The 27th Annual Meeting of

PME-NA

North American Chapter of the International Group
for the Psychology of Mathematics Education

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Roanoke, Virginia

Frameworks that Support Research and Learning

Hosted by

Virginia Tech

VIRGINIA POLYTECHNIC INSTITUTE
AND STATE UNIVERSITY

Preface to the 2005 PME-NA Proceedings

It is with excitement that we at Virginia Tech have brought technological advancement to the 27th annual meeting of PME-NA. For the first time in the history of PME-NA, the conference Proceedings are entirely electronic. Upon registration in Roanoke, conference participants received this CD-ROM containing the Proceedings – instead of the paper volumes traditionally distributed at PME-NA meetings. An extensive program booklet, including abstracts for each session, was distributed at the conference so that participants could make informed decisions about session attendance. In addition, participants were able to browse binders containing complete printed copies of the Proceedings during the meeting. An online set of Proceedings[1] was also available prior to the conference.

Users of this CD-ROM have the ability to access conference reports by viewing and selecting from (1) the daily conference schedules, (2) a list of presenters and their sessions, and (3) a list of session topics and themes. Links allow users to read, print, or download individual reports (.pdf files). It is our hope that the electronic Proceedings (both CD-ROM and online) offer PME-NA participants, and the larger mathematics education community, flexibility and easy access to all reports from the PME-NA meeting in Roanoke.

The contents of the Proceedings relate to the PME-NA 27 theme, "Frameworks that Support Research and Learning." The Proceedings include plenary reports by John Mason and Denise Mewborn, 12 working group and discussion group reports, 99 research reports, 55 short oral reports, and 41 poster descriptions. Proposals for working groups, discussion groups, research reports, short orals, and posters were submitted electronically to All Academic's online system. Proposals were reviewed by 2-3 reviewers and, based on the peer reviews, acceptance decisions were made by the editors and the Virginia Tech Planning Committee. Full papers were submitted electronically and edited for uniform formatting and style throughout the Proceedings.

The Proceedings of PME-NA 27 are dedicated to the memory of James Kaput, an internationally renowned mathematics education leader who died tragically on July 31, 2005. Jim's personal and professional contributions are described in a commemorative paper that begins the Proceedings. In remembrance of Jim, his name appears in all of his accepted sessions and papers in the PME-NA program and Proceedings. Jim's participation in this and future PME-NA meetings will be greatly missed.

The quality of these Proceedings has been enhanced greatly by the contributions and support of many people and organizations. The voluntary PME-NA peer reviewers and the 2004-2005 PME-NA Steering Committee helped to shape a high quality and intellectually demanding conference program. The staff at All Academic provided excellent support throughout the review process, paper submission, and the development of the program and Proceedings. We thank the Virginia Tech School of Education, Department of Mathematics and College of Science, and Continuing and Professional Education at Virginia Tech for the many resources that enabled the development of these Proceedings. Finally, enormous gratitude is owed to the Virginia Tech Planning Committee, particularly to those graduate students who devoted themselves for over a year to the many preparations for the PME-NA conference and the production of these Proceedings.

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Gwendolyn M. Lloyd, Editor in Chief
Melvin (Skip) Wilson
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October 2005

[1] http://convention2.allacademic.com/index.php?cmd=pmena_guest

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JAMES J. KAPUT –
HIS CONTRIBUTIONS OF INTELLECT AND CHARACTER

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August 5, 2005

Jim Kaput liked to tell me stories about his huge lung capacity. He could take in more air than anyone else. As a child, he surprised and delighted others by swimming underwater for longer than anyone imagined possible. Now, like many things, this seems like a metaphor. He lived life more fully than most of us. He took more in, he worked harder, and he gave more to others.

Jim could frame a vision in a phrase. "Democratizing access to the mathematics of change" was the mission I shared with him for twelve years. It's the flag that those who loved him will carry onwards.

Jim's personal mission was to bring much more powerful and meaningful mathematics to many more people. He was a theorist of democratization. In his view, access to concepts was a function of representation and pedagogy. By transforming the way concepts were represented and methods of instruction, many more people would be able to gain access to difficult but important ideas. He wanted to accomplish in mathematics something like the democratization of literacy that began with the printing press. Before the printing press, scribes were thought of a very special people. Who could imagine that someday everyone would read and write? Surely many people thought, "I'm just not good at reading." Jim believed in a future in which everyone would be able to access the mathematical jewels of their cultural heritage -- the jewels that lay beyond the "basics" of shopkeeper arithmetic. He believed it was the responsibility of an advanced civilization to make powerful mathematics learnable, meaningful, and useful.

Jim believed deeply in democratization and acted on his beliefs with passion. For decades, he taught a class at the University of Massachusetts, Dartmouth for academically disadvantaged freshman preparing for technical majors. Many accomplished professors want to work with only the best students. Jim, in contrast, dedicated his thoughtful class preparation to students who were underprepared. He constantly sought to improve his course, striving to help these students go on to further coursework in science and engineering. Often when I called, he would start the conversation by telling me excitedly about the innovation he planned for the next day's class.

When Jim was with students, he would coax thinking out of them. He celebrated each little step a student made as a major advance. "Notice what is going on here!" he would say, drawing the class's attention to the student's idea and embellishing it, making it more precise, more mathematical, more fruitful for later growth. "Ordinary kids can do extraordinary things," was one of his core beliefs.

Likewise, Jim's research projects always committed to do research in schools that had very little. To him, it would be meaningless to show that technology could help elite students; he wanted to prove technology's potential for helping the least advantaged to master mathematical concepts. His signature technology was SimCalc, an approach to introducing the mathematical ideas of rate and accumulation through more intensive use of computer-based graphs and

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animation, used alongside conventional tables and algebraic symbols. Many people think of SimCalc as a piece of software. There were many pieces of SimCalc software, and these were always works in progress. There were also many curricular units to accompany the software, and these too were always in a state of perpetual improvement. SimCalc was an idea that Jim was constantly refining with the best and brightest team he could inspire to join him.

I still have the original video that Jim created to communicate his vision of SimCalc. It's a sign of how he worked that there are almost no features of that video are part of today's MathWorlds software. SimCalc was really a process of constant iterative improvement towards the goal of democratizing access to the mathematics of change. Jim's teams proposed, debated, tested and implemented new features all the time. Gradually, in the crucible of classroom experience, his teams separated the wheat from the chaff. Some half-baked ideas died quickly; some transformed slowly, but in the end Jim never held onto a feature or idea that didn't prove out in the classroom. And he was always open to yet more powerful capabilities of technology that would require rethinking everything all over again but might lead to a quantum leap. The most recent instance of this was his passionate work on taming classroom networks to become instruments of a participatory, engaged, emergent mathematical experience for all his students. Jim's vision will only become a concrete thing when every student has in hand a powerful combination of representation and communications that enables him or her to participate meaningfully in expressing, constructing, modeling, and analyzing concepts using the mathematics of change.

One pervasive character of Jim's work was a drive to scale. Once he accomplished something, he always raised the stakes. He wanted to take it to the next level on the path to massive impact. And so SimCalc went from studying a few students, to studying a few teachers, to studying teachers in a few regions of the country, to statewide studies scale up among many teachers in Texas. He also pushed hard to get elements of his designs on calculators, thereby bringing his mathematical representations into widespread use.

Jim acted locally in his region; he acted nationally to influence key reports and standards documents; he acted globally wherever mathematics educators met. A distinguished National Science Foundation (NSF) program officer once challenged the field to imagine how it could fruitfully spend $1 billion to make an impact on math and science education through technology. The assembled room of 100 distinguished educators collectively hid under their chairs mumbling, "we're not ready. We haven't done enough research yet." Not Jim. He lept out his chair and proclaimed, "Wait a minute! I think I could use at least half of it!"

Jim always took the long view. He refused to work on fads or political imperatives designed for short-term impact. He reminded anyone who would listen that 100 years ago only 3 percent of students studied algebra: today we expect algebra for all. But that is only today's problem, Jim would remind us. Schooling is only slowly catching up to 19th-century mathematics. Jim was addressing the problem of how to teach everyone 21st-century mathematics. He wanted to transform the current curriculum, in which calculus is icing on the layer cake of mathematics education into a continuous strand of mathematical concepts that are introduced to students throughout the math curriculum as they advance from grade to grade. He believed that we live in a time of change and every student could benefits from an understanding of the mathematics of change. Because he had a long-range view, Jim refused to be pinned down to research questions that could be answered in a year or two. "You can test this little fraction of my ideas," he'd say, "but there isn't any way to test the whole thing." I always teased him that his epitaph should say, "we will know if he was right in 50 years." Indeed, it would take 50 years to fully test the scope of Jim’s vision for improving mathematics education, which stretched from elementary grades though university education.

The huge upswell of feeling for Jim as people learned of his death was not, however, merely a reflection of his beliefs, his ideas, or his projects. Jim was there for so many people at crucial moments in their careers; he shared their personal hardships and help them at the junctures and gauntlets of their lives. Whether in the role of mentor, colleague or boss, he was unbelievably generous with his time. James Burke told me, "he had the most compassionate intelligence." Jim slaved over letters of recommendation to help junior faculty whom he believed were qualified for tenure. He helped colleagues reframe their ideas to achieve publication and develop their funding proposals.

A virtual college formed around Jim. He connected people and helped them work together successfully. This college had no formal organization, no web site, no e-mail list. But it was no less real. He brought people together, from local schools, national universities, and major technology companies. Jim worked through powerful connectivity as well as powerful representation of ideas.

Jim was committed to his family. He would always get home from his busy travel schedule by Friday night or early Saturday morning, to spend the weekend with his family and to read aloud to his disabled son. Despite offers from universities all over the country, he wanted to stay in Massachusetts, close to family and friends. He was legendary not just for his research but for his Super Bowl parties.

More than anyone I know, Jim maintained intense working relationships with people across time and space. His town was home base, but only a starting point. He did not allow his location to limit his work.

Jim was a character from toe to head. He perpetually wore running shoes and a yellow, orange or red shirt. He had an Abe Lincoln beard. But what I will miss most is his eyes. Jim had extraordinarily expressive eyes. When he heard an idea he liked, they would grow huge with excitement and radiate light. His eyes beamed enthusiasm for the contributions he devined in others' thoughts. I will miss that the most.

Jim was my mentor and made me feel incredibly special and important. When, as a green kid, I proposed that I join him full time in 1993, he acted as if I were doing him the greatest honor in the universe. As we worked together, he sang my praises to leaders at NSF, credited project accomplishments to me, and promoted me to co-Principal Investigator. We stayed up nights together despite 2600 miles between us, sometimes passing drafts coast to coast over the Internet to meet a deadline. As he did with so many other young researchers, he poured energy into my career development. As I progressed, the relationship shifted and he became more of a colleague. We worked as co-PIs on eight grants totalling more than $10 million over a time span of 12 years so far (three more years remain of our largest project together). And yet, we did so by expanding the umbrella to embrace all the best people we each could bring to the mission, by intuitively anticipating what the project needed next, by agreeing on what quality meant, not how it was achieved. Our actual acts of overt coordination were surprisingly sparse. This was how Jim led.

Jim relished his upcoming retirement and had many dreams. Among them, he planned to build a tower in his backyard. He wanted to be higher than the trees, to see clearly out to the horizon, to the ocean.

Jim was an engaged visionary, a compassionate intellect, an inspirational poet of mathematics education reform. Although he will be missed by many, I believe his dreams will some day be realized: new curriculum coupled with new technology will someday enable many more ordinary people to accomplish extraordinary things through mathematics.

Many people have asked what they can do. Jim's family would like all contributions to go to a scholarship fund when established. If you wish to make a contribution now please make checks payable to UMDF and write "Kaput Scholarship" in the memo line. These should be forwarded to:

UMass Dartmouth Foundation
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FRAMEWORKS FOR LEARNING, TEACHING AND RESEARCH: THEORY AND PRACTICE

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I have chosen to reverse the order of the key words in the conference title, and to interpolate a third term in order to fit with my view of the role and functioning of frameworks. I begin by introducing a framework for learning in which systematic variation can be used to provoke learners into becoming aware of mathematical structure. Structural Variation Grids have evolved over several years and I indicate some of the history of their development. I then use some frameworks for teaching based on Ference Marton's notion of variation, some based on George Polya's descriptions of mathematical thinking, some based on Jerome Bruner's three modes of re-presentation, and one based on my own work on the structure of attention, in order to provide theory-based justifications for pedagogical and didactic choices that the Structural Variation Grids afford. These frameworks can be used to enhance and enrich the learning potential of particular instances of grids, but also any other mathematical task in any mathematical topic. Like most frameworks for teaching, the ones I will use can be transformed into frameworks for learning through the process of scaffolding and fading (Brown et al *1989), itself a framework for teaching. In the final section I suggest why and how these frameworks work, and this includes a description of the methods used to justify the claims in this paper. My aim is to exemplify what I think is at the heart of learning and of being taught, at the heart of professional development, and indeed at the heart of research as well, namely, the emergence and elucidation of informative frameworks as collections of related distinctions. I try to demonstrate and describe conditions which can make a framework become active for individuals, and I elaborate on what makes them effective for people for whom they are active. I end with some advice on how to cope with new frameworks when they are encountered.*

Structural Variation Grids

Historical Context

Many years ago I experienced a lesson given by Laurinda Brown based on the function game (Banwell *et al* 1972, see also Rubenstein 2002). It was conducted entirely in silence to great effect. Participants were invited to conjecture the result of applying an unknown function to different inputs, based on examples provided by her at the beginning. Everything was done in silence, with sad or happy faces drawn according to whether the *keeper of the rule* agreed or disagreed with the conjecture. The one rule was that no-one was allowed to say what they thought the rule was. Those who thought they knew 'the rule' were encouraged to offer examples which would help others come to the same conjecture, and also to try to test and challenge their conjecture. Apart from the silence, the format has strong resonances with the game *Eleusis* described by Martin Gardner (1977; 2001 p504–512). Gardner observes that the rules provide an analogy with science, because nature never tells you whether your conjectured rule is correct.

I was stimulated to look for the first opportunity to try working in silence and it came in a lecture to 300 Open University students, in which I presented the first few terms of a sequence:

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$$2 + 2 = 2 \text{ x } 2 \qquad 3+1\frac{1}{2} = 3\times 1\frac{1}{2} \qquad 4+1\frac{1}{3} = 4\times 1\frac{1}{3} \qquad 5+1\frac{1}{4} = 5\times 1\frac{1}{4}.$$

I paused at each equal sign, and at the end of each equation, in order to show that I was doing the calculations myself. I have since done this with thousands of people over many years. Each time, no matter who the people are, everyone seems to know what the next term will be even if they struggle with the arithmetic to check the validity of their conjecture. I have used many sequences like this, getting participants to re-present the first term in the format of the others, to go backwards into the negatives (starting with 0, then -1, -2, …), to use not just whole numbers but rationals (starting with _ or _), irrationals (starting with $\sqrt{2}$ or $\sqrt[3]{7}$) and beyond, according to the sophistication of the audience. The main thrust is towards expressing the general equation, and then justifying it using algebra. Sequences like this can be used to provoke learners into wanting a way to manipulate generalities (letters), as well as a providing a source for appropriate rules for that manipulation: the rules of algebra as generalizations of the rules of arithmetic. This contrasts with algebra presented simply as rules for 'alphabet arithmetic'.

In 1998 I was asked by some teachers in Tunja Colombia to suggest how to work with learners on factoring when they did not have facility with or even belief that (-1) x (-1) = 1. My response was what I then called *Tunja Sequences* (Mason 1999, 2001) which used the same principle of a developing sequence of specific instances of a factored quadratic such as

$$1^2 - 1^2 = (1 - 1)(1 + 1) \qquad 2^2 - 1 = (2 - 1)(2 + 1) \qquad 3^2 - 1 = (3 - 1)(3 + 1) \ldots$$

Here learners could be expected to detect the pattern and to express it in general, verbally, and even algebraically. By being exposed to a number of such sequences derived from factored quadratics, learners could be expected to become adept at expressing and justifying generality (the heart, root and purpose of algebra). Having generalised, they can work out the rules for expanding brackets, and for factoring quadratics, simply by using their natural powers to detect what is changing and what is invariant.

Recently, while writing a book on the teaching of algebra (Mason *et al* 2005) I wanted to extend these Tunja sequences to allow a second parameter to vary, and thus was born Structural Variation Grids. Tom Button kindly provided me with a basic Flash template which I then modified to produce different Grids, some of which are described in the next section.

Using these grids briefly with teachers has already generated considerable excitement, and this is what has encouraged me to present them in this forum. I am confident that many of you will have done or used something similar at various times. The reason for presenting them here is to exhibit them as an exemplar of a pedagogic framework for learning.

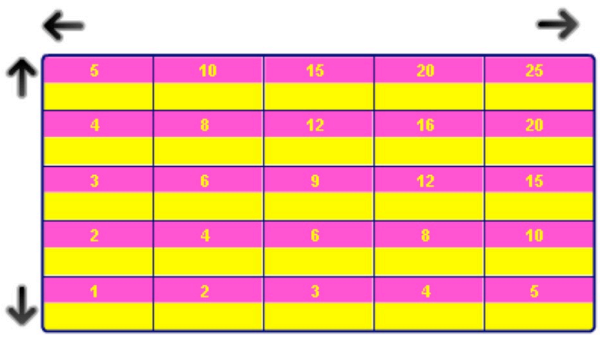
Sample Grids

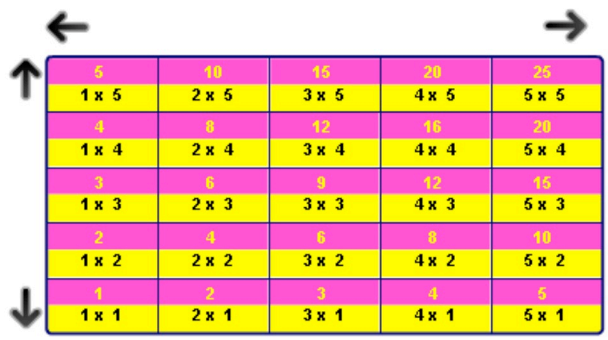
Multiplication Table and (-1) x (-1)

Task 1: Number Grid

Say (to someone else or to yourself) what you see in the left-hand grid below.

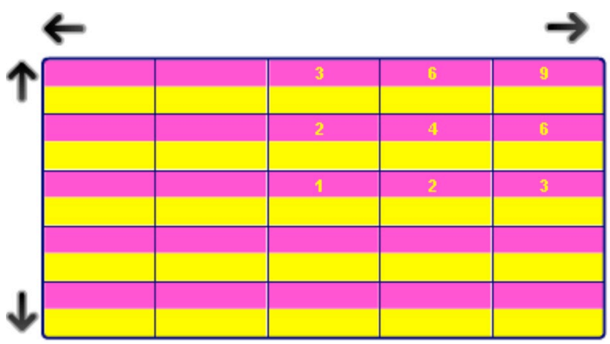
Explain any patterns you see in the following grid. Conjecture and justify the entries in the square which is 20 cells to the right of the bottom left hand corner, and 13 cells up. Generalise.

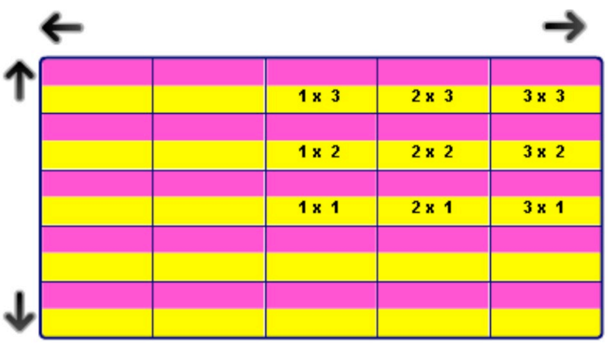




People rapidly come to the conclusion that the grid is actually both a mathematical use of a table format, and an integration of the 'multiplication tables', as shown in the grid on the right. In the Flash version of the grid you can click on any cell to reveal or obscure its contents. Thus you can start with the grid empty and reveal just a few cells. Note that it is also possible to start by revealing multiplications and then revealing the answer.

The arrows shift the window in the direction indicated on an infinite grid, so you can extend the patterns in any direction, but particularly down and to the left. You can then check for consistency between extending down then left, and left and then down starting from the shifted grids below.

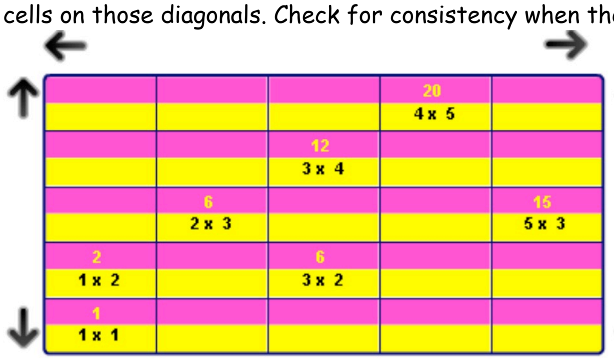


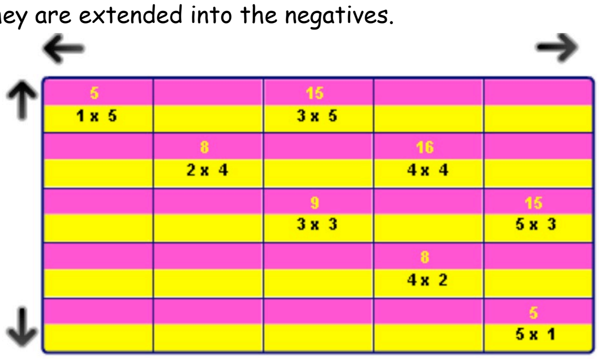


Task 2: Number grid (cont'd)

Get some people to extend the patterns downwards and then to the left; get others to extend it to the left and then downwards. Do you predict the same cell entries?

Describe the entries in a diagonal such as the ones shown below so that you can predict the entries in other cells on those diagonals. Check for consistency when they are extended into the negatives.





Note that again you can choose whether to display both rows in a cell, and the order and speed at which to reveal them. Note also the difference between predicting and checking a new entry, and colouring in patterns on an already extant grid. The empty cells invite learners to anticipate, to imagine and to conjecture, and to reason on the basis of properties they perceive.

Other Number Grids

Numerous variations on the same theme are of course possible. For younger children the entries in a cell could use other operations. In the sample cells here I have used cell (3, 2) as a generic example extracted from different grids.

3 + 2
5

3 – 2
1

3 ÷ 2
1.5

5(3 + 2)
25

5(3 – 2)
5

By predicting (anticipating) the entries in various cells, learners quickly get a sense of the structure of arithmetic by exercising their own powers of observation and generalisation. By locating all the cells with a specified entry they encounter mathematical structure.

Working with Zero

Another variant uses a similar grid structure with fractions in order to expose the reasons for outlawing certain operations using zero. Flash versions of these are in preparation.

Task 3: Zero in Fractions

What values would you expect where the question marks appear in the grid to the right? The … indicates an opportunity to extend and generalise. By following different patterns to predict values for the empty cells, explain why they must remain empty.

…	…	…	…	…
?	1 / 4	2 / 4	3 / 4	4 / 4
?	1 / 3	2 / 3	3 / 3	4 / 3
?	1 / 2	2 / 2	3 / 2	4 / 2
?	1 / 1	2 / 1	3 / 1	4 / 1

Extend the grid to the left and downwards.

Do the same for the following grid. It may be helpful to look for patterns in the format as presented, and then to perform the calculations and look for patterns in the answers. (Much more of the grid is shown in order to be clear about what is possible. Of course in a live presentation this would not be necessary.)

-1 / 1	-1/2 / 1	-1/3 / 1	…	?	…	1/3 / 1	1/2 / 1	1 / 1
-1 / 1/2	-1/2 / 1/2	-1/3 / 1/2	…	?	…	1/3 / 1/2	1/2 / 1/2	1 / 1/2
-1 / 1/3	-1/2 / 1/3	-1/3 / 1/3	…	?	…	1/3 / 1/3	1/2 / 1/3	1 / 1/3
…	…	…	…	?	…	…	…	…
…	…	…	…	?	…	…	…	…
-1 / -1/3	-1/2 / -1/3	-1/3 / -1/3	…	?	…	1/3 / -1/3	1/2 / -1/3	1 / -1/3
-1 / -1/2	-1/2 / -1/2	-1/3 / -1/2	…	?	…	1/3 / -1/2	1/2 / -1/2	1 / -1/2
-1 / -1	-1/2 / -1	-1/3 / - 1	…	?	…	1/3 / - 1	1/2 / -1	1 / -1

Notice the possibilities afforded for working on the division of fractions as well, by rehearsing a column, and then generalising it, then rehearsing a row and generalising that. You can even rehearse diagonals with different slopes, and generalize them. Learners are in this way exposed to the structure of division of fractions. Work on such grids would be presumed to take place in the context of other images such as the division of a rectangle into cells.

Task 4: Zero and Exponents

Extend the left-hand grid outwards. In how many different ways can you use sequences to justify values for the question marks? Explain why one entry must remain empty. Extend it to the left and downwards.

…	…	…	…	…	…
?	3^1	3^2	3^3	3^4	…
?	2^1	2^2	2^3	2^4	…
?	1^1	1^2	1^3	1^4	…
	?	?	?	?	…

?	…	$1^{1/3}$	$1^{1/2}$	1^1
?	…	$(1/2)^{1/3}$	$(1/2)^{1/2}$	$(1/2)^1$
?	…	$(1/3)^{1/3}$	$(1/3)^{1/2}$	$(1/3)^1$
…	…	…	…	…
	…	?	?	?

What is the same, and what is different about the left and right-hand grids? Extend the right-hand grid to the left and downwards to justify values for the question marks. Why are there difficulties extending the upper-right quadrant below the row of question marks but not in extending it to the left of the column of question marks?

Factoring Quadratics

A factored quadratics grid with the upper and the lower cell entries filled in are displayed below.

(x - 1)(x + 3)	x(x + 3)	(x + 1)(x + 3)	(x + 2)(x + 3)	(x + 3)(x + 3)
(x - 1)(x + 2)	x(x + 2)	(x + 1)(x + 2)	(x + 2)(x + 2)	(x + 3)(x + 2)
(x - 1)(x + 1)	x(x + 1)	(x + 1)(x + 1)	(x + 2)(x + 1)	(x + 3)(x + 1)
(x - 1)x	xx	(x + 1)x	(x + 2)x	(x + 3)x
(x - 1)(x - 1)	x(x - 1)	(x + 1)(x - 1)	(x + 2)(x - 1)	(x + 3)(x - 1)

x^2+1x-6	x^2+2x-3	x^2+3x	x^2+4x+3	x^2+5x+6
x^2+0x-4	x^2+1x-2	x^2+2x	x^2+3x+2	x^2+4x+4
x^2-1x-2	x^2+0x-1	x^2+1x	x^2+2x+1	x^2+3x+2
x^2-2x	x^2-1x	x^2+0x	x^2+1x	x^2+2x
x^2-3x+2	x^2-2x+1	x^2-1x	x^2+0x-1	x^2+1x-2

Using pedagogic devices like those suggested for the number grids, learners can work out for themselves from just a few cell entries, what other cells are likely to contain. Negatives can be pursued by shifting the window, and various forms of data can be provided. For example, a diagonal from top left to bottom right somewhere in the extended grid can be used to predict the other two corner entries. Again the aim is to get learners to generalise, not just for predicting what will appear in a specified cell, but how the upper and lower entries relate to each other. Thus they encounter both factoring and expansion of brackets. By looking for patterns they can work out how to factor an expression as well as how to multiply out brackets.

Other Possibilities

Structural Variation Grids can be used for laying out any two-parameter families of operations. A variant form using a spreadsheet to display grids using numbers from arithmetic sequences, and looking at properties of entries forming specified geometric relationships appears in Hewitt *et al* (2005). Other possibilities include pairs of simultaneous linear equations, and pairs of quadratics differing only in the sign of the constant term. In every case the aim is to provoke learners into using their natural powers rather than telling them things they can work out for themselves. The Zero grids show how particular topics require didactic as well as pedagogic decisions.

Structured Variation Grids as a Framework for Learning

There are many formats or layouts which are used with learners and which can develop into a framework for learning. Examples include number-lines (empty or not), Cuisenaire rods, Dienes or Multi-Base blocks, bundles of ten sticks, the balance metaphor for equations, graphical presentation of functions, and grid multiplication which is also known as Gelosian multiplication and is related to Arabic diagrams for the expansion of $(a + b)(c + d)$. There are advantages to using the same structure in different contexts because learners bring to each successive use the way of working used previously. This reduces the overheads in getting to grips with the rules and affordances of a new format or tool. Thus with Structural Variation Grids, once learners have become used to extending and expressing patterns and justifying their expressions of generality, they are likely to behave in the same way again. The grid displays structure and stresses consistency as a driving force in extending mathematical concepts (Mazur 2003 p73). Learners are likely to become imbued with mathematics as sense-making because the grids provide visual access to the structural patterns which justify calling negatives, fractions and the like, *numbers*. Consistency and continuity become second nature rather than new concepts. The grid also provides a format for learners to use for exploring two parameter patterns for themselves.

Frameworks for learning can become tedious, and downright boring if their use is too formulaic and if the pedagogical practices fail to call upon learners' natural and developing powers. Formats and frameworks also need to exhibit sufficient variation so that learners do not make assumptions about the role or importance of invariants which the teacher does not intend learners to include. Exposure to a restricted range of examples can lead to the formation of inappropriate concepts analogous to the *figural concepts* identified by Fischbein (1987, 1993).

Informing Pedagogical & Didactic Practices

As devices these grids have already proved to be attractive to a range of teachers. They are formats which afford the possibility for anticipating, conjecturing, and generalising, but their form also invites pedagogical and didactical choices. *Pedagogical* is used here to refer to general teaching strategies such as beginning in silence, asking learners to say what they see or what is the same-and-different about objects presented to them. *Didactical* is used here in the European sense of specific to the mathematical objects used in the grid and to the mathematical aims of the lesson. Pedagogical and didactical choices are informed by frameworks which arise from research of various kinds, but which act to remind practitioners about choices of actions, and which could inform research into those choices.

In the following subsections I make use of a number of different frameworks in order to highlight features of the grids which seem to make them potentially fruitful as one framework for learning what could be used throughout the school curriculum. Most of the frameworks I shall mention were developed in and for a 200 hour distance-taught course entitled *Developing Mathematical Thinking* which first appeared in 1981. Some 400 mathematics teachers studied it as a form of professional development in the first year. It ran for 6 years (with numbers declining to 180 per year) and profoundly influenced a generation of teachers and teacher educators in the U.K.. It was said at the time that every mathematics teacher-education group in the country had someone who had been either a student or a tutor (or both) on the course. Evidence for the success of the course is that we would (and still do) come across people in other contexts not just using the language introduced in the course, but making use of the language to inform and justify their current practice. The 'language' being referred to was a collection of frameworks: mostly triples of words which could be used to trigger sensitivities to notice opportunities for actions, as well as access to the actions themselves. I begin with a framework that has emerged relatively recently.

Dimensions of Possible Variation

One of the essential features of Structural Variation Grids is that they permit rapid exposure to several examples which display variation in one or two different aspects. These examples can be chosen according to pedagogic and didactical purposes according to the perceived needs of the learners. Human beings have brains which are well suited to detecting variation, especially systematic or structured variation, and the grids enable a quick succession of examples to be presented.

Ference Marton has for some years been developing the observation that what people discern is variation (Marton & Booth 1997, Marton & Trigwell 2000, Marton & Tsui 2004; see also Runesson 2005). This has led him to propose that learning consists of discerning freshly, that is, of becoming aware of new *dimensions of variation*. A *dimension of variation* is an aspect which can vary in an example and still it remains an example. In other words, a concept or technique is understood to the extent that the person is aware of what can be varied and what must nevertheless remain invariant. Often it is relationships which are invariant rather than aspects of objects themselves. Anne Watson and I (Watson & Mason 2002, 2005) extended this idea to *dimensions of possible variation* to indicate that at any time teacher and learner may be aware of different aspects or dimensions which could be varied, even if they are not varied in the current situation. Furthermore, we noted that often in mathematics learners have a restricted notion of the *range of permissible change* in any specific dimension. For example, when generalising sequences and grids, learners typically think of whole numbers while the teacher may be aware of rationals and reals as possibilities.

The vital aspect of variation as a description of learning is experiencing sufficient variation in sufficiently quick succession to be aware of it as variation of some feature, and hence as a dimension of possible variation.

We have found Marton's idea of great interest and use for two basic reasons: first, it fits with our own view that invariance in the midst of change is a central theme of mathematics, and second, it proves to be fruitful for analysing learning and the potential for learning afforded by tasks, including sets of exercises (Watson & Mason in press).

With and Across The Grain

The layout of Structural Grids is designed to afford plenty of opportunity for learners to experience structure through sequential variation. Whereas Tunja sequences are sequential, the two dimensionality of the grid permits many variations in how learners are exposed to enough data to be able to predict the entries in as-yet-unrevealed cells. This in itself can be attractive and motivating to learners who are becoming used to being invited to detect pattern and structure. But the pedagogic significance emerges when learners are asked not only to predict cell entries but to justify why the upper and lower entries in a cell must always be equal. For the Zero Grids, this means making sense of potential values in a particular cell by using the structure of sequence of cells which include that cell. Trying to make sense of the over-all patterns in a grid calls upon mathematical sense-making. It is convenient to describe these two aspects of using grids using a well known metaphor associated with wood: *going with and across the grain.*

Anne Watson (2000) came across some learners asked to copy and complete a table based on the following structure

7 x 1 = 7	1 x 7 = 7	7 ÷ 1 = 7	7 ÷ 7 = 1
7 x 2 = 14	2 x 7 = 14	14 ÷ 2 = 7	14 ÷ 7 = 2
…	…	…	…

She noted that when learners follow a simple number pattern to anticipate the next and future terms, they are acting in a manner which is similar to going with the grain of a piece of wood: fresh wood splits relatively easily along the grain. This matches my experience of offering people sequences of terms in which everyone (mathematicians and non-mathematicians alike) quickly work out the pattern and can predict the next and future terms. Going with the grain on sequences and grids means following simple patterns such as writing all the 7s in the first column, then all the multiplication signs, then the numbers 1, 2, 3, … and so on. Cutting across the grain reveals the structure of wood, so *going across the grain* can be used to refer to the act of making mathematical sense of relationships, here, between the different entries in a row of the table, which is presumably what the authors intended learners to encounter.

Copy-and-Complete has become a classic form of task in UK textbooks, referring to a partially filled in table in the text which learners are expected to copy into their books and then fill out according to some underlying pattern. Copying is a clerical activity. It calls upon some hand-eye coordination but does not draw on mathematical thinking, especially when it is carried out by inserting all the invariant elements first, and then inserting the things that vary. Even if learners end up with a completed table, they may not have encountered the target technique or actually done much thinking. Mathematical thinking only begins when attention is directed to what a specific term is saying, or to what each row or column or other subsequence in a grid is saying, not in particular, but as an instance of a generality, that is, by going across the grain.

The phrase *with and across the grain* can shift from description to action when it reminds teachers to prompt learners to make mathematical sense and so turn copy-and-complete from a clerical exercise into a significant and relevant mathematical experience. Another way of saying this is that in order for doing a task to influence learning, it is necessary to prompt learners to see the general through (each of) the particulars, and then to see each of the particulars in (as instances of) the general. This two way process was summarised by Alfred Whitehead (1932):

> To see what is general in what is particular and what is permanent in what is transitory is the aim of scientific thought. (p4)

I prefer to rephrase it more expansively: ‘to see the general through the particular and the particular in the general’ and ‘to be aware of what is invariant in the midst of change’ is how human beings cope with the sense-impressions which form their experience, often implicitly. The aim of scientific thought is to do this explicitly.

With and Across the Grain, when internalised as a description of actions which a teacher can take to direct learner attention, has become a teaching framework which enhances or structures learning. When taken up by learners, it acts as a framework for learning. As with other teaching-learning frameworks, it serves to bring to mind actions which might enrich learning, but which might otherwise have slipped by unnoticed.

Specialising & Generalising; Conjecturing & Convincing

Structural Variation Grids can be used to prompt learners to make use of their natural powers to detect and express generality, to make conjectures, to test them by specialising or particularising, and to try to justify those conjectures to others. If these powers are seen by the teacher as essential to learners making sense of mathematics and making mathematical sense, then they will seek out opportunities to provoke learners to use those powers. By drawing learner attention to the spontaneous use of those powers, and offering a label by which to refer to them in the future, teachers can promote the development and refinement of those powers. George Polya (1962) promoted them as components of mathematical behaviour, and Mason *et al* (1982) promoted them as processes which contribute to mathematical thinking. As the focus of educators has changed over the years, the collective noun to describe these processes, practices, or powers has to be changed to match current concerns. Speaking of learners' natural powers (Mason 2002) certainly attracts teacher attention and finds resonance with their experience.

Enactive–Iconic–Symbolic and related frameworks for teaching

Some of the patterns in Structural Variation Grids are so elementary that most people find themselves enacting them without even really being aware. Usually this is because the sequence uses something very familiar such as 1, 2, 3, … . There is a sense in which their body responds to implicitly perceived structure rather than passing thorough the intellect, even though it manifests itself through the intellect in words and symbols. Sometimes when a pattern is not immediately detected, people nevertheless have a sense of pattern, even though they cannot immediately articulate it. You might say they have an image or overall shape or sense. Once a pattern starts to emerge, it can be developed and expressed in words, pictures, icons such as clouds to stand for 'the number I'm thinking of but am not going to tell you', and even using letters as symbols for as-yet-unknown or unspecified numbers, as in traditional algebra. This description builds on distinctions proposed by Jerome Bruner (1966) who identified three modes of representation: enactive, iconic and symbolic. In designing the Open University course mentioned earlier, we found that these resonated with our experience, but that we wanted to elaborate on some of the ramifications. The result was a metaphorical interpretation of his distinctions, and three closely related frameworks. The basic framework remains the same as Bruner's but with elaboration:

Enactive mode: manipulating familiar and confidence-inspiring entities, whether they are physical (blocks, sticks, counters, rods, …), as Bruner suggested, or meta-physical (numerals as numbers, letters as variables or as generalities, familiar diagrams, screen manipulable objects, etc.);

Iconic mode: images, pictures and drawings which depict what they are (as in a cloud for a number I am thinking of or don't yet know) as Bruner suggested, but including also a pre-articulated as-yet-inchoate 'sense of'.

Symbolic mode: symbols whose use is a convention and so by their nature have to be explained, as Bruner suggested, but they are abstractly symbolic only so long as they remain unfamiliar.

For example, Helen Drury (personal communication) working with a year 10 top set used the Factor Grid for the first time as a computer display in whole-class mode and then invited learners to fill in a blank grid for themselves. They had a choice either to try to fill in the expanded expressions above the factored versions immediately, or else to begin by writing out the factored cells before completing the expanded expressions.

Filling in the entries themselves afforded an opportunity for enactive subconscious awareness of patterns to be manifested and experienced. This is why paying attention to *how* you fill out a table or draw a picture enactively can be so useful when trying to articulate a generality: so useful in fact that a slogan such as *Watch What You Do*, along with *Say What You See* can be useful for reminding learners to do more than simply try to get answers. Learners who tried to do the expanded expressions immediately mostly struggled to find the complex patterns, especially in the constant term. They tended to enter all the x^2s first, which is efficient, and which may direct attention to significant patterns, but it may also divert attention inappropriately, as with copy–and–complete. One learner spotted that opposite corners had opposite signs, and another described similarities with the arithmetical multiplication grid. Pedagogic decisions had to be made about whether to invite them to report their observations or to leave others to make similar discoveries. The important work involved trying to make sense of the upper and lower entries in each cell, leading to a deeper enactive awareness of how factoring quadratics works.

Developments from E-I-S

Once symbols become familiar (for example, numerals for numbers), the symbols become less symbolic in effect and more enactive. To capture this specifically we developed the triple *Manipulating—Getting-a-sense-of—Articulating* (MGA for short) as a spiral of development (using Bruner's notion of spiral learning). It was used by teachers, and subsequently by their learners, to remind learners to backtrack to something more familiar and manipulable when they get stuck or when something seems to be beyond immediate grasp.

The use of MGA as an acronym illustrates the framework perfectly. Unless you are already familiar with MGA, you are likely to need to expand it in your mind, to read out the full form of words and then think about the meaning. Over time and with use you may find MGA becoming a useful shorthand for triggering actions and awareness in yourself, for making sense of experiences, and for communicating with others. The acronym can actually help you to articulate some observation or to describe some phenomenon.

In relation to Structural Variation Grids, filling out a grid for themselves from the contents of a sequence of cells in one row and another in one column, or even from the content of a few sporadically placed cells, enables learners to work with familiar entities (expressions in cells whose content is known) in order to get a sense of the overall structure of a particular grid. At a more meta-level, familiarity with one grid enables them to recall what they did previously when they tackle a new one, so that over time they get a sense of structure indicated by a two-way grid, and two-way grids as a format for arithmetic structure.

Hand in hand with MGA is the triple *Do–Talk–Record* (in the sense of writing-up not exploratory writing-down). Writing our course in the early 80s we were well aware of the importance of learner-learner talk and collaboration. This framework allowed us to remind teachers that pushing learners to make written records too quickly can be at best unproductive and frustrating for all concerned, and at worst, actually harmful. It is valuable if not essential to allow learners time to talk about what they have been doing, and indeed to get them doing things (enactively manipulating the familiar) so that there is something mathematical to talk about. Talking to others, trying to justify your ideas and conjectures is an excellent way to externalise your thinking, get it outside of yourself so that you can look at it critically. That makes it easier not to be indentified with your idea but to treat it as a conjecture to be modified.

Alongside these three frameworks we also found it useful to include something to remind teachers that learners do not usually master ideas on first exposure. So we suggested a triple of *See—Experience—Master* as a reminder that first encounters are a bit like seeing a fast vehicle go by. It takes repeated encounters to begin to discern details and to recognise relationships amongst those details. Only then does it make sense to try to achieve mastery, to develop facility and fluency and to minimise the amount of attention needed to carry out techniques and procedures.

Finally, we embedded these frameworks in the notion of a *classroom rubric* or *ways of working* which correspond to what is now described as socio-cultural practices of a *community of* *practice* (Lave & Wenger 1991) and as *sociomathematical norms* (Yackel & Cobb 1996). Fundamental to the effective functioning of a classroom ethos is a *mathematical* or *conjecturing atmosphere* (Mason, Stacey & Burton 1982, see also Mason & Johnston-Wilder 2004a). This is a way of working in which everything said is treated as a conjecture, uttered in order to think about it more clearly and to modify it as appropriate. Those who are very confident take the opportunity to listen and to suggest illustrative examples and counter-examples, and those who are not so confident take opportunities to try to express their thinking in order to help them clarify that thinking, just as in the *Eleusis* game.

Structure of Attention

Inviting learners to say 'what is the same and what is different' about several entries in a Structural Variation Grid, or simply to 'say what you see' initiates a movement of their attention. They may gaze at the whole (and be aware that there appear to be missing or hidden entries in a grid) or at the whole of a particular element such as a cell entry; they may discern details such as particular entries, or details within an entry (such as two rows to each cell, or the presence of various mathematical signs); they may recognise relationships within a cell (such as an equation) and between cells (such as all having two factors or the upper part being a calculation and the lower part an answer or vice versa); they may perhaps perceive some relationships as properties which apply across all visible cells and so might apply to all cells; they may even be able to reason on the basis of those properties in order to justify their prediction of what will appear in different cells, or where a particular entry is to be found.

As a teacher with a class, the problem is that different learners may be attending in different ways. If as teacher you are attending in one way, say talking about properties of cells, when learners are busy discerning details or recognising relationships between particular entries, there may be a mismatch and consequent breakdown in communication. By being aware of what you are attending to, and how, you can either direct learner attention appropriately, or put your own focus of attention to one side and try to enter the experience of some of the learners.

What is Attention?

For William James, philosopher and psycholgoist, attention is what makes it possible to perceive, conceive, distinguish and remember. It is the basis of all our psychological functioning (James 1890 p 424). As might be expected, he deals with a number of important issues concerning attention in general. For example, he argues on the basis of experiments that attention is not simply what the eyes are looking at, or indeed any other particular source of sense impressions (p 438). He links attention to anticipative imagination (p 439-411) as a prerequisite for discerning anything at all. James develops this theme of discernment, or discrimination, to make use of what he calls Helmholtz's law, that

> we leave all impressions unnoticed which are valueless to us as signs by which to discriminate things (p 456).

In other words, we notice what we are attuned to discern. James goes on to discuss pedagogic implications such as that it is useful for teachers to work with learners to strengthen and attract their attention in order to improve motivation, since people engage with what catches their attention (James 1890, p 446). To do this requires being aware of what in learners' previous experience can be used as a basis of previous attention-experience, what John Dewey referred to as 'psychologising the subject matter' (Dewey 1902, p 12).

James sees attention as a form of 'free energy', since when you make an effort to attend to something you can sustain it for only very short periods before attention wanders (p 420) requiring a further expenditure of effort, but when attention is engaged it requires no energy expenditure at all for it to remain focused for long periods of time.

I agree with James that *'my experience is what I agree to attend to'* (his emphasis), although his wording might be taken to imply voluntary agreement, which is certainly not always the case. At each moment, as my attention shifts, I am the totality of that attention; the totality of my experience *is* my attention. Attention is not just as what puts me in touch with the world of my experience, but what creates and maintains that world. This is meant to include things of which I am subliminally or covertly aware, sometimes through body awareness, sometimes through social awareness, sometimes through emotional resonance, and sometimes through cognitive awareness. None of these need be conscious. This makes attention difficult to study directly, because it is no good asking people 'what are you attending to?' since the very question alters the focus and locus of that attention.

Where I differ with James is in his metaphor of attention or consciousness as a flowing stream, for it seems to me that his own descriptions (e.g. James 1890 p456 quoting Müller), as well as my observations, lead to the conclusion that attention is briefly sharp and alert, and then slowly declines into absence of awareness until some fresh stimulus wakes it up again. The sense that we have of experience flowing by is actually much more episodic and fragmentary (Mason 1988), as attempts to reconstruct recent and distant experiences demonstrates all too clearly.

Task 5: Focus, Locus, and Multiplicity

Can you gaze into the distance while asking yourself what you think attention is?
Can you be aware while you are reading that this paper is just one contribution to a whole collection of papers, and can you then shift so that you are focusing intently on the wording of the next part of the task, oblivious to the other papers?
How many different things can you attend to at once? For example, in a lecture, can you attend to the speaker's voice, use of display, clothes, and content of what they are saying? Can you at the same time as reading this imagine yourself going and getting something to drink, and being aware of some background music or other sounds?

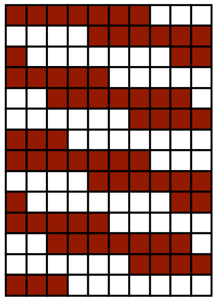
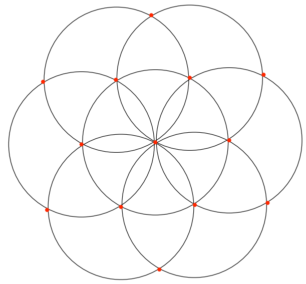
You can attend to things physically present and also to things not physically present (locus); you can 'gaze' while pondering, and you can concentrate very specifically on some small detail (focus). You can be aware of one single detail, and you can be multiply aware cognitively, multiply aware enactively, and multiply aware affectively (multiplicity). Once focused, attention can be diverted by rapid movement within your field of vision, especially if it is peripheral, and changes in other sense impressions can also attract your overt attention.

There are deep physiological questions about whether you actually attend to several things at once, or whether you rapidly cycle through a variety of foci, the way computers now do. There is also an issue about whether consciousness directs behaviour or is subject to a 'user illusion' of being in charge, as Tor Norretranders (1998) proposes. Whatever may be the case, personal experience is sufficient to highlight important aspects of attention which can be used to improve both teaching and learning.

Interrogation of experiences suggests that attention can be focused or diffuse, localised or global, single or multiple. But even focused localised attention has different forms.

Task 6: Say What You See

Say (to yourself, to a colleague) what you see in the three pictures, and try to pay attention to how your attention alters.


Which diagram seems the most complex?
How many different rectangles can you find in the central diagram of the circles, where the vertices have to be on the points of intersection of the circles?

It is quite likely that the middle diagram seemed quite simple compared to the others, at least until you started to work within the diagram looking for distinct quadrilaterals. The counting question serves to focus attention, calling then upon not just discerning particular vertices, but relating their positions so as to form rectangles. To be a rectangle is to impose a property on the specific points so that they satisfy a relationship. This task is just part of a complex of tasks using this same diagram and developed by Geoff Faux (private communication).

The picture on the right makes perfect local sense, but when you gaze at the whole there are some inconsistencies with the way material space is structured: one part of the picture seems to pull against other parts.

By contrast, in the picture on the left, you may have found yourself beginning to count the dark or the white squares in each row, then discovering that there are the same number in each block if you carry over to the next line. You may have gazed at the whole without seeing very much until you noticed a repeating pattern, although this may not have emerged until you did some counting. Did you think to look for patterns in the columns? Recognising possible relationships between alternating columns leads to perceiving a potential property, which can then be justified on the basis of an assumption about how the rows are generated.

Whether attention is the subjective experience of physiological functioning, as Théodule Ribot (1890) would have it, or the engine for physiological response to environment, as William James (1890) proposes, there seem to be quite distinctive if subtly different forms of attention:

Holding Wholes (gazing)
Discerning Details (features & attributes)
Recognising Relationships (part-part, part-whole)
Perceiving as Properties (leading to generalisation)
Deducing from Definitions (reasoning on the basis of explicitly stated properties stated independently of particular objects)

Shifts between these are rapid, often subtle, but vital in order to engage in mathematical thinking. While gazing, some sudden movement, perhaps even apparent motion produced from circadian eye movement can suddenly switch attention to awareness of details amongst a mass of other, undiscerned detail. As details are detected and discriminated, the mind automatically looks for relationships: differences and samenesses. To do this requires something being relatively invariant as a background against which to detect change. Recognising relationships tends to focus on particulars, whereas perceiving properties is a move to the more general, to the particular as exemplary or paradigmatic. Formalising in mathematics is the overt action which accompanies a shift from perceiving properties to taking certain properties as definitive and so as the basis for further reasoning. Discerning these subtle shifts in the structure of attention develops Marton's notion of learning as discerning variation, because it provides a more detailed structure of what might attended to, and how.

Here is an opportunity to 'see' whether you recognise some of the subtle moves being suggested by this framework.

Task 7: Some Sums

What do you make of the assertion that $\sum_{k=\sum_{1}^{n-1} j}^{\sum_{1}^{n-1}(j+1)} (2k+1) = n^3$?

Is your first reaction panic? Did you find your attention drawn to or away from the little summation signs? Did you try some particular values for n? Did you sit back and gaze at the whole, then pick out details when you gained a little confidence, perhaps? Did you recognise a sum of consecutive odd numbers? Were you able to focus on the upper and lower limits of the main sum? Did you detect a relationship between them? That could tell you how many terms there are in the sum. Did you recognise the lower sum so that you could write it in terms of n? That would then enable you to write the upper sum in terms of n as well, making it much easier to try some examples, but you might recognise that the sum of consecutive odd numbers is a difference of the squares of the term preceding the first, and of the last term. So you could actually check it out in general without having to try some particular cases!

Notice that reasoning on the basis of properties of sums of consecutive natural numbers and of consecutive odd numbers is only possible if you can attend to the structure without having to keep in mind a particular example. This requires that you see through particularities of specific symbols (summation signs, bracketed terms), and appreciate what compound symbols are saying as a whole. Recognising a relationship between the shape of the upper and lower summation signs shortcuts the algebra of evaluating the upper sum independently of the lower.

Gazing is an under-rated form of attention. It includes mulling things over as you wait for a bus, take a shower, or wash dishes. Discerning different forms or structures of attention is only useful if it serves to inform future practice, whether by refining research probes, or by suggesting ways of acting with learners so that there is a better match between what learners and teachers are attending to, and how they are attending.

Why might the structure of attention matter?

At a classroom level, if learner attention and teacher attention are significantly differently structured then confusion is a most likely outcome. More particularly,

if some learners are attending gazing when the teacher is discerning specific details;

if some learners are discerning details when the teacher is talking about relationships amongst details;

if some learners are recognising relationships amongst discerned elements when the teacher is talking about properties of objects in general;

or if some learners are thinking about properties when the teacher is reasoning on the basis of, or deducing from those properties;

then here is likely to be a mismatch, a failure of communication. If some learners are focusing on what they are supposed to do, and others on what the object itself is, then again a mismatch is likely. One of the abiding problems in mathematics education is how to promote mathematical reasoning (proof). Colette Laborde (2003) observed in the context of dynamic geometry that

> It is a matter of getting pupils to understand that in geometry they have to rely on what they see to get ideas on how to solve a problem, but they do not have the right to use that when it is a matter of reasoning 'rigorously'; they must then restrict themselves to the theoretical plane.

This is the difficult move, the move which is uncommon outside of mathematics, the move which marks out the natural mathematician from others who have to struggle to make it.

Similar mismatches are likely to occur in an in-service, continuing professional development context. If task-exercises are offered, as in this paper, as an opportunity to engage in mathematical thinking, attention may be fully caught up in the mathematics, leaving little attention free for meta-concerns and the use or influence of some chosen framework. If behaviour such as styles of questioning or prompting is modelled, teachers may attend to specific details rather than recognising relevant relationships, so that they are left with behaviours but no criteria for when to use them, and no sensitivity to features of a situation which might usefully trigger awareness of that behaviour. Where examples of pedagogical decisions are offered, attention may be absorbed by recognising relationships within the particular rather than drawing back to perceive properties which could apply in many different situations. Hence it is less likely that a new situation will trigger relevant awareness. Even where an explicit framework is presented, as here, it may be that most attention is taken up with relationships so that there is insufficient attention available for perceiving those specific relationships as properties, so the label does not become richly imbued with personal meaning.

van Hiele Levels

Anyone familiar with van Hiele levels (van Hiele-Geldof 1957, Burger & Shaunessy 1986) will be aware of close similarities with the structures of attention being proposed. Here is one version (based on Burger & Schaunessy *op cit*), to which I have appended a version cast in terms of what reasoning might look like:

Level 1: Visualization (reasoning based on direct perception)
Level 2: Analysis (reasoning based on relating component parts and attributes)
Level 3: Abstraction (reasoning based on necessary conditions as known facts)
Level 4: Informal Deduction (reasoning based on relating properties)
Level 5: Formal Deduction (reasoning from axioms systematically)

Pierre van Hiele generalised these beyond geometry (van Hiele 1986) but in the process made them even more abstract and, for me, harder to connect to moment-by-moment experience, which is where I think it is important to focus in order to influence and improve learners' experiences of being taught mathematics. I came to the structures of attention through an entirely different route based on Eastern sources (Bennett 1956-1966).

The difference between the structures identified here and the van Hiele levels lies precisely in the notion of levels. Rather than seeing these structures as levels or even as hierarchical qualities in the way researchers have developed the van Hiele ideas to date, I am proposing the radical stance that these so-called levels are actually descriptions of the way that people attend all the time, often with rapid shifts from one to another.

There are of similarities also with the onion model of understanding developed by Pirie & Kieren (1994) but there is not sufficient space here to elaborate on this connection.

Reflection

Notice the steps taken to try to engage the reader with each framework and especially this one concerning the structure of attention: presenting task-exercises designed to highlight aspects of the phenomenon of interest in (different ways in which attention is structured); commenting on that experience, with the hope that there will be resonances with or challenges from the reader's past experience; justifying the potential value of the framework in terms of addressing perplexing questions in teaching and learning mathematics; and linking to the literature.

Theory of Frameworks

The examples of frameworks provided in this paper illustrate my view of a framework as a collection of labels which serve to bring to mind useful distinctions, and relevant relationships as instances or manifestations of properties as well as ways of working with learners. The label acts as a reminder, as a re-sensitisation not only to notice but also to act. Thus *enactive–iconic–symbolic* distinguishes three forms of representation, and *do–talk–record* distinguishes three learner activities. They also trigger possibilities such as re-presenting in a different mode or inviting learners to do this in order to develop flexibility in moving between modes, and such as constructing tasks which call upon learners to try to describe what they are thinking about what they have been doing, before being urged to make written records. But making distinctions is only the beginning, as the *structure of attention* framework indicates. What also matters are relationships between distinguished elements, and the abstraction rendered by perceiving specific relationships as more general properties.

Encountering Frameworks

The exposition given here of a number of frameworks has been necessarily brief and truncated. Working with teachers over many years, and writing materials to try to support their professional development has highlighted again and again the necessity of several experiences before a framework begins to be active. First, it is vital to engage people in immediate mathematical or classroom experience in which they are likely to experience some aspect of a framework to be proposed. In text, this means offering mathematical tasks and then offering comments based on the sorts of things people tend to notice when engaging with those tasks. Labelling salient experiences with what might become a framework then permits later reference back to such experiences. Where classroom video is available, incidents can be shown and then used to resonate personal experiences with what seem to participants to be similar elements. Individuals can be stimulated to report incidents that seem to have something in common, and negotiation can take place as to what similarities and differences people are aware of. It really helps if people have the opportunity to discuss such examples in order to come to some agreement as to what the framework labels refer. The aim is to build up a rich network of associations with past experiences and appropriate actions, triggered by the label. This means that generic labels using words that might even be used in a relevant situation are more useful than labels such as the names of people involved in particular past incidents.

Thus frameworks as labels can act to sensitise people to notice situations that might have gone unnoticed previously. In order for frameworks to become active and informative, whether for teaching or research, it is vital that they also become associated with actions which can be initiated as a result of noticing. Techniques for enhancing and enriching the use of frameworks in this way have been described in detail in Mason (2002) as the *discipline of noticing*.

How Frameworks Arise

Whenever a situation or incident is recognised as not just particular, but as an instance of some general structure, it becomes a representative of a phenomenon. Thus does a phenomenon come into existence in the mind of the observer. If, as is often the case, someone wishes they could have acted differently in the situation, or else wants to remember to act in a particular way in future, then there is an opportunity for a framework to develop. Some word or words which capture an essence of the phenomenon, and which can be strongly associated with a desirable action, produces a framework.

Many frameworks arise out of the literature, as was illustrated by some of those described earlier. Others arise as a result of probing and considering what actions you would like to take in certain situations. As with diagrams and other forms of notation, the ones you create for yourself are the ones which are most vivid, along with frameworks which crystallize awareness you have had but hadn't yet articulated. At the core of any useful framework are distinctions which enable you to discern details and to recognise relationships in the particular situation. These then need to bring to mind actions you would like to initiate. By setting yourself to notice phenomena in the future and to use that noticing to act in fresh ways, a framework of labelled distinctions and actions can inform future practice.

Validity

Frameworks, however publicly negotiated and developed, are essentially personal. Despite widespread currency of the labels, without careful and ongoing negotiation of meaning and interpretation, frameworks can divide as much as unite. Examples abound, such as the many uses of labels such as *ZPD*, *activity theory*, *constructivism*, and *problem solving*.

It is the sensitivity to notice and to act, triggered by resonance between a framework and a particular situation, which renders the framework useful. Frameworks are not then either valid or invalid, but rather informative or not informative for a person at a time in a situation. There is no claim of universality. The deeper issue about validity is whether someone really is sensitised to notice more and to act more effectively, or whether a framework supports the individual in misapprehended solipsism and prejudice. The only way to guard against self-delusion is to engage in ongoing practices of offering incidents and tasks to new colleagues to see if they recognise what is being pointed to, and whether they too find the distinctions informative in their future practice. This is the basis for the discipline in the *discipline of noticing*. Thus validity is at once personal yet in need of frequent testing against an ever widening community.

A Note About Method

My method is and always has been to reflect deeply on my own experience of doing mathematics and of being mathematical with others. I am constantly seeking situations analogous to those of learners so that I can get a taste of what they are experiencing. I use that to inform my pedagogic and didactic choices. I try to stimulate others to notice what I think I am noticing so as to guard against solipsism, and I try out actions which seem to be an improvement, in a ongoing development of noticing, sensitisation and action. Stimuli are constantly being refined and honed to meet new situations. The data I am offering you as reader is the collection of memories and awarenesses which come to mind as you engaged with the stimuli offered in the paper, in the form of tasks and commentary. If you didn't engage in the tasks, you are unlikely to have gained access to much data. Validity for you resides in the extent to which you find your past experience resonated or challenged, and your future actions informed. I do not seek any absolute validity, because it seems to me impossible: we are dealing with human interactions. Human beings are a remarkable combination of predictable mechanicality and creative agency. I see all 'truths' in mathematics education as people, time, place, and situation dependent.

Distinctions are not 'natural cleavages' of the world, but rather lie in the eye of the beholder. They are psychological in the sense that they involve a re-structuring of attention in the mind of the individual. They are social in the sense that they are often encountered in the practices of others, and if they resonate with or challenge previous experience, can be adopted and adapted into a personal practice.

How Frameworks Can Inform Research

I have concentrated here on frameworks for learning and for teaching, confident that any framework which is informative for these purposes will be informative for research purposes as well. However it is important not to allow theoretical frameworks and distinctions to displace careful observations. To be useful as data, observations and transcripts depend on being accounts-of incidents, so that readers feel they could recognise such a situation had they been present, and even that they recognise the type of situation in their own experience. If justifications and explanations which partially account-for the situation are intermingled with accounts-of incidents, then the reader is unable to question or disagree with what is said, and may not be able to recognise similarities and differences with instances from their own experience. Similarly, if analysis is intermingled with accounts, such as substituting framework labels for specific observations, then again it is difficult if not impossible for a reader to question or disagree with the analysis. Distinctions triggered by frameworks need to be justified from observations of behaviour rather than baldly asserted as part of the data.

Attempts to classify people according to distinctions offered by a framework are especially unhelpful, because it is behaviour that is being observed and classified, not the person. But even classifying behaviour can be misleading, because what is observed is only a fragment in time. The person may be experiencing a much richer flow of awarenesses not displayed in behaviour. The best research leads people to reveal their awareness and their dispositions as well.

Frameworks such as those described in this paper can also inform the researcher in the design of their study, such as when seeking tasks to reveal dimensions of variation of which subjects are aware or can access, to get them doing and talking as well as making records, to provoke them into displaying mathematical thinking and to stimulate them to expose the subtle shifts in the structure of their attention.

Dangers

Frameworks are rather like geometric diagrams: they have implicit structure which you need to know about in order to make good use of it. A framework such as *enactive–iconic–symbolic* or the van Hiele geometry levels may be used in a community as if they were agreed and well defined, when actually what is resonated within different people by those terms is markedly different. One has only to look at the use of the *zone of proximal development* as a term in mathematics education papers over the last twenty years to be reminded of this! This is the danger of frameworks. It is a danger inherent in any signifier which refers to abstractions from specifics. To be confined to a world of specifics is to lose the power of generalisation, while to be restricted to a world of abstractions without referents to specifics is to live in an ivory tower. The rich use of frameworks as described here straddles the two worlds of particulars and generalities, of concrete experiences and abstract notions.

Conclusions

What can be learned from these observations? Every time someone offers you a framework, model, or even just a list, ask yourself questions such as the following:

What is being stressed? What is being distinguished? What am I sensitised to notice now that I did not discern before?

What examples come to mind from immediate or recent experience, and from past experience, and how are these experiences informed by the framework?

What is being ignored?

What associated actions would I like to use in such a situation, and why?

What other possibilities are afforded as a result of being sensitised by this framework?

When you are using technical terms with a colleague, offer an account of an incident which you think exemplifies some aspect, and seek agreement as to its appropriateness. In this way mathematics education can develop from a ragbag of distinctions into a theory-based discipline.

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FRAMING OUR WORK

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The theme of this conference, "Frameworks that Support Research and Learning," invites us to take stock of where we are as a field with respect to frameworks, which are a critical element of scholarly inquiry. In an effort to take stock, I briefly review the purpose of frameworks, make the case for why we need more robust frameworks, and suggest approaches that might lead us to more robust frameworks.

Model. Construct. Theory. Paradigm. Framework. To some, these words have vastly distinct meanings, while to others they are separated by shades of grey. Dictionary definitions (Houghton-Mifflin, 2004) of these terms would favor the shades-of-grey school of thought:

Model: A schematic description of a system, theory, or phenomenon that accounts for its known or inferred properties and may be used for further study of its characteristics.
Construct: an abstract or general idea inferred or derived from specific instances.
Theory: The branch of a science or art consisting of its explanatory statements, accepted principles, and methods of analysis, as opposed to practice: a fine musician who had never studied theory.
Paradigm: A set of assumptions, concepts, values, and practices that constitutes a way of viewing reality for the community that shares them, especially in an intellectual discipline.
Framework: A set of assumptions, concepts, values, and practices that constitutes a way of viewing reality.

Given the relative cohesiveness of these terms as defined above and the fact that some of them can be combined (e.g., theoretical framework), I am going to use the term *framework* throughout this paper to be consistent with the conference theme. However, I do want to identify some terms that I am deliberately omitting from this paper. I make a distinction between a theoretical *perspective/orientation/viewpoint* and a theoretical *framework*. A theoretical perspective is a world-view that influences one's approach to professional life, in general. Wikipedia defines a perspective as "the choice of a single point of view from which to sense, categorize, measure or codify experience, typically for comparing with another. Viewpoint is another word for this principle - with a similarly broad interpretation. It may be visual and/or mental, related to cognition." Examples of theoretical orientations include radical constructivism, postmodernism, and interpretivism. While reports of research are certainly strengthened by revelation of the authors' theoretical stance, most people do not find this difficult to do. Thus, I am confining my remarks to what I see as the area in which our field needs more work–both in

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explicitness and robustness–that of theoretical frameworks.[1] I am by no means the first person to suggest that we need to grow as a field in this arena. Indeed, several plenary speakers at the 1991 PME-NA conference addressed this topic (viz., diSessa, 1991; Eisenhart, 1991).

Roles and Purposes of Frameworks

Consider the every day uses of a frame–a picture frame, a bed frame, or the frame for a house under construction. These frames serve various purposes that parallel the use of frameworks in mathematics education research. A picture frame serves to demarcate an image and set it off so that it will be noticed by others and can be easily distinguished from the wall on which it is hung. The material from which the frame is made, the size of the frame, and the placement of the frame in relation to the image all influence how an observer processes the image. A bed frame serves as a base upon which a mattress can be placed without fear of the mattress warping or sagging with time. A house frame provides an underlying structure to support the sheetrock, floor joists, and roof that make up the house. Without the framing, the house would collapse upon itself. The purposes of a theoretical framework are similar. A theoretical framework can help "set off" ideas from other data to draw attention to them, giving them names and robust definitions. It can support the building up and deepening of an idea, or it can provide a structure on which to hang new ideas.

Thus, a framework can serve multiple purposes. In a particular study a framework may serve only one of these purposes, or it may serve several of them. Further, a framework can be somewhat minimalist, as in the case of a picture frame, or it can be quite robust and well-tested, as in the case of a house frame. As with everyday frames, frameworks are not right or wrong. They must fit the researcher and the data; some are a better fit than others, but they are not inherently right or wrong.

As a graduate student, I struggled with the notion of a theoretical framework. I really thought it was something I had to put in Chapter 2 to satisfy my dissertation committee; I did not see its relevance and usefulness as a research tool. The most common and biggest flaw I see in manuscripts that I review for journals is that authors go to some trouble to describe a theoretical basis for their work, but these ideas never appear again later in the paper. They seem to be parading their theory before the reviewers but not actually *using* it. Ideally, theory ought to be the element that undergirds an entire research project from the research questions to the conclusions. In the papers written from that research project, theory should wind and weave throughout a paper and be used to "tie it all up" in a neat package at the end. As a manuscript reviewer, I want to see how the theory relates to the research questions, the data collection methods, and the analysis of results. A framework is what moves a manuscript from an anecdotal account to a scholarly piece of literature. Ideally, I also want to see the author tie the results to the literature, pressing on points of cohesion and divergence, which may help other researchers refine the framework. In the next two sections I provide arguments for why we need frameworks to undergird our research–both as individual researchers and as a field.

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[1] I acknowledge that others offer additional types of frameworks and names for frameworks, such as conceptual frameworks (e.g., Eisenhart, 1991), but I am not going to enter the semantic fray of defining each of these terms.

The Value of Frameworks for Individual Researchers

There are several ways that frameworks are useful to individual researchers. First, a framework serves as a sort of binocular that allows one to narrow down the scope of the research site to focus on particular aspects of the situation. They help us notice things and help us cut out "noise" in our data. Importantly, they also help us know when we have found what we are seeking.

Let me share an example from a study that is currently underway. One of the doctoral students with whom I work, Andrew Tyminski, is studying a phenomenon first labeled by Mary Boole in the early 1900s (Boole, 1931) as "teacher lust." Teacher lust occurs when a teacher acts in a manner counter to his/her intentions by assuming an authoritative stance in the classroom and telling students something about mathematics. As we have read in the literature (e.g., Chazan & Ball, 1999; Lobato, Clarke, & Ellis, 2005), not all telling is inherently bad; some judicious telling is necessary to good instruction. So Tyminski was faced with the challenge of deciding how he was going to recognize instances of telling that qualified as teacher lust. He used Smith's definition of traditional telling (Smith 1996) to describe telling actions that did exhibit teacher lust, but he needed an alternate description of telling actions that did not. This involved getting beyond the trap of labeling all teaching as either "reform-oriented" or "traditional" by conceptualizing various ways that teachers interact with students and mathematics content. To do so, he used Mason's six of modes of interaction (Mason, 1998)–expounding, explaining, examining, exploring, exercising, and expressing. The basis for Mason's modes of interaction is the relationship among teacher, students, and content with regard who/what is affirming, responding, and mediating. Figure 1 depicts the three modes of interaction that Tyminski classified as examples of what judicious telling could look like. As Mason describes these relationships, the teacher's role in these three instances is that of facilitator of knowledge, not as a keeper or provider of it.

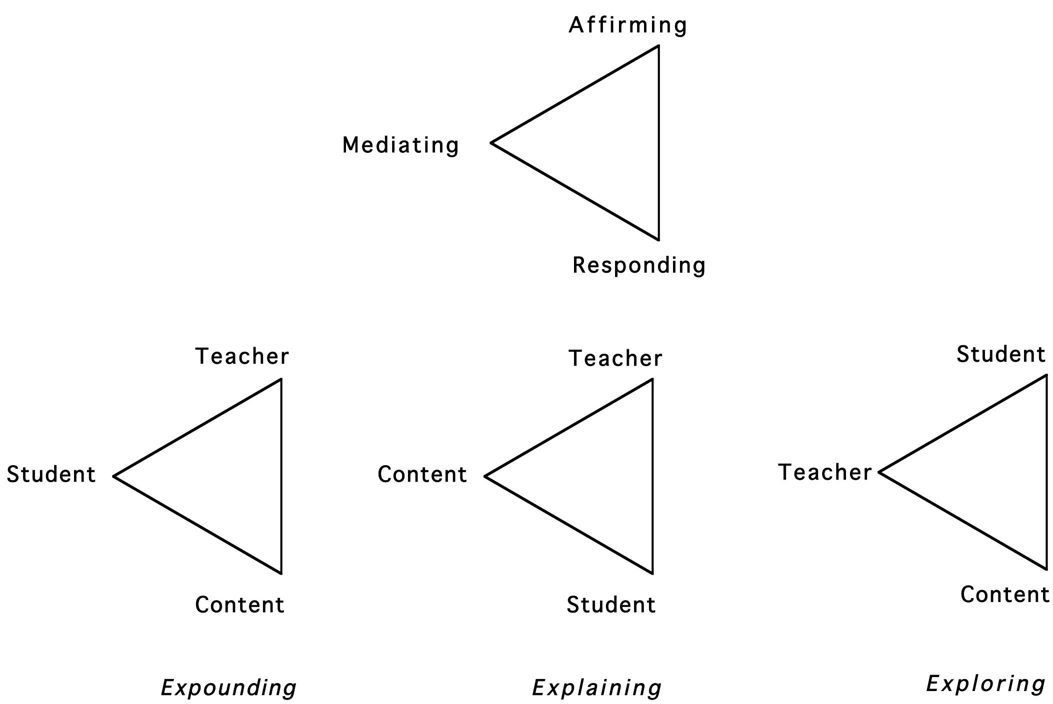

Figure 1: Mason's modes of interaction

Tyminski plans to look for instances where classroom interaction is in the expounding, explaining, or exploring mode and then slips into one of the other modes because such shifts will signal a switch from judicious telling to teacher lust. By analyzing antecedents (from his perspective and the teachers' perspectives) to these shifts, he will be able to enrich the construct of teacher lust by explaining when and why it occurs in mathematics classrooms.

By developing a framework for the notion of teacher lust prior to data collection, Tyminski has positioned himself to better notice instances of the phenomenon and ignore the many things that happen in a classroom that are not related to the phenomenon under study. Further, he will use this framework to analyze data during data collection in order to select instances of teacher lust for stimulated recall interviews with teachers. The framework will also be useful as he conducts a retrospective analysis of his data after it has all been collected because the framework will provide a structure for looking across teachers to offer more general comments about teacher lust. Finally, the framework will be useful in writing up his study as it will provide a structure for reporting his findings. The framework will enable him to communicate his findings in a more general way that transcends the particular teachers and contexts that he studied.

A second purpose for the use of frameworks by individual researchers is what Eisenhart (1991) and others have called sensitizing. Eisenhart noted that frameworks cause the researcher to "tack between the concepts advanced or assumed and the meanings given or enacted in context" (p. 211). Thus, a framework forces a researcher to constantly compare and contrast what the data are saying with what the framework is saying. This notion is commonly referred to as the constant comparative method of data analysis (Glaser, 1965). Eisenhart suggested that this tacking between the framework and data helps guard against poorly warranted conclusions. Note, however, that in order for a framework to serve the purpose of sensitizing the researcher, it must be actively used as a research tool throughout the study. The researcher must be tacking back and forth between the framework and the data during data collection and analysis. The framework cannot be a well-written piece of prose or a pretty diagram that sits on a page; it must be actively used.

As a related side note, I would like to suggest that we make a conscious effort in our field to "clean up our language," with regard to the constant comparison method and building grounded theory (Strauss & Corbin, 1990). Saying that one is using the constant comparison method or that one is building grounded theory has become immensely popular these days, but I question how many of us are really being faithful to these methods. I have already commented above on what it means to use the constant comparative method, and much more can be found in the writings of Glaser. Grounded theory implies that one is doing analysis as the data are being collected so that emerging hypotheses can be tested in subsequent data collection sessions. This is akin to what Steffe (Steffe & Thompson, 2000) strives to do in teaching experiments when he is building second-order models of children's mathematics. He develops a hypothetical learning trajectory for a particular child and poses tasks that will test the path of the trajectory across the course of the teaching experiment. What happens one day is shaped by what happened on the previous days. I think that many of us use the notion of grounded theory to mean that we are building new ideas, but we are not necessarily adhering to the methodological implications of grounded theory where analysis is on-going throughout the data collection process. I think we would be doing a service to the field if we–both writers and reviewers–revisited the original writings on which these methods are based and interrogated our own methods to see if they can legitimately be labeled as constant comparison or grounded theory. Reviewers should expect to see some evidence beyond a mere statement that these methods were used and should push authors to provide evidence if none if given.

A common concern about frameworks is that they can be confining and restrictive. A framework is often not a perfect fit; not every person or instance will map directly onto the framework. In some cases, these misfits provide us with an opportunity to refine the framework. However, in general, frameworks are not meant to be pigeonholes into which we cram data. Rather, frameworks are meant to guide data collection, analysis, and reporting. Frameworks help us move one level beyond the particulars of the study at hand to the more general ideas at play. For example, much research has been done using Perry's scheme for intellectual development (Perry, 1970). While an individual researcher may struggle to classify a study participant as dualistic or multiplistic, it is really not terribly important to the rest of the mathematics education community which classification is a better description of this particular person. What is more important is for the researcher to illuminate the ways in which the intellectual development of college students interfaces with their learning to teach mathematics. Where do preservice teachers who exhibit dualistic tendencies tend to struggle with the ideas presented in mathematics classes? How do preservice teachers who exhibit multiplistic tendencies interpret the messages of the current reform movement? It is in thinking through these kinds of larger questions that frameworks help us progress; the power of a framework is not in labeling John Q. Preservice Teacher as dualistic. I would like to note, however, that frameworks can be confining when used inappropriately. The mindless application of an a priori framework can blind us to certain elements of our data. This is where notions of triangulation, disconfirming evidence, and member checking play a role.

Making the Case for Frameworks at the Level of the Field

In addition to the strength that frameworks lend to individual studies, they also serve a purpose at the level of the field. When frameworks span a number of studies they begin to have a cumulative effect (diSessa, 1991) that leads to predictive power. Frameworks with predictive power transcend the particulars of time, place, context, and participants. When frameworks have predictive power, they also afford us greater credibility for making links to practice.

As we open new areas of research, we generally start with a collection of anecdotal stories that often lack theoretical coherence. Analyzing this collection of stories or anecdotes leads to frameworks that have explanatory power to help us make sense of a particular situation. Continued analysis of these explanatory frameworks and the studies from which they came leads to frameworks that have predictive power. Moving toward predictive frameworks is not going to come from doing more studies alone; it will come from thoughtful analysis of a large collection of existing studies.

I see striking difference in our field between frameworks used by those who study the learning of mathematics and those who study the teaching of mathematics.[2] I will assert that we have predictive frameworks in the area of student learning, but we are still at the stage of explanatory frameworks in the area of teaching. I suspect that if I challenged you to think of a framework in the arena of learning, most readers could produce an example. I suspect the same request for teaching, however, would produce less robust results.

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[2] For simplicity's sake, I am going to stick to these two large arenas of research, although I acknowledge that there are subfields within each and that there are areas of study, such as equity, that cross these boundaries or do not fit neatly into one of them.

Frameworks in the area of learning seem to come in two types–more general frameworks about how learning occurs, and frameworks that are specific to a particular piece of content. For example, Steffe (1988, 1994) and diSessa (1988, 1993) take different approaches to a theoretical basis for learning. Steffe relies on schemes, while diSessa relies on a knowledge in pieces epistemology. Either of these frames is transportable across different content domains and different student populations. In a different vein, there are frameworks that are specific to counting, fractions, geometry, and algebra, among other topics. Perhaps the most widely known of these is the van Hiele levels of geometric thought (van Hiele, 1986) and the corresponding phases of learning (which seem to be largely forgotten!). Such frameworks have been used by multiple researchers across various contexts over time, leading to their predictive power. While there are many idiosyncrasies to student learning, we can make reasonable predictions about how new groups of students will respond to particular kinds of tasks. Thus, we are not surprised when kindergarten students identify a square that has been rotated 45 degrees as a diamond because the van Hiele levels allow us to predict that this will be so. The phases of learning also give us some direction for shaping experiences that will help the child come to see the figure as a square.

In contrast, research on mathematics teacher education was described as a collection of interesting stories (Cooney, 1994) a little over a decade ago. Cooney suggested that we had a lot of local theories that explained the behavior of a particular teacher in a particular classroom, but we lacked more general theories that would "allow us to see how those stories begin to tell a larger story" (p. 627). We have made some progress toward more general frameworks since Cooney wrote those words. For example, as noted above, Perry's stages of intellectual development (1970) have been used by many to describe the growth of preservice teachers. Thus, we are not surprised when preservice teachers come to us wanting to know *the* right way to teach decimals because Perry's scheme suggests that many college students hold dualistic views of knowledge. We are subsequently not surprised when these same preservice teachers later assert that mathematics teaching is simply a matter of finding your own individual style because nothing is clearly right or wrong about teaching. Perry's scheme predicts that as college students mature, they will adopt a more multiplistic view of knowledge. Perry's scheme also gives us some guidance in thinking about experiences that we might provide for teacher education students.

By and large, I do not yet see robust frameworks in the areas of mathematics teaching and teacher education that would parallel the frameworks in the area of learning.[3] One could speculate many reasons for this, but that leads easily into the trap of making excuses for why such frameworks are not possible in the complicated venue of teaching and teacher education–a trap diSessa urged us to avoid (1991). diSessa also noted that there is no shame in the fact that we do not yet have robust frameworks. We were in 1991 and still are today a relatively young field, especially when compared to many scientific fields. Thus, we need not be alarmed by the lack of frameworks, but it is certainly appropriate to work toward their development.

Moving Forward

In this section of the paper I offer some possible avenues for generating predictive frameworks in mathematics education. I first suggest a few places where we need more frameworks and then offer ways that we might develop them.

