accepted from high school students. When they were asked to view the tasks in this way, some said that even though the proofs were valid, they could not be accepted. Some of the teachers cited the lack of explicit reasoning as reason for the proof's unacceptability, and some cited other concerns. In evaluating the proof tasks given in the second interview, Tracy found some to be valid but said that she could not accept them from high school students because she perceived the proofs as written to subvert the implied purpose of the tasks. This implied purpose was different from task to task but in order for the purpose to be fulfilled, it always required the student to adhere to a prescribed line of thought. We will highlight three of the fifteen tasks to which the participants were asked to respond. *Completing the Square* is a visual argument establishing the formula for the completion of the square, *Summation* is a derivation of the formula for the sum of the first n natural numbers, and *Odd Squares* systematically checks cases to establish that if is odd, then *a* is odd as well. For each proof, she believed that the argument was valid but believed that the proof subverted the implicit intent of a proof task assigned to high school students.

Completing the Square uses pictures rather than algebraic symbols and language to arrive at a formula. Tracy believed that *Completing the Square* was a "shortcut" around the intended task. When asked if she would accept this argument from a student, she said, "I don't think it would be the point of the activity so probably not." In her view, the point of proving this formula was not merely to get a formula but to develop one's capability for using algebra in mathematical reasoning. She felt that the intent, implicit in the task, was to derive the formula by algebraic means. She said that if a student had handed in this visual proof, that he probably had not understood how to do it by the intended method and that he might later be unable to complete either similar or more complex tasks that required him to be able to use algebra. Much like her comments regarding understanding the rules of geometry, she perceived that this task asks students to demonstrate an understanding of the rules. She described *Completing the Square* as a use-it-once strategy that gave the right answer but that in deviating from the anticipated line of thought, it ran counter to her perception of the intent of the assignment.

Summation is a familiar derivation of a familiar formula that is often proved by mathematical induction. Tracy would have preferred that this proof provide stepwise justifications, and further, since the proof does not provide a motivation for adding the integers 1 though n in that particular way, she felt that the reasoning underlying this proof was somewhat concealed. These two factors, she said, made the evaluation of student thinking difficult. Consequently, in order for it to be accepted from a student, she wanted to see more of the reasoning underlying the proof. She believed that *Summation* was just as valid as a proof by mathematical induction but similar to *Completing the Square*, she considered it a shortcut around the intended task. She said, “Generally, you would say ‘Prove by mathematical induction’…You would be testing for induction…I don’t think [this] is probably what they would have been going for.” In Tracy’s experience, this claim was most often proved by mathematical induction as an exercise designed to help students gain experience in proving statements by this method and alternate proofs of this claim subverted the intent implicit in the task.

Odd Squares systematically checks five cases (natural numbers with ending digits 0, 2, 4, 6, and 8) rather than employing a more general argument to establish the fact that the squares of even numbers are also even. This fact is then used to establish that an odd perfect square is the square of an odd number. Tracy thought that the case checking subverted the intent of the task. Instead of proving the claim in a general way, Tracy felt that the proof dealt too much with specifics.

> [It’s] covering all the bases, but I just don’t like those much…I would think that a reason for this question…would be to work with something like [a divisibility argument] so that when you get to proofs where you couldn’t [come] up with this, you would be able to…start dealing with [divisibility] and stuff…This [proof] wouldn’t help your long term understanding of proof.

As with the previous tasks, the acceptability depended on the goal of the activity. In this case, the point of the activity, she believed, was not to prove the claim but to let students practice proof techniques on an easy example. “Then [students] can refer back to this and say ‘OK, what we [did] with the easy stuff…we have to apply to [the] more difficult tasks.” Proving this claim by systematically checking cases does not, she believed, develop student’s facility with writing proofs.

From Tracy’s comments during the interviews, and her voiced opinions regarding the tasks, we were able to synthesize certain aspects of her perception of proof and its purposes in school mathematics. Tracy believed that explicit reasoning in a proof was required to document a line of thought. She did not perceive that the specific mathematical knowledge gained by writing proofs to be of long-term benefit to most of her students but the thinking implicit in proof writing carried life-long benefits. Because most students will not continue to take mathematics courses beyond high school that involve proof writing, students who are more interested in correct answers than in careful thought processes might not develop these transferable thinking skills. She said several times over the course of the interviews that the answer is less important than the reasoning. She seems to have viewed the tasks not as mathematical claims which need to be proved as much as she viewed them as activities to give students practice and experience with particular forms of mathematical reasoning or formats of mathematical writing. In her view, *Completing the Square*, *Summation* and *Odd Squares* all subverted an implicit instructional goal and employed strategies that were neither general nor reusable. Students, she argued, might then be unable to prove something by algebraic means, by mathematical induction, or by using divisibility arguments at a later date. The purpose of proof tasks was not merely to derive a new formula, or to prove a new theorem, but to develop students' thinking along prescribed lines of reasoning, and thus to develop their facility for writing proofs. By following the intended pathways, students would develop the skills needed for future mathematical tasks and would develop a pattern of thinking that would benefit them in later life.

DISCUSSION AND CONCLUSION

Tracy's views about proof in school mathematics may be unorthodox but are not naïve. They are well formed and coherent. They stem from a belief that the reason for teaching proof was to enable students to develop thinking skills that were transferable to other areas of inquiry. To that end, a student's proof should explicitly indicate a thinking pathway leading to the assigned conclusion and should demonstrate an understanding of the "rules" (e.g. definitions, axioms, theorems, rules of inference). A student's proof that is not explicit in its reasoning was perceived to be insufficient for failing to document their thinking. Further, a student's proof that deviated from the anticipated line of reasoning might be seen to be a short-cut around the intended task – even if the proof was clear, convincing, well reasoned, explicit, and valid. The way that Tracy negotiated between valuing intended and alternate proofs followed from her belief that the main purpose of proof in school mathematics is to develop thinking skills in her students. She felt that these skills would be best developed when students followed the intended and prescribed formats.

As Herbst (2002a) pointed out, there are competing demands regarding proof writing in school mathematics. A mathematical claim is not only a mathematical for which a teacher might hold students accountable for producing a proof, but also an opportunity for students to demonstrate a mastery of mathematical thinking and technique. The way teachers negotiate between the demands of holding students accountable for finding *a* proof and for finding *the* proof may be influenced by their perceptions of the purposes of mathematical proof in school mathematics. Further, these perceptions may account for some of the differences between the formal and less formal yet valid proofs described by Knuth (2002). It seems likely that proof would be experienced differently in classrooms in which teachers had different perceptions of the purposes of proof in school mathematics. Understanding teachers' perceptions of proof and its purposes in school mathematics may help to understand how teachers evaluate various forms of proof, particularly those that deviate from standard, (usually two-column) proofs.

References

Goldin, G. (1999). A scientific perspective on structured, task-based interviews in mathematics education research. In Kelley, A. & Lesh, R. (Eds.), *Handbook of Research Design in Mathematics and Science Education*, (pp. 517-545). Mahwah, NJ: Lawrence Erlbaum Associates, Publishers.

Herbst, P. (2002a). Engaging students in proving: A double bind on the teacher. *Journal for Research in Mathematics Education, 33*(3), 176-203

Herbst, P. (2002b). Establishing a custom of proving on American school geometry: Evolution of the two-column proof in the early twentieth century. *Educational Studies in Mathematics, 49*(3), 283-312.

Herbst, P. (2003). Using novel tasks in teaching mathematics: Three tensions affecting the work of the teacher. *American Educational Research Journal, 40*(1), 197-238.

Knuth, E. (2002). Teachers' conceptions of proof in the context of secondary school mathematics. *Journal of Mathematics Teacher Education, 5*(1), 61-88

Nelson, R. (1993). *Proofs Without Words: Exercises in Visual Thinking.* Washington, DC: Mathematical Association of America.

ASSESSING PRIMARY STUDENTS' KNOWLEDGE OF MAPS

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This study explored primary students' knowledge of maps through a sample of mathematics test items. One cohort completed these items annually for three years in a mass testing situation. Another cohort was interviewed once on the same map items. Mass testing results revealed that students' performance generally improved over time. However, significant gender differences in favour of boys were noted annually on each item. Interview results highlighted key difficulties experienced by both girls and boys including interpreting vocabulary incorrectly, attending to the incorrect foci on maps, and overlooking critical information. Our results indicate a need for a focus on extracting and reading information from maps, and analysing and interpreting this information. Girls' achievement should be closely monitored.

INTRODUCTION

In contemporary times, the demand and necessity to become proficient with maps has burgeoned as representations become more complex (e.g., Google Earth) and the desire to traverse unfamiliar environments increases. Hence, the acquisition of complex and dynamic mapping knowledge is required in school mathematics (e.g., Lowrie & Logan, 2007). The purpose of this paper is to investigate students' ability to interpret maps and to identify issues that influence this knowledge.

BACKGROUND

Maps and Information Graphics

Maps are one of the six basic types of information graphics that variously represent quantitative, ordinal and nominal information through a range of perceptual elements (Mackinlay, 1999). The other five graphical languages are: Axis *Languages* (e.g., number line), *Opposed Position Languages* (e.g., bar chart), *Retinal List Languages* (e.g., saturation on population graphs), *Connection Languages* (e.g., network), and *Miscellaneous Languages* (e.g., pie chart). In maps, information is encoded through the spatial location of marks, which are made from a range of perceptual elements such as position, length, angle, slope, area, volume, density, colour saturation, colour hue, texture, connection, containment, and shape (Cleveland & McGill, 1984). Although maps provide an authentic context for learning and assessing mathematical knowledge, students do not always find their interpretation straightforward. Wiegand (2006), for example, maintained that there are three levels of sophistication involved in map interpretation. The initial stage involves *extracting information from a map* and generally reading names and attributes. *Analysis* involves ordering and sequencing information. Finally, *interpretation* requires higher levels of problem solving and decision making involving the application of information.

Spatial Tasks, Map Interpretation, and Gender

Interpreting maps is a spatial task. The literature indicates that on spatial tasks males outperform females (e.g., Bosco, Longoni, & Vecchi, 2004) and on map tasks that males and females adopt different strategies. Saucier et al. (2002) suggested that males employed Euclidean-based strategies to describe directions (e.g., north or west) and distance whereas females tended to use landmark-based approaches (e.g., left or right) to make sense of information. They also found that males outperformed females on tasks that were Euclidean in nature but there were no gender differences on tasks that were represented in a landmark-based form. Reasons for apparent performance differences between males and females on spatial tasks are often associated with confidence and attitudes toward mathematics, classroom interactions, psychological factors, the everyday (out-of-school) experiences of students and even the manner in which tasks are represented. However, most gender differences are attributed to general experiences rather than neurological makeup (Halpern, 2000).

To examine possible differences between the performance of males and females in mathematics, Fennema and Leder (1993) have called on studies to be more focused and strategic. They suggest that rather than taking a broad view of mathematics performance, more studies should be framed at a micro level rather than across large populations. In this investigation we focus on students' mathematics performance on map items that have Euclidean and landmark features.

DESIGN AND METHODS

This study is part of a longitudinal investigation of primary students' ability to interpret information graphics. Three research questions are explored:

- Are there differences between students' performance on Map items over time?
- Is there a difference between boys' and girls' performance on Map items over time?
- What difficulties do students' experience on Map items*?*

The Instrument and Items

The Graphical Languages in Mathematics [GLIM] Test is a 36-item multiple choice test that was developed to assess students' ability to interpret items from the six graphical languages including maps. Test items vary in complexity, require substantial levels of graphical interpretation, and conform to reliability and validity measures (Diezmann & Lowrie, 2007). The GLIM items were selected from state, national and international tests (e.g., QSA, 2002a) that have been administered to 10- to 13-year-olds. This paper reports on students' performance on three of six GLIM map items which include Euclidean or landmark features (See Appendix).

The GLIM map items were administered to different cohorts in mass testing and interview situations. The mass testing cohort completed the GLIM test annually for three consecutive years. The selected map items were scored as 1 or 0 for correct/incorrect responses. The interview cohort was presented with 12 graphical language items annually from the GLIM test including two map items. Students were interviewed on one of the three selected map items each year: Item B (Grade 4), Item A (Grade 5), and Item C (Grade 6). In the interviews, students selected a multiple choice answer and were asked to justify their responses. Interviewers encouraged students to explain their thinking but did not provide scaffolding. Students' responses on each item were analysed for difficulties.

The Participants

A total of 476 students from two Australian states participated in this study. The participants comprised two cohorts. Cohort A and B participated in the mass testing and interview components of the study respectively. Cohort A comprised 378 students (M=204; F=174) from eight primary schools (6 New South Wales, 2 Queensland). Cohort B comprised 98 (M=44; F=54) students from five different primary schools (3 New South Wales, 2 Queensland). The students were in Grade 4 or equivalent when they commenced in the 3-year study. (Grade 4 in New South Wales is equivalent to Grade 5 in Queensland. New South Wales grade levels are used throughout this paper for convenience.) Students' socio-economic status was varied and less than 5% of students had English as a second language.