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[3] I would note, however, that Andrew Izsák is preparing to conduct a study to determine whether diSessa's knowledge in pieces perspective is useful for studying teacher cognition.

As a field we have developed a body of vocabulary that is taken-as-shared within the mathematics education community, but we lack conceptual definitions of these terms. For example, many of us glibly use terms such as “traditional,” “reform-oriented,” “standards-based,” and the like without much behind them. We also tend to set up false dichotomies using these and other terms.

In a related vein, there is little in the way of a theoretical framework behind long-used ideas in the field, such as Lortie’s apprentice of observation (Lortie, 1975). Many of us have read and taken for granted Lortie’s ideas, but in the 30 years since *Schoolteacher* was published we have not come terribly far in theorizing about apprenticeship of observation.

Another place where we might strive to develop frameworks is in the arena of linking teaching to learning. In this age where there is such a push for highly qualified teachers and for raising student achievement on standardized tests, it is painfully obvious that we have not come far as a field (nor has the more general education field) in theorizing about the links between teacher knowledge and practice and student learning.

As for where to turn to build new frameworks, I see a number of potentially promising directions. One starting place would be to conduct theoretical meta-analyses of studies in a particular subfield. Authors of handbook chapters typically do a sort of meta-analysis of the findings of studies in a subfield, but this same type of analysis of frameworks could be quite instructive.

We might revisit some of the scholars of yesteryear and see what they have to offer the twenty-first century researcher. In addition to people like John Dewey, we might reconsider the work of earlier scholars in mathematics education. For example, Kenneth Henderson’s work on discourse moves (Henderson, 1965) has largely disappeared from the contemporary research scene. While we might be able to improve upon his methods, and we might even be interested in different questions, it is worth revisiting his framework. So often these days I see literature reviews that contain nothing earlier than 1989 as if the birth of the National Council of Teachers of Mathematics *Standards* obliterated the value of all that came before them. While the field has changed rapidly in the last decade or so, there are perennial issues that date to the earliest beginnings of the field, and the work of earlier scholars may illuminate some of the questions we have now.

There is value in following in the footsteps of others and transporting an existing framework to another setting. For example, Izsák used diSessa’s knowledge-in-pieces framework (diSessa, 1988) to study elementary school students’ understanding of whole-number multiplication in the context of area (Izsák, 2005). Not only was Izsák able to use the knowledge-in-pieces idea to make sense of his data, he was also able to add a layer of robustness to diSessa’s framework because he showed that it is useful beyond the domain of physics and with younger students. There is particularly fertile ground here in looking at frameworks that have been used to study mathematical understanding (such as the one by Pirie and Kieran, 1989) to see if they can be transported to the teacher cognition.

Because mathematics education as a discipline is grounded in so many other disciplines (e.g., mathematics, sociology, psychology, anthropology), we might turn more deliberately to those fields in search of fresh directions. Indeed, many of us have turned to these fields for methodologies, so it makes sense to look there for frameworks as well. However, I offer a point of caution here. I urge us to retain a central place for mathematics in our frameworks. This is obvious in the case of frameworks for learning because the content is critical to what is being studied. However, in studies of teaching, teacher education, and equity, in particular, I often find myself reading studies in these areas where I am left wondering what is so special about mathematics. There is most definitely a place for studies of these topics without regard to content, but I also believe there is a place for studies where the content is a prominent element.

Conclusion

Consider one more real-world use of frames–a series of still pictures that make up a moving picture. In our writing and our presentations, we tend to show the moving picture version of our frameworks. As I have noted earlier in this paper, authors often go to great lengths to describe their frameworks in detail, but they rarely provide a glimpse of how the framework was actually used. Thus, we need to slow down the moving picture and show the individual frames that allow others to see the details of how our frameworks were used in data collection and analysis. As we move toward developing more robust frameworks, our field would benefit from more explicit descriptions of the development and implementation of frameworks. If we take the same care in sharing our frameworks with readers as we do with our data, we will provide fodder for others to consider as they develop their own frameworks.

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WORKING GROUP FOR THE COMPLEXITY OF LEARNING TO REASON PROBABILISTICALLY

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There are several critical aims that guide our work together. In particular, members are interested in examining: (1) mathematical and psychological underpinnings that foster or hinder students' development of probabilistic reasoning, (2) the influence of experiments and simulations in the building of ideas by learners, particularly with emerging technology tools, (3) learners' interactions with and reasoning about data-based tasks, representations, models, socially situated arguments and generalizations, (4) the development of reasoning across grades, with learners of different cultures, ages, and social backgrounds, and (5) the interplay of statistical and probabilistic reasoning and the complex role of key concepts such as variability and data distributions. Recent foci in the working group have been to understand: (1) students' and teachers' reasoning when simulating probability experiments with hands-on materials and computer tools, and (2) connections between probability and statistical concepts such as inference and variability. At PME-NA 27 in Roanoke, Virginia, the group will build on discussions and the elaboration of the research agenda that it began at PME-NA 26 in Toronto. To stimulate and extend discussions, members of the group will be invited to show video clips in order to engage the group in data analysis that can inform work on (1) designing tasks that elicit students' probabilistic thinking, (2) understanding how students learn to reason probabilistically, and (3) outlining implications for teaching, learning, and research. Through these discussions, group members will refine previously posed research questions as well as elaborate additional questions. The group will also make preliminary designs for cross-national, collaborative research to be conducted in 2006. The working-group organizers plan to solicit several papers emerging from the cross-national collaborative work of group members that will lead to a set of papers that describe our work.

Nature and Topic of the Working Session

This Working Group was formed at PME-NA 20 (Maher, Speiser, Friel, & Konold, 1998) and has convened annually at PME-NA each of the past six years (see Maher & Speiser, 1999; 2001; 2002; Speiser, 2000; Stohl & Tarr, 2003; Tarr & Stohl, 2004). During the joint meeting of PME-NA 25 and PME 27 in 2003 (Hawaii, USA) and PME-NA 26 in Toronto, Canada, we expanded our working group to include many more international researchers across 12 different countries. Through shared research, rich and engaging conversations, and analysis of instructional tasks, we continually seek to understand how students learn to reason probabilistically. There are several critical aims that guide our work together. In particular, members are interested in examining: (1) mathematical and psychological underpinnings that foster or hinder students' development of probabilistic reasoning, (2) the influence of experiments and simulations in the building of ideas by learners, particularly with emerging technology tools, (3) learners' interactions with and reasoning about data-based tasks, representations, models, socially situated arguments and generalizations, (4) the development of reasoning across grades, with learners of different cultures, ages, and social backgrounds, and (5) the interplay of statistical and probabilistic reasoning and the complex role of key concepts such as variability and data distributions. Recent foci in the working group have been to understand: (1) students' and teachers' reasoning when simulating probability experiments with hands-on materials and computer tools, and (2) connections between probability and statistical concepts such as inference and variability.

Background on Probabilistic Reasoning

The ways in which students reason about the likelihood of an event can be considered in terms of an objective or subjective view of probability (e.g., see Batanero, Henry, & Parzysz, 2005; Borovcnik, Bentz, & Kapadia, 1991). One cannot precisely determine whether a 4 will appear when rolling a regular six-sided die because of the complex physics involved (e.g., the speed and angle at which the die is thrown, the initial spin of the die, air resistance – see Wolfram, 2002). In the presence of this uncertainty, the construct of probability is formed as a theoretical model of the event. In an *objectivist* perspective, probability is viewed as an inherent property of the event and can be well estimated either through a classical or frequentist approach. We can use a classical Laplacean approach to embody the complexities of the physics and apparent (and probably imperfect) symmetry of the die *a priori* tossing the die and express the probability of rolling a 4 as 1/6. This probability of 1/6 is an estimate of the actual theoretical probability that is unknown to us. If one rolls a die a given number of times, a frequentist approach can be used to hypothesize the probability in terms of the theoretical limit of the observed proportion of 4's as the number of trials tends to infinity. But again, this estimate is bounded by real world constraints and can only describe the probability of getting a 4 based on a finite set of die rolls. A repeated finite set of die rolls would most likely yield a different experimental estimate of the actual probability and may in fact allow one to change the estimate of the probability based on new data.

In a *subjectivist* perspective, probability is viewed as a condition of the information known to the individual assigning the probability and not an objective property of the given event. Thus, two people may assign different probabilities to the same event based on different *a priori* information, even after they observe the same empirical data *a posteriori* trials being conducted. For example, one student might recall instances where there were few outcomes of 4 and thus might infer that all outcomes are not equally likely. Another student might believe that all outcomes on a die are equally likely based on their previous experience of not being able to predict the outcome of rolling a die and noticing no distinct pattern in any number being "harder to get." Upon conducting repeated trials and observing that there were relatively fewer 1's and 6's and more 4's, the first student might state that all outcomes are equally likely since this set of data was different than his belief about 4's from his prior experience. The second student may be perturbed by the low number of 6's as compared to her belief that none of the numbers should be "harder to get" and subsequently believe the die is biased.

The *law of large numbers* is used to interpret empirical results in relation to theoretical probabilities and, thus supports the viability that an estimated probability from a frequentist approach will be reasonably close to the theoretical probability. This principle states that the probability of a large difference between the relative frequency of an outcome and the theoretical probability limits to zero as more trials are collected. Even after a large number of trials, it is possible to have a relative frequency substantially different than the theoretical probability.

A frequentist approach to probability, grounded in the law of large numbers, has only recently made its way into curricular aims in schools (Jones, 2005). Teachers are encouraged to use an empirical introduction to probability by allowing students to experience repeated trials of the same event, either with concrete materials or through computer simulations (e.g., Batanero, Henry, & Parzysz, 2005; National Council of Teachers of Mathematics [NCTM], 2000; Parzysz, 2003). In these types of curricula, a theoretical model of probability based on a classical approach is not the starting point. Rather, a theoretical model is constructed based on observing that the relative frequencies of an event from a repeated random experiment stabilize as the number of trials or sets of trials (different samples) increases. However, there is general agreement that research on students' probabilistic reasoning has been lacking sufficient study of students' understanding of the connection between observations from empirical data (probability in reality) and a theoretical model of probability (e.g., Jones, 2005; Parzysz, 2003).

Guiding Framework for Our Discussions

As a way of framing the discussions for the working group, we are utilizing a conceptual framework developed by Lee, Rider, and Tarr (2005) which was extended from earlier work by Stohl and Tarr (2002). This framework can provide the members of the working group a common starting place and a way of talking about students' probabilistic reasoning. In no way is the group restricted to using this framework, and in fact we hope modifications, extensions, and improvements will emerge out of group discussions and subsequent research efforts.

Considering the bi-directional model shown in Figure 1, students may start from the theoretical side of the model and begin to reason using an image of the theoretical probability of each event developed from either an objective classical view or a more subjective view based on their experiences and knowledge. Their initial assumptions of the probability provide an image of what students expect to observe in empirical data. They may then compare the results (e.g., frequencies or relative frequencies of an event) against their mental image and initial hypothesis about the probability of that event (Watson & Kelly, 2004). Noticing patterns in the data may make them call into question the prior assumptions, or they may not believe the data varies enough from their mental image to contradict their initial assumption. Their reasoning may then lead them to decide to collect and analyze more data to again test the reasonableness of the match between their mental image of the hypothesized theoretical probability and results from repeated empirical trials.

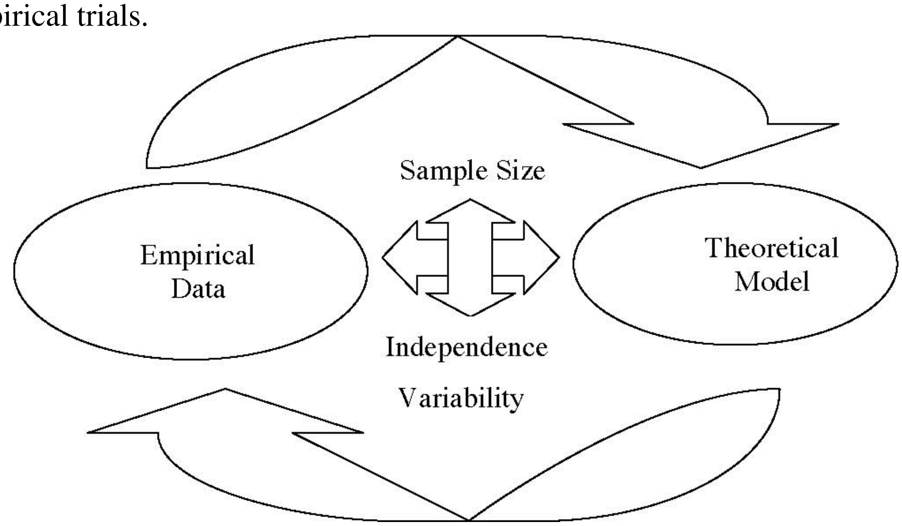

Figure 1. Bi-directional model

Starting from the empirical side of the model, students may reason about the probability of an event where they have no prior experience with the phenomena or can not use a classical approach (e.g., how likely is it that a tack will land on its side when dropped on the floor). Thus, they may start by examining empirical data and using the relative frequencies from that data to inform a hypothesis regarding the underlying theoretical probabilities. The first hypothesis about theoretical probability allows students to form a mental image of the expected results in future data. Students may then use their mental image to inform how (or whether) to collect empirical data about the phenomena.

The robustness of students' reasoning from empirical data back to their initial assumption of the theoretical probability is influenced by the sample size, understanding the independence of trials, and variability of their data. Students need to consider that different trials and different sets of trials (samples) are independent of one another and variability among individual trials and samples is to be expected. They also need to coordinate conceptions of independence and variability with the role of sample size in the design of data collection and interpretation of results. Relative frequencies from larger samples are likely to be more representative of the theoretical probabilities while smaller samples may offer more variability and be less representative. For example, ten rolls may yield no 3's and such data may support a child's notion that rolling a 3 is an improbable (or even impossible) event, although such a claim would likely not be made by someone who had made a more robust connection between relative frequency in empirical data and theoretical probability, and the importance of sample size.

Summary of Activities from 2004

Seventeen researchers (faculty and graduate students) from the United States, Canada, Mexico, and Israel met During PME-NA 26 in Toronto. After analyzing a video of students' work on a computer-based simulation task (Schoolopoly task, see Stohl & Tarr, 2002; Lee, Rider, & Tarr, 2005), the group discussed the different aspects of students' thinking when they are trying to generate and analyze empirical data to make inferences about an unknown probability distribution. This discussion led to different participants expressing interest in conducting various pilot research studies during Spring 2005. Some of the ideas for follow-up research included:

- What are the longitudinal benefits and effects of students' engagement in using simulation techniques to approach probability and statistics tasks— particularly following students from middle school through Advanced Placement (AP) Statistics?
- How do students who are at the end of an Advanced Placement (AP) Statistics course and have been traditionally taught with an emphasis on theoretical statistics and probability apply their understandings to a task like Schoolopoly where students must generate and analyze data to make inferences about an unknown probability distribution?
- How do students' reasoning about the design and results of probability simulations differ when they use hands-on tools (e.g., coins, dice, spinners, bags of marbles) and computer-based tools?
- What role does students' agency play in their understanding of probability concepts when given an open-ended tool like Probability Explorer to design experiments, generate data, and analyze results to make inferences about unknown distributions?

Planned Activities for 2005 Meeting

At PME-NA 27 in Roanoke, Virginia, the group will build on discussions and the elaboration of the research agenda that it began at PME-NA 26 in Toronto. To stimulate and extend discussions, members of the group will be invited to show video clips in order to engage the group in data analysis that can inform work on (1) designing tasks that elicit students' probabilistic thinking, (2) understanding how students learn to reason probabilistically, and (3) outlining implications for teaching, learning, and research. Through these discussions, group members will refine previously posed research questions as well as elaborate additional questions. The group will also make preliminary designs for cross-national, collaborative research to be conducted in 2006. The working-group organizers plan to solicit several papers emerging from the cross-national collaborative work of group members that will lead to a set of papers that describe our work. These papers could be part of a monograph, journal special issue, and many joint presentations at future conferences.

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WORKING GROUP ON GENDER AND MATHEMATICS: RECONCEPTUALIZING DIRECTIONS

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The Gender and Mathematics Working Group has been an active participant of PME-NA since 1998. This working group's history, in brief, is included in this proceedings paper. The most recent work of the group has included a monograph project, now in review, followed by a self-analysis of our work that has brought us to discussion and investigation of new topics. Those topics include: 1) Investigating research and teaching paradigms that develop new understandings of the relationship between gender and mathematics education; 2) Questioning the nature of school mathematics; 3) Problemetizing a (re)definition of the field of gender and mathematics; and 4) Establishing connections across technology, gender, and mathematics. These topics now frame the sessions for the Gender and Mathematics Working Group for the 1005 sessions. Work on the intraconnection of gender, mathematics and technology, on international studies, and implications of the research in application in the mathematics classroom are some of the specific topics under discussion.

Introduction

In this year's PME-NA XXVII meeting in Roanoke, Virginia, members of the Gender and Mathematics Working Group (GMWG) plan to examine the new directions initiated by the group at PME-NA XXVI in Toronto. At that session, the group began to investigate, question, problematize, and establish connections among research and pedagogy paradigms, the nature of mathematics, the intersection of gender, mathematics, and technology, and how critical theory can inform our work. Since the time of those Toronto sessions, members of our working group have presented work that emerged from the GMWG sessions. In this paper, I review the history of the GMWG and then outline some of the work of group members since the Toronto sessions in the section entitled "In the Interim – Work Between Sessions." Finally, in the section entitled "The Gender and Mathematics Working Group and Its Relationship to PME-NA" I describe the relationship of our work to the PME-NA goals and to previous Gender and Mathematics Working Group endeavors; and I discuss plans for the 2005 meeting of the Gender and Mathematics Working Group.

History of the PME-NA Gender and Mathematics Working Group

The Gender and Mathematics Working Group has been meeting annually at PME-NA since 1998 (Raleigh, NC). At that time of our first meeting, the work of the group began with reviews of gender and mathematics scholarship, and sought to identify absences from the research strands reviewed. Committing to an integration of our collective scholarship on gender and mathematics, we defined future directions for research and for the working group. An early result was a visual representation, a graphic, of our conception of the field of gender and mathematics, and the complexity of the elements with(in) which we work (Damarin & Erchick, 1999; Erchick, Condron & Appelbaum, 2000).

Lloyd, G. M., Wilson, M., Wilkins, J. L. M., & Behm, S. L. (Eds.). (2005). *Proceedings of the 27th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.*

After the first meeting of the Gender and Mathematics Working Group, we continued to gather together at each PME-NA meeting, sharing our scholarship on gender and mathematics, redefining our direction and purpose, seeking feedback from the membership at large in PME-NA discussion groups and fine-tuning the focus of our work. Forming peer groups of individuals with common interests and related research efforts, we reviewed, critiqued, and discussed the body of scholarship we were engaged in, including research into both theory and practice.

A guiding project of the working group was the creation of a gender and mathematics monograph. We began, as a working group, in a conversation about the absences in the scholarship on gender and mathematics. As we have continued that conversation and pursued further research on gender and mathematics and the absences, we have been aware of the complexity and non-linearity of the issues we seek to investigate and understand. In our scholarly interpretations, we are committed to a respectful regard for the voices and reflections of all the women and girls who participate in this research: researchers, teachers, students.

In working group sessions we developed a structure for the monograph, selecting themes around which to organize its contents. These include multiple perspectives of researcher, teacher, and student; history, critical theory, and feminism; and methodological, self-reflective, and empirical standpoints. The monograph includes the writings of eight scholars. We await publisher responses.

At the 2004 PME-NA XXVI sessions in Toronto, the Gender and Mathematics Working Group members began moving our work into new spaces. In these sessions we explored ways in which we can more deeply examine the relationship between gender and mathematics in our work, and did so with reflection upon international perspectives and critical theory, connected work in gender and technology, and critical perspectives on pervasive, recurring questions about the place for gender work in mathematics education (Erchick, Applebaum, Becker, & Damarin, 2004).

In the Interim – Work Between Sessions

Shortly after the 2004 GMWG sessions at PME-NA in Toronto, some of the continuing members of the group completed and submitted for review a monograph on gender and mathematics. Eight authorsm working in various combinations, contributed 13 papers for the overall manuscript, with the work being sorted into three sections of the monograph: Setting the/Our Frame, Empirical Work, and Reframing Toward the Future. For those of us participating in the Gender and Mathematics Working Group and the monograph project coming out of it, our work on the monograph verifies for us that we are a grassroots effort, as Peter Appelbaum explained in collaborations with the group, grounded in the forefront of mathematics education, with both a political and academic agenda, with members as activists and scholars, researchers and practitioners.

Several members of the GMWG presented at the National Council of Teachers of Mathematics National Conference in Anaheim in April, 2005, both at the Research Presession and during the full NCTM conference. That work, too, is directly a consequence of the PME-NA Gender and Mathematics Working group scholarship. At the NCTM Research Presession Peter Appelbaum, Suzanne Damarin, and Olof Steinthorsdottir presented their work with Diana Erchick serving as discussant for that symposium.

In his paper, Peter Appelbaum (2005) identified some of the work done by the GMWG, in particular, critical questions on the identification of gender as a "problem" in mathematics and mathematics education. His argument was grounded in the development of the GMWG scholarship which he described as moving from a study of gender and mathematics to a study of women and their mathematics experiences, interpreted through a feminist standpoint lens.

As discussant, Diana Erchick (2005a) identified Peter's work as a foundation for the whole of the body of the work presented in the session, work that assures us that the field is built upon a wealth of scholarship around gender and mathematics, albeit at times with conflicting approaches and findings. Additionally, Erchick recognized that we have come to a time where a scholarly analysis *across* the field is needed in order to appreciate the complexity and value of the studies, especially in terms of how they together make a field of study. She also identified a need for us to critique the needs of the field more comprehensively, not strictly in terms of absences that emerge in 1998, but also in terms of re-defining and re-orienting the field based on that analysis.

The work presented by Olof Steinthorsdottir (2005) and Suzanne Damarin (2005) were excellent examples of the potential of the kind of comprehensive analysis Erchick acknowledged. As Olof spoke to gender and mathematics work presented at the International Congress of Mathematics Education 10 (ICME) and the Programme for International Student Assessment (PISA), she brought together multiple findings from international perspectives. She raised questions about those findings, findings that suggest a "problem" with gender and mathematics, but not always what one would think. She brought us back to gender as a problem and indeed the very problems of identifying gender as a problem.

Suzanne Damarin's work spoke beautifully not only to the content of her inquiry – i. e. the intra-action of gender, mathematics and technology – but also to the rigor, and the value of that rigor, in a thorough, organized methodology. A scholarly, comprehensive inquiry as we see in Suzanne's work reveals not only complexity and absences, but the emergence of a new scholarly agenda, fueled by critique and the search not for more "problems" with gender and mathematics, but more questioned, more self-critique, and more re-defining.

Also at the NCTM conference, Diana Erchick (2005b) co-presented a session on classroom applications of the GMWG research findings. Participants at the session were able to experience activities intended to support middle grades girls as they develop relationships with mathematics. Participants also learned about the ways in which the research on gender and mathematics informed the pedagogy of the activities – research that emerged from the GMWG and our monograph project.

Finally, the group has a website developed about a year ago, that now has a GMWG logo and has recently been updated http://www.newark.osu.edu/derchick/pmena.htm. An effort to use the site more fully to keep the members connected is in progress, as is an effort to make accessible the individual work of members of the group. Collectively, the on-going scholarly initiatives of the GMWG and its members - from research to monograph, from discussion at PME-NA to distribution through presentation at other conferences, and in the utilization of technology to make more accessible the work of all of our members - serves to institutionalize and publicize our work, taking the grassroots, political, activist agenda, into the mainstream of the academic environment in which we work.

The Gender and Mathematics Working Group and Its Relationship to PME-NA

Our Work and the Goals of PME-NA

Since its inception, the GMWG has had a goal of impacting classroom practice in positive ways. This goal is directly related to the PME goals to further a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implications thereof;

and to promote and stimulate interdisciplinary research, with the cooperation of psychologists, mathematicians, and mathematics teachers. Also, technology-related issues are embedded in our work. Technology is an increasingly present and important component of the mathematics classroom. Thus, research on gender and technology informs and contributes to our work in exciting and meaningful ways.

We also are committed to another goal of PME, that of promoting international contact and exchange of scientific information in the psychology of mathematics education. In terms of further and broadening growth and exchange of ideas, inclusion of international perspectives of gender and mathematics is crucial, and participation of international colleagues is not only welcome, but essential. We address this goal in this year's GMWG agenda with the expectation of developing collegial relationships and integrating diverse perspectives into our research agendas.

Our work is also further connected to the PMENA XXVII conference theme of "Frameworks that Support Research and Learning." That our research is steeped in classroom practice, both as a site of study and as a site in which to apply results, as well as emerging as theory, contributes to the development and explication of frameworks that support research and learning. Additionally, the commitment of the Gender and Mathematics Working Group to broadening the scope of its perspectives to include international experiences contributes to the construction of frameworks supporting research and learning.

Plan for Active Engagement of Participants

As has always been the case with our Gender and Mathematics Working Group, the sessions we conduct this year are intended to be active with discussion, decision-making, and work activities. As a group we remain committed to an initiative that depends upon participant voices for direction and support. In this year'' sessions, we begin with introductions and a short synthesis of the work to date, as well as updates on current projects and recent presentations of GMWG participants.

One of two major components of the working group sessions this year is an introduction of the topics that emerged from last year's work (listed below) and whole group sharing and discussion of on-going work on the topics. Within that discussion some of the members will hare on-going work. Suzanne Damarin will share her work on the intraconnection of gender, mathematics and technology; Olof Steinthorsdottir will discuss her work on the international perspectives; and Diana Erchick will bring work on taking the research into the classroom.

The second major component of the working sessions is discussion on, and decisions about individual participants' and the whole group's commitment to moving forward on this agenda.

Topics Grounding This Year's Work

The following topics from the GMWP 2004 sessions grounding this year's work:

- Investigating research and teaching paradigms that develop new understandings of the relationship between gender and mathematics education.
- Questioning the nature of school mathematics.
- Problemetizing a (re)definition of the field of gender and mathematics.
- Establishing connections across technology, gender, and mathematics.

As mentioned earlier in this paper, discussion around the on-going work on these topics will be followed by work sessions organized around the topics. These working subgroups will study, plan for independent work for the coming year, and determine additional work session activity, both electronically and through other professional organization meetings such as IGPME, NCTM, and AERA.

When Suzanne Damarin and Diana Erchick started this project in 1998, an early result of the working group sessions was a graphic, cited above, that revealed two conceptions determined by the scholars working within the group. One determination of the group was that the structure of our examination of the scholarly work of gender and mathematics was nonlinear and very complex. The other determination of the group was that there were absences in the field of study, and it would be part of our mission as members of the working group to pursue scholarly inquiry in directions that would begin to contribute to the field in the areas of those absences. Our monograph project, currently under review, satisfies a part of that agenda. However, reflection upon that project now reveals more absences, all of which are foundational to our exploration of topics in this year's sessions.

Closing

In pursuing inquiry around Gender and Mathematics, the PME-NA Gender and Mathematics Working Group participants have committed themselves to an interpretation of the field of gender and mathematics as complex and nonlinear. We have also chosen to investigate the absences we encounter with a respect for the reflective voices of the researchers, teachers, students, women and girls who contribute to the work. In the papers and processes of this project, we work consistently to respect the structure and voices that emerge. Original absences apparent in 1998 have grounded our work since then. Newly apparent absences now ground our new directions, and our commitment to addressing absences in the field continues.

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EMERGING AGENDAS AND RESEARCH DIRECTIONS ON MATHEMATICS GRADUATE STUDENT TEACHING ASSISTANTS' BELIEFS, BACKGROUNDS, KNOWLEDGE, AND PROFESSIONAL DEVELOPMENT: WORKING GROUP REPORT

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Teaching assistants (TAs) play vital roles in the mathematics education of undergraduates and may go on to become professors of mathematics. From the K-12 literature, it is clear that patterns of teaching practice, as well as beliefs about teaching and learning, form early in teachers' careers. Here we document an emerging body of scholarly inquiry into the TA experience and the professional development needs of TAs. The working group exists to foster collaboration between K-12 and undergraduate mathematics educators in framing and carrying out this research. Meeting time will be devoted to discussion of participants' research projects at various stages of development. Participants will provide feedback on research in the planning, data collections, data analysis, and reporting stages. These discussions will serve as the basis for the group's goals of building a community of researchers interested in TA issues, the analysis of similarities and differences with K-12 mathematics education, and the development of an agenda for continued work.

Introduction

Mathematics graduate student teaching assistants' (TAs) professional lives and development represent an area of growing research interest within the mathematics education community. As summarized in Speer, Gutmann, & Murphy (2005), TAs provide the lion's share of teaching contact hours for undergraduate mathematics students and go on to become faculty members teaching mathematics. Thus, TAs' importance as current and future educators and the importance of providing informed professional development opportunities for them cannot be denied.

Researchers have begun to inquire into various aspects of the TA experience from several theoretical perspectives. Some, working within socio-cultural traditions, are examining characteristics and the nature of identities of beginning TAs. Others seek to understand the structure and features of the communities in which TAs participate. Taking more cognitive approaches, others are investigating TAs' knowledge and beliefs, particularly those related to student thinking. Another area of current activity is curriculum development for TA professional development (PD) and the adaptation of PD materials and programs from K-12 contexts for use with mathematics graduate student TAs.

In addition to continued work on the research and development programs described above, the TA researchers are furthering their goals by expanding on traditions from K-12 research and pursing new methodologies. For example, development of traditional PD is now being augmented with videocases, opening up new issues for design and research on TA development.

Studies of knowledge are being extended to include investigations of how TAs acquire the pedagogical content knowledge necessary for teaching. Studies of teaching are now being augmented with studies of how TAs plan for teaching and how knowledge and beliefs shape the decisions TAs make while planning their classes. The agenda of understanding the TA

Lloyd, G. M., Wilson, M., Wilkins, J. L. M., & Behm, S. L. (Eds.). (2005). *Proceedings of the 27th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.*

experience is being extended to include more detailed examinations of the challenges faced by first-year TAs as well as more extensive inquiry into the complexity of the context in which TAs work. In addition, while the TAs in most researchers' studies are teaching or preparing to teach calculus, recent contributions take the work into the arena of statistics education.

Against this backdrop of developing interest, the Mathematics Teaching Assistant Preparation Working Group has attempted to fulfill three main goals: (a) to help mathematics educators interested in the TA experience and TA professional development to connect and collaborate; (b) to provide critical, informed support and feedback for researchers considering TAs; and (c) to organize a research agenda of relevant, common concerns. Here we summarize the work of the group to date and present a list of five central issues that constitute the research agenda as determined during the 2004 meeting. Further, this paper includes summaries of ongoing projects from several contributing authors to be discussed in detail at the 2005 meeting.

The working group met during two PME-NA conferences. In 2002, time was divided between two activities. First, participants shared backgrounds and interests in TA issues. Second, participants discussed issues and potential research directions. In addition to furthering community development by engaging in substantive discussion, these activities provided organizers with insight into participants' areas of interest. The discussion also began our efforts to identify key research issues and to form a research agenda to which all group participants can contribute. At the 2004 meeting, time was devoted to individual project presentations and whole-group discussion of cross-cutting issues. Individual presentations were "working sessions" where participants presented plans for research or artifacts from research-in-progress. Participants received feedback on plans, data collection instruments, theoretical approaches, and data analysis methods. Discussions focused on assessment of projects as contributions to the field and consideration of how projects might be advanced.

Issues in the Psychology of Mathematics Education for the Discussion Group

Broadly speaking, the group's work concentrates on issues of teacher development and practices. More specifically, research centers on mathematics TAs and factors that shape their teaching and their learning to teach. The group's work has a broad focus in the psychology and sociology of mathematics education, from a variety of theoretical and methodological perspectives. Rather than concentrating on a single issue or a particular perspective, the group exists to serve the needs of its members and to provide a forum for discussion and collaboration on research from their varied perspectives. One of the developing aims of the group, however, is to generate and pursue a coherent research agenda building on existing TA research as well as connecting to K-12 educational research.

The following five points summarize the 2004 group's progress toward defining a research agenda:

1. In creating professional development (PD) experiences for TAs, we need answers to questions related to (a) the nature of TAs' thinking about teaching and learning, and (b) how TAs process and learn from PD experiences and curriculum materials. What experiences and materials do they need and want?
2. We should develop baseline information about how TAs work and learn as part of a community and what motivates them. We need to understand the norms related to valuing teaching and studying mathematics and how these norms are communicated. In doing so, our theories must address TAs' backgrounds and their long- and short-term goals. We should not assume all TAs are Ph.D.-bound. Rather, we must also develop models to

explain the cultural implications of being a transient TA—a TA planning to become something other than a research mathematician.

3. Research should exploit the discipline-specific nature of being a mathematics TA. While many universities have PD programs that assume teaching is teaching, whatever the discipline, we should emphasize ways in which this is not true for mathematics TAs and incorporate ideas specific to the learning of mathematics. How are mathematics TAs challenges and needs different from those of TAs in other fields?
4. Our work will be done with, for, and in support of faculty and TAs in mathematics departments. As such we have a special responsibility to (a) meet their needs and (b) set our work solidly within psychological and epistemological frameworks that guide mathematics education. Doing so, we must be mindful of how the two communities define *validity*. Results must be presented in frameworks acceptable and useful to both client communities.
5. Existing mathematics education research related to preservice and in-service teachers' thinking and practices is rich. In working with TAs, an important task is to consider what pre-K-12 teacher research has to tell us about TAs and to consider what this research does not address.

Current Working Group Projects

Members of the working group have contributed synopses of seven projects to be discussed during the group meeting time. These projects fall into four categories, with some in multiple categories: creation of curriculum material for professional development (Hauk et al, Noll); examinations of TAs' knowledge and/or beliefs about student thinking (Kung, Noll, Speer et al); investigations of TAs' planning practices (Winter, Speer et al); and study of characteristics of TAs and their adjustments to challenges encountered in their teaching (Meel, Belnap).

During the group meeting each researcher will share the project as described below and solicit feedback. For each project, the researcher(s) describes the work, indicates what "stage" of development the project is in (planning, data analysis, reporting, etc.), and sets out how they intend to structure their portion of time during the working group meeting. Projects in earlier stages of development (e.g., planning) are described first, followed by those in later stages.

Video Cases for Novice College Mathematics Teacher Development

Shandy Hauk, David T. Kung, Nikita Patterson, Angelo Segalla, & Natasha Speer

This project addresses two major challenges facing undergraduate science, technology, engineering, and mathematics education: building college students' understanding of mathematics and enhancing the teaching efficacy of new college faculty. The proposed work folds together cognitive and psychological theories on mathematics learning and teaching with lessons learned from successful K-16 practice to encourage growth among undergraduates in mathematics service[1] courses and sustainable professional development among the graduate students who teach them.

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[1] "Service courses", (representing 85% of mathematics course enrollment nationally (NCES, 1999), are mathematics courses taught to non-mathematics majors. Usually referred to students as "the last math course I'll ever take", they include prospective elementary school teacher content courses, college algebra, finite mathematics, elementary statistics, business calculus, and other courses that satisfy general education/breadth requirements.

As the undergraduate population grows more diverse, so do the graduate student and faculty populations (NCES, 1999). This project, grounded in attention to acculturative issues, supports the expansion of a diverse professoriate. One strand of basic research will center on the evolution of novice college mathematics instructors' cultural repertoires and resolution of cultural dissonance. Additionally, applied research, via teaching experiment, will aim to improve college mathematics teaching and learning through teacher-scholar development of participating TAs and through self-regulatory development among TAs and their students. The research strands will inform the design and use of video-case materials.

The FIPSE-funded Boston College Case Study (BCCS) Project produced a book of 14 fictionalized written accounts of college mathematics teaching interactions (Friedberg, 2001). Building on this project, and the proven efficacy of case use in K-12 teacher preparation, the project's curricular goal is to create a collection of video cases from college mathematics classes. Video-cases will be chosen for their power to illuminate or stimulate reflection and discussion. Accompanying materials will include notes on case use for TA trainers, problem sets, writing and grading rubrics, comments by and for TAs, and an independent reflective learning guide to facilitate distance course use. Field-testing of case materials will inform an annual reflective cycle of development, field-testing, evaluation, research revision and reimplementation.

The DVD created will be similar to the Integrating Mathematics and Pedagogy (IMAP) Project materials for K-12 teachers (Phillips & Cabral, 2005). Differences between IMAP and the proposed cases are: (a) the video-case tools and accompanying text will be for an audience with mastery of mathematics but little or no formal training in pedagogy; (b) case tools organization will allow use in distance-learning; (c) materials will include in-class video-clips and video vignettes and/or textual materials about out-of-classroom interactions such as office hours, email communication, undergraduate and graduate student advising, communicating with junior and senior colleagues about teaching, and interview clips with TAs.

Status. The research strands of the project are still in development. To strengthen a grant proposal to the NSF for a three-year project combining research and curriculum development, pilot video-case materials are being collected and a pilot DVD interface created in Summer 2005. Some field-test agreements are in place, as are initial publication agreements with the Conference Board for the Mathematical Sciences and the American Mathematical Society.

Working group plans. The grant proposal associated with the project will be developed and available for comment by the working group members. In addition to feedback on the grant proposal, the working group can provide support for this project in several ways:

- Viewing and commenting on pilot video-clips and DVD user-interface.
- Reviewing, discussing, and offering suggestions for pilot textual materials.
- As a potential source for volunteers to join the project as researchers, evaluators, video-case data generators, and/or field-testers for materials.

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Teaching Assistants Learning How Students Think

David Kung

The goal of this project is to understand the process by which TAs gain knowledge of student thinking about calculus and how they use that knowledge in the course of teaching. We take a cognitive perspective shared by much of the work on teacher cognition and pedagogical content knowledge (Borko & Putnam, 1996; Shulman, 1986). In particular, we assume instructors' teaching decisions are influenced by their understanding of student thinking. Improving teacher knowledge of student thinking has proved to be a powerful tool of professional development (PD) at the elementary level (Fennema, Carpenter, Franke, Levi, Jacobs, & Empson, 1996; Cobb, Wood, & Yackel, 1990). We hypothesize that the same holds true at the college level, and that improvements can be made in college calculus teaching by finding ways to improve TAs knowledge of student thinking. We see this project as laying the groundwork for research-based PD materials for TAs.

In trying to better understand the process of TAs learning about student thinking, we focus our work around three main questions:

1. How do calculus TAs' various experiences contribute to their learning about student thinking?
2. What types of knowledge of student thinking are gained and through what types of activities?
3. How do calculus TAs use their knowledge of student thinking in their day-to-dayteaching activities?

Status. In an initial study, eight former Emerging Scholars TAs were interviewed about their knowledge of student thinking and how they gained that knowledge. Those interviews indicated that different activities lead to different types of knowledge. For instance, observing students working on problems provides very fine-grained information about their thought processes (including their misconceptions and solution strategies), but grading exams and homework led to knowledge of what students perceived to be the correct answers – knowledge not available in the process of observing students.

This work has been submitted for publication, but several parts of the main questions remain unanswered. To what extent were TAs' recollections an accurate portrayal of their actual learning process? Was their knowledge gained through a few specific incidents, or was it gained more gradually through years of teaching? What types of knowledge of student thinking do TAs already possess when they enter graduate school? The question of how TAs use their knowledge of student thinking in the course of teaching remains completely unanswered.

Working group plans. Several people are working to provide insights into the questions posed above. This working group will allow us to coordinate our efforts more fully. This might take the form of simply informing our research more fully or even sharing instruments and assessment tools. In addition, I would like feedback and assistance about planning the next step in this research program. What instruments are needed to detail the learning process TAs go through *while* it is happening? What types of studies could determine how calculus teachers are using their knowledge of student thinking while they teach?

Using a CGI Professional Development Framework for Improving Statistics TAs' Pedagogical Content Knowledge

Jennifer Noll

Probability and statistics education and the promotion of statistical literacy have received increased attention in the mathematics education community in recent years (National Council of Teachers of Mathematics, 2000). Undoubtedly the increased use of statistics and graphical displays of data in today's media is one of the driving forces behind the mathematics education community's concern with the teaching and learning of probability and statistics. Furthermore, at the college level more and more undergraduates are being required to take introductory statistics in their degree programs. In fact, enrollment in elementary statistics courses (non-calculus based) at four-year colleges and universities rose 18% from fall 1995 to fall 2000 and by 45% from 1990 levels (Luzter, Maxwell, & Rodi, 2000). At many universities, TAs teach the bulk of the introductory statistics courses or teach recitation sections for large lecture classes. Thus, TAs have the potential to play a vital role in undergraduate statistics education and the promotion of statistical literacy among college students.

Unfortunately, many colleges and universities lack professional development (PD) opportunities for TAs. Beginning TAs typically participate in orientation programs, where the focus is to help them become acquainted with the university, fill out paper work and provide them with general rules of thumb in the classroom (Speer, Gutmann, & Murphy, 2005). However, PD is and should be different than orientation programs. Whereas orientation programs provide students survival skills, PD should provide opportunities for TAs to examine and discuss course content, teaching practices, and theories of learning and teaching before and during their first teaching assignments.

Misconceptions in reasoning about probability and statistics are common even for those with considerable statistical training (Kahneman & Tversky, 1972; Konold, et al., 1993; Tversky & Kahneman, 1971). Thus, because PD opportunities for TAs are lacking and TAs are susceptible to these common misconceptions, an understanding of how TAs think about statistics, and their beliefs about how students learn statistics is badly needed. Additionally, PD programs are needed to create opportunities for TAs to examine, discuss, and reflect on their own understanding of statistics, how students come to learn statistics, and practices for teaching statistics.

The overarching goal of my project is to broaden the developing base of research concerning graduate teaching assistants by initiating research on the statistical knowledge of TAs. I plan to investigate the impact of a PD program on TAs' knowledge of statistics, their beliefs on the nature of statistics, and their beliefs on teaching statistics to undergraduates. In particular, the following research questions will be investigated:

If TAs participate in a PD course that focuses on research-based studies on how students learn and think about statistics:

1. Will TAs' beliefs about teaching statistics and their role as teacher change?
2. Will TAs' understandings of the role of statistics in undergraduate education change?
3. Will TAs' pedagogical content knowledge of statistics change/grow?
4. Will TAs who participate hold a different view of statistics (what it is and why it is important) than TAs who do not participate?

Using elements of the Cognitively Guided Instruction (CGI) framework I plan to develop a PD course for TAs grounded in research on students' thinking in statistics in three content domains: center and variation, bivariate relationships, and sampling and sampling distributions.

The PD course will provide TAs the opportunity to reflect on their own understandings of these concepts, learn how students understand these concepts, and reflect on methods for teaching these concepts.

Working group plans. During the working group session I would like feedback on (1) instruments for measuring teaching assistants' beliefs about the teaching and learning of statistics (2) refining and narrowing my content domains (3) ideas for developing PD activities centered around my three content domains, and (4) refining my research goals.

Influences of College Mathematics Teachers' Knowledge and Beliefs about Student Understanding on their Plans for Instruction

Natasha Speer, Sharon Strickland, & Nicole Johnson

The goal of this project is to use findings from K-12 research in the design, implementation, and research of professional development (PD) for TAs. Elements of PD programs with proven results at the K-12 level, such as Cognitively Guided Instruction (CGI), will be adapted for use with TAs. CGI approaches PD by emphasizing development of teachers' knowledge and beliefs related to student understanding for particular mathematics concepts. The PD activities in our project will create opportunities for TAs to learn about student understanding of college mathematics concepts, focusing on limit, derivative, and function.

We are currently in the first phase, focusing on the development and refinement of data collection methods and materials. The long-term research goals center on understanding and improving the development of mathematics TAs' teaching practices and examining how development of such practices shape the learning opportunities of students. Hence, we aim to develop methods permitting us to coordinate the data we gather on TA learning, TA instructional practices, and student learning opportunities.

The objective of the development of these methods is to enable investigation of the following research questions: (1) What knowledge and beliefs do TAs possess and how do those factors shape their teaching practices, particularly their planning, instructing, and reflecting? (2) How do TAs engage with PD activities and, as a result, are TAs able to learn about student understanding of mathematics concepts? Does this learning change TAs' planning, instructing, and reflecting practices? (3) As TAs attend more to issues of student understanding, how is that reflected in students' learning opportunities?

Status. We have conducted pilot interviews with TAs to document their knowledge and beliefs related to student understanding of our focal concepts by engaging them in discussions of related tasks. These interviews probe TAs' general understandings of the concepts, their solutions to the tasks, their planning for a lesson or lessons to introduce students to these concepts, and their knowledge of student understanding related to the concepts and tasks. In addition to these task-based interviews, we also interviewed TAs as they planned for teaching an upcoming class, observed that lesson, and conducted a post-teaching videoclip-based interview.

Working group plans. During the working group session, in addition to providing more details on our methods, analysis, and preliminary data, we will focus on the data collected during the planning portions of the interviews. We are particularly interested in identifying aspects of lesson planning and teaching where TAs decisions appear to be based on their knowledge and beliefs about student learning. We are seeking feedback on how to modify our interview procedures to obtain richer data on TAs' use of their knowledge of student learning while planning lessons.

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Lesson Planning Practices of Graduate Student Instructors in Mathematics

Dale Winter

The mathematics department at the University of Michigan is well known nationally for its large-scale, innovative introductory mathematics courses (Brown, 1996) and for the student-centered style of teaching encouraged in these courses (DeLong and Winter, 1998). The current project seeks to understand the lesson planning practices of TAs as they learn to function within this instructional environment.

The decision to focus on planning is rooted in research conducted with K-12 teachers suggesting improvements in planning can lead to enhanced classroom learning environments and improved student learning (Zahorik, 1970). In fact, some research into teacher decision-making (conducted with high school teachers) suggests most of the decisions instructors make to substantially affect the quality of the classroom environment are made during the "pre-active, planning phase" (Bush, 1983). Several research studies on beginning teachers (including K-12 teachers and novice graduate student instructors) have noted that novice teachers do not always implement innovative courses and pedagogies optimally (DeLong & Winter, 1998; LaBerge & Sons, 1999), and studies of teacher planning at the K-12 level have linked teachers' difficulties with innovative curricula to their planning processes (Yinger, 1980; Zahorik, 1970).

Finally, the Ph.D. program in mathematics at the University of Michigan typically graduates between 20 and 30 students per year. Of these, more than 80% accept an academic position with a substantial undergraduate teaching component as their first appointment. What seems clear from these statistics is that the instructional practices of TAs at an institution like the University of Michigan are also the instructional practices of individuals for whom undergraduate teaching will be a major, lifelong occupation (Speer, Gutmann, & Murphy, 2005).

This project is being conducted using clinical interviews of novice and expert TAs, and content analysis of the interview transcripts. The principal research questions that the project will attempt to explore are as follows.

1. What do novice TAs in mathematics actually do when they prepare for their lessons?
2. Are there any typical planning procedures that TAs in mathematics follow when preparing their lessons? If so, what are they?
3. What sources of information and resources do TAs use when they prepare for their lessons?
4. Do TAs consider assessment of student learning as a part of their planning process?

Status. We have conducted seven pilot interviews with beginning TAs. In this pilot study, the TAs were each interviewed once. The current study is in the data collection stage. In this larger study, fifteen TAs have been included and each TA will be interviewed three times during the course of the semester.

Working group plans. During the working group session, I will present the interview questions that my group has been working with, along with some of our pilot data and preliminary plans for data analysis. I am seeking feedback on how to integrate the responses from individual TAs over the course of a semester. For example, what forms of additional data could prove helpful in trying to distinguish between genuine shifts in TAs' approaches to lesson planning and habitutation to the questions asked during the interviews? I am also seeking participants' thoughts on our plans for analysis and suggestions for additional approaches for examining our data.

Acknowledgement. The research undertaken in this project has been supported in part by the Office of the Provost and the Office of the Associate Dean for Undergraduate and Graduate Education at the University of Michigan.

Exploring First-Year TA Experiences Through Weekly Reflective Writing Assignments

David E. Meel

TAs hold an important role in undergraduate student development. Bender (2004) stated, "Whether they want to be or not, TAs are important role models for undergraduates and often serve as influential mentors for the students in their classes. TAs make a significant difference in the lives of undergraduates. An enthusiastic and committed graduate student can help to transform an undergraduate student not only into a major in the field but also into a potential graduate student. The reverse is also true: when graduate students fail in their teaching duties, undergraduate learning suffers. A disorganized, ill-prepared, and ineffective classroom instructor can undermine the hopes of even the most dedicated undergraduate to pursue the discipline in future semesters" (p. 267). Gaining an understanding of issues that novice TAs face when entering an undergraduate mathematics classroom for the first time is essential to developing ways of enhancing development programs to help them anticipate their role as a future faculty.

One possible way of gathering information on the struggles TAs face is to engage the TAs in journal writing activities. Not only can journal writing elicit information on problematic issues encountered by TAs but journal writing has been found to impact understanding (Birken, 1989; Porter & Masingila, 2000; Powell & Lopes, 1989; Pugalee, 2001; Shepard, 1993; Wahlberg, 1998) and enhance metacognitive abilities (Kreeft, 1984; Linn, 1987; Nahrgang & Petersen, 1986; Stanton, 1984). In particular, Linn (1987) identified that journaling actively involves participants in their own learning process, forces synthesis of information, and causes reflection on strengths and teaching and learning styles. Evoking reflection on effective practice was the goal of incorporating journal-writing activities into TA training activities for novice TAs.

The project began in Fall 2004 with 19 new TAs entering the graduate program at a Midwest regional state university. Each week of the Fall semester, the TAs were expected to provide an email journal response to one of the following four prompts: (1) This week in teaching I struggled with…; (2) I was flabbergasted when I read a student's response which said…; (3) I have to tell you what my student did… and (4) A really great conversation was created when… Another part of the data collection, although not necessarily a component of this particular study was the requirement that the TAs observe one novice TA and two experienced TA's or instructors and then provide written observation reports on what they saw and what they might consider doing differently. Seventeen of the 19 participants provided the requisite number of journal entries and therefore analysis will be restricted to their responses. Specifically, analysis of the data will focus on the issues and problems the TAs have in their classrooms, preparing for class, or balancing teaching and school work. The goal is to determine the coping strategies these novice TAs bring to problematic situations they encounter and to continue to develop a repertoire of techniques to help novice TAs build appropriate coping mechanisms prior to becoming involved in such problematic situations.

Status and working group plans. The research is in the data analysis stage and there are three tasks for which I would like feedback and assistance during the working group session: (1)

reacting to current framework for analysis; (2) reflecting on appropriate coping mechanisms; and (3) exploring possible linkages between TA struggles and their observation reports. In particular, item (3) focuses on whether the observation of other TAs and instructors helped the novice TAs to reflect on their own teaching practice and whether they gleaned useful coping strategies from such observations. As the working group assists in addressing these three tasks, I believe they will help in obtaining better analysis of the data, improved insights, and enhance future TA training.

Illustrating the Complexity and Variety in the Graduate Mathematics Teaching Assistant Experience

Jason Belnap

Over the past decades, many programs and methods have been developed to prepare TAs for teaching responsibilities (Feiman-Nemser & Remillard, 1996; Friedburg et al., 2001; Gray & Buerkel-Rothfuss, 1991). Recent research raises the question of whether these programs are having an impact and if so, how (Shannon, Twale, & Moore, 1998; Defranco & McGivney-Burelle, 2001).

As many of us are now focusing on these concerns as specifically related to mathematics TAs, it becomes important that we understand the challenges and factors that TAs in other fields experience, what differentiates mathematics TAs from other TAs, and how these impact the development of TAs' teaching views and practices. To begin to describe the mathematics TA experience and the complexity of this context, I conducted an investigative, year-long qualitative, multi-case, dissertation, interview study involving seven TAs who differed by gender, teaching background, and area of study; interview results were substantiated by observations and written assignments (Belnap, 2005).

The study demonstrated the complexity of the TA experience and illustrated the diversity of the TAs we seek to prepare. Several TA prototypes were identified with diverse views and backgrounds, responding to and implementing the preparation they received quite differently. Consequently, a variety of challenges and factors were identified influencing and impacting their teaching development.

Status and working group plans. This study has been completed and planning for subsequent research is now underway. From the working group, I seek ideas and feedback in two main areas. First, I seek ideas on research directions that would build on this study and its results, including possible collaborative efforts. Second, I welcome ideas regarding publication and other venues for dissemination of results, and ideas on ways of breaking-up the results for publication.

Conclusion

Compared to the number of school teachers and preservice teachers who might serve as research informants, the number of TAs available at any one site is often small. Further, each university has its own professional requirements and PD programs for TAs. As a result, validity of research results in this field will require the collaboration of professionals across institutions, even across types of institutions. This working group aims to help interested researchers form partnerships that will lead to collegially-accepted valuable contributions to the field.

The 2004 working group meetings moved the group forward significantly beyond where it had been after the discussion group meetings of 2002. While the 2002 meetings served to help members of the community meet, the 2004 meeting marked a point where researchers were able to draw upon a more involved community and present projects, both theoretical and applied, with specific goals. In 2005, the working group expects to be able to track how projects underway in

2004 have developed and to identify which areas are proving especially fruitful. Furthermore, as a more established working group, the organizers hope graduate student members of PME-NA will now see more opportunities to build on promising projects discussed at the conference.

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MODELS AND MODELING WORKING GROUP

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The Models and Modeling Working Group has provided participants with a setting to reflect on models and modeling perspectives to understand how students and teachers learn and reason about real life situations encountered in a mathematics and science classroom. From these perspectives, a model is considered as a conceptual system that is expressed by using external representational media, and that is used to construct, describe, or explain the behaviors of other systems. There are different types of models that students and teachers develop (explicitly) to construct, describe, or explain mathematically significant systems that they encounter in their everyday experiences, as these models are elicited through the use of model-eliciting activities (Lesh, Hoover, Hole, Kelly, & Post, 2000). During the workshop we will continue to explore these aspects of learning, teaching, and research.

New directions for Models and Modeling Perspectives will be the topic of discussion and dissemination during this year's Working Group. Current and important achievements include collaborative work that has been done in innovative research design and assessment. These attainments include the soon release of several publications that will focus on Real-World Models and Modeling as a Foundation for the Future of Mathematics Education, Design-based Research, and Assessment Design. These three will be the main themes addressed during this year's Working Group.

Introduction

The Models and Modeling Working Group at PME-NA XVII has the following goals:

- To disseminate and contribute to the research on the use of models and modeling in school mathematics, with a focus on students, teachers, researchers, and policy makers.
- To create and support collaborations among researchers to build international communities of practice.
- To extend the field of mathematics education towards new directions on assessment, problem solving, research design, learning environments and complexity; as it relates to the use of models and modeling in school mathematics.

Highlights of a Models and Modeling Perspective

The Models and Modeling Working Group has provided participants with a setting to reflect on *models and modeling perspectives* to understand how students and teachers learn and reason about real life situations encountered in a mathematics and science classroom. From these perspectives, a model is considered as a conceptual system that is expressed by using external representational media, and that is used to construct, describe, or explain the behaviors of other systems. There are different types of models that students and teachers develop (explicitly) to construct, describe, or explain mathematically significant systems that they encounter in their everyday experiences, as these models are elicited through the use of model-eliciting activities

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(Lesh, Hoover, Hole, Kelly, & Post, 2000). During the workshop we will continue to explore these aspects of learning, teaching, and research.

For several years, the Models and Modeling Working Group at PME and PME-NA has been a productive setting for participants to present results, develop new ideas, and create new directions for the use of models and modeling in school mathematics. In 2003, the book *Beyond Constructivism: Models and Modeling Perspectives on Mathematics Problem Solving, Learning, and Teaching*, edited by Richard Lesh and Helen Doerr, crystallized many of the outcomes and collaborations that emerged from participants in this Working Group.

Models and modeling perspectives have proved to be a rich context for research and development. One of the main points of convergence from the conclusions achieved by each of the groups in the past workshops resided on the need for innovative designs for research and assessment that can help answer questions that involve the understanding of complex situations that are dynamic and iterative, like the ones needed to study advances in: Student Development (Aliprantis & Carmona, 2003; Johnson & Lesh, 2003; Kardos, 2003; Dark, 2003; Lehrer & Schauble, 2003; Lesh & Carmona, 2003; Lesh, Zawojewski, & Carmona, 2003; Oakes & Rud, 2003), Teacher Development (Hjalmarson, 2004; Doerr & Lesh, 2003; Koellner Clark & Lesh, 2003; McClain, 2003; Schorr & Lesh, 2003; van Reeuwijk & Wijers, 2003), Assessment (Trigueros & Carmona, 2005a, 2005b; Carmona, 2004; Lesh & Lamon, 1993), Curriculum Development (Cramer, 2003; Carlson, Larsen, & Lesh, 2003; Lamon, 2003; Lesh, Cramer, Doerr, Post, & Zawojewski, 2003; Shternberg & Yerushalmy, 2003), and Problem Solving (English & Lesh, 2003; Harel & Lesh, 2003; Kehle & Lester, 2003; Lesh, Lester, & Hjalmarson, 2003; Lester & Kehle, 2003; Middleton, Lesh, & Heger, 2003; Zawojewski & Lesh, 2003; Zawojewski, Lesh, & English, 2003).

New directions for Models and Modeling Perspectives will be the topic of discussion and dissemination during this year's Working Group. Current and important achievements include collaborative work that has been done in innovative research design and assessment. These attainments include the soon release of several publications that will focus on *Real-World Models and Modeling as a Foundation for the Future of Mathematics Education*, Design-based Research, and Assessment Design. These three will be the main themes addressed during this year's Working Group.

Real-World Models and Modeling as a Foundation for the Future of Mathematics Education

Discussion and presentations on *Real-World Models and Modeling as a Foundation for the Future of Mathematics Education* will be guided by the following questions: *How can research investigate systems of interacting systems –in situations where students interact with one another, students interact with teachers and students, teachers interact within continually evolving learning communities, and the learning activities are themselves continually evolving situations? What steps can be taken to develop a research community that is more than just a community of isolated individuals?*

During panel discussions and presentations, emphasis will be made on the research needs in mathematics education; and how fields like engineering or other design sciences can help inform research methods and models to explore and better understand the types of settings encountered in our field. In particular, participants will be able to examine the distinction between single-theory-based research and problem-based research. Single-theory-based research starts with a theory and seeks to establish principles within it –usually by testing hypotheses in "real world" situations. In contrast, problem-based research starts with a problem that needs to be solved, or an artifact that needs to be designed; and the results typically draw on more than a single theory. "Real life" problems often involve too much and not enough information –as well as too little time, too few resources, and conflicting goals (such as low costs versus high quality, time-efficient versus competence, to mention a few).

Most of the systems that are priorities for math educators to understand and explain are complex systems; and one of the distinguishing characteristics of mathematically complex systems is that the system-as-a-whole have "emergent properties" which cannot be deduced from properties of isolated elements of the systems. These "emergent properties" cannot be described by using single-function models –or even using lists of single function models. Rather, these complex systems are more similar to the problem-based systems that engineers need to understand and explain –such as: complex programs of instruction, interacting with complex learning activities, in which complex conceptual systems of students, teachers, and researchers are functioning, interacting, and adapting. Within such a systems, feedback loops and systems-as-a-whole develop patterns and properties in which results among elements of the systems cannot be derived or deduced from an aggregation or collection of individual properties of elements, or from properties of elements themselves plus properties of any "treatment" that might be used.

Design-Based Research in Mathematics Education

Appropriate research methodologies need to be explored and used in order to study and better understand educational phenomena as complex systems of the sort previously described. The book *Design Research in Education* (Kelly & Lesh, in press), a sequel to the earlier *Handbook of Research Design in Mathematics and Science Education* (Kelly & Lesh, 2000), is a significant and innovative effort to explicate, adopt, and extend the use of this type of research design from fields like engineering and technology, to the field of education; with an overall goal of understanding and improving teaching and learning.

Based on the term and the characteristics described by Brown (1992), Collins (1992), and Collins, Joseph & Bielaczyc (2004), we will call such research design a "design experiment" or design-based research. This type of research design can be characterized through four general principles, which focus on the development of constructs and conceptual systems used by students, teachers, or researchers.

1. The Externalization Principle

Situations should be identified in which the relevant ways of thinking that are desired to investigate (and/or develop) are expressed in forms that are visible to both researchers and to relevant participants. Design activities naturally tend to lead to *thought-revealing artifacts*, like the model-eliciting activities (Lesh, Hoover, Hole, Kelly, & Post, 2000). For these activities, the underlying design often is apparent in things that are designed; the underlying constructs often are apparent in complex artifacts that are constructed; and, the underlying models often are apparent in conceptual tools that embody them. In other words, in the process of designing complex artifacts and conceptual tools, participants often externalize their current ways of thinking in forms that reveal the constructs and conceptual systems that are employed. Therefore, as the tools or artifacts are tested, revised, or refined, the underlying ways of thinking are also tested, revised, and refined; and these cycles are made visible, leaving trails of documentation of the designer's development over time.

2. The Self-Assessment Principle

Design "specs" should be specified as criteria that can be used to test and revise trial artifacts and conceptual tools (as well as underlying ways of thinking) –while discerning products that are unacceptable, or that are less acceptable than others. The design "specs" should function as Dewey-style "ends-in-view". That is, they should provide criteria so that formative feedback and consensus building can be used to refine thinking in ways that are progressively "better" based on judgments that can be made by participants themselves. In particular, ends-in-view should enable participants to make their own judgments about: (a) the need to go beyond their first primitive ways of thinking, and (b) the relative strengths and weaknesses of alternative ways of thinking that emerge during the design process. Productive ends-in-view also should require participants to develop constructs and conceptual systems that are: (a) powerful (to meet the needs of the client in the specific situation at hand), (b) shareable (with other people), (c) re-usable (for other purposes), and (d) transportable (to other situations). In other words, both the tools and the underlying ways of thinking should be shareable and generalizable.

3. The Multiple Design Cycle Principle (or the Knowledge Accumulation Principle)

Design processes should be used in such a way that participants clearly understand that a series of iterative design cycles are likely to be needed in order to produce results that are sufficiently powerful and useful. If design processes involve a series of iterative development>testing>revision cycles, and if intermediate results are expressed in forms that can be examined by outside observers as well as by the participants themselves, then auditable trails of documentation are generated automatically; and, this documentation should reveal important characteristics of developments that occur. In other words, the design processes should contribute to learning as well as to the documentation and assessment of learning.

4. The Diversity and Triangulation Principle

Design processes should promote interactions among participants who have diverse perspectives; and, they also should involve iterative consensus building –to ensure that the knowledge, tools, and artifacts will be shareable and reusable- and so that knowledge accumulates in ways that build iteratively on what was learned during past experiences and previous design cycles. In general, to develop complex artifacts and tools, it is productive for participants to work in small groups consisting of 3-5 individuals who have diverse understandings, abilities, experiences, and agendas. By working in such groups, communities of relevant constructs tend to emerge in which participants need to communicate their current ways of thinking in forms that are accessible to others. Once diverse ways of thinking emerge, selection processes should include not only feedback based on how the tools and artifacts work according to the ends-in-view that were specified –but also according to feedback based on peer review. In this way, consensus-building processes involve triangulation that is based on multiple perspectives and interpretations. So, the collective constructs that develop are designed to be shareable among members of the group; and, they are designed in ways so that knowledge accumulates.

In addition, design-based research should maintain the following goals. First, to radically increase the relevance of research to practice –often by involving many levels and types of practitioners in the identification and formulation of problems to be addressed- or in the interpretation of results, or in other key roles in the research process. Second, to acknowledge that most of the things that need to be understood and explained in mathematics education are complex systems –not necessarily in the strict mathematical sense, but at least in the general sense that they are dynamic, interacting, self-regulating, and continually adapting. Third, to recognize that the mathematical models that are needed to describe and explain the preceding systems are not restricted to linear equations or other kinds of simple input-output rules that presuppose the existence of independent variables that can be isolated, factored out, or controlled (Lesh & Lamon, 1992). Fourth, to acknowledge that research is about knowledge development; and, that not all knowledge is reducible to a list of tested hypotheses and answered questions. In particular, in mathematics and science education, the products that require emerging new research designs are intended to emphasize the development of models (or other types of conceptual tools) for construction, description, or explanation of complex systems. When producing these latter types of products, distinctions are being made between: (a) model development studies and model testing studies, (b) hypothesis generating studies and hypothesis testing studies, and (c) studies aimed at identifying productive questions versus those aimed at answering questions that practitioners already consider to be priorities.

Some of these questions that will lead to the re-forming and informing the nature of how innovations, infrastructure and implementations in education should be constructed; and which will be approached during this year's Models and Modeling Working Group include: *What are appropriate models to understand the behavior of complex, self-organizing systems? How can we design powerful artifacts that will help in this modeling process? How can we gather information that will advise the development of a science of learning? What are appropriate methods for data collection of a design study, so that other communities can learn from and be persuaded by the study (including researchers, teachers, administrators, parents, students, and policy-makers)? What constitutes evidence that an intervention needs refinement or revision? How can we conduct educational research on innovative practices using novel technologies that can be viewed as accumulating science and providing scalable artifacts to positively influence future practice?*

Models and Modeling in Assessment Design

Similar to the previous discussion about the need to make a shift on the types of models and research design to better reflect the current changes and development in social sciences, *models and modeling perspectives* present the need to shift views in the development of assessment designs that more accurately reflect the types of knowledge development in students and teachers. This will be the third focus for discussion in the Models and Modeling Working Group during this year's PME-NA. Presentations and discussion panels will describe new types of dynamic and iterative assessments that are especially useful in design research –where rapid multi-dimensional feedback is needed about the behaviors of complex, dynamic, interacting, and continually adapting systems.

Most of the work being done in assessment includes the production of tests, whose outcome is a number – instead of thinking about knowledge development, and how it can be improved. Assessment needs to be more complex than explaining knowledge development by just producing a number. A focus shift needs to be made in Assessment Design towards producing conceptual tools that provide useful information for decision makers.

From *models and modeling perspectives*, a more appropriate Assessment Design is such that incorporates a holistic view of the education, considering it as a complex system, as previously described (where students, teachers, policy makers, and other educators interact; changing dynamically, modifying and being modified by the curriculum, assessment, classroom environment, and other factors). Among the things that need to be considered in the Assessment Design, and that will be part of the discussion include the following questions:

1. Focus on the decision-makers. Different decision-makers will need and value different information in order to make decisions. Thus, it is important to identify: *Who will be the decision makers? Teachers? Policy makers? Parents? Tax Payers? Administrators? Students? Learning Communities?*
2. Establishing Goals. *What decisions are priorities for these decision makers? For what purposes? What are their ends-in-view?*
3. Operational Definitions. *What is it that needs to be described, assessed, or measured? What is understood by "good" and "better"? Under what circumstances? What attributes should be valued? How can relevant conceptual systems be understood without partitioning into meaningless pieces to be measured, but providing a holistic and systemic view?*
4. Designing Tools. *What conceptual tools should be designed in order to document participants' development and interactions? How can models be designed so that knowledge is documented at the same time as it is being developed?*

The Assessment Design that will be considered from *models and modeling perspectives* is such that allows for the different participants (students, teachers, researchers, policy makers, to mention a few) to design artifacts or models that document their knowledge development at the same time as it is being created. This documentation should also elicit the interactions among different participants, and how these interactions contribute to the development of the whole system (Lesh & Kelly, 2000). In addition, following advances and new developments in technology and design, it is no longer necessary for educational decision-makers to rely on reports that involve nothing more than simple-minded unidimensional reductions of the complex systems that characterize the thinking of students or teachers –or relevant communities; rather, capabilities exist to use graphic, dynamic, and interactive multimedia displays to generate simple (but not simple minded) descriptions of complex systems (similar to weather systems, traffic patterns, biological systems, dynamic and rapidly evolving economic systems, to mention a few examples). Thus, new reporting artifacts should be developed that allow a better depiction, assessment, and evaluation of the phenomena.

The Working Group at PME-NA XXVII

For the PME-NA XXVII Models and Modeling Working Group, several sessions will be organized throughout the Conference. In particular, there will be two main working group sessions. For each session, after a general introduction on different topics is provided, participants will be invited to select one, and smaller groups will be formed. Each sub-group will have a panel of discussants, and a discussion leader, who will approach the selected theme. In addition, participants will be encouraged to attend to other sessions that will be offered throughout the Conference, and that will further support and enrich the discussion that will take place during the two Working Group sessions.

The topics that will be discussed during the Working Group Sessions are:

Working Group Session 1

Discussion Group Topics:

- Modeling Students' Modeling Abilities – New Directions for Research Collaborations
  Panel: Les Steffe, Tom Kieren, Tom Post, Jeremy Roschelle
  Richard Lesh (discussion leader)

- Modeling Perspectives on New Directions for Research on Problem Solving?
  Panel: Judi Zawojewski, Caroline Yoon, Frank Lester, Eric Hamilton
  Margret Hjalmarson (discussion leader)
- Modeling Perspectives on Design Research Methodologies for Assessing Complex Achievements
  Panel: Eamonn Kelly, Finbarr Sloane, Lyn English, Roberta Schorr
  Lupita Carmona (discussion leader)
- Modeling Perspectives – New Directions for Research on Teacher's Knowledge
  Pat Thompson, John Mason, Kay McClain
  Helen Doerr (discussion leader)

Working Group Session 2

Discussion Group Topics:

- New Directions for Research in the Primary Grades
  Panel: Lyn English, Shweta Gupta, Jennifer Fonseca
  Joan Moss (discussion leader)
- New Directions for Research at the University Level
  Panel: Eric Hamilton, Judi Zawojewski, Sally Berenson, Marilyn Carlson
  Maria Droujkova (discussion leader)
- Learning Environments where the Problem Solver is a Group
  Panel: Jim Kaput (Andy Hurford), Walter Stroup, Eric Hamilton
  Jim Middleton (discussion leader)
- Modeling Perspectives on What's Needed for Success beyond School?
  Panel: Jim Kaput (Jeremy Roschelle), Jerry Goldin, Pat Thompson
  Richard Lesh (discussion leader)

The additional sessions include six discussion panels on the following topics:

Panel 1: Modeling Students' Modeling Abilities – New Directions for Multi-Site Research Collaborations
Les Steffe, Tom Kieren, Jeremy Roschelle
Richard Lesh (discussion leader)

Panel 2: Modeling Perspectives on New Directions for Research on Problem Solving?
Judi Zawojewski, Caroline Yoon, Frank Lester, Eric Hamilton
Margret Hjalmarson (discussion leader)

Panel 3: Modeling Perspectives on New Directions for Research on Teacher's Knowledge
Pat Thompson, John Mason, Kay McClain
Helen Doerr (discussion leader)

Panel 4: Modeling Perspectives – What's Needed for Success beyond School?
Jim Kaput (Walter Stroup), Jerry Goldin, Eric Hamilton
Richard Lesh (session coordinator)

Panel 5: Modeling Perspectives on Design Research Methodologies to Investigate or Assess Complex Achievements
Eamonn Kelly, Finbarr Sloane, Lyn English, Roberta Schorr
Lupita Carmona (discussion leader)

Panel 6: Modeling Perspectives on Students' Developing Mathematical Knowledge
Jim Middleton, Marilyn Carlson, Tom Kieren, Les Steffe
"Tom Post (discussion leader)

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VIDEO-BASED RESEARCH ON MATHEMATICS TEACHING AND LEARNING: RESEARCH IN THE CONTEXT OF VIDEO

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This discussion forum, an outcome of fruitful collaboration between researchers in the U.S and Germany, focuses on mathematics education research in the context of teaching videos and analyses common findings and their implications for research, teaching and learning of mathematics. We address several important issues such as the use and analysis of videos in conjunction with available theoretical frameworks for (a) the teaching and learning of mathematics, (b) for mathematics teacher education, and (c) for the training of future university educators in the field. Among the various contributions will also include a paper, which identifies culture as a dependent variable in teacher's learning processes. The forum will also be open to a discussion of the use of teaching videos as a research tool. The co-ordinators have gathered teaching videos from the U.S., Germany and the Netherlands to serve as basis for discussion as well as to highlight similarities and differences in mathematics teaching methods across cultures.

Background

Our conception and preference for a particular style of mathematics teaching is the confluence of prior experiences as learners of mathematics. We have a natural tendency to mimic our teachers and adapt teaching practices which are compatible with our beliefs about mathematics and the learning of mathematics. In spite of teachers' best intentions to cultivate a true reform oriented, constructivist classroom where knowledge is constructed via discourse and the give and take of negotiations of meaning, such changes can be quite difficult to achieve. Often there is a tendency to "fall back" on a mental template or script of teaching that is deeply ingrained in our sub-conscious (see Stigler & Hiebert, 2000).

We propose that video can be used to help teachers explore this issue in their own practice and the practice of others. In particular, the purpose of this forum is to stimulate critical debate concerning the use of videos to promote changes in the practice of university educators and teachers and in their conceptions about the teaching and learning of mathematics.

As the limited literature list in this proposal indicates, videos have been used as a medium for teacher training (Sherin & Han, 2004) as well as a research tool in general. The usefulness of videos in research was especially revealed in the video studies conducted in TIMSS, where one could analyze teaching in classrooms in the U.S, Germany and Japan (Stigler & Hiebert, 2000 and the literature quoted). These studies consolidated the fact that mathematics teaching is a culturally based activity, which follow certain scripts which are resistant to alterations. These observations correlate with recent research findings on teacher knowledge and beliefs (e.g.,

Lloyd, G. M., Wilson, M., Wilkins, J. L. M., & Behm, S. L. (Eds.). (2005).
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Sherin et al., 2000). For example, teacher knowledge has been classified as consisting of two interrelated systems, namely knowledge of lesson structure and content knowledge (Leinhardt & Greeno, 1986; Leinhardt & Smith, 1985). Lesson structure knowledge is described as an understanding of how to plan and implement a lesson, whereas content knowledge consists of understanding the specific mathematics to be taught.

The idea is that these schemata consist of sequences of goals and actions-goals and actions that correspond to what the teacher does in the classroom. Thus, Leinhardt et al. posit a direct relationship between a teacher's knowledge of a lesson and the teacher's behavior in the classroom. Furthermore, Leinhardt and her colleagues do not simply refer to all of these different types of knowledge as schemata; instead, they give names to the various types of schemata, according to the jobs they do and the time scale at which they structure behavior.

There are schemata for:

- mundane activities, such as handing out papers, as well as
- schemata for complex subject matter-specific behavior, such as explaining a difficult concept, repeating standard content, doing calculations (and many more) and
- content-specific schemata (introduction of functions, Pythagoras Theorem, quadratic equations)

Furthermore, these schemata vary greatly in the time-scale at which they structure behavior. While some schemata set a broad plan for a large portion of a classroom session, others are associated with low-level, short duration activities.

According to Schoenfeld (as summarized by Sherin et al (2000)) teaching typically consists of a series of episodes which in turn correspond to a set of actions, termed action sequences. Analogous to zooming into a fractal, each of these episodes are further breakable into more fine tuned action sequences. The underlying skelatal structure of the teaching model manifests when one focuses on these fine tuned action sequences, which include classroom routines, scripts, mini lecturing and talk. Central to the model is the claim that there is a correspondence between these finely tuned actions sequences and classroom goals. Schoenfeld explains that teachers hold multiple goals at multiple grain sizes. Therefore, an action sequence may be related to an overarching goal, a content and/or social goal, as well as more local goals. Similarly, the model elaborates the beliefs and knowledge that influence each action sequence, along with the triggering and terminating events.

In this forum we examine video as a tool for reflection in general, and more specifically, how videos of teaching can elucidate the above model of teaching. Furthermore, we will explore the ways in which understanding teaching as a set of episodes that relate to particular goals and actions can help to promote change both among teachers and among graduate student researchers.

While it is easy to be critical of classroom teachers actively involved in changing their teaching practices and to suggest changes, the university educator must be sensitive to the difficulties and frustrations experienced by teachers during this process. In the U.S., there is a growing trend of doctoral recipients in mathematics education without public school teaching experiences (or even self-reflective teaching experiences), finding themselves in university positions, which invariably involves some training of pre-service and in-service teachers. Yet, many doctoral programs do not involve reflective and critical teaching experiences, necessary to understand the complexities of teaching and the experiences necessary to cultivate the sensitivity required for implementing change in reform oriented classrooms. In the absence of required teaching experiences as requirements in such doctoral programs, it becomes critical that university educators involved in the training of future university educators create the experiences necessary to understand the complexities of teaching. The use of videos in experimental courses on teaching school and college mathematics can greatly aid in this endeavour. Videos are an efficient way of generating data with the caveat that the data although comprehensive in one sense, is also very complex.

Overall Objectives

1. To construct a comprehensive review of the different applications of videos in mathematics teaching.
2. To review the relevance and usefulness of portfolio projects in teacher training.
3. To establish international networks to implement the use of videos for conducting a uniform analysis of teaching in different nations.
4. To explore the use of videos with mathematics education graduate students promote sensitivity about the difficulties of teaching advanced mathematics (Calculus, Abstract Algebra, Analysis) constructively?
5. To classify the types of "mental scripts" that become transparent to graduate students when they critically analyse their teaching videos (a) separately and (b) in collaborative groups.
6. To study the relationship between beliefs about the nature of mathematics and the preference for a particular "underlying" teaching script.
7. To further develop the classification of teaching scripts by generating different examples. For instance classifying seemingly different teaching scenarios which in essence follow one underlying script.
8. To identify beliefs about mathematics which underlie teaching scripts.

It is evident that also technical questions will be addressed (software tools, experiences with these tools, etc.) The discussants have contributed short theoretical reports which will serve as the basis for the discussion. They will also present videos which are part of their research reports. Last not least, we hope to discuss two videos in detail in order to elaborate adequate categories of an analysis.

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BELIEFS ABOUT MATHEMATICS AND TEACHING SCRIPTS FROM THE PERSPECTIVE OF SCHOENFELD'S THEORY OF TEACHING-IN-CONTEXT

Günter Törner Bharath Sriraman

1. Terminology: Following Schoenfeld, we understand beliefs as mental constructs representing the codification of people's experiences and understandings (Schoenfeld, 1998). The problem of identifying mathematical beliefs is well known. The question is how can we actually observe the mental constructs of people? Surveying beliefs via questionnaires only reveals professed beliefs. However there is a major difference between professed beliefs and attributed beliefs. On the other hand, following the theory ‚Teaching-In-Context' (Schoenfeld,

1998) beliefs are an ingredient parameter (among three) to understand the decision-making of a teacher in an actual teaching situation. If one subscribes to the Leinhard Greeno theory that in teaching, certain stable action routines stand out, then we do not completely understand how these action routines found in teaching scripts, are linked with teacher beliefs.

2. An actual teaching situation as starting point: Based on the preceding notes is the analysis of a detailed and limited phase within a video-taped mathematical lesson. The goal of the lesson in grade 8 was to introduce linear functions. The lesson started in an open-ended and problem centered manner, however, in the development of the lesson the teacher changed his/her plan and favored a more classical procedure, namely by communicating definitions as autonomous with the setting of mathematics. The interesting question is what led to this change, which visibly carried itself out as a clear break from the original open-ended delivery of the lesson. One explanation is that due to the over-ambitious agenda of the teacher to deliver the lesson in an open-ended and problem centered manner, time pressure developed which forced the teacher to resort to definitions (safety devices) to meet the end goal of the lesson which was to introduce linear functions. Subsequent interviews confirmed that switching to this classical mathematically structured form of instruction appeared to be the only way out for the instructor to meet the end goal of the lesson. Andelfinger & Voigt (1986) describe a drastically similar procedure in a different context: After an apparent open ended investigative opening of the lesson, the instruction rapidly changed over to the classical form in which the „thing“ to be discovered was told directly to the students as well as ways to procedurally manipulate it. This observation raises several questions:

3. General Observations and Questions: One can suppose that content-specific schemata (introduction of functions, the introduction of the derivative, Pythagoras theorem etc..) from the teacher contain prior basic ideas (concept images). These content-specific schemata for the teaching of complex mathematical ideas are not pedagogically motivated, but contain clues which are traceable to beliefs. Such teacher scripts (with classical safety devices) critically influence the development of the mathematical material in a lesson, whose prior actual goal is to encourage mathematical processes such as argumentation and proof. This radical conversion which occurs during instruction, is inevitably loaded with beliefs about the nature of mathematics.

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VIDEOS IN COLLEGIATE MATHEMATICS TEACHING

Bharath Sriraman

Undergraduate and graduate mathematics courses in the United States are typically taught in the traditional format of direct instruction lecturing. Student understanding of concepts covered in advanced mathematics courses is assessed via timed paper and pencil tests and periodic homework assignments. This traditional delivery of mathematical knowledge with the view of a simple transfer of concepts from teacher to student is based on the Thorndikian premise, that direct instruction plus hard work results in success in advanced mathematics coursework. Yet the attrition rates of numerous minority groups in mathematics, engineering and sciences has been the cause of considerable concern for The Mathematical Association of America (MAA), and resulted in numerous publications (e.g., Gold, 1999; Hibbard & Maycock, 2002; White, 1993; which emphasize alternate methods of teaching and learning in undergraduate and graduate mathematics courses aligned with findings in mathematics education research.

In this report I will discuss the need to emphasize a new approach to the teaching and learning of advanced mathematics with future university mathematicians and mathematics educators enrolled in graduate courses on collegiate mathematics teaching. The objective of one such graduate course taught in Spring 2005 was:

1. to survey in depth recent efforts to reform college mathematics content and teaching.
2. to familiarize students' with special topics within Calculus (Analysis), Abstract Algebra, History of Mathematics, and Statistics.
3. to create "micro" teaching experiences, opportunities for individual reflection on teaching experiences and group critique and analysis of teaching experiences.
4. to familiarize students with frameworks for analyzing collegiate teaching.

Although adopting alternative approaches to the teaching and learning of advanced mathematics was difficult for these students to embrace, the use of current research on collegiate teaching and videos to provide feedback for changing their traditional approach to subjects like Advanced Calculus and Abstract Algebra resulted in a gradual shift from a traditional mode of delivery to a more discovery oriented and humanistic approach to the teaching of these subjects. Student teaching of advanced mathematical topics was video-taped and analyzed (both individually and as a class). Individual reflection and group feedback was used to modify, re-teach and re-re-teach the same topic. Here is a short case summary of the shifts in teaching that occurred with one graduate student (henceforth the "teacher") enrolled in this course. The focus of lesson was on Groups in general. In particular the goal was to introduce the matrix group GLN2 via MATLAB. The first lesson was taught in a very traditional way. That is, the focus of the lesson was on verifying the definition by introducing it at the very beginning and then having students use MATLAB to verify the group axioms for various matrix groups. Several students in the class had difficulty in understanding the various MATLAB commands, which were handled very efficiently by the teacher. One question that was raised was the motivation for computing/generating matrix groups to simply verify the definition as well as its place in mathematics history. Some students were not comfortable with the MATLAB setting and it took considerable amount of time for them to generate the elements of the first two matrix groups and fill out the Cayley table.

Some of the feedback received by the teacher was

(1) To introduce some historical origins and motivation for studying groups.

(2) To have students generate examples of familiar matrix groups to motivate/discover the definition.

(3) To be more involved with what students were doing and increase student-teacher interaction while the lab was being completed.

The second lesson started with a brief excursion into the motivation for using MATLAB to study matrix groups. The lesson began without a foray into outlining the objectives for the particular lesson, and without restating (or having students restate) the definition of a group. In this iteration the class was more comfortable with using MATLAB and students were able to "crank" out the Cayley tables for the first 4 groups. When questioned about the motivation for going through this seemingly procedural exercise, the teacher replied that most extant books on Abstract Algebra only provide "lip-service" to matrix groups and students are left without a good feel for understanding its inherent group structure. This deficiency is remedied by creating a lab where students actually perform a computation via which they "discover" its group structure. The lesson was vastly different from the first lesson because the focus was on discovering the group structure via computation as opposed to simply verifying it. Some of the feedback given to the teacher at the end of the second iteration as:

(1) to delve deeper into the historical origins of matrix groups

(2) to shorten the computations to allow time for class discussion of what had been performed and whether or not a groups structure was visible.

In the second iteration, the teacher was more comfortable in the lab setting, often elicited student comments about what they were performing, and more attentive to students that were still having difficulty with the MATLAB commands.

The third iteration began with a brief outline of the objectives. The machines were already set-up a priori which allowed students to directly jump into the lab activity without going through the routine of getting set up first. The lab activity was considerably shortened which allowed for students to finish the computations. Due the severe time restrictions the teacher was not able to lead a discussion of what the students had done and the results of filling out the Cayley tables. The teacher encouraged students to look up the historical background of matrix groups in general and invoked Heisenberg's work which required the use of matrices to facilitate computations. This got the student's attention. In the three iterations there was a dramatic shift in the focus of the lesson. The shift was measurable in terms of the time allotted to discovering the definition via computation and examples as opposed to the initial (n=1) lesson where the definition motivated the computation simply for verification purposes. The teacher also appeared more relaxed towards the third lesson and there were more incidences of student-teacher interaction. The study shows that systemic change in the teaching of advanced mathematics at the collegiate level is possible via the use of videos in graduate courses on collegiate mathematics teaching.

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UNDERSTANDING THE ROLE OF VIDEO IN TEACHER LEARNING

Miriam Gamoran Sherin

Since video technology became more portable and less costly in the 1960s, numerous activities for teachers have been designed that make use of videotapes of teaching practice. Yet, changes in the ways in which we use video with teachers during this time have not generally been driven by empirical results. Rather than coming about because of an increased understanding of how and why teachers best learn from video, changes have been driven by broader theoretical trends in the educational research community and by advances in technology. This situation is made worse by the lack of theoretical frameworks for describing the process through which teachers learn as they reflect on video.

To address these issues, I explore three themes concerning the role of video in teacher learning. First, I look closely at several key properties of video itself. Specifically, I attempt to identify those features of video that make it particularly useful for teachers (Sherin, 2004). For example, rather than simply use video as a substitute for live classroom observations, I argue that we must consider how video might provide a different perspective than is possible during a live observation. Along these lines, I propose three key affordances of video for teacher education:

(a) video provides a lasting record of classroom interactions; (b) video can be collected and edited; and (c) video affords a different set of practices for teachers. For instance, when watching a pedagogical dilemma on video, one does not have to respond with the immediacy required during teaching. Similarly, video provides teachers with the opportunity to engage in fine-grained analysis of classroom practice, something that is not often possible during the moment of instruction. These affordances have implications both for considering how current video-based programs support teacher learning as well as how future program can effectively leverage what video has to offer teachers.

Second, I examine a particular video-based program called video clubs, professional development cooperatives in which teachers gather to watch and discuss video excerpts of their teaching. Through empirical studies of video clubs, I have sought to understand both what and how teachers learn in this type of setting. For example, Sherin and Han (2004) describe the learning that occurred as four middle-school mathematics teachers participated in a year-long series of video club meetings. Over time, discourse in the video clubs shifted from a primary focus on the teacher to increased attention on students' actions and ideas. In addition, discussions of student thinking moved from simple restatements of students' ideas to detailed analyses of student thinking. Furthermore, teachers began to reframe their discussions of pedagogical issues in terms of student thinking. These types of shifts in the teachers' thinking are particularly important to uncover as they are widely reported to positively influence teachers' ability to implement reform (Ball, 1993; Franke et al., 2001).

Third, my analyses of teacher learning via video focus on the notion of teachers' professional vision (Goodwin, 1994; Sherin, 2001; van Es & Sherin, 2002). The idea is that professionals in a discipline develop specialized knowledge in order to interpret the phenomena of interest to them. Thus, archeologists have professional vision for examining sand and stones, and meteorologists have professional vision for examining the shape and movement of weather systems. Because the phenomena of interest to teachers are classrooms, we can think of teachers as having professional vision for identifying and interpreting significant features of classroom interactions. I believe that much of the important teacher learning that occurs during video clubs can be understood in terms of changes to the teachers' professional vision. Specifically, as discussed above, I found that, over time, the teachers began to notice new aspects of classroom interactions. They also developed new strategies for making sense of what they noticed. Furthermore, there was a dynamic interaction between the teachers' noticing of key events and their interpretations of these events. Thus, what teachers noticed influenced their interpretative strategies, but in addition, as they developed more sophisticated interpretive strategies, what they noticed became more complex. Future research will examine the extent to which a focus on professional vision provides a way to characterize key components of teacher learning in other kinds of video-based programs for teachers.

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OBSERVING THE PROVING PROCESS IN MATHEMATICS CLASSROOM

Aiso Heinze

In the last years several studies on the proving process in the mathematics classroom were conducted (e.g. Herbst, 2002; Heinze, 2004). These studies analysed the question how mathematical proof is taught in school from different methodological perspectives.

The investigation of Herbst (2002) revealed different kind of problems the teachers are faced with in the mathematics lessons when they are organising the proofs in a two-column format. Herbst (2002) identified a double bind on the teacher, because, on the hand, she/he has to create a learning situation that is based on certain didactical ideas to facilitate the content for the students. On the other hand, by this kind of teacher support the students are obliged to understand the didactical situation and to follow the prepared way the teacher has planned. The consequence is that the teacher cannot identify if student problems in the lessons are based on an inadequate mathematical understanding or on an inadequate understanding of the didactical situation.

The situation in Germany is somewhat different, because there is no strong classroom tradition regarding the two-column format. However, in Germany we find a tradition regarding a general teaching style for mathematics lessons which is called the “fragend-entwickelnde” teaching style. It is a kind of classroom discourse, in which the teacher tries to develop the content by asking questions to the students. The interaction between teacher and students is framed by the social norms of this discourse. There are several studies on this teaching style which identified certain scripts like the funnel-shaping of the teacher questions, i.e. if the students’ answers are not suitable to the questions the teacher will more and more close the question till the answer is more or less trivial.

In a video study based on 20 lessons on geometry proofs in grade 8 we analysed how the proving process in the German mathematics classroom is organised (Heinze, 2004). As a basis for this investigation we took a model of the proving process which consists of five phases: (1) finding of the hypotheses, (2) formulation of the hypothesis, (3) exploring the hypothesis and generate a proof idea, (4) creation of a sketch of the proof, (5) formulation of the proof and retrospective summary. In contrast to the proving process in academic mathematics, which is divided in a private and a public part, in the geometry classroom we found hardly any private parts in the proving processes. In general, the teacher asks the students to make a geometrical construction and to measure some angles and lines. The results were collected and a hypothesis was derived. After that the students got shortly the possibility to give some ideas for a proof. These ideas were in general not successful, because they were spontaneous and not based on a deeper investigation of the problem. Then within the “fragend-entwickelnde” teaching framework the teacher worked out the proof idea and the final proof step by step at the black board.

Our results indicate that particularly the exploration phase in the proving process is neglected by the teachers. In the observed proof lessons the students were hardly able to get any idea of how to explore a proof task systematically and how to generate a first approach to a proof. The consequences are that students get an inadequate understanding of mathematical proofs and lack of strategies to solve proof problems.

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CROSS-CULTURAL ELEMENTS OF LESSON STRUCTURE IN CLASSROOMS FROM GERMANY, HONG KONG AND THE USA

Eva Jablonka

The findings reported in this section rest upon six case studies that used the data produced in the Learner's Perspective Study (LPS). The LPS studies the practices and associated meanings in eighth-grade mathematics classrooms in 13 countries (see http://extranet.edfac.unimelb.edu.au/DSME/lps/). Each county participating in the LPS used the same research design to collect videotaped classroom data for ten consecutive mathematics lessons and to conduct post-lesson video-stimulated interviews with two students after each lesson in each of three participating 8th grade classrooms. In part, the LPS study is motivated by the postulated cultural specificity of teacher practice and by the belief that the characterization of the practices of a mathematics classroom must attend to learner practice with at least the same priority as that accorded to teacher practice. The methodology of data production in the LPS aims at documenting not just the obvious events that might be recorded on videotape, but also the participants' construal of those events.

One rationale for studying lesson structure internationally is to identify elements of lesson plans and classroom interaction that grew out of local traditions of curriculum and teacher training and are evidenced in distinct organizational forms. It can be assumed that the availability of a repertoire of specific forms of activities shapes the process of learning/teaching.

On the other hand, it is interesting to see how rather cross-national elements of mathematics lessons turn out to be enacted in distinct ways in different classrooms within and across national traditions. Such a study could offer insights into the extent to which these forms are shaped by similarities of teachers' and students' perceptions of mathematics and its teaching/learning.

For example, in all six classrooms from this study, teachers frequently employ a form of interaction and talk that consists of addressing the whole class by a series of connected questions; the students volunteer for getting a turn by raising their hands; the teacher selects the turns, opens and closes the discussion and evaluates the contributions of the students. The questions do not aim at eliciting information from the students that is not accessible to the teacher, but at eliciting information from the students in order to incorporate it into a collective development of the topic. Microanalyses of these episodes show how students adapt to this form of interaction.

Searching for cross-cultural similarities in mathematics classrooms can help to identify different ways of how teachers and students deal with the same problems and which modes of teaching/learning constrain or afford students' learning.

EFFECTIVENESS AND QUALITY OF ALTERNATIVELY PREPARED MATHEMATICS TEACHERS

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Through this discussion group on effectiveness and quality of alternatively prepared mathematics teachers, we will provide a forum for discussing ongoing research studies in this area. Focal discussion points of interest to participants well be use to guide the direction of the session. After the organizers of the session share their ongoing research, the working group will engage the participants in discussion based upon the participants' research and interests. It is expected that major themes will evolve. These themes will guide the future directions for the continuation of this group within PME-NA.

Introduction

Given the influx and impact of alternatively prepared teachers in K-12 mathematics classrooms across this nation and the documented lack of research on the effectiveness and quality of these teachers (Goldhaber & Anthony 2003; Wilson, Floden, & Ferrini-Mundy, 2001; Zeichner & Schulte 2001), we are initiating a discussion group to explore research efforts with respect to alternative preparation of mathematics teachers. Within this discussion group, researchers in mathematics education who are interested in conducting studies or have ongoing research with respect to the preparation of mathematics teachers through alternative pathways will have the opportunity to share and receive feedback on their research. It is expected that a critical mass of researchers coming together in this area holds the promise of becoming a formal working group within the Psychology of Mathematics Education-North American (PME-NA) chapter.

We, the organizers of this proposed working group, have been involved in the alternative teacher preparation program in our institution and are conducting research on alternatively prepared teachers for urban classrooms. Our research is longitudinal and includes focused studies across various stages of the continuum--from recruitment into the preparation program through induction into the profession. While the context of our work is focused on urban environments, this discussion is open to all environments and contexts of alternatively prepared mathematics teachers.

Rationale for Research and Working Group

According to Wilson, Floden and Ferrini-Mundy (2001), there are higher percentages of alternative certified teachers teaching in urban settings or teaching minority children than in other settings. There is also a mixed record of the quality of teachers recruited and trained. Research about the impact of alternative certification of teachers is also limited. As a result, Wilson et al (2001) found a need for studies that are designed to include more sensitive measures (e.g. content and quality) that describe specific features (e.g. subject matter, pedagogy and clinical experiences) of alternative teacher preparation programs. Among the gaps identified, they suggest need for research across:

Lloyd, G. M., Wilson, M., Wilkins, J. L. M., & Behm, S. L. (Eds.). (2005). *Proceedings of the 27th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.*

Pedagogical Preparation:

- For systematic and comparative research on the content of pedagogical preparation beyond lists of course titles and on the instructional methods best suited for professional teacher preparation;
- To know more about the preparation of teachers to teach diverse student populations, including those in urban and poor rural settings;
- To know more about the actual knowledge and skills that teachers acquire in their education coursework and associated experiences;
- To know more about what teachers learn in subject matter education courses and how that professional knowledge compares to subject matter preparation of an academic major;

Clinical Experiences:

- To know more about the impact of innovative field experiences on new teachers' effectiveness;
- To know more about the relative impact of various types of field experiences: early field experiences, field experiences integrated into particular university courses, student teaching, and yearlong internships;
- To know more about the relative contributions of coursework and fieldwork to a teacher's progress in learning to teach, more about the ways in which the coursework integrates into the fieldwork, and under what fieldwork conditions the novice teachers are most likely to continue to learn productively;

Alternative Certification:

- To describe the content and components of high quality alternative certification programs;
- To document and analyze the professional knowledge (both of subject matter and of teaching) that graduates of alternate routes acquire, and how they acquire it, and relate that knowledge to teaching practice;

Teacher Preparation:

- To know much more about how to prepare teachers for urban and poor rural areas and how to create policies that ensure that those children get highly qualified teachers; and
- To assess the impact of teacher preparation programs to include designs that examine impact longitudinally.

Goldhaber and Anthony (2003) claim teacher quality is the most important educational input predicting student achievement. Further, they claim that teacher quality has historically been synonymous with personal traits, such as high moral character and intellectual curiosity, while today it tends to encompass structured standards developed by the Interstate New Teacher Assessment Support Consortium (INTASC) and National Board for Professional Teaching Standards (NBPTS). The National Council for the Accreditation of Teacher Education (NCATE), INTASC and NBPTS, though they differ in some respects, share common themes about teacher quality. However, despite thinking of teacher quality as an immutable characteristic, Goldhaber and Anthony (2003) state that it is possible for a teacher who is highly effective in one setting, say a highly structured environment with explicit standards and accountability measures, to be ineffective in a more flexible environment.

In the report, *No Dream Denied: A Pledge to America's Children* (2003), "highly qualified teachers" have been benchmarked by a set of criteria that are aligned with the Interstate New

Teacher Assessment and Support Consortium (INTASC) and the National Board for Professional Teaching Standards (NBPTS). The report has also indicated "American students are entitled to teachers who know their subjects, understand their students and what they need, and have developed the skills to make learning come alive" (p. 7). Despite these claims, the report states that the nation is far from providing every child with quality teaching.

Current Work of Focus Group Organizers

Our current study is situated within the recruitment of students into alternative preparation for secondary mathematics. The purpose of our current research is to develop a research-based recruitment tool that can be used to identify potential teacher candidates who possess attributes or characteristics of becoming high quality mathematics teachers for urban schools. Our research question is: *How can a recruitment tool be developed to facilitate identification of potential high quality mathematics teachers for urban environments?*

Ultimately, our focus is to increase the number of high quality urban mathematics teachers who seek jobs in urban school districts and are committed to remain. In this light, it is vitally important for us to make informed decisions in selecting students to prepare as mathematics teachers for urban schools. Therefore, we need to understand: (1) the characteristics of high quality teachers that are pertinent to the willingness, the stamina and longevity of commitment to the urban classrooms, and (2) how we prepare teachers to acclaim those qualities. Having an understanding of these characteristics in ways that lead to the development of a recruitment tool that will assist us in identifying potential high quality urban mathematics teachers is the goal of our research. We are asking questions such as: *What does it mean to be a high quality urban mathematics teacher? What are the experiences in a typical day for a high quality urban mathematics teacher? What does the high quality urban mathematics teacher do differently from teachers in other settings?*

Focused group discussions addressing these types of questions have facilitated our progress in establishing an initial set of characteristics of high quality urban mathematics teachers. In the acquisition of characteristics of high quality urban mathematics teachers, we chose to focus on individuals who had lived-experiences through which they had developed beliefs and perceptions with respect to the phenomenon under investigation, "high quality mathematics teachers in urban environments." In particular, this current study engages urban mathematics teachers who are considered to be high quality.

We applied phenomenology using focus groups as the setting for data collection. Researchers who use phenomenology are interested in showing how complex meanings are built out of simple units of direct experiences. That is, a phenomenological study follows the format of explicitly examining one particular phenomenon to allow carefully chosen participants to make meaning out of it (Creswell, 1998). Further, the use of focus groups allows for explicit interactions that produce data and insights that would be less accessible without the interaction (Morgan, 1988). This reliance on interaction between participants is designed to elicit more of the participants' point of view in the context of the views of others, which would be evidenced in the more researcher-dominated interviewing (Mertens, 1998; Patton, 2002). As a result of this initial phase in the research process, we have produced and categorized a preliminary list of characteristics of high quality urban mathematics teachers.

While the initial step has produced pertinent information that will be used in the design of the recruitment tool, this research study includes several stages that will build upon our initial work over the next three years. Participants are engaged in the research through: (1) focused group interviews for the purpose of defining characteristics of high quality urban mathematics teachers, (2) collaborative efforts for developing and refining items for the recruitment tool, and (3) validation of the recruitment tool. This study uses a mixed-methodological analysis that began with qualitative methods and will lead sequentially to quantitative methods in the development and validation of the recruitment tool.

Plans for Engagement of Participants

In this first session, the organizers will begin by sharing the background of the alternative preparation program of their research, the research design, and the progress of the current research on recruitment. During PME-NA XXVI Toronto, the organizers engaged in informal conversation with conference presenters whose presentations focused on alternatively prepared teachers. Through informal conversation, we found an interest among the researchers for this discussion group. In this session, researchers will be invited to share their work with the whole groups as well as in small group discussion.

After ongoing research has been shared and discussed, participants will have time for questions and answers. The participants' areas of research and interest will be recorded prior to establishing breakout sessions. Therefore, there will be an attempt to form as best as possible small discussion groups by topics within a manageable numbers. After the breakout sessions, group reporters will share pertinent points discussed, issues raised, and the group's decision for the future directions of this working group. Some of what is expected to be accomplished in the discussion group includes:

- Engagement in a format for rich discussion across common themes
- Sharing descriptions of alternative pathways into mathematics teaching
- Sharing ongoing research and research interest areas of common interest
- Reporting out from the discussion groups
- Examining theoretical conceptual frameworks
- Examining research questions and research designs
- Discussing ways in which research can impact the quality and effectiveness of alternatively prepared mathematics teachers
- Sharing ways in which we can move this discussion forward and establish steps for further dissemination of the ongoing research of this discussion group
- Establishing next steps for the longevity of our work and this discussion group within PME-NA.

Types of question that may be used to open and guide discussions are:

- What is alternative certification?
- What are common components across alternative certification programs for mathematics?
- Should an alternative certification program place more emphasis on content knowledge or pedagogy in the preparation of mathematics teachers?
- Is alternative preparation an effective means to addressing the problem of teacher shortage in mathematics?
- How do alternative certification programs prepare high quality mathematics teachers?
- What examples of effective alternation preparation programs are in existence and what are the features of the effective programs that prepare teachers of mathematics?
- What is the duration of an alternative certification program that prepares high quality mathematics teachers?

- What does research about retention report about the longevity of alternatively prepared teachers?
- Should alternative preparation programs be context specific?

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FRAMEWORKS THAT SUPPORT RESEARCH ON MATHEMATICS EDUCATION WITH COMPUTER ALGEBRA SYSTEMS: REFLECTIONS ON CAME SYMPOSIA

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Since the Hawaiian conference in 1995 (Berry et al, 1997) Computer Algebra in Mathematics Education *(CAME) has been established. Further to this (but not because of this!!) some of us have 'lost our innocence' along the way; youthful dreams of enhanced learning have been replaced by realism that all tools have their constraints as well as their enablements. CAME Symposia have been forums for these discussions. The rationale and goals for CAME Symposia are to serve as a bridge between two communities, the CAS research community and the main mathematics education community; to facilitate the dissemination and exchange of information on research and development in the use of computer algebra in mathematics education; to facilitate access to international expertise in the use of computer algebra in mathematics education; to promote the study of the use of computer algebra in mathematics education. The rationale for this discussion group is to present the products of CAME Symposia and to discuss their potential contribution to the PME community. In this paper we first describe briefly issues that were dealt with in each symposium. We then outline the specific topics for discussion.*

The First CAME Symposium

The first CAME Symposium was held at the Weizmann Institute of Science, Rehovot, Israel in August 1999 (see link to Weizmann, 1999) following on from PME 23. The theme was: *Exploring CAS as a pedagogical vehicle towards expressiveness and explicitness in mathematics*. Plenary speakers were: Jean-baptiste Lagrange, Steve Lerman , Edith Schneider, Ted Eisenberg, Paul Drijvers, Kaye Stacey, Nurit Zehavi and Anna Sierpinska, John Berry, Richard Noss and Amitai Regev. We wanted to address two issues in particular:

1. the links (or lack of) between "theoretical" work in mathematics education and classroom practice, with particular respect to the role of CAS;
2. the place of CAS research and CAS-related activities within mathematics education research as a whole.

It is a long-standing problem in mathematics education to connect research and classroom practice; as CAS technology increasingly impacts on mathematics curricula, the challenges and opportunities for mathematics education to inform curriculum change are considerable. The CAS-related research questions that we wished to highlight at the workshop had to do with what we labeled explicitness and expressiveness. In using a CAS, a particular explicit symbolism is forced: each input requires a particular forced way of viewing things and expressing relationships and the output needs to be interpreted similarly. CAS has the potential to provide expressive powers for its users; it is possible to express ideas (mental objects) in a concrete form (visible objects). It seems reasonable to assume that in the tensions set up between thoughts and explicit expressions there is considerable scope for researchers to understand mathematical learning better. At the conference presenters from four countries shared their experience and research direction in using CAS for teaching. Colleagues from the PME community reacted to the papers and together with the participants the ground was set for further work.

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Proceedings of the 27th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.

In this first Symposium Jean-Baptiste Lagrange introduced many people to a new approach to CAS work. This approach used the anthropological framework of Chevallard and an "instrumentation" approach to tool use. This theme has been a constant feature of all subsequent CAME Symposia.

The Second CAME Symposium

The second CAME Symposium was held in July 2001 at the Freudenthal Institute, Utrecht, The Netherlands with the theme: *Communicating Mathematics through Computer Algebra Systems* (see link to Freundenthal, 2001). The symposium examined research on the relation between techniques and conceptual understanding, on the role of the teacher, and on the affordances of technology in realizing specific pedagogical approaches. Plenary speakers were: Michèle Artigue, Neil Challis, Koeno Gravemeijer, Kathleen Heid, Kenneth Ruthven, Kaye Stacey, Michal Yerushalmy and Rose Mary Zbiek. As a natural continuation of the previous symposium we began with the issue of the subtlety of the relationship between paper-and-pencil techniques, CAS techniques and conceptual understanding. The nature of paper-and-pencil techniques is different from that of CAS techniques. How do these different kinds of techniques interact with concept development and understanding? What is the nature of the "instrumentation" process, during which a tool gradually develops into an instrument, for learning to do mathematics using a computer algebra system? Another perspective was presented by Kathleen Heid who shared with the participants her view of how theories about the learning and knowing of Mathematics can inform the use of CAS in school mathematics.

Friendly personal relationships contrasted strongly with deep theoretical divisions between many of the participants (you don't come to CAME for an easy intellectual ride). Even with a topic as focused as 'CAS and teachers' (presented by Stacey and Zbiek) it was clear that the different interests of the group members created different foci and questions. For example, members who were involved in in-service work with teachers wanted to know how best to train teachers whilst curriculum developers focused on how teachers might help students. At the core, however, everyone was united in trying to understand the phenomena of teachers using CAS.

The Third CAME Symposium

The third symposium was held at the IUFM in June 2003 in Reims, France with the theme: *Learning in a CAS Environment: Mind-Machine Interaction, Curriculum & Assessment (*see link to Reims, 2003). Plenary speakers were: Lynda Ball, Roger Brown, Al Cuoco, Peter Flynn, Celia Hoyles, Colette Laborde, David Leigh-Lancaster and Luc Trouche. The themes were a little bit more discrete than they were in CAME 2 reflecting, perhaps, work that different groups were immersed in.

Most of the members of the working group on 'Assessment' were also involved in CAS-related curriculum development that involves assessment with CAS and so a good deal of discussion was on very practical matters. Five members of the group were from Victoria, Australia, where the *Computer Algebra in Schools: Curriculum, Assessment & Teaching Project* has been ongoing. This project has been followed with interest by a number of people and there were many questions about this in the working group. For example: How CAS in assessment forces us to examine the goals of teaching and learning mathematics?

Strangely enough the mathematics curriculum itself emerged as a point of heated debate (strange that this division had not occurred earlier). Some argued that there is a strong need for a general outline of a curriculum where the enhancements in technology, especially the presence of computer algebra systems and dynamical geometry environments, will be incorporated. Others suggested that we need a curriculum where the choice of technology used is not the central factor.

This ends our brief description of the first three CAME Symposia. We now explore various themes in greater depth. These are: CAS-based curricular materials; CAS and teachers; CAS and assessment; CAS, the instrumental approach and orchestration. We end by outlining our intentions towards the Fourth CAME Symposium.

CAS-based Curricular Materials

The availability of CAS for microcomputers in the mid-eighties led some teachers and researchers to explore new teaching methods that utilize CAS to enhance the teaching and learning of mathematical topics. One notable strategy was to allow students to concentrate on conceptual aspects as they learn a topic. Early CAS-based studies by Heid (1988) and others indicated that re-sequencing of the content, so that the concepts are taught before the manipulation skills, was effective for achieving a greater understanding of concepts without decreasing the achievement of manipulation skills. As more mathematics teachers became familiar with CAS, a new area surfaced, namely the design of CAS-based written support materials. Researchers and curriculum designers realized, through experimental projects, that learning to utilize CAS to construct mathematical meaning is both complex and insightful. As more studies were carried out, more questions were asked: What tasks involving CAS engage students in conceptual aspects? How do we assess their quality? What is the role of CAS in emerging curricula? Such questions were addressed in the first CAME Symposium. Didactical works with CAS in France, Austria, The Netherlands, Australia and Israel were reported and analyzed.

Jean-baptiste reported on his involvement in an experimental project of the French ministry of Education for the teaching of pre-calculus with CAS. He argued that focusing on an opposition between skills and concepts does not help to understand changes introduced by CAS use in teaching and learning. He proposed to think rather of the contribution of techniques to students' understanding and to consider the impact of CAS at this level: obsolescence of paper/pencil techniques, potential contribution of new CAS techniques. He presented the notion of "praxeology" as means to consider tasks, techniques and theorizations in students' processes of conceptualization. This point was further elaborated by Steve Lerman who reacted to the French work. Lagrange's paper has been a prelude to ongoing analysis and discussion of the role of techniques in CAS work.

Edith Schneider and her Austrian colleagues were trying to get mathematics teachers interested in using CAS in their classrooms by working with the individual teacher to develop materials specifically for them to use. She reported an effective change of atmosphere of learning in the experimental classes, where the role of the teacher was less as the source of knowledge and more as a collaborator in helping students discover knowledge for themselves. However, the researchers realized that the teachers were unable to take a deeper, more critical examination of the fundamental didactic questions on the potential of CAS in teaching. Ted Eisenberg, in his reaction paper, emphasized that teachers are not curriculum developers, thus they should get mock lessons from experts. He expressed his skepticism strongly by stating that the Austrian team was years ahead of themselves.

Paul Drijvers from the Freundenthal Institute described the role of the theoretical framework of Realistic Mathematics Education and developmental research in developing and performing an experiment using symbolic calculators in senior secondary mathematics classes. He provided rich data that helped him address the following questions:

1. Is it possible to re-sequence a course using CAS, so that concept development precedes the solving techniques and algorithms?
2. Can algebraic insight improve because of CAS use?
3. What obstacles do students experience while working with computer algebra?

Kaye Stacey discussed these questions, drawing on experiences from experiments that she and her students carried out at the University of Melbourne. We will say more about Australian work in the next section.

Nurit Zehavi and Giora Mann from Israel presented a study that attempted to investigate (a) the role of CAS in modeling word problems, and (b) the role of algebraic expressions involving parameters in making explicit the underlying structure and constraints of a family of story problems. The research was done within the formative development stage of the MathComp project that was initiated in 1996, and aimed to integrate CAS into teaching in junior and senior high school. The learning process was organized through a careful design of problems and tasks for the students as a group and planning of the activities with the teachers. The students' tasks were not only to solve problems given by the teacher but also to invent their own problems. Anna Sierpinska, in her reaction paper, proposed that it was this feature of the didactic situation, and not so much the availability of a CAS, that was responsible for the students' progress in their thinking about modeling word problems using equations and conditions on variables. At the end of the symposium it was clear that the mutual impact of CAS use and task design would be further explored in the following symposia.

The group discussion that followed the plenary lectures by Michèle Artigue and Ken Ruthven in the 2nd symposium, dealt with the difficulties in developing criteria for CAS-based tasks, and proposed several suggestions to the question: What are the implications of the complexity of instrumentation for task design and research in CAS environments? Here are some of the participants' suggestions for designing tasks and the associated research.

TASKS	*RESEARCH*
Tasks designed to address known difficulties of students	Are the difficulties encountered in a paper and pencil environment actually present in a CAS environment?
Diverse tasks designed for conceptual development	Examine the relationships between techniques and concepts when using CAS and other technologies. Encourage the implementation of tasks under different conditions (to obtain an international perspective).
Tasks designed to integrate the different types of representations available in a CAS environment	What role does visualization play in different environments? Under what circumstances is it appropriate to work with two or more representations?

Tasks that encourage a good classroom discussion and for which the resulting discussion opens 'webs of meaning' (Noss & Hoyles (1996)) that go beyond that which occurs within the regular curriculum.	Exploring the opportunities to extend the curriculum. Carry out classroom experiments with rich tasks that would not really be feasible without the CAS.
Tasks that motivate socio-mathematical norms that arise in the new mathematical environment	Examine the types of communication and accounting in a CAS classroom. Examine the tension between the cognitive and cultural aspects.
Tasks for assessment of the impact of integrating CAS into teaching	Identify those difficulties created by the complexity of instrumentation when students are using the CAS. Compare students' achievement when implementing alternative approaches to the use of CAS.
Tasks specially designed for teachers to experience doing mathematics with CAS	Examine the process of teachers' development in introducing CAS. Examine the transition of teachers from graphic calculators to the CAS culture. Classify and analyse tasks developed by teachers who are experienced users of CAS.

Consequently, at the 3rd symposium one of the themes dealt with Curriculum and Task Design. In the plenary session for this theme, Al Cuoco presented his and Paul Goldenberg's paper, *CAS and Curriculum: Real Improvement or Deja Vu All Over Again,* and Collete Laborde presented a reaction paper *The design of curriculum with technology: lessons from projects based on dynamic geometry environments*. Both enriching talks presented a solid framework for a very lively and practical discussion.

Perhaps the most important observation was that nowadays we see just the fragments of a new technology-based curriculum that is emerging. The group agreed that technology makes changes in the curriculum necessary. "Are we better with gradual changes of the existing curricula or should we try to make a new curriculum from scratch?" Of course a fully "revolutionary" approach is not possible due to social, economic, political and other factors, but nevertheless it could be interesting and beneficial if we could see what such a curriculum looks like. This discussion will be taken one step further by the topic group on "The impact of CAS on our understanding of mathematics education", at the 4th CAME Symposium. The theme will deal with questions such as: is the cognitive availability of operative knowledge and skills essential for mathematics education in spite of CAS? have the meanings of operative knowledge and skills changed so that these aspects of mathematics now require an ability to work with CAS? Getting a better understanding of these issues may enable the CAS-in-education community to better communicate with teachers, who are often reluctant to integrate CAS in teaching in spite of its availability and student familiarity with technology.

CAS and Teachers

CAS and Teachers was a theme of the 2001 Symposium. Teachers, of course, have a crucial mediational role in students' learning (with or without CAS). However, and as the meta study of Lagrange et al. (2001) note, CAS studies in the period 1994-1999 generally do not discuss the role of the teacher. For our Symposium we wanted speakers who could broaden our understanding of teachers and CAS. We were aware of interesting work going on in Australia and in the USA and this is where we turned.

Both Australia and the USA have federal governments[1]. Within Australia, Victoria was the centre of a synergy between local government CAS initiatives (see the section on Assessment for more on this) and academic study around these initiatives. At the University of Melbourne Kaye Stacey was supervising the PhD theses of two experienced teachers, Margaret Kendal and Robyn Pierce. Kaye's presentation acknowledged the contributions of Margaret and Robyn. In the USA Kathy Heid's work stands out for its longevity and productiveness. Kathy is clearly someone who likes to work in a team and an important co-worker was Rose Mary Zbiek. We were aware of Kathy and Rose Mary's work on CAS and teachers from their PME-NA papers (Heid, 1995; Zbiek, 1995) and thought that Rose Mary would be an excellent reactor to Kaye's presentation.

Kaye's presentation focused on case studies of two teachers who had "adopted CAS as an extra technique for solving standard problems". Her assumption was that integrating CAS into teaching changes many aspects classroom practice "which teachers will make from the base of their prior teaching styles and their beliefs about mathematics and how it should be taught". One teacher focused on student understanding and restricted students' use of CAS. The other focused on CAS as an additional (to standard paper and pencil) technique for solving standard problems and emphasized time-saving routines. Issues raised included: different forms of classroom organization; the variety of approaches available to teachers; the increased range of methods available for solving problems and the tensions this raises for teachers; the different ways that teachers use graphic and symbolic calculators; implications for changes to the curriculum and assessment.

Rose Mary's reaction noted that "the route a teacher's journey takes depends on the individual's answers to a few non-trivial questions:

1. What is school mathematics?
2. What is mathematical understanding and how does it develop?
3. What is the teacher influence?"

Rose Mary introduced two other pairs of teachers, from the USA and from the UK, and noted situational constraints and enablements. However:

> Common to all scenarios is a professional development and research design that included observation data and artifacts as well as data about the teacher's belief, conception, and understanding of CAS, mathematics, and teaching.

She concludes that the critical essence of teaching with CAS in a time of transition is not CAS itself but "the extent to which it challenges values, abilities, practice and assumptions".

The day of discussion on the themes raised in the presentation and reaction noted that introducing CAS into teaching involved many groups of people, other than teachers, with distinct interests: students, parents, teacher trainers and curriculum developers. Further to this, but still regarding distinct interests, it was notable that individuals within the discussion group had different agendas with regard to teachers and CAS, e.g. members who were involved in in-service work with teachers wanted to know how best to train teachers whilst curriculum developers focused on how teachers might help students. These varied "interested parties" and the varied agendas strongly suggest that future considerations of teachers and CAS should be

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[1] Do federal systems present fewer constraints on educational innovations such as the introduction of CAS?

placed within a wider "institutional" analysis. This links to a theme raised by Michèle Artigue at the Symposium "the life of a mathematical object in an institution" and the group recognised that it was important to understand teachers' "positioning" within their institutions.

CAS and Assessment

CAS and assessment was a theme of the 2003 Symposium but "assessment" is really too wide a term to describe the presentations and the debate. Important current themes in assessment such as formative assessment, dynamic assessment, adaptive assessment and e-assessment were not addressed. Further to this assessment of school and university level mathematics generally differ: in the classroom where schools generally have much smaller student numbers, and in final examinations where universities generally have greater freedom, including teachers setting the questions. The focus of the 2003 Symposium was really "CAS and senior high school high stakes examinations". We deviated from the CAME norm of presenter-reactor and had four presentations; all from Australians and three of these concerned with work going on in Victoria, Australia[2]. We briefly describe the four papers and then the group discussion.

Roger Brown presented a paper "Comparing system wide approaches to the introduction of Computer Algebra Systems into examinations" which looked at CAS-assumed examinations in Denmark and Australia. He addressed two questions:

1. How are the examination writers responding to the introduction of CAS?
2. What is the role of the CAS within the examinations?

He concluded that the use of CAS, in these case studies, results in surprisingly little difference in the types of questions used but that CAS greatly enhances the range of solution strategies available to students.

David Leigh-Lancaster reported on a CAS-assumed pilot study from the viewpoint of the Victorian Curriculum and Assessment Authority. This paper covered the design and development of the CAS pilot study and details of student performance in examinations. With regard to performance the CAS-cohort generally did as well or better than the non-CAS cohort. Pilot teachers and students, moreover, generally affirmed benefits of using CAS including greater depth treatment of existing material, access to new and interesting content and enhanced engagement, persistence and confidence of students.

Lynda Ball explored "Communication of mathematical thinking in examinations: A comparison of CAS and non-CAS student written responses". She examined students' written records in examinations including a comparison of written solutions for examination items common to CAS and non-CAS students. She found that CAS students generally gave shorter written solutions and noted a CAS-student tendency towards using a mixture of mathematical notation and words in documenting how problems were solved.

Peter Flynn presented a paper "Using Assessment Principles to Evaluate CAS-Permitted Examinations". The "principles" in question are three that pre-date the Victorian CAS pilot and relate to content, learning and equity in high-stakes assessment. They are, respectively, that assessment should: reflect the mathematics that is most important to learn; enhance mathematics learning and support good instructional practice; support every student's opportunity to learn important mathematics. With regard to CAS-assumed examinations Peter addressed a number of matters of concern. With regard to equity he explored different CAS calculators where there was

[2] http://extranet.edfac.unimelb.edu.au/DSME/CAS-CAT/ see: http://www.vcaa.vic.edu.au/vce/studies/mathematics/caspilot/casbackground.html

a differential effect on some items but, that over all questions, no one brand was of significant advantage. Matters raised in the day of discussion included:

- Whether examinations should have 'CAS free' sections or not.
- The difficulty of separating curriculum and assessment.
- The need to examine the goals of teaching and learning mathematics
- The need for further work on teachers' professional development
- The need to consider all types of assessment and not just timed examinations
- The need to be explicit about why particular questions are being set
- The problem that the aspect of CAS use envisaged in examination questions may be quite different to the aspects of CAS use that a teacher (and his/her students) focuses on.

CAS, the Instrumental Approach and Orchestration

CAS is a tool, a very complex tool, which incorporates various computational media (Cuoco, 2002). In their quest to understand CAS-as-a-tool CAS researchers turned to French research on *instrumentation*. V_rillon & Rabardel (1995) distinguish between a tool, as a material object, and an instrument as a psychological construct: "the instrument does not exist in itself, it becomes an instrument when the subject has been able to appropriate it for himself and has integrated it with his activity".

At the third symposium, Luc Trouche provided a clear exposition of the psychological foundations of instrumentation. He offered to consider an instrument as an extension of the body "made up of a tool component (a tool, or a fraction of a tool mobilized in the activity) and a psychological component." He emphasized the subject-tool dialectic and names the direction of the influences: *instrumentation*, how the tool shapes the actions of the tool using subject; *instrumentalisation*, the ways the subject uses (shapes) the tool. The psychological component was explained via the Piagetian notion of a *scheme* "the structure or organization of actions as they are transferred or generalized by repetition in similar or analogous circumstances" (Piaget & Inhelder, 1969, p.4). The evolution of this dialectic between tool and scheme is called "instrumental genesis". This is a complex process over time which links the enablements and constraints of the tool to the agent's prior understandings and activity. Trouche introduces the notion of "gestures". Gestures are observable behaviours and a "scheme is the psychological locus of the dialectic relationship between gestures and operative invariants, i.e. between activity and thought."

Trouche also introduced the term *instrumental orchestration* "to point out the necessity (for a given institution – a teacher in his/her class, for example) of *external steering* of students' instrumental genesis." He started from a study of a particular technological classroom environment as one of many possible forms of orchestration. This environment included students with TI-92s and exercise books, a rotating (amongst the class) "sherpa" student who operates the viewscreen, a viewscreen, a blackboard, specific tasks and a teacher. He noted the instrumental genesis of students via change in their mathematical behavior and their tool trajectories throughout the instrumental process. In commenting on how to support instrumental genesis he argued for strong teacher involvement in the instrumental process and recognition of the constraints and potential of the artifacts and of student behavior.

Hoyles broadened Trouche's discussion of orchestration to elaborate the role of artifacts in the process, describing how the notion of situated abstraction could be used to make sense of the evolving mathematical knowledge of a community as well as an individual. She concluded by elaborating the ways in which technological artifacts can provide shared means of mathematical expression, and discussed the need to recognize the diversity of student's emergent meanings for mathematics, and the legitimacy of mathematical expression that may initially diverge from that of institutionalised mathematics.

Towards the Fourth CAME Symposium in Tandem with PME-NA

> Computer algebra use in mathematics teaching and learning is in its infancy. Nevertheless there are many teachers and educationalists who have integrated CAS into their teaching or conducted research into student understanding with CAS or who have led curriculum/assessment projects involving CAS use. (*Introduction,* Berry et al., 1997)

These words were written after the 1995 ACDCA symposium in Honolulu. Ten years later, innovative classroom uses of CAS still exists, but progress has not been as rapid as many expected. Successes, and also difficulties, have provided opportunities for conceptualizing the complex impact of technology on classroom processes. CAS is not the most popular technology in classrooms. It is challenged or complemented, even in algebra, by numerical or geometrical environments (spreadsheets, dynamic geometry) and by web based applications. As one technology amongst others, CAS appears to have very rich and complex links to mathematical understanding and practices. It is therefore not surprising that didactical conceptualizations, outlined above, that were necessary in order to analyze these links are now recognized as very useful even with regard to other technologies.

In the 4^{th} CAME Symposium we plan to further examine these conceptualizations, beginning with the issues of instrumentation and praxeologies. We will also deal with the mathematics curriculum, namely, the impact of CAS on our understanding of mathematics education, and again try to examine what teachers learn while teaching with CAS. In the PME-NA discussion group, through this paper and subsequent discussions, we hope to familiarize participants with changes in the goals, content, methods and forms of social interaction in mathematics teaching that computer algebra affords and, at the same time, help the CAME community in its efforts in shaping research and development of computer algebra in mathematics education.

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Reims, 2003 http://www.lonklab.ac.uk/came/events/reims/

DISCUSSION GROUP ON MATHEMATICS CLASSROOM DISCOURSE

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This discussion group will investigate the nature and role of discourse in mathematics classrooms. We will analyze, discuss, and interrogate various frameworks for researching the nature and impact of discourse practices in terms of both social and mathematical aspects. We will address related methodological and analytical challenges and consider ways of connecting research with practice. We will structure the sessions around three framing questions relating to theoretical frameworks, analytic techniques, and impact on mathematics education. The first session will begin with multiple analyses of one classroom episode. The second session will build from discussions begun in the first session and will focus on developing future directions for the discussion group and potential writing projects, with the focus on how we might offer a unique contribution.

Rationale for Discussion Group on Mathematics Classroom Discourse

This discussion group will investigate the nature and role of discourse in mathematics classrooms. We will analyze, discuss, and interrogate various frameworks for researching the nature and impact of discourse practices in terms of both social and mathematical aspects. We will address related methodological and analytical challenges and consider ways of connecting research with practice.

The NCTM *Standards* documents (1991, 2000) stress the role of discourse in the learning and teaching of mathematics, yet the mathematics education research community has far to go in its attempt to understand many aspects of discourse (Steinbring, Bussi, & Sierpinska, 1998). Furthermore, there is evidence that discourse practices have not changed much in the last two decades (Spillane & Zeuli, 1999; Stigler & Hiebert, 1999) and there is little evidence of the connection between the nature of discourse practices and mathematics achievement (Steinbring et al., 1998).

In addition to the need for extending present scholarship relating to mathematics classroom discourse, we need to develop more analytic tools that are specifically geared toward *mathematics* classrooms. While we can learn much about the social order of mathematics classrooms using tools developed by discourse analysts, these tools do not take into consideration the specific mathematical content of the conversations taking place (Steinbring et al., 1998). Additionally, issues associated with social class, gender, and race are rarely examined in discourse studies in mathematics classrooms. Focusing discourse studies on inequities can

Lloyd, G. M., Wilson, M., Wilkins, J. L. M., & Behm, S. L. (Eds.). (2005). *Proceedings of the 27th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.*

help us understand the range of language use and interaction patterns students bring to mathematics learning and illuminate issues of authority and power (Atweh, Bleicher, & Cooper, 1998; Herbel-Eisenmann, 2003; Herbst, 1997; Zevenbergen, 2001).

From a practical perspective, research has shown that mathematics teachers' discourse patterns are quite traditional, including those of teachers who are attempting to change their classroom practices (Cohen, 1990; Herbel-Eisenmann, Lubienski, & Id Deen, 2004; Spillane & Zeuli, 1999) and a broader sample of mathematics teachers in the US (Stigler & Hiebert, 1999). This is important given that the reform movement in North American mathematics education has made some particular demands on teachers.

The *Standards* recommend that teachers orchestrate classroom discourse to provide a context "where students learn to mathematize situations, communicate about these situations, and use resources for mathematizing and communicating" (Moschkovich, 2002, p. 197). In order to orchestrate discourse, teachers will need to have strong pedagogical content knowledge (O'Connor & Michaels, 1996), will need to balance social and mathematical tensions, and must decide which student explanations from which to build discussion (Sherin, 2002). Similarly, demands are also made of students. For example, students are expected to take on more responsibility for their learning, posing questions, explaining their thinking, and offering their own ideas about mathematics (Hufferd-Ackles, Fuson, & Sherin, 2004). Forman, McCormick and Donato (1998) state that "new forms of instruction include more active participation of students in providing explanations, conducting arguments, and reflecting on and clarifying their thinking" (pp. 313-314).

Researchers have used various theoretical perspectives to investigate the nature and role of discourse in the learning of mathematics. For example, O'Connor and Michaels (1996), employing a sociolinguistic analysis, document how the use of linguistic moves termed *revoicing* created participant frameworks which positioned students as producers and evaluators of mathematical ideas. Voigt (1996) uses an interactionist approach as a way to connect the analysis of the individual and collective components of discourse. Forman (1996) discusses the implications of employing a sociocultural framework to analyze classroom discourse. She elaborates on how the concepts of legitimate peripheral participation, activity setting, and instructional conversation can be used to understand mathematics reform.

Format for Discussion Group

This discussion group will structure the conference sessions around three guiding questions. These questions are intended to encompass overarching issues for the study of discourse in mathematics classrooms. Research presentations and ensuing discussions will focus on addressing the three questions:

1 What theoretical frameworks might be used to study classroom discourse in demographically diverse settings?
2 What are the specific mathematical characteristics of discourse, and how do our analytic techniques account for these characteristics?
3 How can the study of discourse help us understand and transform the teaching and learning of mathematics?

The initial session will consist of the analysis of a videotaped classroom episode, from a set of videocases and commentaries of a middle school mathematics class (algebra) developed by Jo Boaler and Cathy Humphreys (Boaler & Humphreys, 2005). In this session, three researchers will analyze and discuss the episode, each using a different framework to address the three questions. The ensuing discussion will focus on the constraints and affordances of each perspective.

The second session will focus on: (1) discussions emanating from the first session; (2) readings selected by the organizers, which attempt to synthesize the research on discourse (e.g., Cazden, 2001; Lampert & Cobb, 2003) or present an international perspective (e.g., Setati & Adler, 2000); and (3) developing future directions for the discussion group and potential writing projects, with the focus on how we might offer a unique contribution.

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NEW FORMS OF TEACHING AND LEARNING WITH NETWORKED CLASSROOMS AND METHODOLOGIES TO EXAMINE THEM

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This discussion group will extend work on the development of networked classroom technologies and related activities, turning to examination of classroom interactions these networks foster. Examining the ways in which deep, conceptual understandings of key mathematical ideas are constructed is a common focus across the several projects included. We are also working to understand such phenomena as group-level mathematical discourse and practice (Stroup, Ares, & Schademan, 2004; Abrahamson & Wilensky, 2004), patterns of participation that foster inclusive learning environments (Stroup, Ares, & Hills, 2004; Abrahamson & Wilensky, 2005), phenomenological and psycho-social dimensions of learning (Kaput & Hegedus, 2004), learning about complex dynamic systems by co-constructing them (Wilensky, 2004; Berland & Wilensky, 2004)), and students' cultural practices as resources for network-mediated learning (Ares & Stroup, 2004). Studies of teachers' evolving pedagogies are also underway. Diverse methodologies are being developed to bridge theory and practice across the various projects. This discussion will deepen our interdisciplinary, multi-site dialogue. The specific focus will be to examine the varied evidence we have that teachers and students are affected in unique or important ways as a result of networked activities and the means by which we are gathering and analyzing that evidence. We will leverage this focus to articulate cross-project design principles for framing diverse content vis-^-vis our evolving technological infrastructures and teachers' network-specific practices.

Key Mathematical Learning

We all examine how important mathematical concepts and skills are developed in networked classrooms. For example, students were found to develop calculus-related reasoning and concepts (e.g., relationships among rate, amount, and velocity) as they explored the motion of elevators whose movements they controlled through velocity graphs (Ares, Stroup, & Schademan, 2004). The emerging real-time graphical representations of position and velocity, along with the elevators' motions, were important resources the class drew on to develop increasingly sophisticated understandings. This was evidenced in part by their discourse moving from more qualitative to more quantitative characterizations of rate. Middle-school students who conducted individual statistical analyses pooled collective numerical values that were plotted as sample-mean distributions. The social–mathematical space enabled students to ground the predictive power of numerous samples in terms of a Law of Large *Social* Numbers (Abrahamson & Wilensky, 2004).

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Changing Pedagogies

Interview data indicate that teachers are working harder and differently in networked activities (Ares, Schademan, Evans, & Postell, in prep.). The work is harder because they can't anticipate where the discussions may head. The need to be "light on their feet" in terms of mathematical concepts to pursue, critical insights to highlight, and connections to the course curriculum requires heightened attention and analytical listening. They also cite the fact that students must work collectively for the activities to proceed as changing the ways they manage student behavior and learning, given that students keep each other engaged, cajoling their peers into participating productively. They also worry that the emphasis on public, verbal contributions hinders some students' participation. This aspect also requires a shift for them in the ways they orchestrate participation. In comparing our networked-classroom designs to other reform-pedagogy design, we are addressing possible tradeoffs inherent in the potential homogeneity the network imposes on student engagement and the impact of this in terms of a desired variability in content and activity entry-points.

Methodologies

The approaches by which the projects are gathering and analyzing evidence of networked classrooms' learning-focused activities vary in complementary ways as well, with work focusing on both individual and group-level interactions and knowledge construction. For example, the Patterns of Participation Project (Stroup, Ares, Hills, & Wilensky, 2005) uses the real-time data collection capabilities of these networks to conduct group level of analyses. Evidence of unique influences on mathematical discourse and reasoning indicates that network-mediated learning fosters academic mathematical discourse (e.g., conjecture, visualization, mathematization, prediction, linking multiple representations) in comparison to more procedural knowledge in textbook-mediated activity (Ares, Stroup, & Schademan, 2004). The Project extends those findings by analyzing network-supported interactivity along three dimensions: (1) a content dimension, (2) a socio-cultural dimension and (3) a behavioral biological dimension where the analytical tools of behavioral biology are highlighted. This work moves notions of inclusion beyond deficit driven models that focus on remediation, to approaches that can explicitly engage the full range of learners.

The SimCalc Project's (cf., Kaput & Hegedus, 2004) attention to cognitive and psychosocial features of learning extends analyses of calculus-related learning to examine affective dimensions of network-supported activity and issues of identity. Students invest personal meaning in mathematical objects they construct, which then are given added significance in relation to the collective object that results from individual contributions being displayed together in a public, visual space. Personal investment in contributions to the construction and discussion of mathematical objects (e.g., "that's me up there," "Joe's function." "I'm going backward") is being examined in relation to such things as academic self-efficacy and identity as related to mathematical learning and practice. This focus on affect and identity is integrally related to a focus on cognition, emphasizing the inter-related processes of social and domain-related learning.

The ISME Project (cf., Wilensky, Stroup et al., 2004) examines students' learning to reason about complex systems. we study how student-initiated inquiry into this challenging domain is facilitated by technological tools, and how these practices and contexts impact the implicit formulation of domain-specific heuristics. Student reasoning is viewed through the lenses of "agent-based" and "aggregate" perspectives on complexity - lenses that informed the design rationale, are embedded in the design of tools and activities, and are fostered as efficacious and complementary cognitive tools. Employed modes of investigation are the cluster of methodologies used in design-research for eliciting individual-student pre/post understandings, microgenetic analysis (Schoenfeld, Smith, & Arcavi, 1993), and a grounded-theory bottom-up/top-down formalization of significant categories of student cognition as expressed in student utterance, gesture, and written work.

The WideNet Project examines networked classroom technologies' potential as culturally relevant technology (Ares, 2004; Ares, Schademan, Evans, & Postell, in prep.). Culturally relevant means design and use of technology in ways that honor the cultural practices of students as valuable, legitimate resources for learning; treat use of those practices as central issues in design and implementation; and scaffold students' learning of rigorous academic content by drawing on those practices in service of generative learning (Stroup, Ares & Hurford, in press). Analyses treat mathematics as discursive practice (Moschkovich, 2002), and examine: "activity building," or interactions that comprise specific activities in specific contexts; "socioculturally-situated identity and relationship building," or beliefs, interaction and communication patterns, and attitudes that comprise participants' identities and relationships in specific situations; and "political building," or valued activities, positions, and/or interactions that accord participants status or power (Gee, 1999, p. 86). Marginalized students' cultural practices are highlighted as resources that may expand both the mathematical and social space of networked classrooms.

Format for Sessions

Day One -- opening remarks about focus and goals, followed by a structured poster format for small group work on the kinds of relationships being fostered with teachers, the types of evidence being generated, theoretical frameworks used, and analytical methods being developed. Each project will bring videos and other artifacts to use as springboards for discussions among participants. Handouts with guiding questions will support audience involvement.

Day Two -- continue with structured poster format, but have people from each research group sitting together at each other's site to foster cross-project discussions to promote a multiple-perspectives approach to looking at data and to collaborating with participating teachers. Day Three -- whole group discussion of what we accomplished with specific attention to how we can further cross-project collaboration and inform classroom practice for practitioners working to implement the networked technologies. This discussion will include co-researcher relationships with teachers.

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INVESTIGATING MATHEMATICS TEACHERS' PROFESSIONAL GROWTH: A DISCUSSION GROUP ON INSERVICE TEACHER EDUCATION

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The focus of this discussion group is on frameworks for studying inservice mathematics teacher professional development. Building on work done in the 2003 PME-NA discussion group, participants will continue to discuss research questions and available data collection instruments. We will further our discussion by dedicating a large portion of meeting time to examining viable frameworks for guiding research on professional development.

A Brief History of the Group

At the 2001 NCTM Research Presession, the facilitators of this discussion group led a session titled "Studying Professional Development is Messy Work. What are the research issues?" Approximately 50 people attended. At the 2002 PME-NA meeting in Georgia, the same facilitators began a PME-NA-based discussion group to address continued interest in the issues surrounding research on professional development for teachers of mathematics (Arbaugh, Brown, & McGraw, 2002). Approximately 70 people attended the discussion group, which met twice during the conference.

The 2003 PME/PME-NA discussion group picked up where the 2002 group left off (Arbaugh, Brown, & McGraw, 2003). We continued to focus on our goals of productive conversation and products. Over the two days that this discussion group met, approximately 30 people engaged in a lively exchange of ideas that focused on issues that are central to studying professional development for mathematics teachers. In the end, the group generated information about three important components of studying professional development: essential questions for understanding teacher learning and change in practice; instruments to use while undertaking investigations; and research designs that would support addressing questions of importance.

Focus for the PME-NA 27 Discussion Group: Frameworks for Studying Mathematics Teacher Professional Development

What do we mean by "Frameworks"? Building on the work begun in prior discussion groups, the focus of the 2005 discussion group is on frameworks for studying mathematics teacher professional development. For our work in this discussion group, we "define" frameworks as guiding lenses through which we study teacher development. We adopt this definition from Eisenhart (1991), who argues that educational researchers need to consider a specific framework that guides their research efforts for any particular study. She perceives of research as having three steps that require thoughtful planning. First is the identification of the problem to be studied. Second is the choice of perspective through which to study that problem. The third step begins with data analysis. Eisenhart argues that a framework is critically important beginning with the second step, for it is here that the adopted perspective or framework begins to guide decisions concerning data collection. In the third step, data analysis, the framework maintains its importance in helping the researcher "decide how to reduce the empirical data collected into meaningful categories, how relationships among categories of findings will be specified, and

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what form the explanation for the empirical data will take" (p. 204). Ultimately, a framework provides "a coherent way of thinking about how to organize and interpret the data" (p. 204).

Eisenhart calls these types of frameworks "conceptual frameworks":

> a conceptual framework is an argument including different points of view and culminating in a series of reasons for adopting some points—i.e., some ideas or concepts—and not others. The adopted ideas or concepts then serve as guides: to collecting data in a particular study, and/or ways in which the data from a particular study will be analyzed and explained. Crucially, a conceptual framework is an argument that the concepts chosen for investigation or interpretation, and any anticipated relationships among them, will be appropriate and useful, given the research problem under investigation. (p. 209)

Stein and Brown (1997) provide a useful example of studying teacher learning through two frameworks:

1. Lave and Wenger's (1991) theory of learning through legitimate peripheral participation in communities of practice; and
2. Tharp and Gallimore's (1988) model of learning as movement from assisted performance to unassisted performance through a Zone of Proximal Development (ZPD). (p. 155)

Both of these frameworks are grounded in a sociocultural perspective (we call this the "theoretical perspective" or "theoretical framework") as opposed to a psychological perspective.

The 2005 Discussion Group Agenda

Over the course of the 2005 PME-NA meeting, we intend to address the following:

1. What additional essential questions need to be added to the list generated at the 2003 meeting?
2. What frameworks exist that are useful in addressing the "Essential Questions" generated at the 2003 PME-NA discussion group? Do we have questions about teacher learning and change in practice that cannot be addressed using existing frameworks?
3. What instruments, from the list generated at the 2003 PME-NA discussion group, are appropriate data sources for each framework?
4. What instruments are missing from this list? What needs to be developed?

The Discussion Group's Future

At the end of the PME-NA 27 discussion group session, we will spend time planning for future working groups. This work includes:

1. Setting goals for future meetings.
2. Generating possible products that can come from our work.
3. Committing to participation at future meetings.

An important long-term goal for the group will be to develop and support leadership in the area of research on mathematics teacher professional development. Individuals who are beginning work in this field should benefit from engaging with a community of researchers and examining and discussing the usefulness and limitations of various frameworks and research methods. In addition, this working group will provide a much-needed arena for cross-pollination of ideas among both senior and junior researchers and encourage movement toward a coherent and conceptually rich research base in mathematics teacher professional development.

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RESEARCH ON TEACHING AND LEARNING MATHEMATICS WITH TECHNOLOGY: WHERE DO WE GO FROM HERE?

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The NCTM Standards (2000) state, "In mathematics-instruction programs, technology should be used widely and responsibly, with the goal of enriching students' learning of mathematics" (p. 25). Although many mathematics educators believe in this vision, the research base we have for justifying such a belief is incomplete. Much of what we know about the use of technology in the teaching and learning of mathematics is anecdotal and might be referred to as "possibility" research. We believe there is significant interest in research regarding the use of technology in mathematics teaching and learning and propose the formation of a PME-NA discussion group to investigate frameworks which can move this work beyond "possibility" research. The overall purpose of the discussion group is to address the following questions: What do we really know regarding teaching and learning mathematics with technology? What questions do we ask from here (what more do we want/need to know)? What frameworks, methodologies and collaborations will support the research that will produce this knowledge?

In 1991, the National Council of Teachers of Mathematics (NCTM) stated in their *Professional Standards for Teaching Mathematics* that mathematics teachers should "help students learn to use calculators, computers, and other technological devices as tools for mathematical discourse" (p. 52). This position was a weak though admirable endorsement for the use of technology in the teaching of mathematics. By contrast, the NCTM's *Principles and Standards for School Mathematics* (2000) devoted one of its six overarching principles wholly to technology. The four principles addressing curriculum, teaching, learning, and assessment have long been pillars of their recommendations for educational reform (e.g., NCTM, 1961, 1980, 1989, 1991, 1995). The other two, addressing equity and technology, are not new to NCTM's vision, but their prominence is. The Technology Principle states: "Technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances students' learning" (NCTM, 2000, p. 11). The statement "technology is essential" is strong language. That technology can enhance learning is commonly accepted, although less commonly translated into practice; the claims that technology might influence the very mathematics that is taught is not mainstream thinking in U.S. mathematics classrooms.

The following excerpt further illustrates NCTM's (2000) strong commitment to a reform-oriented approach to teaching with technology:

> Students can learn more mathematics more deeply with appropriate use of technology (Dunham and Dick 1994; Sheets 1993; Boers-van Oosterum 1990; Rojano 1996; Groves 1994).[1] Technology should not be used as a replacement for basic understandings and intuitions; rather, it can and should be used to foster those understandings and intuitions. In

[1] Note that the five sources cited in support of these statements were published since the release of the 1989 NCTM *Standards*.

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> mathematics-instruction programs, technology should be used widely and responsibly, with the goal of enriching students' learning of mathematics. (p. 25)

Although many mathematics educators believe in this vision, the research base we have for justifying such a belief is far from complete. Although some research (as cited above) has made a case for the positive impact technology can have on the learning of mathematics, much of what we know about the use of technology in the teaching and learning of mathematics is anecdotal. That is, we have seen incredible ways technology *can* be used, but "the effects are not yet visible to many" (Kelly, 2003, p. 1038). Thus, we might refer to much of what has been written about technology and mathematics as "possibility" research.

Based on experiences at previous PME-NA conferences, we believe there is significant interest in research regarding the use of technology in mathematics teaching and learning. We propose the formation of a PME-NA discussion group to investigate frameworks which can move research on the teaching and learning of mathematics with technology beyond "possibility" research. The overall purpose of the discussion group is to address the following questions:

What do we really know regarding teaching and learning mathematics with technology?

What questions do we ask from here (what more do we want/need to know)? What frameworks, methodologies and collaborations will support the research that will produce this knowledge?

What follows is an outline of the agenda for the proposed discussion group. This agenda is based on the past pattern of meeting three different days for between 1.5 and 2.5 hours, but could be adjusted as needed.

Day 1

Purpose

To introduce the overall purpose of the discussion group and facilitate an initial discussion surrounding that purpose.

Plan

The organizers of the group will begin the discussion group with a brief presentation, in which they will introduce the proposed purpose and vision of the group. Group participants will then be formed into small groups, in which they will discuss several questions, including those outlined in the purpose statement above and prepare to share the results of their discussion with the full group. The remainder of the time will be spent sharing, comparing and contrasting the small group reports. In a broad sense we hope to address questions such as the following:

- How important is it that research on the use of technology in the mathematics classroom focus time and effort on justifying such use?
- How do varying degrees of access to technology effect the learning that occurs?
- How can we assess the learning that occurs in a technology environment?
- What experiences do teachers need in order to be prepared and motivated for andsupported in productive use of technology in their teaching?
- When considering the NCTM principles, why should technology be given such a prominent position?

Day 2

Purpose

To discuss researchable questions that can move the field forward in the directions outlined on day 1, and to discuss what frameworks exist or need to be developed in order to carry out such research.

Plan

The organizers of the group will begin the discussion by summarizing the results of the previous day's discussion. Group participants will then again be formed into small groups. Each group will be given an area of possible research interest and several related articles (e.g., Burrill et al., 2002; Cadiero-Kaplan, 1999; Doerr & Zangor, 2000; Goos, Galbraith, Renshaw, & Geiger, 2000; Heid, 1997; Kaput & Thompson, 1994; Pierce & Stacey, 2004; Schwarz & Hershkowitz, 1999; Zbiek, 1998) on teaching and learning mathematics with technology, from which they will be asked to extract "where do we go from here?" questions. These sample research areas will focus on the differences that are available (also probable, possible, desirable or undesirable) when one compares learning a given mathematical concept with or without the influence of technology. Examples include the following:

- The geometric understanding (of construction, proof, circles, loci, etc…) that is facilitated using a compass and straightedge versus using dynamic geometry software.
- The algebraic reasoning facilitated by manual versus technological manipulation of algebraic expressions and equations.
- The statistical understanding (of randomness, variability, sampling, etc…) that is facilitated by tactile manipulatives versus computer applets or dynamic statistics software.

As groups consider these questions, they will be encouraged also to discuss the frameworks that might be used, adapted, or created in order to conduct the research that is designed to answer these questions.

Day 3

Purpose

To discuss and plan how the discussion group, as an eventual PME-NA working group, could productively organize and collaborate so as to begin to produce research that will contribute to the research base we have on teaching and learning mathematics with technology?

Plan

Much of this day's activity will be an outgrowth of what is accomplished on the previous two days. Based on those discussions, the group will collaborate in articulating an agenda for the continuation of the group. We hope to encourage members of the group to consider ways in which they might pool resources throughout the coming year in order to move toward the goals of this agenda.

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HAS PME-NA BECOME SUPERSIZED? WHAT CAN WE DO ABOUT IT?

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This year marks the 27th annual conference of PME-NA. Since the 1980 meeting in Berkeley, California, we have grown considerably as a group, reflecting the incredible expansion that has taken place in the field of mathematics education research. It seems appropriate to review the organization of the conference's scientific program at this time and to consider whether the present format meets the present needs of PME-NA. The purpose of this discussion group is to examine data from past conferences, debate the issues, and make recommendations for constructive changes to the Steering Committee. One motivation for this discussion can be found in our present level of success. At the 2004 conference in Toronto there were a total of 139 Research Reports, organized into 8 time slots, of 18 parallel sessions. In addition, there were 92 Short Oral Reports organized into two time slots of 16 parallel sessions each. With this large number of choices, it was difficult to make a selection. Could it be that we are getting too large? This discussion group provides an opportunity to seek a balanced resolution to scheduling problems and to consider issues related to the quality of the presentations, as well as the review process. Alternative venues for information exchange besides those of paper presentation and working groups will also be examined. It is important to the on-going vitality of the annual conference that we engage in this discussion at this time.

Participation Growth

PME-NA has grown since the last time the conference was hosted by Virginia Tech in 1991. At that meeting, 56 research papers were scheduled into 13 different time slots consisting of no more than six parallel sessions. In addition, three non-concurrent "symposia" provided opportunities for group discussions (there were no working groups). The symposia, as well as two non-overlapping poster sessions, were scheduled concurrently with the research reporting sessions, providing the approximately 180 participants with up to seven choices at any one time.

Attendance and participation has increased since 1991. Table 1 shows the location, total number of presentations, and attendance (where figures were available) for all PME-NA conferences from 1999 to 2005.

	Location	# presentations	Attendance
1999	Cuernavaca, Mexico	133	
2000	Tucson, Arizona	191	
2001	Snowbird, Utah	158	390
2002	Athens, Georgia	246	
2004	Toronto, Canada	288	391
2005	Roanoke, Virginia	214	
`03	PME/PME-NA: Hawaii (July)	358	
`05	PME: Melbourne, Australia (July)	226	

Table 1. Recent PME-NA and PME Conferences

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Although there is some variation in the numbers from year to year due to the location, there is a noticeable upward trend in the data. Information for the 2003 conference in Hawaii, which was held in July concurrently with PME, and the 2005 PME conference in Melbourne, Australia are included. (It should be noted that presentations are spread out over five days at PME, as opposed to two and a half days at PME-NA.)

Table 2 shows the distribution, across all eight conferences, of the number of presentations broken down by Research Reports (RR), Short Orals (SO), posters (Post), and working and discussion groups (WG, DG).

	1999	2000	2001	2002	2004	2005		03 Ha.	05 Aus.
RR	83	75	85	122	139	104		176	131
SO	37	54	29	70	92	55		81	61
Post	13	51	34	40	48	43		87	24
WG		8	6	6	8	7		6	2
DG		3	4	8	1	5		8	8

Table 2. Distribution of Types of Session for both PME-NA and PME Conferences

Not only has the number of presentations increased over the years, but the number of papers that are submitted for consideration has risen as well. Table 3 presents acceptance rate data for several conferences. These data must be interpreted with caution since some Research Report proposals that are not accepted in that category are recommended and accepted later as Short Orals. The following table reflects some of this "passing along" from one category to another. Note the similarity in submission levels between PME-NA and PME for these conferences.

	2002		2004			`03		`05	
	subm	accept	subm	accept		subm	accept	subm	accept
RR	199	122	250	139		280	176	184	131
SO	24	70	50	92		92	81	78	61
Post	32	40	91	48		44	87	34	24

Table 3. Number of Papers Submitted and Accepted for each Presentation Category

The increase in the number of papers that are submitted over the years can be interpreted as an indication of growth both in interest in the conference and in the number of professionals in the field. At the same time, constraints of time and available meeting rooms place upper limits on the number of papers that can be accepted each year. The growing success of PME-NA has implications both for reviewing guidelines and for a participant's realistic expectations for presenting a paper at a conference.

Schedule Comparisons

A key measure of the size of a conference is the number of parallel sessions that are required to schedule all accepted papers. The following table shows the number of time slots allocated for Research Reports and Short Orals and the number of parallel sessions scheduled within each time slot for four recent conferences. Also indicated are the number of separate slots allocated for Working and Discussion Group sessions and the time allotted for these special sessions. (The total number of presentations for each year is shown in parentheses in the first row of the table.)

	2000	(180)	2001	(148)	2002	(232)	2004	(279)
	# slots	// sess.	# lots	// sess.	# slots	// sess.	# slots	// sess.
RR	12	5-8	10	8-10	9	13-16	8	18
SO	12[a]	1-4	9[a]	1-3	3[a]	3-4	2	16

	# slots	time/slot	# slots	time/slot	# slots	time/slot	# slots	time/slot
WG	2	2 hrs	2	2 hrs	3	1.5 hrs	3	1.5 hrs
DG	2[a]	1.5 hrs	1	1.5 hrs	2	1.25 hrs	3[b]	1.5 hrs

[a]Sessions ran concurrently with RR sessions. [b]DG ran concurrently with WG sessions.

Table 4. Number of Time Slots and Parallel Sessions Scheduled in each Slot

The trend in Table 4 not only reflects the growing success of PME-NA but also the increasing challenge of concurrently scheduling all of the accepted presentations. As long as the conference remains two and a half days long, participants will have to make choices among many alternatives. It can, indeed, be frustrating to be forced to pick only one session out of eighteen, especially when several cover the same research area. (For example, at the 2004 conference in Toronto, 49 Research Reports touched on some aspect of teacher education, knowledge, or beliefs.)

Even with a large focus on teacher issues, the research that is reported at each conference covers a wide range of interests. Table 5 shows the distribution, by percentage of total Research Reports, for these papers across the different research categories. (The absence of papers in a particular category may reflect the fact that different conferences used slightly different categories to classify the presentations.) Information in each *row* reflects the relative popularity of a topic across time. The numbers in each *column* indicate the concentration of interest, within any one year, among the different research categories.

	99	00	01	02	04
Adv. Math. Thinking	10.8	4.0		10.7	8.6
Algebraic Thinking	13.3	14.7	4.7	9.0	7.9
Assessment	2.4	2.7		4.9	2.9
Geometry	4.8	8.0		6.6	2.2
Learning & Cognition	2.4	9.3	31.8	16.4	
Probability & Statistics	9.6	2.7	5.9	4.9	
Problem Solving	12.0	2.7	12.9	4.1	7.9
Reasoning & Proof			3.5	4.1	10.8
Research Methods		1.3		0.8	1.4
Social-Cultural Issues	9.6	8.0	7.1	7.4	12.9
Teacher education	24.1	17.3	24.7	23.0	23.7
Teach. Knowledge/Beliefs	1.2	17.3	5.9	6.6	11.5
Technology	1.2	1.3	3.5		8.6
Whole #. Rational #	8.4	10.7		1.6	1.4

Table 5. Percentage Distribution of Research Reports by Topic and Conference Year

The table indicates several small shifts in research interests over the years. Most noticeably, there has been a decrease in presentations on whole and rational numbers and a growth in those covering technology. On the other hand, the papers with a teacher-centered focus continue to maintain a high proportion of the total number of conference presentations, varying from 25 to 35 percent each year.

Things to Consider

This discussion group is intended to be an open forum for a constructive evaluation of the existing scientific program of the annual PME-NA conference. Participants are expected to contribute to a vigorous exchange of ideas and engage in imaginative thinking. This group will also make recommendations to the Steering Committee on how to continue the work begun by the discussion group, with the goal of implementing future changes.

Large Numbers of Presentations

The data from the past six years indicate some of the ways in which PME-NA has changed. One of the most salient aspects is the proliferation in the number of parallel sessions. At the same time as the opportunity increases for more people to present papers, attendees find it more difficult to choose among the many presentation options. A goal of this discussion group is to consider ways to maximize the former while minimizing the latter situation.

If the number of parallel sessions is set at a reasonable limit, this puts an upper bound on the number of Research Reports and Short Orals that can be accepted. At issue here is the notion that any and all reports that meet a certain standard of reporting should be included. However, limiting the number of presentations may limit the potential for many individuals to participate. Should the conference be a venue in which any quality report may be presented – giving opportunities for more to participate, or should it become a more manageable vehicle with selected presentations? If the criteria for acceptance become more stringent, does this push for higher standards of presentation, and is this a good thing?

Instead of limiting the number of presentations, an alternative is to increase the number of separate sessions that can be scheduled. A possible suggestion is to limit each Working Group session to three 40 minute periods. (At present the groups meet twice for two hours each.) Limiting the sessions to 40 minutes each would force the groups' organizers to make them more efficient, i.e., spend little time on individual presentations and maximum time on group participation. It is also appropriate to consider whether a conference that lasts only two and a half days can have the luxury of dedicating such a large proportion of its time to working and discussion groups at the expense of the many overlapping paper sessions.

Other ways to increase the number of available sessions include scheduling Short Oral presentations concurrently with Research Reporting sessions, and cutting the number or duration of Plenary Presentations. Past conferences have increased the number of available time slots by scheduling evening sessions. There may be innovative solutions to deal with the growing size of each conference that can maximize participation by the members, not only through presentations but also via scheduled free time for collegial exchanges. Lengthening the conference beyond its present two-and-a-half-day format is also an option.

Different Types of Session

It is important to consider the general goals of those who attend the conference. For some, attendance depends on having a paper accepted. For others, a paper is presented in order to solicit constructive critique from experts in the field. The conference may viewed as an opportunity to listen and learn new ideas about a particular area of research. Groups of colleagues may use the opportunity for cooperative work in an area of shared interest. The conference is also a time in which to renew professional contacts and exchange ideas. PME-NA has always been a very graduate-student friendly organization and the needs of this group must also be kept in mind. It is important to include opportunities for students to meet with each other as well as interact with researchers established in the field.

The alternative modes of presentation at each conference have solidified over the years into plenary sessions and panels, research reports, short orals, posters, working groups, and discussion groups. Given the size of the conference, it may be appropriate to consider other types of information exchange, especially those adopted by larger conferences such as AERA and the joint mathematics associations' winter meeting. For example, research presentations could be organized, in a way similar to the present Short Oral organization, into parallel thematic sessions that run for several hours. Less total time than the present 40 minutes would be allocated for each presentation and its follow-up questions, with the idea that interested persons could engage in further discussions one-on-one during conference breaks.

How effective are the present formats used by working and discussion groups? Are there alternative ways that the needs of the organizers and participants of such groups can be met? Asking the question, "Who benefits, and in what ways?" may allow us to design different modes of information exchange that better fit the time constraints and scientific goals of a PME-NA conference.

Going "electronic" may have implications for the conference organization. With proceedings available electronically before the annual meeting and available on CD after the conference, reporting opportunities may change. If more information about each presentation can be downloaded before the conference, less time may be necessary during the actual meeting for public processing of the report. If this is the case, Research Reports and Short Orals could become more information exchange sessions with less time allocated to a formal presentation. In a similar way, the working and discussion groups could require less meeting time, due to electronic pre-conference networking.

Goals of PME-NA

At present the goals of PME-NA are:

1. to promote international contacts and the exchange of scientific information in thepsychology of mathematics education;
2. to promote and stimulate interdisciplinary research in the aforesaid area with thecooperation of psychologists, mathematicians, and mathematics educators;
3. to further a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implications thereof.

It is appropriate for the discussion group to consider these goals, as well as to question whether they adequately reflect the reality that the annual conference has become. Perhaps a refined set of goals better meets the present needs of the membership.

The way in which this discussion group approaches it work is affected by how participants view the goals and aims of PME-NA. By thinking about the purpose of the annual conference in a different way, innovative alternatives to the status quo may become more evident. PME-NA is more than simply a venue that allows participants to add to their bibliography. The organization needs to feel a sense of mission. For example, the premise that students should, and can, do better than they do at present – and, in association, how can research help? What should the researcher's role be in mathematics education? How does, or should PME-NA facilitate important goals of the field?

What type of research should be presented at the conference? Should papers be accepted because they exhibit sound methodology, but little else? It may be that reports should be valued for reasons other than simply their content - for instance, papers that provide examples of exemplary research for beginners, that push the envelope of what is acceptable, that provide new insights, or new ways of looking at established research. Thinking deeply about what we are and where we want to go are important in order for this organization to maintain, and even expand, its significant role in the area of mathematics education research.

RATIOS AND PROPORTIONS: COMPLEXITY AND TEACHING AT GRADES 6 AND 7

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Introduction

This study relies on a previous survey that I conducted with 120 seventh-graders two years ago. A questionnaire presented for solution five ratio problems based on the same mathematical framework: working out a fourth proportional in combining the same type of data. But each problem referred to a different physico-empirical context (e.g. mixture and enlargement). For a given student, significant variations were found, from one problem to another, both in the success rates and in the procedures used. It seems therefore necessary to better take into account the physical context, and to classify the different types of ratio problems one student can face at this scholastic level, according to physical references. It seems also desirable to better organize, in the teaching, the articulations between and within the physical and mathematical domains. Now, in order to better articulate, it is first advisable to better separate, that is, better point out: differences between physical investigations (multi-sensorial with or without the real or virtual use of instruments) and mathematical ones (formal expressions and processing); differences between the diverse ratio problems on one hand and between the means of expression of rational numbers on the other hand. So that the complexity of ratio problems at Grades 6 and 7 is declined in three levels of separation/articulation. Kieren and Noelting (1980), or Vergnaud (1983) emphasized the importance of physical references in solving ratio problems. I just try here to specify these references and how the mathematical ways of expressing ratios match to them.