RESULTS AND DISCUSSION

Part A: Grade and Gender Differences in Map Performance

Questions 1 and 2 relating to grade and gender differences were investigated through an analysis of Cohort A's responses to three map items (See Appendix) that were presented annually in a mass testing situation when students were in Grades 4 to 6. Students' performance on these items was analysed using a multivariate analysis of variance (MANOVA). The dependent variables were Grade (Q. 1) and Gender (Q. 2). The MANOVA indicated statistically significant differences between the two dependent variables across the items with *Grade* [$F(6, 2092)=11.28$, $p \leq .001$] and *Gender* [$F(3, 1045)=9.91$, $p \leq .001$]. Table 1 presents the means (and standard deviations) for grade and gender over the 3-year period.

	Grade 4			Grade 5			Grade 6		
	Total	Male	Female	Total	Male	Female	Total	Male	Female
Item A	.78 (.42)	.79 (.41)	.76 (.43)	.93 (.30)	.96 (.20)	.91 (.29)	.92 (.28)	.94 (.23)	.89 (.31)
Item B	.79 (.41)	.81 (.39)	.77 (.42)	.87 (.34)	.92 (.28)	.81 (.39)	.91 (.29)	.92 (.27)	.89 (.31)
Item C	.63 (.48)	.70 (.46)	.55 (.50)	.73 (.44)	.80 (.40)	.65 (.48)	.75 (.43)	.80 (.40)	.70 (.46)

Table 1. Means (and Standard Deviations) of Student Scores by Grade and Gender

Are there differences between students' performance on Map items over time?

ANOVAs revealed statistically significant differences between the performances of students on each of the three map items *Item A* [$F(2, 1053)=24.7$, $p \leq .001$]; *Item B* [$F(2, 1053)=9.3$, $p \leq .001$]; and *Item C* [$F(2, 1053)=7.9$, $p \leq .001$]. Post hoc analysis showed that mean scores, in all but one case, increased across each of the three years of the study for all three items (See Table 1). It is noteworthy that there were statistically significant differences between the performance of the students between Grade 4 and Grade 5 (across all three items) but differences were not significant between Grades 5 and 6 (on any items). This may be due, in part, to the fact that the improvements in scores from Grade 4 to Grade 5 were substantial (with increases from 10%-20%) — and thus ceiling effects are evident, especially for Items A and B.

Is there a difference between boys' and girls' performance on Map items over time?

There were statistically significant differences between the performance of boys and girls across all three items: *Item A* [$F(1, 1053)=4.89$, $p \leq .03$]; *Item B* [$F(1, 1053)=7.6$, $p \leq .001$]; and *Item C* [$F(1, 1053)=23.5$, $p \leq .001$]. For each item, across each year of the study, the mean scores for the boys were higher than that of the girls. These results indicated that the boys' capacity to interpret these map items was approximately twelve months ahead of that of the girls (with Grade 6 girls' means between 3%-14% below Grade 5 boys' means). Generally, girls' mean scores improved at a constant rate across the three years of the study while the boys' mean scores plateaued from Grade 5—although this may be due to very high scores achieved by the boys in Grade 5 (particularly on Items A and B with means of .96 and .92 respectively).

Our finding that gender differences in favour of boys were evident on map items in the middle to upper primary years is consistent with our previous findings on structured number-line items (Diezmann & Lowrie, 2007). This trend of gender differences on spatial tasks is not confined to the later years in primary school but seems to be apparent from the early years of school (Levine, Huttenlocher, Taylor, & Langrock, 1999). This study and Levine et al.'s study suggests that girls need to be provided with early and ongoing support to develop their map knowledge to a similar level to boys in the primary years.

Part B: Students' Difficulties with Maps Items

The final question was explored through an analysis of unsuccessful students' responses from Cohort B on the same three map items as in Part A (See Appendix).

What difficulties do students' experience on Map items?

Students were unsuccessful on these items in the interviews for various reasons including guessing responses, misunderstanding the question, interpreting vocabulary incorrectly, attending to the incorrect foci, and overlooking critical information. The first two reasons for a lack of success are generic errors and are not discussed here. Examples of the latter three reasons are presented to provide some insight into students' thinking on map items. Due to performance differences in favour of boys identified in Part A this paper, examples of each of these errors were sought from Cohort B girls' responses. Although a full gender comparison of responses is beyond the scope of this paper, differences were consistent across cohorts.

Interpreting Vocabulary Incorrectly: Some students were misled by their interpretation of a key spatial term in the text. For example on Item A (See Appendix), Noni incorrectly interpreted "through" as relating to being "included in" or being "outside of the bike track" in *What part of the Park won't she ride* <u>*through*</u> (emphasis added)*?*

> Noni: Because at first I went through all of them and B4, A4 and B5, like, is <u>included</u> in the bike track and I stuck with A5 and B5 and I just pick (sic) A5 because I thought it's more <u>outside</u> of the bike track (emphasis added).

Attending to Incorrect Foci: Although students' counting was generally accurate, they sometimes counted an incorrect item or action. In addition, they did not use the map as a referent in their counting. For example on Item B (See Appendix)—*How many times did he (Ben) cross the track?*—Bree focussed on the <u>movements</u> between landmarks on the map rather than <u>the number of times the track</u> was crossed. Thus, she selected the incorrect response of 'three' rather than the correct response of 'two'. Her response indicated no reference to the landmarks in relation to the track.

> Bree: I reckon it was three because he went from the gate to the tap (one) and then he went to the tap (two) and then to the shed (three) (emphasis added).

Overlooking Critical Information: Some students only paid attention to part of the information given in their responses. For example, on Item C (See Appendix) some students focused on the numerical and directional information in isolation rather than in combination in a set of instructions. On this item, students were required to identify the "second road on the left" rather than simply the second road and the left and right turns independently.

> Ellen: Post Rd (her incorrect answer). Started at the pool, then took right turn (Wattle Road) then left turn and it's Post Rd.

Analysis of students' difficulties on the three map items suggests two points of interest. First, students' difficulties relate to each of Wiegand's (2006) levels of sophistication in map interpretation. *Extracting information* from a map requires knowledge of vocabulary (e.g., Item A - "through"). *Analyzing information* requires knowledge of how to interpret complex information (e.g., Item C - "second on the left"). *Interpreting information* requires knowledge of what can and should be counted (e.g., Item B). Second, girls' difficulties on Items B and C suggest that Saucier et al.'s (2002) proposal that gender differences can be explained by girls' use of landmark-based approaches (e.g., left or right) and boys' use of Euclidean-based strategies (e.g., north or west) is not supported. Both genders (Cohort B) experienced difficulties with these items. Boys also outperformed girls on these items (Table 1).

CONCLUSION

Our study revealed that some students are making errors on relatively simple map items. Difficulties with mathematical know-how (of maps) indicate a need for a focus on mathematical practices (Ball, 2004). This focus should address each level of sophistication in understanding map information (Wiegand, 2006): extracting and reading, analysing, interpreting. Learning opportunities related to these levels need to be provided and achievement monitored throughout the primary years especially for girls. Our identification of gender differences in the middle to upper primary years suggests that research is needed in the early primary years to identify and ameliorate early differences and in high school to establish the impact of these differences.

References

Ball, D. L. (Chair) (2004). *Mathematical P for All Students: Toward a Strategic Research and Development Program in Mathematics Eeducation*. RAND Mathematics Study Panel. Retrieved January 1, 2006, from http://Fwww.rand.org/pubs/monograph_reports/MR1643/MR1643.pref.pdf

Bosco, A., Longoni, A.M., & Vecchi, T. (2004). Gender effects in spatial orientation: Cognitive profiles and mental strategies. *Applied Cognitive Psychology, 18,* 519-532.

Cleveland, W. S., & McGill, R. (1984). Graphical perception: Theory, experimentation, and application to the development of graphical methods. *Journal of the American Statistical Association, 79*(387), 531–554.

Diezmann, C. M., & Lowrie, T. J., (2007). The development of primary students' knowledge of the structured number line. In J-H. Woo, H-C Lew, K-S Park & D-Y Seo (Eds.), *Proceedings of the 31st Annual Psychology of Mathematics Education Conference* (Vol 2, pp. 201-208). Seoul: PME.

Fennema, E., & Leder, G. (Eds). (1993). *Mathematics and gender.* Queensland, Australia: University of Queensland Press.

Halpern, D.F. (2000). *Sex differences in cognitive abilities.* Mahwah, NJ: Lawrence Erlbaum Associates.

Levine, S. C., Huttenlocher, J., Taylor, A., & Langrock, A. (1999). Early sex differences in spatial skill. *Developmental Psychology, 35,* 940–949.

Lowrie, T., & Logan, T. (2007). Using spatial skills to interpret maps: Problem solving in realistic contexts. *Australian Primary Mathematics Classroom, 12,* 14-19.

Mackinlay, J. (1999). Automating the design of graphical presentations of relational information. In S. K. Card, J. D. Mackinlay, & B. Schneiderman (Eds.), *Readings in information visualization: Using vision to think* (pp. 66-81). San Francisco, CA: Morgan Kaufmann.

Queensland Studies Authority [QSA] (2001). *2001 Queensland Year 5 test: Aspects of Numeracy.* Victoria, Australia: Australian Council for Educational Research.

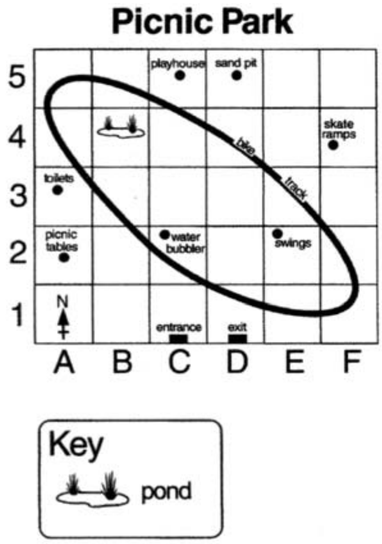
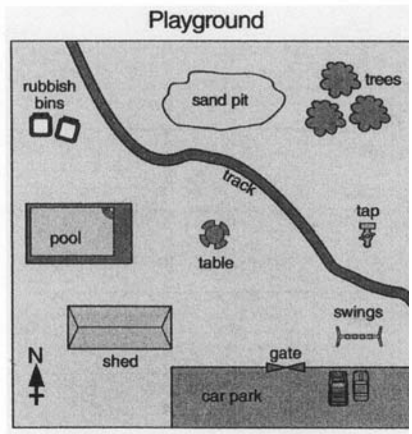
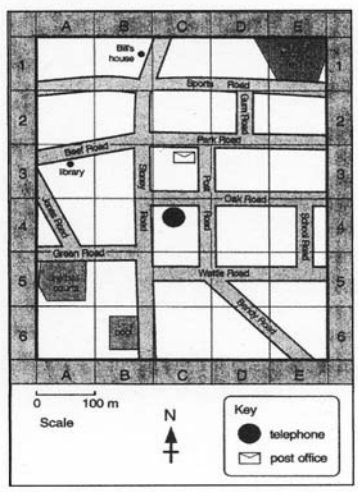
Queensland Studies Authority (2002a). *2002 Queensland Year 3 test: Aspects of Numeracy.* Victoria, Australia: Australian Council for Educational Research.

Queensland Studies Authority (2002b). *2002 Queensland Year 5 test: Aspects of Numeracy.* Victoria, Australia: Australian Council for Educational Research.

Saucier, D. M., Green, S. M., Leason, J., Macfadden, A., Bell, S., & Elias, L. J. (2002). Are sex differences in navigation caused by sexual dimorphic strategies or by differences in the ability to use the strategies? *Behavioural Neuroscience, 116,* 403–410.

Wiegand, P. (2006). *Learning and teaching with maps.* London: Routledge.

Appendix: Map Items

<table>
<tr>
<td>Deb rides her bike along the bike track. What part of the park won’t she ride through?

</td>
<td>Ben went from the gate to the tap, then to the shed, then to the rubbish bins. How many times did he cross the track?

</td>
<td>Bill leaves the pool. He drives north and takes the first road on the right, then the second road on the left. Which road is he in?

</td>
</tr>
<tr>
<td>Item A (QSA, 2001, p. 16)</td>
<td>Item B (QSA, 2002a, p. 11).</td>
<td>Item C (QSA, 2002b, p. 7)</td>
</tr>
</table>

CONCEPTUAL INTEGRATION, GESTURE AND MATHEMATICS

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The research reported here focuses on an examination of the conceptual underpinnings of two areas of mathematical thought, fractions and proof. The analysis makes use of the theoretical framework of conceptual integration, and draws on spontaneous gesture as an important data source. The question of how gestures evoke meaning is addressed within the context of two studies, one involving prospective elementary school teachers discussing fractions, and the other involving doctoral students in mathematics talking about and carrying out proofs. In both situations, gestures and their accompanying language are analyzed in terms of conceptual mappings from more basic conceptual spaces.