Hypotheses

At the considered scholastic level (grades 6 and 7),

1. The complexity of ratio situations can be described by two variables, one referring to the physico-empirical domain[1], the other to the mathematical[2], and the relations between and within their values.
2. Because of its physical and mathematical features, the graduated line, in a specific computer environment, allows to actualize the three levels of separations /articulations within and between the physical and the mathematical domains.

Methodology and Findings

Two groups of pupils were followed during two school years at grades 6 and then 7. The teaching sessions, in the group referenced as the Ful-experimental group, took into account the

[1] Values (the thinkable types of ratio situations): ratio of two heterogeneous quantities" (e.g. speed), "measurement", "mixture", "frequency", "enlargement", and "change of unit".

[2] Values (the considered semiotic registers in the sense of Duval – 2000): Graduated line, fractional and decimal writing.

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three levels of separation/articulation described in the introduction. Furthermore, the mathematical part of the teaching mainly relies, in a computer environment, on the graduated line, according to. In the second group, referenced as the Partial-experimental group, the teaching was based on the same corpus of ratio problems, but was led in a paper-pencil environment and without stressing the different articulations and separations. The findings show that variations (success rate and procedures used) are important from one group to the other. The Ful-experimental has a more complete evolution, which is a better acquisition of fractions and their use for solving usual proportionality problems.

WHAT COUNTS AS "PRODUCTIVE" DISPOSITIONS AMONG PRE-SERVICE TEACHERS

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Mathematics teachers' dispositions may influence the dispositions of their students, which presumably influence students' interactions with mathematics. It seems reasonable then to consider that teachers' dispositions may influence their students' interactions with mathematics. It is therefore important to try and understand the dispositions that pre-service teachers [PST's] may carry with them out into their classrooms. As teacher educators, we must clarify what composes a productive disposition for PST's before we develop more effective ways to nurture the development of their dispositions.

A productive disposition [PD] is defined to be a "habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy." (NRC, 2001, p. 116). In other words, according to the NRC (2001), a PD comprises components of a student's 1) beliefs and attitudes about mathematics, and 2) mathematics self-concept. A student's beliefs and attitudes about mathematics support the inclination to see mathematics as sensible, useful, and worthwhile. Mathematics self-concept constitutes the student's belief in his own efficacy and influences his belief that diligence leads to successful learning.

While the NRC (2001) defined dispositions in terms of students' attitudes and beliefs about mathematics, coupled with their mathematics self-concept, as mentioned above, it is less clear what these dispositions might look like in terms of how PST's talk about their beliefs, attitudes, and mathematics self-concept. In order to operationalize a productive disposition toward learning mathematics among PST's, I designed an interview protocol around these themes. The purpose of the interview is to examine the relationship between the connotations of students' responses regarding their attitudes and beliefs about mathematics, as well as their mathematics self-concept and their past achievement in mathematics. Eight first-year, elementary education majors at a mid-Atlantic state university were randomly selected for the interview. These students participated early in their degree program, while taking the first of three mathematics content courses for elementary education majors. Demographic data were compiled for each respondent. The interviews were audio taped and transcribed. The transcripts were analyzed thematically to build a composite disposition toward learning mathematics [DTLM] of PST's. Trends in the data were compared with the students' demographic data to distinguish a productive DTLM from non-productive dispositions. The hypothesis was that PST's whose responses communicated negative connotations about the sensibleness, usefulness, and worthwhileness of mathematics, as well as themselves as learners of mathematics would tend to have lower math SAT scores, grades in mathematics classes, as well as exposure to mathematics coursework beyond minimally required coursework than those students whose responses communicated generally positive connotations about the same topics. Data did not provide sufficient evidence to support the hypothesis unequivocally; however, trends in PST's responses suggest that PST's DTLM may be dependent on the content, in that, the connotations of some PST's responses regarding the sensibleness, usefulness, worthwhileness of mathematics, or themselves as learners of mathematics differed according to the mathematical topics being discussed.

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VIEWING PROFESSIONAL DEVELOPMENT THROUGH DIFFERENT LENSES: EXPERIENCES OF TWO MAJOR GRANT PROJECTS

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The National Council of Teachers of Mathematics (1991) recognizes the complexities of teaching mathematics well and the need to provide teachers with professional development (PD) opportunities that bridge theory with practice. Some characteristics of successful PD models are that they are long-term, school based, allow teachers to revisit their understanding of mathematical content, and encourage teachers to be active learners (Mewborn, 2003). This presentation offers how two major university grant projects are providing PD that employ these characteristics, among others, for teachers implementing reform-based practices in the teaching of mathematics.

Project Descriptions and Professional Development Opportunities

Project M^3: Mentoring Mathematical Minds is a \$3,000,000, 5-year Jacob K. Javits grant funded by the U.S. Department of Education with the aim of designing challenging curriculum units, increasing attitudes towards and achievement of math for diverse grade 3-5 students with math potential, and providing ongoing PD. Ten schools of varying socioeconomic levels in CT and KY are participating. Teachers take part in an intensive two-week summer training session, PD inservices during the year, and weekly collaborations with a PD Team Member. *Beyond Access, to Math Achievement (BAMA)* is a \$3,600,000, 3-year NY State Department of Education grant with the aim of raising teacher's math proficiency, increasing student math knowledge, and reducing achievement gaps among students. *BAMA* staff is serving 24 schools and 300 students in grades 3-8 in Syracuse City Schools, including 10 instructional support teachers providing PD.

Conceptual Framework

In an effort to better understand how both grant projects are addressing teachers' needs, Bolman and Deal's (1997) frameworks are being implemented. The purpose of their frameworks is to provide leaders with a set of lenses in an effort to consider and implement different approaches to address issues and better meet an organization's needs. The *structural frame* addresses the goals and formalized roles among an organization. The *human resource frame* views the organization as a social network where individuals have particular "needs, feelings, prejudices, skills, and limitations"(p. 14). The *political frame* sees the organization as an arena where groups and individuals compete for power and limited resources. Lastly, the *symbolic frame* considers how an organization also is a cultural phenomenon that has its own way of working and individuals or groups take on certain parts they are expected to play. This framework helps highlight the interactions between the universities and schools collaborating on *Project M^3* and *BAMA*.

Project M^3 and *BAMA* both have encountered accomplishments and challenges in their PD goals. Noteworthy accomplishments and challenges will be presented, including a discussion about the decisions made that addressed them, which frameworks were used to make sense of them, and our reflections about them. District-, school-, grade-, and classroom-level events will be discussed.

Conclusion

Both schools and universities are bestowed the responsibility of providing support for teachers to continue to grow professionally in their teaching of mathematics (NCTM, 1991). Suggestions on how to plan for and improve upon professional development will help facilitate this process for other professional development endeavors aimed at improving field-based mathematics education.

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ISSUES IN COLLECTING AND ANALYZING STUDENT REPRESENTATIONS OF DATA

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This poster session highlights the issues that arose in analyzing student-generated representations of data. The data was gathered from an elementary school while piloting a data unit from *Investigations in Number, Data, and Space*. In order to allow collaboration among the teachers, grades 1, 2, and 3 piloted a second grade unit, and grades 4, 5, and 6 piloted a fourth grade unit. In the beginning of the pilot, the research team determined a subset of activities involving written student work that would be collected to provide information on students' thinking of different aspects of data analysis – representation, description, comparison, and interpretation. This paper focuses only on representations.

The analysis of student work was an interactive process of individual analysis of student work, followed by group analysis, and subsequent revision of the analysis scheme. In the early stages of this process, only portions of the data were analyzed, until the group was sufficiently satisfied with the analysis scheme to warrant analysis of the entire data set. There were four main phases of progression of analysis scheme for student-generated representations. These phases will be presented at the poster session with detailed descriptions and sample student work. While the open-ended nature of the activities made these tasks a rich source of information about students' thinking, the wide variety of the responses made it challenging to create an analysis scheme that could capture the richness of the data. Different grade levels provided information on different aspects of representation of data, therefore the analysis scheme needed to describe and illustrate these differences in an efficient way. Two examples follow.

Consider the following set of data: 13, 1, 12, 0, 4, 13, 4, 4, 6, 0, 14, 12, 12, 6, 2, 4. A common representation for lower-elementary students involved drawings of cube towers to represent each value. In many cases, the towers were not ordered according to height (and thus data value), and in some cases, students chose not to represent the zero values. The research team considered the specific exclusion of zero values as significant, and thus created a criterion for it. However, after analyzing the representations of upper-elementary students (where this was no longer an issue), the team chose to simply include this under the more general criterion of *including all data*. Another issue arose when analyzing student-generated representations that generally resembled a bar graph. Most of the upper-elementary students using this general form of representing the data not only ordered the data, but also indicated holes in the data. For example, when representing the same set of data listed above, many of these students would leave a large gap between bars for the data values of 6 and 12, indicating that there was no data in that interval. In this case, the representation of this hole can significantly impact the way one makes sense of the data, thus the inclusion of this criterion in our analysis scheme was deemed

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essential. While representing holes had not been an issue with the initial lower-elementary student work involving cube towers, it did appear in later samples involving bar graphs.

The student work collected as part of this project was predominantly responses to relatively open-ended tasks, intended to prompt a variety of responses from students. Such a rich and varied data set is, by nature, difficult to analyze by reducing it to a small number of attributes. This process is made even more difficult by the attempt to create a framework that is applicable for student work in grades 1 through grade 6. The value in attempting to create such a framework is that it serves to highlight the issues that students face when they work to construct their own understandings of mathematical ideas (like representing data), rather than simply learning to apply taught understandings (like creating particular types of graphs).

INVESTIGATING VARIATIONS IN PROBLEM-SOLVING STRATEGIES FOR SOLVING LINEAR EQUATIONS

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To investigate variations in students' strategy development, this study engages students in problem-solving interviews, which have been widely used in research on mathematical problem solving (e.g., Star, 2001; Hunting, 1997). Specifically, study participants were prompted to share and explain their ideas before and after they solved problems. Data from these problem-solving interviews were used to identify, categorize and analyze students' developmental changes of strategies in problem solving.

Videotaped problem-solving interviews were conducted with twenty-three 6th grade students (12 males and 11 females). Students participated for a total of five hours over five consecutive days. Each student was given a pretest, twenty minutes of instruction, three one-hour videotaped problem solving sessions, and a posttest. In each of the three one-hour problem-solving sessions, students were asked a series of questions as they solved linear equations, including prompts to explain their choice of problem-solving strategies.

Of particular interest here is the level of sophistication of students' utterances relating to their written strategies. In order to analyze variations in students' strategies (as evidenced by students' utterances), several coding categories were employed. These categories include the consistency between utterances and written strategies, relations between actions and subgoals, goal-subgoal structure, certainty of utterances, speed of utterances, students' justification of strategy choice (e.g., quickest, most accurate, more familiar). Together these categories were aggregated to provide a measure of the sophistication of students' utterances.

There are three main results. First, students' utterances got more sophisticated as they gained problem solving experience. Students gradually increased the detail and rationale included in their descriptions of strategies as they engaged in more problem-solving practice. Second, students' written strategies got more sophisticated as their utterances got more sophisticated. Several students changed or added new written strategies when their utterances displayed more detail about their choices on problem-solving strategies. Third, students became more successful problem solvers (getting more correct answers) as their written strategies and utterances became more sophisticated.

The research reported here can extend our understandings of the developmental stages of problem-solving strategies for solving linear equations that have been highlighted in recent research on mathematical learning (Star, 2001; Catrambone, 1998).

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PRE-SERVICE MATHEMATICS TEACHERS' VIEWS OF THE NATURE OF TECHNOLOGY

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A philosophy of technology is important yet a neglected area in educational research (Flick & Lederman, 2003). The purpose of this study was to explore pre-service mathematics teachers' views of the nature of technology and how such views were related to their pedagogical beliefs in mathematics teaching and learning.

Theoretical Framework

Feenberg (2002) argues that there are three major schools of thought on the nature of technology. (1) An instrumental theory of technology argues that technology is merely a tool or device that is ready to serve the purpose of its user; technology is seen as neutral and apolitical. (2) A substantive theory of technology argues that technology represents an autonomous cultural system and fundamentally controls human thoughts and actions. Technology has been transforming our society to a more technically oriented system where values and questions are re-defined and solutions are directed to technical ones. (3) A critical theory of technology argues that technology is ambivalent in nature and the choice of civilization can be effected by human action. Humanity's future can be found in a democratic advance.

Research Context and Mode of Inquiry

The participants of the study were four pre-service secondary mathematics teachers. Data consisted of (1) a philosophy statement on technology use in education from each of the participants; (2) their electronic portfolio containing lesson plans, papers, project reports, etc. (3) three audio-taped semi-structured interviews in which participants elaborated their notions of the nature of technology and their perceptions of using technology in mathematics education.

Findings and Implications

The four pre-service teachers had homogeneous conceptions of the nature of technology and the role it plays in both education and society. They generally subscribed to the instrumental theory of technology described above. They viewed technology as a neutral tool that is under human's control. In education, they believed that technology could be used to aid learning or it could be used to expose kids to violence, depending on how teachers use it. They also believed that technology had the ability to empower teachers and enhance students' achievement. These findings are consistent with Fleming's (1992) study of 596 Canadian teachers' views on technology. These teachers overwhelmingly took an artifact or tool perspective on technology.

Constructed in this way, technology was treated as transparent. In their discourse on the role of technology in the teaching and learning of mathematics, the pre-service teachers in the study were not aware of the thought-mediating and culture-shaping characteristics of technology as observed by substantive and critical theorists of technology.

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THE USE OF ALTERNATE BASE SYSTEMS IN THE PREPARATION OF ELEMENTARY TEACHERS

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Understanding of the concept of ten is important for pre-service elementary teachers (PSETs). PSETs may not appreciate the complexity of this concept for students since they are familiar with it as adults. However, it is critical that PSETs see that student understanding of this concept (demonstrated by counting by tens, counting on by tens, and trading up and down in base ten) opens the door to many important mathematical concepts such as multi-digit addition, multi-digit subtraction, and multiplication. Pengelly (1990) states that "once all the ideas that characterize the number system are mastered, the structure [of the number system] becomes apparent, incorporating all the ideas that have gone before" (p. 376). Developing this appreciation and understanding of the aforementioned complexity is challenging for PSETs.

The purpose of this poster is to examine the progression PSETs make as they transition from content to methods courses. At Purdue University, PSETs are required to pass a series of three undergraduate (100 level) courses in mathematics during the first and second years of the program. While in the third year of the program, students are required to pass a methods course called Mathematics in the Elementary School. This course centers on using a problem solving approach to teaching mathematics. In both courses the use of base eight is implemented. In the first course of the mathematics series, different bases are introduced to help PSETs understand the difficulties their students may encounter while learning different operations in base ten. In the methods course, PSETs experience base eight while thinking about problem solving strategies that they may observe students using in the field experience component of this course.

This poster presents findings from a small scale research study of PSETs in these two courses. The questions guiding this study are: As PSETs progress through their mathematics content course into their elementary mathematics methods course, does their understanding of why the use of alternate base systems is included in the respective course curricula become more consistent with the stated course learning goals? To what extent does this dual approach, from methods and content courses, support PSETs in their development of this understanding? It is our intent that these questions will give us insight into the PSETs development of student understanding of the concept of ten and reveal connections PSETs are making between content and methods courses.

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EXAMINING PROSPECTIVE TEACHER SUBJECT MATTER KNOWLEDGE THROUGH STUDENT QUESTIONS

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Focus and Background

An essential part of teachers' knowledge that goes beyond specific topics within a curriculum is the subject matter that is to be taught. It includes, in very broad terms, the topics, facts, definitions, procedures or algorithms, concepts, organizing structures, representations, influences, reasons, truths and connections within the area of study and the connections outside the area of study to other areas. Leinhardt and Smith (1985) defined mathematical Subject Matter Knowledge (SMK) as the knowledge of "concepts, algorithmic operations, the connections among different algorithmic procedures, the subset of the number system being drawn upon, the understanding of classes of student errors, and curricular presentations" (p.247). This definition suggests that SMK has several influences that shape the learning and teaching of prospective teachers. Ma (1999) found several parallels between elementary teachers' SMK and the ability to function as an effective teacher within the classroom. A teachers' SMK influences both their actions in the classroom and their interactions with students. To further emphasize the importance of SMK, Shulman (1986) stated that a "teacher need not only understand *that* something is so; the teacher must further understand *why* it is so" (p. 9).

The Study

Thirty-one prospective teachers (PT) were asked a series of questions related to content that is usually taught in an Algebra 1 or Algebra 2 high school course. The questions were designed as student questions and the PT did not have prior knowledge as to the question being asked. At first the PT would be given 5 minutes to answer the question, as if the student had just asked the question in a class they were teaching. These responses were then collected and the PT were then allowed to form groups of 5-6 and then try and answer the question as a group. Then after about 10-15 minutes of discussion the groups were asked to present their ideas to the class.

Discussion

This poster will display several examples of answered questions. The purpose of the student questions were used as a reflective tool to examine the PT SMK and use these questions as opportunities to recognize that they need to know more than just the facts, terms, and concepts, but *how* and *why* they work.

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EFFECTIVENESS OF THE 'CHANGE IN VARIABLE' STRATEGY FOR SOLVING LINEAR EQUATIONS

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In this research, students' strategies for solving linear equations were examined. Of particular interest was the strategy referred to as "change of variable" or CV. CV was found when students rewrote terms such as 3(x+2) + 6(x+2) as 9(x+2). There are very few research studies which attempt to understand students' strategies to solve linear equations (e.g., VanLehn & Ball, 1987; Pirie & Lyndon, 1997). In prior research in this area, researchers have not commented on the use of the CV strategy to solve linear equations.

157 students who had completed 6^{th} grade participated in five one-hour problem-solving sessions on linear equation solving. Students were given a pretest and then a short lecture (20 minutes) in which the researcher introduced four different steps for solving equations (adding to both sides of equation, multiplying both sides of the equations by the same constant, distributing, and combining variables or constants) to solve linear equations. After that, students worked to solve a series of linear equations for three one-hour sessions.

25% of students used CV at least one question throughout pretest and posttest. An analysis of the time that students spent solving each problem indicated that students who used the CV strategy spent less time than students who did not use CV. For example, the average time for all CV users was 3 min 41 seconds to solve CV questions throughout the pretest and posttest, while the average time for non-CV users to solve the same questions is 4 min 53 seconds, a difference that is significant. Not only did students who use CV solve problems faster, but they also used fewer steps to solve each problem, on average. While CV users typically solved problems in 3-5 transformations, non-CV users solved the same problems in 4-7 steps. The use of CV enabled solvers to solve problems both quicker and in fewer steps. Finding shorter and quicker solution paths is not only important for solution efficiency but it reduces the chance of error (VanLehn and Ball, 1987). Non-CV users had significantly higher rates of error on CV questions as compared to CV users.

CV is an example of an innovative strategy for solving linear equations, but it has received little attention in prior research on linear equation solving. This study represents an initial attempt to investigate the prevalence and use of CV among beginning algebra learners.

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MATHEMATICS COMMUNICATION AS EARLY FIELD EXPERIENCE

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Dewey (1904/1964) believed that in order to be able to hear and extend students' thinking in a given subject matter, ones own study of the latter should include reflecting on where students of different ages might be in relation to understanding the necessary pre-concepts and how one would build on this understanding. A century later, the U.S. Department of Education identified this kind of integrated learning – "developing teachers' mathematical knowledge in ways that are directly useful for teaching" – as the first of three areas for a proposed long-term research and development program (RAND Mathematics Study Panel, 2003), while new guidelines for teacher preparation also call for such integrated experiences (CCTC, 2003). However, given the sheer number of teachers prepared, limited personnel, and limited access to K-12 students, many mathematics teacher educators find it difficult to provide such structured early field experiences, let alone to integrate it into the mathematics content courses.

Online mentoring (OM), offered through Drexel University's Math Forum, provides a way to engage large numbers of future teachers in doing mathematics, communicating mathematics, and mentoring students who submit solutions to problems. Although OM does not let future teachers interact with students face to face, its ability to provide them time for reflection in the process of mentoring, is an advantage as an early field experience, since it allows novice teachers to focus on the mathematical content and on individual student thinking. As one future teacher stated, "I learned how to identify patterns of reasoning, differences in approach, and clarity of ideas." While student teaching, Jiang credited her experience with OM as the reason why she tried to anticipate different ways her students might solve a problem and why she encouraged them to communicate their thinking, both verbally and in writing. A year earlier, Jiang reflected on her OM experience as follows: "From the in-class discussion I saw that even my own classmates had different methods for finding the answer. This helped me understand that my students will also have different methods for finding the answers. Furthermore, the different methods were proven to me so now I can see for myself how and why [their] methods work."

Does participating in mathematics courses that integrate early field experiences emphasizing communication have a significant impact on the pedagogical content knowledge of teachers? The integration of online mentoring into mathematics courses for prospective teachers and a preliminary analysis of data related to the latter's pedagogical content knowledge, while in the course, and as novice teachers, will be discussed.

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SOFTWARE FOR THE YOUNGEST MATHEMATICIANS: CONNECTING QUALITATIVE, MULTIPLICATIVE AND ADDITIVE WORLDS WITH METAPHORS

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This poster focuses on children two to four years old, working with software modules. Each module is based on a metaphor mathematizing everyday actions. The recursive metaphoric process (Sfard, 1997) connects a source, consisting of more concrete, better understood images, and a target, which is the new, more formal concept being constructed. Children start from qualitative work, and then move to additive and multiplicative worlds within the same metaphor (Droujkova, In review). This approach allows systematic, "algebrafying" approach to different expressions of the fundamental mathematical ideas (Carraher, Brizuela, & Schliemann, 2000). Here are examples of modules, each coordinating several games and activities.

Hide-and-seek Equations

The image of *hiding* serves as a source of the metaphor that targets the idea of *unknown*. From the qualitative actions of figuring out *which* of different cartoon characters are hiding, children gradually move to quantitative actions of *how many* of similar characters are hiding. These actions, which can be additive or multiplicative depending on the game setup, are supported by the same hiding metaphor.

Grid Road Tables

The image of a grid of *roads* running at right angles to each other serves as a source for the metaphor that targets the ideas related to *tables*, such as covariation. Delivery cars children drag along each row or column "road" distribute qualitative features to be combined within cells, such as shape and color. The same metaphor is extended toward quantities combined in additive or multiplicative operations.

Function Machines

The image of a *machine* transforming the input serves as a source for the metaphor that targets *function*. Initial rules are qualitative, such as a machine transforming baby animals into adults. The metaphor of the machine coordinates additive and multiplicative operations. For example, "mirror machines" double or triple images symmetrically.

Fractal Power

The image of an *iterated splitting action* such as folding, branching or fragmenting (Droujkova, 2003) serves as a source corresponding to the target of unitizing and powers. The software supports the iteration of children's drawings, and the transition toward quantifying and representing the splitting actions.

There is an increasing need to support early mathematics education. Learning before the age of five may determine children's future success. Computer games can be a tool changing the ways young children and their parents or caregivers approach mathematics (Clements, 2002).

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A STUDY ON THE USE OF NETWORKED TABLET PCS IN THE ELEMENTARY SCHOOL CLASSROOM

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Research on networked Tablet PCs (Simon 2004) and other classroom networking devices (Kaput 2000) has focused on experiences in mathematics or computer science classes at the high school and college level. Although elementary school students have different needs from their older counterparts, they may also benefit from this emerging technology.

This preliminary study looks at two small K-8 Charter schools using Tablet PCs with wireless connection, enhanced with the software package Athena® (excelleworks, Inc.; Red Bank, NJ). Tablet PCs are laptops with the usual functionality, but with the additional capability of being used with a stylus pen as an individual whiteboard. The Tablet screens lay down flat so it is not unlike writing on a piece of paper. The Tablets are connected with a wireless network, and Athena allows the teacher, on his/her computer, to see each student's screen as they work. The teacher can use this copy of the student's screen to annotate his work and make comments and/or suggestions, which the student then sees on his own Tablet. Also, an individual student's work can be easily displayed as an overhead, via the wireless connection.

Teachers can use prepared lessons provided with the software, create their own lesson ahead of time, or use Athena® as they would use a set of whiteboards. As the class moves through each screen of a lesson, previous screens are saved for future reference. Thus, in effect, the Tablet PC is being used as an electronic notebook.

The research question is: To what extent are Tablet PCs being used in elementary classrooms to enrich the mathematical experiences? We look at ways in which the technology is used effectively towards this goal and outline ways it could be use more effectively. Informal discussions with teachers and administrators at the schools indicated more initial success at the younger grade levels. Thus, as a starting point the study focuses on Tablet PC-based Math lessons in Kindergarten through 3rd grade classes.

Observations take place once a week for several weeks; the tasks of the lessons vary. Preliminary results suggest:

- Students as young as Kindergarten use all the options offered with no difficulty.
- Students are attracted to the use of the technology and are focused on lessons.
- Teachers do not take full advantage of the capabilities of the technology, e.g., they often miss opportunities to display student's work to demonstrate different ways of thinking.

We will report on follow-up interviews with each of the classroom teachers which include questions focusing on why the teachers did certain things and why they did *not* do certain things during the observed classes. We will also report on teacher expectations, discoveries, and frustrations with the new technology.

An understanding of current practices may enable researchers to influence both the development of this emerging technology and the role it plays in elementary mathematics education.

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GENDER DIFFERENCES IN CHILDREN'S ARITHMETIC STRATEGY USE AND STRATEGY PREFERENCE

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In recent years, research revealed gender differences in first, second, and third grade children's strategy use for addition and subtraction problems (Fennema et al., 1998). More girls than boys used modeling, counting, and standard algorithms, whereas more boys than girls used invented strategies that take advantage of place-value properties in the base-ten number system. The current research study was designed to replicate and extend the findings from Fennema et al. (1998). The replication study re-examined existing data collected by Hiebert & Wearne (1992) from 72 children as they progress from first through third grade. Only children's strategies for multidigit addition and subtraction story problems that required regrouping were reanalyzed. The extension study used data collected from 15 second-grade children to explore their strategy preferences and their rational for the preferences.

The findings from reanalyzing the Hiebert and Wearne (1992) data suggest that gender differences in strategy use exist but the size of the differences are smaller than those reported by Fennema et al. (1998). No differences were found between first grade boys and girls addition and subtraction strategy use. In second grade, 1) more boys (100%) than girls (70%) used invented strategies to solve multidigit addition and subtraction story problems, 2) boys used invented strategies more frequently than girls did with a moderate effect size of 0.69, and 3) boys obtained more correct solutions than girls when using invented strategies with a large effect size of 1.13. No differences were found between second grade boys and girls use of counting strategies or standard algorithms. In third grade, 1) slightly more girls (96%) than boys (87%) used standard algorithms to solve multidigit addition and subtraction story problems, 2) girls used standard algorithms more frequently than boys did with an effect size of 0.55, and 3) girls obtained more correct solutions than boys when using standard algorithms with an effect size of 0.57. No differences were found between third grade boys and girls use of counting or invented strategies.

Findings from the exploratory extension study of 15 second graders showed most children preferred easy-to-use strategies. However, ease of use meant different things to different children. Several children claimed a counting strategy using manipulatives was easier because the demand on their memory was minimal and they could see and touch the items that need to be counted. Some children thought using the standard algorithm was easier because they knew the procedure, could keep track of sums or differences by writing down quantities, and it was faster than counting cubes. One child believed using an invented strategy was easier because it was less time consuming than counting cubes and it required less physical labor than writing an algorithm. In this small sample, the children's explanations did not appear to be related to gender but rather to their understanding of place value. Whether or not understanding of place value is related to gender will be investigated with a forthcoming study using a larger sample.

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TEACHER LEARNING THROUGH THE STUDENTS TRANSITIONING TOWARD ALGEBRA PROJECT

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The Students Transitioning Toward Algebra project is a partnership between the Florida State University Mathematics Department and Middle and Secondary Education Department in collaboration with the North East Florida Educational Consortium [NEFEC], and six member districts identified as high-need in accordance with project criteria. The goals of Students Transitioning Toward Algebra include the following: (1) the enhancement of middle grades (4-8) teachers' ability to prepare students for success in high school mathematics, particularly algebra-based courses; (2) the development of a learning community of middle grades teachers knowledgeable in mathematics content and instructional practices consistent with national, state, and district standards and curricula, as well as the Just Read Florida! initiative; (3) the enhancement of middle grades teachers' use of technology and manipulatives for promoting students' mathematical development in their transition toward algebra; and (4) increased mathematics achievement of middle grades students.

Project participants are 46 middle grades (4-8) mathematics teachers selected in teams of at least two per school in vertical groups across elementary and middle schools. Project teachers participated in two weeks of summer institute and six face-to-face and web-based meetings across the academic year. A matching control group of teachers was selected in order to investigate teacher learning and the resulting effect on students of teacher participation in the project. Data was collected through multiple data sources including surveys, content and pedagogical knowledge instrument, classroom observations, and interviews.

Students Transitioning Toward Algebra is significant in various ways. It will provide insight into how teachers develop their knowledge and use of instructional practices in ways consistent with national, state, and district standards and curricula through collaboration with mathematicians, mathematics educators and curriculum specialists; how to facilitate the development of a learning community of teachers that can help other teachers develop instructional practices that facilitate middle grades students' transition toward algebra; and how these instructional practices impact on students' understanding of concepts that support success in algebra based courses.

Initial results reveal project teachers’ growth in algebraic thinking and the implementations of lessons that foster their students’ algebraic thinking. Teachers are broadening their view of the definition of algebraic thinking and our building confidence in their own knowledge and teaching in this area. Teachers report that their students are enjoying project tasks and are developing their own algebraic thinking.

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YOUNG CHILDREN'S MATHEMATICAL PATTERNING

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Mathematical patterning is fundamental to the development of mathematics. Steen (1990), in fact, argued that “Mathematics is the science and language of patterns” (p. 5). The years prior to formal schooling (pre-compulsory education and care services) are widely recognized as a period of critical development where the salient role of patterning features significantly.

In a multi-case study children’s engagement in mathematical patterning experiences was investigated as was the teachers’ involvement in, and influence on these experiences. The study was conducted in one preschool and one preparatory year setting. These sites were typical learning environments for Queensland children in the year prior to compulsory schooling. Multiple sources of data were collected. These data comprised semi-structured interviews with each teacher, copies of their daily programs and video-taped observation of the classes. Ten episodes of mathematical patterning were identified and categorized as teacher-planned, teacher-initiated, or child-initiated. Two episodes were initiated by children and the other eight were guided by the teachers. The nature of the teacher intervention in the child-initiated activities was of particular interest. Frameworks were developed to guide the examination of these episodes, with these frameworks being informed by the conceptual framework of Stein, Grover and Henningsen (1996).

The findings of this case study suggest that child-initiated episodes containing mathematical patterning are productive learning occurrences. During unstructured play times, children initiated activities that explored repeating patterns, pattern language, and the elements of linear patterns. These episodes were rich opportunities where children shared, refined, and developed their knowledge of patterns. Thus, child-initiated experiences can be powerful learning opportunities with the potential to develop children’s knowledge of mathematical patterning in meaningful contexts.

The findings also suggest that teachers’ understanding of patterning as well as their engagement in, and influence on child-initiated episodes impacts significantly on the outcomes of the event. Teachers play a myriad of salient roles to assist the development of mathematical patterning. The role of the teacher in questioning, providing resources, being involved, and offering encouragement has the potential to enrich mathematical patterning experiences and extend the children’s existing knowledge. Likewise, teachers’ limited knowledge of patterning concepts and processes, and the confines of their teaching competencies can hinder the outcomes of patterning events.

The poster will illustrate some of the above findings and will include a focus on how teachers’ intervention can either extend or inhibit children’s development of mathematical patterning.

Many early childhood professionals now agree that children should be “guided if not taught” to do some mathematics (Ginsburg et. al., 1999). When teachers understand what to teach, when to teach, and how to teach, they can provide rich opportunities for children to engage in patterning experiences, and capitalize on child-initiated learning activities.

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EXPERIENCE AS A POWERFUL TOOL FOR MEANINGFUL LEARNING OF PROBABILITY

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Ordinary problems using standard algorithms do not enable students to understand probabilistic situations meaningfully. Like in other branches of mathematics, students learning probability need to be involved in authentic situations that motivate their way of thinking. Littlewood (1953) declared that a good mathematics riddle (or joke in his words) is worth more than a dozen fair exercises. Probability is very much connected to every day life, but the synthesis between determinism and uncertainty makes it difficult to understand. The theoretical models used in explaining probabilistic thinking sometimes contradict intuition, which is based on every day experiences. Our experiences are deterministic, not continuous, and usually not guided (Rokni, 2001). Using guided experiences concerning probabilistic situations which derive from authentic problems may result in meaningful understanding of probabilistic principles.

The Research Question

What are the sources of mistakes in solving probabilistic problems and how does experience help lead to meaningful understanding of the correct answers.

Subjects

16 pre-service junior high school teachers.

Instrument

A questionnaire with three authentic situation probabilistic problems:

a. What is the probability of finding at least two people whose birthdays are on the same date, among 30 random participants?
b. There are three doors. Behind one door there is a prize. You are asked to guess where the prize is. After you guess, one of the other two doors is opened and you see that the space is empty. Now you are given the opportunity to change your guess to the remaining door. Will this increase your chance of winning? If so, what is the probability?
c. You have 2 discs: one is red on both sides and the other is red on one side and blue on the other. You choose one disc at random, put it on the table and you see red. What is the probability that the other side of this disc is also red?

Procedure

***Step I*.**

a. 15 of the 16 participants wrote that the probability of finding at least 2 people whose birthday is on the same date is very small and may be 30/365, because there are 365 days in a year. Only one gave the correct answer based on previous learning of such a situation.

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b. All the participants wrote that changing the guess does not increase the probability. The only difference is that the probability changes from 1/3 to 1/2 in both cases.
c. 15 of the 16 participants wrote that the probability that the other side is red is 1/2, because the are only 2 discs, one with red on both sides. Only one participant (not the same one who answered item (a) correctly) gave the correct answer using an intuitive explanation based on 4 sides.

Step II

After sharing the results with the subjects, the researcher gave intuitive-logical explanations using demonstration and modeling for the 3 problems.

Step III

Some of the participants experienced cognitive dissonance after hearing the explanation because it did not fit their intuition and/or past experience.
The researcher then performed 3 experiments, one for each question above. The students participated directly and individually in all the experiments:

a. Collecting birthday data.
b. Simulating the 3 door game.
c. Playing a game with 2 discs.

Results

After step 3 (the direct experience), all 16 participants were convinced about the correct answer and changed their way of thinking about uncertainty.

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INTERVIEW EFFECTS ON THE DEVELOPMENT OF ALGEBRAIC STRATEGIES

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This research explored the effect that interviews and the presence of an adult 'helper' had on novice algebra students' work solving linear equations. 84 students participated, one hour each day, in a weeklong summer 'camp' before entering seventh grade. On the first day, they completed a pretest and were introduced to four operations to solve algebraic equations (adding/subtracting to both sides, dividing both sides, distributing, and combining like terms). Students spent three days working through linear equations before completing a posttest on the last day. Of the 84 students, 23 were randomly selected to work individually beside an interviewer. These students performed similarly on several pretest measures as non-interview students. For each interview student, an adult prompted the student to explain his/her work, reasoning, and strategies before or after solving selected problems. The adult did not provide assistance in completing problems but supplied encouragement and prodding ("What do you think you should do next?" "Nice work!"). The other students worked individually on the problems, were not interviewed, and were essentially provided no feedback as they worked.

The literature is clear on how the presence of an adult helper, even one who does not provide explicit help but merely words of encouragement, impacts student learning. Studies have documented the positive effect an adult helper can have in one-on-one learning situations (e.g., Bloom, 1984). In addition, the self-explanation literature (e.g., Chi, Bassok, Lewis, Reimann, & Glaser, 1989) suggests that students who are asked to verbalize or explain their problem-solving steps are more likely to develop deeper knowledge. Even when students are not self-explaining but merely describing the reasons behind their choice of strategies, beneficial effects have been found (Aleven & Koedinger, 2002; Stinessen, 1985).

However, in this study, students who worked with an adult did not benefit as much as other students. Interview students were *less* likely to get three of the eight post-test problems correct; a similar, although not significant, trend was observed with the remaining problems. Interview students used *more* problem-solving steps, and were therefore defined as less efficient, in correctly solving two post-test problems; a similar trend was seen for the other problems. These results raise questions about the benefits of self-explanation and an adult presence. Additional work should explore whether interviews may actually lessen algebra students' efficiency and their likelihood of solving problems correctly, particularly for novice learners.

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ENGAGING MATH FACULTY IN TEACHER PREPARATION

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The process of preparing prospective teachers needs to engage experts in the content areas, usually housed at the Colleges Arts and Sciences. Most of the time, the College of Education and the College of Arts and Sciences do not work collaboratively in this process but both groups could bring to the table important insights for the successful preparation of teachers. At The University of Alabama a group of faculty from both colleges has been working for three years, redesigning courses, adapting materials, and engaging other faculty in this endeavor. With the support of the Mathematical Association of America (MAA) through the Preparing Mathematicians to Educate Teachers (PMET) program, we have sponsored two workshops for faculty at math departments in our and other local institutions. The main purpose of these workshops is to have a dialogue among math faculty and math educators about courses for pre-service elementary school teachers. Additionally, we have two more goals, to get more faculty at our institution involved in teaching these courses and to provide us with feedback on the design of the courses. Both one-day workshops have been organized with the same structure. The topic and the relevant literature are presented in the morning, followed by a demonstration class with volunteer students from the teacher education program and a discussion of the lesson. In the afternoon the discussion is more open, having more input from participants and creating an opportunity for school teachers to share their experiences. The first workshop was held in Spring 2004 with 23 participants. The main topic for this workshop was the first course designed for pre-service teachers. The content of the course is Numbers and Operations. We used this workshop to present the recommendations from CBMS-MET report and some of the leading research in the area of teachers' content knowledge. We also discussed the role of the teacher in a classroom that promotes understanding and the demonstration class showed how this role could be modeled in a teacher preparation course. The second workshop was held in Fall 2004 with 30 participants. In this case the main topic was the use of technology in the preparation of prospective teachers. The two software programs used in the other two courses for teachers were presented with examples of how we use them in the courses. Geometer's Sketchpad and Fathom are used in the Geometry and the Data analysis courses. We discussed the technology standard from the *PSSM* (NCTM, 2000) and research related to the use of technology in math courses. Again, the demonstration class showed how the use of Geometer's Sketchpad supports student's development of mental images of geometric shapes. Teachers who participated in the Data analysis course shared their experiences and the influence the use of technology had in their understanding of statistical concepts. Both workshops were very well received by the participants and the evaluations show the need for more activities like these. As a result of the workshop, we have one new faculty member teaching the courses for prospective teachers and two ready for next fall. We have also organized a cadre of faculty from other local colleges and universities interested in teaching these courses. One of the projects for the near future is to provide all the resources and training for offering the courses at their institutions.

MULTIPLE SOLUTION STRATEGIES FOR LINEAR EQUATION SOLVING

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Although an algorithm (referred to here as the "standard algorithm" or SA) exists for solving linear equations, its use does not always lead to the most efficient solution (VanLehn & Ball, 1987; Star, 2001). For example, several possible solution strategies for solving an equation are shown in Table 1:

Table 2: Several possible solution strategies for a sample linear equation

Strategy I	Strategy II	Strategy III
3(x+2) + 5(x+2) = 8 3x + 6 + 5x + 10 = 8 8x + 16 = 8 8x = -8 x = -1	3(x+2) + 5(x+2) = 8 8(x+2) = 8 8x + 16 = 8 8x = -8 x = -1	3(x+2) + 5(x+2) = 8 8(x+2) = 8 (x+2) = 1 x = -1

The first strategy is the SA. The second strategy ("change in variable" or CV) uses an alternative in which (x + 2) was treated as a unit and then combined in the first step. The last strategy ("divide not last" or DNL) uses both CV and another transformation in which the equation is divided by 8 as an intermediate step, rather than as a final step (as is the case in the other two strategies). We will consider strategy III (which uses both CV and DNL) to be the most efficient, given that it involves the application of the fewest transformations. Of interest in the present research is how students learn to use and be flexible in their use of multiple strategies for solving linear equations. In this study, we were particularly interested in the effect of direct instruction of multiple strategies on students' ability to be flexible.

Method

153 sixth-grade students participated in the study. In the first one-hour session, students completed a pretest and were then introduced to the steps that could be used to solve equations. Students then spent three one-hour sessions working individually through a series of linear equations (similar to the one in Table 1). In the last session, students completed a posttest. Half of the students received an eight-minute presentation on how to use CV and DNL strategies (the "strategy instruction" or SI condition). The other half of students saw no examples of solved equations (the "strategy discovery" or SD condition).

Results

Although the SI and SD conditions had a similar effect on students' use of SA, *all* of the students who used CV in the post-test were in the SI condition. However, even after receiving a demonstration of the most efficient strategy (strategy III), *all* students who used CV only did so using strategy II. We interpret these results to suggest that students were able to initiate the most efficient strategies only through direct instruction, which is consistent with work by Schwartz and Bransford (1998).

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A VISION OF A THREE-DIMENSIONAL RE-CONCEPTUALIZATION OF MATHEMATICAL KNOWLEDGE

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Traditionally, mathematical knowledge has been classified along a single dimension: procedural knowledge vs. conceptual knowledge, as laid out by Hiebert and LeFevre in 1986. I propose that a three-dimensional model is more appropriate for classifying students' mathematical knowledge.

Hiebert and LeFevre defined conceptual knowledge to be knowledge that is rich in relationships; "A unit of conceptual knowledge cannot be an isolated piece of information; by definition it is a part of conceptual knowledge only if the holder recognizes its relationship to other pieces of information" (1986, p. 4). On the other hand, they defined procedural knowledge to include knowledge of the formal symbol representation system and of the rules, algorithms, or procedures for completing mathematical tasks. Hiebert and LeFevre suggested that the primary relationship between procedural knowledge units is "after," indicating that procedural knowledge is comprised of sequences of linearly related steps. Hence, conceptual knowledge was viewed as understood (well-connected), while procedural knowledge was not.

Star (2000) argued that the traditional usage of the terms procedural knowledge and conceptual knowledge obscures the myriad ways procedures and concepts can be known. He added a depth dimension to Hiebert and LeFevre's (1986) classification system to account for his observations. The traditional definitions do not easily accommodate all units of mathematical knowledge. For example, it is difficult to categorize the memorized facts of mathematics (such as the definition of slope or the commutative property of addition) as conceptual knowledge. After all, conceptual knowledge is supposed to be understood; yet facts can be memorized without being understood. Consider also the long-division algorithm as a classic example of procedural knowledge. An advanced student might know not only how to do the procedure, but also when to apply the procedure, how to predict the answer, and how to interpret the answer in a meaningful way. Adding a depth dimension allows us to more precisely classify students' procedural and conceptual knowledge. Well-memorized (but disconnected) procedures and concepts are labeled shallow; well-understood procedures and concepts are labeled deep.

There is still another dimension of mathematical knowledge that is of interest. During students' initial interactions with a concept or procedure, their knowledge can be considered tentative. In this early stage, execution of procedures may be error prone and require great cognitive effort, and conceptual facts may be recalled slowly or inaccurately. In time, this tentative knowledge becomes automatized and efficient, so that procedures can be executed fluently and facts and connections can be recalled on demand. This suggests the need for a third dimension to account for students' developing aptitude.

Therefore, I propose three dimensions of mathematics knowledge: type (procedural vs. conceptual), depth (shallow vs. deep), and aptitude (novice vs. practiced). My poster presentation will include a visual representation of the proposed three-dimensional model, as well as additional examples to illustrate the model and my vision of how this model can inform mathematics research and everyday classroom practice.

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PRE-SERVICE MATHEMATICS TEACHERS' CONTENT TRAINING: PERCEPTIONS AND THE "TRANSFORMATION" OF MATHEMATICS KNOWLEDGE FOR STUDENT TEACHING

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The mathematical knowledge required for effective teaching is a topic currently receiving much attention (e.g., Wilson, Floden, & Ferrini-Mundy, 2001; Ball & Bass, 2000; Ball, Lubienski, & Mewborn, 2001). Correspondingly, questions are being asked about "how many" and "what kind" of coursework can best serve mathematics teacher education students (TESs) as they begin their journey to become increasingly proficient instructors. In an effort to gain insight into this important area of inquiry, we conducted a research study to examine TESs' course taking patterns, their perceptions and understandings of their formal training, and how this training was a resource for them during their student teaching experiences and their future work as full-time teachers.

Subjects for this study were a cohort of mathematics education majors (n=16) at a large public university. Data were collected in the weeks prior to and during the student teaching experience. The data collection included surveys, transcript reviews, and in-depth interviews about their mathematics course taking, feelings of preparedness to teach various subjects and topics, and their understandings of how their formal mathematics training served as a resource for their teaching. Additional interviews and classroom observations were conducted with four participants to further explore, in situ, relationships between their formal training and their teaching.

Analyses revealed a wide range of perceptions regarding the role of formal mathematical training across the cohort and the differential valuing of various aspects of the disciplinary knowledge they held. Importantly, TESs identified differences between their knowledge *of* the discipline and their knowledge *about* the discipline, as well as an array of "gaps," which they felt affected their current efficacy as teachers. Despite nearly identical coursework in mathematics, TESs' feelings of preparedness to teach particular subjects varied across students and from subject to subject. Connections between these perceptions and other factors, such as the TESs' mathematical attainment, orientation towards students' thinking, and vision of themselves as a teacher, are examined.

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LESSON STUDY: A CASE OF THE *INVESTIGATIONS* MATHEMATICS CURRICULUM

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Generally, in the USA, concerted efforts are under way to move students' thinking from an instrumental and procedural understanding of mathematics to a relational and conceptual understanding (Senk & Thompson, 2003). This study explores how practicing fifth grade teachers' past instructional experiences impact their present teaching practices especially when implementing the prescribed *Investigations* curriculum. This study views the instruction of mathematics as the negotiation of practices of school mathematics with the teacher as initiator. Negotiation in this study involves reasoning, interpreting, and making sense of mathematical meanings. The study employs negotiation of meanings in its theoretical considerations and employs analytic induction in its data analysis (Bogdan & Biklen, 2003).

A qualitative case study research design was used to explore the teachers' practice of *Investigations'* mathematics in fifth grade classrooms. Data were collected through lesson plans, classroom observation for a semester and through audiotape and videotape of lesson study meetings. Three lessons for the lesson study meetings were planned, and only two were taught. The participants identified the lead teacher who went ahead and planned on an agreed-upon *Investigations* lesson. Even though different teachers led in planning a lesson, they all discussed it at length beforehand. The discussions focused on several issues e.g. what page they were on with the *Investigations* curriculum, classroom climate, students' work, or on the teacher. The lead teacher then conducted the instruction of the lesson as others observed and took notes. Finally the teachers met to debrief on instructional and content issues.

This study found that lesson study is a powerful intervention that influenced how the participants implemented the inquiry-based instruction in their classes. Holding a positive perception of *Investigations* is important but this alone does not inculcate a teacher's ability to use inquiry-based approaches. There should be an in-build framework to support reform efforts for practicing teachers. A structured, stable, and supportive environment is healthy for in-depth learning, work, and professional growth. This study found that professional development programs that are teacher-led and immersed in actual classroom lessons are effective (Fernandez & Yoshida, 2004), as seen in how these teachers changed their practice as they engaged and committed themselves more to the lesson study meetings – the teachers' instruction reflected more of constructivist's learning perspectives.

Working collaboratively, raising questions, or just hearing what others suggest about a lesson makes the participants' rise above self. To effectively implement the *Investigations* curriculum in schools that have adopted it, this study found that teachers must collaborate with each other and establish new classroom instructional approaches.

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THE "INSERTION" ERROR IN SOLVING LINEAR EQUATIONS

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This proposed research investigates a particular phenomenon that occurred during a study of students' flexibility in solving linear equations (Star, 2004). 160 6th graders participated in five hours (over five days) of algebra problem solving. In the first hour, the students were given a brief lesson on four different steps that could be used to solve algebraic equations (adding to both sides, multiplying on both sides, distributing, and combining like terms). Students then spent three hours solving a series of unfamiliar linear equations with minimal facilitation. 23 students (randomly selected from all participants) were interviewed while working individually with a tutor/interviewer. On the last day of the project, students completed a post-test.

Analyses of students' work made apparent an interesting type of error, named "insertion", in 12 (7.5%) students' of which three were interviewed. The insertion error was evident when 2x + 10 = 4x + 20 became **4x –** 2x + 10 = **4x –** 4x + 20. Similarly, 2(x + 5) = 4(x + 5) became 2 **– 2**(x + 5) = 4 **– 2**(x + 5). Interestingly, this error has not previously been reported nor classified in the literature on linear equation solving (e.g., Matz, 1980; Payne & Squibb, 1990).

Out of the many proposed classifications of students' rule-based errors in computational or algebraic problems (Matz, 1980; Payne & Squibb, 1990; Sleeman, 1984), Ben-Zeev's (1998) classification is the most relevant here. Its context of solving unfamiliar problems is very similar to the context of the present research. In this framework, the errors are classified into two major types: critic-related and inductive. Critic-related errors are due to the students' failure to signal a violation of a rule while inductive errors are due to student's over-generalization or over-specialization of conceptual interpretations or surface-structural features of worked examples.

The interview transcripts of three students suggest that they have over-generalized the procedure of subtracting the same term on both sides in order to eliminate a term of a linear equation. As a result, two erroneous procedures are created – one that violates the subtraction law by inserting "TERM –" to both sides and the other that violates the distributive law by inserting " – TERM" in between a coefficient a and its associated term $(x + n)$. However, they stopped making these errors after they were made aware of the violation of such rules.

The data analysis thus proposes to include the following into Ben-Zeev's classification: 1) another critic-based failure whereby prior rules can be suppressed by the desired effect of a new procedure, and 2) errors which are generated by the confluence of over-generalized rules and critic-based failures even though this may be rare in the case of the "insertion" error.

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AN INVESTIGATION OF TEACHERS' MATHEMATICAL KNOWLEDGE FOR TEACHING: THE CASE OF ALGEBRAIC EQUATION SOLVING

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Solving algebraic equations is a central topic in traditional school algebra curricula. Although there have been extensive studies on students' understanding of equations and equation solving, few has been conducted with mathematics teachers. The knowledge of equations and equations solving that teachers employ for teaching becomes a particularly important issue for inquiry when the function-based approach has been reshaping school algebra curriculum, teaching, and learning in the past decade, and challenging the conventional, formal rule-based approach to equation solving.

Meanwhile, several groups of researchers (e.g., Ball, Bass, Hill and colleagues, Ferrini-Mundy and colleagues) have devoted to conceptualizing and measuring teachers' mathematical knowledge for teaching, both in general and in particular content areas (such as number concepts and operations, reasoning and proof, and secondary algebra). Most of their published work is about developing the measures. More detailed findings are in progress in terms of the component and characteristics of knowledge for teaching in a branch or special area of school mathematics.

This presentation is a summary of the initial stages of the presenter's ongoing dissertation research on secondary mathematics teachers' knowledge for teaching with specific focus on algebraic equation solving. A central piece at display is a conceptual framework for examining teachers' content knowledge for teaching mathematical concepts and procedures. The framework is constructed based on a summary of related theories, a review of research on student understanding of equations and equation solving, a mathematical analysis of equation solving process, analysis of algebra curricula, and the presenter's own empirical experiences in working with mathematics teachers. It incorporates five dynamic and interactive components:

1. *Knowledge of the core*. The typical or formal definition of a concept, standard or general algorithm for a procedure, and the mathematical rationale underlying them.
2. *Knowledge of alternatives*. Alternative definitions of a concept, alternative strategies for a procedure, the contrasts and connections between the standards and the alternatives.
3. *Knowledge of connections*. Definitions and properties of the same concept, algorithms of the same procedure, that are taught at different levels of mathematical study; other concepts and procedures that are connected to the one in focus (horizontally and vertically).
4. *Knowledge of presentations*. Effective ways of introducing a concepts or an algorithm, explaining related mathematical ideas to a specific group of students.
5. *Knowledge of learners*. The concepts that students bring into classrooms; typical student conceptions (misconceptions, mistakes, and difficulties) in learning a certain concept and procedure; effective ways of probing and assessing student understanding.

After being introduced, the framework is applied to analyzing the details of the mathematical procedure in focus: algebraic equation solving.

Two research instruments are being developed: A set of multiple choice and written-response items combined and embedded in teaching and learning scenarios, and a semi-structured

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interview protocol. The presentation discusses some design issues and also demonstrates sample items and questions from the instruments.

THE POSSIBLE CURRICULUM: ENCOMPASSING INTENDED AND ENACTED VERSIONS OF MATHEMATICS CURRICULUM

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This poster displays analytic tools for understanding the subtle differences between the mathematical goals of reform-based curriculum materials and the mathematics that teachers actually draw out and teach in their classrooms. The tools, once applied, provide ways to usefully describe the *possible curriculum*—the range of valid mathematical goals that can be addressed through the same material, and different routes teachers and students could move along to fulfill the goals.

Drawing on and adding to current research on curriculum enactment, we focus on the design and implementation of a 7th grade mathematics "replacement" unit which was developed in the context of a statewide experiment in Texas, and consists of SimCalc technologies together with written curricula and instructional materials specially designed to meet Texas standards. Specifically, we examine the ways three experienced teachers implemented a lesson focuses on a problem situation that involves motion, constant speed, unit rate, and proportionality, and with which students are engaged in collecting information, generating and analyzing data table, and formulating functional relationship.

Based on a through analysis of all mathematical concepts and processes involved in the lesson, as well as a preliminary review of observation notes and videotapes for classroom teaching, we developed an conceptual instrument that demonstrate two basis ways of working on the data table (down the rows and across the columns), and four ways of describing relationships observed from the scenario (physical formulas, arithmetic equalities, proportions, and function rules).

While the students in all three classes eventually arrived at a linear equation describing the motion of the runner in the scenario, each of the teachers focused their lesson on a different aspect of proportionality, some that the developer had not imagined as part of the intended lesson. We choose not to view this as an implementation failure. Indeed, these teachers got very high and significant student gains from pre- to post-test. Rather, we use their enactments and the conceptual instrument to create a "composite image" of the possible curriculum. Once articulated, this image can be utilized to help teachers and professional developers negotiate the mathematical terrain of conceptually rich mathematics instructional units. This approach can also be used to inform the development and use of materials within research projects designed to bring innovations to a wide variety of classrooms.

The presenters will be available to discuss the implications of possible curricula for the mathematics education community — researchers, curriculum developers and practitioners. The original curriculum materials and computer software used will also be on display.

UNSTRUCTURED VS. STRUCTURED FREE-WRITING IN HIGH SCHOOL APPLIED MATHEMATICS

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Free-writing, which can promote self-assessment, was incorporated as part of a larger study on writing in a grade 10 applied mathematics class. Modifications to free-writing were conducted the next year in a grade 11 applied class. Schoenfeld (1985) states that students often do not self-monitor and self-evaluate; hence, teachers need to provide such opportunities for students. Rolheiser & Ross (2000) believe that through self-evaluation, students develop self-efficacy and motivation, which can lead to confidence, greater responsibility, and higher achievement.

In the initial study, students engaged in weekly 5-minute free-writes of what came to mind about mathematics (i.e., no specific questions were provided). Data analysis revealed that reflections were similar throughout the study; namely, the importance of maintaining/raising their mark and the past week's progress. Only 1 student (out of the class of 12 who participated in the study) provided evidence of devising action plans. As the study progressed, entries became shorter and student resistance (e.g., "Do we have to do free-writing?") was more prevalent. In the third anonymous questionnaire, only 3 of the 10 students understood the purpose of free-writing.

Reflecting on the initial study, free-writing was too open-ended. Lacking purpose, the assessment was invalid. That is consistent with the students in Broadfoot et al.'s (1988) study, who found self-assessment difficult since criteria was vague (in Gipps, 1999).

For the second study, the first author developed 13 prompts to allow students to reflect and develop action plans. The purposes of free-writing, along with a rubric, were shared at the beginning of the study. Students selected 1 prompt and spent 10 minutes engaged in reflection. Unlike the first study, descriptive feedback was provided after each entry, with acknowledgment of what is good and questions or suggestions to help students advance their thinking. Students wrote a total of 8 entries. In the 2 take-home assignments, they completed reflection sheets to identify up to 4 action plans stated in their entries, evidence of implementation, and 1 additional plan to be implemented in the future. Students did not resist, and at mid-term, 16 of the 21 students stated free-writing is beneficial (1 student provided no response).

In this poster session, the prompts and rubric, along with students' writing from both studies, will be shared. Also, questionnaire responses from the two studies will be compared.

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USING MATHEMATICS STORIES TO UNDERSTAND WHAT ELEMENTARY TEACHERS BRING TO REFORM

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Math stories like can provide important insights into what teachers value with regards to teaching strategies, how mathematical understanding develops, and, more generally, what teachers like and dislike about mathematics. Much has been written about how teachers' *past experiences* affect how they teach (Ball, 1997; Smith III, 1996). Teachers teach as they've been taught in what Lortie (1975) describes as an "apprenticeship of observation." At the same time, researchers have often described the role of *beliefs* in teachers' decisions about how and what to teach. Beliefs about the nature of mathematics, beliefs about how students obtain knowledge, and beliefs about how teachers convey knowledge are some of the kinds of beliefs that have been written about (e.g. Ernest, 1988; Thompson, 1992). In this poster, we will present the use of *mathematics stories* as both a conceptual and methodological tool for understanding the complexity and contexts of teachers' beliefs and experiences and for describing the frameworks, or lenses, through which teachers are seeing, interpreting, and implementing mathematics education reform.

The mathematics story interview asks teachers to consider all of their experiences learning and teaching mathematics. Teachers are asked to identify several key events within these experiences, including the high point, low point, and any turning points in the story as well as any challenges they may have encountered. They are also asked to describe a positive and negative future for themselves and mathematics.

In particular, this poster will address the following research questions:

- What types of stories do elementary teachers tell about their experiences learning and teaching mathematics?
- How are patterns in these story types related to patterns in teachers' grade level, years of experience with mathematics education reform, and teaching practices in the context of reform?
- How can mathematics stories be used as tools for research as well as for pre-service and in-service teacher education?

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NUMERATION CARDS: INNOVATIVE CURRICULUM MATERIALS FOR THE PRIMARY SCHOOL

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Introduction

It is very important that young students explore mathematics concepts using a variety of concrete materials. They help students to have initial opportunities to explore for themselves followed by careful guidance into an understanding of the abstract mathematics involved. Student-participants attending an urban elementary school in Edmonton were introduced to the numeration cards to help them do subtraction that involves renaming in problem solving. The purpose of the study was to make the learning of compound subtraction visible to participants by way of base complement additions strategy (BCA) using numeration cards.

The Numeration Cards

Trying to research into a problem may sometimes result in other discoveries (Gyening, 1993). According to history of mathematics education, "equal additions" was an approach for solving subtraction that involves renaming but it was very difficult to make its teaching and learning visible by using manipulatives. In an attempt to objectify equal additions "base complement additions" strategy has rather been discovered by researchers using numeration cards. Numeration cards are a set of innovative curriculum materials that was used for problem solving in addition and subtraction. The target task, notwithstanding, the numeration cards were found to have other uses in problem solving: numeration, addition and subtraction of all types, place values among others.

Methodology

This poster reports on a descriptive qualitative study on effective teaching and learning of compound subtraction using the numeration cards. Pre-intervention, intervention and post-intervention design was used as main source of data collection; and tests, students' scripts and semi-structured interviews were the evaluation instruments for the study. Two students in an urban elementary school, grades 4 and 5, volunteered to participate in the study. Each student was initially tested with the same test items and then engaged in an interview to find out entry-level strategies for compound subtraction before the intervention. The pre-intervention, intervention and the post-intervention tests were identical for each has the same number of items for two, three and four digits, vertical and horizontal digits, money and word problems. The same set of questions was given to both participant for the pre-intervention, intervention and the post-intervention tests. The post-post intervention tests were different, but parallel in form. The structure of the interviews was open-ended and the questions were developed in order to focus on participants' thoughts. Interviews between researcher and the students were audio-recorded; the researcher took detailed field notes during sessions with the students.

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Findings

Findings from the interviews implied that the cards helped the students to have better understanding of operations on numbers. From the students the numeration cards were useful manipulatives for effective learning of both decomposition (borrowing) and base complement additions strategies for subtraction that involves renaming. The students continued that activities using the numeration cards engaged them and motivated them to learn mathematics by seeing and doing.

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THE PROCESSES OF LEARNING IN A COMPUTER ALGEBRA SYSTEM (CAS) ENVIRONMENT FOR COLLEGE STUDENTS LEARNING CALCULUS

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Introduction

This study is a qualitative case study focusing on the question "What are the processes of learning in a Computer Algebra System (CAS) environment for college students learning calculus?" The study is designed to research the impact on student learning of particular software available for mathematics education and aims to provide insight into the nature of learning in a technology-rich environment.

Motivation for the Study and Theoretical Framework

There is research on student achievement in examinations after they have used CAS during their course of study. However, there is a gap in the research on CAS in the area of investigations into the processes of student learning and students' development of concepts while learning using CAS. Among my research questions are How do students use technology? Does using CAS effect students' learning strategies? Do students use the opportunities afforded by CAS to experiment with mathematical objects? Do students using CAS develop a different epistemological sense of mathematics compared to students not using CAS?

This research employs two theoretical frameworks through which to approach student learning while using CAS. The Rotman Model of Mathematical Reasoning (1993) is used as a macro-framework for the place of technology in the learning of mathematics. This framework is useful for addressing the question of the effect of technology on learning by positioning technology in the activity of mathematical reasoning. The Pirie-Kieren Model of Mathematical Understanding (1990) is used as a micro-framework and as a lens through which to interpret and analyse specific learning episodes as they take place in the classroom. The two frameworks together provide a vehicle for understanding learners' mathematical activity, reasoning and development in a CAS environment across a period of time.

Design of the Study, Methods, and Results

The primary data I used for the study are audio and video tapes of students in a college course learning calculus using CAS software. These data are supplemented by interviews I conducted with the students. My study provides a detailed description of the process of learning mathematics with the use of CAS and the emergence of conceptual development arising from collaboration among the students in the collective interaction with the software.

My principal findings are that (i) that students are aware of multiple strategies facilitated by CAS but often do not implement those strategies very well, and (ii) that the framing of technology in the learning environment has a considerable effect on how students approach their work and can be a hindering factor on their willingness to experiment in their learning. My study also shows that the adapted Rotman Model of Mathematical Reasoning is an accurate model for understanding the place of technology in the learning of mathematics; and that the Pirie-Kieren Model of Mathematical Understanding can be fruitfully applied to learning in a CAS

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environment. The significance of my work lies in the provision of a far richer picture of the CAS classroom than has been available before.

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ONLINE DISCUSSION IN A MATHEMATICS CONTENT COURSE FOR PRESERVICE ELEMENTARY TEACHERS

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Mathematics content courses for preservice elementary school teachers aim to promote deep understanding and clear communication of mathematical concepts, as well as familiarize future teachers with multiple approaches to math problems. These goals are achievable through reflection upon course material and the sharing of interpretations; journal writing and group collaboration have been suggested for encouraging these activities in college math courses (Beidleman, Jones & Wells, 1995; Dees, 1983). Online discussion boards (ODBs) enable both collaboration and reflection as an extension of classroom activities. This study fills a void in the literature by focusing upon a college mathematics course, the needs of elementary teachers, and the connections between course content and ODB use. Studies in other academic disciplines have shown that ODBs can encourage students to reflect upon and develop deeper understandings of course material, collaborate with peers, strengthen communication skills, and improve academic performance (Hofstad, 2003; Wickstrom, 2003). These studies have documented positive student experiences and have provided course-specific ODB implementation strategies.

This research studies the use of ODBs in a mathematics content course for preservice elementary school teachers, examining ways in which students use the ODBs, the impact of online collaboration and reflection upon students' mathematical learning and understanding, and students' attitudes and perceptions regarding ODB use. Students' perceptions of the online technology and their methods of use are determined via analysis of survey responses and patterns of interaction extracted from the ODBs. Cognitive and metacognitive activity, as well as the social construction of mathematical knowledge and understandings, are examined within ODB postings using modified versions of the content analysis models proposed by Hara, Bonk, & Angeli, (2000) and Gunawardena, Lowe & Anderson (1997). These models are also used to analyze student responses to exam questions, examining the transfer of deep understanding of discussion topics and the improved clarity of explanations beyond the online forum. Survey responses and measures of academic performance are compared between a course section using ODBs and another conventially taught section lead by the same instructor.

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SUPPORTING THE MIDDLE SCHOOL MATHEMATICS TEACHER IN PURSUIT OF NATIONAL BOARD CERTIFICATION

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The North Carolina Middle Math Project (NCM^2) is a state-wide professional development project for middle school teachers funded by the National Science Foundation. It is a collaboration between the North Carolina Mathematics and Science Education Network (NC-MSEN) and the North Carolina Department of Public Instruction (NCDPI). The major goals of the project are two-fold: to improve mathematics education in middle school, and to retain and support teachers in their professional development. Nine participating NC-MSEN centers assembled a team of university mathematicians, mathematics educators, school district administrators, and mathematics teachers to carry out the goals of the project. NCM^2 created three graduate-level courses for teachers, focusing on the content areas of statistics and probability, geometry and measurement, and number and algebra. Approximately one hundred thirty teachers took these courses. NCM^2 teachers are using this coursework and other initiatives of the project in the pursuit of National Board Certification in Early Adolescence Mathematics. The NCDPI and the North Carolina State Board of Education recognize this certification as an indicator of an accomplished teacher. Many teachers are also applying these courses to requirements for a Master's degree at several of the participating universities.

Our qualitative study was based on eighteen teacher interviews and classroom observations. In 2003, nine teachers, one from each NC-MSEN center, were interviewed. A classroom observation was also conducted. The following year, nine different teachers were interviewed and visits were made to their classrooms. All eighteen interviews were recorded and transcribed; field notes were taken of the classroom observations. In 2004, all teachers participating in the project were surveyed to measure the degree of the project's impact on their National Board Certification process. Analysis of this data indicated that the three graduate courses developed by NCM^2 had an impact on the teachers' National Board Certification process. They claimed that their preparation was influenced by two major components of the courses offered through the NCM^2 project: increasing teacher content knowledge and providing teachers with standards-based tasks. Overall, the NCM^2 participants specified that they were better prepared for completing the Portfolio Entries than the Assessment Exercises. More detailed results from the interviews, classroom observations, and surveys will be shared during the poster session.

Projects that increase teacher content knowledge can have a beneficial impact on a teacher's professional development as he or she undertakes the National Board process. To achieve National Board Certification status, a teacher must demonstrate he or she knows the mathematics being taught. Projects like NCM^2 can provide these learning opportunities. Successful teachers demonstrate a second type of knowledge that is equally important: the understanding of how to communicate the mathematics to their particular students (Wilson, Shulman, & Richert, 1987). NCM^2 courses and activities not only provided teachers with standards-based tasks, but showed teachers how to implement these tasks with middle school students.

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MATHEMATICAL CONNECTIONS IN OPEN-ENDED PROBLEM-SOLVING ENVIRONMENTS

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Cohen and Ball (1999, 2001) emphasized that instruction is a function of interactions among teachers, students, and content as mediated by instructional materials. Instructional materials shape what teachers and students do through the problems they use, their development of ideas, and the representations they contain. Teachers use their knowledge of the content and experience with students to interpret the materials and mediate students' opportunities to learn. Students use their prior knowledge to comprehend, interpret, and respond to materials and teachers. Moreover, students' prior knowledge and responses to the materials and content help determine what teachers can accomplish. In this study, we attempt to understand these interactions by looking at two classrooms in which the instructional materials are the same, but the teachers and students are different. Specifically, we explore what previous experiences and understandings students relied on in solving one problem and what experiences and understandings the teachers wanted the students to elicit as they attempted to solve the problem. In short, our research questions were (1) How do students solve open-ended problems, specifically, what connections to prior learning do they make; (2) Do the approaches the students choose to take match those recommended by their teachers; and (3) How do student approaches impact the goals of the activity?

Data for this study were collected in two sixth-grade classrooms across two days of instruction for each class. The primary data source was videotape, with two cameras being used in the classroom: one following the activities of the teacher and one focused on student and teacher writing. Additionally, videotaped interviews with students from each class asked students to explain their reasoning as they solved this problem and to comment on segments of videotape from the classroom. Videotaped teacher interviews relied on classroom and student interview data to engage the teacher in prompted reflection on her teaching strategies and goal setting and analysis of student thinking. Data were analyzed using a fine-grained analysis of talk, hand gestures, and drawings. The mathematics problem of interest came from the *Connected Mathematics Program* (Lappan, Fey, Fitzgerald, Friel, & Phillips, 2002) and asked students to determine the fractional portion of land owned by each of several landowners in a fictitious town. In each of the classrooms, students were allowed a variety of exploration approaches and each teacher worked with a somewhat different goal.

This poster presentation will highlight the different approaches students took to solving the problem and the mathematical experience they drew on to solve the problem. It will also draw conclusions about the mathematics goals addressed by the different approaches the students took within the framework of the goals the teachers wanted the students to meet.

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INVESTIGATIONS OF HOW AN IN-SERVICE TEACHER VIEWS HERSELF AS A LEARNER

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What does learning mean for teachers who are learners in a course devoted to the reconstruction of mathematics curriculum? How do they think about their own learning? How do they act as a learner? Are they reorganizing their mathematical thinking? Do they make connections between their experiences in this class and their constructed mathematical realities whatever they are (von Glasersfeld, 1985)? How do they become a viable member in the learning community?

These were the questions that framed my interactions with a high school in-service teacher, Tamara, a student in the curriculum course I served as an assistant. Throughout my interactions with Tamara, I have developed series of questions. As Stake (1995) wrote, "the best research questions evolve during the study" (p. 33). Those questions were how Tamara recorded her learning experiences, and how she reflected about them during the course, and what kinds of things she was paying attention to when she was a learner in a class context. As Mason (2003) said, "how attention is structured is crucial to what can be noticed, and what can be learned" (p. 13), so how Tamara was talking about those issues were important clues about my model of her as a learner.

The course was taught at a southern university in USA as a 7-week summer graduate level curriculum course. The course instructor had a purpose of the students reconstructing the basics of middle and high school curriculum: emphasizing combinations, permutations, counting, multiplicative reasoning, unknowns, binomials, fractional operations, Pythagorean theorem, and quadratic formulas. During the course, students used Geometer's Sketchpad (Jackiw, 1995) and JavaBars to investigate problems. I met with Tamara seven times during this course for one-hour long sessions. Those sessions were video-taped and partially transcribed. Two of these meetings were for interview purposes, but the remaining five were sessions in which I investigated Tamara's learning while also offering help for her homework.

During the poster presentation, I will discuss how Tamara viewed herself as a learner in this class, what kinds of responsibilities she took, and how she reconstructed mathematical concepts using technology. I will discuss Tamara as a learner using two problems she solved related to fractions where she was supposed to use the concept of "co-measure" that was developed in the class.

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DEVELOPING TEACHER PRACTICE THROUGH VIDEO ANALYSIS

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Professional development is one of the primary vehicles through which teachers are provided opportunities to develop their knowledge for teaching mathematics. This knowledge includes understanding how each of their students learns and being able to design lessons that support and build on these understandings. The importance of meeting all students' needs is supported by the Equity Principle put forth by the National Council of Teachers of Mathematics (NCTM, 2000) which states, "Excellence in mathematics education requires equity – high expectations and strong support for all students" (p. 12).

This pilot study takes place in an ongoing professional development program in which teachers are studying their at-risk students' problem solving. The goal of this professional development program is to develop interventions to help support the mathematics problem solving abilities of at-risk students. In this program, at-risk students are students identified by their teacher as having difficulty problem solving. The participant is a middle school teacher in an urban school teaching one of the reform curricula. The question driving this pilot study is: In what ways can analyzing videos of at-risk students' problem solving provide opportunities for teacher learning? More specifically, how might this analysis help teachers to better understand how their at-risk students learn mathematics? Also, how might this analysis help teachers to better understand the effects of their teaching on at-risk students' learning?

The main source of data is three interviews with the teacher. One interview occurred at a professional development retreat, the next interview occurred after I observed one of her classes that was videotaped as part of the professional development, and a third interview was conducted after she was able to view and reflect on the video.

Preliminary results from analysis of the interviews suggest that that the teacher found the video analysis instrumental in guiding her teaching. In an interview she mentioned that if it was not for the professional development she would not be teaching the curriculum the "correct way" (interview 2), that she would be teaching it more traditionally. She also stated that observing the video enabled her to realize that her students knew more then she was giving them credit for, and in turn she adapted her classroom practice based on this information.

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A COMPARISON OF LEARNING SUBJECTIVE AND FREQUENTIST PROBABILITY

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Ever since the publication of *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989), probability and statistics have been prominent in the K-12 curriculum. A recent summary of research on human judgment and decision-making under uncertainty (Shaughnessy, 2003) addressed how humans rely on certain judgmental heuristics. Research shows that children have a subjective approach of playing out hunches, beliefs, and intuitions about what might occur during a probability experiment, yet the school curriculum does not consider subjectivity.

The purpose of this study was to explore the subjective theory (de Finetti, 1974) of probability with children. A mixed methodology study included students in grades 4, 5, 6 (n=87) who were engaged in a teaching experiment to compare learning traditional probability concepts (n=44) to learning traditional and subjective concepts (n=43). Pretest and posttest scores were analyzed using a MANOVA, while researcher observations from classroom lessons, teacher journals, and researcher interviews with students were coded for themes. All students improved significantly in probabilistic reasoning ($p<.01$). The combined fifth and sixth grade experimental groups who were exposed to subjective probability concepts improved more than the traditional group students ($p=.096$). Qualitative data showed that students have beliefs about probabilistic situations based on their past experiences and prior knowledge. This research adds to a growing body of literature about probability and statistics, and suggests that exposure to subjective probability concepts enhances students' reasoning skills.

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TESTING THE GRADUATED LINE AS A SEMIOTIC REGISTER FOR RATIONAL NUMBERS

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Introduction

Among the three registers of expression of rational numbers, described in Adjiage's poster (also presented to PME-NA 2005), the graduated line, seen as a semiotic register (Duval, 2000), is central in the teaching/learning process of fractional and decimal expressions. In the physical domain, the graduated line, equipped with a single or a double regular scale, permits to represent and to solve any of six considered types of ratio problems (Adjiage, 2000; 2005). This poster reports a research that tries to explain some of the difficulties that pupils face in using this semiotic tool in order to express and process rational numbers.

Main Hypothesis

- Using the graduated line (**GL**) for expressing and processing rational numbers requires the mobilization of a complex thinking, which includes not only mathematical knowledge, but also capacities of spatial structuring necessary to select and to organize the relevant information.
- Actualizing these competencies depends on contextual factors linked to the task, and on individual factors as former experiences in using a graduated ruler, inhibitions in former mathematical learning …

Methodology

Twenty-one 7th-graders of the same class-room are presented with ten mathematical exercises. Five of these exercises require to express rational numbers (e.g. "*Drop* $\frac{4}{5}$ *on a graduated line segmented in tenths*") using a GL. The results are related to those obtained with psychological tests (Rey' Figure and GEFT) and with other mathematical exercises: identifying different expressions of a given fraction, representing a fraction on a segmented surface, and thus mastering the operating mode of the denominator and numerator (Streefland, 1991).

The mathematical test is followed by individual "explicitation interviews" (Vermersch, 1993) which have two goals. The first one is related to the task: students have to make explicit the procedures they used and the obstacles they encountered when solving the exercises. The second one is related to the former students'experiences in mathematics, including their learning inhibition (Ancelin-Schutzenberger, 1993; Metz, 1999).

Results

An implicative analysis (Gras, 1992) made it possible to select in the mathematical questionnaire some items so that succeeding these items tend to imply succeeding to others.

Global results at the GL items are correlated with the GEFT results: pupils who succeed in operating the graduated line tend to be those who are able to get free from perceptive factors.

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An analysis of the interviews shows that a persistent fixing to the decimal system inhibits the capacities for using the GL properly. That allows to make a new assumption: some pupils remain hung on the decimal system because of the persistence of the “percept” related to the decimal graduated ruler as a well-known measuring tool. Many pupils fail to master the graduated line because of this “percept” which functions then as a “Gestalt”. This inhibits the implementation of their capacities to properly operate the numerator and denominator of a fraction, and thus makes them incapable to represent this fraction on a graduated line, even if they succeed to represent it on a segmented surface.

APPLY WHAT YOU KNOW TO A PROBLEM YOU HAVE NEVER SEEN: HOW AP STUDENTS APPLY STATISTICAL REASONING TO SOLVE A TASK USING EMPIRICAL DATA

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Statistical reasoning has been defined as how one makes sense of statistical information and makes inferences using statistical concepts (Garfield & Gal, 1999). Students who have been classically educated in statistical inference techniques such as hypothesis testing and confidence intervals may perform well in statistics classes but can they use their statistical reasoning when faced with making an inference based on empirical data that they collect? This study examined the statistical reasoning displayed by students in two US high school Advanced Placement Statistics classes.

The task used in this study was adapted from a research study with sixth grade students (Stohl & Tarr, 2002) in which students used a computer simulation (Probability Explorer, Stohl, 2002) to generate data to determine if a fictitious company produces fair dice and to estimate the probability of each outcome if the given die is not fair. Our analysis focuses on how students approached the task, the size of the samples they chose to collect, and whether they applied statistical inference techniques to provide compelling evidence or used other non-standard evidence. We also examine the students' prediction of the probability of outcomes of the die they are working with and what statistical reasoning the students employ in their estimation. To examine the effects of the computer simulation on students' statistical reasoning, a second class of students also conducted the same experiment with physical dice, which had been weighted. Each pair of students investigated an assigned company, made a decision on whether that company produced fair dice or not, and estimated the probability of each outcome. Each pair then presented these results to their classmates in the form of an oral presentation and some type of display of their evidence. The class was then able to ask questions about the presentation and the students had to support their reasoning.

The data for this study is currently under analysis, though preliminary data shows that although some students confidently complete the task, others struggle to apply the methodology they have learned in a practical sense using empirical data. Our poster will contain a description of the task the students had to complete, analysis of students' statistical reasoning using Models of Statistical Reasoning (Garfield, 2002), and examples of students' work.

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USING EPIDEMIOLOGY TO MOTIVATE ADVANCED MATHEMATICS

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The quality of secondary mathematics instruction has on the decline due to increase demand to meet national education standards. In response to this growing problem, the presenters developed and implemented a mathematics workshop for secondary mathematics teachers and high school students to stimulate interest in advanced mathematics. The workshop was designed to introduce mathematical biology, an emerging discipline within the field of mathematics, and to demonstrate innovative ways of teaching mathematics with graphical modeling software.

Mathematical modeling was the primary topic of the workshop in which the participants explored basic concepts in abstract algebra and graph theory. The participants were engaged in discovering applications of mathematics through a hypothetical epidemiological problem based on actual data. This experience fostered the educators' appreciation of innovative ways of teaching mathematics using advanced mathematics and mathematical modeling software, with an emphasis on integration of these topics into the Standards of Learning mathematics curriculum. In addition, the activities generated interest of mathematical modeling in the students, while maintaining relevance of advanced mathematics.

During this workshop, we found that the students extended their problem solving and communication skills. All participants established a connection between mathematics and biology and discovered novel applications of technology to solve mathematical problems. Through the creation of an intergenerational environment in which educator worked alongside student and illustration to creative approaches to instruction, the high school teachers developed novel methods to integrate technology and advanced mathematics concepts into the standards of learning mathematics curriculum.

The presenters will introduce the framework of the workshop, including a description of the mathematics topics and the mathematical software. We will also present the epidemiological problem given to the participants. We will close with a discussion of the implications of this outreach experience.

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GEOMETRY CONNECTING RATIO UNDERSTANDING: REPRESENTATIONS, STRATEGIES, AND DISCOURSE

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Research indicates that people perceive of ratios in two ways—between ratios and within ratios (Lamon, 1994). Grounding these ideas in rectangles was central to one teacher's ability to orchestrate sixth-grade students' problem solving and subsequent metarepresentational competence in a two-year design experiment investigating geometric similarity as a means to promote algebraic understanding (Boester & Lehrer, in press; Lehrer, Strom & Confrey, 2003).