INTRODUCTION

The analysis of mathematical thinking has long distinguished between publicly shared and "private" mathematics (e.g., Tall & Vinner's (1981) distinction between "concept definition" and "concept image"). The goal of understanding how learners as well as mathematicians conceptualise mathematics is an important aim of the field of mathematics education. Over the past decade, research in mathematics education has made contact with recent developments in cognitive science that offer fruitful methods of both collecting and analyzing data in order to characterize the conceptual underpinnings of mathematical thought. The research described here utilized one such framework, cognitive linguistics, to examine the ideas of two different groups of people involved with mathematics: undergraduate prospective elementary school teachers on the one hand, and doctoral students in mathematics at a major research university on the other. In each study, the students were interviewed in pairs and asked to solve one or more mathematical problems; for the undergraduates, the topic was fractions, and for the graduate students, the topic was mathematical proof. The overall question guiding the research was: How does gesturing contribute to and/or express the ways in which a learner understands a mathematical concept? More specifically, the two studies addressed the question of how mathematical meanings are conveyed with the help of gestures, how we are able to interpret the meaning of gestures, and whether gesture can be used as a source of information in uncovering the more basic understandings out of which mathematical ideas are constructed (Lakoff & Núñez, 2000).

THEORETICAL FRAMEWORK AND RELATED RESEARCH

The overarching theoretical framework utilized in the study is embodied cognition (Varela, Thompson & Rosch, 1991), that is, the principle that thinking and reasoning are grounded in physical experience and the particularities of biological existence. As applied to mathematics, research from an embodied cognition perspective investigates how humans utilize their embodied capabilities to construct both concrete and abstract mathematical understandings that can be shared within classrooms, real world settings, and the professional mathematics community.

Within the embodied cognition perspective, the current study draws on two specific lines of work. The first is the theory of conceptual integration and conceptual or mental spaces. Mental spaces, as defined by Fauconnier and Turner, are "small conceptual packets constructed as we think and talk, for purposes of local understanding and action" (Fauconnier & Turner, 2002, p. 40). Conceptual integration or blending is a cognitive mechanism or mapping that "connects input spaces, projects selectively to a blended space, and develops emergent structure" (Fauconnier & Turner, 2000, p.89). Conceptual integration can be seen as a general mechanism that encompasses more specific mappings such as conceptual metaphor; the latter have been used in the analysis of mathematical ideas ranging from arithmetic to calculus (e.g., Bazzini, 1991; Lakoff & Núñez, 2000; Núñez, Edwards & Matos, 1999). One of the goals of the current research was to propose conceptual mappings that could account for the mathematical ideas discussed by the participants.

The second line of work related to embodied cognition that is integral to the research is the investigation of gesture as an important modality of communication and cognition. Gesture studies has emerged as an interdisciplinary enterprise drawing from linguistics, psychology and other cognitive science fields, and has recently attracted the interest of mathematics education researchers. Investigations of gesture and mathematics have addressed activities ranging from counting to differential equations (e.g., Graham, 1999; Rasmussen, Stephan & Allen, 2004); have been examined through time and synchronously (e.g., Arzarello, 2006); and have focused on individuals, pairs, small groups and entire classrooms (c.f. Roth, 2001 for a review of gesture studies in mathematics and science). Findings of research on gesture and mathematics include evidence that gesture and speech can "package" complementary forms of information, and can be utilized by the speaker to support thinking and problem-solving (Arzarello, 2006; Goldin-Meadow, 2003; Radford, 2003). In several studies, learners are able to express their understanding of a new concept through gesture before they are able to express it in speech; that is, gesture seems to be an indicator of "readiness to learn" the new concept (Goldin-Meadow, 2003). The research described here involved participants in both talking about mathematical ideas and solving mathematical problems; gesture was used as a clue to how they were thinking about the mathematics, and as a modality of expression complementary to speech.

METHODOLOGY

There were two groups of participants in the study. The first were twelve female undergraduate students, approximately 20 years of age, taking a required course in number systems, algebra and problem solving for prospective elementary school teachers. The course was taught by the author, who offered extra course credit to the students who volunteered to participate in the study.

The second group of participants were twelve graduate students enrolled in a doctoral program in mathematics at a large research university. There were three women and nine men; all had had experience as teaching assistants in various undergraduate courses.

The participants were interviewed in pairs; the interviews were videotapes. Each interview consisted of a set of questions about the mathematical topic, followed by joint problem solving by the pair of students, and concluding with additional questions. The undergraduate students were interviewed about fractions, and the questions they were asked included: How did you first learn about fractions? Was there anything particularly difficult for you in learning about fractions? What is the definition of a fraction? How would you introduce fractions to children? The problems presented included four problems involving arithmetic with fractions and one problem comparing two fractions.

For the doctoral students, the topic of the interview was mathematical proof. The students were asked questions such as: Are there any kinds of proofs that your students have difficulty with? Would you say there are different kinds of proof? What makes a proof difficult or easy for you? The students were then given an unfamiliar conjecture and asked to find a proof for it, and were also asked to judge whether a particular mathematical argument presented in visual form constituted a proof.

The sessions were videotaped, and the tapes were transcribed in order to document the students' speech. In addition, the physical gestures displayed by the participants were tabulated and categorized, utilizing the dimensions identified by McNeill (200*). These dimensions included iconicity (resemblance to the object that is the referent of the gesture), metaphoricity (when the referent is an abstraction, thus the gesture cannot display physical resemblance directly), and deixis (indication of a location either in physical space or "gesture" space, that is, the virtual space constructed via gesture and concurrent speech).

A comprehensive analysis of all of the gestures displayed by the participants in each study will not be presented here (see Edwards, in press, for such an analysis of the fractions data). Instead, examples will be provided from each study that address the central research questions, that is, how do gestures convey mathematical meanings, and, along with speech and other modalities, can they provide information on the students' conceptualizations of mathematical ideas?

ANALYSIS

The analysis will focus on two examples, a simple iconic gesture about learning fractions, and a gestures displayed when discussing proof. The analysis draws directly on the theory of conceptual integration and in specific on work by Parrill and Sweetser (2004) applying conceptual blending to the interpretation of gestures.

Example 1: An iconic gesture for "cutting"

Figures 1a and b shows LR, a prospective teacher, describing how she first learned about fractions, utilizing a simple "cross-cutting" gesture.

Figure 1a. “Like pies, like cutting the pie in like // pieces”.

Figure 1b. “and you cut in like eighths”.

Given that we share common physical and cultural experiences with LR, we easily interpret this iconic gesture as referring to the action of cutting or slicing with a knife. However, the theory of conceptual integration explains *how* we are able to make this interpretation, and, furthermore, how we are able to understand that LR’s slicing gesture does not refer to a culinary activity, but instead to a mathematical idea.

From the perspective of conceptual integration, LR’s gesture is a blend that draws from two input spaces: first, her conceptual understanding of the immediate physical world (including the shapes that her hand can make), and second, her mental model of the act of cutting with a knife.

Figure 2 illustrate the conceptual blend that gives rise to this gesture. The two inputs are shown on the left and right sides of the diagram. Above, the “generic space” refers to elements that the two spaces have in common; these commonalities allow our minds to construct the blend, shown in the bottom circle. In this case, the generic space includes such features as the perpendicularity of both the hand and the knife to the surface of the table, the fact that both are narrow relative to their lengths, and that both can be moved up and down. In utilizing the affordances of her hand and arm to highlight these commonalities, LR evokes a conceptual blend that allows an interlocutor to “see” her hand as a knife being used to cut or slice something.

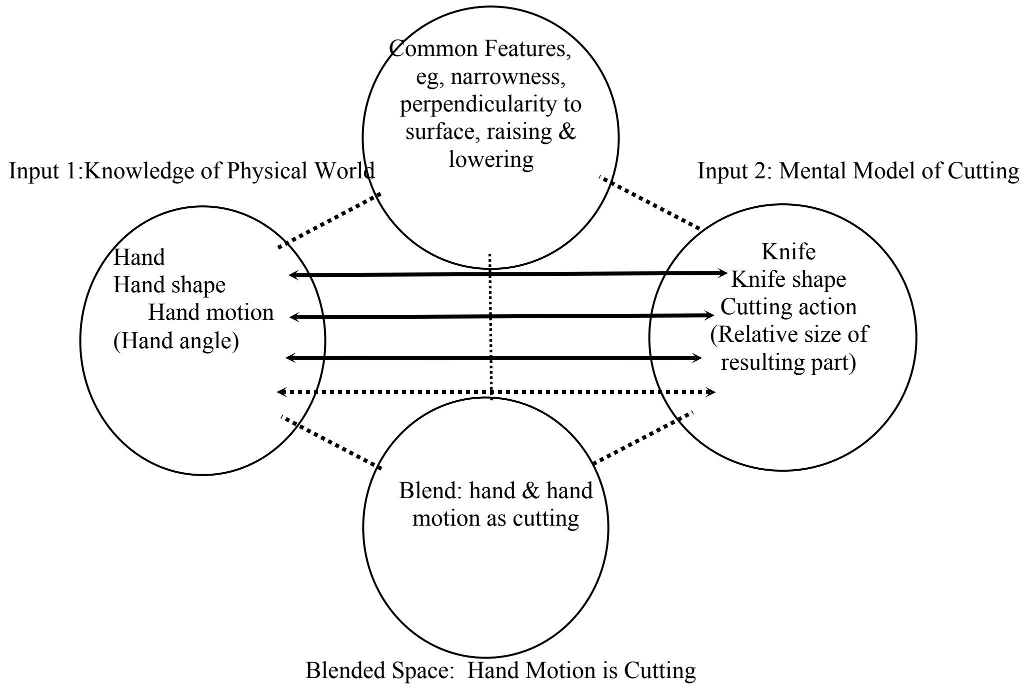

Figure 2. Conceptual blend for the iconic gesture of "cutting".

Although the blend for this iconic gesture is straightforward, it is notable that most participants' gestures for "cutting" or "splitting" were not as precise as LR's. In Figure 1b, the 45° angle that LR made with her hand was a meaningful part of the gesture, resulting in a blended space in which the amount "one-eighth" was embodied visually and concretely. These "optional" visual elements, "Hand angle" and "Relative size of resulting part," are shown in parentheses in the blending diagram, in order to indicate that they are not found in all gestures for cutting.

Of course, in the given context, we are not interested in the gesture of cutting in and of itself (although in a different context, it might have an important meaning in terms of a recipe). In this context, the cutting gesture itself refers to the act of dividing a whole (which LR gestured by tracing a circle on the desktop) into equal-sized pieces named by specific fractions (hence her precision about the angle of her hand). LR produced the gesture when describing how she first learned about fractions; thus, the gesture was meant to evoke cutting a pie into equal pieces, which itself was meant to evoke the abstract mathematical concept of a fraction. Figure 3 illustrates this "chain of signification" (Walkerdine, 1988), where the gesture is one of the inputs to a further conceptual blend, in this case, a simple one-way metaphor in which the source (cutting a pie) is mapped to the target (a fraction conceived as part of a whole).

Thus, LR's gesture of slicing a pie into equal pieces has an iconic dimension, since the slicing gesture intentionally resembles the action of cutting with a knife. But it also displays metaphoricity, because it ultimately refers to an abstract mathematical idea, that of a fraction. The gesture arose through a memory of a classroom learning experience, in which realia or manipulatives were used to help students construct an understanding of fractions.

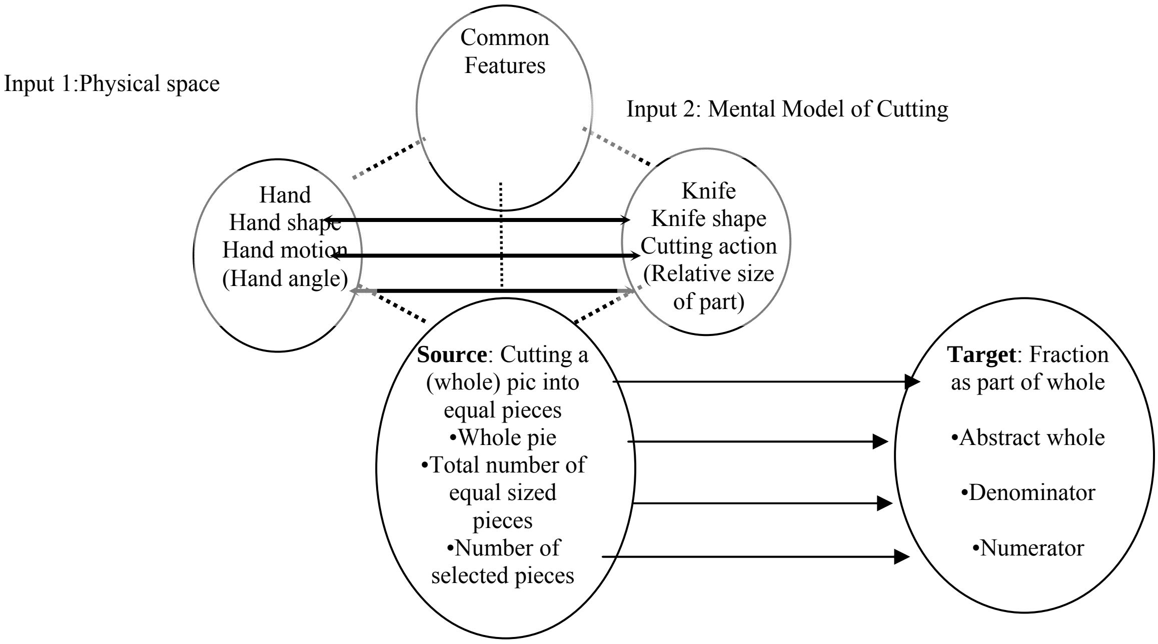

Figure 3. "Chain" of conceptual blends for fraction concept.

The role of tangible materials in this context seems not to be as "representations" of mathematical ideas, but rather as objects on which the students act, and from which they abstract salient characteristics. Conceptual integration provides a mechanism for explaining how this abstraction occurs.

Example 2: A metaphor for proof

Figure 4 illustrates a still from a gesture sequence displayed by WG, one of the graduate students in the second study. When asked what kinds of proofs he found difficult or easy, in part of his reply, he said:

> cause you start figuring out, I'm starting at **point a and ending up at point b**. There's gonna be **some road**//**where does it go through**? And can I show that **I can get through there**? (**bold** text indicates synchronization of speech with gesture).

Figure 4: "Proof is a journey" gesture.

He began the full gesture sequence by closing the fingers of his left hand and touching a location near the top of his thigh ("**point a")**, then opening his right hand and pointing as he moved it away from his body ("**point b**"). He then traced a fairly straight path through the air with his right index finger, returning and pausing briefly after "some road." He then made a small horizontal circle with the same figure, and retraced the path between the origin and end of the gesture.

The metaphor underlying both the gesture and the speech in this example is clear: WG is conceptualising proof as a journey. Table 1 summarizes this metaphor (also known as a single-scope conceptual blend).

Source: A Journey	**Target:** A Mathematical Proof
Starting point	Givens
Destination	To prove/conclusion
Possible routes	Possible sequences of statements
"Dead ends"	Sequences that don't result in the conclusion

Table 1. "Proof is a journey" metaphor

The "journey" metaphor was not the only way that this student spoke (and gestured) about proof. Just prior to this example, WG said, "And then the question is, well, can I fill in **those steps** that I have ?", while displaying a series of gestures in front of him, with his right hand held horizontal and dropping vertically below itself three times. Although his speech, on its own, might be interpreted as referring directly to a journey ("steps" could refer to walking), his gesture made it clear that the "steps" he was talking about were statements within a proof, written from top to bottom either on a piece of paper, or on a blackboard. The underlying metaphor of a journey is arguably still there, in that the socially common use of "steps" to indicate logical inferences in a proof betrays a grounding in thinking about carrying out a proof in terms of motion or travel. However, the most immediate input space for the conceptual blend is a written inscription, which in turn refers to the recording of a sequence of logical statements, in a second example of a "chain of signification."

DISCUSSION

Clearly, the "journey" metaphor does not provide all of the essential conceptual elements of a mathematical proof, nor does the input of "cutting equal parts of a whole pie" support a complete understanding of fractions. For one thing, in the proof situation, the logical necessity that makes one statement "follow" another (note the ubiquity of the metaphor) is not part of the input space or source domain of a journey – the steps of a journey are not determined by the prior steps. In fact, the cognitive phenomenon of "logical necessity" may originate not (or not only) as a conceptual blend, but from more basic cognitive capabilities. Any individual conceptual blends or metaphor is not intended to fully account for the richness of a given mathematical concept. Yet the framework of embodied cognition, and the tools of cognitive linguistics and gesture analysis can help us discover the ways that both novices and more experienced students build and conceptualize mathematical ideas.

References

Arzarello, F. (2006). Semiosis as a multimodal process. *Revista Latinoamericana de Investigacion en Mathematic Educativa, Numero Especial,* 267-299

Bazzini, L. (1991). From grounding metaphors to technological devices: A call for legitimacy in school mathematics. *Educational Studies in Mathematics, 47,* 259–271

Edwards, L. (in press). Gestures and conceptual integration in mathematical talk. *Educational Studies in Mathematics.*

Fauconnier, G. & Turner, M. (2002). *The Way We Think: Conceptual Blending and the Mind's Hidden Complexities.* (New York: Basic Books)

Goldin-Meadow, S. (2003). *Hearing Gestures: How Our Hands Help us Think*. (Chicago: Chicago University Press)

Graham, T. (1999). The role of gesture in children's learning to count. *Journal of Experimental Child Psychology, 74,* 333-355.

Lakoff, G. & Núñez, R. (2000). *Where Mathematics Comes from: How the Embodied Mind*

McNeill, D. (2005). *Gesture and Thought*. (Chicago: Chicago University Press)

Nuñéz, R., Edwards, L., & Matos, J. (1999). Embodied cognition as grounding for situatedness and context in mathematics education. *Educational Studies in Mathematics, 39*(1-3), 45-65.

Parrill, F. & Sweetser, E. (2004). What we mean by meaning. *Gesture, (4)*2, 197-219.

Radford, L. (2003). Gestures, speech and the sprouting of signs. *Mathematical Thinking and Learning, 5*(1), 37-70.

Rasmussen, C., Stephan, M., & Allen, K. (2004). Classroom mathematical practices and gesturing. *Journal of Mathematical Behavior, 23*, 301-323.

Roth, W.-M. (2001). Gestures: Their role in teaching and learning. *Review of Educational Research, 71*, 365-392.

Tall D. & Vinner S. (1981). Concept image and concept definition in mathematics, with special reference to limits and continuity. *Educational Studies in Mathematics, 12,* 151-169.

Varela, F., Thompson, E., & Rosch, E. (1991). *The embodied mind: Cognitive science and human experience*. (Cambridge, MA: MIT Press)

Walkerdine, V. (1988). *The mastery of reason*. (London: Routledge)

TEACHING THE SAME ALGEBRA TOPIC IN DIFFERENT CLASSES BY THE SAME TEACHER

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Ruhama Even — Weizmann institute of science

The aim of this study is to examine how a teacher enacts the same written algebra curriculum materials in different classes. The study addresses this issue by comparing the types of algebraic activities (Kieran, 2004) enacted in two 7th grade classes taught by the same teacher, using the same textbook. Data sources include lesson observations and an interview with the teacher. The findings show that overall the three types of algebraic activity were enacted in both classes in similar proportions. But an examination of the whole class work only shows that there was more emphasis on global/meta-level activities in one class than in the other. Thus, students in one class were learning a different algebra than students in the other class during whole class work. This difference is linked to students' behavior.

BACKGROUND

Are students exposed to the same mathematical ideas when a teacher enacts the same written curriculum materials in different classrooms? Previous studies of curriculum enactment have suggested that different teachers enact the same written curriculum materials in different ways (e.g., Cohen & Ball, 2001, Manoucheri & Goodmann, 2000; Tirosh, Even, & Robinson, 1998). Studying the enacted curriculum in different classes of the same teacher, however, has only now started to be the focus of research studies (Even & Kvatinsky, 2007; Herbel-Eisenmann, Lubienski, & Id-Deen, 2006; Lloyd, in press). These studies highlight contextual factors that contribute to the enacted curricula (e.g., student/parent expectations). Still, seldom did the teacher in these research studies use the *same* written materials in the different classes, and the focus in these studies was mostly on pedagogy and rarely did they examine the mathematics in the enacted curriculum in different classes of the same teacher. This study addresses this deficiency in the context of school algebra.

Kieran (2004) developed a model of algebraic activity that we find to be useful as a framework for organizing school-level algebra activities. The framework distinguishes among three types of school algebra activities:

- *Generational activities.* These activities involve the forming of expressions and equations that are the objects of algebra (e.g., writing a rule for a geometric pattern). The focus of generational activities is the representation and interpretation of situations, properties, patterns, and relations.
- *Transformational activities.* These include 'rule-based' algebraic activities (e.g., collecting like terms, factoring, substituting). Transformational activities often involve the changing of the form of an expression or equation in order to maintain equivalence.

- *Global/meta-level activities.* These are activities that are not exclusive to algebra. They suggest more general mathematical processes and activities in which algebra is used as a tool. They include activities that require students to problem solve, model, generalize, predict, justify, prove, and so on.

Exploring “match trains” (see Figure 1), the following problem (see Figure 2) illustrates the three types of algebraic activity described above.

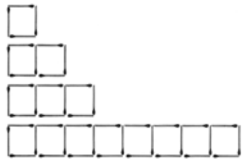

Figure 1. "Match trains".

> Doron said: "For the number of matches required to build a train with r squares, the algebraic expression $4+3\cdot r$ is suitable." Is this algebraic expression suitable? Use substitution to check. How many numbers need to be substituted to determine that this algebraic expression is not suitable? (Robinson & Taizi, 1997, p. 10)

Figure 2. "Match train" problem.

Analysis of the types of algebraic activity shows that the potential of this problem is all three types. To check the suitability of the algebraic expression $4+3\cdot r$ one may, for example, reconstruct the hypothetical process Doron used to form it: four matches for the first wagon, and three matches for each of the other wagons, resulting with an extra set of three matches (generational). Another way to check would be to substitute a specific number in the given expression, build and count the number of matches in the corresponding train, and compare the two results *(*transformational). The last part of the problem calls for an examination of the role of examples and counter-examples in mathematics proof and refutation (global/meta-level).

The aim of this paper is to examine, using Kieran’s framework of generational, transformational, and global/meta-level algebraic activities, how a teacher enacts the same written algebra curriculum materials in two different classes.

METHODS

This is a case study of one teacher, Sarah (pseudonym), who taught two 7th grade classes, each in a different school, Carmel and Tavor (pseudonyms). Sarah used the same curriculum materials (i.e., textbook and teacher guide) in both classes (one of the innovative 7th grade mathematics curriculum programs developed in the 1990's in Israel).

Sarah, the teacher, has a B.Ed with emphases in mathematics and biblical studies from a teacher college. In the year preceding the research she worked with the team that developed the curriculum materials, which are the focus of this study, as part of a professional development program. She became very fond of the curriculum and decided that she wanted to use it in her teaching. The year of the study was a year of several new beginnings for her. It was her first year teaching in 7th grade, and the first year of teaching the new curriculum materials. It was also her first year teaching in the two schools, Carmel and Tavor. Carmel is a selective single-gender (girls only) Jewish religious school. The 7th grade class (with 20 students) that participated in the research was characterized by a learning atmosphere with rich and meaningful classroom talk. Tavor is a secular junior-high school. Mathematics lessons in the 7th grade class (with 27 students) which participated in the research were characterized by lack of cooperation – the class was very noisy and there were many disciplinary problems.

Data collection was conducted during one school year (2002-2003). The main data sources included video-taped observations of the teaching of the beginning of the topic *equivalent algebraic expressions* – nineteen 45-minute lessons in Carmel, and fifteen 45-minute lessons in Tavor (where the first author was a non-participant observer), and an audio-taped interview with Sarah that was conducted after all observations were completed.

The data were analysed both quantitatively and qualitatively. First we coded the written curriculum materials. The beginning of the topic *equivalent algebraic expressions* was divided into 15 units in the curriculum materials, each suggested for a 45-minute lesson. Eleven of these units were enacted (fully or partially) in Carmel; ten of them in Tavor. For the purpose of this study, only the 11 units that were enacted in at least one of the classes were analysed. In general, each unit started with a multi-task assignment for small group work, followed by another multi-task assignment for whole class work. Some of the units included also single- or multi-task assignments to be assigned by the teacher as needed, either to the whole class, or to specific students, sometimes in parallel (e.g., to low or high achievers, to slow or advance students). The 11 units analyzed included a total of 46 assignments. We coded these assignments into one or more of the following categories: generational, transformational and global/meta-level algebraic activity, by analysing their potential. (Note that only the potential type of a written item can be analysed because the enacted activity may not realize its potential, e.g., justification may not be provided even tough was asked for.) We also added the time suggested for class work on each assignment, as indicated by the written materials. Almost all the assignments were composed of several related smaller tasks; the 46 assignments included a total of 367 tasks. We coded also these 367 tasks into one or more of the above categories.

After analysing the types of algebraic activity in the written curriculum materials, we analysed the types of algebraic activity in the enacted curriculum in the two classes. Using a Chi-square test, we then compared between the distributions of algebraic activity types:

a) in the curriculum materials and in the enacted curriculum, for each of the two classes; and b) in the enacted curricula in the two classes. Comparisons were made, taking into account that the categories are not distinct, between the total number of assignments and tasks in each category, and on the total time spent on assignments in each category, as all are important indications of the nature of students' algebraic experiences in class. Finally, we examined the nature of the class activity and the realization of the potential of the suggested algebraic types as well as Sarah's view on that.