Combinations of student work and classroom discussion document how students and teacher measured similar rectangle sides to visually and verbally connect seven successively abstract representations of ratio: a) paper strips for depicting linear measure, (b) physical rectangle cutouts, (c) equations, (d) ratio tables, (e) co-ordinate graphs illustrating linear groupings of similar rectangles, (f) stair-steps drawn between points along graphed lines, and (g) slope. Fractions, ratios, and quotients were integrated alternative forms of ratio (Lamon, 2001).

In each section, the poster demonstrates that using rectangles with each representation enabled mutually sensible or interanimated conversations (Seymour & Lehrer, submitted). As a whole, it is a story of a teacher orchestrating understanding of limitations of the discrete between-ratio strategy, and virtues of the continuous within-ratio strategy for determining the slope of a line of an infinite group of similar rectangles.

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PROBLEMATIZING WRITING IN THE LEARNING OF MATHEMATICS

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Objectives

The purpose of this poster is to (a) explore the importance of teaching students how to write mathematics, (b) identify some of the challenges to researching and implementing writing instruction in mathematics classrooms, and (c) suggest directions for research.

Theoretical Framework

Communication has long been recognized as being important in the learning and teaching of mathematics. Policy statements such as the *Principles and Standards for School Mathematics* (NCTM, 2000) suggest that communication should not only include spoken discourse, but also writing. Aspects of writing in the mathematics classroom, such as representing and symbolizing, have been heavily studied. Much less research has been devoted to studying broader aspects of writing, such as the writing of explanations, justifications, conjectures, and descriptions. Given the important role that writing can play in the development of understanding, and given the importance of being able to write mathematics as a requirement for participation in legitimate mathematical activity, it is important to study these broader aspects of writing.

Methods and Results

We conducted a literature review to assess the challenges currently facing the research and implementation of writing instruction in mathematics classrooms. We encountered three assumptions among researchers about writing that inhibit the research and implementation of writing instruction in mathematics classrooms: First, writing is merely a tool to learning and understanding mathematics, and not a worthwhile end goal in and of itself. Second, if students are able to successfully vocalize their understanding, then they will have not trouble writing. Third, if students cannot write, it is because they do not understand, and not because they may not know how to write mathematics texts.

In addition to these three assumptions, we noted three additional challenges to the research and implementation of writing instruction: First, there is currently no common written discourse in reform-oriented mathematics classrooms. Consequently, it is unclear exactly what writing should be researched and taught. Second, due to the lack of a common written discourse, current attempts to research and implement writing instruction often rely upon the use of written genres from outside of mathematics, such as poems, songs, stories, historical reports, or philosophical arguments. Such efforts may be misguided, because these different genre are not particularly effective for communicating and developing mathematics. Third, it is unclear how to explicitly teach writing, because attempts to help students to write may be interpreted by students as providing them with the ideal explanations or solutions they should be memorizing.

Conclusions

Research is needed to identify the types of writing that are being done in current reform-oriented mathematics classrooms, to judge if these types of writing are appropriate for learning

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and participating in mathematics, and to document how teachers are promoting these types of writing. Further research is also needed to identify other types of writing that are currently not being used in classrooms, but that could enhance learning and students' ability to participate in mathematical activity. We anticipate that the teaching and learning of mathematics will benefit as the field moves toward establishing a written discourse for students' mathematics and methods for facilitating fluency in this discourse.

ASSESSING PRESERVICE TEACHER HABITS OF MIND WHEN ATTEMPTING AND PLANNING A MODEL ELICITING ACTIVITY

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Focus of Study

The purpose of this preliminary study is to examine the actions of several preservice teachers when working on a model-eliciting activity, involving slope and compare their actions to their ideas about students' actions when planning a similar lesson.

Conceptual Framework

This research is based on the frameworks of Pirie & Kieren (1994) and Berenson et. al (2001) who propose that the growth of mathematical understanding and pedagogical content knowledge can be observed progressing through iterative levels. The theory, developed to offer a language for observing this progression, contains eight potential levels for understanding, the first five of which we use in this study: *primitive knowing, image making, image having, property noticing* and *formalizing*. Berenson expands the Pirie-Kieren model to account for tasks involved in the preparation of teachers. Our interest in this study is to compare the preservice teachers' knowledge of slope when solving a model-eliciting activity as measured by their attempts to fold back, and their awareness when planning lessons of this need in students. Lesh and Doerr (2003) refer to model-eliciting activities as those that involve making mathematical descriptions of everyday situations.

Methodology and Results

The researchers acted as co-teachers of a larger teaching experiment occurring over a 14-week mathematics education methods course, at a large southeastern university. Weekly class meeting were designed to encourage preservice teachers to use multiple representations within and between selected mathematical topics, all pertaining to proportional reasoning.

We found that these preservice teachers relied heavily on images and representations, marked by various levels of rigor, in order to complete their work. This lends itself to the Pirie-Kieren model (1994) in that *image making* preceded a more formal, symbolic result. These preservice teachers folded back to images that are more elementary to assist in their work. On the other hand, when preparing an introductory lesson on slope, a significant number planned to use their students' presumed knowledge of linear functions and coordinate geometry to teach a symbolic representation of slope, thus overlooking the need of their students to make images and notice properties, before formalizing.

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UNDERSTANDING TEACHING AND LEARNING OF FRACTIONS IN A SIXTH-GRADE CLASSROOM

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Teaching and learning of mathematics have been studied separately most of the time in the literature. There is research either about teachers' beliefs, mathematical content knowledge, or pedagogical knowledge or studies that combine just-mentioned phenomena with teachers' teaching practices. In the literature there is also research done solely in K-8 students' cognition for understanding mathematics of students (Olive & Steffe, 2002; Schoenfeld, Smith, & Arcavi, 1993). These studies are related to students' mathematical learning, and they are either unconnected with the school mathematics or the experiences students are having in the classrooms.

However, there is a gap in the literature where classroom instruction, student thinking, and teacher practice are combined to understand the interaction of teaching and learning (Izs‡k, Tillema, & Tun_-Pekkan, 2004): specifically, how students' mathematical learning is shaped through classroom instruction. Cohen and Ball (2001) defined instruction as a function of interactions among teachers, students, and mathematical content, and I will use "instruction" with a similar meaning, but investigate the parts in the definition more, and interactions between the parts for this presentation.

In this interpretive study, I interviewed a pair of sixth-grade, female, white students, who were learning fractions in a classroom that used reform-oriented curricula in a middle school located in a small USA southern town. For a semester when classroom fraction instruction took place, I conducted five interviews each lasting approximately an hour. During interviews, I frequently used classroom video clips taken as a part of a bigger research project (Coordinating Students and Teachers Algebraic Reasoning) to remind interview students some of the classroom instruction. Fraction instruction included number line representations of fractions, comparing fractions and equivalent fractions.

During poster presentation, I will present data and analysis of the interview students' understanding of equivalent fractions on the number line and will discuss how their understandings were supported and shaped by the classroom instruction.

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MULTIMEDIA INSTRUCTIONAL PRESENTATIONS ON LIMIT: EXPERT EVALUATIONS

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Multimedia instructional environments are a new and evolving instructional context that incorporate multiple modalities, e.g. verbal presentations (on-screen text or narration) as well as pictorial presentations (including static and dynamic illustrations). In recent years, the availability of sophisticated technological tools has not only enabled novel presentations of traditional instructional materials but has also shaped the development of novel instructional materials, such as interactive exercises and simulations. In addition, the Internet now allows vast audiences access to these educational materials. As a consequence, online courses and course materials have become publicly available in a number of subjects, including statistics, economics, science, and mathematics.

What factors distinguish different realizations of these courses and determine their effectiveness? The shaping of a framework to guide the systematic evaluation of multimedia instructional messages is on the current agenda of media and educational research (Mioduser, Nachmias, Oren, & Lahav, 1999). With respect to presentational quality, media research has focused on the construction and evaluation of short, context-independent multimedia instructional presentations (Mayer, 1999). With respect to educational quality, interest is focused on longer instructional messages and the extension of cognitive learning theories to multimedia settings (Larreamendy-Joerns, Leinhardt, & Corredor, 2005). However, the overall goal of assessing effectiveness in the larger educational sense remains a challenge (Larreamendy-Joerns & Leinhardt, in press). Our research addresses this challenge by collecting and analyzing the perspectives of experts from relevant disciplines on instructional materials that are part of an online college-level mathematics course. For the context of the instructional message, we chose a foundational yet notoriously problematic mathematical concept that calculus students encounter, namely the limit. Evaluations of two contrasting online course presentations of limit were solicited from experts in mathematics, mathematics education, human-computer interaction, and psychology.

Presentational coordination was a common theme across all experts. The experts focused on the coherence of the instructional exposition both within and between the course components (e.g. exercises, examples, and explanation). In addition, each expert contributed a unique critical perspective: mathematical nuances (mathematician), conceptual connections (mathematics educator), online affordances (human-computer interaction specialist), and associated discourse practices (psychologist). The impact of these findings extends beyond the development of effective online mathematical presentations to the shaping of a genre of critique for online educational materials.

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LEARNING MATHEMATICS IN CENTRAL APPALACHIA: LIFE HISTORIES OF FUTURE ELEMENTARY TEACHERS

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While a perpetual mathematics achievement gap for students in areas of poverty has been clearly documented (Secada, 1992), few studies have explored those results in rural areas (Silver, 2003). Even less research exists for mathematics achievement of students in Central Appalachia, an area of persistent rural poverty, despite the 1983 call for "ethnographic research which examines…internal and external factors shaping school experience" by Keefe, Reck, and Reck (p. 218) who reviewed variables and interactions that comprise the educational experience in Appalachia.

As a native Appalachian with mathematics teaching experience in middle school, I was recruited to help teacher candidates at a Central Appalachia college who were struggling with *Praxis I* – Mathematics, a test required for licensure. In my work with them, I observed that their difficulties appeared to stem from the mathematical content of their public school years.

This poster reflects the research I conducted for my doctoral dissertation. Using an oral history approach, I interviewed three individuals who had all attended public school in Central Appalachia, who were good students in terms of grades, behavior, attendance, and test scores, yet who struggled with the mathematics on the *Praxis I* test. In addition to the interviews, each participant created a graph of their own mathematics self-image for each year of public school.

Life histories for Laura, Faith, and Peyton were constructed. From these narrative data, themes of positive primary grade experiences, shyness, loss of confidence in middle school mathematics, struggles with geometry, choices to take fewer mathematics courses in high school, success in college mathematics, and Appalachian self-image arose. These themes relate to the literature on rural poverty and help to understand the impact of Appalachian culture on mathematics achievement.

Questions were raised regarding effective mathematics teaching for Appalachian students and how it may differ than what is recommended for the mainstream. Future research could include additional life stories from a variety of native Appalachians including those who did not attend college or graduate from high school, as well as those who excelled in mathematics in college and careers, to better understand the various mathematical experiences that students have in school and which were beneficial or detrimental to their learning. Future students could benefit from research that focuses on improving mathematics education in areas of rural poverty, such as Central Appalachia, to narrow the achievement gap and to increase opportunities for success in college and careers.

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MATHEMATICS REASONING HEURISTIC (MRH): WRITING-TO-LEARN

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Mathematical problem solving requires a complex set of cognitive actions with many connections to cognitive structure and to the context of the situation. When solving problems, students practice cognitive activities that can enhance their mathematical thinking and reasoning by discussing different solutions for a mathematics problem. Scholars (Bereiter & Scardamalia, 1987) have argued that writing, as a tool for developing and communicating ideas, helps learners make the connection between existing knowledge and newly encountered information and experience in an organized manner because writers (learners) engage in a dialogue between their thoughts and their written statements. Writers must take into account multiple factors such as the topic, the audience to whom the text is written, the writing type used, and their knowledge about the topic (Bereiter & Scardamalia, 1987).

The mathematics reasoning heuristic (MRH) is a pedagogical tool for supporting mathematics classroom discourse by engaging students, through problem solving, in reasoning and communicating their ideas through dialogical interaction and writing tasks. Writing supported with public negotiation before production of a text encourages students to acknowledge the social and interpersonal dimensions of knowledge. There are two templates: teacher and student templates (see Table 1). The student template outlines a series of questions that students consider through problem solving; whereas, the teacher template guides teacher for the preparation of a unit and the implementation of the unit. The idea of the MRH teacher and student templates comes from the Science Writing Heuristic (SWH) (Hand & Keys, 1999).

The MRH refers to a conceptual framework that explains the relationship among students' knowledge of mathematics, teacher's knowledge of mathematics, interaction with students, negotiation of ideas, writing, and process of students' problem solving.

The teacher's knowledge of mathematics and students' knowledge of mathematics interact in the course of learning. However, prior to actual students' learning process, teacher has some initial understanding of students' mathematics to which he/she relates his/her own mathematics (Simon, 1995). According to the MRH, teacher defines big ideas and, at the same time, anticipates students' prior knowledge. This planning phase for learning goals and activities is crucial in implementing the MRH.

Method

The purpose of this study was to examine the effect of writing on students' understanding of mathematical content and their reasoning skills within these writing tasks for a "real numbers" unit. The main data sources were students' writing samples, videotapes, on-site observations, field notes, and students' pre- and post-test scores (for the statistical analysis). This study was conducted in a high school with an algebra teacher who taught three Algebra I classes divided into one control group (17 students) with no extra writing tasks and two treatment groups (24 and 25 students in each) who completed extra writing tasks. Students were asked to write a letter to a construction company about the area of a rectangular ranch style house.

Teacher Template	Student Template
<u>Preparation</u>: - Identify the big ideas of the unit. - Make a concept map that relates sub-concepts to the big ideas. - Consider students' prior knowledge **<u>During the unit</u>:** **1. Students' knowledge of mathematics** - Give students opportunity to discuss their ideas - Have students put their ideas on the board for exploration **2. Teacher's knowledge of mathematics** - Use your knowledge to identify students' misconceptions - Guide students to the big ideas identified earlier during the preparation **3. Negotiation of ideas** - Create small groups and whole class discussion - Encourage students to reflect on each other's ideas **4. Writing** - Have students write about what they have learned in the unit to real audiences (teacher, parents, classmates, lower grades, etc.)	**1. What is my question (problem)?** - Specify what you are asked (What is the question asking?) - Outline the information/data given (What information is given?) **2. What can I claim about the solution?** - Use complete sentences how you will solve the problem - Tell what procedures you can follow **3. What did I do?** - What steps did I take to solve the problem? - Does my method make sense? **4. What are my reasons?** - Why did I choose the way I did? - How can I connect my findings to the information given in the problem? - How do I know that my method works? **5. What do others say?** - How do my ideas/solutions compare with others? a. My classmates b. Textbooks/Mathematicians **6. Reflection** – How have my ideas changed? - Am I convinced with my solution?

Table 1: MRH teacher and student templates

Results and Discussion

One-way analysis of variance (ANOVA) showed a non-significant result for the pre-test scores between control and treatment classrooms, (F (1,63) = .43, p= .515) and a significant difference between the control group and the treatment groups in favor of the treatment groups for the post-test (F (1,63) = 4.82, p = .032).

Students' reasoning varied from a trial-error process (e.g., *The best way to maximize area is to make it 40ft x 40 ft <u>a perfect square.</u> I found this by t<u>rial-error</u>*) to a mathematical demonstration (*It can have any sized length and any sized width <u>only if</u> it equals 160ft. It can be 10x70, 20x60, 30x50, and so on. My best opinion for the size of this ranch is 40x40 <u>because the rectangle ranch can be a square</u>…*). The analysis of students' written text suggested that the students appeared to solve the problem by intuition. Students' misconceptions also appeared in their writings: "*if we make it 40 feet on each side, it will be a square, not rectangle*." Some students recognized the condition for the problem (*any sized length and any sized width <u>only if</u> it equals 160ft*.) This is important for problem solving: stating which information can be used. Although this student did not provide evidence why the area would be the largest, she was aware that 'a square is a special case of rectangle' (i.e., *because the rectangle ranch <u>can be a square</u>*.)

The use of the writing task, while helping students on their test performance, also helped the teacher better understand students' understanding of the topic. While this is a pilot study, it does provide some encouragement to continue this line of research in the area of secondary schools.

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ROUTINE AND ADAPTIVE EXPERTS IN PROPORTIONAL REASONING

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Among the problems calling for proportional reasoning, those in which the task is a comparison of ratios can be classified according to several issues. One of the issues is the context; a possible classification according to it is in rate (extensive quantities), part-part-whole, and geometrical problems; in turn, part-part-whole problems can be mixture (e.g. the classical juice problem) or probability (e.g. double urn) problems. (Freudenthal, 1983; Tourniaire and Pulos, 1985; Lesh, Post and Behr, 1988; Lamon, 1993).

Another issue is the numerical structure. In a ratio or rate comparison there are four numbers in two "objects", each of which has an antecedent and a consequent. There is a classification of all possible such foursomes in 86 different situations that in turn can be grouped in three difficulty levels, L1, L2, and L3 (Alatorre, 2002; Alatorre and Figueras, 2003, 2004).

In the cited papers there is also a proposal for a classification of the strategies used by subjects in their answers to such problems; a brief description follows. Strategies can be simple (centrations or relations) or composed. Centrations can be on the totals, on the antecedents, or on the consequents. Relations can be order relations, subtractive relations, or proportionality relations RP; of the latter, five types are considered: three semi-formal (recognition of multiples, groupings, and equalizing), and two formal (quotient comparison and fractions properties). Composed strategies can take four forms of logical juxtapositions of two strategies. Strategies may be labeled as correct or incorrect, sometimes depending on the situation where they are used. Most strategies are incorrect; correct strategies are the proportionality relations and two kinds of informal strategies: some order relations RO, and some composed strategies that can be considered as theorems in action, TA (see e.g. Vergnaud, 1981).

The difficulty levels mentioned before refer to which correct strategies may be applied. L1 consists of all the situations where in addition to RP, RO and/or TA may be used. In L2 and L3 the only possible correct strategies are RP; L2 consists of situations of proportionality (both ratios or rates are the same), and L3 consists of situations of non-proportionality.

A case study was conducted in Mexico City with 23 subjects, aged from 9 to 65 and with schooling from 0 (illiterate adults) to 23 years (PhD). During the interviews, subjects were posed several questions in each of 8 sorts of problems, which were 4 Rate problems, 2 Mixture problems, and 2 Probability problems. Each of the problems was posed in different questions according to numerical structure. Each time, the subjects were asked to make a decision (object 1, object 2, or "it is the same") and to justify it. A total of 2152 answers was thus obtained, of which 80% were classified using the strategies system mentioned above.

In level L1 most subjects showed a good achievement. Here they used different correct strategies: almost never RP, almost always RO or TA, varying the strategy according to the context and the numerical structure, apparently searching each time the easiest way to solve the problem, in a manner that reminds of the adaptive experts described by Hatano (see e.g. Hatano and Oura, 2003). Thus, most of the subjects could be classified as locally (L1) adaptive experts.

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However, this kind of expertise is the only one that about half of the subjects could attain. Some of these subjects continued in L2 and L3 questions to use simple or composed centrations and incorrect order and subtractive relations, and thereby obtained incorrect answers. Only in scarce occasions of L2 and L3 questions did these subjects use some RP strategies.

If one considers that an expert in the kind of problems contemplated in this research is someone who solves a high percentage of questions using correct strategies, then the second half of the subjects can be considered as experts in at least one context. The easiest contexts were rate problems and the most difficult contexts were the probability problems. Only one of the subjects could be considered an expert in all contexts and difficulty levels.

The behavior of this last subject will be compared to the other ones. Given that he has a PhD in Chemistry, it would have been expected that Vicente (50 y.o.) was an adaptive expert, but he behaved as a routine expert: He used the same strategy almost in all the questions, monotonously calculating quotients. He did so even in most of the facile L1 questions.

Some of the other subjects had favorite strategies, but even if they did, they used at least two different kinds of RP strategies. The case of Dalia stands out as opposite to Vicente's. Dalia (25 y.o., with 3 years of schooling) can be considered an expert in all the contexts with the exception of the probability problems. In L1 questions she used mostly TA strategies, but also some RP strategies. In L2 and L3 she used an assortment of RP strategies. Dalia's behavior could be described as a constant search for the most comfortable strategy, taking into account the four numbers as well as the context. She prefers the use of informal or semi-formal strategies but uses formal calculations whenever she feels necessary. Dalia is an adaptive expert.

In conclusion, proportional reasoning can cover a range from informal strategies to formal calculations. In Vicente's case it seems that school favored a rigid use of formal strategies, while life has taught Dalia the ability to flexibly search for the most comfortable strategy covering the whole range of possibilities, although not in probability problems.

It has been argued (Hatano and Oura, 2003) that school is biased toward routine expertise even though adaptive expertise is more desirable. Dalia's case shows that in this sense life may be closer to the desired school than school itself, while Vicente's case is a confirmation of the routine expertise acquired at school, although this does certainly not prevent him from being a highly successful and creative teacher, lecturer, and scientist.

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CURRICULUM-SPECIFIC PROFESSIONAL DEVELOPMENT: A PHENOMENOGRAPHICAL STUDY OF TEACHERS' PERSPECTIVES

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To date, little research in the mathematics education literature addresses curriculum-specific professional development programs (i.e., professional development programs designed specifically for addressing the issues that arise when using a particular mathematics curriculum). As the use of reform curricula becomes more prevalent at all grade levels in the U. S., curriculum-specific professional development becomes critical. Consequently, understanding how teachers experience curriculum-specific professional development is vitally important; the more we know about how teachers experience professional development, the better we can support their learning as well as changes in classroom practice.

"Getting to the Core" was an NSF-funded professional development project that occurred from 2002-2004. During that time, 30 grades 8-12 teachers in one mid-western school district participated in approximately 200 hours of professional development focused on implementing *Contemporary Mathematics in Context* (Core-Plus) (Coxford et al., 1997, 1998, 1999, 2001). The professional development activities included: a) summer workshop-like sessions developed and implemented by a Core-Plus author, a university mathematics education faculty member, and the district 6-12 mathematics coordinator; and b) study groups throughout the two academic years.

In an attempt to focus on teachers' perspectives of the professional development, we utilized a phenomenographical approach in this study. "Phenomenography investigates the qualitatively *different* ways in which people experience or think about various phenomena" (Marton, 1996, p. 31, emphasis added). The goal of a researcher working within this approach is to "uncover all of the understandings people have of specific phenomena and to sort them into conceptual categories" (p. 32), and "these categorizations are the primary outcomes of phenomenographic research" (p. 33).

Methods of Inquiry/Data Sources

The primary data source in phenomenographical research is the interview. Interview transcripts are analyzed by first identifying and coding passages that relate to the research question. The marked passages are then "interpreted and classified in terms of the contexts from which they are taken" (Marton, 1996, p. 42). It is at this juncture that the unit of analysis shifts from individual subjects to interview passages or quotes. The quotes are then sorted into categories based on similarity. The resulting discrete categories are then used to describe the participants' different perspectives of a phenomenon.

We analyzed 29 semi-structured interviews from 21 teachers; 12 of these interviews occurred after the first year of the project, the other 17 occurred at the end of the project. Our analysis

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followed the sequence as described above (Marton, 1996). At the end of analysis, we had three discrete categories that described the participants' different perspectives of their participation in "Getting to the Core." Those categories comprise our findings.

Findings

The teachers experienced the professional development as an opportunity to *build knowledge* about the Core-Plus curriculum. They became more aware of strengths and weaknesses of Core-Plus; they learned more about how Core-Plus is different from traditional textbooks, both in philosophical underpinnings about learning and in structure and content. Most frequently, they spoke of knowing the curriculum itself much better after the professional development -- they understood better the articulation of the mathematical content within and across the four courses. Some teachers claimed that their new knowledge influenced their beliefs about the curriculum.

The teachers experienced the professional development as having an influence on their *classroom practice*. Many teachers spoke of attempts, successful and not, to transition to a more reform-oriented practice or philosophy.

Some teachers reported that they were more focused on their students than before the project, using student thinking to drive instructional decisions, and trying to understand what students understood about a topic. Many teachers attempted a new classroom physical layout – some of them grouping students for the first time. They spoke of explicitly establishing classroom norms with regard to teacher-student and student-student communication. The teachers found that they were trying new ways of managing the everyday work in the classroom – ways of collecting and distributing work and materials, as well as employing different methods for grouping students. Some teachers reported that they had been influenced in similar ways in their non-Core-Plus classes as well.

The teachers experienced the professional development as an opportunity to *collaborate* with other teachers in the district. They worked together, across the district, to share teaching ideas and plan lessons. A number of the teachers met in study groups to work through the Core-Plus curriculum in some manner. By collaborating to work through the curriculum, teachers told us that they deepened their understanding of mathematical content. Further, the teachers said that collaborating with other teachers as they worked through the curriculum allowed them to take on the role of the student.

Overwhelmingly the teachers talked about the opportunity to collaborate with different people over the course of the project. Along with teacher-to-teacher collaborations, the teachers also talked about the importance of collaborating with project staff. They perceived as particularly important the involvement of the Core-Plus author and experienced Core-Plus teachers, as well as local university faculty and project staff.

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INTEGRATING MATHEMATICS OF MEASUREMENT INTO ELEMENTARY TEACHERS' PEDAGOGY: COLLABORATIVE DESIGN AS A PROFESSIONAL DEVELOPMENT TOOL

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We report on a collaborative design project addressing the mathematics of measurement situated within elementary classrooms. We are working on a contextually-detailed account of teachers' growth in their pedagogical-content knowledge (PCK) of measurement topics, especially by selecting and implementing tasks (Simon & Tzur, 2004); we are beginning to sketch out a hypothetical learning trajectory on teaching measurement. (Empson & Turner, 2004; Hill, Schilling, & Ball, 2004).

Our understanding of the process of professional development of teachers is based on Piagetian concepts of *reflection* and *abstraction* (cf. Simon & Tzur, 2004). Ball et al. (2004)suggest that the work of teaching mathematics includes: giving and evaluating explanations, modeling operations as they link to concepts, judging representations, and interpreting students' mathematical ideas. We characterize teacher's developing abstractions along these critical aspects of teaching to elaborate on PCK of measurement. Children's understanding of measurement demands both mathematical and cognitive sophistication: for example, one must establish and extend units by clarifying the association between zero and one as locations along a line (Lehrer, 2003; Stephan & Clements, 2003). Lehrer and his colleagues (1998) observed a threefold improvement in teachers' classroom practices while engaged in a design experiment: (1) shifting from isolated task implementation to integrated sequences of tasks addressing themes, (2) shifting toward coordinated representations (3) shifting from gestures and brief verbal accounts of geometric ideas toward elaborate and specific verbal accounts of geometric concepts. Here, we examine the growth of a case-study teacher's thinking while engaged in a three-year professional development project (Thornton & Barrett, 2000)* centered on a reform curriculum.

How does a primary teacher's awareness of children's conceptualization of units and unit iteration contribute to changes in the teachers' explanations that fit children's strategies and ideas more closely, to improvements in task design, including the construction of representations that correspond to the mathematical foundations of measurement? We set up a teaching experiment using accounts of practice (Simon, 2000) to address this question; we examined the teacher's reflection while engaged in task selection and development, situating our analysis within the teachers' own classroom interaction patterns (D. Schifter, Bastable, & Russell, 2002; D. E. Schifter & O'Brien, 1997). We gathered three sources of data: (1) our own articulation of the learning trajectory throug written field notes, (2) the teachers' own reflective statements about lessons and (3) classroom videos.

To sketch some of our findings, we focus on the teachers' modeling of operations in relation to the concepts *unit* and *iteration* during the early period of the case study.

Fall 2001, Day 1

Mark's explanations and discussions of measuring topics did not relate to mathematical notions of a unit or to an iterative pattern. He only used the word *unit* as a reference to standardization (cm or in).

Reflective Discussion after Day 1

The researcher (first author) proposed a task with non-standard rulers, labeled with number sequences beginning with an integer greater than one; the researcher sought to draw attention to students' understanding of units along the edge of a ruler within an iterated sequence.

Fall 2001, Day 2

Mark constructed such rulers and set up measuring activities based on these rulers alone. He found some of the students thought these novel rulers had different lengths depending on what number label they found marking the end of the ruler, even though they compared the rulers directly and found them to fit exactly alongside each other. Mark was puzzled when the students did not rely solely on direct visual comparison.

Post Lesson Discussion

The researcher suggested that such children may not have established a unit concept, nor had they iterated units to interpret the rulers. Mark suggested these students would understand if they were only shown how to re-number the inch marks along their rulers. He continued to model measure operations by verbally rehearsing his own actions but resisted suggestions that he talk to students about length units. He assessed students' knowledge by checking for a fit between students' statements and his own wording.

A Modification of Rule-Following During Fall 2002

Mark began to emulate the first author in using a tile-rolling procedure to help students use tiles to find perimeters for tile-based polygons; he centered his own instruction around a demonstration of an edge-rolling procedure, telling students to "go around the perimeter you are measuring and find the number of tile edges." While Mark was still asking students to learn his wording, he was now using words to identify successive iterations of a unit along a path to measure. We believe he was beginning to model operations that corresponded closely to conceptual aspects of measurement (unit-length edges of tiles), building a linkage between students' strategies and those conceptual aspects.

In this period, Mark transitioned from gestures and non-conceptual verbal accounts of geometric ideas toward elaborate and specific verbal accounts of units and iteration sequences, which he promoted as common procedures for measuring perimeter among his students. This is one of several changes in Marks' practices we attribute to collaborative participation in task design cycles. Across the entire teaching experiment, we observed a significant and positive development of PCK for measurement. Our findings suggest consistent reflection on task-effect relations directed at units, and unit iteration (Simon & Tzur, 2004) promote teacher growth.

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PERCEPTIONS OF THE MATHEMATICS ACHIEVEMENT GAP: A SURVEY OF THE NCTM MEMBERSHIP

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There is increasing concern among educators about the disparities that exist among ethnic groups in mathematics achievement. There is no simple explanation for the achievement gap. However, it is important to recognize that the achievement gap is not a result of membership in any group, but is instead a result of the conditions of education (Thompson & O'Quinn, 2001). Consequently, a variety of school, community, and home factors seem to underlie or contribute to the gap. For example, the lower mathematics achievement levels of minority students, particularly Black students, may be indicative of the curriculum and instruction these students receive (Lubienski, 2003). Other researchers have highlighted differences in teachers' expectations of students as a function of race, gender, and social class which influence achievement (Berry, 2004; Ferguson, 1998). However, there are few studies that have surveyed educators to explore their explanations of the achievement gap in mathematics.

Therefore, the purpose of this study was to survey the perceptions of members of the National Council of the Teachers of Mathematics (NCTM) on the achievement gap in mathematics education. For the purposes of this study the achievement gap was defined as an indictor of disparities between groups of students usually identified (accurately or not) by racial, ethnic, linguistic or socio economic class with regard to a variety of measures (attrition and enrollment rates, drug use, health, alienation for school and society attitude toward mathematics, as well as test scores). More specifically the following research questions were addressed:

(1) What do respondents perceive to be the most important contributors to the achievement gap in mathematics?
(2) Do these perceptions vary as a function of personal characteristics of the respondent (i.e., gender, ethnicity, or age)?
(3) Do these perceptions vary as a function of characteristics related to employment (i.e., position held, years of experience, or educational degree)?

Method

Data was collected via an online survey sent to a random sample of the NCTM membership. At the time of the survey, there were a total of 41,508 NCTM members in the population to draw the sample from. The random sample was composed of 5,000 non-student NCTM members. On March 9, 2004, the sampling of the NCTM membership received an email containing the URL link that opened the online survey. The online survey closed on March 29, 2004. Eight hundred seventy members from the random sample visited the website and 623 members completed the survey.

Data Sources

The data source was the questionnaire developed by the researchers. The first section contained items requesting information on demographic and employment characteristics. The next sections presented 23 rating scale items pertaining to factors contributing to the

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achievement gap. The items were organized into five sub-areas or scales and included (1) Background and Societal Influences, (2) Student Characteristics, (3) Curriculum and Instruction, (4) Politics and Policy, and (5) Language. Respondents were asked to rate the extent to which they agreed with the statement on 5-point Likert-type scale, ranging from "strongly disagree" (1) to "strongly agree" (5).

A factor analysis (principal components extraction method with varimax rotation) was conducted to empirically investigate the validity of the rating scale items. The results supported only 4 scales. The component matrix did not support the original scale called Background and Societal Influences. The final solution of four factors, all with eigen values greater than one, accounted for 52 percent of the variance. Reliability coefficients (Cronbach's alphas) for each of the scales ranged from a low of .61 to a high of .85. More detailed results and explanation of the factor analyses supporting the subscales appear in an article by Bol & Berry (2005).

Highlights of the Results

The results pertaining to the first research question highlight the complex nature of peoples' perceptions of the achievement gap. When looking at the items that were most strongly endorsed as contributors to the achievement gap, educators endorsed items related to student characteristics that focused on family support, student motivation, peer pressure, and intellectual ability. The mean ratings on these items were 4.00 or above. This is important because these factors can be perceived as primarily non-school factors that are more resistant to educational interventions. The mean ratings obtained on the other three scales were similar and somewhat lower, suggesting moderate levels of agreement.

The second research question addressed whether perceptions differed as function of personal characteristics of the respondents. Minority respondents were significantly more likely to agree that factors related to curriculum and instruction contributed to the achievement gap. A significant effect for gender was observed on the Language scale. Females were more likely to attribute the achievement gap to language differences or difficulties.

When examining the variation in factor scores by employment position, we found significant differences between mathematics supervisors and teachers across all grade levels on the Student Characteristic scale. This presents a dilemma because teachers who interact with students on a daily basis perceive that factors such as peer pressure, family support, motivation, intellect, and interest in mathematics are more contributory to the achievement gap than do mathematics supervisors. Perhaps, daily contact with students makes teachers more attuned to student characteristics as a contributory factor on student achievement.

Our findings illuminate mathematics educators' perceptions of the causes of the achievement gap. Additionally, they may inform future studies on interventions or strategies aimed at alleviating this gap.

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UNDERGRADUATE STUDENTS' INTERPRETATIONS OF MATHEMATICAL PROOF

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Objectives

Because of the central role proof plays in mathematics, scholars have called for the learning of proof to become a central goal of teaching mathematics, especially at the college level (e.g., RAND Mathematics Study Panel, 2002). However, in order to design instructional interventions that address this, we first need to understand what students know about proof. Research has begun to address issues such as students' difficulties with proof (e.g., Coe & Ruthven, 1994) and their cognitive proof schemes (Harel & Sowder, 1998). This study extends current work by exploring undergraduate students' competencies in evaluating mathematical proofs.

Methodology

Participants were 400 undergraduate students from six academically and geographically diverse US universities. Participation was voluntary and based on the criteria that (1) participants had not previously taken formal courses in mathematical proof, (2) had not completed beyond first semester calculus, and (3) were enrolled in a course for whom the instructor had agreed to administer data instruments during class instruction. While it was not possible to randomly select student participants, effort was made to use a variety of educational settings in order to have a sample that could be considered representative with respect to demographics and university type. Student self-reported statistics on factors such as gender and ethnicity indicate that the sample was representative of the overall US college student population.

An instrument consisting of a survey questionnaire and multiple-choice items was designed and administered during one classroom instructional period at the beginning of the Fall semester. The 45-question Likert-scale survey focused on students' proof construction and understanding, attitudes and beliefs about proof, and classroom experiences with proof. The 25-item multiple-choice test examined students' ability in evaluating simple proofs and elicited their personal views on the role of these proofs. Prior to the large-scale study, the instrument was pilot-tested and a task analysis (item difficulty, item discrimination and distractor analysis) was conducted in order to establish its reliability and validity. Using Cronbach's α, reliability was computed at 0.75 for the multiple-choice test and 0.77 for the survey questionnaire. Descriptive statistics based on frequency tables, simple correlations, and tests of significance were used to compare and interpret the data. This study reports findings from student responses to the multiple-choice portion of the instrument.

Results

The multiple choice test included 3 conjectures and 4 supporting arguments (empirical, narrative, visual, and deductive) from which students were asked to select the one they felt was (a) closest to the one they would construct, (b) the most rigorous, and (c) the one they would use to convince a peer. Results (see Table 1) suggest that students primarily chose an empirical

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approach for conjectures (1) and (3), but a deductive approach for (2) as the one closest to the argument they would construct. This discrepancy, which could be due to the statement of conjectures (1) and (3) in natural language and that of (2) in more symbolic form, or to potential learning effects of the instrument, is an area for further research. For all conjectures, students were also more likely to chose a narrative approach (over deductive) as the one closest to their own solution or to use to convince a peer.

For all conjectures, a significant majority of students (≥ 63%) chose the deductive argument as the most rigorous approach, while no more than 15% chose an empirical approach. In conjunction with this, responses to other test items indicated that students were aware of the limitations of an empirical approach, with at least 40% interpreting the empirical argument to be "true for only a few cases". This suggests that, while students may have difficulty in constructing deductive arguments, they can identify more mathematically rigorous arguments. More work is needed to determine what salient features of deductive arguments lead students who do not have formal training in writing proofs to select these as more rigorous.

Mathematical Conjecture	*(a) Solution closest to student approach*	*(b) Solution chosen as most rigorous*	*(c) Solution chosen to convince a peer*
(1) The sum of any two even numbers is even.	**empirical - 42%** narrative - 30% visual - 5% deductive - 23%	empirical - 15% narrative - 15% visual - 4% **deductive - 66%**	**empirical - 35%** narrative - 31% visual - 21% deductive - 13%
(2) For any integers a, b, and c, if a divides b with no remainder, then a divides bc with no remainder.	empirical - 31% narrative - 23% visual - 10% **deductive - 36%**	empirical - 15% narrative - 17% visual - 5% **deductive - 63%**	empirical - 25% **narrative - 29%** visual - 22% deductive - 24%
(3) The supplements of two congruent angles are congruent.	empirical - 12% narrative - 28% **visual - 42%** deductive - 18%	empirical - 7% narrative - 17% visual - 11% **deductive - 65%**	empirical - 15% narrative - 25% **visual - 52%** deductive - 8%

Table 1. Results of student selection of argument type.

Relationship to Goals of PME-NA

By offering insight into undergraduate students' interpretations of mathematical arguments, this study contributes to our understanding of the issues associated with the teaching and learning of proof, a domain central to students' mathematical development.

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THE ROLE OF EXAMPLE-GENERATION TASKS IN STUDENTS' UNDERSTANDING OF LINEAR ALGEBRA

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Background and Theoretical Framework

Examples play an important role in learning mathematics. Students are usually provided with examples by teachers or textbooks, and very rarely are asked to construct examples themselves, especially in postsecondary level courses. Research has shown that linear algebra is one of the postsecondary mathematics courses that students are having difficulty with (Dorier, 2000; Carlson et al, 1997). Part of the difficulty is due to the abstract nature of the subject. Dubinsky (1997) points out that the overall pedagogical approach in linear algebra is that of telling students about mathematics and showing how it works. There is a lack of pedagogical strategies that give students a chance to construct their own ideas about concepts in the subject.

As research shows (Hazzan & Zazkis, 1999; Watson & Mason, 2004), the construction of examples by students contributes to the development of understanding of the mathematical concepts. Simultaneously, learner-generated examples may highlight difficulties that students experience. The analysis and investigation of learner-generated examples has been guided by APOS (Action-Process-Object-Schema) theoretical framework for modeling mathematical mental constructions (Asiala, et al, 1996). This framework was developed for research and curriculum development in undergraduate mathematics education.

Methods or Modes of Inquiry / Data Sources or Evidence

Participants in this research were students enrolled in elementary linear algebra course. The data was collected through students' written responses to the example-generation tasks. These are non-standard questions that require understanding of the concept rather than merely demonstrating a learned algorithm or technique. The written questionnaires were administered to the participants during the course. The topics addressed in the questions included linear (in)dependence of vectors, matrix algebra, and linear transformations. Several examples of the tasks are listed below:

(Q1) Give an example of a 3x3 matrix A with nonzero real entries whose columns $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$ *are linearly dependent. Now change as few entries of A as possible to produce a matrix B whose columns* $\boldsymbol{b}_1$, $\boldsymbol{b}_2$, $\boldsymbol{b}_3$ *are linearly independent, explaining your reasoning. Interpret the span of columns of A geometrically.*

(Q2) Give an example of a matrix for which the corresponding linear transformation maps the vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ *to* $\begin{bmatrix} 3 \\ 7 \end{bmatrix}$.

Results

The study discussed students' difficulties with constructing examples, and also suggested possible correlations of students' understanding with the generated examples. Furthermore, it

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showed that the example-generation tasks reveal students' (mis)understanding of the mathematical concepts. In particular, generating examples for the mathematical statements require more than just procedural understanding of the topic.

Using the APOS theoretical framework for analyzing students' responses to (Q1), one can identify different levels of students' understanding of the linear dependence concept. When students construct examples using random guess-and-test strategy, they operate with an action conception of linear dependence. They have to perform row reduction on a matrix to find out if its columns are linearly dependent. Students that construct examples of matrices with the same rows or rows being multiples of each other, i.e. inverting the row reduction procedure mentally, understand linear dependence as a process. Students that emphasize relations between column vectors have encapsulated linear dependence as an object, and consequently will be able to construct any set of linearly dependent vectors.

This research provides a better understanding of the role of example-generation tasks in students' understanding of linear algebra. It analyzes students' difficulties involved in generating examples and how students' examples correlate with their understanding. It is a novel study on example-generation tasks as a research and pedagogical tool in postsecondary mathematics education. It provides a variety of tasks for implementation in Linear Algebra course, and opens opportunities for future research and development of pedagogy.

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INVESTIGATING DISTANCE PROFESSIONAL DEVELOPMENT: LESSONS LEARNED FROM RESEARCH

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Effective professional development (PD) is critical in retaining rural teachers (Storer & Crosswait, 1995). Although all school districts are charged with providing teachers access to high-quality PD (National Staff Development Council, NSDC, 2005), rural school districts increasingly face budgetary reductions for PD exacerbated by rising travel costs. Emerging technologies and alternative delivery methods, such as videoconferencing and web-based media, are enabling the implementation of novel and attractive distance professional development (D-PD) opportunities for rural teachers and districts.

The current study focused on a D-PD initiative in mathematics and science education and examined how face-to-face, web-based, and video conferencing technology delivery affects the learner-learner/learner-instructor environment, communication and interaction. The study sought to answer the questions: 1) how does D-PD via electronic modalities differ from traditional face-to-face instruction; 2) how can D-PD allow for these differences; 3) what affect does video conferencing and web-based technologies have on reducing isolation of rural teachers; and 4) in what ways are participant-participant and participant-instructor communication enabled and constrained by videoconferencing and web-based technologies?

Theoretical Framework

Within the construct of D-PD, PD providers in this project sought to provide rural teachers socially negotiated (Cobb, 1994) and authentically constructed (Brown et. al., 1993) learning environments. Researchers sought to facilitate and observe the construction of communities of practice (Wenger, 1998) and the interaction and communication within and among those communities. Participants were observed in communities of practice, working together towards a common goal of studying visualization technologies and their classroom applications.

The complexity of learning to pedagogically employ visualization technologies necessitated that participants continually collaborate and communicate. Participants simultaneously grappled with learning the underlying mathematics, to interpret the visualization in the context of the science, learning the epistemological and pedagogical use of the visualization technologies. Through various methods of instructional delivery, teachers in Illinois and North Carolina interacted with others (local and remote), and utilized various communication technologies.

Methodology

This initiative purposed to enhance the retention and renewal of rural mathematics and science teachers through community building by delivering content, providing mentoring, and creating virtual teams among teachers in different states. Through an environment which was designed to be rich in communication and interaction, teachers used visualization and immersive technologies to deepen their understanding of core mathematics and science topics. Data was collected on the effectiveness of individual instruction sessions, type of instruction, importance of communication, use of technology, and impact on the participants. Baseline questionnaires at the beginning, halfway through, and post institute questionnaires were administered. Changes in the distribution patters were measured using Wilcox on Signed Rank Tests.

Data Sources/Results

Teachers found videoconferencing effective as a D-PD delivery method, although participant satisfaction was higher for sessions where the presenter was live and local than when the presenter was remote and participants watched the electronic presentation. Although a class culture formed at each location through formal and informal interactions among professional developers and teachers, it did not translate well from onsite to the remote site. Therefore, if PD is limited to information dissemination, videoconferencing was effective; however, if PD connotes the creation of communities of practice, videoconferencing may yet be limited.

Typically, via electronic communication, natural, spontaneous, and voluntary communication between remote participants remained minimal. The most effective method of D-PD was the small group (by subject/grade) parallel session via videoconferencing. Participants were very likely to communicate and build communities of practice among teams of teachers from the same school and/or when they had common teaching interests (subjects/grades). Proximity seemed to be the decisive factor in whether a community of practice was formed.

Conclusions

Conducting high quality D-PD in mathematics and science using videoconferencing and web-based tools presents new challenges to providers. Particularly difficult is facilitating the social interactions between distance learning sites to create communities of practice among educators at remote locations. The researchers felt that a stronger emphasis on communication between participants at remote sites may have a greater impact on fostering communities of practice. These ideas have been implemented and are currently being studied under cohort II. As a methodology to enhance rural teacher renewal and retention, D-PD continues to be investigated through this project.

Relationship of paper to goals of PMENA

This research attempts to deepen understanding of D-PD using video-conferencing and web-based technologies and attempts to further understand how professional learning communities of rural teachers can be formed using D-PD to reduce teacher isolation in rural settings.

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TEACHER CANDIDATE EFFICACY IN MATHEMATICS: FACTORS THAT FACILITATE INCREASED EFFICACY

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The research objective of this study was to identify the specific experiences of preservice teachers that influence their efficacy in mathematics teaching. Research in the area of teacher efficacy (Gibson & Dembo, 1984; Bandura, 1997; Tschannen-Moran & Woolfolk Hoy, 2001; Goddard, Hoy & Woolfolk Hoy, 2004) has produced a solid body of literature that examines how teachers judge their own capability to bring about student learning. The teacher assesses his or her ability to perform a given task based on an analysis of what is required to accomplish the task, reflection on past similar situations, and assessment of resources available (Bandura, 1986). Those teachers who believe they are effective are more likely to set high goals and persist to meet those goals even when faced with obstacles. Those teachers are also willing to experiment in the classroom (Allinder, 1994) with instructional strategies and student-directed, activity based methods (Riggs & Enochs, 1990).

Elementary teachers are at a critical juncture where their sense of efficacy teaching mathematics has the potential to increase. Researchers in the field have called for studies that illustrate methods which promote the enhancement of teacher efficacy. The most powerful source of information on teacher efficacy is mastery experience. Self-efficacy generally rises with experience, particularly following practice teaching. However there have been few qualitative studies to examine the factors that facilitate increased efficacy, particularly in mathematics teaching. The case of mathematics teaching is complex because preservice teachers are placed in classrooms where there is a limited range of reform-based mathematics instructional practices being used (Ross, 1999). Yet the reform movement clearly indicates that student directed conceptual approaches and paradigms are more effective and impact positively on student achievement (Ross, McDougall & Hogaboam-Gray, 2002; Simon, Tzur, Heinz, Kinzel, 2000). Further, preservice teachers have reported that they experienced traditional programs as mathematics students (Bruce, 2004). Thus, preservice teachers are attempting practices they have not experienced as students or as observers of students in host classrooms. To understand the mechanisms through which preservice programs might influence teacher efficacy and math reform implementation, intensive case studies are required. Therefore, the research questions of this study were: (i) what is the nature of the learning trajectories of preservice teachers in a Bachelor of Education program; (ii) which methods contribute to preservice teacher development of efficacy related to mathematics reform based teaching; and, (iii) what are the implications for preservice programs?

The site of this study was a newly established Bachelor of Education program in Ontario, Canada. Participants in the study were 12 preservice teachers enrolled in an elementary mathematics methods course. Data sources included open-ended inventories, focus group and individual interviews, observations, the Teachers' Sense of Efficacy Scale (Tschannen-Moran & Woolfolk Hoy, 2001), and participant math log entries. A Constructivist Grounded Theory approach (Charmaz 2000, 2003), with a zig-zag method for data collection and analysis (Creswell, 2005) were used. Methods of open, active and axial coding were combined with

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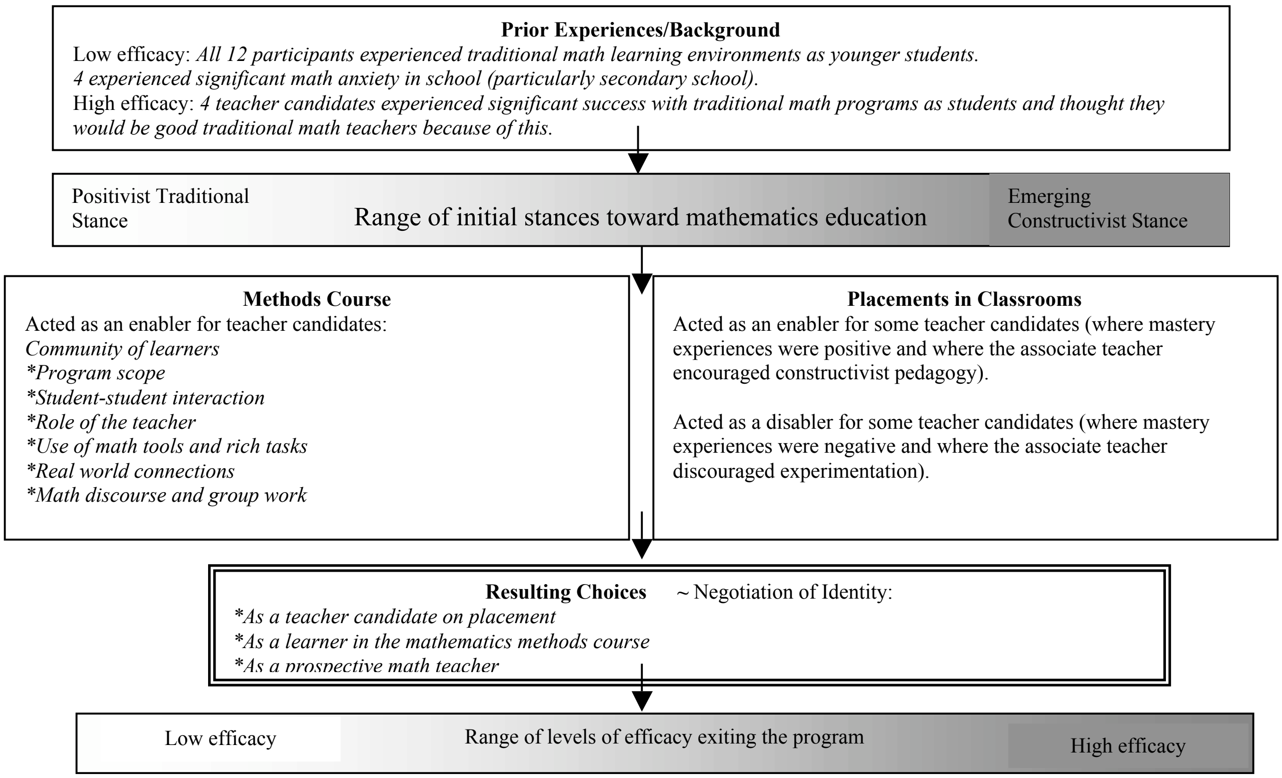
theory notes and visual maps to clarify and confirm understanding of the data. In order to fully mine the data, two case studies were completed to illustrate extreme trajectories: One extremely positive, the other very challenging. Other cases were used in a cross-case analysis to examine the range between the two extremes. For all participants, common stages in the trajectory were identified as were methods for enabling increased efficacy teaching mathematics.

The findings of all 12 participants are summarized in a diagram that illustrates the trajectories of preservice teachers, the influences on teacher efficacy in mathematics and the outcomes (see figure).



Factors that contributed to increases in teacher efficacy included features of both placements in schools and strategies used in the mathematics methods course. In most cases, opportunities to teach mathematics while on placement proved to be a tremendous confidence builder. However, in those cases where preservice teachers were strongly discouraged from experimenting with reform based methods of teaching math, teacher efficacy decreased. In the methods course, a combination of modelling reform based practices, developing a community of learners, encouraging student-student interaction, and guiding discourse were identified as enabling increased efficacy for all participants.

This study demonstrates that qualitative descriptions of shifts in teacher efficacy ratings experienced in a Bachelor of Education program are useful in identifying enablers of confidence teaching mathematics. Although challenging, mathematics methods courses can be structured and delivered to enhance preservice teacher efficacy. Further, teaching placements which support and encourage the use of reform based teaching strategies are strong influences on increased efficacy. The full findings of this study are important because the education community needs to establish and communicate researched effective strategies that enhance teacher efficacy at the preservice level in order to increase sustained implementation of mathematics reform.

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MODIFICATIONS GONE AWRY?: EXCLUDING FORMAL PROOFS TO ADDRESS EQUITY

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Mathematics education is widely viewed as an opportunity for students to develop skills and habits that will benefit them in the school mathematics curriculum, other subject areas, and in present and future life experiences. The National Council of Teachers of Mathematics (NCTM) (NCTM, 2000) describes not only the mathematical content that students should know but also ways through which content knowledge should be acquired and identifies reasoning and proof as two such skills that students should develop through their mathematical learning experiences. A high school geometry course *can* provide opportunities to develop systematic reasoning skills – a component of developing deductive reasoning and writing formal proofs. Students' justifications of mathematical arguments enhance their understanding through clarifying ideas and concepts. Thus, formal proofs are not to be viewed merely as an end product. Moreover, these skills are valued for their roles in preparing students for mathematics-intensive fields. Yet, students' overall performance in geometry, (Blank & Wilson, 2001), use of formal deduction (Burger & Shaughnessy, 1986), and ability to write formal proofs (Senk, 1985) are low.

Purposes

This research addresses two areas that are rarely considered concurrently – geometry and equity. Rather, at the secondary level algebra is viewed as a minimum requirement, and calculus is often viewed as an indicator of how successful one's mathematics education is. Although these perspectives are vital, an examination of the role of geometry in facilitating (or obstructing) equitable mathematics education is needed. The reasoning and logical thinking that accompany the construction of formal proofs in geometry are valuable skills that benefit students in future courses, real world applications, and access to advanced mathematics and mathematics-based majors and careers. Perhaps, greater concern is warranted for underrepresented students in that weakening the curriculum further impedes the realization of their mathematical potential.

I examine the influences and impact of the exclusion of formal proof from a high school geometry course disproportionately taken by students of color. This course, Modified Geometry, was designed to address inequities among students taking low level mathematics courses. I discuss how the course design addressed (1) the department's goals to increase the number of students who take more advanced mathematics courses and improve standardized test scores and (2) the department's quest to make mathematics education more equitable. While I assert that the exclusion of formal proofs negatively impacted students, I do not argue that the mere inclusion of formal proofs ensures desirable or equitable mathematics education.

Methods

This case study was conducted in a high school mathematics department in a school with approximately 1500 students. Approximately 50% of the students in the district received free or reduced lunch, and the racial demography was 65% White, 28% African American, 5% Hispanic, and 2% Asian/Pacific Islander. The graduation rate was 68% for the entire student

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body, 75% for Whites, 55% for African Americans, 41% for Hispanics, and 67% for Asian/Pacific Islanders.

Data were collected for eight months and included a focus survey, interviews, field notes from department meetings, and school documents. Twelve of the 13 members of the department including the chair and the district's curriculum coordinator participated. Their teaching experience averaged 10.8 years. Three key-informants participated in three interviews. The other teachers were interviewed once.

Data analysis consisted of triangulation of data and a search for disconfirming evidence. Transcripts of each interview and field notes were coded using an initial list of codes constructed from the research literature. I identified emergent themes after each iteration of data analysis and revised the list of codes, accordingly.

Findings

The department designed Modified Geometry for students who had completed Modified Algebra or who had passed the traditional algebra course with a C or below. Modified Geometry omitted formal proofs and was offered in addition to a traditional high school geometry course. A goal of Modified Geometry was to increase access to geometry topics to students who had previously stopped taking mathematics upon completing the four credit requirement. Equity is a multi-faceted construct that involves an examination of access/inputs, practices, and ends/outcomes. This analysis focuses on access, although access is not taken as a more important component. Greater access to geometric topics resulted. However, Secada (1989) outlined a second component to consider when examining how equitable mathematics education is – "is that which is being distributed worth having?" In this context, the exclusion of formal proofs and the accompanying deductive reasoning, yielded a more deprived course that impeded students' access to future mathematics courses and careers. Despite students' increased access to some geometry topics, the design modifications – which limited access beyond geometry, eliminated opportunities for developing critical skills (e.g. deductive reasoning), and perpetuated low expectations – had gone awry.

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HELPING PRESERVICE TEACHERS TO DEVELOP INSIGHT INTO CLASSROOM PRACTICE

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The use of videotaped classroom practice as an as an instrument of professional development has been well documented (for example: Davis, Maher & Martino, 1992; Fosnot & Dolk, 2002; Powell, Francisco, & Maher, 2003; Schmidt, McKnight & Raizen, 1996; Tarlow, 2004; Warner & Schorr, 2004;). However, in these studies, a fine-grained analysis of the data was utilized, which is something that may not be practical for classroom teachers or college students studying to become teachers. PRIVATE UNIVERSE (HSCFA, 2001), includes classroom clips to promote discussion. Although student artifacts and additional reading opportunities accompany this series, these supports are separate entities. The design of the project described here is distinctive in that supports are integrated directly into the actual software and can be viewed concurrently with the observation of the teaching and learning of mathematics.

The video vignettes that were examined for this study are part of a Virtual Learning Community (VLC) website. The VLC was created to support novice and mentor teachers as well as preservice teachers (Fraivillig, Wish & Bulgar, 2004). The question under study in this research is the following. How did the preservice teachers interpret the mathematical activity they observed through the use of supported videotaped vignettes and demonstrate their understanding of the interest and sustained engagement of the 2nd graders they observed?

The subjects of this study were full-time undergraduate students in two sections of a Mathematics Methods course at a small private university in NJ. They looked at vignettes from a 2nd grade elementary classroom as part of their regular university class experience. The data consist of undergraduate student work related to the project, including the answers to four questions about the vignettes. Undergraduate students viewed the vignettes prior to coming to class. In class, they were provided with laptop computers, working in groups of three or four to re-examine the vignettes, reflect upon their observations, discuss what they observed and answer four questions about the classroom mathematical activity. The following is an example of one of the questions: What is it about this task that makes it so engaging? The intent of this question was to focus attention on what led to student engagement in the hope that this would impact task design and selection for the future teachers.

The following excerpts from the undergraduate students' work products provide examples of the prospective teachers' responses.

Group 1: There are no limitations set for the way that they go about executing this task. The use of manipulatives kept the children engaged because it enabled them to see abstract ideas more [sic] concrete. The fact that they were able to talk to each other about the project also excited the children.

Group 2: They are extremely anxious to participate because of the atmosphere in the classroom and the interest in the problem.The problem is not traditional practice and drill exercise.

Group 3: …Students are given independence to complete the task using their ideas…Talking, using manipulatives, and conversation all allow the students to become engaged…

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Group 4: …They aren't concentrating on the math, but they are concentrating on the task at hand.

Group 5: …They each pick a strategy that works best for them.

The compiled responses indicate that the following criteria resulted in the young children's engagement: ownership of the task; authenticity; becoming decision-makers, empowerment; child-centered; enjoyment; teacher's encouragement and support; task relating to real-life; small group setting; collaborations; lends itself to the use of manipulatives; students have control over their ideas; students have control over their choices.

For future teachers to understand how to create and select mathematical tasks that will be engaging, they must first recognize the characteristics of engaging tasks. These pre-service teachers' responses indicated that viewing video vignettes with scaffolds embedded directly into the software provided them with an opportunity to identify criteria for engaging tasks.

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"MAKING MATHEMATICAL CONNECTIONS" IN THE TEACHING OF SCHOOL MATHEMATICS

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Problem Statement and Research Questions

The National Council of Mathematics Teachers' (NCTM) document, *Principles and Standards for School Mathematics* (2000) and its earlier versions in 1989 and 1991 establish a framework to guide improvement in the teaching and learning of mathematics in schools. The documents identify "mathematical connections" as one of the curriculum standards for all grades K to 12. In this framework, "… mathematics is not a set of isolated topics but rather a web of closely connected ideas" (NCTM, 2000, p. 200). Both the research literature and the pedagogical literature stress the value and importance of making mathematical connections, the rationale being that making connections will allow students to better understand, remember, appreciate and use mathematics. Teachers are exhorted to teach in ways that will encourage the making of useful mathematical connections by their students. Learners might make connections spontaneously, but "we cannot assume that the connection will be made without some intervention" (Weinberg, 2001, p.26). The implied role for teachers is to act in ways that will promote learners' making of mathematical connections (Thomas & Santiago, 2002). Studying teachers' pedagogical efforts to promote the making of mathematical connections necessitates considering the intersection of three frameworks – their own understanding of mathematics, their general pedagogical knowledge, and their specific pedagogical content knowledge (Shulman, 1986).

This report describes an exploratory study designed to identify emerging themes in teachers' thinking related to making connections by probing their own perceptions. The study addresses the questions:

- how do mathematics teachers conceptualize "making connections", and
- how do they see themselves attending to making mathematical connections in their teaching.

Research Setting

The participants are three secondary mathematics teachers with 5-8 years of teaching experience, all acknowledged to be excellent teachers. Each teacher participated in an audiotaped semi-structured interview lasting 30-60 minutes. Prepared questions focused on mathematical connections and how they are used in teaching practice, for example:

- What sorts of things come to mind for you when you hear the term "connections"?
- In addition to connections "to the real world", another interpretation of connections is "connections within mathematics". To what extent are these kinds of connections part of your teaching? Please give an example.
- What kinds of things do you do to show students connections within mathematics?

Follow-up questions were not pre-planned and probed teachers' individual initial responses Schram (2003). I transcribed the interviews and developed a coding scheme (Gall, Borg and Gall, 1996). Some categories of responses could be identified a priori because I had asked

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specific questions about certain topics – for example, mathematics background, beliefs about connections, teaching goals. Other general categories emerged from a reading of transcripts – for example, beliefs about the nature of mathematics, beliefs about learning and teaching, teaching strategies.

Data and Interpretations: Views of "Making Connections"

All three teachers reported similar views with respect to "making connections". The main issues that emerged in the interviews were:

- Making connections in mathematics means making connections to the "real world", specifically to finances, games and using mathematics as a tool in other subjects.
- Making connections can be useful in assisting students' memory and increasing their motivation. The teachers saw making connections as particularly important with younger students and students in non-academic math courses.
- None of the three teachers spontaneously included concept-to-concept connections within mathematics in their views. With further probing, they equated making connections between mathematical concepts to recalling previously learned ideas and procedures.
- The teachers found it difficult to present examples from their own teaching of specific events that they viewed as "making connections".

These teachers saw themselves as paying little attention to making connections in general, and even less to mathematical connections. A crucial next step in the research is to establish whether this belief is an accurate perception of their practice – do teachers *really* not attend to making connections? Or, might they be acting automatically and unaware of actions that others might identify as "making connections"? In either event, in the context of the NCTM's emphasis that understanding develops "only if students grasp the connections" (NCTM, 1989, p.147), this finding raises a crucial question about the nature of the connections that students make if the connections are not dealt with *explicitly* during their mathematics lessons.

Conclusion

This exploratory study identifies some emergent themes in the way that practicing teachers view "making mathematical connections" that appear to differ from the conceptualizations in the literature. The differences noted should be explored by a more in-depth study in order to verify these emerging themes with larger and broader samples of teachers, and using a range of methods, and then to build a model of the interaction of teachers' pedagogical beliefs, their own knowledge of mathematics, and their specific pedagogical content knowledge.

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THE EVOLUTION OF MATHEMATICAL EXPLORATIONS IN OPEN-ENDED PROBLEM SOLVING SITUATIONS

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The purpose of the study was to examine the problem solving processes that solvers use to solve open-ended mathematics problems. Subjects were interviewed as they solved a set of open-ended problems. Drawing from the episodes of two students solving a Number Array task, the analysis explains how the solvers' self-generated problem posing actions help to establish and then extend conceptual boundaries for their solution activity. Our on-going work in problem solving is to develop a model of the solvers' general exploration processes. The current study is part of this overall effort and is an extension of an earlier study (Cai & Cifarelli, in press).

Subjects and Analysis

The subjects were two secondary math education majors, Sarah and Gavin. The students were interviewed as they solved a Number Array task that required them to find relationships within a square array of numbers (Figure 1).

The data consisted of video protocols, written transcripts of the videos, the researchers' field notes, and the subjects' written work. We hypothesized that problem solving in open-ended tasks involves varying degrees of problem posing (that involves the solver's interpretations and how they give meaning to the tasks) and problem solving (in the sense that once goals are developed, the solver shifts his/her focus to the carrying out of action designed to achieve his/her goals). We believe this process to be recursive in nature, with each self-generated question indicating acts of problem posing that the solver initiates to frame and structure their subsequent actions.

Figure 1: Number Array Task

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
2	**4**	**6**	**8**	**10**	**12**	**14**	**16**	**18**	**20**
3	**6**	**9**	**12**	**15**	**18**	**21**	**24**	**27**	**30**
4	**8**	**12**	**16**	**20**	**24**	**28**	**32**	**36**	**40**
5	**10**	**15**	**20**	**25**	**30**	**35**	**40**	**45**	**50**
6	**12**	**18**	**24**	**30**	**36**	**42**	**48**	**54**	**60**
7	**14**	**21**	**28**	**35**	**42**	**49**	**56**	**63**	**70**
8	**16**	**24**	**32**	**40**	**48**	**56**	**64**	**72**	**80**
9	**18**	**27**	**36**	**45**	**54**	**63**	**72**	**81**	**90**
10	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**

Results

Working in the number array task provided the subjects with opportunities to explore and develop a variety of mathematical relationships; a subset of these is summarized in Table 1.

Table 1: Mathematical Relationships Generated by Solvers	**Sarah**	**Gavin**
Relationships about the arrangement of the numbers	**4**	**4**
Relationships about the sums of the numbers	**7**	**2**
Relationships about the products of the numbers	**3**	**2**
Relationships about number sequences	**5**	**0**
Total	**19**	**8**

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The results summarized in Table 1 indicate some compatibility in the relationships developed by the students. However, Sarah developed several more sophisticated relationships than did Gavin. These included an informal 'skipping' method to find the sums of the entries of all NXN blocks containing the square numbers on the diagonal.

> ***Sarah**: So, for a 1x1, I get a sum of 1. For a 2X2 (**Points to 2X2 block [1,2:2,4][1]**) I get a sum of 9 … but what happened to 4? It has been skipped! (**reflection**) Okay, let me try this, I will write down the sequence of squares of all numbers, all in a row (**She writes the sequence: 1, 4, 9, 16, 25, 36, 49, …, 2251).** The first number, 1, is the sum of the first matrix, a 1x1. And the first 2x2 has a sum of 9. So, I skipped over 4 to get the sum for the 2X2 in the upper-left (**crosses out the 4 in the sequence of squares**), going from 1x1 to a 2x2, a sum of 9. The 4 is skipped! Interesting!*

Sarah was able to generalize her 'skip' method to all NXN blocks.

> ***Sarah:** So, going from the 2x2 to the 3x3 (**Points to 3X3 block [1,2,3:2,4,6:3,6,9]**), we go from 1, to 9, to 36 – so we skipped over the next two numbers, 16 and the 25 (**crosses out 16 and 25 in the sequence**), a skip of 2 in this sequence! Okay, then we will skip over the next 3 square numbers, and that should tell us the sum for a 4x4 should be 100 (**crosses out the next 3 in the sequence after 25: 36, 49, 81)** – that is what I have over here !! Cool! So, for a 5X5, we skip over the next 4 numbers in the sequence, the number 121, 144, 169, 196 and get 225 – yes!*

Sarah had developed an in formal 'skipping' method for computing the sum of the entries of NXN matrices down the diagonal of the array. She generalized a more efficient algorithm that involved operations of the row and column numbers of each NXN.

> ***Sarah**: I wonder why this skipping works? Let's see, for the 6X6, we add the rows of the block, 21+42+…+126= 21(1+2+3+4+5+6)= 21x21=441. Do we get 441 by skipping the next 5 in this square sequence? (**She checks her original sequence and crosses out the corresponding 'skips', and gets 441 as the next number in the sequence!**) I notice that 21 over here (**she points to the factored form 21(1+2+3+4+5+6)** is the sum of the 6 numbers in that first row. Yes! So to find the sum of these NXN blocks, I bet you just need to look at the sum of 1 to N and then square the total to get the sum. Let's try 8x8 … it would be 1+2+…+8=36, and then I take 36^2 …1296. And does it check with my skipping sequence over here? So for 8x8 I first skip 6 over 21 to get 28^2 for 7x7, and then skip 7 more to get the one for 8x8, so 7 more is 35, and the next one is 36! So my algorithm works! It is efficient for large numbers – how about a 100x100 grid? – but the skipping relationship was pretty cool!*

Summary

The findings are consistent with research on open-ended problem solving that posit conceptual benefit when students solve open-ended problems (Becker and Shimada, 1997; Cai & Cifarelli, in press; Cifarelli & Cai, in press; Silver, 1994). Moreover, Sarah's evolution of her solution activity from informal methods to her invention of a sophisticated algorithm for NxN cases illustrates an important way that solvers stretch their conceptual boundaries when engaged in open-ended problem situations.

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[1] In order to refer to various blocks of numbers in the array, we use a notation that lists the top-to-bottom rows of the block. For example, the 3X3 block in the upper left position is denoted by [1,2,3:2,4,6:3,6,9].

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ANOTHER HIDDEN CURRICULUM: EAVESDROPPING ON STUDENT GROUPS

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Purpose

This paper reports the results of a pilot study in which I use ethnographic methods to demonstrate that even with reform-oriented, student-centered teaching methods, there exists another "hidden curriculum" in the math classroom. Historically, the hidden curriculum has focused on how teachers, administrators, and the institution of schooling perpetuate particular roles for students. As classrooms shift from being teacher directed to student centered, I show that there is *another hidden curriculum* – one where the responsible parties are the students themselves: as the classroom control is turned over to students, the hidden curriculum is being constituted by them. I highlight the ways female students position themselves within groups to be dually powerful and powerless.

Theoretical Framework

Critical multicultural mathematics education is "concerned with the social and political aspects of the learning of mathematics" (Skovsmose & Borba, 2004). The alienation of non-whites and women as a causal factor in their historical lack of success and participation in mathematics is indicated in numerous studies (Kincheloe & Steinberg, 2001). In the US, the National Council of Teachers of Mathematics *Standards* document supports a critical multicultural view of mathematics education through its mission to provide opportunities for "every child" to be successful in mathematics.

In this paper, I use a critical qualitative analysis to demonstrate the existence of *another* "hidden curriculum." This analyses benefits from a "Foucauldian gaze" which recognizes that power is present in all human relationships (Walshaw, 2001). Foucault argues that power is not solely owned by one person or one group, but rather exchanged and reformulated. In this paper, I focus on objectification and competing discourses in small group interactions to examine power relations. Objectification is defined as "a process whereby a powerful group establishes and maintains dominance over a less powerful group by teaching that the subordinate group is less than human or like an object" (Gamble, 1999, p. 286). Judith Baxter (2002) "shows the complexity of how girls are multiply positioned by competing classroom discourses as at times powerful and at other times powerless" (p.5).

Context of the Study and Methods

In this ethnographic pilot study, I observed a middle school mathematics classroom of 26 students with an equal number of males and females who, according to their teacher, Grace, represented "a mix of kids and abilities." I visited the classroom seven times during fall, 2004, to observe Grace, an award winning mathematics teacher, and her students. Field notes were taken and then coded to identify themes, of which the power and positioning of the female students in small groups became apparent. To illustrate the findings from this study, I present two situations from my data that occurred with the same students (working in a group of four) who had clear direction from their teacher to complete a certain task.

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Results

Objectification of the Female

In the first vignette, Janelle, described by Grace as a good student who cares about her grades, is trying to get her work done. While she is not the designated "quality control person," she becomes the group member who tries to focus the group to accomplish the task. While doing so, she is objectified by the boys in the group who say things like, "now she's acting motherly" and "now she's serious." Objectifying illustrates how men are socialized for gender dominance (Kincheloe & Steinberg, 2001). Although this interaction does not have explicit sexual overtones, the treatment of women as objects could be viewed as a source out of which grows masculine domination. This dominant vision of men, Bourdieu (1999) says, leads women to find their place in the social order and see it as normal. Carspecken (1996) contends that when subordinates accept their social status as natural or inevitable, oppression is reproduced.

Competing Discourses

In another situation, Janelle tries to focus the group, but after several attempts, she decides to do her own work and "ace the test." After making fun of one of her answers in the small group, one of the boys, Tim, uses her answer when called on later in a whole class discussion. When a male student takes a female student's answer, he is making a "claim to knowledge" (Kincheloe & Steinberg, 2001, p. 139). Male authority appropriation over what women have said is a power dimension that is illustrated daily on the individual level, e.g. during board meetings, union meetings, and teachers' meetings (Kincheloe & Steinberg, 2001, p.139).

When the group continues their off-task behavior, Janelle proceeds to work quietly on her own. When girls try to be nice, kind, and helpful – characteristics that teachers publicly hold up as good – they put themselves in psychic and social double-binds (Walkerdine, 1998). In this case, Janelle feels confident that she is capable of getting the work done by herself, but allows herself to be dismissed by Matt, saying "You **told** me to do my own [work]." Baxter (2002) demonstrates how and why girls can be silenced in classroom contexts by examining the *contradictory* positioning of the student. On one hand, Janelle is a capable, rule follower; on the other, her response is passive aggressive and later causes the teacher to reprimand her for not working with the group. This power struggle positions Janelle both as powerful (she'll "ace the test") and powerless (she was "told" what to do and gets in trouble with Grace).

Summary and Implications

Even an award-winning teacher has limited social control. She is practicing methods deemed useful in promoting equity, yet the gender roles are clearly apparent in the small group interactions. Kincheloe and Steinberg (2001) note that the "conditions under which knowledge is produced have changed dramatically over the past twenty years" resulting in new forms and guises of power and hegemony. Teachers interested in social justice need to be aware of another hidden curriculum—one that plays out in small groups-- and work to promote equity for historically subjugated groups.

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THE TEACHER AS A BROKER IN ESTABLISHING A CLASSROOM COMMUNITY OF PRACTICE

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Theoretical Framework

A classroom community of practice is used to describe the social setting of a classroom in which students come together to work towards a communal goal. It is within this classroom community of practice that the teacher acts as a broker (Wenger, 1999) to help guide the students to engage in activities in a way that is commensurate with the larger mathematical community. The term broker is used to recognize that the teacher was a member of both the classroom community and the larger community of mathematicians. Brokers thus have the role of aligning the communal goals of the classroom with those of the larger mathematical community. They encourage this alignment by initiating certain social norms which allow the sociomathematical norms (Yackel & Cobb, 1996) to emerge. They also aid this alignment by introducing tools and formal conventions accepted by the larger mathematical community.

Methods

This data was collected during a semester long teaching experiment (Cobb, 2000) in a mathematical structures course where students' transition from the computational mathematics of Algebra and Calculus to the more rigorous proof writing required for upper level mathematics. The students engaged in activities that encouraged them to use formal deductive arguments to justify their conclusions. Videos of the class sessions were analyzed and then transcribed. The transcripts were coded for areas in which the students and teacher were observed contributing to the norms of the classroom and points where the teacher's influence helped student learning.

To allow the sociomathematical norms to emerge, the teacher initiated the social norms of the classroom including that the students were to engage in classroom discussions, were to share their thinking about their solutions, were to try to make sense of other students' solutions, and were to challenge solutions they did not agree with. She did this by asking questions that encouraged students to participate, asking questions that encouraged students to check the validity of other students' solutions, and by inviting comments and critique from the students. Through these interactions the students became aware that she valued student participation.

Results and Discussion

The primary sociomathematical norm contributed to by the teacher and the students in this study was the criteria for what constituted a sufficient mathematical argument. The teacher served in her role as broker to encourage the students to create arguments that would be acceptable to the mathematical community while supporting students' emerging activity and arguments. The primary criterion for a sufficient mathematical argument became that the argument be deductive, meaning that it showed why a claim must be true.

At the beginning of the semester the teacher used the fact that students were explaining their solutions to the class and evaluating each other's solutions (social norms) to engage students in reflecting on what would count as a sufficient mathematical argument. She also supported this norm by pushing students to provide a more deductive or more complete argument when students did not at first provide one. Later in the semester students were more likely to provide such arguments or to require such arguments of others. Another important role of the teacher as broker was that she had to introduce the conventions and symbols that have been established by the mathematical community. Often students will have difficulty deriving these conventions or recognizing a need for them. It is the teacher's job to make sure the students use the same symbols as the mathematical community.

Finally the teacher as broker is responsible for helping students create the formal tools into the classroom community that are used by the community of mathematicians. While modeling the students thinking on a problem, the teacher created a grid that resembled the way they were discussing the problem. Noticing its similarity to a truth table, the teacher took the opportunity to introduce a formal truth table which is a tool of the mathematical community. The students were then able to use this tool in working on problems in order to make sense of them. In order to become a member of a community, whether it is the classroom community or the community of mathematicians, it is important for the individuals to be able to use the tools of that community.

In the examples above the teacher serves as a broker by helping to align the sociomathematical norms, tools and conventions of the class with that of the mathematical community. On the one hand the teacher has the role of engaging students in the classroom community of practice and working with in the boundaries of that situation. On the other hand the teacher has the responsibility as a representative of the mathematical community to work to align student activity with that of practicing mathematicians. These dual roles are illuminated by the use of Wenger's (1999) term broker and bring out the relationship between a classroom community and the mathematical community. This research is part of a larger study on the emergence of a classroom community of practice (Clark, 2005) and was supported in part by the National Science Foundation under Grant No. 0093494. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the NSF.

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COMPARING THE MATHEMATICAL THINKING OF GRADES 2, 5 AND 8 STUDENTS ON IDENTICAL MATHEMATICAL TASKS

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Introduction

Some studies have found that as students progress through the grades, at least some of their mathematical thinking deteriorates (Burns, 1994; Kamii, 1989; Kamii, Lewis, & Livingston, 1993; Reid, 1995). Although students enter school as eager and capable mathematical thinkers, as they progress through the grades they appear to give up their own sense-making capacities and learn to rely on memorized procedures (McGowen & Davis, 2001a; 2001b; Romberg, 1992**.** This study explores grades 2, 5 and 8 students' mathematical thinking when completing identical mathematical tasks, in structured task-based interviews. The objective of the study is to develop a conceptualization of student mathematical thinking across these grades.

Methodology

Twenty students from each of grades 2, 5 and 8 in the same school were randomly selected from class lists. The interviews focused on the three tasks shown in Table 1. Task 1 was used in the study by Reid (1995). Task 2 is similar to questions used by Kamii (1989) to compare grades 2 and 4 student thinking. Task 3 offered students the opportunity to explore patterns in the context of making sense of division by zero.

Table 1. The mathematical tasks

TASK 1: Your parent gives your teacher \$6.25 to buy popsicles for your class. Popsicles cost 25 cents each. Will there be enough money to buy one for each student?
TASK 2: Think of the numbers 176 and 58 and 5. (*Written form: 176 + 58 + 5*) Please add these numbers in your head.
TASK 3: • What is 2 times 0? What is 2 divided by 0? • What is 2 divided by 2? Or in other words, how many twos are there in 2? • What is 2 divided by 1? How many ones are there in 2? • What is 2 divided by 1/2? How many halves are there in 2? How did you figure this out? • What is 2 divided by 1/4? How many quarters are there in 2? How did you figure this out? • What is 2 divided by 1/8? How many eighths are there in 2? How did you figure this out? • What would be the next question if we continue this pattern? What would be the answer? • What is happening to the numbers that we're dividing by? • What is happening to the answers to these division questions? • What do you think now about the question of 2 divided by 0?

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The interviews were tape-recorded and transcribed. Students were given positive feedback for their answers/contributions using phrases such as ‘thank you’ and ‘that’s very interesting’, but they were not told whether they were right or wrong. Scaffolding was provided for grade 2 students for Task 3, as needed. A content analysis was then conducted of the transcribed interviews and field notes (Berg, 2004). The content analysis focused on the level and nature of students’ mathematical thinking, using 4 levels of performance ‘similar’ to those found in the Ontario Mathematics curriculum documents (Ontario Ministry of Education, 1997).

Findings and Discussion

TASK 1. Unlike the study by Reid (1995), we did not find that grade 5 students performed less successfully than grade 2 students. The grade 5 students demonstrated the greatest variety of solutions. The grades 2 and 5 students were more likely than grade 8 students to rely on solutions that made use of graphical representations – all students who used such methods were successful in solving the problem.