TYPES OF ALGEBRAIC ACTIVITY IN THE ENACTED CURRICULA

Analysis of the curriculum materials shows that most assignments and tasks in the written materials - about three-fourths - were transformational, and a similar part of the class time was suggested to be devoted to these assignments. Still, the written curriculum materials included quite a few generational and global/meta-level activities (note that the categories are not distinct). About one-half of the assignments were generational, and a similar part of the class time was suggested to be devoted to them. Moreover, almost one-third of the assignments were global/meta-level, and more than 40% of the class time was suggested to be devoted to them.

In the following we present first the types of algebraic activity that characterize the assignments and tasks that the teacher chose to *assign* students. For this we combine small group and whole class activities. Yet, classroom observations suggested that in Tavor students often did not work on their assigned small group work, but instead, engaged in various non-mathematical activities. Also in Carmel some of the students were not always task oriented during small group work. Thus, an analysis that combines small group and whole class activities does not necessarily reflects the activities that were *actually worked on*. Therefore, in the second part of this section we examine separately the whole class work, which includes only activities actually worked on in class. The whole class activities are especially important because, according to the written materials, their aim was to advance students' mathematical understanding and conceptual knowledge. Whereas the first part of the section includes findings from a quantitative analysis only the second part reports findings from both quantitative and qualitative analyses.

Types of assigned activities

Analysis of the enacted curricula in each of the two classes showed that Sarah used only assignments from the curriculum materials, and rarely used tasks that were not from the curriculum materials (only in a few cases of whole class work). Still, not all of the assignments and tasks from the written curriculum materials were used, either in Carmel or in Tavor. About two-thirds of the assignments and the tasks from the written materials were used in Carmel and about one-half of them were used in Tavor. Although not all of the assignments and tasks included in the written materials were used in the classes, in Carmel statistically significant more time was devoted to the teaching of the materials than either the time suggested in the curriculum materials or the time devoted to the teaching in Tavor.

Analysis of the types of algebraic activity of the assignments and the tasks used in the two classes, and of the class time devoted to the different types, showed that in spite of the differences in materials' coverage, there were no statistically significant differences between both Carmel's or Tavor's assigned assignments and tasks and the written curriculum materials, in their overall emphasis on the different types of algebraic activity. All three types of algebraic activity appeared in both enacted curricula in a similar proportion to that of the written curriculum materials. Transformational activities were again more dominant (about three-fourths of the activities), and generational and global/meta-level activities also played a considerable role, with generational activities being more frequent. Thus, overall, the relative distribution of the types of algebraic activities assigned was similar in the two classes and it was also similar to the distribution in the curriculum materials.

Types of whole class activities only

Analysis of the whole class work showed that statistically significant lesser percentage of global/meta-level activities was enacted in Tavor during whole class work (three out of 10 assignments, and one out of 48 tasks) compared with Carmel (six out of 11 assignments, and nine out of 51 tasks). Moreover, Tavor's students not only worked during whole class work on less global/meta-level activities than Carmel's students, but they did it only during the first part of the teaching sequence whereas Carmel's students did it throughout the teaching of the topic.

In addition to omitting the global/meta-level activities from the whole class work during the second part of the teaching sequence in Tavor, there were several cases when the same assignment or task was enacted in Carmel as a global/meta-level activity but not so in Tavor (Eisenmann & Even, in press). For example, the whole group work in Carmel on the problem in Figure 2 included all three algebraic activity types. Led by the teacher, the class examined the situation, formed suitable expressions, a generational activity, and by analysing the hypothetical process Doron used to form his algebraic expression, showed that his suggestion was inappropriate. Working on the task also included substitution in Doron's expression (r=5) to enable a comparison between the numerical result of the substitution (19) and the actual number of matches in a five-wagon train (16), a transformational activity. Finally, the teacher explained and named an important method of refutation in mathematics (counter example), which also made this activity a global/meta-level one.

In contrast with Carmel, in Tavor the whole group work on this problem included only two algebraic activity types. Again, led by Sarah, the class examined the situation, formed suitable expressions, a generational activity, and by analysing the hypothetical process Doron used to form his algebraic expression, showed that his suggestion was inappropriate. An important component of the work on the task in Tavor was substitution in Doron's expression (r=6) to enable a comparison between the numerical result of the substitution (22) and the result of the actual counting of the number of matches in a six-wagon train (19), a transformational activity. However, unlike the work in Carmel, the class activity did not include a global/meta-level aspect. Neither

Sarah nor the students incorporated more general mathematical processes and activity, such as the role of examples in mathematical proof and refutation.

The difference in emphasis on global/meta-level activities between the two classes seemed to be related to the different characteristics of the two classroom environments, namely, discipline problems and lack of student cooperation with Sarah at Tavor. In her interview at the end of the observation period, Sarah explained how this caused her to change her instructional strategy to implement less thinking-related activities during whole class work:

> If I had to choose whether to do something or not, there are things, there are things that require more thinking and more, eh. In *Tavor* sometimes I gave up on them. More so, later in the year…
>
> I knew that not everything could work there… Because of the problems that, discipline problems, problems of students' cooperation.

Observations at Tavor indeed indicated that, during the whole class work, there were many discipline problems that caused interruptions in the mathematics activity. An examination of the percentage of time in the whole class work devoted to mathematical activity vs. non-mathematical activity (mainly discipline interruptions) showed that in Carmel, there were rarely any discipline problems (about 2% of whole class time) that caused interruptions in the mathematical activities. In Tavor, however, the case was quite different; in every lesson during the whole class work, there were interruptions to the mathematical activities, totaling 20% of the whole class work time.

Furthermore, as mentioned earlier, Tavor's students, in contrast to Carmel's, often did not complete the assigned small group work. Therefore, at Tavor, tasks intended for the small group work were repeated during whole class work. Since mathematical work at *Tavor* was interrupted many times, either because of discipline disruptions or because of unfinished small group work, Sarah found it more difficult to enact whole class activities that required higher-order thinking. Some of these activities were of the global/meta-level type. For example, the class in *Tavor* did not get to generalize all the algebraic expressions that the students generated during the small group work to a "family" of algebraic expressions with the same character and/or structure, nor did they get to demonstrate general mathematical principles, such as refutation by using counter examples. Consequently, most of the global/meta-level activities recommended to be enacted during whole class work were enacted only in Carmel and, as we saw earlier, there were cases when the same assignments/tasks were enacted in Carmel as a global/meta-level activity but not so in Tavor.

DISCUSSION

Sarah taught the topic *equivalent algebraic expressions*, using the same curriculum materials and teaching sequence, covering by and large the same teaching units, in two 7th grade classes in two different schools. Even though significantly fewer activities were enacted in both classes than recommended in the written curriculum materials, all three types of algebraic activity were enacted in both schools in similar proportions and order, the same as their proportion and order in the written curriculum materials. Transformational activities were more dominant, but generational and global/meta-level activities also played a considerable role.

An examination of the whole class work only, which role is to advance students' mathematical understanding and conceptual knowledge, showed that both generational and transformational activities were given relatively a similar emphasis in the two classes, but in Tavor Sarah enacted less global/meta-level activities during the whole class work than in Carmel. Generational and transformational activities are often considered to be the heart of school algebra and are the main focus of school algebra textbooks. Thus, it may seem that Sarah provided students in the two schools with similar algebraic activities. However, the fact that Tavor students had less opportunities to engage in global/meta-level algebraic activities during whole class work cannot be ignored. This type of algebraic activity is an integral component of algebra. Knowledge about mathematics (i.e., general knowledge about the nature of mathematics and mathematical ways of work) is not separate from but rather is an essential aspect of knowledge of any concept or topic (Even, 1990). Thus, Tavor students were learning a different algebra than Carmel students during whole class work; algebra that, in contrast with Carmel's algebra, included less hypothesizing, justifying, and proving.

Several research studies linked between the curriculum materials enactment and the teacher's perception of the curriculum materials and of mathematics teaching and learning (e.g., Even & Kvatinsky, 2007; Manoucheri & Goodmann, 2000). Some studies added more factors that impact and shape the curriculum material enactment, such as, the school's support of the pedagogical approach of the curriculum materials (e.g. Cuban, Kirkpatrick & Peck, 2001), parental expectations and demands of their children mathematics studies (e.g., Herbel-Eisenmann, et al., 2006), the need to prepare for external evaluation exams (e.g., Freeman & Poter, 1989), and classroom norms (e.g., Yackel & Cobb, 1996). This study adds to this growing literature on curriculum enactment, by showing that various factors (such as, discipline problems) may cause the mathematical ideas dealt in class to change even when the same teacher enacts the same written curriculum materials in different classes.

REFERENCES

Cohen, D. K., & Ball, D. L. (2001). Making change: Instruction and its improvement. *Kappan, 81*(1), 73-77.

Cuban, L., Kirkpatrick, H., & Peck, C. (2001). High access and low use of technologies in high school classrooms: Explaining an apparent paradox. *American Educational Research Journal, 38*, 813–834.

Eisenmann, T. & Even, R. (In press). Similarities and differences in the types of algebraic activities in two classes taught by the same teacher In J. T. Remillard, B. A. Herbel-Eisenmann, & G. M. Lloyd (Eds.), *Teachers' Use of Mathematics Curriculum Materials: Research Perspectives on Relationships Between Teachers and Curriculum*. New York: Routledge.

Even, R. (1990). Subject matter knowledge for teaching and the case of functions. *Educational Studies in Mathematics, 21*, 521-544.

Even, R. & Kvatinsky, T. (2007, August). *Teaching Mathematics to Low-Achieving Students: Enhancement of Personal or Traditional Teaching Approach?* Paper presented at the 12th Biennial Conference of the European Association for Research on Learning and Instruction (EARLI), Budapest, Hungary.

Freeman, D. J. & Porter, A. C. (1989). Do textbooks dictate the content of mathematics instruction in elementary schools? *American Educational Research Journal, 26*, 403-421.

Herbel-Eisenmann, B.A., Lubienski, S. T., & Id-Deen, L. (2006). Reconsidering the study of mathematics instructional practices: the importance of curricular context in understanding local and global teacher change. *Journal of Mathematics Teacher Education*, *9*(4), 313-345.

Kieran, C. (2004). The core of algebra: Reflections on its main activities. In K. Stacey, H. Chick, & M. Kendal (Eds.), *The Future of the TReaching and Learning of Algebra; The12th ICME Study* (pp.21-33). Boston, MA: Kluwer.

Lloyd, G. M. (in press). Teaching mathematics with a new curriculum: Changes to classroom organization and interactions. *Mathematical Thinking and Learning*.

Manouchehri, A. & Goodman, T. (2000). Implementing mathematics reform: The challenge within. *Educational Studies in Mathematics, 42*, 1-34.

Robinson, N., & Taizi, N. (1997). *Everybody Learns Mathematics: Mathematics for Heterogeneous Classes, on the Algebraic Expressions 1*. Rehovot, Israel: The Weizmann Institute of Science (in Hebrew).

Yackel, E. & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education, 27,* 458-477.

Tirosh, D., Even, R., & Robinson, N. (1998). Simplifying algebraic expressions: Teacher awareness and teaching approaches. *Educational Studies in Mathematics, 35*, 51-64.

FROM COUNTING BY ONES TO FACILE HIGHER DECADE ADDITION: THE CASE OF ROBYN

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The Numeracy Intervention Research Project (NIRP) aims to develop pedagogical tools for use with low-attaining 3rd- and 4th-graders. NIRP involved 25 teachers and 300 students, 200 of whom participated in an intervention program of approximately 30 25-minute lessons over ten weeks. The paper overviews the need for intervention in the learning of addition and subtraction. The paper describes one intervention student's progressive mathematization, from counting to non-counting strategies, and from context-bound to formal reasoning. The paper includes (a) descriptions of the student's knowledge as determined in initial and final interview-based assessments; (b) excerpts from three teaching sessions which highlight the student's progress; and (c) insights into instructional procedures and materials used.

In early number learning, children use strategies involving counting by ones (Carpenter & Moser, 1983; Fuson, 1992; Steffe & Cobb, 1988), for example solving 8+7 by counting on seven from 8, using fingers to keep track. Children make a qualitative advancement when they solve additive tasks without counting by ones (Carpenter & Moser, 1983; Fuson, 1992; Steffe & Cobb, 1988; Wright, 1994), for example 8+7 as '8+8 is 16, less 1 is 15'. As in this example, facile additive thinking involves four interrelated aspects: (a) the use of non-count-by-ones additive strategies, such as near-doubles; (b) a part-whole construction of number; (c) a rich knowledge of number combinations, such as knowing 8+8=16; and (d) relational thinking, such as connecting the unknown 8+7 to the known 8+8. The development from counting strategies to facile non-counting strategies for addition and subtraction in the range 1 to 20 is regarded as an important accomplishment of early childhood mathematics (Wright, 1994; Young-Loveridge, 2002). As well as facilitating calculation in the range 1 to 20, the non-counting strategies are required in efficient calculation in the higher decades (Heirdsfield, 2001; Treffers, 1991), for example, in calculating 38+7, or indeed 38+27. Further, part-whole thinking, relational thinking, and knowledge of number combinations are important aspects of number sense (Bobis, 1996; McIntosh, Reys, & Reys, 1992; Treffers, 1991). In short, facility in adding and subtracting without counting is a critical goal in early numeracy.