TASK 2. Unlike the Kamii (1989) study, none of the grade 2 students were able to solve this problem. The grade 8 students performed less successfully than the grade 5 students. The grade 5 students were more likely to visualize the numbers in a column and apply the standard addition procedure. The grade 8 students used a greater variety of methods.

TASK 3. The level of mathematical engagement did not seem to vary among grades 2, 5 and 8 students. Although older students were typically more successful and needed less scaffolding in completing the questions involving operations with fractions, none of the students were able to engage with the task beyond the level of answering individual questions: They did not search for and did not identify patterns, and they did not see any connections to the final question of “What do you think now about the question of 2 divided by 0?”

In conclusion, unlike previous similar studies by Kamii (1989) and Reid (1995), older students in our study were generally more successful with problems involving number operations. At the same time, older students did not appear to have matured in their mathematical thinking, and could not see nor conjecture beyond small, individual tasks.

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EXAMINING PROSPECTIVE TEACHERS' GROWTH IN UNDERSTANDING SIMILARITY USING LESSON PLAN STUDY

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Focus of Study

The investigation of prospective teachers' (PT) knowledge of similarity was part of a 3-year study on PT change in understanding of the high school content, during Lesson Plan Study (LPS). This paper discusses the influences on PT preparation and the instructional teaching activities associated with the content of high school mathematics (Berenson, Cavey, Clark, & Staley, 2001).

Conceptual Framework

The growth of the PT knowledge of similarity was assessed within the Pirie-Kieren (1994) model of growth in understanding as adapted to teacher preparation by Berenson, Cavey, Clark and Staley (2001), while noting instances of folding back, and collecting (Pirie & Martin, 2000). The teacher preparation model is a framework for studying PT understanding of *what* and *how* to teach. The *what* of teaching includes representations, and the knowledge and understanding of the essential features of the concept. The *how* of teaching incorporates the ways of approaching the topic, the basic repertoire in teaching the topic, and the prospective teachers' knowledge about mathematics. Within the discussions and presentations of a lesson to introduce the topic of similarity, images and growth of PT knowledge of similarity were examined.

Methodology

Five participants (Alice, Anne, Ava, Rose, and Mary) were chosen for an analysis of their growth of understanding of similarity. They participated in the LPS on similarity, during their first of four mathematics education courses. The LPS contained four distinct stages occurring over a six-week period of time. The first stage was an individual interview in which a researcher got an initial understanding of what a PT knew about similarity and how they might teach it. The second stage was a group interview, the five participants together, were asked to construct a group presentation on similarity, and discuss their ideas. The third stage was the presentation of the group lesson to the methods class. In the last stage, the PT produced a reconstructed view of their individual lesson plans (Reference withheld).

Results

The PT images of *what* and *how* to teach similarity changed while involved in the LPS. At first, the PT related images of similarity with proportional sides and congruent angles in triangles. The group planning allowed the participants to make new images of *what* and *how* to teach by listening, discussing, and reflecting on their ideas. After the group planning stage, everyone collected and formalized Ava's images of the similarity postulates and Mary's images of modeling and indirect measurement. The following is a direct quote from Anne's final individual lesson plan:

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I will explain to the class that we have short cuts to find out if two triangles are similar or not. If it would take to long to see if all the angles are congruent or if the sides are proportional. What would we do if we were missing some information from the triangles? That is why we have conjectures.

Many of the PT made images of activities that allowed their students to derive mathematical similarity. Rose moved from a procedure-only approach of teaching to having an image of the importance of a conceptual activity.

Anne: What should be our hands-on activity?

Rose: I like her idea [*points to Ava, Anne nods*], I really do…

Anne: Yeah.

Alice: Me too.

Rose: Cause hers would take away my need to go, say okay well, "If you have two triangles and try and figure out if they are similar or not, and that actually teaches them.

While she did notice the importance of this activity, she never formalized this teaching strategy within her knowledge of how to teach and still believed that the best approach was a teacher-led lecture.

Towers (2001) stated that students' understanding is partly determined by teacher interventions. Interventions such as lesson-planning activities may allow PT to reflect on their own knowledge (Berenson, 2002; Davis, 2004; Davis & Staley, 2002; Staley & Davis, 2001). Through these reflections and discussions with colleagues and teachers, students could realize some of their own limitations which gave them the opportunity to improve upon their understandings.

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ON THE USES OF TREES AS REPRESENTATIONAL TOOLS IN ELEMENTARY PROBABILITY

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Difficulties of students in understanding and using tree diagrams are well-known of teachers, but researchers have extensively documented the problem. To cite one example, Green (1983) has provided extensive evidence that a great number of pupils who have been taught tree diagrams do not use them with success. Studies on use of tree diagrams in probability education have examined how to better teach these diagrams (Totohasina, 1994), and the effectiveness of trees as a pedagogical tool (Bernard, 2003; Dupuis & Rousset-Bert, 1996).

In this communication I carry out a discussion of the semantic transparency of tree diagrams by analyzing related literature and by adding to it observations from classroom experience.

What is So Sacred About Tree Diagrams?

Every representation has its history, and as with any idea, person or society, some may have a history of more conflicts, some may have had a brief existence, some may have been brought to existence rather recently and survived with no major difficulties due to favorable conditions and environment. It is not an accident that the Hindu-Arabic numeration system, for example, is such an effective notation: it has competed with dozens of other systems that were invented throughout a history spanning thousand of years and taking place in a vast geographical arena. But some notations do not have the "evolutionary pedigree" of the Hindu-Arabic numeration system (Cheng, 2003, p. 234).

What is the history of the tree diagrams? Hacking, who at the present is researching "the cultures and uses of tree-diagrams" (2005) but has not yet published on this topic, says that, while there are cognitive scientists who argue strongly that arranging hierarchies, taxonomies, or temporal processes in the form of tree-diagrams may be an innate tendency in humans, the use of tree diagrams seems very recent in history. At the moment, the earliest Western record of tree diagrams is from the 8th century; in the East, in Syriac, they go back to 5th century. Hacking doesn't have information on the earliest use of the diagrams to represent probabilities, though (Hacking, personal communication).

A history of the growth in the scope of use of tree diagrams may offer insight into what kinds of relations may be more naturally represented by trees (if we correspond "more natural" with that which was done earliest in history) and which ones took longer to be associated to trees. At least one study done in classroom seem to agree that some features may be more readily represented by tree diagrams than others: Pesci (1994) examined how tree graphs were used in eight third year Junior Middle School classes, and found that students immediately used tree graphs for solving a problem in which the various phases of the random experiment in the problem occurred successively in time. Tree graphs did not "come as naturally" in another problem proposed which had no explicit temporal sequence (p. 32).

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Alternatives to Tree Diagrams

Since representations naturally “compete for survival”, falling in disuse when a better one is created, people should not refrain from designing alternative, or competitive, representations for the outcomes of multi-stage random experiments and their probabilities.

Konold has proposed a variation of the tree diagrams, which he has called *pipe diagrams*, with two major distinctions from the former: The branches, or pipes in his metaphor, are tagged with joint rather than conditional probabilities; and the values of these joint probabilities are graphically represented by pipe widths (1996).

Cheng (2003) designed “Probability Space” diagrams (PS diagrams) as an alternative to the current representations in probability theory, which he claims to combine the functions of Venn diagrams, tree diagrams, set theory notation, outcome tables and algebra.

Adding to Transparency

My experience suggests at least three reasons why tree diagrams do not have much semantic transparency:

Multiplicative Representation

One thing is to use tree diagrams to represent hierarchies, genealogies, or taxonomies, and quite another to represent the multiplicative principle. In a hierarchy tree or a genealogical tree, each item is represented in one and only one node. On the other hand, if you use a tree to denote possibilities, that changes. For example, in a family tree every node represents one and only one person: You can have kids make their family tree by pasting a picture of each relative on a branch. Now, take for example the task of making a tree for the different combinations of outfits we can have with two shirts, three skirts, and two pairs of sandals. Could we give students a picture or a template of the two different shirts, the three different skirts and the two different pairs of sandals and ask them to paste them on a tree? Would that amount of templates suffice? No, we would have to create, for each different shirt, templates for the 3 different skirts, etc. We would end up with two “copies” of the 3 different kinds of skirt and 6 copies of each kind of shoes (assuming we built the tree in the order “shirts, skirts, sandals”). This stems from the multiplicative principle, but students seem to have an awful hard time with that. The tree has “too many branches” for them. There inevitably is in every classroom of elementary education majors that I teach a great number of constructed trees that have too few branches – even after direct instruction on the issue.

Interestingly, the argument that tree diagrams “have too many branches” was heard from a student in a study by Figueiredo (2000), although she interpreted the assertion differently then I did above – she took that as evidence that we should avoid situations that would yield trees with a great number of ramifications.

We cannot rule out the possibility of these difficulties being due to an underlying weakness in combinatorial thinking, more than to the characteristics of the representation itself. Navarro-Pelayo, Batanero, & Godino (1996) have showed that combinatorial reasoning cannot be taken for granted, and have highlighted the importance of teaching that focuses on recursive thinking and systematic enumeration.

Where Is Each Outcome Represented?

Another problem I see my students (elementary education majors) with using trees to represent outcomes of multistage experiments is visualizing where each outcome is – in a path, not in a branch – especially not in the final branches, as students tend to think. I have found that asking student to highlight different paths in a tree and to write out the outcome that the path represents, as well as placing a tag next to every final branch on the tree where students are to list the outcome corresponding to following the path from the root of the tree to that final branch, has led to more successful uses of the trees.

Visualizing each Phase of the Experiment as a Level Across the Tree

If you make the tree from left to right, for example, each level or phase should be "read" vertically in columns. Vice-versa, a tree that is written vertically has a horizontal dimension. There is nothing particularly representing this on a tree, and students often have difficulty in adopting this convention. I have found that adding dashed lines to delineate the different levels of ramifications in a tree and labeling each of them, has made the representation more semantically transparent to students.

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COMMUNITY MATHEMATICS EDUCATION AS A FRAMEWORK FOR ELEMENTARY MATHEMATICS METHODS

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Purpose

In preparing elementary teachers to teach mathematics in urban schools, a tension exists between facilitating candidates' development of content and pedagogical content knowledge and helping candidates to be advocates for social justice in their schools and communities. Oakes, Franke, Quartz, & Rogers (2002) highlight this tension in their discussion of "high-quality urban teaching":

> An effective urban teacher cannot be skilled in the classroom but lack skills and commitment to equity, access, and democratic participation. Likewise, if one is to be a teacher, a deep caring and democratic commitment must be accompanied by highly developed subject matter and pedagogical skills. (Oakes et al., 2002, p. 229)

At the same time, those who are preparing to teach in high-poverty schools are often confronted by the lack of material resources- including curriculum materials, manipulatives, and technology- available for teaching mathematics, raising significant questions about equity and about the effectiveness of methods classes based on the use and availability of these resources.

The purpose of this paper is to describe a theoretical framework for an elementary mathematics methods course that seeks to address not only issues of equity, but also the tensions among content knowledge, pedagogical content knowledge, and a commitment to social justice. This framework builds on the work of Oakes et al. (2002) as well as a number of currently disconnected bodies of research highlighting the *human* and *social* resources available for the teaching of mathematics– including teacher candidates' mathematics identities, the mathematical thinking of K-5 students, and the mathematics "funds of knowledge" (Moll, 1992) and problem-solving opportunities available through parents, families, and communities. Furthermore, this framework provides an ecological perspective on mathematics instruction that prompts students, regardless of their teaching context, to explore and understand the mathematics resources available in any school or community and the value of connecting mathematics instruction to those resources.

Theoretical Framework

Teacher candidates often arrive in elementary methods classrooms aware of general feelings of like or dislike of mathematics, but with little idea how to use and build on specific mathematics pedagogies they have experienced or observed. Eliciting candidates' mathematics stories (LoPresto & Drake, 2004/2005) or autobiographies (Guillaume & Kirtman, 2005) can make these experiences more accessible as resources for teaching.

Similarly, teacher candidates are typically not aware of the variety of strategies and mathematical knowledge that students bring to the classroom. Explicit instruction and discussion about this variety and its usefulness as a pedagogical resource (e.g., Empson, 2002), as well as problem-solving interviews with children and the collection of student work around a common problem (e.g., Kazemi & Franke, 2004) illustrates the value of students' mathematical thinking as a resource for instruction.

Finally, teacher candidates are generally more aware of ways to use community and family resources in content areas such as literacy and social studies than in mathematics. Assignments asking students to design mathematics activities that could be completed during trips to various community sites, as well as interviews with parents eliciting the "funds of knowledge" (Moll, 1992) available within any group of parents help address this gap.

Pre-service teachers can better understand and utilize this framework not only by planning lessons that incorporate personal, student, and community resources, but also by observing examples of classroom practice and identifying the variety of ecological resources utilized in those lessons.

Implications and Future Research

To be clear, the four sources of human and social resources identified here are certainly not the only human and social resources available to urban teacher candidates. However, they were chosen because they are resources that are available in every context, whether urban or not, and because there is substantial research supporting the separate roles of each of these resources in elementary mathematics education. The focus on these four sources is not a replacement for discussion of curriculum, manipulatives, and technology as resources, but is instead presented as a complement to the use of these more traditional material resources.

Helping teacher candidates learn to utilize personal experiences, students' thinking, families, and communities as resources for mathematics instruction also highlights the importance, particularly in urban schools, of using mathematics and mathematical resources as *tools* for creating change within classrooms, schools, and communities. Much work remains to be done in the development and implementation of this framework. Nonetheless, the promise of this framework for teacher candidates, as well as the relationship of the framework to the PME-NA goal of more deeply understanding aspects of the teaching and learning of mathematics, is clear.

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BLOGGING PENTACUBES: ENHANCING CRITICAL READING AND WRITING SKILLS THROUGH COLLABORATIVE PROBLEM SOLVING WITH MATHEMATICS-BASED WEBLOGS

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Introduction

A number of educational studies have explored writing activities and their impact upon mathematical understanding of students. Researchers have investigated the use of journaling and "diaries" (Clarke, et al., 1993), submission of manuscripts to peer-reviewed journals (Brown, 1990), and mathematics "penpals" (Phillips & Crespo, 1996) to enhance student reflection on a variety of mathematical tasks. These and other studies suggest that mathematical writing has the potential to improve the communicative abilities and mathematical understandings of students.

Among the innovative approaches to writing in mathematics classroom, some, like diary-type journaling, require personal reflection on the mathematics at hand. Other approaches, such as peer-reviewed journals and math penpals, are based on critical peer interaction. Most of the epistemological work in the past two decades suggests that effective learning is inherently social (Ernest, 1998). Nevertheless, high-stakes testing pressures, saturated teaching schedules, and insufficient teacher training inhibit the realization of critical interactive writing in many classrooms. Despite NCTM recommendations, writing often takes a backseat to multiple choice assessments.

In this paper, we propose that weblogs (i.e. "blogs") are tools that may be used to address such limitations. Because blogs encourage academically-oriented interactions and mentoring among students and teachers *outside the traditional classroom*, use of the tools supports enhanced reading, writing, and mathematics skills without adding to time pressures that teachers typically face. In our study, pre-service teachers explore problems involving pentacubes as they compose initial drafts of problem solutions – then revise the drafts using a modified "writing workshop" model (Ray & Laminack, 2001) within an on-line blogging environment.

Framework for Understanding the Role of Blogging in the Study

In typical mathematics courses, students are assigned numerous homework problems of relatively low quality compared to those assigned to students in higher achieving countries. To help students develop conceptual understanding of mathematics rather than merely teaching them "how to obtain answers," we provide students with opportunities to submit multiple drafts of problem solutions. In peer-revision groups, students read and discuss mathematical work with others, provide encouragement and revision suggestions, and learn multiple solution strategies for a relatively small number of engaging problems. A model of this interaction is depicted in Figure 1.

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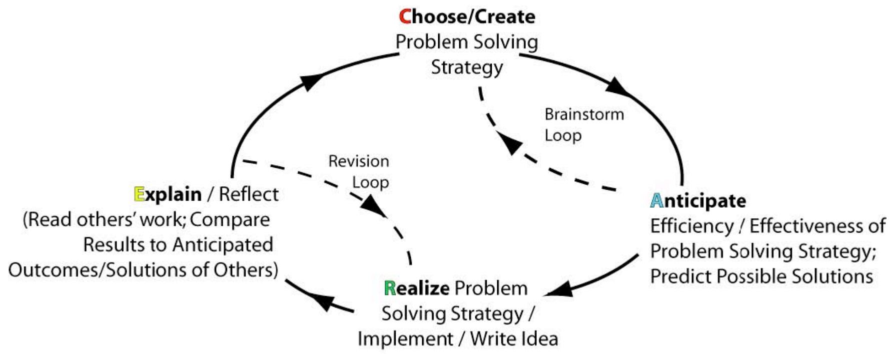

Figure 1: Collaborative Problem Solving Cycle (CARE)

In the cycle, brainstorming and revision mini-loops are implemented in traditional classroom and weblog settings. The model provides preservice teachers with the types of feedback and encouragement that supports quality teaching and learning.

Methodological Concerns

To informally measure the extent to which collaborative, "writing workshop"-style mathematics instruction impacts the reading and writing skills of preservice teachers, we collected and analyzed data from several sources: (1) Mathematics writing samples (draft and final responses to open-ended mathematics problems); (2) Short attitudinal questionnaires dealing with writing in mathematics classes; and (3) Writing samples taken from a class weblog. For all writing, structural and stylistic qualities of the samples were assessed using the "Holistic Rubric for the Ohio Graduation Test: Writing" (ODE,2005). The samples were analyzed both in terms of clarity and grammatical precision. Mathematics-specific work was assessed independently by two math educators to ensure inter-rater reliability.

Preliminary (Anticipated) Findings

Although the data analysis phase of the study is incomplete at the time of this writing, several interesting trends appear in the data. First and foremost, blogs appear to be useful tools for building student writing and reading skills in content areas outside of language arts. Assessment of student writing samples indicate a growth in students' writing during the study period -both stylistically and in terms of mathematical sophistication.

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A LOOK AT GENDER DIFFERENCES IN THIRD GRADERS' MATHEMATICAL PROBLEM SOLVING

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There has been much research done during the past twenty-five years on the differences in mathematics achievement of females and males (Hyde, Fennema & Lamon, 1990; Leder, 1992). Girls' achievement in mathematics, as measured by standardized tests, has generally been found to fall below that of boys by early adolescence, although there is evidence that this difference is becoming minimal (Cole, 1997; Hyde, Fennema & Lamon, 1990).

More recently, there has been a much smaller body of literature which presents compelling evidence that girls and boys are using different strategies to solve mathematical problems and computations in the early elementary grades (Carr & Davis, 2001; Fennema, Carpenter, Jacobs, Franke & Levi, 1998; Ricard, Paredes, Miller & Boerner, 1990; Zhang, Wilson & Manon, 1999). Additionally, Van den Heuvel-Panhuizen (2004) presents evidence that boys and girls perform differently depending on the mathematical nature of the tasks presented.

I will present the results from a study, the purpose of which is to identify and describe gender differences found in third grade students' work on a written mathematics assessment. In particular, comparisons will be made between the work of students in classrooms using *Investigations in Number, Data, and Space*, a reform-minded mathematics curriculum whose development has been funded by the National Science Foundation, and students in classrooms not using *Investigations*. A smaller sample of students will also be given task-based interviews.

The research questions that are the focus of this study are the following: (1) what strategies are third grade children using to solve specific mathematical tasks? (2) are there gender differences in the solutions or the strategies used or in the types of problems solved successfully? and (3) do these differences vary depending on whether or not the children are in *Investigations* classrooms?

This research for this study is being conducted as a piece of a much larger study. Diana Lambdin and Indiana University were awarded a subcontract to evaluate TERC's current revision of the *Investigations in Number, Data and Space* curriculum. The larger study looks students' mathematical growth and understanding of over a three-year period and compares the mathematical achievement of students in classrooms using the *Investigations* curriculum with students in classrooms that are not using *Investigations* ("comparison" classrooms).

The subjects for this study, who are participating in the larger *Investigations* evaluation study, are approximately 400 third grade students from a large, urban school district in the Midwestern United States. A smaller group of approximately 20-40 students will be asked to participate in task-based interviews.

The data is being collected using two methods: reviewing and analyzing the students' work on a written assessment instrument, and reviewing and analyzing children's responses to the interview questions and tasks. The written assessment was given to students in both the fall and the spring of the 2003-2004 school year, as a part of the larger *Investigations* study. The instrument includes six sections focusing on number and operations and two focusing on algebraic reasoning. The interview protocol will be used in the fall of 2005 with the smaller group of students to gather more in-depth data about the tasks and strategies used on the written assessment.

Coding for the data collected on the written assessment has been developed and defined by the I.U.-TERC research team as a part of the larger evaluation study. Students' answers have been coded as correct or incorrect, as well as by the types of strategies used to solve the problems. The results are being analyzed to look for growth in the students' achievement and differences in the work of children in *Investigations* and comparison classrooms. More detailed analyses will look at the strategies that girls and boys use to solve the problems and at the types of problems that girls and boys are able to answer successfully. Interview questions and tasks will be used with the smaller group of students to further investigate the similarities and differences found.

Preliminary results indicate that the gains made from fall to spring by *Investigations* classes were greater than those of the comparison classes. There also appear to be an indication of some differences between the percentages of boys and girls doing better on individual problems, depending on whether or not the children were in *Investigations* classrooms. Further analyses will be conducted throughout the spring and fall of 2005, and the results will be reported.

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TEACHER EDUCATORS' ACTIVITIES FOR TEACHING BASIC MATHEMATICAL CONCEPTS

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This study had two purposes: 1) to investigate pre-service elementary school teachers representational fluency and use of different kinds of manipulatives to solve problems in fraction equivalence, addition and subtraction and 2) to develop instructional activities that surround the Animated Fraction Addition and Subtraction Tool (A-FAST), that teacher educators can use to enable teachers have different kinds of representational and modeling experiences, so that they can use more modeling activities to introduce basic mathematical concepts. Seventy preservice elementary school teachers from three classes participated in this study. A pre-test was conducted which guided the researchers to design the instructional activities to be used with the preservice teachers. The methodology of "Multi-tiered Teaching Experiments" (Lesh & Kelly, 2000) with two levels (researchers and teacher educators, and preservice teachers) was used to conduct the study. At the end of the study the teachers' conceptual and pedagogical understanding was assessed using a post-test and lesson plans which they prepared in groups of three. Results from the study are presented below.

Participants

The participants for this study were seventy preservice elementary school teachers enrolled in a mathematics methods course at Indiana University School of education. These teachers have already completed their mathematics content courses offered by the mathematics department as a pre-requisite for this methods course. The assumption for this mathematics methods course is that after completing their mathematics content courses, the teachers have sufficient content knowledge to start thinking about how children learn mathematics and how to teach them for better understanding. The teachers have either done T104 (Teaching Elementary Mathematics via Problem Solving) or T101 (Mathematics for Elementary Teachers- I).

Method

At the beginning of the study the preservice teachers' were assessed using a paper-pencil pre-test and interviews regarding their knowledge of different representations for fractions, use of manipulative materials, and their reactions to common misconceptions of elementary school students. The results from these assessments led to the development of instructional activities around the software which were later used with the preservice teachers in two class periods of one hour and fifteen minutes each. After work with A-FAST and after the implementation of the instructional activities, the preservice teachers were asked to prepare lesson plans on "how to teach fractions" and write individual reflections on the thinking process they went through while working with the software and instructional activities, and while writing the lesson plans. A post-test was implemented with the same purpose as the pre-test.

Lloyd, G. M., Wilson, M., Wilkins, J. L. M., & Behm, S. L. (Eds.). (2005). *Proceedings of the 27th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.*

Analysis I

Pre-tests and interview questionnaires were used to collect the data for a preliminary analysis. Data analysis based on the preservice teachers' answers is presented: (1) Preservice teachers have a poor understanding of fractions. Concepts such as whole, part, whole-part, and part-part are not clear for them; (2) Preservice teachers aren't familiar with or able to use other manipulatives (square, Cuisenaire rods, or counters) except pies- and not in its totality; (3) Preservice teachers understand the importance of manipulatives in the classroom; (4) Preservice teachers don't feel prepared to teach fractions at schools-pedagogical ideas are their more important concerns; (5) Preservice teachers have a brief idea of how to add fractions where one of the denominators is a multiple of the other (e.g. 4 and 12) using manipulatives (pies most of the time) but they are not able to show this when the denominators aren't multiples (e.g. 4 and 7); (6) Preservice teachers ignore the concept of "common unit" while thinking about equivalent fractions and comparison of fractions.

Analysis II

The lesson plans, reflections, and post-tests were analyzed. The analysis is presented next:
(1) Preservice teachers support the view that models and manipulatives help in learning. Most of the teachers wished that if they had been exposed to fractions using models during their own elementary education or even during their more advance mathematics classes, they would have had a better understanding of fractions today; (2) Preservice teachers claim that their own knowledge of fractions has increased after they used the software and the instructional activities, and with this knowledge they feel better equipped to answer students' misconceptions about fractions; (3) Preservice teachers claim that their pedagogical knowledge of fractions has increased. The instructional activities put the preservice teachers in some real classroom situation that helped them to think about teaching fractions, and that helped them to come up with good teaching strategies; (4) Preservice teachers found the linear model most difficult to understand. The area models were the easiest for them to understand followed by the set model. This difficulty can be attributed to what research already showed- it is difficult for students to work with discrete objects (set model) than the region models like circles and squares (Post, 1981). Teachers' difficulties with the linear model can be explained by the design structure of the software and the lack of real manipulatives for simulating the actions of the software.

Results

Assertion 1: The instructional activities designed around the software influenced preservice teachers' beliefs about the use of different kinds of models and manipulatives for teaching fractions.

Assertion 2: The instructional activities designed around the software did have some positive influence on preservice teachers' representational fluency and fraction knowledge but not to the desired extent. *Assertion 3:* Although the preservice teachers claimed that will use different kinds of models and manipulatives, their own knowledge of using appropriate models for fractions and their representational fluency is still limited.

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EXAMINING PRESERVICE TEACHERS' KNOWLEDGE OF ALGEBRA

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Many researchers agree that subject matter knowledge is an important factor affecting classroom instruction. Ball (1990) analyzed preservice teachers' understandings of mathematics and concluded that subject matter knowledge should be a central focus of teacher education in order to teach mathematics effectively. However, understanding of subject matter is not sufficient condition to teach. Teachers should possess a representational repertoire, because teachers need to generate representations to facilitate students' learning (Wilson, Shulman & Richert, 1987). Their conceptions of knowledge may limit their ability to present subject matter in appropriate ways or give helpful explanations (Even & Tirosh, 1995).

In this study, false algebraic statements, taken from the study of Marquis (1988), were used. Our goal was to investigate whether or not preservice teachers would be able to determine why the statements were false. In addition, the researchers wanted to analyze preservice teachers' alternative solutions for common algebra mistakes.

Theoretical Framework

The data in this study was analyzed using the framework described in the study of Ball (1990). She stated that understanding mathematics for teaching requires both knowledge of mathematics (i.e., understanding of principles and meaning of underlying mathematical procedures) and knowledge about mathematics (i.e., understanding of the nature of knowledge in the discipline: where it comes from, how it changes, and how truth is established; the relative centrality of different ideas as well as what it is conventional or socially agreed upon in mathematics versus what is necessary or logical, p.6).

Methods

In this study, three audiotaped interviews were conducted with three preservice teachers. Harry was a sophomore and a middle school preservice teacher. Both Ashley and Jane were seniors and secondary preservice teachers. Each participant was given three algebraic statements with common mistakes and asked how they would respond to students who were making the mistakes.

Task 1: $\sqrt{x^2 + y^2} = x+y$. Task 2: If $2(2-z) < 12$ then $z < -4$ Task 3: $\dfrac{xa + xb}{x + xd} = \dfrac{a + b}{d}$

Data Sources

When they were given Task 1, Harry easily showed that the statement was false. He first took square of both sides. Then, he said $(x + y)(x + y)$ was equal to $x^2 + 2xy + y^2$. However, he failed to explain why there should be 2xy. Ashley did not know how to solve the equation at all. Jane said students could not multiply $(x + y)$ and $(x + y)$ and explained why students made this mistake. Besides this approach she also suggested that students did not know the difference between the expressions $\sqrt{x^2}\ \sqrt{y^2}$, $\sqrt{x^2} + \sqrt{y^2}$, and $\sqrt{x^2 + y^2}$ and demonstrated the difference among them.

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The participants solved the inequality for z correctly. However, they could not explain why they needed to reverse the inequality symbol, when both sides were divided by -2. Jane said, "Once, they divide by negative 2, you have to flip the inequality. It is probably because the reason I almost forgot. I myself do not have good understanding of why you have to flip it… You are hanging on the fact that you have to remember."

When Harry and Ashley were given Task 3, they said that the expression in parenthesis should be 1 + d, when x was factored out. However, they could not explain why it should be 1 + d. They suggested that writing one next to x might help students factor out the algebraic expression correctly. The following is an excerpt from Harry's interview.

Harry: You just have to put something in there. In this case is one… you can put the one in front of the x. See actually there is one in there.

After solving Task 3 correctly, Jane demonstrated an example that would help students understand why they need to place 1 when x is factored out. She said, "When students factor out x, they forgot they are supposed to leave 1 in there. If students did this, I would show them. When you factor out something, you should be able to reverse and get to the same thing. So if we take x+xd and x (d) and you reverse it. This is not the same thing (she wrote x + xd = x (1 + d) and x(d) = xd)."

Results

All the participants solved Task 2 correctly. However, they could not explain why the rule worked. For Task 3, Jane was the one participant who could provide an alternative approach by reversing the algebraic operation. She multiplied (1 + d) and (d) by x in order to show that they were not the same. She could not make any connection with other mathematical ideas (or properties). Ashley could not solve Task 1 at all. Harry solved it. However, he could not explain why the mistake occurred. After solving Task 1, Jane demonstrated three possible factors behind the mistake and showed the connections among them.

Conclusion

The results of this study revealed that the participants had difficulty solving the tasks as well as providing alternative solutions for common algebraic mistakes, because they had limited knowledge of mathematics. Their explanations were mainly based on algebraic procedures and arbitrary facts. They did not know the underlying meaning of algebraic procedures they used. They need to know more than describing steps and procedures. They should be able to give meaningful explanations and develop appropriate strategies for common algebraic mistakes.

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HOW ONE-TO-ONE TUTORING EFFECTS PROSPECTIVE MATHEMATICS TEACHERS' REFLECTIONS ON LEARNING MATHEMATICS

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While teachers do not necessarily have the time to work for extended periods one-on-one with students, pre-service programs can include these opportunities as part of the field-work. It has been shown that changes in teacher beliefs about learning may occur when given the opportunity to engage in one-on-one experiences (Pinnell, Lyons, Deford, Bryk, & Seltzer, 1994). Gipe & Richards (1992) found that reflective practices in pre-service education students could lead to improved performance in the field. With one-on-one teaching or learning situations considered being an effective method of instruction (e.g., Cohen, Kulik, & Kulik, 1982) tutoring of middle or high school students and their tutors (prospective mathematics teachers) could both benefit from the experience. Not only is the middle or high school student afforded the opportunity to get assistance with the mathematics s/he may be having trouble with but also the prospective teacher is provided an opportunity to develop the questioning and teaching skills needed in a classroom. Hedrick, McGee, & Mittag (1999) found that through tutoring sessions pre-service teachers experienced growth in understanding about the instructional cycle that included a consideration of cognitive, emotional and environmental factors.

The purpose of this study was to examine the qualitative nature of prospective mathematics teachers' analysis of and reflection on middle or high school students' mathematical thinking in a tutoring context. This research helps in the understanding of the process of how one comes to be a mathematics teacher and the effects of a specific "field-experience", namely, one-on-one tutoring, on the maturity of prospective mathematics teachers pedagogical and pedagogical content knowledge.

Methods

The mathematics education program at a large research institution now includes a year of tutoring as part of the field component for both middle grades and secondary mathematics education majors. This occurs in the junior year of preparation. Since fall 2001 a tutoring center for middle and high school students has been provided on the university campus as a service to the community. After each tutoring session prospective teachers are required to complete a reflections log to document what assistance was provided, what is recommended for subsequent visits, and their reflections on the mathematical thinking of the middle/high school tutee. Reflection logs from thirty-five tutors were collected. Interviews (either email or face-to-face) with ten prospective teachers constituted a second data source. Questions asked in the interview included "describe your typical tutoring session", "describe your interpretations of student difficulties", and "describe how you approach planning for teaching" (for those who were doing a field experience). The reflection logs and follow-up interviews constitute the data analyzed.

Data for two years (2002-03 and 03-04) were analyzed for evidence of the types of tutors' instructional strategies during the sessions; ways tutors overcame students' difficulties in understanding mathematics; and what was learned from the experience. Data were coded and

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categories for each area were developed. Using analytic induction assertions were made and data re-examined to determine if there was any evidence that would refute the assertion (Erickson, 1986).

Results

One-on-one tutoring experiences provided prospective mathematics teachers with an opportunity to work more closely with students who were not successful with mathematics and to develop a set of skills that could be used with this group of students.

As reported to one of the researchers during class, most of the prospective teachers had been in advanced or gifted mathematics classes in high school. Therefore, their perspective on teaching mathematics had been shaped by these classes. For the most part in their classes students were motivated to be successful regardless of the teaching strategies used by the teacher. Thus, as a prospective teacher each was challenged by interactions with middle or high school students who were not necessarily successful in mathematics or who were struggling. What might seem like a straightforward procedure or concept for the prospective teacher was a complicated and/or meaningless process or idea for the student. This was compounded by the less than enthusiastic attitudes held by the students for learning mathematics.

In reflecting on their tutoring sessions prospective teachers indicated areas they felt they struggled in, such as, finding appropriate problems, finding additional ways to explain a procedure (different from the book because it might not be helpful), and finding fruitful ways to ask questions rather than telling a student how to do something. These became critical teaching moments in the methods courses so that multiple strategies for dealing with these situations were explored.

Conclusions

The inclusion of year-long tutoring experience appears to have provided an environment for the prospective mathematics teachers through which they are able to develop skills for working with traditionally unsuccessful students in mathematics. Given that most of them have been successful and have a minimal awareness of difficulties someone might have in learning mathematics, the one-on-one opportunity provides real-life examples to situate what is being learned in the methods courses.

Further examination of the reflections shows how over the course of a year the nature of the reflections changes. Additional analyses revealed through the reflections the progression of the prospective teacher on the development of a more conceptual approach to teaching mathematics.

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DEVELOPING THE CRITICAL LENSES NECESSARY TO BECOME A LESSON STUDY COMMUNITY

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Conceptual Framework

Becoming a successful Lesson Study community requires developing three critical lenses: the researcher lens which requires teachers to design classroom experiences that explore questions they have about their practice; the curriculum developer lens which requires concern about how to organize, sequence and connect children's learning experiences; and, the student lens which requires the teacher to examine all aspects of the lesson through the eyes of the student. This is not a simple task. There are significant differences in the curricula and cultures of Japanese and U.S. teachers. This study raised the following question. Would U.S. teachers new to the Lesson Study process develop the critical lenses necessary to become a highly functioning Lesson Study group?

The School System

The CSD school system is a small urban system in the south with six elementary schools. Thirty-eight percent of the CSD students are on free or reduced lunch, 53% are African-American or other minority, and 47% are Caucasian. According to the mathematics coordinator, a teacher-directed model continues to be the primary mode of instruction in mathematics.

Participants

In an effort to find a self-sustaining model for change the system implemented a Lesson Study project during the 03-04 school year. Third grade teachers in the district were selected to participate. With only two teachers at that grade who were new to the system but not new to teaching, and a strong group of experienced teachers, third grade was perceived as a relatively stable grade level. Participation was voluntary and eight of the ten teachers opted to join the project, representing five of the six elementary schools. There were two African-American females, one Asian male, and five Caucasian females. Teachers ranged from 3 to 15 years experience in the elementary classroom.

Methods

In this study I looked for evidence that teachers were developing the three critical lenses necessary to become a functioning Lesson Study community. Videotapes were made of each planning and debriefing session. For this research, tapes from the initial and ending planning/debriefing sessions were selected for transcription. In consultation with the mathematics coordinator for the school system, a rubric was developed prior to coding. The rubric was used to describe possible language patterns or dialogue that we anticipated would be indicators of evidence of each of the three critical lenses. Each transcript was coded using R for the researcher lens, S for the student lens and CD for the curriculum developer lens, e.g., if teachers talked about what students said or did, we would code the event with an S. If teachers talked about how the mathematics in the lesson was presented or how it developed, we would code the event CD, or if the teachers talked about questions they had about how to best teach a particular topic we would code the event with an R. The clips were then sorted to determine if the sum of the clips in each category actually reflected the essence of the category. Finally, clips within each category were compared for qualitative differences in the conversation.

Results

Because of space limitations, I will give only two examples from the debriefing sessions; one at the beginning of the year and one at the end of the year. The clips are typical of teacher conversation with no probing or prompting by the facilitators.

Clip #1 ***[September- Third grade lesson on addition and subtraction word problems]***

T1: On the first one (problem) they didn't use the manipulatives at all . . .one strategy was used until you came over and asked "can you think of another way of doing this?" Then they suggested. . .let's draw a picture to answer the question and that's what they did. [S]

T2: They didn't really draw much of a picture [S]

T3: They used little models, little things to represent their thinking [S]

T1: and, each time they spoke they were very respectful of each other. They didn't say no, that's wrong. They were very calm in the group. [S]

Clip #2 ***[April – Lesson on identifying unshaded regions as fractional parts of a square]***

T1: I have down the word denominator in big letters because I really think that the concept of denominator is just hard. [CD] It seemed like my group could divide anything into one-eighth or one-sixth. They can do the ones but when you show them a larger region . . I think that was the big challenge. When they see a large region in their minds they see one-fourth, one-fourth, one-fourth instead of three-fourths. [S]

T2: I wonder about shading, if there was something about how she shaded. [CD]

T3: That makes it more challenging, not being shaded because they are use to that. [S]

T1: I guess it was two things (that made it hard), not shading it and giving them a larger region, so maybe one or the other would have been good to do. [CD]

Discussion

In the September clip, less than 10% of the comments were made from the curriculum developer's lens. Comments about the students [S] were primarily about what they did or said and about their behavior. In the second clip, the teachers are still talking about the students, but they are trying to take the perspective of the student (what was hard for them and what they were familiar with) and they were thinking about the organization of the learning [CD] indicating first that the concept of denominator is hard (although they don't indicate why) and that the instruction decision to give them a large region and one that was not shaded made the problem more difficult. Interesting, neither transcript provided evidence of the researcher lens.

There is no doubt that differences in the curricula and cultures of Japanese and U.S. teachers impacts successful adaptation of Lesson Study with U.S. teachers. Given the possible benefit for teacher learning through Lesson Study by gaining deeper insights into seeing the mathematics through the eyes of the student and thinking about the organization and presentation of the curriculum, further research in the area is clearly warranted

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A DISCURSIVE FRAMEWORK FOR EXAMINING THE POSITIONING OF A LEARNER IN A MATHEMATICS TEXTBOOK

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Personal Positioning in Relation to Mathematics

In *TMM*, first person pronouns[1] are entirely absent. Such an absence obscures the presence of human beings in a text. The second person pronoun ***you*** appears 263 times in *TMM*. Two forms are especially relevant: 1) ***you*** **+ a verb** (165 times); and 2) **an inanimate object + an animate verb +** ***you*** **(as direct object**) (37 times). The most pervasive form, ***you*** **+ a verb**, includes such phrases as *you find, you know,* and *you think*. In these statements, the authors tell the readers about themselves, defining and controlling the 'common knowledge' (Edwards & Mercer, 1987), and thus use such control to point out the mathematics they hope (or assume) the students are constructing. In *TMM*, the other common *you*-construction (**an inanimate object + an animate verb +** ***you*** **(as direct object**)) provides a striking example of obscured personal agency*:* inanimate objects perform activities that are typically associated with people – e.g., "The graph shows you…". In reality graphs cannot "show" you anything.

The modality of a text also points to the text's construction of the role of humans in relation to mathematics. The modality of the text includes "indications of the degree of likelihood, probability, weight or authority the speaker attaches to an utterance" (Hodge & Kress, 1993, p. 9). One set of modal forms, hedges*,* describe words that point at uncertainty. The most common hedge in *TMM* is *about* (12 instances), followed by *might* (7 instances) and *may* (5 instances). Modality also appears in the authors' verb choice: *would* (55 times), *can* and *will* (40 times each), *could* (13 times), and *should* (11 times). The frequency of these different modal verbs indicate an amplified voice of certainty because the verbs that express stronger conviction (*would, can*, and *will*) are much more common than those that communicate weaker conviction (*could* and *should*). The strong modal verbs, coupled with the lack of hedging, suggest that mathematical knowledge ought to be expressed with certainty, which could suggest that the knowledge is not contingent upon human relations.

Student Positioning in Relation to Peers and the Teacher

Pictures alongside verbal text can impact the reader's experience of the text. In *TMM*, for example, there are 24 pictorial images. Of these, only 7 are photographs. The textbook's preference for drawings, which are more generic than photographs, mirrors its linguistic obfuscation of particular people. Furthermore, only a quarter of the images show people, and among these we find only one image of a person doing mathematics – a drawing of a hand conducting a mathematical investigation. The disembodied, generic hand parallels the lost face of the mathematician in agency-masking sentences such as the ones discussed above.

Morgan (1996) asserts that imperatives (or commands) tacitly mark the reader as a capable member of the mathematics community. However, we suggest that such positioning is not clear

[1] Tools and concepts from discourse analysis are underlined here.

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from the mere presence of imperatives. Rotman (1988) distinguishes between what he calls inclusive imperatives (e.g. *describe, explain, prove*), which ask the reader to be a thinker, and what he calls exclusive imperatives (e.g. *write, calculate, copy*), which ask the reader to be a scribbler. The thinker imperatives construct a reader whose actions are included in a community of people doing mathematics, whereas the scribbler imperatives construct one whose actions can be excluded from such a community. The student who 'scribbles' can work independent from other people (including her teacher and peers).

Student Positioning in Relation to the World

Most of the prompts in the analyzed textbook are referred to as 'real life', 'applications', and 'connections' (connections between mathematics and real life). Though the textbook consistently places its mathematics in 'real' contexts (with few exceptions), linguistic and other clues point to an inconsequential relationship between the student and her world. When we compare the instances of low modality (expressing low levels of certainty) with those of high modality, we begin to see what experiences the text foregrounds. The text refers with uncertainty to the student's experiences outside the classroom using hedging words like *probably* or *might*. However, the text expresses certainty about the student's abstract mathematical experiences, as in "In your earlier work, you saw that …" (p. 9). Because the authors know what the curriculum offers, they work under the assumption that the student has learned particular mathematical ideas. Yet, the authors cannot really know what their readers have seen. Students might be led to think that their everyday experiences matter less than their mathematical experiences?

Revisioning Mathematics Text

We were surprised by the results of our analysis of this textbook that we both appreciate for its constructivist approach to mathematics. The language forms and images suggest a different view of mathematics, one in which the student works independently from a pre-existent mathematics. How then does such a text become a tool for constructivist-informed education?

We see room for mathematics textbook writers to change the *form* of their writing to recognize the connections between readers and their world, which includes the people around them. Until such textbooks appear, we note that any textbook is mediated through a person (the teacher) in a conversation amongst many persons (students). In such a community, there is room to draw awareness to relationships between particular persons (historical or modern, professional or novice mathematicians) and the apparently abstract, static discipline of mathematics.

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LINGUISTIC INVENTION IN MATHEMATICAL COMMUNICATION AMONG PRACTICING ELEMENTARY SCHOOL TEACHERS

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Introduction

For many students, learning mathematics involves learning to communicate using conventional mathematics terminology (Moschkovich, 2003). However, as discussed by Brown (2001), students of mathematics should not only learn conventional methods of communication, but should also be able to verbally describe mathematical situations in relation to themselves. The practice of describing a mathematical situation in relation to oneself may be viewed as linguistic invention (Brown, 2001, p. 76). While conventional language identifies abstract mathematical concepts, linguistic invention draws on personal experience to provide those concepts with richer meaning.

For example, a student solving a story problem concerning a generic reservoir of water changes the context to that of a bathtub because she is more familiar with filling up bathtubs than filling up reservoirs. Meanwhile, another teacher communicates the conventionally stated idea of “the rate is increasing” by specifically saying that “the reservoir is filling up faster.” In both cases, students of mathematics use linguistic invention, language that relates concepts to their personal experience, to describe unfamiliar or abstract mathematical concepts. The purpose of this study is to describe how practicing elementary school teachers optimally utilize linguistic invention to communicate mathematical ideas when working together to complete a mathematical task.

Method and Analysis

Twenty-four practicing elementary school teachers from one school district participated in a calculus course for elementary teachers as part of a larger research project in professional development. Participants worked together to determine how the volume of water in a reservoir changes when supplied with only a graph showing the rate of water entering the reservoir versus time (Connally et al., 1998, p 53).

Researcher field notes, participants’ class notes and submitted solutions to the problem supported transcription and coding of approximately four hours of videotape showing classroom discourse related to the “Reservoir Task.” Discourse turns taken by the participants were coded according to the types of language used in the turn and the effect of the turn on the conversation as observed in the reactions of the participants. Language codes ranged from conventional language (“the rate is increasing”), to uses of units and numerical values (“from one gallon a minute to two gallons a minute . . .”), to references to generic reservoirs (“the water’s coming in faster”), to very personal references (“this is where you’re actually turning the knob [on a bathtub faucet]”). Effect codes described whether each turn was (a) immediately verified, (b) skeptically questioned and/or clarified, (c) acknowledged in non-mathematical manners, (d) acknowledged with frustration, or (e) not acknowledged at all by the other participants.

Student work and transcripts were coded to identify personal linguistic invention that comprised five specific interpretations of the task. Interpretations that received the greatest

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amount of verification from the participants were viewed as optimal uses of linguistic invention. The developmental stages and public presentation of the various interpretations were studied to determine differences in language and effect code patterns between optimal and other instances of linguistic invention. Differences in the development and presentation of optimal and other instances of personal linguistic invention were analyzed to develop a theory of how linguistic invention was used in conjunction with other language forms to successfully create and communicate meaning for mathematical concepts.

Findings and Implications

Data analysis showed that optimal uses of linguistic invention involved five major elements. First, the linguistic invention was developed verbally by more than one participant. Second, overarching principles and assumptions which guided the interpretation of the graph given in the Reservoir Task were verbally identified and reiterated often. Third, during the development of linguistic invention, the personal situation was modified to fit the mathematical concepts represented on the graph in the Reservoir Task. Fourth, the linguistic invention was presented in conjunction with conventional language, which explicitly identified individual mathematical concepts common to both the given Reservoir Task and the personal situation. Finally, the participants seemed make the most progress with a simplest-case scenario involving a bathtub that involved the essential mathematical concepts present in the Reservoir graph yet eliminated the complicated details inherent in a reservoir situation.

In order to connect personal experience to mathematical situations in meaningful ways, students must identify key mathematical concepts common to both situations. A possible method for ensuring identification of key concepts is through the use of conventional language. As students describe what the given mathematics (in this study, a graph of rate versus time) and their personal experience (for our participants, filling up a bathtub) have in common, they further increase their understanding of the mathematical concepts involved (such as increasing, constant, and decreasing rate). If students are not familiar with conventional terms, their attempts to describe these mathematical concepts using personal language may provide a context for meaningful introduction of more conventional language.

Furthermore, teachers can use this information to monitor their students' linguistic invention. For example, if one is interpreting a graph of rate, linguistic invention that makes reference to volume rather than rate may not be as effective as a situation where rate is more prominent. Teachers can guide their students in identifying and differentiating between essential concepts and entertaining details in order to use linguistic invention more effectively in the classroom.

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THE DEVELOPMENT OF COMBINATORIAL THINKING IN UNDERGRADUATE STUDENTS

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Background and Objectives

This study examines the development of combinatorial reasoning and understanding among undergraduate liberal arts and social sciences students. My objective is to analyze students' difficulties in combinatorics and get a better insight into the development of understanding of elementary counting problems. Combinatorics is the study of ways to list and arrange elements of discrete sets according to specified rules (Cameron, 1994). Combinatorics is foundational in computer sciences, and it is used in many other fields of science such as chemistry and physics. In NCTM 2000 Principles and Standards, we see an explicit attention paid to the teaching and learning of combinatorics. Combinatorics is a growing field of mathematics and further attention has been paid to it in school curriculum in recent years. Unfortunately, mathematics education research has not yet caught up to this trend, and not much research has been done in this field.

On of the tasks in this study is students' generation of examples for particular combinatorial structures. These examples help us get a better understanding of students' difficulties with particular of the structures in elementary enumeration problems. Teachers use examples often in the mathematics classrooms to help students understand and explore different topics. When learners are invited by the teacher to construct their own examples, it helps them to think about the topic in a different way. According to Watson and Mason (2004) learner-generated examples promote reflection on the concept, encourage creative thought, and help learner to reason and communicate their understanding in more depth.

Theoretical Framework

Classification of problems and students' difficulties in each class of problems has been studied in detail for elementary arithmetic, and has been proven to be very useful for teaching and learning (Fennema et al., 1992). Batanero et al. (1997) and Rosen (2000) have presented a categorization of basic enumeration problems. I have modified their classification to design a new classification, which is more suitable for the purpose of this research:

1. Arrangement: Order of the elements within the configuration matters.
   - Unlimited repetition allowed: 'How many 4 digit passwords can you make?'
   - No repetition allowed (permutation): 'How many ways can 5 people sit in a row?'
   - Limited repetition allowed: 'How many 3-letter words can one make with the letters FINITELY?' (Note that there is no repetition except for the letter 'I', of which we have two.)
2. Selection: Selection of elements from a set such that the order of the elements within the configuration (selection) does not matter.
   - Unlimited repetition allowed: 'How many ways can you choose 3 roses if there are red and white roses available?'
   - No repetition allowed (combination): 'In how many ways can Kim choose 3 of her friends to invite for dinner if she has 10 friends?'

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- Limited repetition allowed: 'How many different fruit baskets can one make with at least one fruit in it from 5 oranges, 3 peaches, 10 bananas?'

In this research, I have designed a variety of tasks, some of which requires the generation of examples, from different types of combinatorial structures to examine students' difficulties with different types of combinatorial configurations.

Modes of Enquiry and Data Source

The participants in this study are liberal arts students enrolled in an undergraduate mathematics course. The data is gathered from their written responses and a set of clinical interviews. Participants were presented with a variety of tasks in which they were asked (1) to solve given combinatorial problems (2) to generate an example of a combinatorial problem given the presented solution. The following is the example of (2).

Write three scenarios that can be modeled and solved using each of the following calculation:

a. 30*29*28*27*26

b. 30*29*28*27*26/5!

c. 30*29*28*27*26/4!

The goal of such questions is to invite students to think about two categories of enumeration problems (permutation in part a and combination in part b) and to explore their ability to generalize their examples to include a mixed configuration (as in part c). From the preliminary data, it appeared that students had most difficulty with tasks similar to the questions of part c. The data revealed students' difficulties strongly depended on the combinatorial structure of the task, particularly with the mixed configurations.

Conclusion

This study investigates students' development of combinatorial reasoning, and examines their difficulties in solving different types of combinatorial problems. I will suggest possible ways to help learners acquire understanding, by describing the source of their difficulties in this topic. Furthermore, I will categorize different enumeration problems. The systematic classification of different enumeration problems will help teachers to develop "a taxonomy of problem types" (Fennema et al., 1992). The distinction between the different types of problems will also reveal students' difficulties with particular problem types and assists teachers to help students overcome the obstacles particular to a certain problem type.

Relationship of the Paper with the Goals of PME-NA

Combinatorics is being taught in high schools and universities in North America, yet there are only a few studies dedicated to this growing field of mathematics. This study allows us to get a better understanding of students' difficulties with elementary enumeration problems, which are the basis of combinatorics. This research will also have pedagogical implication for teachers, in providing them with a deeper insight into challenges that their students face in this field and guiding how to design their teaching accordingly.

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WHAT ARE THEY LEARNING?: GRADE 3 RESULTS OF A FOCUSED LONGITUDINAL COMPARATIVE MATHEMATICS CURRICULA STUDY

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In 1990, the mathematics research and development group known as TERC (of Cambridge, Massachusetts) was funded by the National Science Foundation (NSF) to develop a complete K-5 mathematics curriculum. Developing and publishing the complete curriculum known as *Investigations in Number, Data, and Space* took eight years, with the first units appearing in 1994 and the final units appearing in 1998. *Investigations* was one of the first curricula to be funded by NSF in that agency's attempt to develop K-12 options that embodied the *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989). The curriculum is now in use by well over 1,500,000 students in several hundred school districts in over 40 states.

In response to the new NCTM (2000) publication, *Principles and Standards for School Mathematics,* the *Investigations* authors applied for and received funding from NSF for a 5-year-long revision process, which began in 2001. At the same time that the *Investigations* revision was funded, NSF also funded a subcontract (to Indiana University) to design and carry out a focused longitudinal comparison study of the impact of the revised *Investigations* materials on student achievement.

This study involves approximately 2,000 students in three different geographical areas and drawn from a wide range of schools. The students are in two cohorts, a grade-one cohort and a grade-three cohort, each of which will be followed, in parallel, for three years. The grade three cohort is composed of a study group of schools using *Investigations* and a comparison group of schools not using *Investigations*. The comparison schools were selected to match the study schools as closely as possible on criteria of socio-economic status, racial/ethnic composition, and previous achievement on reading and mathematics tests.

The grade-three cohort will be studied to continue tracking the development of mathematical understanding, examined in the grade-one cohort, through grades three, four, and five, but of equal importance is the comparative dimension which allows us to evaluate the relative achievements of students using the *Investigations* curriculum and those who are not.

In order to address TERC's interest in selected aspects of the revised curriculum, to keep the study feasible, and to minimize the disruption of normal school routines by additional assessments, we have focused our study on two content areas: number sense & operation, and algebraic thinking. Within each of these areas we employ authentic tasks when possible and emphasize problem solving and open-ended responses over rote exercises; there are no multiple choice items.

We use three different kinds of assessment instruments to collect our primary data. First, the bulk of the data is obtained from instruments that we design for each grade level in the study. These instruments are administered in the fall of the first year of the study, and in the springs of

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all three years of the study to both cohorts. Second, the third-grade cohort completed the Iowa Test of Basic Skills Survey instrument to help us account for initial differences in achievement between the *Investigations* group and the comparison group so that we can better determine what, if any, differences in achievement are due to the *Investigations* curriculum. Third, we collect relevant state- or district-mandated standardized test results for all students in the grade-three cohort. This provides a second basis for comparative analysis of student achievement between the *Investigations* and comparison groups.

In addition to the instruments mentioned above, we collect curriculum-implementation data through surveys and teacher-completed curriculum logs. These data help us account for the inevitable variations in mathematics instruction that take place over the three years of the study.

This study builds on the work collected in Senk and Thomson's *Standards-based School Mathematics Curricula* (2003), and in particular on Kilpatrick's (2003) chapter in that volume, as well as on previous studies of *Investigations* represented by the ARC Center study (2001) and the several studies reviewed by Mokros (2003); and it is guided by other current curriculum evaluation projects, such as the University of Wisconsin study of the *Mathematics in Context* curriculum led by Romberg and Shafer.

In this short oral session we discuss the comparative results found in the third-grade data from the first year of the study. A hierarchical linear modeling analysis reveals that in general, *Investigations* students demonstrated normalized gain scores that were as strong as or stronger than those of students in matched comparison groups.

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WAYS OF ASSESSING A PROFESSIONAL DEVELOPMENT PROGRAM

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The purpose of this paper is to assess a professional development program's affect on the participants' growth as teachers. The professional development programs in this paper were supported by three externally funded projects for grades K-8. These programs were designed based on the teacher needs-based (TNB) model (Lee, 2005) with the purpose of improving teachers' content knowledge and pedagogical content knowledge.

The TNB model was designed to fulfil the participants' needs, and the needs were examined before, during, and after each meeting throughout the project period. This model used a workshop form that took place during regular school days as well as some Saturdays over the course of a whole year. The advantage of having workshops during school days was that teachers were able to make connections with classroom teaching and their own needs and goals (Ball, 1996; Darling-Hammond, 1997; Desimone et al., 2002; Garet et al., 2001; Stile et al., 1996). Having them for a whole year provided the opportunity to hold in-depth discussions, obtain closer familiarity with new strategies; it also allowed teachers to try out new practices in their own classroom (Desimone et al., 2002; Garet et al., 2001; Speck, 2002; Shields, et al., 1998). Some participants of the professional development were members of a collective group from their school. It is reported that this cohesiveness helps teachers keep their enthusiasm about new knowledge and novel applications, as well as to have these take hold and endure (Belcastro et al., 1992; Garet, 2001; Langberg, 1989).

The effects of a professional development program can be assessed in the areas of teacher's beliefs, knowledge, practice/implementation, influence on other educators, and sustenance of learning. Teacher participants in an efficient professional development program reported improvement in understanding curriculum, teaching practice, approaches to assessment, use of instructional technology, strategies for teaching diverse student populations, and the depth of knowledge of mathematics (Desimone et al., 2002; Garet et al., 2001). In evaluating the effectiveness of the professional development programs, the current study took into consideration these areas. The programs were evaluated by teacher participants, program instructors, curriculum supervisors, and the project evaluator. The following tools were used to evaluate the effectiveness of the program:

- *Pre- and post-study questionnaires* to evaluate the overall project impact on the participants' practice.
- *Follow-up interviews* to evaluate sustenance of the project impact on participants' teaching.
- *Concept maps* to compare participants' entering and exit knowledge, beliefs, and attitudes toward teaching mathematics.
- *Reflective journals and one minute papers* to prepare the following sessions and to fulfill participants' needs.
- *Observing* participants' teaching, to evaluate the project contribution to participants' teaching.
- *Visiting* family math nights provided by participants, to evaluate the project's indirect contribution to other teachers, students, and parents.

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- *Reflecting on practice* by participants, to monitor and evaluate their own teaching. Participants recorded their own teaching in audio and/or videotapes for self-monitoring and self-evaluation of their own lessons. Participants exchanged audiotapes and/or videotapes (recorded their lessons) and provided comments on the teaching performance and classroom management of others.

Evaluation by project participants, instructors, and the curriculum supervisor was an ongoing process. Each session was designed to supplement participants' needs and areas of weakness, which had been identified by the instructors and/or expressed by the participants. Instructors visited all the participants' classrooms. These visits provided instructors with information about the teaching situation/status of each participant, which helped the instructors understand each teacher's immediate needs. These communications with teachers and site visits helped in assessing the project's direct and indirect contribution.

The participants reported that they became less afraid of adapting invented pedagogical strategies and introducing innovative ways of solving mathematics problems, and alternative ways to assess their students' learning, which are recommended by the NCTM Standards (2000). The teachers also started to investigate and modify resources based on the students' level and the objectives of the lesson. It was reported that the teachers asked themselves 'why' and 'how' questions more often. By the end of the program, the teachers' concerns were not limited to personal matters or short-term solutions, but widened and deepened to find long-term solutions. The teachers reported that they now allow more time for students to think about a problem and to discuss why their solutions work or do not work. In addition, it was also observed that participants took time to determine if their teaching goals and approaches met the standards recommended by NCTM and the State, which was also one of the project goals.

This study suggests that professional educators include the participants as decision makers and consumers; that they recruit teachers from the same context, connect professional learning and professional practice, and build a partnership between university, public schools, and local education agents. Overall, for the best outcomes, a PDP should have an appropriate level of challenge and support, provide activities demonstrating new ways to teach and learn, build internal capacity, use a team approach, provide time for reflection, and evaluate the effectiveness and impact of the activities.

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WHAT TWO MATHEMATICS EDUCATORS LEARNED ABOUT THEIR INSTRUCTION AND PRE-SERVICE TEACHERS' MATHEMATICAL UNDERSTANDING

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The results of our self-study show preservice teachers' content knowledge, self-efficacy, and pedagogy can be improved. Each of the research questions and the results of the measures needed to answer each question are enumerated below:

1. How do preservice teachers' scores on a mathematics assessment compare before and after intervention in a mathematics methods course?

Two different versions of the same type of mathematics content test were given as a pre-post measure to preservice teachers during the fall. Fall data show preservice teachers (n = 24) scored 61.2% (SD = 16.42) on the pretest and 80.04% (SD = 12.07) on the posttest. These scores were analyzed using a paired t-test. Results show a significant improvement in preservice teachers' content knowledge ($p < .0001$, two-tailed). These results validated the use of instruments as a pre-post measure.

The same mathematics assessments were given to preservice teachers in Carol's methods courses during the spring semester. Spring scores were consistent with scores obtained in the fall. The pre-post assessment was analyzed using a paired t-test. Results show a gain of 13.5 points in overall score. Pretest scores improved from M = 62.5 (SD =17.9) to M = 75.9 (SD =15.6) on the posttest. The results of the paired t-test show a significant difference between preservice teachers' pre-post scores, $t(46) = 8.27$, $p <.0001$, two-tailed. From the descriptive analysis, we learned that 2 (4%) preservice teachers answered all 18 fraction problems correctly on the pretest. Seventeen (36%) answered 15 or more of the problems correctly at the outset. This number increased to 26 (55%) students on the posttest. Five (11%) students responded correctly to all the problems (100%) on the posttest. Five (11%) students answered all 4 of the percent problems correctly on both the pre- and posttests. Nine (19%) students responded correctly to 3 of the problems on the pretest. This number increased to 13 (27%) on the posttest. These results support our assumption that specific interventions improve preservice teachers' content knowledge in mathematics methods courses. However, additional studies are needed to compare the results with a control group.

2. How does preservice teachers' self-efficacy change as a result of the interventions?

We compared pre- and post-survey scores to determine whether preservice teachers' self-efficacy improved. The pre-survey was given during the first week of class and the post-survey during the last week of class. Students had completed the mathematics assessment prior to the post-survey but had not yet received their scores to minimize the effect the test scores may have had on the students' rating of their confidence levels. Because of the nature of the survey, a quantitative analysis was not used to analyze these data.

When we analyzed the survey, we found preservice teachers' responses to teaching specific mathematics content in the "very confident" category increased in 9 out of 10 topics areas. The

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areas where preservice teachers' confidence increased the most in the "very confident" category are: Metric Measurement (30%), Addition/Subtraction of Whole Numbers (21%), Probability (21%), Geometry (18%), and Multiplication/Division of Whole Numbers (17%). The areas with the lowest increase in this category are: Algebra (15%), Ratio/Proportion (13%), Decimals (11%), and Fractions (2%). The only topic where confidence decreased in this category was percents, falling from 25% to 23%.

However, when we analyzed both the "very confident" and "moderately confident" levels the results were more impressive. After examining the pre- and post-surveys, we found 63% of the preservice teachers were very confident or moderately confident in teaching decimals at the beginning of the semester. After taking the methods course, 85% were very confident or moderately confident in teaching decimals. Sixty-five percent of prospective teachers were very confident or moderately confident teaching fractions at the start of the spring semester. The post-survey indicates 85% of the students felt very or moderately confident teaching fractions when the semester concluded. Sixty-nine percent of the students were very or moderately confident teaching percents at the start of the spring semester while 74% felt very or moderately confident at the end. While we can conclude an increase in teacher efficacy after taking the reformed methods course, additional studies are needed to track these changes statistically and to compare them with a control group.

3. How do preservice teacher's microteaching lessons change as a result of the intervention?

During the first semester, Carol found preservice teachers' microteaching lessons mirrored her own teaching style. Carol realized her own instruction needed to change in order for her preservice teachers to implement effective lessons. During the second semester, Carol saw noticeable improvement in both the quality of the lesson plans and the teaching that she observed. What follows is one example of a lesson taught during the spring semester that stands out in terms of creativity and difficulty.

Discovering Pi

This lesson was designed by a group of preservice teachers who were completing their practicum in a sixth-grade classroom. Prospective teachers used several circular objects, string, a ruler, and a calculator to generalize the relationship of the circumference divided by the diameter, or Pi. After measuring several objects, students found the average of the measurements. The preservice teachers in this group discovered as the number of objects measured increased, the closer the average came to 3.14.

One preservice teacher expressed disappointment that the average did not equal exactly 3.14. However, doing this activity helped the preservice teachers to develop an understanding of what an irrational number is and what is meant by the term "approximation" in mathematics. What Carol found surprising was most of the preservice teachers knew the value of Pi, but only a few knew the value represented a physical relationship. Thus, there was an "aha" moment for the preservice teachers and for Carol as well, who was reminded not to assume preservice teachers possess all of the content knowledge necessary to teach mathematics.

Those teaching the lesson also realized one must be careful in choosing objects to use when teaching about Pi. For example, a CD was one of the objects measured, but preservice teachers had great difficulty measuring the circumference because the string kept slipping off the edge.

The preservice teachers also realized the importance of practicing a lesson before attempting to teach it to students in the classroom.

Instructor Reflection

Carol believes the improvement in the lessons she observed during the second semester are due in part to her change in teaching style. Additionally, using the jigsaw method for group work had a twofold benefit. For many preservice teachers, the methods course provided their first experience teaching a math lesson. The ability to collaborate on a lesson allowed prospective teachers to share both their insights and misconceptions about certain mathematics concepts. This helped each planning group to develop a solid lesson that was standards-based and to incorporate good questions into the lesson. This planning group work also help preservice teachers to anticipate some of the difficulties that might arise when they taught the lesson. These prospective teachers were able to plan ahead, making modifications to ensure the lesson would be successful even if difficulties in implementation should occur.

THE EFFECTS OF COACHING ON A TEACHER AND HER COACH: A MULTI-LEVEL ACTION RESEARCH STUDY

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The discourse in most United States classrooms has long been little more than a speech on a topic and one-sided at that (Kilpatrick, 2001). Teaching mathematics for many is simply showing students what to do and how to do it, with very little interaction of adult and student thinking. Kilpatrick suggested that discourse not be limited to answers only but should include discussion of connections to other problems, alternative representations and solution methods, and the nature of justification and argumentation. Thompson (1992) concluded that, "virtually nothing is known about whether students' views of the subject matter influence teachers' instructional decisions and actions as well as their views of the subject" (p. 142). A review of the literature (Cohen & Hill, 2002) revealed a paucity of research that examined the relationship between teachers' and students' learning. This study examined effects of my coaching on the teacher's pedagogy, in particular her communication with her students and views of mathematics held by the teacher, her students, and me. The specific research question addressed in this paper was, "How does instructional coaching influence the teacher being coached, her students, and the coach?" The results of this study indicated that the coaching influenced the teacher's and students' classroom discourse.

Theoretical Framework

The theoretical framework for this study used activity-reflective cycles (Tzur & Simon, 1999) and modifications to these cycles by Olson and Barrett (2004). Following each lesson, the teacher and coach analyzed the lesson for evidence of learning and student engagement in their learning. The activity-reflective cycles included analyses followed by adapting the learning trajectory theory, creating new learning activities, and observing the interactions between students and teachers (Olson & Barrett). In this study, the teacher, her students, and the coach gradually became a learning community (Lave & Wenger, 1991) and the observations, albeit over a limited time, revealed changes in the classroom discussions of mathematical ideas. This report focuses on these changes.

Methods

The research methodology was a single case study of one teacher, Jane. Jane taught 8th grade mathematics in an urban middle school that was collaborating with my university in the second year of a three-year grant investigating ways to improve the academic engagement of African American and Latino students. Jane, who had earlier participated in a best practices workshop led by me, was excited about the prospect of having a coach and agreed for me to observe two of her classes. Audio and video recordings of classroom sessions and teacher debriefings were recorded from March to May 2003. Each session was transcribed and the data from the observations and interviews were analyzed from a systems perspective, (e.g., Jenlink, 1995, and Clark, 2003), by examining patterns of activity. Debriefing sessions followed the observations.

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Results and Discussion

Jane encouraged student presentations in both classes, but initial observations revealed that these presentations were generally procedural in nature. Most of Jane's questions focused on the procedures used by the students, as in the following excerpt:

> Lindsey: So cross multiply. You get 2x + 14 = 3y + 27. You want to get the variable term by itself, so subtract 27 from both sides and get 2x - 13 = 3y. And then you want to get y by itself, so you divide everything by 3 and you get y = 2/3 x - 13/3. (Lindsey went on to explain each step of her equation-solving process)
>
> Jane: But, what are you solving for?
>
> Lindsey: I'm solving for y.
>
> Jane: So her equation ends to be y equals? (To the class) Have you got the equation part? (Observation March 2003)

Throughout the debriefing sessions, Jane and I discussed how each lesson might be enhanced to include more student discussion of the mathematics. Gradually, the questioning of and by the students and Jane changed, as in the next excerpt:

> Jane: Wait a minute. Where's the 3i come from?
>
> Linnen: Think about what you did if you multiplied 3i times 3i.
>
> Camille: Wouldn't that be 9i-squared?
>
> Linda: Would it be equal to 9i-squared?
>
> Linnen: Do you agree with what she said?
>
> Jane: Is that making sense to you? OK. Chris.
>
> Chris: Multiplying i by i is basically undoing the imaginary number.
>
> Jane: Why?
>
> Chris: If you start with an imaginary number, then anything times anything is going to be a positive number. (Observation May 2003)

Presentations that had initially been characterized by merely the reading of the written procedural steps now took the form of questioning and wondering about the mathematics.

Summary

Olson and Barrett (2004) found that providing three first grade teachers with rich mathematical tasks and discussing the embedded mathematical concepts did not promote the anticipated professional growth and resulted in the teachers' use of innovative materials in traditional ways, as though mathematics contained only right and wrong answers. These three first grade teachers managed the discourse in ways that discouraged exploration of students' understanding. In contrast, this study found, that Jane used traditional materials in innovative ways by encouraging and modeling a classroom based on mathematical inquiry. Clearly, these diverse findings point out a need for further research of the effectiveness of coaching.

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UNDERSTANDINGS OF MARGIN OF ERROR

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Research Topic

Margin of error is the signature index of sampling variability in poll results that appear in non-technical publications such as newspapers and magazines. Yet it is also one of the least understood statistical concepts by the public. There is abundant confusion in both the lay and technical literature about margin of error (Saldanha, 2003). For example, the writings of ASA (1998) and Public Agenda (2003) misinterpreted margin of error as "95% of the time the entire population is surveyed the population parameter will be within the confidence interval calculated from the original sample". Against this background, the goal of our study was to address a series of interconnected questions: What does it mean to understand margin of error? How do people understand it, and how might we support people's development of a more coherent understanding of margin of error? We view the answers to these questions crucial not only for creating models of understanding margin of error, but also for supporting instructional design intended to promote the learning of margin of error. To tackle these questions, we examined a set of data collected from a professional development seminar that we had conducted with a group of eight high school teachers that aimed to investigate their understanding of probability and statistical inference (Liu & Thompson, 2004).

Background Theories & Methodology

Our study was guided by a radical constructivist perspective on human knowledge and human learning. Radical constructivism entails the stance that any cognizing organism builds its own reality out of the items that register against its experiential interface (Glasersfeld, 1995). As such, in our study that aimed to understand others' mathematical understanding, it is necessary to attribute mathematical realities to subjects that are independent of the researchers' mathematical realities. This is what Steffe meant when he described the researcher' activity in a constructivist teaching experiment as that of performing the act of de-centering by trying to understand the *mathematics of [others]* (Steffe, 1991).

To construct models of others'/teachers' understanding, we adopted an analytical method that Glasersfeld called conceptual analysis (Glasersfeld, 1995), the aim of which is "to describe conceptual operations that, were people to have them, might result in them thinking the way they evidently do." Engaging in conceptual analysis of a person's understanding means trying to think as the person does, to construct a conceptual structure that is isomorphic to that of the person. In conducting conceptual analysis, a researcher builds models of a person' understanding by observing the person' actions in natural or designed contexts and asking himself, "What can this person be thinking so that his actions make sense from his perspective?" In other words, the researcher/observer puts himself into the position of the observed and attempt to examine the operations that he (the observer) would need or the constraints he would have to operate under in order to (logically) behave as the observed did (Thompson, 1982).

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Research Design & Data Analysis

The seminar, which lasted two weeks, was conducted in the summer of 2001. Table 1 presents demographic information on the eight selected teachers. None of the teachers had extensive coursework in statistics. All had at least a BA in mathematics or mathematics education. Statistics backgrounds varied between self-study (statistics and probability through regression analysis) to an undergraduate sequence in mathematical statistics.

Table 1: Demographic information on seminar participants

Teacher	**Years Teaching**	**Degree**	**Stat Background**	**Taught**
John	3	MS Applied Math	2 courses math stat	AP Calc, AP Stat
Nicole	24	MAT Math	Regression anal (self study)	AP Calc, Units in stat
Sarah	28	BA Math Ed	Ed research, test & measure	Pre-calc, Units in stat
Betty	9	BA Math Ed	Ed research, FAMS training	Alg 2, Prob & Stat
Lucy	2	BA Math, BA Ed	Intro stat, AP stat training	Alg 2, Units in stat
Linda	9	MS Math	2 courses math stat	Calc, Units in stat
Henry	7	BS Math Ed, M.Ed.	1 course stat, AP stat training	AP Calc, AP Stat
Alice	21	BA Math	1 sem math stat, bus stat	Calc hon, Units in stat

Each session began at 9:00a and ended at 3:00p, with 60 minutes for lunch. All seminar sessions were led by a high school AP statistics teacher (Terry) who had collaborated in the seminar design throughout the planning period. We interviewed each teacher three times: prior to the seminar about his or her understandings of sampling, variability, and the law of large numbers; at the end of the first week on statistical inference; and at the end of the second week on probability and stochastic reasoning. This paper will focus on day 3 & 4, in which we focused on parameter estimation.

Results

Part I: Theoretical Framework for Understandings of Margin of Error

Margin of error (for a population with known standard deviation), when centered around a population parameter, yields an interval that captures a certain percentage of sample statistics collected from repeatedly taking samples of a given size. Expressed symbolically, this interpretation is:

$$\textit{The interval } p\pm r \textit{ captures } x\% \textit{ of } s_{i,} \quad x\in[0,100]. \qquad (1)$$

Reciprocally, when margin of error is centered around the sample statistics, it yields confidence intervals $x\%$ of which contain the population parameter.

$$x\% \textit{ of intervals } s_i\pm r \textit{ contain } p. \qquad (2)$$

Although typically, report of margin of error follows a sample estimate of an unknown population, margin of error in fact does not communicate to us how far off that sample statistic is from the population parameter. Rather it tells us that if we were to repeat the same sampling method, a certain percentage of all sample statistics will be within a given range of the population parameter. Therefore, with respect to one particular confidence interval, the best we can say is:

We don't know whether the interval $p \pm r$ captures s, and (3)
we don't know whether the interval $s \pm r$ contains p
(but we do know that x% of intervals $s_i \pm r$ contain p). (4)

Understanding of margin of error is not complete until one also understands that

x% is the statistic's confidence level. (5)

In other words, the percentage of sample statistics captured by $p \pm r$ is the confidence level of a sampling method. The combination of interpretations 1&3&5 conveys the definition/ways of thinking about margin of error. The combination 2&4&5 conveys a conventional interpretation/understanding of confidence interval.

Analysis of literature as well as data from the teachers seminar and prior teaching experiments found interpretations or ways of thinking that are incompatible with understanding margin of error. A classic misunderstanding of margin of error is:

The interval $s \pm r$ contains p. (6)

This interpretation is completely devoid of the idea of confidence level and a distribution of sample statistics. It exhibits a perspective that focuses on the accuracy of one individual sample statistic, and takes the margin of error as a measure of the distance between the sample statistic and the population parameter. Note that (6) is the direct opposite of the idea expressed in (4).

There are three other interpretations that indicate either a lack of or an erroneous understanding of margin of error. One interpretation is:

There is an x% probability that the interval $p \pm r$ will contain s. (7)

This interpretation is not wrong in itself, but it is vague. "*x%* probability" could mean *x%* of sample statistics, in which case (7) is the same as (1). It could also denote a subjective belief, which means it does not convey a distribution of sample statistics. In this paper, we will remove the ambiguity by assigning a subjective meaning to the word, "probability". That is, if a teacher says (7) but we have evidence that she is thinking (1), and we would assign (1) to her thinking.

The second interpretation is

The interval $s \pm r$ captures x% of s_i, (8)

The interpretation conveys a distribution of sample statistics. However, it says that *x%* of the sample statistics would be captured by the confidence interval constructed from the sample statistics, instead of the confidence interval centered on the population parameter. The difference between (8) and (1) is the center of confidence interval constructed from the margin of error.

The third interpretation is

The interval $p \pm r$ contains x% of the intervals $s_i \pm r$. (9)

This interpretation is incoherent because all confidence intervals are of the same width (*2r*). It does not make sense to think that one interval will contain other intervals. Note that the interpretations 1, 2, 8, and 9 are all interpretations of margin of error that contain an image of distribution of sample statistics.

The above interpretations, taken together, constitute a theoretical framework/coding scheme (Figure 1) for understanding teachers' conceptions and interpretations of margin of error.

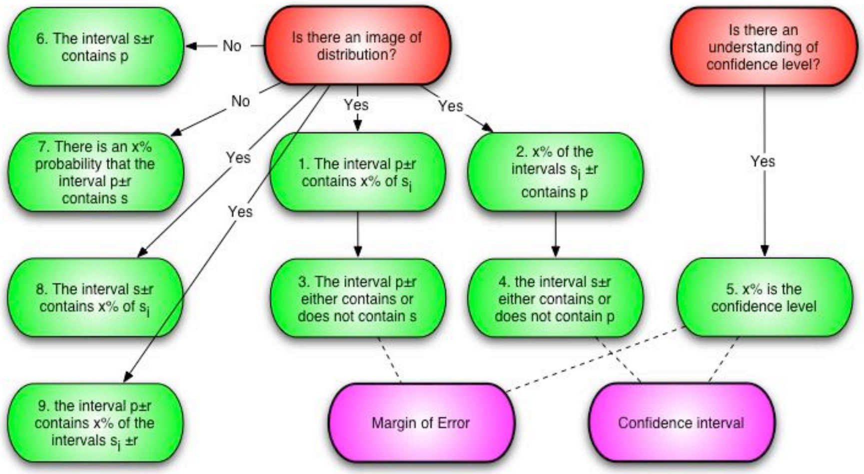

Figure 1: Theoretical framework for understandings of margin of error

Part II: Teachers' Understandings of Margin of Error

In our attempt to explore teachers' understandings of margin of error, we provided the following table of results obtained by resampling samples of size 500 from various populations with known parameters.

Percent of Yes in Population	Number of People in a Sample	Number of Samples Drawn	% of Sample Percents within 1 Percentage Point of Population %	% of Sample Percents within 2 Percentage Points of Population %	% of Sample Percents within 3 Percentage Points of Population %	% of Sample Percents within 4 Percentage Points of Population %
65%	500	2500	36.7%	64.5%	84.8%	91.5%
32%	500	2000	37.1%	65.8%	83.9%	91.1%
57%	500	6800	36.2%	64.9%	84.2%	91.3%
60%	500	5500	36.1%	65.2%	84.3%	91.4%

Afterward, we asked teachers this question:

Stan's statistics class was discussing a Gallup poll of 500 TN voters' opinions regarding the creation of a state income tax. The poll stated, "… the survey showed that 36% of Tennessee voters think a state income tax is necessary to overcome future budget problems. The poll had a margin of error of ±4%." Stan said that the margin of error being 4% means that between 32% and 40% of TN voters believe an income tax is necessary. Is Stan's interpretation a good one? If so, explain. If not, what should it be?

This question queried teachers' understandings of margin of error by having them comment on a particular interpretation of the reported margin of error for a public opinion poll of 500 people. We coined the scenario so that the information on the table could determine the confidence level associated with the scenario's sampling method and the reported margin of error. A "conventional" interpretation of the reported margin of error is: *The margin of error ±4% means that if we were to repeatedly sample 500 TN voters, around 91% of the sample statistics will be within ±4% of the true population proportion. We don't know if 36% is within that range*. The same interpretation expressed with the idea of confidence interval is: *We don't know if the interval 36%±4% will contain the true population proportion, but we do know that if we were to repeatedly sample 500 TN voters, around 91% of the intervals constructed like this* *will contain the true population proportion*. This question was given as a homework on day 3 of the seminar. Teachers were asked to give a written answer. After a 2-hour discussion on day 4, we asked the teachers to give a second answer to the question.