Some children do not achieve this facility. Instead, they persist with strategies involving counting by ones for addition and subtraction in the range 1 to 20, and in turn use counting strategies in the higher decades. Persistent counting is characteristic of children who are low-attaining in number learning (Denvir & Brown, 1986; Gervasoni, Hadden, & Turkenburg, 2007; Gray & Tall, 1994; Treffers, 1991; Wright, Ellemor-Collins, & Lewis, 2007). Low-attaining 3rd and 4th grade students might typically solve 17-15, for example, by counting back 15 counts from 17. They often show little knowledge of number combinations, for example, finding 8+8 by counting, rather than as a known doubles fact. Further, they typically do not relate unknown number combinations to known combinations: for example, knowing 6+6 is 12, but finding 6+7 by counting. Such persistent counting strategies cause inefficiency and error (Ellemor-Collins, Wright, & Lewis, 2007), and disable further generalisation of arithmetic strategies: persistent counting is a mathematical dead-end (Gray & Tall, 1994). Numeracy is a principle goal of mathematics education (e.g. *Numeracy, a priority for all*, 2000), and there are calls for intervention in the learning of low-attaining students to bring success with numeracy (Bryant, Bryant, & Hammill, 2000; Rivera, 1998). In developing numeracy intervention, there is a pressing need to design instructional sequences that are likely to progress students from counting strategies to strategies that do not involve counting. Designing such sequences is a central goal of the present study.

NUMERACY INTERVENTION RESEARCH PROJECT

The Numeracy Intervention Research Project (NIRP) has the aim of developing assessment and instructional tools for intervention in the number learning of low-attaining 3rd- and 4th-graders (Wright et al., 2007). The NIRP adopted a methodology based on design research (Cobb, 2003), with three one-year design cycles. In each year, teachers and researchers implemented and further refined an experimental intervention program with students identified as low-attaining in their schools. The program included individual interview assessments, and an instructional cycle consisting of approximately 30 25-minute lessons across ten weeks. The assessment and instruction addressed several key aspects of number knowledge, including number word and numeral sequences, structuring numbers to 20, addition and subtraction in the range 1 to 100, conceptual place value, and multiplication and division (Wright et al., 2007). Each lesson typically addressed three or four aspects. In total, the project has involved professional development of 25 teachers, interview assessments of 300 low-attaining students, and intervention with 200 of those students. Interview assessments and lessons were videotaped, providing an extensive empirical base for analysis. The analysis of the learning and instruction is informed by a teaching experiment methodology (Steffe & Thompson, 2000).

Instructional design

We find it helpful to describe an intended learning trajectory as *progressive mathematization* from informal, context-bound strategies to more formal, generalised strategies (Gravemeijer, Cobb, Bowers, & Whitenack, 2000; Treffers, 1991). In accord with the emergent modelling heuristic (Gravemeijer et al., 2000; Wright et al., 2007), we seek to devise instructional settings in which students can first, develop their informal strategies, and then, reflect on their activity, and generalize toward more formal reasoning about numbers. For addition and subtraction in the range 1 to 20, informal non-counting strategies commonly develop around doubles combinations, combinations with 5 and 10, and tens-complements (9+1, 8+2, 7+3, 6+4, 5+5)(Gravemeijer et al., 2000). We describe the development of knowledge of these combinations and the relationships between them as *structuring 1 to 10* (Gravemeijer et al., 2000; Wright et al., 2007). The *ten frame* is a useful setting for these combinations (Bobis, 1996; Treffers, 1991; Young-Loveridge, 2002). The *Bob card* setting can extend ten frames into the range 1-100 (Wright et al., 2007).

Settings

Ten frames 1-10. A 2x5 frame with a standard configuration of dots for a number in the range 1 to 10 (such as five dots on one row and two on the other).

Ten frame addition cards. The 36 frames having 0-5 red dots on one row and 0-5 green dots on the other.

Bob card. A full ten frame, that is, a frame with 10 dots.

Expression card. Two addends in the range 0 to 9, in horizontal format (such as 2+7). The set of expression cards includes all 100 such expressions.

Numeral roll. A long strip of card with the numerals from 1 to 100 in sequence.

Focus of the current study

The focus of this paper is a case study of a child (Robyn) from the second year of the project who progressed from counting to non-counting in her addition and subtraction strategies. The purpose of the case study is to document Robyn's development, and to highlight significant aspects of instruction such as settings and tasks. We believe such exemplars are of interest to practitioners and researchers.

THE CASE OF ROBYN

Robyn was nine years old and in the 4^{th} grade when she participated in the study. Her intervention teacher was Anne. Robyn's initial assessment was in May; the intervention included 29 lessons across 10 weeks from July to October; the final assessment was in October. Below, we discuss Robyn's addition and subtraction in the ranges 1-10 and 1-100 (a) in her initial assessment; (b) in episodes from weeks 3, 5, and 6 of her instruction; and (c) in her final assessment.

Initial interview assessment

Robyn did not have automated knowledge of tens-complements, or of double 7, 8 or 9; she attempted these tasks using counting by ones. For one-digit written tasks 6+5, 7+6, 9+3, 9+6, and 8+7, she solved all by counting on by ones, the last task incorrectly. She was not successful with the following tasks presented in a horizontal written format: 43+21, 37+19, and 86-24. Her thinking took a long time, she could not coordinate the units of tens and ones, and her strategies included some counting by ones with her fingers (Ellemor-Collins et al., 2007).

Lesson 10, week 3

Tens-complements with ten frames. Anne used a set of ten frame 1-10 cards. She flashed a card, and Robyn's task was to say the number of dots, and the number needed to make ten. By and large Robyn was successful and fluent with these tasks. A 3-dot card and 2-dot card appeared to be harder for her than others. Robyn reasoned with known combinations and patterns. For example, for the 3-dot card, Anne asked "How do you know it was seven?", and Robyn answered "Because there's five" sweeping the empty row "five empty, and two empty there" pointing to the other two empty boxes. A segment followed in which Anne did not use the frames. Rather, she stated a number and Robyn's task was to say the number needed to make ten. Robyn was successful and facile on these tasks as well.

Subtracting from a decuple with Bob cards. Next, Anne presented higher decade subtraction tasks using Bob cards and an upright screen. She placed out eight cards, and briefly unscreened the cards. With the cards screened again, she covered three dots on one card and said "80 cover up 3?". After Robyn answered, Anne unscreened the cards. Robyn then solved 30 take 4, and 50 take 3. For 20 take 8 she first answered "22", then Anne lifted the screen, Robyn looked at the cards, and said "12". Her solutions to the next two tasks are described in the following.

50 take 7. Anne lays out five Bob cards, and then lifts the screen. Robyn looks, nods, and Anne replaces the screen.

Anne: How many dots- how many tens?

Robyn: Five.

Anne: How many dots?

Robyn: Fifty.

Anne: Cover up…seven. (She covers seven dots on one card.)

Robyn: (After 10 seconds) That's forty-six…(shakes her head) for...forty-t, -three.

Anne: (Lifts the screen. Robyn looks at the cards, and nods.) Well done.

80 take 2. Anne lays out eight Bob cards, and then lifts the screen. Robyn looks, nods, and Anne replaces the screen.

Anne: How many tens?

Robyn: Eight.

Anne: How many dots?

Robyn: Eighty.

Anne: Covering up two. (She covers two dots on one card.)

Robyn: Eighty-er…sixty-…-ni, -eight. Er, wait, seventy-eight.

Anne: (Lifts the screen.)

Robyn: (Looks at the cards and nods.) Yep.

Anne: Yep. How did you know that was eight so quick? (Indicates the partially covered Bob card.)

Robyn: Cos I know that eight…plus two equals ten. (Taps her forefinger on the desk with each of the five words "eight", "plus", "two", "equals", "ten".)

The final task that Anne presented was 60 take 6 for which Robyn answered 53.

We contend that the particular way Anne used the Bob card setting enabled Robyn to reason facilely about subtracting numbers less than 10 from a decuple. Robyn used her knowledge of tens-complements in solving these tasks. Crucial to her doing so, was Anne's instructional strategy of covering some of the dots on one Bob card to correspond with the subtrahend. It was not necessary for Robyn to see Anne cover those dots, or to see the cards. With the setting of Bob cards used in this way, Robyn could construct a model for reasoning (Gravemeijer et al., 2000). By contrast, in a subsequent segment of the lesson, Robyn worked on related higher decade tasks without the Bob card setting, and used counting-by-ones strategies. Thus, the Bob card setting was important for Robyn's progress with non-counting strategies.

Lesson 15, week 5

Combinations < 10. The first instructional activity involved flashing ten frame addition cards. Robyn's task was to say the number of dots (a) on the top row; (b) on the bottom row; (c) in all; and (d) needed to make 10. Cards presented were 1+1, 3+2, 2+1, 4+1, 5+4, 5+1, 4+4, and 1+4. Robyn responded facilely. In the second activity Robyn put expression cards with sums in the range 1 to 9, into columns according to their sums. She was generally successful. Her responses on these two activities indicated that she could now calculate combinations less than ten without counting by ones, both in a bare number setting and in the ten frame setting. She had consolidated her knowledge of structuring 1 to 10.

Jumping across decuples. In this lesson segment Anne presented four subtractive tasks involving two 2-digit numbers with an unknown difference less than 10. Robyn used a *jump-through-ten* strategy to solve each task and after each solution she used a numeral roll to check. After Robyn solved and checked the tasks of 28 to 34 and 39 to 45, Anne posed the task of 53 to 47.

Robyn: (Looking ahead for four seconds.) Six.

Anne: Check it.

Robyn: (Unfolding the numeral roll.) 53…wait what was it, 43?

Anne: 53.

Robyn: 53. That's 3 jumps (traces an arc from 53 to 50) and another 3 (traces an arc from 50 to 47).

Anne: Six?

Robyn: (Nodding) Six.

Anne: Good. 82 to 75.

Robyn: (Looking ahead for six seconds.) Seven.

Anne: Check it.

Robyn: (Unfolding the numeral roll.) Umm, that's two (traces an arc from 82 to 80) and five (traces an arc from 80 to 75) is seven.

For these verbal tasks in the range 20-100, Robyn used her structuring 1 to 10 knowledge. The jump-through-ten strategy had become the accepted practice (Gravemeijer et al., 2000) for solving and checking.

Lesson 17, week 6

Anne used expression cards to present tasks such as 9+6, 7+9, 4+9, 8+9, 7+6, 4+8 and 5+8. Robyn gave facile responses to these tasks using strategies such as jump-through-ten, compensation, and using a double. She reasoned flexibly about number relationships without reference to the ten frame imagery.

Final assessment

Robyn's final assessment included the same additive tasks as her initial assessment. In her final assessment, Robyn had automated knowledge of tens-complements and of double 5 through double 10. She solved the same one-digit written tasks (6+5, 7+6, 9+3, 9+6, and 8+7) using the non-counting strategies of near-doubles and compensation. She solved the same two-digit written tasks (43+21, 37+19, and 86-24), and also 50-27 and 138-24, using non-counting strategies. For example, she solved 37+19 using a jump strategy: 37+10→47+3→50+6→56.

DISCUSSION

Comparing her initial and final assessments, for additive tasks in the range 1-20, Robyn progressed significantly from counting to non-counting strategies, developing facility with structuring 1 to 10. In the range 20-100, Robyn progressed from unsuccessful strategies to successful non-counting strategies. The progress is due in part to her use of structuring 1 to 10 knowledge in the higher decade calculations, as in the jump strategy for 37+19 above, knowing that 47 to 50 is 3, and that 9 partitions into 3 and 6. Referring to Gray and Tall (1994), while Robyn's additive strategies were initially constrained by a preference to think procedurally, her number knowledge was able to develop sufficiently for her to think proceptually.

Important progressions in this learning trajectory are now described. In structuring 1 to 10, Anne focused on tens-complements, then on combinations less than ten, in each case progressing from ten frames to bare number settings. From week 6 onward, with written tasks in the range 1-20, Robyn's thinking took for granted knowledge of a network of number relations and combinations (Gravemeijer et al., 2000).

In higher decade subtractive tasks, Robyn progressed as follows: (a) Weeks 2 and 3 – for verbally-stated tasks without materials, (such as how far from 82 to 75), she used counting by ones with some errors; (b) Week 3 – in the context of Bob cards, tasks subtracting a number less than 10 from a decuple, she solved without counting by ones: (c) Week 5 – for verbally-stated tasks without materials, she used jumping-through-ten. Her progressive mathematization in the range 20-100 was from counting to non-counting, and from ten frames to bare numbers.

We propose an emergent modelling description of the learning trajectory. In the early weeks, when Robyn was using counting strategies in bare number contexts, we contend that the ten frames and Bob cards provided settings in which Robyn could develop a model of non-counting reasoning (Gravemeijer et al., 2000). Robyn's reflections and explanations consolidated her reasoning in each setting. With Anne's judicious distancing of the settings through flashing, screening, and removal, Robyn was generalising her activity toward independence from the settings.

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References

Bobis, J. (1996). Visualisation and the development of number sense with kindergarten children. In J. T. Mulligan & M. C. Mitchelmore (Eds.), *Children's Number Learning* (pp. 17-33). Adelaide: Australian Association of Mathematics Teachers/Mathematics Education Research Group of Australasia.

Bryant, D. P., Bryant, B. R., & Hammill, D. D. (2000). Characteristic behaviors of students with LD who have teacher-identified math weaknesses. *Journal of Learning Disabilities*, 33, 168-177, 199.

Carpenter, T. P. & Moser, J. M. (1983). The acquisition of addition and subtraction concepts. In R. Lesh & M. Landau (Eds.), *Acquisition of Mathematics Concepts and Processes* (pp. 7-44). New York: Academic.

Cobb, P. (2003). Investigating students' reasoning about linear measurement as a paradigm case of design research. In M. Stephan, J. Bowers, P. Cobb & K. Gravemeijer (Eds.), *Supporting Students' Development of Measuring Conceptions: Analyzing Students' Learning in Social Context (Journal for Research in Mathematics Education, Monograph Number 12)* (pp. 1-16). Reston, VA: NCTM.

Denvir, B. & Brown, M. (1986). Understanding of number concepts in low attaining 7-9 year olds: part 1. Development of descriptive framework and diagnostic instrument. *Educational Studies in Mathematics*, 17, 15-36.

Ellemor-Collins, D., Wright, R. B., & Lewis, G. (2007). Documenting the knowledge of low-attaining 3rd- and 4th- graders: Robyn's and Bel's sequential structure and multidigit addition and subtraction. In J. Watson & K. Beswick (Eds.), *Proceedings of the 30th Annual Conference of the Mathematics Education Research Group of Australasia, Hobart* (Vol. 1, pp. 265-274). Adelaide: MERGA.

Fuson, K. C. (1992). Research on whole number addition and subtraction. In D. A. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 243-275). New York: Macmillan.

Gervasoni, A., Hadden, T., & Turkenburg, K. (2007). Exploring the number knowledge of children to inform the development of a professional learning plan for teachers in the Ballarat Diocese as a means of building community capacity. In J. Watson & K. Beswick (Eds.), *Proceedings of the 30th Annual Conference of the Mathematics Education Research Group of Australasia, Hobart* (Vol. 1, pp. 317-326). Adelaide: MERGA.

Gravemeijer, K. P. E., Cobb, P., Bowers, J. S., & Whitenack, J. W. (2000). Symbolizing, modeling and instructional design. In P. Cobb, E. Yackel, & K. J. McClain (Eds.), *Symbolizing and Communicating in Mathematics Classrooms: Perspectives on Discourse, Tools, and Instructional Design* (pp. 225-273). Hillsdale, NJ: Lawrence Erlbaum Associates.

Gray, E. & Tall, D. (1994). Duality, ambiguity, and flexibility: A "proceptual" view of simple arithmetic. *Journal for Research in Mathematics Education*, 25(2), 116-140.

Heirdsfield, A. (2001). Integration, compensation and memory in mental addition and subtraction. In M. Van den Heuvel-Panhuizen (Ed.), *Proc. 25th conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 129-136). Utrecht, Netherlands: PME.

McIntosh, A. J., Reys, B. J., & Reys, R. E. (1992). A proposed framework for examining basic number sense. *For the Learning of Mathematics*, 12, 2-8.

Numeracy, a priority for all: Challenges for Australian schools. (2000). Canberra: DETYA.

Rivera, D. P. (1998). Mathematics education and students with learning disabilities. In D. P. Rivera (Ed.), *Mathematics Education for Students with Learning Disabilities* (pp. 1-31). Austin, TX: Pro-ed.

Steffe, L. P. & Cobb, P. (1988). *Construction of Arithmetic Meanings and Strategies*. New York: Springer-Verlag.

Steffe, L. P. & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In A. Kelly & R. Lesh (Eds.), *Handbook of Research Design in Mathematics and Science Education* (pp. 267-306). Mahwah, NJ: Lawrence Erlbaum Associates.

Treffers, A. (1991). Didactical background of a mathematics program for primary education. In L. Streefland (Ed.), *Realistic Mathematics Education in Primary School* (pp. 21-56). Utrecht: Freudenthal Institute.

Wright, R. J. (1994). A study of the numerical development of 5-year-olds and 6-year-olds. *Educational Studies in Mathematics*, 26, 25-44.

Wright, R. J., Ellemor-Collins, D., & Lewis, G. (2007). Developing pedagogical tools for intervention: Approach, methodology, and an experimental framework. In J. Watson & K. Beswick (Eds.), *Proceedings of the 30th Annual Conference of the Mathematics Education Research Group of Australasia* (Vol. 2, pp. 843-852). Adelaide: MERGA.

Young-Loveridge, J. (2002). Early childhood numeracy: building an understanding of part-whole relationships. *Australian Journal of Early Childhood*, 27(4), 36-42.

AN OPPORTUNITY LOST IN THE HISTORY OF SCHOOL MATHEMATICS: NOAH WEBSTER AND NICOLAS PIKE

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Nicolas Pike is often incorrectly credited with being the first person born in the United States to write and have published an English-language arithmetic textbook. Although Pike's (1788) arithmetic text was applauded by numerous dignitaries, some later scholars maintained that Pike missed the opportunity to revitalize school mathematics. Our paper contrasts the impact of Pike's book on school arithmetic with that of Noah Webster's texts on school English language studies. We argue that, whereas Webster seized the moment and thereby effected lasting change, Pike, by proceeding cautiously, held back progress in school mathematics. Another issue, concerned with principles of historiography, is discussed briefly: Under what circumstances is it fair to criticize a writer for "silence"?

1780S – THE CHALLENGE TO CHANGE US SCHOOL MATHEMATICS

The 1780s was a decade of optimism and opportunity so far as the history of schooling in the United States was concerned. Having just emerged victorious from the Revolutionary War with its former colonial master, England, the fledgling nation now looked forward to facing, and conquering, many educational challenges (Ogg, 1927). There was a strong national consciousness, and a feeling that from that moment onwards the nation's schools should reflect the achievement of independence and the opportunity to create a unique and model democracy.

George Washington, in writing to Nicolas Pike in June 1788, could not have been clearer on the matter. With respect to Pike's *Arithmetic,* he wrote:

> It seems to have been conceded, on all hands, that such a System was much wanted. Its merits being established by the approbation of competent Judges, I flatter myself that the idea of its being an American production, and the first of the kind which has appeared, will induce every patriotic and liberal character to give it all the countenance and patronage in his power. (Washington to Pike, June 20, 1788)

In those years the young nation's leaders were prepared to accept structural alterations to school curricula which would have been entertained by only a vanguard of reformers in the colonial era. Thus, for example, in 1786 Congress officially introduced decimal currency, with the United States becoming the first nation in the world to decimalize its currency (Pike, 1788; Robinson, 1870; Schlesinger, 1983). What was needed in the young nation's schools was an arithmetic curriculum that supported such an important change. One might have expected that textbook authors, and those in schools and colleges who were responsible for developing school arithmetic curricula, would have thought carefully about how to assist their students to make decimal currency "normal". Furthermore, although Congress had decided only to decimalize currency, one might have expected Pike to have scrutinized the old system of measuring lengths, areas, volumes, capacities, and time (Cohen, 2003).

THE TEXTBOOK CHALLENGE

Before the Revolutionary War almost all arithmetics used in North American[1] schools were written by European authors. In questions involving money, these European texts used European currencies and measurement systems for lengths, weights, volumes and capacities, time, etc. In the 1780s, then, a major challenge for North American teachers, scholars and writers was to produce more authentic textbooks that could replace those previously used in their schools and academies.

Following the Revolutionary War, school texts written by North American writers began to appear. Perhaps the most important of the publishers of these texts was Isaiah Thomas (Tebbel, 1972), who would publish Noah Webster's (1787) famous *The American Speller* and later editions of Pike's *Arithmetic*.

Greenwood's (1729) Arithmetic, and Other Early English-Language Arithmetics

Isaac Greenwood, the first Harvard University mathematics professor, is generally believed to have authored the first arithmetic text written in English by an American and printed in America. Greenwood's (1729) arithmetic seems to have been used very little, if at all, outside of Harvard. In addition to Greenwood's text, several other arithmetics written in English, by American authors, appeared before Pike's (1788) *Arithmetic.* For example, Alexander M'Donald's (1785) *The Youth's Assistant: Being a Plain, Easy and Comprehensive Guide to Practical Arithmetic*, a text with 102 pages, had five editions between 1785 and 1795 (Karpinksi, 1980).

Pike's (1788) Arithmetic

The first major North American school arithmetic to appear was Pike's (1788) *A New and Complete System of Arithmetic Composed for Use of Citizens of the United States.* Pike (1743-1819), a native of New Hampshire, graduated from Harvard College in 1766 and in the 1780s was a school teacher in Newburyport, a seaport northeast of Boston (Albree, 2002),

"Old Pike", as Pike's *Arithmetic* came to be known, went through six editions between 1788 and 1843 (Karpinksi, 1980). It sold initially for about $2.50 – a price which placed it out of the reach of most pupils and teachers (Monroe, 1917). The original 1788 publication was a portentous volume of 512 pages. Besides arithmetic proper, it introduced algebra, geometry, trigonometry, and conic sections. Applications of the arithmetic were made to problems in mechanics, gravity, pendulum motion, mechanical powers, and to astronomical problems requiring calculations of the moon's age, the times of its phases, and the date of Easter.

Most of the text was devoted to narrow forms of traditional arithmetic set out in the book's 200 sections. The book began with rules for elementary operations on integers, together with many examples worked out in detail. Then followed sections on vulgar fractions, decimal fractions, rules for exchanging currency, tricks for rapid computation, extraction of square roots, computation of interest, commissions, annuities, the volumes of particular solids, arithmetic and geometric progressions, permutations and combinations, and topics from elementary mechanics. The book provided a detailed compendium of techniques, formulas, and worked examples, in a wide diversity of practical applications. There were very few, if any, formal proofs. The formulae presented for the slightly more advanced topics appeared without any detail of their origins. An abridged version aimed at schools, which first appeared in 1793, omitted any discussion of logarithms, trigonometry, algebra and conic sections.

In his choice and ordering of content and his methods of handling various topics, Pike leant heavily on school arithmetics written and published in England but widely used in the American colonies – especially those written by Cocker (1738), Dilworth (1762) and Bonnycastle (1778). Naturally, those English texts assumed that English "pounds, shillings, pence" currency would be solely used in the schools. Although Pike's (1788) *Arithmetic* devoted 28 pages (pages 96-123) to currency conversion, only three of these (pages 96-98) were concerned with the new Federal currency – even though the 1788 edition included a copy of the 1786 Act of Congress which created the US Federal Money System. None of the problems involved the new North American currency; rather, they were based on the English system. Units used in other sections of Pike's book included measures for cloth, wine, and beer – beer measures consisted of pint, quart, gallon, firkin, kilderkin, barrel, hogshead, puncheon, and butt – and both troy and apothecary weights.

Pike's (1788) *Arithmetic* offered few examples on how the new Federal currency should be applied in farming, trade and business transactions. With a view to supplying information needed by merchants, Pike discussed such subjects as United States Securities, and rules adopted by the United States, and by State governments, on partial payments – topics of only peripheral relevance to most school students.

AMERICAN SCHOOLS IN THE 1780s AND THE COPYBOOK TRADITION

Pike's *Arithmetic* was superimposed upon an established system of school arithmetic that had relied heavily on what has come to be known as the "ciphering" tradition. Prior to 1800, most North American schooling took place in one-room school-houses with limited resources. Very little of the arithmetic in Old Pike would have been useful for instruction in these schools, for often, the teachers were women who had never studied arithmetic beyond the four operations. There were no blackboards, slates, or maps, and almost all of the school supplies were homemade. The pens were goose-quills, and families supplied their children with homemade ink (Cajori, 1890). Entries in ciphering books often featured beautiful penmanship and calligraphy, for these would be featured on special occasions, especially at the end of a term of work, when local committees and parents met to assess the work of the teacher and pupils.