Teachers' first written answers (Table 2) showed that none of the teachers agreed with Stan's interpretation. Three teachers, John, Betty, and Alice, interpreted the margin of error ±4% as meaning "95% of sample statistics fall within ±4% of the unknown population parameter". Henry believed that the margin of error ±4% meant, "95% of the confidence intervals constructed from this margin of error will contain the unknown population parameter". These two interpretations of margin of error, conveyed by codes 1 and 2, are two coherent interpretations of margin of error, both of which build on an image of a distribution of sample statistics. Nicole had the misconception that the *interval s±4% contains x% of the sample statistics* (code 8). Three teachers, Linda, Lucy, and Sarah, used the word "probability" to relate the sample statistic and the population parameter (code 7). These interpretations of margin of error were not built on an image of a distribution of sample statistics. Although none of the teachers agreed with Stan's interpretation, only one teacher, Henry, explicitly stated the idea that countered Stan's interpretation, that *the interval s±4% does not necessarily contain p* (code 4). With respect to the idea of confidence level, all teachers used the number 95% where they hoped to convey their subjective level of confidence. Only three teachers, John, Sarah, and Linda, stated that the 95% was the "confidence level". None of the teachers utilized the table to infer that the confidence level (standard sense: number of sample statistics that are within the interval *p±r*) was 91%.

Table 2: Teachers' initial interpretations of the ±4% margin of error

	1	2	3	4	5[1]	6	7	8	9	1or2or8or9	1&3&5	2&4&5
John	√				*					√		
Nicole								√		√		
Sarah					*		√					
Lucy							√					
Betty	√									√		
Linda					*		√					
Henry		√		√						√		
Alice	√									√		
Counts	3	1	0	1	0	0	3	1	0	5	0	0

Table 2 shows that five teachers' interpretations of margin of error were built on an image of a distribution of sample statistics (code 1or2or8or9). Codes 1&3&5 or 2&4&5 are used to denote two different ways of understanding margin of error that are both coherent and complete[2]. As we

[1] We assign √ when an answer indicates that the confidence level is 91%, and assign * when a teacher uses the phrase "confidence level" to refer to the percentage of samples that are within the interval *p±r*.

[2] By "coherent", we mean understanding margin of error as "95% of sample statistics are within the interval [population parameter ± margin of error]". By "complete", we mean understanding of margin of error that also include an understanding of confidence level, and an understanding that "a particular sample statistic might be one of those sample statistics that are not within the interval [population parameter ± margin of error]".

can see from the table, none of the teachers understood margin of error as indicated by either combination.

Teachers' second answers were summarized in Table 3.

Table 3: Teachers' second interpretations of the ±4% margin of error

	1	2	3	4	5	6	7	8	9	1or2or8or9	1&3&5	2&4&5
John	√	√			*					√		
Nicole	√					√				√		
Sarah			√		*	√						
Lucy	√	√								√		
Betty	√	√				√				√		
Linda	√	√		√						√		
Henry	√	√		√						√		
Alice	√									√		
Counts	7	5	1	2	0	3	0	0	0	7	0	0

Table 3 shows that all the teachers, except Sarah, understood the margin of error ±4% to mean "95% of sample statistics fall within ±4% of the unknown population parameter". Five teachers also interpreted the margin of error ±4% as "95% of the confidence intervals constructed from this margin of error will contain the unknown population parameter". None of the teachers used the word "probability" to relate the sample statistic and the population parameter, or had the misconception that *the interval s±4% contains x% of the sample statistics*. All teachers except Sarah had an image of distribution of sample statistics in their understandings of margin of error. Compared to their first written answers, 3 additional teachers, Nicole, Lucy, and Linda, had a coherent image of the distribution of sample statistics and understanding of how it relates to margin of error.

Three teachers, Sarah, Linda, and Henry, stated explicitly that *the interval s±4% does not necessarily contain p*, or *the interval p±4% does not necessarily contain s,* as opposed to only one teacher (Henry) in prior answers. However, a conflicting result was while no teacher agreed with Stan's interpretation in prior answers, three teachers, Nicole, Sarah, and Betty, held the same interpretation as Stan's interpretation this time around.

With respect to confidence level, only John and Sarah mentioned the phrase. Like in the prior answers, all teachers used the number 95% where they needed to convey their confidence level. None of them utilized the table to infer that the confidence level was 91%. As a result, once again none of the teachers had a complete understanding of margin of error.

In the post-interview, we asked the teachers the following question: A Harris poll of 535 people, held prior to Timothy McVeigh's execution, reported that 73% of U.S. citizens supported the death penalty. Harris reported that this poll had a margin of error of ±5%. Please interpret "±5%. How might they have determined this? How could they test their claim of "±5%"? Table 4 summarized the teachers' answers to this question.

Table 4: Teachers' interpretations of the ±5% margin of error

	1	2	3	4	5[3]	6	7	8	9	1or2or8or9	1&3&5	2&4&5
John	√	√								√		
Nicole								√		√		
Sarah			√									
Lucy	√		√		√					√	√	
Betty	√	√								√		
Linda	√		√							√		
Henry	√	√		√						√		
Alice	√									√		
Counts	6	3	3	1	1	0	0	1	0	7	1	0

As we can see from Table 4, all but two teachers, Nicole and Sarah, understood the margin of error ±5% to mean "95% of sample statistics fall within ±5% of the unknown population parameter". Three teachers, John, Betty, and Henry, also understood the margin of error ±5% as "95% of the confidence intervals constructed from this margin of error will contain the unknown population parameter". Nicole took up again her understanding that *the interval s±r contains x% of the sample statistics*. All teachers except Sarah built their interpretations of margin of error on an image of a distribution of sample statistics. Four teachers, Sarah, Lucy, Linda, and Henry, stated explicitly that *the interval s±4% does not necessarily contain p*, or *the interval p±4% does not necessarily contain s*.

With respect to confidence level, all teachers used the number 95% where they needed to convey their confidence level (Note that the question did not specify a confidence level). Only Lucy explicitly assumed a confidence level of 95% before using it to refer to the percent of samples what are within the interval *p±r*.

Table 5 compared the teachers' interpretations of margin of error in both questions.

Table 5: Comparison of teachers' interpretations of margin of error

Counts	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**1or2or8or9**	**1&3&5**	**2&4&5**
Table 2	3	1	0	1	0	0	3	1	0	5	0	0
Table 3	7	5	1	2	0	3	0	0	0	7	0	0
Table 4	6	3	3	1	1	0	0	1	0	7	1	0

Table 5 shows that there was a significant increase in the number of teachers who interpret margin of error coherently (captured by codes 1 and 2). However, there were no significant changes in teachers' understanding of confidence level, and of the idea that *the interval s±4% does not necessarily contain p*.

Conclusion

1) Understanding margin of error entails an image of repeated sampling, and knowing that margin of error, when centered around the population proportion, captures a portion of all sample statistics. Analysis of teachers' interpretations of margin of error showed that more teachers understood this idea towards the end. However, we also found both inconsistencies and instability in teachers' images. 2) Understanding margin of error entails knowing that margin of error has nothing to do with the particular sample statistics. It is, rather, about the sampling

[3] In this particular situation, we assign √ only when a teacher explicitly assumes a confidence level when talking about a percentage of samples that are within the interval *p±r*.

method. We found that only a few teachers understood this idea. 3) Understanding margin of error entails knowing that the proportion of sample statistics is the confidence level of the sampling method. It tells us the percent of times we obtain a sample result that is within a certain range of the true population proportion. Results showed that this idea was particularly hard for the teachers to understand.

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RESEARCHING TEACHERS' USE OF MATHEMATICS CURRICULUM MATERIALS: ADVANCING THE RESEARCH AGENDA

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Recent years have seen the development of a vital community of researchers committed to studying the role of curriculum materials in mathematics education. While current emphases on accountability and student outcomes have focused national attention on efforts to measure students' learning from curriculum materials, a parallel body of research on teachers' experiences using the materials continues to develop. This work has offered important insights into the mutually constituted relationship between teachers and curriculum materials. Curriculum materials can influence teachers' beliefs, knowledge, and classroom practices; however, teachers change the recommendations of curriculum materials through selective use and interpretations. Continued work on the relationship between teachers and curriculum materials would benefit from focused attention to critical issues that are emerging from the existing research. In the sections below, we propose three areas of research that merit further study.

Expanded Notions of Fidelity

The term "fidelity of implementation" is frequently used in policy arenas to describe efforts to measure the extent to which teachers use curriculum materials as intended (National Research Council [NRC], 2002). However, research revealing the ways that teachers shape or transform curriculum materials raises questions about the possibility of curricular fidelity (Remillard, in press). At the same time, it would be inaccurate and irresponsible to conclude that all interpretations of a written curriculum are equally valid. The field is in need of ways to characterize reasonable and unreasonable variations or instantiations of a particular curriculum that are tied to features most central to its design. Research on variations in teachers' use of curriculum materials is critical to these efforts, but it must occur in conjunction with analyses of curriculum materials themselves.

Preservice Teacher Education

We have much to learn about how preservice and beginning teachers develop the inclination and ability to use, adapt, and develop mathematics curriculum during teacher education and initial classroom experiences. Although calls have been made for the inclusion of curriculum analysis in mathematics teacher education (Ball & Cohen, 1996; Lloyd, 1999; Remillard & Bryans, 2004), the literature contains only limited information about ways that preservice teachers have been engaged in thinking about curriculum (Lloyd & Behm, 2005) and student teachers' uses of curriculum materials (Van Zoest & Bohl, 2002; Wang & Paine, 2003). There remains a need for more detailed examples and systematic exploration of what novice teachers encounter as they use mathematics curriculum materials in the classroom for the first time.

Voice of the Text

A critical dimension of understanding how teachers use curriculum materials is the written materials themselves (Brown & Edelson, 2003). Remillard (in press) argues that the *subtle and* *unintended* messages that are “not directly associated with the content or pedagogy of the curriculum” (p. 42) warrant investigation. Departing from more prevalent analyses of textbooks, studies of the "voice of the text" focus centrally on ideological and epistemological questions like, “How does this textbook position the reader?” or “What values are communicated through this mathematics textbook?” Future work in this area may draw on and extend the range of sociological, postmodern, and linguistic perspectives that have allowed researchers to identify underlying messages in mathematics texts (e.g., Dowling, 1996; Gerofsky, 1996; Herbel-Eisenmann & Wagner, 2005; McBride, 1994; Morgan, 1996). If we advocate for shifts in teachers’ use of curriculum materials, we need to investigate what the materials themselves bring to teacher-text interactions.

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A CURRICULUM ANLAYSIS FRAMEWORK FOR CONCEPTUAL UNDERSTANDING OF MATHEMATICS

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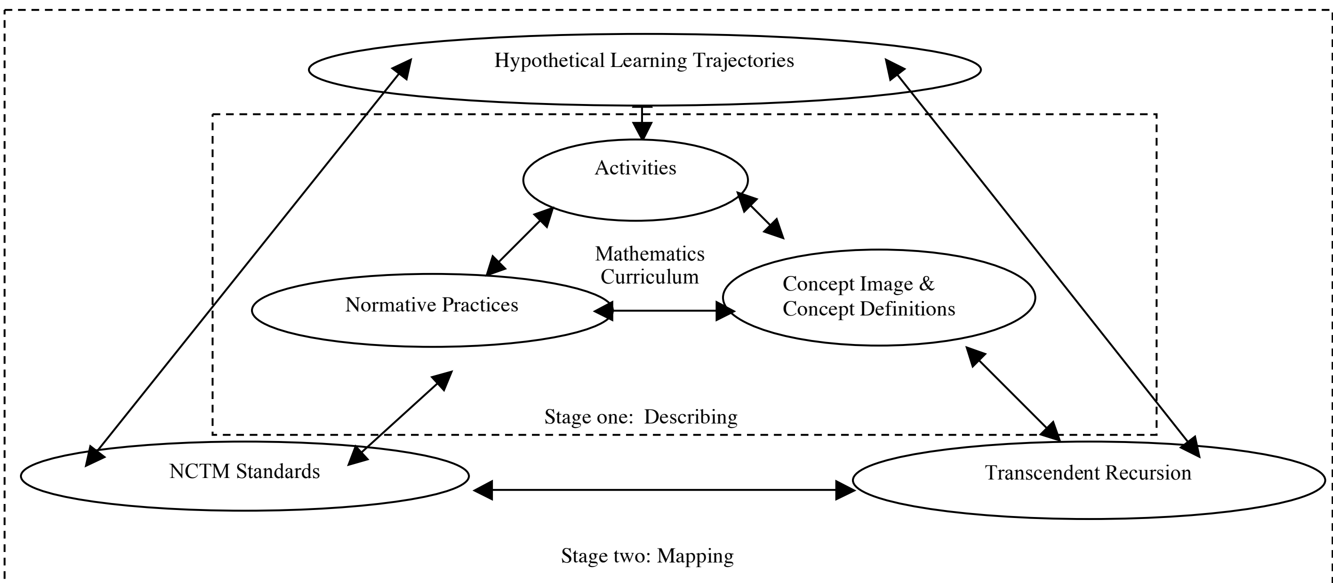
Purpose

The main purpose of this project is to develop a framework to analyze curriculum for its potential to support conceptual understanding. The framework contributes to the on-going effort to improve the quality of mathematics curricula through research-based curriculum development and evaluation processes (Clements, 2002).

The Framework

This framework consists of two stages of analysis. In the first stage, we analyze the teacher manuals and student workbook to describe the concept definitions, concept images, types of activities, and normative practices suggested by the curriculum authors. The second stage of analysis consists of various mappings to examine the curriculum's potential to support conceptual understanding. The figure illustrates the main components of the proposed framework.



Concept definitions are words used to specify a concept whereas concept images are all the mental images and associated properties and processes (Tall and Vinner, 1981). A limited, fragmented concept image is of little use in subsequent learning or for practical use in real life contexts. Individuals' concept images vary widely in the degree of richness and connectedness

Lloyd, G. M., Wilson, M., Wilkins, J. L. M., & Behm, S. L. (Eds.). (2005). *Proceedings of the 27th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.*

as a result of school, extracurricular, and life experiences. For a particular concept included in a curriculum, a concept definition is given either explicitly or implicitly to the students. Mathematical definitions vary greatly depending on the context. It is important to examine the mathematics curriculum and to determine the concept definitions and concept images it attempts to help students build.

Simon (2003) proposes a distinction between empirical activity—whose primary goal is to engage students in identifying certain numerical patterns to support the conclusion of certain generalized relationships—and logico-mathematical activity—whose primary goal is to help students build a conceptual understanding of the mathematical logic and necessity behind a concept. The proposed framework seeks out those activities that have the characteristics of logico-mathematical activity.

Furthermore, students' learning in the classroom must be supported by appropriate normative practices that form the basis for interactions in the classroom and govern the nature of reasoning used to justify the conjectures and conditions. Thus, it is important to examine the normative practices suggested by the textbook authors as an indication of their potential to support conceptual understanding.

The second stage of analysis draws from the theory of mathematical understanding: transcendent recursion (Pirie & Kieren, 1989) and the construct of abstractive reflection (Piaget, 2001; Simon, 2003). Using these, the strength and connectedness of the concept image and concept definitions, the levels of understanding the curriculum intends to achieve, and whether the intended level was supported with the establishment of lower levels of understanding can be determined.

Researchers have proposed the use of the hypothetical learning trajectory (HLT) as the basis for the design and sequencing of activities (Clements, 2002). The HLT consists of the identification of specific concept goals, the sequence of tasks that is used to support student's learning of those concepts, hypotheses about these developmental processes, and a mechanism to monitor and revise the trajectories. With the HLT of each main concept in a curriculum, analyses can be carried out to answer questions such as: " Is the activity sequence compatible with the developmental models identified by research studies?" or "Have the designs of the activities taken the research findings on children's notions of the concept into consideration?"

Finally, we need to examine the curriculum for its compatibility with the NCTM *Standards* (2000). For example, we will analyze the student text and the teacher's manual to see if there is any attention to the process standards such as reasoning and proof or to the various principles.

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UNDERGRADUATE MATHEMATICS COURSES FOR PROSPECTIVE ELEMENTARY TEACHERS: WHAT'S IN THE BOOKS?

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Analysis of 20 textbooks[1] written specifically for undergraduate courses for prospective elementary teachers suggests the following:

1. There is general consistency about the topical content of the textbooks. That is, with a couple of exceptions, the tables of contents suggest that the books "cover" similar mathematical territory.

2. There is a range of approaches to the depth and breadth of content in the books. One type of textbook is encyclopedic, covering rather uniformly (i.e., in the same number of pages) most topics that could be taught in elementary classrooms. Another type of textbook is problem-based, covering topics by including them in a problem or problem set, but not necessarily providing instruction or explanation of every topic and not treating all topics as equally important. A third type of textbook includes explanation of many elementary topics, but is selective in what is included and how much depth is provided for each topic.

3. There is much variety in the rhetoric of these textbooks, in how mathematics is presented, where mathematical authority lies, and how the student is expected to engage with the mathematics. Three examples illustrate this variety. One type of textbook presents the mathematics much as an advanced mathematics text proceeds, from definition to theorem to proof. There is little additional information beyond the mathematical essentials, and the authority clearly lies in the mathematics itself. There are few references to the students. The book may use "we" to mean "mathematicians" or "those with mathematical knowledge." Another type presents extensive explanations with many references to student understanding and many examples and alternative mathematical representations. These books often use "you" to talk directly to students, as in "you may find it helpful to…" Another type of book uses neutral language, as if mathematical authority could passively exist almost without a source.

4. There is also variety in how problems are used. Some books have traditional (and in some cases extensive) problem sets at the end of each section and review problems with each chapter. The problems reinforce material in the chapter. Others use problems as a primary source of mathematical content – without the problems, the mathematics is incomplete.

5. Finally, looking specifically at the topic of fractions, there are a range of approaches to defining the concept of fraction and operations on fractions. There is no clear consensus on whether fractions and rational numbers define the same set of numbers; on whether a fraction is a number or a representation of a number; or on which of a number of possible definitions of

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[1] (Bassarear, 2005; Beckmann, 2005; Bennett & Nelson, 2003; Billstein *et al.*, 2003; Center for Research in Mathematics and Science Education, 2000; Darken, 2003; Jensen, 2003; Jones *et al.*, 1998; Krause, 1991; Long & DeTemple, 2003; Masingila *et al.*, 2002; Milgram, 2004; Musser *et al.*, 2003; O'Daffer, 1998; Parker & Baldridge, 2004; Sgroi & Sgroi, 1993; Sonnabend, 2004; E. Wheeler & Brawner, 2005; R. E. Wheeler & Wheeler, 2002; Wu, 2002)

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fraction is the best place to start in helping prospective teachers learn what they need to know about fractions.

BREAKING THE CYCLE: MODIFYING PRESERVICE ELEMENTARY TEACHERS' ATTITUDES ABOUT MATHEMATICS

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Introduction

The mathematics department in collaboration with the College of Education designed a sequence of mathematics content courses based on the National Council of Teachers of Mathematics (NCTM) five content standards (2000). The five NCTM process standards were used as unifying strands for the material in the courses and emphasized in the delivery of the curriculum. Brownlee (2003, p. 87) states that while preservice teacher beliefs are not often explored, "these beliefs will influence their teaching practices in the classroom." Bischoff and Golden describe a general disconnect between preservice elementary teacher concept knowledge and procedural knowledge (2003). Judy Johnson remarked that, "we teach who we are" in research course in mathematics and science curriculum (personal communication, 2001). Ambrose echoes this remark writing "because of the important role beliefs play in the teaching and learning of mathematics, mathematics educators need to consider ways to assess beliefs and belief change" (2004, p. 56). Thus this study endeavors to look at if poor or average preservice elementary teacher attitudes about mathematics can be modified.

Purpose Statement

The purpose of this study was two fold. First the researchers wanted to identify preservice elementary teachers' attitudes about and past experiences with mathematics before beginning two mathematics content courses especially designed for them. Upon completion of each course the researchers wished to identify what changes in attitude toward mathematics, if any, would be reported by these preservice teachers. The two courses used for this study were Mathematics Inquiry (MATH 2160) and Mathematics for Early Childhood Teachers II (MATH 3162). Both courses included a laboratory component which involved the use of hands-on materials.

The main research questions that informed this study were: 1) what types of previous experiences have preservice elementary teachers had in mathematics, 2) what level of proficiency do preservice elementary teachers have with technology, and 3) how do the NCTM standards-based courses effect/change the attitudes of preservice elementary teachers toward mathematics?

Methods

Both a qualitative and quantitative research approach was used in this study. According to Merriam (1998) qualitative researchers desire to understand the meaning people have constructed about a particular phenomenon, the experiences they have had, and how they make sense of the world around them. Qualitative research integrates multiple methods, involving an interpretive, naturalistic approach to its subject matter (Denzin & Lincoln, 1998).

Students voluntarily answered free-response surveys at the beginning and the end of the semester. Data was compiled using standard statistical methods. Analyses included looking at frequency distributions, cross tab comparisons, and the Wilcoxon signed ranks test (a non-parametric alternative to a paired sample t-test). The results will be discussed at the presentation.

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PAYING MATHEMATICAL ATTENTION

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What does it really mean when students *pay attention?* A few educators study the significance of understanding what students pay attention to. They focus on what students see, experience and communicate. Our research extends this work by studying the dynamics of how students pay attention to mathematical objects. We study how students attend during mathematical activity and how teachers promote mathematical attentiveness.

Our view does not limit itself to visual, conscious and formal ways of attending, much less to individual and sensory attention; we include all sensory modalities as well as habitual, subconscious and collective attention. Mason (2003) and associates maintain that in teaching it is important to develop students' *awareness*. Asserting that "learning consists of shifts in the structure of attention" (1982, p. 9), they focus on what is attended to and on how students attend. They offer methodical ways to provoke students to focus on mathematically significant forms. They infer how students should attend mainly by interrogating their own experiences. In contrast, Sfard and Kieran (2001) analyze students' utterances, inferring what they attend to by observing their communication. They observe that students attend not only to mathematical objects but also to discursive patterns—non-object-level attention. By studying what students attend to, we can explore how students experience concepts, and begin to know what abstract concepts are (Booth et al., 1999).

Our own view is similar to the theorists noted above, except we examine not only the matter (what is attended to) and form (how students attend), but also the coherences and dynamics of students' mathematical attentiveness—the objects enacted, the worlds brought forth and the identities that arise by attending in particular ways. In these 2 pages we can only point to a few sketchy interpretations of students paying mathematical attention.

The Observer and Observing

Human cognition is inseparable from action and perception. But as Maturana (1988) emphasizes, it is observation that is the fundamental human operation. Perceived objects and thoughts arise with *acts of observation*; perceptible worlds are brought forth (Namukasa, 2005). Mathematicians and mathematics learners may be viewed as observers who interact in a particular human domain—the mathematical world. Thus, knowing mathematically, in addition to being synonymous with *speaking, visualizing, representing* and *sensing mathematically*, is synonymous with observing mathematically. Herein lies the point of departure from frameworks that view the role of perception in learning to be peripheral or limited to observing already-formulated mathematical attributes. Attention is participatory. In our work, what counts as mathematical attentiveness spans more than one layer of knowing and distinction-making. It includes embodying mental dynamics with bodily dynamics, and embedding these in cultural contexts as well as extending them to physical and symbolic environments. To explore these dynamics we draw data from two grade 7 lessons on transformational geometry: a lesson on the symmetry of planar figures and one on symmetry of a circle.

Lloyd, G. M., Wilson, M., Wilkins, J. L. M., & Behm, S. L. (Eds.). (2005). *Proceedings of the 27th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.*

What Students Attend To in a Geometrical Task

Studying the lesson transcripts, we observed that the first lesson exhibited interwoven foci of attention, with many foci that had not been directly anticipated by the teacher.

Table 1. Chronology of shifts in attention in a lesson on transformational geometry

A. Teacher [T] directs the class's attention to an object with 3 lines of symmetry	H. SS focus on the lines of symmetry of an octagon
B. T draws attention to lines of symmetry in specific triangles	I. A few SS continue to discuss why a square had 4 and not 8 lines and an octagon 8 not 16 lines of symmetry T draws an octagon to verify
C. T asks about the nature of symmetry in a square	J. A S suggests that he knows an object with lots of lines of symmetry
D. Some students [SS] say "8 lines," teacher draws a distinction between a cube and a square; SS briefly focus on the difference	*K.* SS attending to *a circle as an object with lots of symmetry*
E. Class verifies symmetry of a square	*L.* SS attend to the *nature of symmetry in a circle*
F. T asks if SS can think of an object with 8 lines of symmetry	M. S calls out that he knows something with infinite lines of symmetry
G. T asks why she posed this question but there was no response from the students	*N.* Another S attends to the nature of *symmetry of a sphere in relation to a circle*

The foci of individuals, sub-collectives and the collective drifted to include more than 10 foci of attention in a brief review part of the lesson. Students attended as a whole class to symmetry in polygons (foci A-B). They also attended in sub-collectives (foci L-M) to symmetry in a circle and a sphere. As individuals, some students attended to distinct aspects of symmetry (foci D & I). At many times students did not appear to attend to what the teacher was attending to (focus G). Also, at some points three foci overlapped (foci I-O).

Students habitually attended to regular polygons as they searched for objects with 4 and 8 lines of symmetry. The role of the subconscious is evident when one student interrupted, "I know one with lots" and another student, "I know one with infinite." We attribute the teachers' move to objects with lots of symmetry from objects with 8 or 16 to the collective. It was not the teacher's explicit intention, to explore with seventh graders the symmetrical properties of a circle. But by the time the teacher posed a question about the lines of symmetry a square has, it appears, the teacher and the students were drifting into naming objects with more symmetry. In the follow up lesson on symmetry of a circle, we saw the students attending culturally as they, with the help of the protractor, explained the possibility of 360, 18, 180, 3600 and infinite lines of symmetry. For students who saw 16 and not 8 lines for an octagon, we contemplate the extent to which enumerating lines of symmetry is a cultural way of attending. We have also reflected on what the grade sevens attended to they talked about symmetry using verbs such as cut and fold.

Our analysis explores how the attention of the students focuses, how it becomes re-focused, and how concepts emerge as foci of attention. We assert that how students attend is much more dynamic than paying attention in the restricted sense of keeping quiet and sitting still. Many students in this class who may have appeared to be off-task were more than present for the lesson. They not only *directed their ears and energies* to the teacher, but they *looked out for* mathematical insights. Drawing from their past experiences and current interactions, they *stretched* their understanding. They *awaited* the emergence of mathematical objects. From their ways of engaging and interaction emerged the whole classroom's examination of more abstract symmetrical properties, say of a circle and sphere. Addressing ourselves to how students pay mathematical attention is central if teaching is to help students to observe mathematically.

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EXAMINING THE ROLE OF CONTENT KNOWLEDGE IN DEVELOPING STANDARDS-BASED MATHEMATICS INSTRUCTION

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Objective

Teachers need appropriate experiences and materials from which to build new models of instruction, learning and assessment. Researchers agree that ample opportunities are needed in order for teachers to construct deeper understanding of the mathematical concepts they are expected to teach as well as an increased awareness of the ways in which children learn these concepts is equally important (Carpenter & Lehrer, 1999; Schorr, Maher, & Davis, 1997; Janvier, 1996; Cobb, Wood, Yackel, & McNeal, 1993).

The objective of this research study is to examine the nature of mathematical knowledge middle grades teachers possess; the way they use their knowledge to enact reform-oriented instruction; and how this information might be used to provide professional development to middle grades mathematics teachers. Specifically, we focused on three classroom teachers' development of activities, the construction of concept maps that illustrated the mapping of curriculum, identification of concepts and skills, and documentation of their students' mathematical thinking. These three areas of concept maps revealed important aspects of the role teachers' mathematical content knowledge played in the development of standards-based instruction.

Perspectives

Research studies have documented that teachers' personal beliefs and knowledge about mathematics, and mathematics teaching and learning strongly influence the ways in which they teach (Ball, 1990). These knowledge and belief systems are generally acquired prior to actual classroom experience, and held through years of teaching. The framework that was chosen to examine the role of teacher content knowledge in developing standards-based instruction is the Mathematics Teaching Cycle [MTC] (Simon, 1997). As a conceptual framework, the MTC "describes the relationships among teacher's knowledge, goals for students, anticipation of student learning, planning and interaction with students" (Simon, 1997, p. 76). Simon (1997) further explains that changes in the learning trajectory are based on interactions with students, which impacts teacher's knowledge, thus impacting the goals, plans, and/or hypothesis in a cyclical fashion. Changes in teacher's knowledge impacting the hypothetical learning trajectory [HLT] might occur during a lesson, not just between lessons, particularly if the teacher is reflecting while teaching (Schon, 1983). In this study we specifically focused on the role of content knowledge in developing standards-based activities particularly in their thinking about the underlying skills and concepts found in rich problem solving activities as they develop concept maps.

Methodology

This study examined how three middle grades mathematics teachers use their content knowledge to enact reform-oriented instruction. We used an interpretive case study methodology to focus on how these teachers understand the underlying concepts and skills found in selected problems, design concept maps illustrating their ideas, and execute reform-oriented instruction based on these notions. In addition, we analyzed the teachers' focused attention on their students' mathematical thinking for the purpose of uncovering how this is used in designing their lessons.

Three middle grades mathematics classrooms from a school in Atlanta Public Schools, Georgia were utilized for this project. The teachers met for a week from 9 A.M. to 3:00 P.M. with the researchers (in form of planning times) to discuss underlying concepts and skills, mapped concepts, and identified both Performance Standards as well as important mathematical ideas that were embedded within rich problems. Also, the teachers had an additional 2 weeks of planning times (in form of professional development from Atlanta Public School Mathematics Coordinator). A total of four lessons were used for instruction during this project. The teachers used the national, state, and school standards to further document the types of concepts represented in each lesson. After sharing their own ideas and representations during planning times, they then used these problems in their own classrooms. During classroom implementation with the researchers present, teachers were encouraged to recognize and analyze student interpretations and thoughts about the types of problems presented. Independently the teachers reflected and revised their own concept map. They brought their ideas and thoughts back to share with colleagues and us in subsequent planning times. The teachers (along with other participants not reported in this project) critiqued each other's mappings. This afforded the opportunity to both consider the development of each other's ideas, to discuss students' thinking, and teaching implications. Interviews were conducted before and after instructions with the three teachers and six students (two from each class). The purpose of the interviews was to gain a deeper understanding of (a) how the teachers' content knowledge played a role in their thought processes as they design the standards-based instructions utilizing concept maps and (b) the students thinking about these lessons.

Data Sources

In this study, four lessons were used for instruction. We collected multiple record of practice to use as database for inquiry into the ways in which these teachers used their content knowledge in practice, the ways in which they designed curriculum concept maps, the ways in which they used students' thinking to revise and refine their instructions, and the ways in which they dealt with the dilemmas of teaching. In brief, the data collection for this study include: (a) the teachers' curriculum concept maps, (b) transcripts from pre- and post- semi-structured interviews of students and teachers, (c) transcripts from teacher planning times, (d) teachers' reflections of their own and other teachers' work, (e) students' work on the individual classroom activities, (f) field notes of observation of classroom activities, and (g) field notes taken while working with teachers during planning times. The collection of this data followed the model that Ball and Lampert (1999) used in studying their teaching. By collecting multiple perspectives on classroom practice, including the researchers' perspective, the teachers' perspective, the students' perspective, and the perspective provided by audiotapes, a rich collection of data that showed much of the complexity of teaching via standards-based emerged.

Discussion

The data collection and analysis were driven by the questions: "How does these teachers identify connected concepts and skills in a given problem when developing concept maps?"

"How does these teachers endeavors to teach their students mathematics that is beyond what the students already know?" Results indicate that for teachers the development activity of constructing concept maps that illustrated underlying concepts and skills of particular problems were a valuable form of information about the growth and acquisition of deeper understandings they possessed of the middle school curriculum they currently teach. The three teachers level of detail on the construction of concept maps, organization, and interrelatedness of skills and concepts increased. The depth of thought between the skills and concepts and how they might play out in the classroom was evident. Teachers considered new ways of teaching and learning while collaborating during planning times to discuss content and pedagogy

Also, results show that as teachers listened to students' ideas and documented their thinking, they were able to progressively make sense of student work, make better pedagogical decisions based on their analysis of the problem, and created more detailed concept maps both globally and locally.

Moreover, teachers were able to construct and provide students appropriate problem sets or assessments that more accurately reinforced the problems done in class. As the teachers gained deeper understanding of their curriculum they were also able to explain and justify their curriculum goals and their alignment with textbooks and state standards in a variety of innovative ways. It appeared that the teachers' content knowledge, pedagogical knowledge, and knowledge of student thinking deepened simultaneously. Their concept maps and documentation of student thinking served as conceptual tools that aided in their growth.

By gathering accounts of these middle grades teachers' practice, we developed an understanding of their development as mathematics teachers and added to the larger body of knowledge on mathematics teacher development in work done by Simon and Tzur (1999). Also, by using middle grades teachers, the experience gained would be used in the training of preservice mathematics teachers. Detailed results, conclusion, and implications for teachers and teacher educators will be presented during short oral presentation.

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MESSY LEARNING: PRESERVICE TEACHERS LESSON STUDY WORK

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Current thinking and theories of teacher learning support the notion of situating teacher learning in classroom practice. The benefits of contextual learning in both teacher education and professional development have been theorized widely (e.g. Ball & Cohen, 1999; Hiebert, Morris & Glass, 2003). In particular, lesson study, where teachers collaborate to plan, teach and analyze a lesson, has received a great deal of attention as a collaborative, practice-based form of teacher learning. Researchers have argued that lesson study, if done thoughtfully, could increase teachers' content knowledge, focus teachers' attention on students and help teachers' transition toward reform-oriented teaching (Lewis, 1998; Stigler, 1999).

What often goes unsaid about learning from classroom practice is that the strength of situated learning is also an area for concern. The advantage of situating learning about mathematics or students in the context of the classroom is that complicated interactions can be captured, discussed and analyzed. However, because work in the classroom *is* complicated and unpredictable, it can be difficult for teacher educators to anticipate what their teacher education students will learn when engaged in a practice-based experience like lesson study. In addition, collaboration itself can be problematic in terms of increasing subject matter knowledge and changing beliefs about teaching or students (Fernandez, Cannon, Chokshi, 2003; Grossman, Wineburg & Woolworth, 2001). This study probes the complexities involved in practice-based, collaborative learning by looking at the intended and unintended learning of preservice teachers who engaged in lesson study as part of graduate-level elementary mathematics methods course.

Socio-cultural theories that describe learning as participation in practice (Lave & Wenger, 1991) allow researchers to examine issues of collaboration by looking at how joint work is constructed by members of a community. They also emphasize ways in which learning is contextual, by focusing on the local practices and meanings of individual communities. For this project, Lave and Wenger's theory helped me to see lesson study as a practice that varied across communities, even within my own classroom. To guide my analysis, I asked the following research question: How do beginning teachers talk about mathematics, teaching and students while participating in lesson studies?

As part of a twelve-week, graduate-level math methods course, I asked my students to engage in a lesson study-like project. Students were asked to collaboratively research, plan, teach and analyze a single lesson. I collected data as my students engaged in this work, including field notes based on audio tapes of five class periods, verbatim transcripts of the planning and analysis of four research lessons, field notes of four research lessons taught in elementary schools, and my students written work. I began my analysis of the data with open-coding of the transcripts to identify themes related to my research question. I then did a more fine-grained analysis of a few focal transcripts to look at episode lengths and turn-taking to explore which topics got the most attention during the lesson studies and the varied ways that participants negotiated the project.

Although each of the four lesson study groups received similar support – in terms of the nature of assignments, the amount of time given, and the curricular resources available --- the learning that resulted from participating in the project varied significantly across groups. Two of the four groups I studied developed a mathematical lens for reviewing curricula and instances of teaching; while the other two groups rarely engaged in conversations about the mathematics. Distinctions among the groups could be seen by looking at the number of conversations related to mathematics, the length of conversations and the kinds of questions asked. The use of the mathematical lens helped students to deepen content knowledge, support the learning of less-mathematically-able members, and see mathematical analysis as part of a teacher's job when planning a lesson. However, the use of the mathematical lens also had unintended consequences. For instance, frequent discussions about the purpose of estimation in one group reinforced members' beliefs that "rounding rules" were more important than sense-making. The lesson study experience allowed these preservice teachers to confirm their beliefs about the necessity of providing structured practice rather than open-ended problems.

One of the four groups I studied also developed an equity lens, which they used to evaluate particular curricula and teaching practices for the ability to meet the needs of all students. Although I asked all groups to attend to the learning of all students, only one group did so explicitly and repeatedly. This group modified the lesson suggested in their curriculum to provide multiple access points to the problem, made hundred's charts and multiplication tables available to support struggling students, and introduced group work to provide additional support. However, just as with the use of the mathematical lens, this group's attention to student difference also had unintended consequences. Closely observing differences among students during the research lesson caused the members to strengthen their beliefs in ability grouping. The success of the modifications this group made to the lesson did not encourage them to see the benefits of a heterogeneous class, but instead encouraged them to speculate on the increased benefits they would be likely to see if students were separated by ability more often.

This study suggests that as researchers and practitioners we need to move toward theories about practice-based learning that not only help us develop structures that promote meaningful conversations about mathematics, students and teaching, but also help us think about the unintended learning that is almost certain to occur when practicing or preservice teachers seek to make sense of the unpredictable world of the classroom

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UNDERGRADUATE MATHEMATICS: THE ROAD TO REDESIGN

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Purpose

While the traditional lecture dominates college and university classrooms, research shows that students need to do more than just listen. Much has been written about the need for active learning in postsecondary classrooms (Sutherland and Bonwell 1996; Chickering and Gamson 1991; McKeachie, Pintrich, Lin, and Smith 1987). A student who is *actively* involved in the learning process (rather than sitting in a room *passively* listening while an instructor lectures) will have improved learning and retention of that knowledge. Including instructional technology in this paradigm has led to even more success in improving student learning. In order to implement such a shift in the learning paradigm in our large enrollment mathematics classes, a redesign of how these courses were taught had to be undertaken. The College Algebra course was redesigned based on the mathematics replacement model (Twigg, 2003). This model replaces traditional lectures with a variety of learning resources such as interactive software that encourages active learning, prompts ongoing assessment, and provides individualized assistance.

Theoretical Framework

The major focus of this study is to investigate the effects of the technology-rich environment of the redesigned course on students' learning and retention of mathematical concepts. It was determined that a guiding philosophy was needed to suggest principled changes in the curriculum and effective uses of technology as part of these changes (Forman & Pufall, 1988). Bruner's constructivist theory is the framework that guided these curriculum changes.

Constructivism is a theory of cognitive growth and learning. According to Bruner (1960), one fundamental idea of constructivism is that students actively construct their own knowledge. Students assimilate new information to simple, pre-existing notions, and modify their understanding in light of new data. Educational applications of constructivism exist in creating curricula that match, but also challenge, students' understanding, fostering further growth and development of the mind. Learning must be interactive (Cobb, 1994). The technology used in the redesigned course allows the students to assemble and modify their ideas, access and study information. The instructor engages the students by helping to organize and assist them as they take the initiative in their own self-directed explorations, instead of directing their learning autocratically.

Methods of Inquiry

The study takes place during the spring semester at a large southeastern university. The redesigned course was College Algebra, an introductory course enrolling over 1500 students each year in 41 sections. This study follows a quasi-experimental design because the participants cannot be randomized. The treatment group contains students enrolled in seven redesigned College Algebra sections. These students divide their time equally between a classroom and The Mathematics Interactive Learning Environment (The MILE), a technology-driven facility that provides an array of interactive materials and activities. The control group consists of students enrolled in the three remaining traditional lecture-driven College Algebra courses.

We will employ the following assessment techniques suggested by Peter Ewell, Senior Associate at the National Center for Higher Education Management Systems (Twigg, 2003): matched examinations, student work samples, behavioral tracking, and attitudinal shifts. At the beginning and end of the semester, both experiment groups will complete a survey to assess their attitudes towards mathematics and technology use. Both groups will also complete a pre- and posttest designed to assess changes in their content knowledge. During the semester, the control group attends the regular classroom meetings in which the instructors primarily use the lecture method. The treatment group's intervention employs numerous classroom and web-based activities (available in The MILE).While working in the MILE, students work one-on-one with instructors, graduate research assistants, and peer tutors.

Results and Conclusions

The data for this study is still being collected. The researchers will combine qualitative and quantitative methods to develop the instruments for data collection in future semesters. The results of the research studies will be used for revision of initially developed materials, development of new materials and for assessment of the success of the whole program in general. The researchers will determine the impact of this student-centered learning environment on student achievement.

This study is aligned with the goals of PME-NA to further a deeper and better understanding of the psychological aspects of teaching and learning mathematics. There are opportunities for further studies on topics such as students' understanding of specific algebraic concepts, appropriate and effective technology use in the mathematics classroom, improvement of instruction and undergraduate mathematics education.

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SOCIOMATHEMATICAL NORMS AND MATHEMATICAL PRACTICES OF THE COMMUNITY OF MATHEMATICIANS

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Just as Sociomathematical Norms and Mathematical Practices develop within a classroom community as described by Cobb and Yackel within the framework of the Emergent Perspective (1996), such norms and practices have developed within the community of mathematicians[1] as well. For example, mathematicians, as a cultural community, have developed their own language. The language of mathematics not only differs in vocabulary from the natural language, but also includes different rules for sentence structure than the natural language (Epp, 2003). Also, when considering the history of proof, one realizes that what constitutes an acceptable mathematical explanation for the mathematics community has differed through time and between cultures (Siu, 1993). Furthermore, the validity of a mathematical proof is at times based, in part, on social decisions not specific to logic and mathematics (Segal, 2000). These norms and practices of the community are always changing; new vocabulary and notation are frequently added, new theorems are proved, and new methods of proof become acceptable. The mathematics community decides what mathematical practices are admissible, how mathematical statements should be written and interpreted, and what should be included within a proof.

Unlike classroom communities, however, the community of mathematicians has existed for thousands of years and has consisted of millions of mathematicians. Through time, people enter and leave this community, yet the mathematics community continues to have an entity of its own. Not every member in the mathematics community plays an active role in the molding of the norms, yet this is the case in the classroom setting as well.

Despite the complexity of the community of mathematicians, norms and practices within the community have developed throughout the years. These norms and practices in turn influence the norms and practices in an undergraduate classroom through the instructor, textbooks, and curriculum. As a result, I have found that when analyzing an advanced undergraduate mathematics classroom, it is useful to consider the role that the norms and practices of the mathematics community play in the development of the classroom norms and practices (Piatek-Jimenez, 2004). The first step in doing this is to classify mathematicians' norms and practices into categories. In order to distinguish these from the norms and practices described by Cobb and Yackel (1996) that exist at the classroom level, I will refer to the mathematics community's norms and practices as Community Social Norms, Community Sociomathematical Norms, and Community Mathematical Practices. The intended goal of this work is to develop a framework

[1] Throughout this paper, the terms "community of mathematicians" and "mathematics community" refer to research mathematicians. Though many other people can be included within these terms such as mathematics education researchers, K-12 educators, and those working in industry, I have chosen to think of research mathematicians as the "community of mathematicians" because it is research mathematicians who create new mathematics, in turn directly affecting mathematical practices, such as those dealing with the mathematical language.

based on the Emergent Perspective that will also incorporate the influence of the community of mathematicians into the study of undergraduate mathematics classrooms. In this paper, I discuss what I mean by the terms Community Sociomathematical Norms and Community Mathematical Practices and provide examples of each.

Community Sociomathematical Norms

Sociomathematical norms, as defined by Cobb and Yackel (1996), consist of normative understandings of what counts as "a different mathematical solution, a sophisticated mathematical solution, an efficient mathematical solution, and an acceptable mathematical explanation" (p. 178). Therefore, one sociomathematical norm of the mathematics community is the normative understanding of what methods of proof are considered to be acceptable by the community. Though this norm has evolved with time, it is safe to say that the current mathematics community would agree that empirical evidence does not constitute a valid proof, while deductive reasoning using the axiomatic method is a valid form of proof. Current methods of proof accepted by the mathematics community include direct methods, induction, contradiction, and contraposition.

Another community sociomathematical norm is the normative understanding of what constitutes an elegant proof. It has even been argued that "mathematicians prefer a beautiful proof, even if it contains a serious gap, over a dull, boring, correct one" (Hersh, 1993, p. 394). Of course, not all mathematicians always agree on what makes a proof beautiful, yet certain normative understandings of the elegance of a proof have arisen. Elegant proofs should be terse in nature. Furthermore, proofs that do not just verify, but that also give insight as to why something is true are considered to be more elegant.

Community Mathematical Practices

Classroom mathematical practices are mathematical activities that are used without justification in a mathematics classroom. Therefore, community mathematical practices are any mathematical activities or conventions agreed upon by the mathematical community at large. For example, the mathematics community has agreed that multiplication in the real numbers is a closed operation and therefore one can state these results without warrant. Similarly, with conventions within the language of mathematics, no warrant is needed. For example, in the mathematical language, conditional statements are not equivalent to their converse, however, in the natural language, this is not always the case. If someone were to say, "If I have the money, then I will go to the movies tonight" it is also implied that if they do not have the money then they will not go to the movies. This is not the case, however, for mathematical statements. When a mathematician states a conditional statement in mathematics, it is understood, without warrant, that the converse is not necessarily true as well. Therefore this convention in mathematics is a community mathematical practice.

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PROSPECTIVE SECONDARY MATHEMATICS TEACHERS UNRAVEL THE COMPLEXITY OF COVARIATION THROUGH STRUCTURAL AND OPERATIONAL PERSPECTIVES

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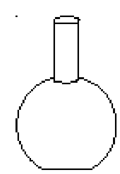
Prospective teachers have dealt with functions in one-variable settings – uncomplicated and direct. It is difficult to determine what students know if we only see what they do in familiar settings or in direct applications of a single concept. We claim that coordination of structural and operational perspectives (Sfard, 1991) can allow students to deal with concepts in unfamiliar and complex settings.

Thompson indicates that requiring the organizations of interrelated quantitative relationships makes a problem complex. He offers two sources of complexity,

> A situation can be complex because it requires a person to possess sophisticated conceptual structures in order to constitute it, such as the concept of rate (Thompson, in press; 1992). Also, a situation can be complex because it requires a person to keep multiple relationships in mind in order to constitute it. (Thompson, 1993, p. 166)

A covariational situation is one that can provide challenging complexity. The *bottle problem* (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002), a version of which also appeared in Shell Centre materials (Swan, & Shell Centre Team, 1999), asks students to draw a graphical representation of the height of water in an irregularly shaped bottle as a function of the amount, or volume, of the water.

The Bottle Problem
Imagine a bottle filling with water. Sketch a graph of the height of the water as a function of the amount of water that is in the bottle.



Methods, Data, and Analysis

Eight junior and senior undergraduates in a secondary mathematics teacher education program participated in task-based interviews. Analysis is based on data consisting of video recordings and verbatim, annotated transcripts of those interviews during which participants engaged in the *bottle problem* for the first time. Researchers interpreted students' reasoning using line-by-line analyses of the data and a range of lenses. We began with the Carlson framework, matching each student's mental actions and reasoning with levels of the framework. Although we found examples that fit each level of the framework, the levels did not seem to capture completely the mental actions we observed. We recognized a complexity in what the students were doing that the framework was not designed to capture. There was a significant difference among students in the extent to which their thinking seemed to be operational or structural, yet those categories also fell short of capturing the complexity. Our subsequent analysis tried to characterize the ways that operational and structural perspectives came into play

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in how they offer different affordances and constraints to individuals as they dealt with complexity.

Results and Discussion

The bottle problem requires solvers to "possess sophisticated conceptual structures" (Thompson, 1993, p. 166) of function, independent and dependent variables of volume and height (as well as, in some instances, an intervening variable of time), rate of change, accrual, accumulation, the geometry of the bottle, and graphical and pictorial representations in order to constitute the situation. Also, a person must keep in mind multiple relationships of the quantities of volume and height and the shape of the bottle at different heights. Following are brief descriptions of students' reasoning and how coordination (or lack thereof) of structural and operational perspectives afforded (or constrained) their dealing with the problem's complexity.

Bob dealt with complexity by focusing only on what he knew (even if, from our perspective, what he knew did not help solve the problem). His structural understanding of the objects, the pictorial representation of the bottle and the function, seemed limited. At first he did not seem to consider the shape of the bottle, instead appealing to a prototypical example of a cubic function because "volume is cubed units." He may have developed the strategy of finding a prototypical example and "going with it." Bob used the parts of the situation that helped him rationalize the prototype and made no reference to parts of the situation that did not support his prototype–he noted that the rate of change at the top of the bottle would be larger, supporting his selection of a cubic function, which increased at a faster rate for larger input values.

Jen also appealed to the prototype strategy, offering linear and exponential functions as possibilities that matched her need for an increasing function. Later, Jen used an operational perspective to reason about the rate of change of height with respect to volume above the middle of the globe. However, she did not coordinate her concepts and the relationships between the quantities in different sections of the bottle, seeming to lack a structural perspective.

Ned started out by reconceptualizing the problem at a structural level – "I'm supposed to … give some representation of what the curve is going to look like, the relationship, as the volume increases, what happens to the level of the water, the height." Ned continued moving smoothly among representations, monitoring one by reference to another, and coordinated, via a structural perspective, the various relationships between height and volume as he used an operational perspective to analyze each section of the bottle and determine how that influences the entire structure.

Instruction often seems to present mathematics as fragmented by focusing on individual topics and providing strategies for students that rely on finding prototypical examples as in the cases of Bob and Jen. Such students seem to be constrained by these strategies as they encounter complex mathematical problems. How does a student, like Ned, develop a larger, structural view, not only of mathematical concepts such as function, covariance, rate of change, etc., but of mathematics as a whole?

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THE RELATIONSHIP BETWEEN THE USE OF REPRESENTATIONS AND THE DEVELOPMENT OF RATIO AND PROPORTION CONCEPTS: NURIA'S CASE

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The case study presented in this report was part of a research carried out for a doctoral dissertation (Ruiz, E. F. 2002). The case study being reported is the one of Nuria (who was eleven years old at that moment) and was chosen to be exhibited in this document, since her performance during the teaching and the final questionnaire showed the strengthening she had at a conceptual level supporting herself, spontaneously, on the three representation modes (the drawing mode, the table and the numerical mode) when solving the problems.

Purpose

To use of different representation modes to favor the construction of internal and external ratio concepts, as well as the proportion.

Theoretical Aspects

Duval (1993) points out that the representations are not only necessary for communication purposes, but also essential for the cognitive activity of the thought and play a primary role in the development of mental representations, the fulfillment of different cognitive functions and the production of knowledge. Therefore, the cognitive activity of the individual needs several representation registers. In the case study presented here, the concepts involved were that of ratio and proportion, and the representation registers used in the teaching were: the drawing, the table and the numerical registers in order to reach the construction of such concepts through teaching situations and activities inducing articulation among them. In that regard, Duval mentions that if this articulation is not brought up by students, it would produce a separate system of symbols that do not have mutual and constructive interaction among them. Another relevant theoretical antecedent regarding the active link kept among the different modes of representation involved and the concepts related to fractions (in the field of ratio and proportion), are included in the researches carried out by Behr, Post, Silver, Mierkiewics (1980), Lesh, Post and Behr (1987), who have established several connections among the concepts and meanings of fractions and the representations of different nature, related to such concepts and meanings.

Research Problem

The problem approached in this case study is referred to the existing *relationship* between the development of *different modalities or representation registers* (involved in the solution of problems ratio and the proportion simple and direct) *and the construction of the corresponding concepts*.

Hypothesis

The representations from different nature used simultaneously allow Nuria to: a) give meanings to the problems presented, b) systematize the information to start a solution process and c) verify ideas or suppositions from which she started when facing problems.

Method

Regarding the interviews, as an example we mention what Nuria worked with. There were five tasks, the three first allude to situations referred to in a carpenter's shop and the last two to activities with which the girl is very familiarized, the use of drawing sheets. The main purpose was to review if Nuria used ratios in the solutions and, if it was the case, if she chose internal or external ratios and why. It was basic to go deeper in the approaches she obtained from the ratio and proportion concepts, what was reflected in the way she interpreted such terms and in the writing she used for that matter. It was also suitable to go deeper regarding the use Nuria gave to the different representation registers and the passage from one to another: the register of the drawing, the one of the table and the numerical one.

Analysis of Results

Before the teaching program, Nuria had an intuitive and weak idea of the proportion concept, though she did not use this term to express it. Likewise, when achieving the articulation of the three registers, Nuria showed a good management of the conceptual (Duval, 1993). The progresses reached by Nuria as a result of the interview were: the integration of the different representation systems, the management of operators with great ductility, link of the measurement figure for the establishment of relations, the use of ratios as fractional expressions and the plain construction of the ratio and proportion concepts. We present as follows an example where the activity and the form Nuria used to solve the problem is described. The task was titled "The photograph of your team". Nuria had to discover how long the girls in the picture measure. She knew how long she measured and could only measure what was in the picture. Nuria initially uses the drawing to solve the problem, what indicates us that the girl gave meaning to the problem. Afterwards, she managed to determine the ratios involved, which in this case were external ratios, when comparing magnitudes of different scales (Freudenthal, 1983) and passing from one representation to another (from the drawing to the numerical). Besides, she explained orally and in written how she solved the problem. An articulation arose between the registers; therefore, she was able to reach the construction of the ratio and proportion concepts (Duval, 1993).

Conclusion

As a conclusion, Nuria used different representation registers to give meanings to the problems presented, to systematize the information and to start a solution process and to verify ideas or suppositions from which she started when facing problems. That permitted Nuria beginning the transition to systematic construction of concepts and the development Nuria's proportional reasoning.

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INVESTIGATING THE RELATIONSHIP BETWEEN PRE-SERVICE TEACHERS' UNDERSTANDINGS OF MATHEMATICS AND THEIR DEVELOPING PEDAGOGY

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In work with pre-service teachers, it became evident that "knowledge of mathematics for teaching," though valuable, was not "fine-grained" enough to be of use. In this short paper, we will discuss our work that examines what it means for pre-service mathematics teachers to develop mathematics content knowledge for conceptual teaching. Our aim in this investigation was to understand the influence of pre-service teachers' [PST] particular understandings of mathematics content (as developed by in a university course setting) on their school-based teaching practices. In doing this, we focused first on PSTs' understandings of mathematics as the primary resource upon which they draw while teaching. The importance of teachers' knowledge of content has been acknowledged by a variety of scholars (Ball, 1993; Grossman, Wilson, & Shulman, 1989; Schifter, 1990). However, it is axiomatic that a teacher's knowledge of mathematics alone is insufficient to support his or her attempts to teach for understanding. In that vein, Shulman (1986) coined the phrase pedagogical content knowledge [PCK]. Ma (1999) and Stigler and Hiebert (1999) further refined the idea of PCK by arguing that teachers need a profound understanding of mathematics – knowledge having the characteristics of breadth, depth, and thoroughness.

In previous work with student teachers (Silverman, 2004) we noted that PSTs na•ve conceptions of "profound" understandings of mathematics are inconsistent with teaching mathematics for understanding. Since teachers' understandings of mathematics enable or constrain their ability to orchestrate mathematical discussions that provide students with opportunities to make sense of advanced mathematical ideas, it is important for teacher educators to understand both the understandings with which PSTs enter our programs and ways in which those understandings can be productively influenced. By teachers' understandings of mathematics we mean "the loose ensembles of actions, operations, and ways of thinking that come to mind unawarely – of what they wish their students to learn, and the language in which they have captured those images" (A.G. Thompson & P.W. Thompson, 1996, p. 16). It is against the background of the images that teachers hold with regard to their own understandings and of the understandings they hope students will have that they select tasks, pose questions, and make other pedagogical decisions

In our recent work with pre-service teachers, we studied a group of PSTs who took part in a course designed to position the PSTs to develop a more advanced, connected understanding of the concept of function (in this case, the more advanced conception is function as covariation of quantities – a conception of function that is consistent with the calls of the NCTM *Standards*), and their subsequent interactions with high school students. Despite the fact that the pre-service teachers did develop a more robust, coherent understanding of functions as covariation of quantities – an understanding of function that supported their ability to speak conceptually about functional relationships – their instruction remained grounded in a variant of the more traditional correspondence conception of function. Analysis allowed us to characterize the pedagogical conceptions of functions that grounded the PSTs plans for instruction and work with the high school students. The conception of coordinate systems, functions and covariation that grounded the course instruction (Covariational Conception) and the PSTs' pedagogical conceptions are shown below in Table 1.

Covariational Conception	PSTs' Pedagogical Conceptions
Coordinate Systems are used to locate points, which are values of the variables that occur simultaneously	**Coordinate Systems** are where you plot points, which tell you information about corresponding values of the variables. The corresponding values are found by using rule for determining the specific coordinates. Once students' know the rule, they need to practice locating the points until they get comfortable with it.
A **Function** is a relationship between two variable quantities	**Functions** are what we use to find the corresponding values of the variables. Sine and cosine are periodic functions, which means that the graph repeats itself. It is important for the high school students to know values of sine and cosine of 30-60-90 and 45-45-90 right triangles so that they can see the values.
Covariation involves making sense of the way that two (or more) variables vary together. This involves considering how a dependent variable(s) vary over intervals of an independent variable	**Covariation** is everything that goes on between the points that you plot.

Table 1: PSTs Pedagogical Conceptualizations

Though space to provide details of the data and analysis is limited, these results are indicative of the study-at-large, which indicates that rather than teaching the way one was taught, one teaches what they know – an individual's understandings of mathematical content and their pedagogical conceptualizations of the content are the lens through which instructional activities with students are conceptualized. Thus, this research indicates that professional development efforts must be grounded in helping the PSTs develop particularly powerful pedagogical conceptualizations of the mathematics that they are to teach. It is only then that the PSTs mathematical and pedagogical understandings can support the orchestration of reflective conversations designed to position students to come to develop true mathematical understanding.

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CONTEXTUALIZING “MATHEMATICS” IN ELEMENTARY TEACHER EDUCATION

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Focusing on the experiences of students enrolled in the graduate school component of a 5-year elementary education program, this paper addresses the question: How do preservice elementary teachers contextualize “mathematics” in the teacher education program by placing it in relations with other program activity? We point to contextualizations made for (1) inquiry-based instruction and (2) the modeling of program emphases. We then highlight for consideration the issue of how these relations trace to schools and to students’ futures.

The Research and Its Significance

This paper is based on ethnographic dissertation research including interviews with 65 preservice teachers and faculty and many informal conversations, approximately 170 hours of observation in and outside of coursework, and the collection of artifacts such as transcripts and program requirement checklists. Where most mathematics education research is bounded, or at least focused, “within” content-specific coursework and fieldwork, this research addressed calls for ecological perspectives on research in teacher education (Wideen, Mayer-Smith, & Moon, 1998), but from a networks perspective—focused on activities and relations between them that we are always part of and producing (cf., Law, 1992; Nespor, 1994). This perspective supports tracing “mathematics” through a more extensive frame of program activity. Analysis was ongoing and drew on the suggestions of Becker (1998) and Coffey and Atkinson (1996).

Inquiry Across the Content Areas

> *We’ve gotten a lot of inquiry-based, child-centered instruction, which is great….It’s such an exciting thing to be able to not do worksheets, but have the children discover things on their own. That’s something that I’ve found has been true in all the classes, especially math and science classes….It’s kind of…shoved down our throats a little bit.* (Marian)

When graduate students talked about teaching mathematics, they often made reference to the “investigative approach” from their Baroody (1998) text in the *Math Methods* course—also discussed frequently by instructor, Paul. Students commonly linked the investigative approach with ways they were learning to teach science, social studies, and language arts. Content-specific graduate courses were also tied back to students’ undergraduate experiences in Early Childhood Education. Preservice teachers often interpreted inquiry-based and child-centered course ideas to represent what they should try to do in future classrooms—experiencing those ideas as closely interconnected and “relevant.” The important point is that mathematics instruction was regularly connected ideologically by students to instruction in other subjects as well as to “good” teaching in general. Typically *not* referenced in students’ making of ideological connections across the program were foundations (also see Goodlad, 1990) and core curriculum courses, among others.

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Modeling in Methods Courses

Paul shows us in the way he would show his students….He'll give us a problem and we have to figure it out on our own—kind of inquiry-based content….I just feel like I'm not learning more about any of these subjects, more…how I can teach [them] the best. (Veronica)

The Mathematical Sciences Education Board (National Research Council, 1989) is one of many groups to have posited a need for modeling in teacher education program coursework. Many graduate students made similar comments to Veronica. It was common for them to describe how, by participating as students in modeled settings in all their methods courses, they were learning the "best" ways to teach these subjects, and they often connected this back to program discourses related to "inquiry-based instruction" or the "investigative approach" in mathematics. However, students also commonly suggested faculty modeling and these program discourses to typify an "ideal" instruction that may or may not be possible in "real" schools.

Tracing Extensions to Schools and Futures

It's kind of hard now…because my [cooperating] teacher, even though he went through the same program as I did, he doesn't really teach the same way as they're teaching us to teach, like my professors are teaching me how to teach. (Veronica)

Students experienced diversity in field placements—spending time in suburban and rural county schools near the university and also in urban schools, and suggesting each to be very valuable. We took interest in how students more often suggested conceptual links between the program and county schools, but still, in looking for "inquiry" teaching, observed little "real" instruction anywhere to tie closely to the program. In a sense, students viewed kids, classroom management, and political and social dynamics as "context" for the "good" teaching modeled and supported in the program. This at times produced the program as flawed for being "ideal" or teachers as flawed for using practices other than "inquiry." Critical interpretations of political and social difference were limited in the program discourses and practices students connected.

This research supports the need for mathematics teacher educators to consider how and to what their work is linked and how to help students use *multiple* frames of reference, beyond "inquiry" (or "best") practices, to interpret fieldwork. Particularly in times of standards-based education emphasizing a de-contextualized pedagogy, important questions are: What and whose political and social purpose does the production of "context" serve, and what can or should be mathematics teacher educators' roles in helping more substantively integrate curriculum?

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QUASI-EMPIRICAL APPROACHES TO MATHEMATICS: RECREATING THE LAKATOSIAN CLASSROOM

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Science is characterized by physical principles inferred via systematic observation, hypothesis generation and testing through experiment. In typical high school science lab the validity of a scientific principle is put to test by performing a structured experiment. Some innovative teachers reverse this process by setting up a structured science experiment, in which students gather data and then try to infer the principle that works. Can the scientific method be somehow adapted to the learning of mathematics? In other words can teachers simulate the discovery of mathematical principles by using structured problem situations that result in the construction of particular cases and observations that eventually result in student discovery of the principle that applies to the various situations? In this short oral, several examples of the use of combinatorial mathematics in research studies will be used to illustrate the strengths and weaknesses of this approach for research that supports learning. The research findings indicate that problem selection is crucial if the teacher wishes to encourage a quasi-empirical methodology that culminates in a discovery in a manner akin to science. Several approaches to problem selection will be presented. Mitchelmore (1993) proposed a theoretical model of conceptual development in mathematics consisting of two phases: *abstract-general* and *abstract-apart*. When abstraction is linked to a large number of diverse situations, it is called an *abstract-general* concept. The links are crucial because they indicate "that the learner is aware that whatever properties or relations a concept summarizes are present in a large number of other situations" (Mitchelmore, 1993, p.49). This is analogous to a professional mathematician using proof techniques from his or her particular area to solve problems in a different area. *Abstract-apart* concepts are those in which an abstraction is developed only in reference to a few situations from which it originated, and hence it never becomes linked to any situations other than those from which it originated. The pedagogical goal of such research is to simulate Mitchelmore's (1993) theoretical model with the hope that students' who do discover mathematical principles via quasi-empirical experimentation realize the wide ranging and diverse applicability of these mathematical principles. Such a quasi-empirical approach to mathematics also meets several instructional goals outlined by the *Principles and Standards for School Mathematics*, chief among which are to encourage problem solving, mathematical reasoning, and encouraging an exploratory approach to mathematics. It is the author's hope that secondary school mathematics teachers will encourage such an approach in the classroom.

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RENEGOTIATING STUDENTS ROLES AND LEARNING PRACTICES IN MATHEMATICS CLASSROOMS

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Perspectives

How to support students' participation in collaborative learning environments is a topic receiving continued attention in mathematics education. Reform efforts focus on "math for all" and promote student engagement in meaningful discussions and sense making activities. (NCTM, 2000). Despite these efforts, we have yet to realize this vision. Reform-aligned classrooms demand considerable changes in students' participation (Corbett & Wilson, 1995; Heaton, 2000). Students are expected to listen to each other's ideas, justify their thinking and engage in more complex, open-ended problem-solving tasks. Such changes are not simply accomplished by telling students what to do or how to do it. From a sociocultural perspective, participation in any community, including a mathematics class, is an act of self (Wenger, 1998). Students understandings of their competencies, the meaning of their participation, and themselves as learners are relevant to engaging them in collaborative inquiry activities.

Data and Methods

As part of a research study focused on understanding the organization and development of collaborative learning environments, I examined changes in students' participation over time and factors that supported them in taking on new roles in the mathematics classroom. The focal case for this study was a ninth-grade pre-algebra class comprising twenty underachieving students. Data for this analysis comprised interviews with students at the beginning and end of the school year, student questionnaires, and stimulated recall sessions with groups of students using video clips from lessons. Fieldnotes and videotapes from lessons also informed the analysis.

Results

Changes in students' participation over the course of the year correlated with changes in students' interpretations of various learning practices. Teacher pedagogical moves seemed to play a critical role in fostering the observed changes in student participation. These moves explicitly attended to the meanings and purposes of various practices and created opportunities to negotiate their meanings. The (re)negotiation of meanings was particularly salient during the first several months of the school year.

Students' Interpretations and Understandings of Practices

There were two main categories of changes in students' understandings of various practices and their participation in these practices. First, students shifted in their orientation towards others, reconceptualizing the role of peers in their learning. At the beginning of the year, the role for others was limited and primarily social or affective. For example, students reported that working with others made math less boring, so they were more likely to do their work, and thus learn. Peers also could answer questions when the teacher was not available. At the end of the year, students reported a much broader role for others, encompassing social and intellectual

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dimensions. For example, working with others was generative and peers might explain something a new way. Their preferences for group members shifted as well, from friends to those who participated and were willing to discuss math.

Second, students reconceptualized the meaning of various classroom practices. Some of these practices were already familiar to students. However, the meanings students held for these practices did not necessarily facilitate their participation. For example, a teacher's question was interpreted as "I must be wrong" instead of "the teacher is interested in my thinking". Other desired practices were unfamiliar to students and needed to be established as part of how they participated in math class. For example, asking a "good question" and going to the board to share one's work were not ways of doing mathematics familiar to students. Not only were some of these practices unfamiliar, many did not necessarily even *make sense* to the students as ways to go about learning mathematics.

Teacher's Role: Negotiating the Meaning of Practices

The teacher actively negotiated the meaning of these familiar and unfamiliar practices with students. There were two identifiable patterns. First, the teacher helped students make sense of practices by offering verbal descriptions and her interpretations of what was happening and its value. For example, near the start of the year, Ken volunteered to do a problem. As he worked at the overhead, his pace was fairly slow and deliberate. The teacher offered intermittent commentary: "He is noticing a pattern over here" and a few turns later, "So now we're going to watch him brainstorm, let's watch him problem solve." In this way, the teacher brought into relief the important mathematical work taking place—Ken was being given time to "figure out," "brainstorm," and "problem solve." Thus, these times that could be construed as "just waiting" were recast as times when productive thinking and purposeful observing was taking place.

Second, the teacher negotiated meanings by making salient to students how their engagement advanced their mathematical understanding or problem solving. For example, a student Dontay had shared his thinking, which was erroneous, but which created a learning opportunity for the class. Later, the teacher recapped the events:

T: Now I like what Ron said: two corners make a line. And I like what Jay said, which was that it's a continuing of the same line, so that it's not a separate diagonal. And what was good about this is that Dontay came up here to show-

D: -and messed up

T: No. See you came up here that showed where some confusion, where half the class is, so that Ron and Jay could make a point, and I want to thank you for that Dontay.

Importantly, the teacher's stated interpretations were not made generically, referencing abstract situations. It was not simply that "mistakes are OK." Rather, she *demonstrated* the value of practices by identifying specific examples of success.

Conclusions and Implications for Practice

These findings broaden our understanding of how a community can expand its shared repertoire and implicate ways teachers can support students' participation in collaborative learning. The particular set of practices that needs to be negotiated depends upon students' past experiences and the desired participation structures (e.g., groupwork, whole class discussion). Findings suggest that teachers need to be encouraged to understand students' perceptions of practices and actively work with these meanings during lessons. Making explicit their own interpretations of practices and demonstrating the value of students' participation in these practices seems to also support collaborative learning environments.

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NUMBER LINES: STUDENTS ACROSS GRADE LEVELS MAKING MEANING THROUGH METAPHOR

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Introduction

Number lines can be a taken-for-granted part of the mathematical landscape in many classrooms. They are taped to desks, posted on walls, printed in books, and shown on rulers. Because number line representations are ubiquitous from early elementary through high school, it is easy to assume that students and teachers have shared understandings of what these lines represent and of what sorts of mathematical thinking they make possible. This paper seeks to complicate the idea that number lines have intrinsic meanings or singular interpretations. It does this by exploring the variety of meanings students have of the number line and by investigating the metaphors they generate and draw upon when working on problems that involve representations of the number line. Our project looks across grade levels, allowing us to consider the role that student development and experience with mathematics curricula may play in making sense of the number line. Doing so also allows us to challenge an assumption that flexibility between metaphors is a natural process and need not be addressed explicitly in classrooms.

Perspective/Theoretical Framework

The notion of metaphor used here comes from the theory of embodied cognition articulated by Lakoff & Núñez (2000). For them, metaphors are "the basic means by which abstract thought is made possible" (p. 39), and are developed though our lived experiences in the physical world. In working with number lines, children may draw on two distinct metaphors: one of the line as continuous and another of the points on a line as discrete. We used this theory to look at how students from first grade to high school made sense of the number line. However, when we talk about students drawing on discrete metaphors of the number line, we are not suggesting that they see the line as gapless and comprised of infinitely many discrete points. Rather, we mean that students see the line as a series of points separated by gaps. This image of the number line conflicts with metaphor of the number line commonly used by elementary teachers and textbooks, which portray the line as a continuous path where numbers are locations on the path and arithmetic operations involve movement. These notions of number as location and calculation as motion can be very different from the metaphors required by other representations of number (like base ten blocks, tallies, or algorithms). Yet, teachers and curricula may not explicitly address the differences between metaphors required by number lines, those required by other models, and ones used by children. The theory of embodied cognition, which emphasizes metaphor, allowed us to probe differences in students' interpretations of the number line and to explore the varied ways they made sense of number lines while solving problems.

Methods/ Data Sources

The researchers individually interviewed a total of six students in 1st, 4th, and 10th grades. The interviews ranged between 30-60 minutes and were video-recorded before being transcribed. Data sources include videotapes of the interviews, transcripts of the interviews and student work produced during the interviews. Students were presented with number line tasks designed to explore their thinking about quantity, addition, subtraction, and equality. Transcripts of these interviews were analyzed thematically. In all transcripts, researchers sought to identify the metaphors used by the child and the interviewer, and focused particularly on confusions that might be related to "conceptual opposites." The researchers asked the following questions to guide their analysis: What mathematical metaphors did children draw on and produce in order to make sense of the number line? In what ways were the metaphors productive or unproductive in developing their understandings of arithmetic and number concepts?

Findings and Implications

The students in this study utilized multiple metaphors, although most seemed to rely heavily on number as a collection of individual objects. In first grade, students demonstrated this by pointing to individual dots on the number line; in fourth grade, a student called the slashes on the number line "tallies;" and in high school, the student's use of the collection metaphor showed up in representations of the number line where order did not matter (e.g. he placed 11 *after* 12). This use of the metaphor of object collection caused difficulty for some children when it was ill-suited to the demands of a particular task. Some students did not adopt new metaphors even when the object collection metaphor was no longer helpful; however, a few students were able to use language about travel as a metaphor for movement on a continuous line move (e.g. "how far I can go"). Moving flexibly between metaphors of both continuousness and discreteness allowed some students to solve and discuss a wider range of problems.

Students also drew on unexpected metaphors to solve problems. Some of these metaphors were quite productive, such as a fourth-grade student's use of the teeter-totter to explain the equality of two series of hops on a double-sided number line or a first grade student's use of bears running errands to different stores to explain negative numbers. Students also drew on some unexpected (but useful) metaphors related to their social world rather than their physical bodies. They talked about numbers being "friends" or "nice" to help them understand problems. Other unexpected metaphors seemed unproductive for particular tasks. When solving 16 + 8 on the number line, one fourth grade student said she didn't count the 16 because "it wasn't there." This was her metaphor to explain why she had to start on the 17 to get the correct answer, but she seemed to have difficulty holding this idea, because she solved similar problems incorrectly.

Mathematically meaningful use of a number line requires students to draw on different metaphors depending on the given task. Teachers and curricula designers may need to make explicit efforts to help children recognize and draw on both continuous and discrete metaphors for the number line and also may need to pay greater attention to the individual interpretations students bring to the number line, especially those involving their social worlds.

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STUDENTS' USE OF REPRESENTATIONS IN THEIR DEVELOPMENT OF MATHEMATICAL REASONING AND MEANINGFUL PROOFS

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The purpose of this study is to document how a group of students built representations to solve challenging combinatorics tasks and then modified those representations in order to develop convincing arguments to justify their ideas to themselves and to others. In doing so, they built and extended their mathematical reasoning and developed meaningful mathematical proofs.

The perspective underlying this research is based on the view that when children are presented with challenging problem-solving tasks in an appropriately supportive environment, they have opportunities to build, refine, and link representations for their ideas, and these in turn can lead to the development of elaborate justifications (Warner & Schorr, 2004). As children build connections between and among different representational systems, their thinking and reasoning also develop (Davis & Maher, 1990). Classroom interventions that invite children to revisit, modify, and extend their earlier-built representations support children's development of argument, justification, and proof (Maher, 1998). In this way, fostering the ability to reason mathematically becomes a long-range goal rather than merely the objective of a unit of instruction.

Methods

As part of a sixteen-year longitudinal study involving the development of children's mathematical ideas, initiated in 1989, students have been engaged in problem-solving explorations in which they worked together to find solutions to problems and to build justifications for their ideas.[1] At the time of this component of the longitudinal study, nine eleventh-grade students volunteered to participate in after-school problem-solving sessions. Five of these students were a subset of the original group that had been involved in the longitudinal study in grades one through eight.[2] In grades three through five, they explored the same tasks that are the basis of this study, the Tower Problem and the Pizza Problem.

In the Tower Problem, students are asked how many different towers four cubes tall they can build with unifix cubes when there are two colors to choose from. In the Pizza Problem, students are asked how many pizza choices a customer has if a shop offers plain pizza and there are four pizza toppings to choose from. In the Tower Problem, the actual objects, towers, represent themselves. However, in the Pizza Problem, the actual objects, pizzas, are unavailable, and a

[1] This work was supported in part by National Science Foundation grants MDR9053597 (directed by R. B. Davis and C. A. Maher) and REC-9814846 (directed by C. A. Maher) and by grant 93-992022-8001 from the NJ Department of Higher Education. Any opinions, findings, conclusions or recommendations expressed in this work are those of the author and do not necessarily reflect the views of the National Science Foundation or the NJ Department of Higher Education.

[2] For further discussion of the earlier work of these students, see Tarlow, 2004.

representation of the problem must be constructed. The Tower and Pizza Problems have isomorphic mathematical structures, and their solution can be represented by a generalization that may be justified by either a proof by cases or a proof by induction.

Videotapes of each session, students' written work, field notes, and videotape transcripts provide the data for this research. A qualitative methodology for data analysis was employed. Students' representations, strategies, justifications, connections, and interactions, as well as the role of the teacher/researcher were coded, and the codes were used to identify and trace the students' development of representations, mathematical reasoning and proof making.

Results and Conclusions

When presented with the Pizza Problem, one student, Stephanie, asked her partner Shelly, "Um, do we just want to, um, plot out the pizzas, like, how we would do, like with, um, shirts and pants or towers [referring to problems that they had explored several years earlier]? Do you know what I'm talking about?" Shelly responded, "The tree diagram type thing." Stephanie and Shelly then used tree diagrams to represent topping combinations in their pizza choices. As they continued their investigation, they modified their earlier-built representation and used lists of letter codes for topping combinations and then began to formulate more symbolic representations, including numbers on Pascal's Triangle, to represent pizza choices. They developed a progression of representations that became increasingly abstract and symbolic.

The students moved back and forth between their representations as they developed their mathematical ideas. This ultimately aided them in finding a solution to the Pizza Problem together with a justification. They organized their pizzas by cases according to the number of toppings and then used a proof by cases to justify their solution. They also connected their cases to the numbers on Pascal's Triangle and explained the addition on Pascal's Triangle using pizzas as a metaphor. Furthermore, they noted the doubling pattern as the number of available toppings increased, explained their reasoning for the doubling rule using both pizzas and towers, and generalized the solution to the problem as 2^n *for n* toppings. Finally, they used their representations to construct a three-way isomorphism between the Pizza Problem, the Tower Problem, and the numbers on Pascal's Triangle.

Given challenging combinatorics tasks in an appropriately supportive environment, these students were able to build, revise, and refine representations to find solutions and to develop mathematical reasoning and proofs as convincing arguments to justify their ideas. This has important implications for teachers and researchers who wish to incorporate mathematical reasoning and proof making into the curriculum in a meaningful manner.

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MIND THE TEMPO: CHILDREN'S CONCEPTIONS OF SYMMETRY IN TIME

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Introduction

Testing the current conceptual/cultural boundaries of symmetry may result in a more nuanced, and arguably messy view of the concept. Studies examining basic understanding of symmetry are few and far between, and attaching symmetry to the larger idea of time is simply not done within the general confines of mathematical education research. Therefore, a non-traditional, tactile approach for examining children's thinking about symmetry is pursued as it goes beyond traditional conceptualizations of symmetry into an embodied, temporal one.

A Framework for Temporal Symmetry

Symmetry is an example of a concept that has strong common usage outside of the mathematical community. As a result of it too-often being "viewed as a collection of disconnected concepts," Leikin and colleagues (1998) define symmetry as "a triplet (S, Y, M) consisting of an object (S), a specific property (Y) of the object, and a transformation (M) satisfying the following two conditions: i) The object belongs to the domain of the transformation; ii) Application of the transformation to the object does not change the property of the object" (p.4). This is the basis on which I discuss the overall concept of symmetry; however, the type of symmetry examined herein is not one that is readily found in textbooks. The notion of temporal symmetry, where time is the medium through which the property of an object is transformed, is only one of many variations on the theme of symmetry: consider Polya's (1973) use of symmetry to explain the interchangeability of variables in some algebraic expressions, or Dhombres (1993) invocation of the concept to describe the relation between any two distinct mathematical proofs of one statement.

Yet, the messiness of this idea (or perhaps the decided shift away from the Gestalt-like, holistic view of symmetry experiences) does raise questions about the nature of its domain: is this a primitive conception of symmetry, or some sort of metaphor by which symmetry can be understood. Lakoff and Nuñez (1999) would suggest that even the most abstract mathematical concepts arise from basic human experience – from the way the body interacts with the world. In inventing mathematics, they contend, humans use metaphors to connect sets of ideas. It is my contention that children's conceptions of the traditional, static symmetry can be understood through their use of the metaphors of time and tempo.

Methodological Considerations

This study examines the conceptions of temporal symmetry through the use of linking metaphors of two elementary school children (Abe, age 8, and Ben, age 10), both of similar ethnic and socioeconomic background. The participants were selected due to their attendance in a non-school related tutorial program, and were volunteers. These particular boys attend the same school (different by two grades) in a small Midwestern city, and are of particular interest for one main reason. They come to the study with different formalized mathematics training in the concept of symmetry (Ben has explicitly studied the concepts of translation, reflection and

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rotation), which allows for commentary on how the primitive metaphors of time and tempo operationalize the symmetry concept for each participant.

Participants were interviewed in an individual setting. The interviews were both video and audio taped so that repeated observations and careful discursive analysis of their responses, as well as of their physical motions (in itself, a discourse), was permitted. Participants were given a small, one-octave keyboard with two mallets to explore and play with for a few minutes prior to the tasks. After this time, they were introduced to each picture and asked to play the shapes (Figures 1-9, with varying levels of traditional and temporal symmetry) on the keyboard; a brief interview followed, with probing follow-up questions. Both participants had at least 1 year of piano lessons, which made them conversant with the basic keyboard layout. Rather than use a hierarchical, level-based framework such as van Hiele levels or the SOLO taxonomy, I choose a more naturalistic, ethnographic analysis of the words and actions used by participants to represent the figures, as well as a simple measurement of any consistent tempo in what they play.