We have examined about 150 ciphering books generated by individual students attending schools in North America between 1702 and 1860. In each manuscript many pages related to practical topics such as currency conversion, multiplication of money, profit and loss, length measure, tare, discount, simple and compound interest, and annuities. Analysis of entries in these ciphering books suggests that school arithmetic did not deal with subject matter of immediate relevance to students. We have been able to link many of the ciphering books to actual textbooks – including one to “Old Pike” – from which, presumably, students copied.

THE MARTIN (1897) VERSUS CAJORI (1907) DEBATE

Martin’s Case Against Nicolas Pike

Many critics of Pike’s (1788) *Arithmetic* (e.g., Cobb, 1835) pointed out that the numerous rules Pike gave were not comprehensible to most school students. In 1897 George H. Martin launched an attack on what he perceived to be enduring negative effects of Pike’s *Arithmetic* on schooling in the United States. Martin (1897) stated:

> The money units were the English; two pages only are given to Federal money, as it was called, which the Congress had just established but which had not come into general use. Nine kinds of currency were in use in commercial transactions, and the students of this arithmetic were taught to express each in terms of the others, making 72 distinct rules to be learned and applied. (p. 102)

An examination of passages in Pike (1788) suggests that these criticisms were warranted. For example, under the title Practice, which is described as “an easy and concise method of working most questions which occur in trade and business”, the learner is expected to commit to memory a page of tables of aliquot parts of pounds and shillings, of hundredweights and tons, and a table of per cents of the pounds in shillings and pence. These tables contain more than a 100 relations, and the application is in more than 34 cases, each with a rule. The following is Case 12:

> When the price is shillings, pence and farthings, and not an even part of a pound, multiply the given quantity by the shillings in the price of one yard, etc., and take parts of parts from the quantity for the pence, etc., then add them together, and their sum will be the answer in shillings, etc. *Or,* you may let the given quantity stand as pounds per yard, etc., then draw a line underneath, and take parts of parts therefrom; which add together, and their sum will be the answer. (Pike, 1788, p. 169)

After that statement Pike advised the learner “to work the following examples both ways by which means he will be able to discover the most concise method by performing such questions in business, as may fall under this case” (p. 169). Under the topic “Tare and Trett” the following rule is given as Case 4, which is meant to relate to the situation “when Tare, Trett and Cloff are allowed”:

> Deduct the Tare and Trett … divide the Suttle by 168, and the quotient will be the Cloff, which subtract from the Suttle, and the remainder will be the Neat. (Pike, 1788, p. 194)

Martin (1897) maintained that Pike’s text “gave tone to all the arithmetic of the district-school period” (p. 104), and was “responsible for that excessive devotion to arithmetic which has of late been the subject of just complaint” (p. 104). He stated that the text had “an almost endless elaboration of cases and prescription of rules” (p. 104). For example, there were 14 rules under simple multiplication, and in all there were 362 rules in the book. According to Martin, no hint of a reason for a rule was given, except in an occasional footnote, and often the problems were very difficult.

Cajori's (1907) Defence of Pike

Florian Cajori, a respected historian of mathematics and mathematics education, reacted sharply to Martin's (1897) criticisms of "Old Pike". Cajori (1907) argued that Pike's emphasis on local, non-Federal currencies was appropriate because those were the kinds of calculations people needed to know how to do if they were to survive with dignity in everyday life at a time when the different currencies of the North American colonies resulted in much confusion. Cajori pointed out that there was, in fact, an abridged version for schools (Pike, 1793). Referring directly to Martin's (1897) criticisms of Pike, Cajori (1907) wrote:

> To us, this [Martin's] condemnation of Pike seems wholly unjust. … Most of the evils in question have a far remoter origin than the time of Pike. Our author is fully up to the standard of English authors to that date. He can no more be blamed by us for giving the aliquot parts of pounds and shillings, for stating rules for "tare and trett", for discussing the "reduction of coins", than the future historian can blame works of the present time for treating of such atrocious relations as that 3 ft. = 1 yd., $5\frac{1}{2}$ yds. = 1 rd., $30\frac{1}{4}$ sq. yds. = 1 sq. rd., etc. So long as this free and independent people choose to be tied down to such relics of barbarism, the arithmetician cannot do otherwise than supply the means of acquiring the precious knowledge. (p. 218)

Cajori (1907) added that, in the early 1800s, there were three great US arithmeticians – Nicholas Pike, Daniel Adams, and Nathan Daboll. He claimed that the arithmetics of Adams (1801) and Daboll (1800) paid *more* attention than Pike did to Federal Money, and said that teachers could choose the text they wanted. Cajori also pointed out that Pike's "abridged version" for schools continued to be published until the 1830s. The abridged versions had about 200 pages less than the original *Arithmetic*, and the publisher's preface stated that, whereas the original *Arithmetic* was used as a classical book in all the New England universities, the abridgements were intended for schools. However, it could be argued that the very fact that an abridgment needed to be published at all testified to the unsuitability of the original (1788) *Arithmetic* for schools. From this perspective it should be noted that all four recommendations written by eminent citizens of Boston and printed in the front of the 1788 edition indicated that the text would be very useful in *all* schools. A certain Benjamin West stated, for example, that the 1788 edition would be read "by great advantage by students of every class, from the lowest school to the university" (p. 4). But not everyone would agree with that assessment. Monroe (1917), for example, in his history of the development of arithmetic as a school subject in the United States, stated that Pike's (1788) *Arithmetic* was "not a text for young pupils" (p. 18).

Cajori (1907) pointed out that Pike was a practising teacher, a product of a system transported from England by which a textbook was expected to state rules which students would copy, and attempt to remember. That is what "Old Pike" was intended to do. For Cajori, Pike's (1788) summary of relationships between local currencies was meritorious - indeed, such was the detail provided that the book became an authoritative reference for numismatic experts (see, e.g., Mehl, 1933).

NOAH WEBSTER – VISIONARY, ENTREPRENEUR, AND PATRIOT

Noah Webster was a contemporary of Nicolas Pike. In the 1780s, he was alarmed by the fact that in many Old World countries different dialects had developed to such an extent that people in one region could barely understand those in a neighbouring region. He saw similar trends in the North American colonies. He recognised that the post-Revolutionary period provided the perfect time to develop and publish a scheme for standardising the spelling and pronunciation of North American English. He drew on his teaching experience, his academic training (at Yale University) and his entrepreneurial nature to write and publish spellers and dictionaries that provided the foundation for "American English". In so doing, he risked bankruptcy, for he was not a wealthy man. In short, by seizing the moment, he changed the face of the English language in the United States of America forever (Morgan, 1975).

The contrast between Webster's and Pike's actions, and the consequences of those actions, carries a message for contemporary mathematics educators. Pike had the opportunity to lead the new nation by providing a text which could have achieved for arithmetic what Webster achieved for American English. The citizens of the United States of America needed educating with respect to the new Federal decimal currency which had been approved by Congress in 1786. In addition, he missed the opportunity to support leaders like Benjamin Franklin and Thomas Jefferson, who were strongly inclined towards the proposed French metric system of measurement.

In attempts to achieve educational change, vision, timing, and willingness to take calculated risks are as important today as they were in the times of Webster and Pike.

SOME FINAL COMMENTS

Cajori (1907) believed that it was not an arithmetic author's task to seek to change the way people used currencies within society. Rather, an author's fundamental task was to make sure that students learned to cope, arithmetically, with the many and varied problems associated with everyday life. Furthermore, Pike's (1788) emphasis on rules was in line with the "best thinking" on teaching and learning at that time. At issue was whether Pike, given his contextual constraints, was right to accept the existing education settings of his day, and to proceed cautiously; or whether he, as a person acting at a pivotal period of history, should have provided leadership by seizing the moment and attempting to achieve fundamental change in the arithmetic curricula of schools.

It should be noted, however, that Pike knew that his would be a landmark text, and so also did all the notable personalities who provided supporting statements in the front of the book. Pike wanted *his* arithmetic to be the first English-language arithmetic text written by a North American citizen. He wanted it to be widely used in the schools and colleges in the new nation. One could argue, however, that, as with Noah

Webster, his was the responsibility to set a new standard, to break away from colonialist fetters that had strangled teaching and learning of arithmetic in the schools before the Revolutionary War. But, he failed to grasp his opportunity. Furthermore, the abridged versions of Pike's arithmetic "for schools" were little better than the original 1788 text.

Was it unreasonable to have expected Pike to see beyond the horizons surrounding his world and context in the 1780s? That question raises intriguing issues of historiography. What principles can historians look to if they want to generate faithful, historically accurate accounts of events, and penetrating and insightful interpretations of those events? Under what circumstances is it fair to criticize a writer for "silence" about ideas and practices of which he was only dimly aware? Those kinds of questions are fiercely contested within the world of academic history today (see e.g., Macintyre & Clark, 2004; Windschuttle, 1996).

Endnote

1. In this paper, the term "North America" refers only to States that ultimately became part of the United States of America.

References

Adams, D. (1801). *The Scholars Arithmetic or Federal Accountant.* Leominister, MA: Adams and Wilder.

Albree, J. (2002). Nicolas Pike's Arithmetic (1788) as the American *Liber Abbaci,* In D. J. Curtin, D. E. Kullman, & D. E. Otero (Eds.), *Proceedings of the Ninth Midwest History of Mathematics Conference*.(pp. 53-71). Miami, FL: Miami University.

Bonnycastle, J. (1778). *The Scholar's Gguide to Arithmetic* (5^{th} ed.). London: J. Johnson.

Cajori, F. (1890). *The Teaching and History of Mathematics in the United States* (Circular of Information No. 3, 1890). Washington, D.C.: Bureau of Education.

Cajori, F. (1907). *A history of Elementary Mathematics with Hints on Methods of Teaching*. New York: Macmillan.

Cobb, Lyman (1835). *Cobb's Ciphering Book*, *No. 1*. Elmira, NY: Birdsall & Co. Philadelphia: James Kay Jn & Brother, stereotyped by J. S. Redfield.

Cocker, E. (1738). *Cocker's Arithmetick.* London: A. Bettesworth & C, Hitch.

Cohen, P. C. (2003). Numeracy in nineteenth-century America. In G. M. A. Stanic, & J. Kilpatrick (Eds.), *A history of school mathematics* (Vol. 1, pp. 43-76). Reston, VA: National Council of Teachers of Mathematics.

Daboll, N. (1800). *The Schoolmaster's Assistant, Being a Plain Practical System of Arithmetic*. New London, CT: Samuel Green.

Dilworth, T. (1762). *TheSschoolmaster's Assistant. Being a Compendium of Arithmetic, both Practical and Theoretical* (11^{th} ed.). London, Henry Kent.

Greenwood, I. (1729). *Arithmetick, Vulgar and Decimal, with the Application Thereof to a Variety of Cases in Trade and Commerce.* Boston: Kneeland & Green.

Karpinski, L. C. (1980). *Bibliography of Mathematical Works Printed in America Through 1850.* New York: Arno Press.

Macintyre, S., & Clark, A. (2004). *The Hstory Wars.* Melbourne, Australia: Melbourne University Press.

Martin, G. H. (1897). *The evolution of the Massachusetts Public School System: A historical Sketch.* New York: D. Appleton and Company.

M'Donald, A. (1785). *The youth's Assistant Being a Plain, Easy and Comprehensive Guide to Practical Arithmetic.* Norwich, CT: John Trumbull.

Mehl, R. M. (1933). *The Star Rare Coin Encyclopedia and Premium Catalog.* Fort Worth, TX: Numismatic Co. of Texas.

Monroe, W. S. (1917). *Development of Arithmetic as a Shool Subject.* Washington, D.C.: Government Printing Office.

Morgan, J. S. (1975). *Noah Webster.* New York: Mason/Charter.

Ogg, F. A. (1927). *Builders of the Republic.* New Haven, NJ: Yale University Press.

Pike, N. (1788). *A New and Complete System of Arithmetic Composed for the Use of the Citizens of the United States*. Newbury-Port: John Mycall.

Pike, N. (1793). *The new Complete System of Arithmetic Composed for the Use Of The Citizens of the United States (abridged for the use of schools).* Newbury-Port: John Mycall, Isaiah Thomas.

Robinson, H. (1870). *The Progressive Higher Arithmetic for Schools, Academies, and Mercantile Colleges.* New York: Ivison, Blakeman, Taylor & Co.

Schlesinger, A. M. (Ed.). (1983). *The Almanac of American History.* New York: Putnam.

Tebbel, J. (1972). *A History of Book Publishing in the United States.* New York: Bowker.

Washington, George to Nicolas Pike. (1788, June 20). Electronic Text Center, University of Virginia Library. Retrieved January 7, 2008, from http:etext.virginia.edu/etcbin/ toccer new2?id=WasFi30.xml&ima.

Webster, N. (1787). *The American Speller*. Boston: Isaiah Thomas.

Windschuttle, K. (1996). *The Killing Of History: How Literary Critics and Social Theorists are Murdering our Past.* San Francisco: Encounter Books.

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