Findings

Overall, a subconscious awareness of symmetry was present. Tempo seems to be subconscious for the two participants; which is supported by the strong presence of a consistent tempo (16 out 18 tasks had less than 5% overall variance in speed), and their lack of concern for whether changes in tempo alter the "sameness" of two interpretations. The fact that both used language to evoke equality of some aspects within the same Figures suggests that there is a latent awareness of symmetry within them – perhaps the use of tempo is a subconscious externalization of this awareness. If that is the case, then referring to this awareness as a kind of temporal symmetry may be extraordinarily valuable in allowing the explicit inclusion of dynamic, holistic and kinesthetic discourse into the typically static geometry found in textbooks.

In terms of temporal symmetry, Abe presented more references to concrete objects, time, and motion as reasons for his playing. This certainly could be due to a lack of technical vocabulary, yet I find it heartening for the future of temporal symmetry that both participants were quick to play and describe tasks 8 and 9 ('bubbles' and 'feet') in those same types of terms. Using the language of motion may simply be something that is increasingly reserved for 'real' objects as students of geometry get older – increasing abstraction calls for decreasing realism. Yet, more evidence is certainly required for making any broad claims.

Figures

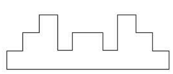
Figure 1: "stairs"


Figure 2: "crown"


Figure 3: "star"

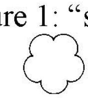
Figure 4: "flower"

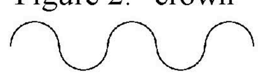
Figure 5: "wave"

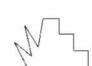
Figure 6: "semi-symmetry"

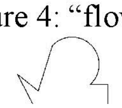
Figure 7: "non-symmetry"

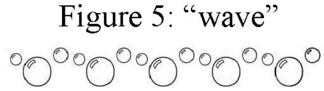
Figure 8: "bubbles"


Figure 9: "footsteps"

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PRE-SERVICE TEACHERS' VIEWS OF THE TEACHING OF MATHEMATICS: THE IMPACT OF A PEDAGOGY COURSE

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Objectives

Available literature indicates that prospective teachers enter formal teacher education programs with ideas and thinking about subject matter, teaching and learning and about schools which they have formed from their long experience as students. The need to challenge these preconceived ideas stems from their potential to influence what pre-service teachers learn from their courses and field experiences (e.g., Calderhead, 1991) by causing them to dismiss alternative ideas about the teaching of mathematics provided in teacher education as theoretical and unrealistic. Pre-service mathematics teacher education therefore needs to take into account ideas teacher candidates bring with them to teacher education programs, in order to develop ways of challenging their views and extending what they know, believe and care about. In the light of this, the study reported in this paper was designed to investigate ideas about mathematics and the teaching of mathematics which pre-service elementary teachers brought to a pedagogy course in mathematics and how these ideas changed or did not change during the semester, as well as, what was responsible for any possible changes.

Theoretical Framework

The design of the study warranted a framework that would aid effective conceptualization of the baseline data. Thompson (1984) investigated three teachers and conceptualized their attitude towards mathematics, their views about mathematics, their beliefs about the teaching of mathematics and how these affected the decisions they made in their classrooms. Thompson (1984) was used as a framework because his conceptualizations were very useful in characterizing the views held by participants of the present study.

Modes of Enquiry

This study took place during the Fall semester of 2003. It involved pre-service elementary school teachers in a pedagogical course in mathematics at the Michigan State University. To document their initial ideas about mathematics and what they perceived mathematics teaching involved, students in this class were asked at the beginning of the semester to write brief philosophy statements highlighting their ideas about mathematics and the teaching of mathematics at the elementary school level. Then in the middle of the semester and during the last week of the semester, students were asked to reflect on these initial philosophy statements and revise them according to whether they felt their initial ideas were changing or not, and to describe what might be responsible for any changes, if any, that they experienced. To get students to freely write about their true experiences students were assured that these philosophy statements were not going to be graded but were needed to study how the course would benefit them or otherwise; beneficial information for future decisions about course content and practices.

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Data Sources

As already mentioned, participants for the first part of this study were prospective elementary teachers in a mathematics pedagogy course at the Michigan State University in the Fall of 2003. In all 26 students took the class. However, only 10 of these students were selected for the study because they were the ones who voluntarily wrote initial philosophy statements as well as the two reviews required during the semester. The data therefore consisted of self reports of the selected students about their initial ideas about mathematics and the teaching of mathematics prior to the taking of the course, how these ideas changed in the course of the semester and the source of this change.

Results and Conclusions

Analyses of data revealed that on the whole, very small things changed. Specifically, some students still maintained static and more prescriptive view of mathematics but changed only slightly from an emphasis on rote memorization. Others showed slight movement toward socio-constructive ideas of teaching, which they attributed to a combination of their interaction with peers in course discussions and their experiences in their field placements. At the same time, there were those whose ideas did not change during the semester; they believed in emphasizing procedural understanding right from the beginning to the end of the semester. Students' reference to their experiences as a source of change in their ideas is consistent with the call for the need to provide students with examples of teaching practices under realistic conditions (Fieman-Nemser & Remillard, 1996) as a means to challenge their preconceived ideas. In addition students' reference to peer discussions of course readings as another source of change in their perception is consistent with socio-constructivists notions of situated cognition and supports the need to provide opportunities for pre-service teachers to engage each others thinking in a critical and reflective manner.

Relationship of Paper to the Goals of PME-NA

Currently there is a growth in interest in discussion of the type of experiences needed to challenge pre-service mathematics teachers' initial ideas and improve their knowledge base for teaching (e.g., Mousley & Sullivan, 1997). This study contributes to this literature by adding to existing knowledge of the kind of ideas teacher candidates bring to their pedagogy courses and the type of experiences they perceive as effective for challenging their views. It also contributes to on-going discussions about improving pre-service mathematics teacher education.

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EVOLVING RESEARCH FRAMEWORKS:
VIDEOTAPES AS A TOOL FOR DIALOGUE

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Researchers at Project 2061, the University of Delaware and Texas A&M University are in the third year of a five year large-scale research project. The purpose of the project is to learn how middle grades mathematics teaching can be improved through professional development. There are several assumptions that undergird the study. First is that the goal of mathematics teaching is "teaching for understanding" (Stigler & Hiebert, 1995). Effective teaching is defined by a set of instructional criteria that are adapted from those used in Project 2061's evaluation of middle grades mathematics textbooks (American Association for the Advancement of Science, 2000). The design of the study is also based on the assumption that teachers can improve their practice through opportunities to analyze and reflect on their own practice. Using a teacher-as-researcher model, teachers are encouraged to examine evidence and develop conjectures about instruction, in a collaborative problem-solving environment.

We have collected videotapes of lessons from more than 50 teachers over three school years, across several sites in both Delaware and Texas. All of the teachers involved in the study are using curriculum materials carefully examined as part of Project 2061's middle grades mathematics textbook evaluation. The videotapes represent a total of four to six lessons recorded in each teacher's classroom each year. The taped lessons were carefully selected to align with specific learning goals, one in number (grade 6), one in algebra (grades 7 & 8), and one in data (across all three grades).

As we study the videotapes of these sets of lessons, we are looking at patterns in teacher behavior and connections to student learning. These findings are the basis for the teachers' professional development. In turn, we are attempting to determine what teachers are learning from this professional development, as illustrated in follow-up classroom videotapes, survey results, and specially-designed tasks. In addition, we have collected data on student learning outcomes for each teacher on each learning goal for which we have videotape data.

In the course of our study, we have found the use of teachers' own videotaped lessons to be a compelling and stimulating resource for both the teachers and the researchers. Although initially, and predictably, intimidated by seeing themselves on camera, teachers quickly became engaged in the actions in the classroom. Through multiple observations of a lesson or set of lessons, teachers have identified evidence of particular instructional practices aimed at specific learning goals, and of student learning. Examination and analysis of the videotapes, for both the teachers and the researchers, have been based on a subset of the 24 Project 2061 instructional criteria. Specifically, teaching is analyzed for how effectively teachers employ mathematical representations. It is also analyzed for the questions teachers use to uncover students' prerequisite knowledge, to encourage students to express, justify, and clarify their ideas, to guide student interpretation and reasoning, and to probe student understanding.

Lloyd, G. M., Wilson, M., Wilkins, J. L. M., & Behm, S. L. (Eds.). (2005). *Proceedings of the 27th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.*

Our research questions for this study included considering how professional development and ongoing support—focused on specific mathematics learning goals— build teacher knowledge and lead to more effective teaching practices. As we began to see the impact of video data on teachers, our use of videotapes evolved and more specific questions developed, including questions about what both researchers and teachers can learn from videotaped observations of teacher practice and its effect on teachers' instructional practice.

The teachers were engaged in three years of professional development, in the form of summer institutes and follow-up sessions during the school year. The focus of the work was on the mathematical ideas in the three learning goals, as well as a small number of the instructional criteria referred to above. Teachers used classroom videotapes and student assessment work to look for evidence of student learning.

We developed instruments for the purpose of assessing student learning for each of the three learning goals. Students of the teachers in the study were given pretests prior to the lessons selected for videotaping, and posttests following those lessons. The assessments provide information about the breadth and depth of students' procedural and conceptual knowledge of each of the learning goals.

To analyze the classroom videotapes, we developed a web-based utility for documenting analysis of the lessons. Trained analysts look for instances in the lessons of enactment of the instructional criteria used in the study and identify some aspect of the learning goal that is addressed in that instance. Each criterion has a small number of indicators that specify some aspect of the criterion. Analysts rate each of the indicators according to a predetermined scale and provide justifications for their ratings.

With the tools at hand we are searching for patterns in teacher classroom behavior across the three years of the study, and expect to be able to link instructional practices with student learning outcomes. Our preliminary analysis lends credence to several potential findings. The first is that engaging teachers in the work of analyzing their own videotapes is a powerful tool for influencing how teachers think and talk about their instructional practice. We have found that the teachers involved in this research routinely report that they have reconceptualized important aspects of both the mathematical content they teach and their daily pedagogy. Most importantly, they describe developing a habit of seeing that content through the lens of their students' thinking. This in turn compels them, they say, to become more skillful at eliciting that thinking within the context of instruction. Determining whether or not this reconceptualization results in measurable changes in daily instruction and student learning is the ultimate goal of our project's analysis.

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PREDICTORS OF SUCCESS IN COMPUTER AIDED LEARNING OF MATHEMATICS

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Purpose of the Study

Mathematics achievement has been of great concern to researchers involved in mathematics education. This concern led to the seeking of the factors that affect positively or negatively student performance in mathematics. The factors relevant to mathematics teaching and learning have been studied in general. In this study, we investigated the factors contributing to student achievement in mathematics when learning takes place in a computer-aided environment.

Theoretical Framework

Several variables have been identified as predictors of students' achievement in mathematics. It has been observed that many of these variables reside within the student himself (Begle, 1979). In his meta-analysis, Begle (1979) identified and categorized students' variables into six, but concludes that the variables that have been studied for the prediction of success are mainly the *affective*, *cognitive* and *non-intellective* variables. We selected seven variables from these three main categories of student variables with the aim of investigating their effects on students' achievement in a pre-calculus course supplemented with a computer lab program. The selected variables were: mathematics attitude, mathematics aptitude, computer attitude, computer prior experience, computer ownership, proficiency in language of instruction, and learning style. The selection of these variables was informed by their recurrence in the theoretical and empirical literature on the subject.

Methods of Inquiry

The methodology of the research was experimental and quantitative in nature. The participants of the study consisted of 120 students sampled from a population of students enrolled in a second pre-calculus course at King Fahd University of Petroleum & Minerals (KFUPM, Saudi Arabia) during 2003/2004. The participants were all male bilingual Arabs who were learning English as a second language. They all had undergone a complete semester of learning pre-calculus in a blended mode of teaching. The weekly schedule consisted of three normal classroom lectures and one computer lab session. The lab was based on MATLAB and WebCT, the former as a problem solving tool and the latter as an online course development, delivery and management tool. Students used MATLAB to solve problems and do weekly homework. On the other hand, WebCT was used for the submission of homework, receiving homework solutions and having an access to additional course material along with cyber discussion on mathematical issues. Prior to the start of the experiment, a comprehensive

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students' lab manual was developed by the researcher with the help of two colleagues under the Summer Special Assignment Grant of KFUPM.

Data Sources

Data was collected with the aid of five instruments. These were: the mathematics attitudes scale (Aiken, 1979), the computer attitudes scale (Loyd & Gressard, 1984), the learning styles questionnaire (Honey & Mumford, 1992) and two others developed by the researchers to measure computer prior experience and computer ownership. Data were collected twice, at the beginning and at the end of the experimental semester (Spring, 2003-2004). The data collected at the beginning of the experiment (independent variables) were relevant to the characteristics of students based on the seven selected variables. It was collected at the first week of the semester through the structured questionnaires for all but the variables: mathematics aptitude and English language proficiency. These two variables were measured from the students' performance in preparatory Math I (MATH 001) and preparatory English I (ENGL 001) in the preceding term of the experiment. Data (dependent variables) concerned with the students' achievement was collected from the letter grades of the sampled students at the end of the experimental semesters.

Results and Conclusions

Hypotheses formulated in the study were tested by using multiple regression. The results indicated that the participants had positive attitudes toward both mathematics and computers. The students with high proficiency in English outperformed the others. The variables mathematics aptitude and English language proficiency were the most significant contributors to students' achievement in mathematics. Together, both variables explained about 41% of the total variance of students' achievement. Other factors that contributed included Activist (among the learning styles) and computer prior experience of 4-6 years. None of the other variables showed any statistically significant effect on the students' achievement.

Relationship of the Paper to the Goals of PME-NA

The present investigation is an extension of numerous studies that have been conducted in the USA and other developed countries. Nevertheless, the academic literature on computer aided learning (CAL) still lacks sufficient information about CAL in many developing countries; therefore, making a generalization of the available knowledge on CAL is difficult. Hence, extending the research on computer usage in mathematics to developing countries will provide us with the information needed to improve our understanding and to make a considerably accurate generalization about the impact of CAL.

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THE RATE OF CHANGE FROM THE NUMERIC POINT OF VIEW

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Introduction

Several authors have confirmed the significance of introducing the derivative concept to students through the use of rates of change. The software proposed in this study, and the activities it entails, embraces this approach, highlighting its visual aspects. Deborah Hughes (1990, 1-8) has observed that many students may algebraically calculate the derivatives of several functions, however, they are incapable of identifying in which points of the graphic the functions have a positive derivative and in which a negative one. In addition, this author notes that only in rare occasions the numeric approach has been used to teach the derivative concept. Jere Confrey (1993) indicates that the presence of numeric tables may illuminate (1) the functional connection among the values they contain and (2) the algebraic presentation. Daniel Scher (1993) has studied the multiple representations to conceptualize the derivative. The author concludes that it is necessary to promote the use of such representations in order to provide the student with an effective understanding of calculus concepts. He mentions, for example, that "the notion of rate of change must be accessible to all students" (Scher, 1993, 16).

In the development of this work, I was able to identify that the idea of the increment of a variable is not easily understood by students. The software, by which the experimentation performed here was conducted, presents activities that highlight the numeric register of functions and the graphic register in which families of functions are presented and can be manipulated to obtain information.

Experimentation

The experimentation was performed with five mathematics high school students, during twelve hours divided in four sessions. They worked in a room equipped with three computers, one blackboard and two video cameras. Three teams were formed (two with two students and one with one); each team worked in a computer with the developed software. In the first session, the students learnt how to navigate in the software's package; with this knowledge, they would freely navigate through the allowed contents of the software in the next sessions. The instructor, present during the experiment's sessions, was basically an observer, however, he could intervene to answer questions, when these were required, or to pose questions that allowed the students to found by themselves the correct strategy. The students could freely communicate their ideas or their solving strategies, all of which was recorded in video.

The experiment involved the part of the software that corresponds to numeric treatments (progressions, increments and rates of change). Firstly, it entailed the work with arithmetic progressions. The software generates, randomly, a table with empty spaces and the task of the user is to fill and complete them (see figure 1); the software evaluates the introduced data and shows if it is correct or incorrect. This first feature of the program has an introduction and four levels.

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General Observations

The students did not have any problem with the software's navigation and rapidly understood the task they confronted. They had some conflicts in defining their appropriate strategy; however, in the end, they succeeded.

Analysis of the Experimentation in Relation with the Contents Presented

Based on the records on video, the analysis of this experiment will try to explain if the students understood the ideas of increments in variables and rate of change. Moreover, this analysis will be a first step to glimpse the possibility that an approach through the rate of change' function allows the students to transit to the derivative concept.

Option of Progressions

In this first task (or option) of the software, the students are presented with an introduction of what is an arithmetic progression and with four levels of exercises. All the students understood well the introduction and the task they had to perform. Level I and Level II did not present any problem for the students and they easily found the required solutions. However, in Level III and Level IV it was very difficult for the students to find a correct solution for the type of exercise proposed. Only one team found a strategy to solve the required exercises of Level III. Next, there is a description of how the team performed in this task.

Elizabeth and Leticia are trying to solve the following exercise:

Position	1	6	25	42	52	53	81
Value	4	14	52				

Elizabeth: "let's see how much is… [*she starts writing in a notebook, counting*] 17 by 2,are 34 and then we add 52".

Researcher: "Can you explain me how did you obtain it?"

Elizabeth: "There are five spaces between 1 and 6. I know that if 1 equals 4, and there are 2 spaces between one and the other, and they are incrementing 2 by 2, then we need to subtract 42 minus 25 to find the spaces; then we multiply by 2 and add the value of 52."

Leticia: "We find the space of one side and the other and, as we already know they go 2 by 2, and [given that] from 52 to 53 there is only one space, we multiply it by 2."

Researcher: "That number you obtained is very important [number 2]. How did you find it?"

Elizabeth shows me a table and explains it to me:

Position	1	2	3	4	5	6
Value	4	6	8	10	12	14

They had completed the table with the values that lacked between 1 and 6, and determined that there was an increment of 2 by 2 in each position. It was suggested to them that they should revise the option of increments and the definition of rate of change. After they had reread the option and the explanation of what was a rate of change, they concluded:

Elizabeth: "I got it! What is happening is that with the increment we give it, we can find it by dividing the increment of x between the increment of y."

Leticia: "It is the opposite way".

Elizabeth: "And we save ourselves what we were doing."

As is possible to observe, in order to solve this type of exercises, it was necessary to use the concept of rate of change—all of which was achieved by this team. In the next tasks they already had this idea and they could apply it.

Conclusion

Through the use of tables representing values of functions, the students can understand and use the rate of change. With this knowledge, they begin to construct a new function from which we, as educators, may introduce the derivative function.

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UNDERSTANDING CHANCE: FROM STUDENT VOICE TO LEARNING SUPPORTS IN A DESIGN EXPERIMENT IN THE DOMAIN OF PROBABILITY

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Six middle-school students participated in pre-intervention interviews that informed the design of learning tools for a computer-enhanced experimental unit on probability and statistics. In accord with the PME-NA XXVII conference theme, we elaborate our methodological frameworks and design principles. Our design was in response to students' failure to solve compound-event problems. We characterize student difficulty as 'ontological fuzziness' regarding the stochastic device, its combinatorial space, and individual outcomes. We conclude that students need opportunities to concretize the combinatorial space. Also, we conjecture that, given suitable learning tools, students could build on their comfort with single-outcome problems to solve compound-event problems. To those ends, we designed the '9-block,' a mixed-media stochastic device that can be interpreted either as a compound sample of 9 independent outcomes, a single independent event, or as a sample out of a population. We explain activities around the designed tools and outline future work on the unit.

Introduction

This paper reports on an empirical study of student mathematical cognition in the domain of probability (see the '*Connected Probability'* project, Wilensky, 1995, 1997). In accord with the PME-NA XXVII conference theme, "Frameworks That Support Research and Learning," the paper foregrounds the methodology employed for this study. Specifically, we discuss student difficulty with problems involving compound events, e.g., three coin tosses. We explain how a *design-research* framework enabled us to respond to student difficulty with tools designed to support students in building from what they know towards fluency with this class of problems.

We begin with the methodological frameworks and design principles of our research, then lay out the theoretical background and data resources of the study. Next, we describe a set of pre-intervention interviews, in which students worked on probability problems. We explain how our interpretation of student difficulty in these problems shaped a design rationale for innovative tools and activities, which we developed and then implemented in an experimental classroom unit. In this unit, students: (a) analyze stochastic devices that produce compound events; (b) classify the combinatorial space of these devices into subclasses and produce and assemble this space into a physical structure (a *combinations tower*); (b) interact with computer-based simulations of probability experiments, which draw from the same sample space and stack outcomes by the same subclasses; (c) compare products of these activities, i.e., students compare the shapes of the 'theoretical' combinations tower and the 'empirical' outcome distribution; and (d) participate in statistics activities in which samples are items from the same combinatorial space of the original stochastic device. The paper ends with future directions for this research.

Methodological Frameworks

At the Center for Connected Learning and Computer-Based Modeling at Northwestern University, we study student mathematical cognition and develop learning tools and activities.

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Design research (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003) is a useful paradigm for our research work, in that it affords us immediate and rich feedback on the efficacy of our tools in supporting student learning. Our goals are both pragmatic and scholarly: (a) our theoretical perspectives, methodologies, and design principles are all aimed at probing student domain-*specific* challenges, to which we attempt to respond by engineering, creating, and field-testing educational artifacts; yet (b) we take from our studies domain-*general* insights—"humble theories" (Cobb et al., 2003)—which we share with the community of education researchers and practitioners and which we incorporate as new theoretical lenses upon data harvested in future studies. In sum, our studies investigate: "What is needed?," "What works?"; and "How does it work?," "What does this mean?" Also, we investigate how new technology may shape content.

An integral methodological component of design-research studies is the elicitation of student understanding and difficulty before, during, and after implementations of the designs. We interview students (Ginsberg, 1978), listening closely to their 'voice' (Confrey, 1991) and, in response, we create, modify, and introduce learning tools into implementations, often while the implementations are underway. We aim these tools as "Vygotskiian" supports—they constitute forms for students to articulate their understandings so as to hone, express, and struggle with their difficulties. That is, we embrace 'difficulty' as a positive cognitive and motivational factor stimulating individual problem solving and communication in the classroom forum.

Toward designing new learning tools, we survey literature on: (a) ontogenetic; (b) philogenetic; and (c) urban/rural ethnomethodological aspects of the target mathematical concepts; as well as (d) previous studies that evaluated, analyzed, and responded to student difficulty with these concepts; and (e) national and state standards and high-stake assessment studies. These resources inform an emergent domain analysis that situates the concepts vis-à-vis students': (1) familiar situational contexts; (2) cultural practices in which the concepts are (implicitly) embedded; (3) mathematical representations students are likely to recognize; (4) the vocabulary of the domain; and (5) K-12 roots and trajectories leading to and from the concepts.

Design Principles

Our educational design work is informed by reform-mathematics pedagogy and agenda (e.g., von Glasersfeld, 1990; Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997). We embed these constructivist and social-constructivist perspectives into the learning tools and classroom facilitation infrastructures we design so as to promote high-level mathematical discourse with an eye on engagement and equity. Toward these goals, we create project-based collaborative-construction activities, using both traditional and computer-based tools, so as to foster student personal and interpersonal construction of knowledge. Our design is further guided by the following perspectives and principles.

a. Constructionism. Students will best learn through voluntarily developing new skills to problem-solve the design, engineering, and construction of their own artifacts (Papert, 1991).

b. Connected mathematics, connected probability. Wilensky (1997) argues that standard mathematical curricula are ahistorical and 'acognitive'—they do not enable learners to experience the problems that stimulated the invention of current mathematical solution procedures (see also Lakatos, 1976). This results in *epistemological anxiety*—students know that their procedural knowledge is correct, yet they do not know why. Specifically for the domain of probability, Wilensky demonstrated how even mathematically-informed adults greatly gained from activities that enabled them to 'connect,' i.e., to ground the content in their intuition.

c. Learning axes and bridging tools. A *learning axis* (Abrahamson & Wilensky, 2005a), a theory-of-learning construct, is a 'space' extending between two necessary and complementary components of a mathematical concept. These conceptual components, each within the learner's comfort zone, are concurrently afforded by a single *bridging tool* (Abrahamson, 2004; Fuson & Abrahamson, 2005), an "ambiguous" artifact designed to support students in linking up previous understandings, situational contexts, procedures, and vocabulary. Students learn by reconciling the *learning issues*—tensions between the bridging tool's competing conceptual components.

d. Stratified learning zone. A *stratified learning zone* (Abrahamson & Wilensky, 2005b) is an emergent and undesirable pattern in classroom participation in collaborative construction projects, in which a subset of high-achieving students have greater learning opportunities, and the more "menial" roles are assigned to other students. We seek to promote more equitable participation through implementing specialized facilitation infrastructures that sustain all students' engagement in the core problem-solving tasks without forsaking collaboration.

e. The resourceful classroom. Optimally, a teacher should be: (a) familiar with all media and tools; (b) fluent in the domain and design; and (c) flexible in navigating between resources in response to student initiative. Implications for professional development are that teachers should have opportunities to practice operating the various media and to anticipate student response.

f. Mixed-media learning environments: Abrahamson, Blikstein, Lamberty, and Wilensky (2005) describe projects in which a range of technologies and expressive tools are integrated into learning environments so as to enable multiple entry points. Students, who come to these flexible environments with different skills, inclinations, literacies, tastes, working habits, and passions, have increased opportunities for expression and for development of expertise. The artifacts students create in these environments reside after the implementation ends, acting as classroom referents in future discussion toward the social construction of further knowledge.

(For domain-*specific* design principles emerging form our research, see Abrahamson, 2005.)

Theoretical Resources of the Study

In their influential *prospect theory* papers, Tversky and Kahneman (e.g., 1974) demonstrated the fallibility of human probabilistic reasoning in decision making. For example, in one set of studies, their participants were asked to consider a battery of hypothetical situations and then evaluate the likelihood of statements pertaining to each situation. The experimental findings did not augur felicitous prospects for students studying probability and statistics. Yet, a tangential perspective on human reasoning posits that mental capacities pertinent to the study of probability may have evolved under conditions far removed from the ecologies of managers reading texts (Wilensky, 1991). From such a perspective, Gigerenzer (1998) advocates a reformulation of curricula so as to accommodate humans' *ecological intelligence*. For example, the formal notation of probability, e.g., ".7," pithily captures the anticipated ratio of favored events out of a set of random events, but a *natural frequency* representation, such as "70 out of 100," would better accommodate the way organisms encounter information. In like vein, this paper describes a study designed to identify breakdowns in students' informal probabilistic reasoning; examine the possibility that such breakdown is due to shortcomings in the formal expressive tools available to the students; and create tools that support learning trajectories. Specifically, we aim to support students in sustaining an understanding of—a *connection* (Wilensky, 1991) to—new procedural skills, as they learn probability (see also Konold, 1994; Metz, 1998).

Our research questions, coming into this study were: (1) What are 8th grade students' entry understandings that are relevant to the subject of probability?; and (2) What learning supports—

objects and classroom activities—could capitalize on students' entry understandings so as to help students develop understandings of more advanced situations? The paper discusses a set of interviews we conducted prior to a teaching–learning intervention in a middle-school classroom. We analyze student utterance and explain the learning supports that we designed in attempt to respond to our interpretation of this input in light of reasonable learning objectives.

Data Sources

The data are from an intervention that is part of a sequence of design-research studies conducted in urban middle-school mathematics classrooms, where we are investigating student learning of probability and statistics using learning supports of our design. Participants were representative of school demographics (25% White, 24% Black, 25% Hispanic, 24% Asian, 2% Native American; 26% ESL; and 63% eligible to free or reduced lunch).

Methods

We conducted pre-intervention interviews both to gauge G8 students' entrance knowledge into the experimental unit and as a datum line for measuring student gain. We selected 6 students (3 male, 3 female) sampled from three teacher-reported performance groups. All pre-interviews were conducted by the first author and lasted 14 minutes on average. In these *semi-clinical interviews* (Ginsburg, 1978), the student was given a problem to solve, and then the researcher and student discussed the student's reasoning. The problem was, "A fair coin is to be tossed three times; What is the probability that 2 heads and 1 tail in any order will come up?" (NCES, 2004; the solution is 3/8 or a .375 probability). The coin item was chosen both for its content and because only 3% of G12 USA students had solved it correctly (NCES, 2004). We wished to investigate the sources of student difficulty and probe for student understanding that could potentially be leveraged. The interviews were videotaped. Following, we conducted *microgenetic analyses* (Schoenfeld, Smith, & Arcavi, 1993) to identify and typify when and why each student moved from secure to tenuous grounds in attempting to solve the problem. Next, we compared students' problem-solving paths to reveal cross-student similarities and patterns. This analysis generated conjectured learning trajectories through the subject matter and informed a domain analysis towards a design rationale for developing tools, which this paper will overview.

Results and Discussion

Pre-interviewed students could not complete the solution of the compound-event item. Yet, they solved correctly single-outcome problems that each of them initiated in various forms. We will now demonstrate a data sample and then analyze and discuss all the data.

Data Sample—A Student Discusses Probability Problems

In working on the coins problem, a student, described by the teacher as below average, said:

> I think you should add more information about the math problem itself. Like, not just say, 'A fair coin is to be tossed three times.'

When asked what additional information she needs, she answered:

> Well, I don't know how to *say* this… that's my problem…[11 sec silence] Ok, I think that… Ok, the first line's perfect. But, I don't know how to tell you what I want in the second… That's what… It's getting me. Uhhm… [8 sec silence] I don't know what to say. I don't know, I don't know… I don't have the right words to come out.

She suggests a simpler problem, in which there is a box of 10 candies—6 caramel and 4 chocolate—and one is to determine the chance of getting a chocolate. She solves this problem ("4-out-of-10 chance"). When asked how this compares to the coin problem, she said:

> In the box of sweets, you know how many there is. You know that there are 6 of those and 4 of the rest….But here you don't know….I mean, you know there's 3 chances that you can get heads and 3 chances that you can get tails….But you're not as… you're not, like, as… as descriptive as you are on the other one….It *is* the same problem, because you're talking about the same concept….And, uhhm, but, it's not the same... well, you *could* say [that in the coin problem, analogously to the candy problem] there's a box, because it's like you're inferring what would happen.

From a Domain-Analysis Perspective on Student Voice Toward a Design Rationale

Ontology of probability. Student utterance revealed "ontological fuzziness" (see also Piaget & Inhelder, 1975; Wilensky, 1991) regarding three key elements of the domain of probability: <u>the stochastic device</u> (e.g., the coin[s]), the <u>combinatorial space</u> of all possible outcomes, including favored and unfavored events (e.g., HHH *HHT HTH THH* HTT THT TTH TTT), and <u>specific outcomes</u> of the sampling action (e.g., THT). Of these three elements of stochasm, only the stochastic device is an a priori substantive tangible object of consistent appearance. The other two elements—the space of all outcomes and the specific outcomes—either cannot be directly seen (the combinatorial space) or they are constantly changing (specific outcomes). One source of student confusion could be that they had only worked on single-outcome problems, in which there is congruence between items in the sampling "population," e.g., each and all of the green and blue marbles in a box, and the space of all possible outcomes of the stochastic action, e.g., all single marbles that can be drawn from that box. Also, note that constructing a combinatorial space wherein symbols replace icons demands advanced representational skills. Informed by this analysis, we concluded that students need opportunities to 'concretize' (Wilensky, 1991) both the combinatorial space and specific outcomes.

Modeling compound-event situations on single-outcome situations. Students' relative comfort with single-outcome probability problems suggested they may be able to use the *single*-outcome model recursively in problem-solving *compound*-outcome problems. For instance, to concretize the combinatorial space in the 3-coins problem, one could first determine all eight possible outcomes and write each outcome on a slip of paper. Then, one could put these eight slips in a box and select one at random. Thus, the "candy-box model" can apply to compound-event situations. Following, we describe an analogous stochastic object that uses marbles.

Design Solutions

In this section we present some of the stochastic objects and activities we designed in response to student difficulty, as evidenced in the pre-interviews and analyzed above.

Figure 1. Design, development, and use of the marble scooper. From left: an image of a computer model for the 3-D "print"; the final product; scooping from a box of marbles; a teacher shows the classroom a scooped sample of 9 marbles; a "9-block" for combinatorial analysis.

The marble scooper and the 9-block. The *marble scooper* (see Figure 1, above) is a device for sampling a fixed number of marbles out of a vessel containing many marbles, e.g., an equal number of marbles of two colors. We have built a scooper that samples exactly nine marbles. A sample (Figure 1, second from the right) may have, e.g., 5 green marbles and 4 blue marbles.

The scooper is a unique stochastic object—unlike in coins or dice, a particular sample is not an inherent physical aspect of the device but is constituted only through an interaction between the device and the "population" of marbles. Decoupling the stochastic *object* from its *outcomes*, the scooper may help in conceptualizing the combinatorial space of the 9-marble compound event as a spatial "variable" with color "values." Also, the intrinsic 2-D spatial form of the scooper, an array, suggests a topology for managing the construction of the space—an outcome is not just, e.g., "5green/4blue," but a specific arrangement of these. To help students concretize and build the space of all different possible 9-marble combinations, we designed the *9-block* (Fig. 1, on the right), a 3-by-3 square grid in which each small square can be either green or blue.

Figure 2. 9-block activities. From left: a sample 9-block; assembling 9-blocks, classified by the number of green squares in each; the "combinations tower"—the combinatorial space of the 9-block; a computer experiments that produced the distribution of the 9-block; a statistics activity.

Related activities. We ask students to determine the chance of getting exactly 5 green marbles. (There are 2^9 = 512 unique items in this space, and a 5green/4blue combination is expected to occur 126/512 = ~.25 of the time). Students literally build the combinatorial space (the *combinations tower*), working in crayon-and-pencil and computer environments (Figure 2, above). Operating with these 512 9-blocks is functionally analogous to operating in a *single*-outcome space ("candy"), yet conceptually this recursively models a *compound*-event situation on a single-outcome situation. Next, students work with computer-based simulations that generate distributions of compound events (Figure 2, second from right). The visual resemblance of this distribution to the combinatorial structure (compare to Figure 2, center) stimulates inquiry into the law of large numbers. The 9-block also features in statistics activities (Figure 2, on the right), where it constitutes a sample taken from a large blue-and-green "population" of squares.

Situating the Study in the Larger Project (ProbLab) and Future Work

Both combinatorial analysis and computer-based experimentation contribute to student understanding of probability (Abrahamson & Wilensky, 2005b), and student conceptual learning may be understood as a synergy or reconciliation of these complementary activities (Abrahamson & Wilensky, 2005a). That is, the cognition of 'probability' can be seen as a theory–process dialectic: students learn to "proceduralize" combinatorial analysis and, in turn, to ground in the products of this analysis an anticipation of empirical distribution. Further work is needed to develop and evaluate a unit that promotes such learning-as-reconciliation in a resourceful classroom by using the bridging tools we have described. Such a unit (see *ProbLab*, Abrahamson & Wilensky, 2002) will be guided by activity-design principles including juxtapositions of: (a) different stochastic devices; (b) different embodiments of the same devices; and (c) different data-analysis perspectives on experiment outcomes (Abrahamson, 2005).

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ROBOTICS AS A CONTEXT FOR MEANINGFUL MATHEMATICS

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This naturalistic phenomenological study looked at the emergence of mathematical understanding in middle school students as they engaged in open-ended robotics activities. The study chronicled the opportunities for the embodiment of mathematics understandings as they engaged in meaningful problem solving activities using robots and sought to understand how the students cooperatively organized their efforts and negotiated meaning. Such activities exemplify rich tasks that appear accessible to students of varied mathematical abilities and may provide an avenue for addressing equity issues in education, such as those related to gender, minority status, and learning disabilities. Students' choices influenced the complexity of the mathematics that emerged from their activities. Robotics seems to exemplify an appropriate use of technology to create meaningful, open-ended, problem solving activities involving significant mathematics.

Purpose of the Study

The purpose of the study was to explore the emerging mathematical understandings and approaches to learning of middle school students as they engage in robotics activities. The study focused on self-selected middle school student members participating in robotics activities. In particular, the following questions were addressed:

1. What mathematical understandings emerge as students engage in robotics activities?
    a. What mathematics are the students using?
    b. What mathematics do they perceive they are using? Do their perceptions change in the course of the activities?
    c. What are the opportunities for mathematical embodiment in robotics activities?
2. How do students working cooperatively organize their efforts and negotiate meaning as they solve complex, open-ended robotics tasks?

The results of this study were used to inform speculation on how complex, meaningful, open-ended activities involving robotics might have an effect on educators in their practice, specifically with the goal of improving mathematics education. In particular, the findings may inform ideas about rigorous, relational curriculum (see Doll, 1993), rich in interconnections and potential for personal meaning-making in mathematics.

Theoretical Perspectives

A number of theoretical and research perspectives inform this study. Ideas of meaningful activities originate in Whitehead's process philosophy (1929) and emerge in postmodern conceptions of learning. Theory and research in process and enactivist approaches to learning emphasize the emergence of understanding in a fertile, open-ended inquiry-based environment. The robotics activities described in this study are proposed as a potential source of such activities for mathematics learning.

Doll (1993) describes learning as self-organization, a process where learning and understanding come through dialog and reflection. From this perspective, challenge and perturbation give rise to organization and reorganization. Since this process of challenge, perturbation, organization and reorganization is taking place within the individual, the outcome is not determined. The educational stage can be set, so to speak, but it is individuals in interaction with their context that co-create the script and bring forth the play.

Doll identifies four key aspects of curriculum to support self-organization/reorganization: *Richness, Recursion, Relations, and Rigor* (Doll, 1993) which are foundational for his curriculum matrix, grounded in process philosophy and chaos theory dynamics, and its implications for ideas about learning. In this light, curriculum emerges and unfolds from within the process rather than being imposed by external authority (Fleener, 2002). Evolving from Piagetian constructivism, postmodern learning theories also are foundational for considering the impact and potential of technology, in general, and robotics, in particular, on student mathematics learning.

Seymour Papert emphasizes that students' intellectual structures are not built from nothing. Instead, children appropriate for their own use what they find at hand—a *bricolage* of models and metaphors suggested by the surrounding culture (1980) and the expanded context, the new possibilities that technology can provide. Papert further echoes Whitehead and complexity theory when he expresses delight that a system (i.e., gears) could be lawful and comprehensible without being rigidly deterministic. His ultimate concern is the interaction of technological and social processes and how they influence the construction of ideas about human capacities. He is concerned with how a culture, a way of thinking, an idea comes to inhabit a young mind. The child is in control of the process. Through programming the computer to think using Logo, they problematize and explore how they themselves think and, in so doing, become epistemologists (p.19).

The learning experience is more than purely cognitive. Learning is very personal and cannot be assumed to be repeatable for others in exactly the same form. Computers and technology can act as transitional objects to translate body knowledge into abstract knowledge (i.e., to translate embodied understandings into more generalized forms of understanding). As such, technology is a tool that "…instantiates the living bond between finite human being and environing world " (Blacker, 1993, Making Connections section, ¶ 19) and provides a way of revealing or opening up of the conceptual or contextual environment to new possibilities.

This view is consistent with that of Davis (1996) and other enactivist theorists and educators. It is not enough for students to program the procedures and observe the results on the computer screen. Action/enaction and opportunities for embodiment are required. The enactivist perspective as articulated by Varela, Thompson & Rosch (1991) views cognition as "the enactment of a world and mind on the basis of a history of the variety of actions that a being in the world performs" (p. 9). The body and mind are seen as indistinguishable and structurally coupled with the environment. Individual and environmental structures co-emerge in interaction with each other (Reid, 2002). As Seitz (2000) puts it, "We do not simply inhabit our bodies; we literally use them to think with" (p. 23). Edwards (1998) extends these notions of embodiment. In her view, education is seen as providing environments that afford learners opportunities to embody concepts, i.e., to kinesthetically and intellectually interact with the environment designer's (e.g., the teacher's) construction of conceptual entities.

If embodiment is critical to understanding, the opportunities to embody are markedly lacking in typical education settings. Berthelot and Salin (1994, p. 74) describe three ways in which we experience our space. *Micro* space is the intimate space of interactions that can be affected without moving; e.g., a book/notebook, a desk or personal computer. It is space mainly composed of objects and it is difficult to distinguish distance from spacing. *Meso* space is the intermediate space of domestic moves and interactions through choice of position. Moves within this space are mastered using intellectual representations of the space. A classroom would be an example of this space. The distance concept is more developed and measured in small units. *Macro* space consists of areas so large that information can only be obtained through successive moves. It is built of a collection of local views connected through travel. Distance measures are correspondingly larger. Berthelot and Salin (1994) noted in their studies of elementary school students that lack of experiences in *meso* and *macro* spaces inhibited the construction of meaning in *micro* space; the space where students typically operate in the classroom. Conversely, students can act as if they have *micro* spatial conceptions of, say, a rectangle. Yet, if called upon to use their notion of a rectangle in a different space, they are unable to recognize, utilize or access corresponding manifestations at that spatial level.

The robotics activities described in this study may be a potentially rich source of meaningful opportunities to embody mathematical concepts. However, robotics activities have not even begun to be used in ways that fully take advantage of the technology to enable the embodiment and emergence of meaning.

Methodology

This naturalistic phenomenological took place entirely at a suburban middle school. Sixteen male and female middle school students participated. The participants consisted of a purposeful sample of volunteer students in grades six through eight; although the team primarily consisted of sixth and seventh graders. Their academic backgrounds were diverse as academic standing was not a condition of participation, neither in the robotics activities nor in the study itself. The sample also included students identified with learning disabilities as well as some that were talented and gifted,

Data was collected from multiple sources. Video and audio recordings of participant activities, interactions, robot programs, and participant surveys and interviews were the primary sources of data. Field notes were taken during each robotics activity session to supplement the recordings and capture my perspective of the activities. Study participants were informally interviewed on an ongoing basis throughout the course of the study in order to explore their mathematical constructions. Data collection continued until there was an exhaustion of sources and a clear emergence of conceptual categories.

A constant comparison method of analysis guided the investigation of socio-mathematical interchanges that might lead to individual mathematical constructions (Strauss & Corbin, 1990). After each session, I examined the data, which I first separated into specific event sections as frames in which to focus subsequent observations, interactions, and interviews. I coded and categorized each of these data sources within each event, looking for regularities and patterns in the ways students and teacher or students and students mathematically interacted within and then across sets. As new ideas, questions, and areas of interest emerged from the data, they were folded back into key informant interviews for checking.

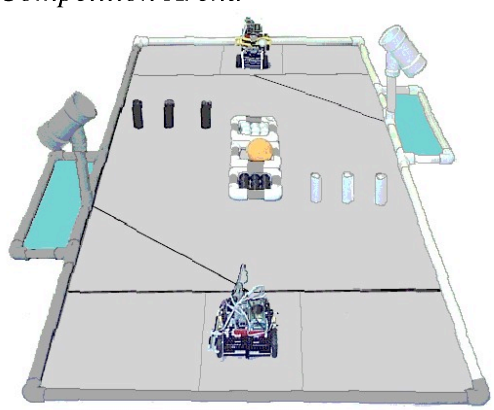
Robotics Activities

Students in this study were from a team involved in an intermural competition called Botball. Their task was to design, build, and program autonomous robots to compete against other schools' teams in a 4 foot by 8 foot competition arena surrounded by 1.5 inch plastic pipe such as seen in Figure 1. The robots had to activate upon a light signal run through their programmed activities, and shut down at the end of 90 seconds; all without any external guidance. The teams could field up to two robots simultaneously and scored points by manipulating various objects in the arena. The teams had six-weeks from the time contest problem was released to prepare for the competition.

From the data collected, there appears to be a wealth of mathematics involved in the robotics activities of this study. The problem of navigation is representative of not only an aspect of the mathematics that the participants experienced but also illustrates the richness of mathematics potentially accessible through the robotics activities; depending upon the participants' design choices as they developed their robots. Navigation involves aspects of algebraic reasoning, proportional reasoning, and geometric interpretation.

Navigation is one of the major problem hurdles the students face in programming their robots. The basic question that they have to address is, "How does a robot know where it is in the arena so that it performs the correct action in the intended location in keeping with the team's competition strategy?" In this competition, one robot (*X-Terminator*) was targeted at the near nest (see Figure 1) to lift up one side, drag it back into our end zone and free its balls before putting it down to go back to get the center nest. Meanwhile, the second robot (*Fluffy II*) was to go down the left side of the board and knock over the cardboard tubes of our team's color for that round, freeing the balls inside, thus scoring additional points. To do this, each robot had to exit the starting box without interfering with its partner robot. There are multiple levels of complexity in terms of how the team could choose to address this navigation problem and, correspondingly, multiple levels of mathematical complexity that emerged from the participants' decisions. The students preferred to use dead reckoning to address the navigation problem.

Figure 1 *2002 Botball Competition Arena*



Note: From "Botball 2002 Teachers Workshop" presentation by D. Miller, KISS Institute for Practical Robotics, January 2002, Norman, OK. Copyright 2002 by KISS Institute for Practical Robotics (www.botball.org). Adapted with permission.

Dead reckoning means navigating only on the basis of time, rate (in terms of motor speed), and distance traveled much as ancient mariners once determined the position of their vessels. This is the simplest means of programming the robots to navigate the arena. However, while dead reckoning is easily accessible to middle school students, it has its drawbacks in terms of reliability because it fails to take into account and respond to changes in environmental and contextual factors.

Dead reckoning in robotics enabled participants to embody the relationship between distance, velocity, and time; a perennial topic in middle school curricula; through the robot's actions in a dynamic way in the *meso* space of the competition arena in a manner unlike the typical mathematics classrooms where the relationship is limited to the two-dimensional *micro* space of the desktop. In practice, the participants navigated the robot by controlling motor speed and specifying the duration of time at that speed. The students could choose between two types of commands to affect motor speed. One type of command consisted of, either *fd (motor number)* or *bk (motor number)* depending on whether the motor was required to rotate forward or backward. Using the *fd* or *bk* command set the motor speed to its maximum rotation speed. The alternate command was a *motor (motor number, rotation speed)* command where the range of values for rotation speed was +/-100 with the sign of the integer determining direction of rotation. In a dead reckoning sequence, these commands would be accompanied by a *sleep (float)* where the *float* is a decimal value indicating the number of seconds to perform the commands in between the current *sleep* command and the preceding *sleep* command.

The use of these motor commands involves algebraic reasoning in that the students are essentially manipulating up to three variables: motor direction, time, and, in the case of the *motor* command, motor speed. Moreover, it also involves proportional reasoning. The distance traveled by a robot is directly proportional to the motor direction and time at that direction and inversely proportional to motor speed and time at that speed. Students directly observed the results of their manipulations of these variables in the actions of the robots. While the students did not directly articulate these relationships, it was clear from their enactions (via the robots) that they understood the relationship.

Each robot's builders chose different approaches in their use of these commands. The *X-Terminator* group coordinated all three variables in their efforts to navigate the robot. The *Fluffy II* group took a simpler approach preferring to reduce the number of variables to two. They chose fix the motor speed at the maximum value by using the *fd* and *bk* commands exclusively.

The students were aware of the tradeoffs in the two methods. The *motor* command allowed better accuracy, provided a means to compensate for drift through the use of differential motor speeds and was easier on the drive train of the robot. However, it was more difficult to coordinate the variable values to achieve the desired effect when using the *motor* command. In contrast, the *fd* and *bk* commands were simpler to coordinate, having two instead of three variables. However, because the motors were commanded to rotate at maximum speed, they tended to stress the robot's drive chain to a greater extent and cause gears to slip or pop out of place, especially in turns. When I asked Gary about his team's decision to use the *fd* and *bk* commands exclusively, he said that they didn't "…want to mess with motor speed in case we have to reprogram during the competition. It takes too much time to get it right."

Proportional reasoning is another area where the robotics activities exhibited significant potential for emergent mathematical ideas. The participants themselves recognized several ways that proportional thinking was involved in the robotics. In the following vignette, the participants discuss the mathematics that they see in their robotics activities.

Question: *Do you guys like math? Is there any way you use math doing this?*

Frank: *Yeah a lot!*

Oscar: *Yeah, it is easy.*

Tom: *Sure!*

Tom: *Yeah gears have to be set a certain way.*

Afterward, Tom indicated that he was referring to the matching of a large gear to a smaller gear or vice versa depending upon whether power (torque) or speed was desired. This is a proportional reasoning problem involving gear ratios. The students were very conversant with which gear ratio to select, although they did not call it that, to achieve a desired outcome and regularly discussed the pros and cons of various gearings. Moreover, they began to relate their experiences with gearing to other problems they were encountering. Frank continued the discussion.

Frank: *Just like when I was setting the servo. It had to be set to 0, then I had to use 180, and then compare the angle and stuff like when it is all straight lines it is like 1000, 2000 and so on, the degree to the amount.*

Here is a second and separate proportional reasoning problem involving programming a servomotor. Frank is attempting to describe the reasoning involved in coordinating the desired position of the system of angular measure that he knows (degrees) with the system required in the programming language, *Interactive C*, by comparing it with the reasoning required in the gearing referred to above. Other students joined Frank in extending the ideas

Oscar: *And like light sensors there are so many degrees wide that it sees. So you like got to decide and figure and make decisions on degrees.*

Frank: *And like especially the sonar, it shows in this book how many degrees the range of it should be. And you have to know how far it goes and reads.*

Oscar: *Look, see here are the standard gears. There is a 40, 32, 24, 16 and an 8.*

Frank: *I don't know what these ones are.*

Question: *If you add a 40 one to an 8 gear one. How many times does the 8 have to go around to make it a 40*?

Frank: *5 times.*

Oscar: *5 times and then the 16 and 24 would have to be odd. For they would have to be different, not whole numbers they would have to be integers.*

Question: *Are you talking about if they were geared with a 40?*

Frank: *Yeah.*

Oscar: *Yeah, but the 8 goes into everything on here.*

Frank: *It goes into 16 and 24 and 32 and 40.*

The students in this vignette mentioned two additional aspects of robotics where proportional reasoning is important. The first is in the gearing of the motors to the drive wheel of their robot. When asked about a 40 toothed gear paired with an 8 toothed gear, Oscar exhibits some playfulness in considering various gear combinations in extension of the question. "5 times and then the 16 and 24 would have to be odd. For they would have to be different, not whole numbers they would have to be integers." Moreover, both Frank and Oscar recognize 8 as the greatest common factor of the 16, -24, -32, and -40 toothed gears. The proportional reasoning involved in gearing becomes even more complex when encoders are used as sensors. Encoder output (tics) has to be related though and the gear train via the program to distance traveled in one rotation of the drive wheel in order to determine the robot's position.

Frank described a second problem that required the use of proportional reasoning. The problem involved programming the servomotor controlling the forklift arm on *X-Terminator*. Servomotors are designed to rotate within a range of 0--180 degrees and hold any commanded position within that range. This makes a servomotor useful to position a device like the forklift arm on X-Terminator. The *Interactive C* language, on the other hand, allows servo commands in the range of 0—4000. To program the servomotor, Frank had to relate the degree range of motion of the servomotor to the servo command range of the programming language. This coordination became even more dynamic as the servomotor had not been set to either end of its range when it was glued to the forklift arm. Frank had to determine the initial starting position within the servomotor's range in order to coordinate his programming commands.

Conclusions

The mathematical understandings of the study participants appear to be enriched through the robotics activities. The robotics activities seem to contextualize the typically decontextualized mathematical abstractions that students encounter in the classroom. From an enactivist perspective, the choices that the participants made in strategy and robot design affected the negotiation of meaning through changes in their personal structure as well as the team-as-a-system structure. The changes in structure in turn constrained both the further choices that might be made and mathematics involved, both from the perspective of mathematics that is used as well as the mathematics understandings that could potentially emerge.

The participants' negotiation of meaning was often played out through action or doing. Being able to reference the individual components of the robot to their own body functions and experiences enabled them to think about decomposing the task to the level necessary to program the robot. This enabled them to use their bodies as objects to think with (Papert, 1980) in coming to understand the level of complexity of instruction required to successfully program a robot.

The participants reported that working with the robots helped their understanding of mathematics. However, they reported that they didn't see a relationship to the mathematics they encounter in school. This raises the question of the implications these types of robotics activities with respect to mathematics instruction, especially to affording students choice while being able to meet instructional or curricular goals. The robotics activities appear to exemplify the notions richness, recursion, relations and rigor elaborated in Doll's curriculum matrix. Activities such as those involved in this study may have the potential to be employed in meaningful ways within the school environment to meet mathematics curriculum objectives and may provide an avenue for addressing equity issues in education, such as those related to gender, minority status, and learning disabilities. Further study is required to determine how this may be accomplished.

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PROSPECTIVE TEACHERS' PERCEPTIONS OF TEACHING AND LEARNING MATHEMATICS THROUGH IMAGES AND DRAWINGS

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The major purpose of this research was to discern teacher candidates' perceptions of teaching and learning mathematics through a series of drawings and narratives in which they illustrated mathematics classrooms of their past, present, and future. This study of 180 drawings and accompanying narratives by teacher candidates was conducted in an undergraduate elementary mathematics methods course. The findings provided a basis for understanding how teacher-generated drawings and narratives can serve as "communicative tools" for mediating inner thoughts about mathematics teaching and learning. The most significant conclusion is that drawings can become a substantial tool that may assist in raising the quality of teacher preparation as the teacher candidates are developing their conception of themselves as mathematics teachers.

Purpose

Teacher candidate perceptions of what constitutes good mathematics instruction pose great influence on the type of mathematics instruction they will deliver in their classrooms. We suggest that teacher candidates' drawings and narratives about mathematics generated during their academic experience may play a role in their preparation and development as effective mathematics teachers. Tools such as drawings and narratives may uncover perceptions of their prior personal teaching/learning experiences in mathematics and provide rich material for self-reflection and analysis of the effectiveness of their teaching strategies. This study examined teacher candidates' perceptions of teaching and learning mathematics through a series of self-made drawings and narratives that asked them to illustrate mathematics classrooms of their past, present, and their idealized classroom of the future.

Theoretical Framework

From the elementary school to the college classroom, students have depicted images in drawings that provide insight into their perceptions of teaching and learning processes (Black, 1991; Goodenourh, 1926; Gulek, 1999; Weber and Mitchell, 1995; 1996; Wheelock, Bebell, and Haney, 2000). The research on these images depicted in drawings has demonstrated the emergence of a consistent assertion that metaphors and images permeate our daily existence. "If we want to understand how [mathematics] teachers make sense of their work – to acquire an empathetic understanding from within," argue Efron and Joseph (1994), "then we must explore an artistic form of image that can grasp and reveal the not always definable emotions" (p. 55).

The epistemological approach of Vygotsky may assist us in understanding teacher candidates' drawings. Vygotsky conceived the role of "tools and signs" (1978, pp. 52-55) as a reflection of the psychological process, assisting the development of the human mind in its learning and thinking. Vygotsky theorized that such tools and signs are derived from our culture and serve to direct us to a deeper understanding of the activity in which we engage. The zone of proximal development permits growth of independent intellectual functioning through the "actual verbal interaction with a more experienced member of society via the richness and substantiveness of verbal dialogue" (Manning & Payne, 1993 p. 364). Vygotskian theory presents a powerful image of human learning, whereby, the sociocultural context influences teacher-generated drawings and narratives. Drawings, as a tool, like text, can communicate and reflect the subtleties of the emerging understandings of teacher candidates' conception of themselves as mathematics teachers. Narratives, as a tool, can assist them in clarifying and organizing their knowledge and perception of themselves as mathematics teachers.

Modes of Inquiry and Data Sources

This study of 180 drawings and accompanying narratives by teacher candidates was conducted in an undergraduate teacher education elementary mathematics methods course. The teacher candidates' were asked to draw three pictures in response to three different prompts. The prompts were given to all of the students in the class at once and were delivered in a set order. An important difference between these prompts is that they represent different time periods and different stages in the candidates' lives. The first prompt asked students to conjure up an image of the past wherein they were the ones receiving the instruction. The passage of time as well as the depiction of a candidate as a student likely caused the creation of images that were *reflections of traditional mathematics* curricula and instruction. The second prompt asked the candidates to depict something that was currently happening while they worked in the role of apprentice in their pre-practicum experience. The immediacy of these drawings likely evoked images of *teaching in action*, where reality was more likely depicted than in the other drawings. The third prompt asked students to imagine what their classrooms would be like in the future where they are completely in charge. Therefore, this prompt would likely produce an idealized vision of teaching and learning mathematics that speaks to some future *potential*. Another essential feature of the timing of the prompts regards the interaction of the drawings with the teacher candidates' learning in their class. This mathematics methods class was designed to allow students to experiment with various pedagogical and curricular forms, thereby influencing candidates' perceptions of deliberate practice in mathematics. Therefore, it is expected that an evolution in candidates' perceptions would occur.

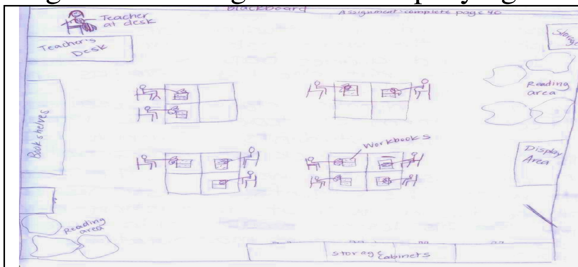
Coding Drawing

This analysis applies the steps outlined by Haney et al (1998) in which the first step in this process is the creation of a coding scheme. The initial step allows researchers to separately view the drawings and develop the codes to create an inductive scheme. These individuals then met to discuss the similarities and differences in the list before agreeing on one set of code definitions. The researchers then coded the drawings separately to test the consistency of coding when using the code checklist. Inter-rater reliability estimates were then calculated for each section of the checklist and the checklist was adjusted, accordingly, to improve on these estimates. When the checklist was finalized, both inter-rater and Cohen's Kappa reliability estimated were calculated on a random sample of twenty drawings, again to ensure rater consistency. Coefficients of simple inter-rater reliability ranged from a low of .81 for *desks* and a high of .96 for *manipulatives*. Finally, the two raters who conducted the reliability study also coded all of the student drawings. These codes were entered into an Excel database, and frequencies and percentages were calculated for each of the codes, as well as for code groupings (i.e., composite). Analyses of coded data were conducted by converting counts into percents and comparing the incidences of codes across each set of drawings (heretofore called *Rounds* as in, *Round 1).*

Findings and Discussion: The Changing Roles of Teachers and Trends in Content

Drawings in Round 1 showed many teachers at the chalkboard or in the front of the room (66%), with most teachers instructing the whole class (55%). By Round 3, teachers were seen instructing groups and individuals more often than whole classes (29% vs. 21%) – only 18% were depicted at the chalkboard or in front of the class – and 45% of the teachers depicted were shown moving in the classroom or walking toward students (up from 8% in Round 1). Here we see the changed perceptions of teaching among these students – from teachers as knowledge transmitter to teacher as facilitator as depicted in Figure 1.

Figure 1 Drawings and Accompanying Narrative for Round 1 and Round 3



In my third grade classroom, there was only individual math work. When I think back to my third grade math lessons, I remember completing workbook assignments while sitting at my desk. In this scene, I drew such an assignment. The students are sitting at their desks silently completing the workbook pages. The teacher sits at her desk and grades papers. The students can ask the teacher questions but she never walks around the room to check on their work.

In five years, I plan to be teaching in my classroom. My class will be organized in islands of student desks with learning centers and reading tables. I drew my students working in groups on the floor. They are using manipulatives because [these] will be a big part of my program. My attitudes [toward] math has changed a little. I now better understand the importance of manipulatives in math. I also understand the significance of collaboration and scaffolding in math.

In the latter picture one can see the teacher in the middle of the room, apparently working with a group of students. The words "collaboration" and "scaffolding" hold great significance in this student's depiction of an ideal classroom. The classroom layout, as well as its mission is geared toward students learning together. For scaffolding to occur, students must enter the *zone of proximal development*, (Vygotsky, 1978) in which they are just beyond their independent capabilities. Once the student is in this zone, a capable adult or more knowledgeable peer can support the student to move to the next level of learning. This is one of the main reasons why this teacher candidate places the teacher amidst a group of students, and also why the class' progression of teacher positions from chalkboard to moving with students is such an important development.

In an effort to capture the changing roles of teachers, composite variables were created to capture both "TRADITIONAL INSTRUCTION" and "COLLABORATIVE INSTRUCTION" pedagogies. The TRADITIONAL INSTRUCTION variable was created by adding the number of times candidate drawings were coded with the following: Teacher at chalkboard/in front of room, Teacher instructing whole class, Teacher at desk, Students sitting at desks/tables, Abstract representations, Individual seatwork, Worksheets, Flashcards, Competition, Clock/time. All of these codes were thought to represent traditional instruction. For example, many students associated the traditional instruction from Round 1 with the use of competition, worksheets, and individual seatwork. Also apparent in these drawings was the use of abstract representations.

Representations in mathematics instruction are ways to display mathematical concepts. For example, one could show multiplication through just its numerical form: 5 x 5 = 25. This example is an abstract representation wherein the learner is provided with numerical data only, and must associate a meaningful mental image with the numbers in order to fully comprehend the number sentence. By contrast, concrete representations are those that connect real objects with the mathematical concept. An example of this is using tiles to form a square that has, as its dimensions, 5 x 5. Concrete models allow students to understand mathematical concepts through actual physical experience, and not through vicarious experience, only. Pictorial representations are two-dimensional depictions of concrete representations. The types of representations shown in the drawings were coded, as well as whether or not drawings showed more than one representation.

A COLLABORATIVE INSTRUCTION composite variable was created to capture pedagogies that supported social-constructivist learning. Social learning contexts are those in which students are encouraged to work cooperatively together toward a learning goal. Constructivist methodologies allow students to form their own understandings of what is learned. Both of these methods emphasize active student participation and teacher facilitation. Therefore, the COLLABORATIVE INSTRUCTION composite variable contains the sum of the following codes: Teacher in the center of the classroom, Teacher instructing a group/individual, Teacher sitting in a chair instructing students, Teacher walking around the room, Students at centers, Students talked to each other, Students walking around the room, Active learning tasks, Cooperative learning, Interdisciplinary learning, Tasks with manipulatives, Two-plus activities at once, Concrete representations, and Two-plus representations. To place the two variables on the same scale, the number of students in each round divided not only by this number, but each sum also. While this did not yield a percent, it did create a type of "standard count." Figure 2 shows the standard counts of the two composite variables at each round. Figure 2 also shows the consistent drop in pedagogies representative of traditional instruction in each round. Likewise, collaborative instructional methods depicted in the drawings show a sharp increase from Round 1 to Round 2, and a slight increase from Round 2 to Round 3. This graph supports the candidates' progression from traditional to collaborative/constructivist pedagogies.

Figure 2 Traditional vs. Collaborative Instructions

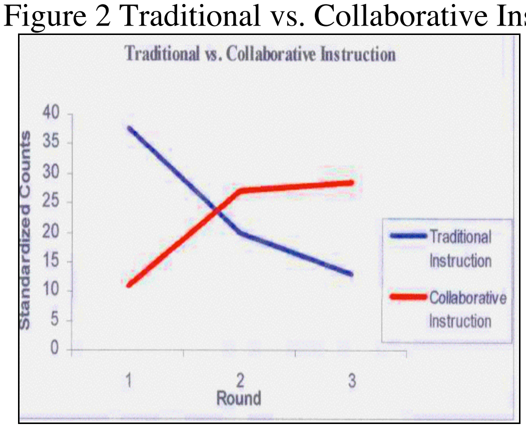
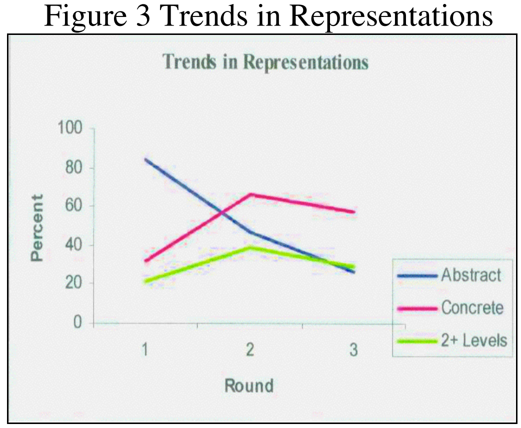
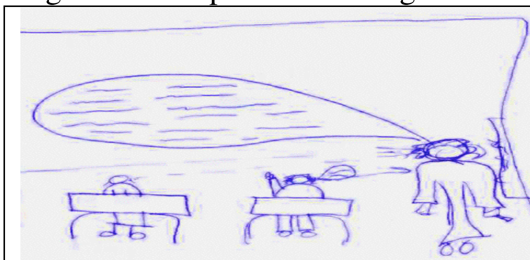


Figure 3 Trends in Representations

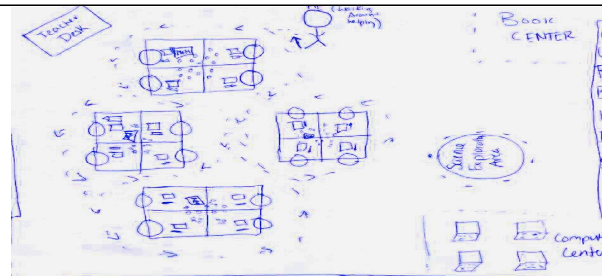


Another finding was the decreasing trend noted in abstract representations along with an associated uptrend in concrete representations (Figure 3). As indicated in Figure 3, typical Round 1 drawings showed primarily abstract representations while drawings in Rounds 2 and 3 showed more concrete representations. In addition, drawings depicting more than one representation increased slightly from Round 1 to Round 2. To better illustrate the two disparate teaching methods and the trends in representations, one drawing each with a high sum in either composite variable is shown in Figure 4, along with excerpts of the accompanying text. The contrasts between the drawings and the excerpts point to a crucial difference between the two approaches. Here, traditional methodologies are associated with student isolation. In these drawings, students tend to be on their own, adrift in the classroom. Learning and engagement are associated with aptitude and inner drive in these classrooms, and are not associated with pedagogy. Collaborative drawings were much more inclusive of all students, regardless of ability levels. Candidates cited pedagogical practices – cooperative learning, the use of manipulatives, teacher facilitation – as the force for the inclusion of these students. Thus, the drawings and accompanying narrative may have provided opportunities for candidates to gain insight into their individual mathematical learning experiences and teaching practices. This awareness and knowledge may have affected their comprehension of their future students' learning processes and thus illuminating the process of effective mathematics teaching they need to develop.

Figure 4 Examples of Drawings Showing Round 1 and Round 3



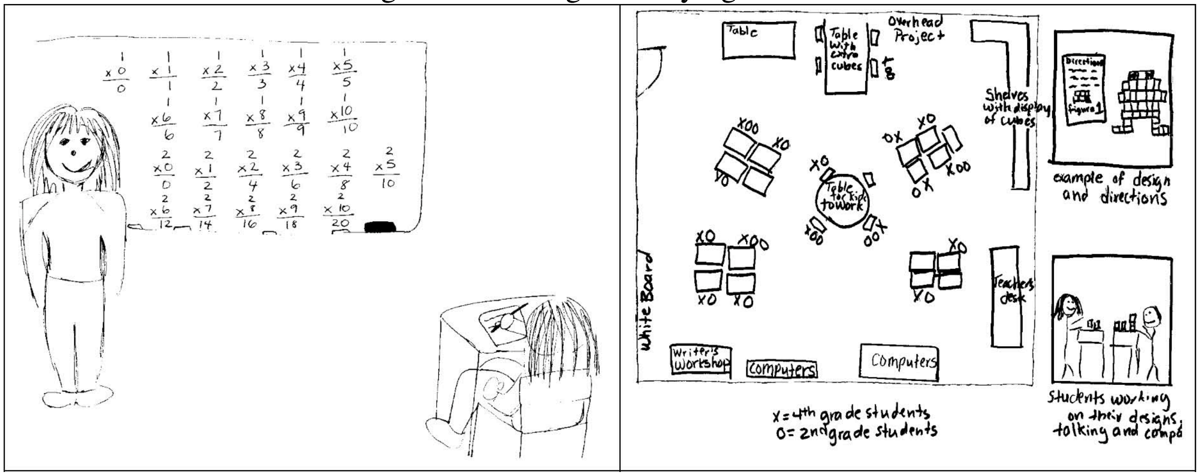
In my class, math was basically a "listen & learn" subject. The teacher would stand in the front of the room and model the concept while the students watched…I felt a little distant from the learning and although I always followed what the teacher was demonstrating, I was reluctant to raise my hand because I was shy. Worksheets were boring…



In the picture, the children are working in groups to complete the different activities. I am walking around the area to answer any questions the children may have. I will obviously not provide answers but I will guide students so that they understand concepts. It is important for them to see that I am available for help – [that] I am attentive to their needs.

The typical abstract representation depicted in Round 1 was multiplication problems written on the chalkboard as illustrated in Figure 5, Frame 1. This drawing from Round 2 shows more than two representations of the same concept: concrete, pictorial, and abstract. In this lesson, students are designing toys with multilink cubes by both building the toys and writing directions for other students to create the same toys (Figure 5).

Figure 5 Drawings Portraying Content



These drawings also underscore a major difference between the drawings of Rounds 1 and 2: differences in content. As shown in Table 1, multiplication was the dominant content portrayed in the drawings from round one, while a greater variety of content was shown in the drawings from round 2. Lessons with more than one type of mathematics content depicted per drawing also increased in Round 2 – as noted by the totals for coded content of each round.

Round 1	Total Count	Round 2	Total Count
Multiplication	**63**	**Addition**	**25**
Addition	**11**	**Classification/Sorting**	**25**
Division	**11**	**Counting**	**25**
None Observed	**8**	**Decimals**	**22**
Subtraction	**8**	**Division**	**22**
Unclear Math Content	**8**	**Fractions**	**17**
Decimals	**5**	**Geometry**	**11**
Fractions	**5**	**Graphing**	**8**
Money	**5**	**Money**	**8**
Place Value/Number	**5**	**Multiplication**	**8**
Counting	**3**	**Patterning**	**8**
Classification/Sorting	**0**	**Place Value/Number**	**6**
Geometry	**0**	**Subtraction**	**3**
Graphing	**0**	**Unclear Math Content**	**0**
Patterning	**0**	**None Observed**	**0**
Total Coded Content	**116**	**Total Coded Content**	**181**

The increases in mathematics content did not carry through to Round 3 (total for round three coded content = 61). This fact coincides with other characteristics of Round 3 drawings (fewer students depicted than in first two rounds and fewer levels of representations presented than in first two rounds). These aspects of round three drawings suggest that candidates' final drawings were of more general mathematics situations – again, underscoring the differences between *teaching in action*, which is quite specific and *potential*, which is a more amorphous idea.

The last set of drawings (Round 3), with emphasis on the structure of the classroom, documents the candidates' strategies for incorporating the new strategies and content into their future mathematics classrooms. While these drawings are light on specifics, they are heavy on pedagogy these candidates have learned to value. In particular, the drawings overwhelmingly indicate a predilection for student collaboration, interactive teaching, and the support of diverse student learning needs. The candidates' desire to improve mathematics instruction is perhaps best appreciated in the context of their less-than-ideal early mathematics experiences. By engaging in the activities from their weekly mathematics methods course, candidates with poor early mathematics experiences have been given the opportunity to address early feelings of failure and to understand that, perhaps, it was not they who failed, but rather the instruction that failed them. By changing instruction in their own classrooms, these candidates have the prospect of improving mathematics experiences of a new generation of pupils – and some of them also have the opportunity to revise their own mathematics histories from mathematics phobic to revolutionaries. These images, therefore, may reflect the existence of certain values in our culture and standards and norms in education, thereby unmasking the conflicting realities therein.

Conclusions

If we can use drawings and narratives as journals/texts to assist teachers in questioning the tacit assumptions that underlie their pedagogical practices, then those practices may change in a substantial manner as they make progress in understanding real-world aspects of mathematics teaching. Manning and Payne (1993) suggest the development of higher cognitive processes within a teacher "is not simply quantitative increments but qualitative shifts as the unique past experiences and previous knowledge of individuals interact with the present learning event" (p. 362). The authors propose the development of a teacher learning theory supported by Vygotsky's sociohistorical perspective on knowledge construction and understanding. Such a theory would enables teachers to acquire self-reflection based on their own personal sociocultural perspective of their individual learning experiences. The drawings themselves can become a substantial tool that can assist in raising the quality of teacher preparation as the teacher candidates are developing their conception of themselves as mathematics teachers. It is our belief that this work contributes to contemporary discussion in this new century about improving the quality of mathematics teachers and their preparation in the sense that this work places teachers' conceptions of themselves and their practices in a broader sociocultural context and as they become aware of their own learning, they are more likely to become aware of the influence of the child's learning context within the mathematics classroom.

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AN ANALYSIS OF TEACHERS' MATHEMATICAL AND PEDAGOGICAL ACTIVITY AS PARTICIPANTS IN LESSON STUDY

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The context for this study is a professional development project, using Japanese Lesson Study as a guide, involving teachers from grades 6-12 in an urban school district in New Jersey. The teachers were supported by a knowledgeable "outside specialist" (Lewis, 2002, p.67) as they defined particular goals for their students. The group then collaborated in planning, implementing, debriefing and modifying an open ended "research lesson" developed to address the goals that they had defined for their students. This research reports on the analysis of the teachers' discussions about mathematical and pedagogical ideas as they planned the lesson and their reflections and perceptions about their students' actual mathematical activity during the lesson implementations.

The Focus of the Study

Of special interest to this analysis is the teachers' attention to the importance of "tools" for learning mathematics: how and when students select and use particular tools in building representations and solutions for problems, and implications of these choices for teaching. By "tool", we refer to any concrete aid offered by the teacher and/or selected by the learner for the purpose of building a representation or constructing a solution to a particular mathematical problem. Examples in this study include measuring devices, graph paper, grids of different sized squares on overhead transparencies, calculators, and mathematical formulas. In particular, we explore the following questions:

- How does the teachers' discourse and reflection about their students' mathematical activity and their own roles as facilitators of this activity develop during the series of working sessions and implementations?
- What particular issues do the teachers note about the importance of tools within the learning and teaching of mathematics and how, if at all does this attention change over time?
- What, if any, is the impact of the lesson study collaboration on the development of this reflective process among the teachers?

Theoretical Background

Lesson study is a collaborative activity among teachers, with university educator-researchers supporting the teachers' efforts to promote their students' mathematical development (Lewis, 2000). The context encourages new forms of discourse where the researchers contribute the critical and reflective stance of the academic community and the teachers set the agenda and bring craft knowledge about pedagogical practices and students' needs (Ruthven, 2002). This form of collaboration offers a unique opportunity for developing a community of practice through mutual engagement, the negotiation of a joint enterprise, and the development of a shared repertoire (Gomez, 2002). Increasingly, since the publication of results of the TIMSS study (Stigler and Hiebert, 1999) researchers have begun to consider the effectiveness of Lesson Study as a context for professional development. Their studies indicate that there is a positive

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Proceedings of the 27th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.

impact on teachers understanding about learning and teaching mathematics (Fernandez, Cannon and Chokshi, 2003; Lewis, Perry and Murata, 2004). The collaboration described in the present study provides an example of bridging the gap between theory and practice within a substantial learning environment (Wittman, 2001). Teachers' learning is viewed as a transformation of participation where the group activity is the primary unit of analysis with the focus on changes in understanding, facility, and motivation as they are documented in an unfolding event (Kazemi & Franke, 2004). This focus allows us to examine the teacher's collective engagement while planning a lesson, and can reveal deepening knowledge about mathematics learning and teaching. Mathematics learning is considered here as a process where the learner builds understanding of mathematical ideas by constructing increasingly powerful representations in the process of solving meaningful problems and sharing and justifying these ideas with others (Davis and Maher, 1993). Our focus on mathematics teaching is concerned with the actions of teachers whose aim is to establish a classroom atmosphere that supports such learning and in which teachers are constantly paying close attention to their students' mathematical thinking (Martino and Maher, 1999). The tasks and the tools provided by the teachers play an important role in the creation of such atmosphere. McClain (2002) shows that the teacher's mathematical agenda is constantly revised and modified based on students' actions which are grounded in the students' use of tools and the resultant inscriptions that they produce.

Methodology

Participants

The data comes from a Lesson Study group including eight middle and high school teachers (6-12 grade). Seven of the eight teachers had participated during the previous year in at least one Lesson Study Cycle. The teaching experience of the teachers varied, some were novice and others quite experienced.

Procedures

Data was collected during a 4-month Lesson Study cycle in the fall of 2004. The first author was the "outside specialist" for the group. The second author was a visiting scholar at Rutgers University. The third author is a graduate student and also a participating teacher in the group. The sources of data were from four, 2-hour planning sessions, and four lesson implementations with their subsequent1-hour debriefing sessions. The group meetings were audio-taped and field notes of the teachers and researchers were collected. The lesson implementations and debriefings were videotaped.

Data Analysis

The data was analyzed systematically with a grounded theory approach (Strauss & Corbin, 1998). Analysis included the identification of critical issues negotiated in the meetings concerning mathematical ideas, children's mathematical learning and implications for teaching. A primary issue identified in preliminary analysis was the role of tools as a means to encourage the development of students' mathematical strategies and representations. Instances from each activity referring to this issue were identified and coded in terms of emergent pedagogical and mathematical issues. Analysis of segments over the period of the study focused on shifts of the group's attention from practical, management aspects pertaining to the role of tools within the lesson toward the teachers' thoughtful attention to the mathematical activity and thinking of the students and how their selection of a tool enhanced or constrained this activity.

Results

In the first session of the cycle the teachers agreed that all of their students needed to explore situations that involved proportional reasoning. The teachers had begun implementing the Connected Mathematics Project at the beginning of the year and noted an increased expectation for students to analyze and solve open-ended problems and think about mathematical ideas flexibly, in various contexts, both spatial and symbolic. The task that the teachers proposed and developed in subsequent sessions asked the students, working in pairs, to figure out the ratio of the red portion to the total area in a particular national flag. Each pair of students was given a different flag, and, after establishing the ratio for that flag, the group was asked to order the flags according to the fraction that was red. The final challenge to each pair was to construct a new flag with a given ratio of red. In tracing the teachers' ideas about the tools that were to be offered by the facilitator and the probable selection and use of particular tools by the students, and their discussion during the lesson debriefings, we documented a number of shifts in the way that the teachers considered this issue.

From Tools as a Means of Implementing the Lesson to Tools as Resources for Thinking

In the third planning session, the discussion was about the materials that the students would be given for the task. Initially, the focus was on planning the management of the task. At this point, the teachers referred to various tools as supporting the implementation of the task but they did not reflect about their potential as aids to promote the development of students' strategies. They had a sense of the materials at a surface level based on their tacit knowledge as mathematics learners and teachers. One teacher (Te) proposed a ruler and centimeter graph paper as the primary tools to be given to the students. The outside specialist/researcher suggested that it might be helpful to also provide both centimeter and inch-square transparent grids. A teacher (Te) responded:

Te: I don't like mixing up the inches and the centimeters.

The researcher (A) pointed out the possible potential for students to build flexible mathematical thinking.

A: OK. The reason I say that is that for some of them a little tiny grid is going to work, for others a larger grid might be more efficient and …. I'm wondering - are you getting the same sense that they are going to get from an inch and from a cm ruler?

The teacher acquiesced but appeared unconvinced by the researcher's arguments.

From Tools as a Means for "Demonstrating" or "Teaching" to Tools for "Exploring" Mathematical Ideas

Later in the discussion another teacher (So), new to the project, challenged the group's decision to use both a cm and an inch ruler:

So: I think that we shouldn't give them the choice right away. We should give them either a ruler with inches or centimeters. Because when they see that they have a different answer they get confused.

The group responded that, to the contrary, a teacher's intervention might either promote or prevent students' autonomy toward mathematical understanding. In the following excerpt Ta, So, and Do are teachers, A is the outside specialist.

Ta: That's what we want them to get. They constantly wait for us to tell them what is ok and they are not making a decision …

So: You are talking about high school.

Ta: No, I am talking about children in general. They get trained that way. "Somebody tell me which one to choose. Someone tell me what to do".

So: They need it. I see them in my class. Sometimes they need to be told.

Ta: They are waiting for us.

So: They need to compare. To see both measurements

A: But I think that one of our ways of combating - you have decided that they will be in pairs. One of our goals which is more an affective than content knowledge goal is that they have to decide together.

Ta: But believe me.

Do: I know what you are saying .. but we always say to them you have to use the ruler to … what if we do not give them any directions, I do not know. I haven't seen this problem myself. I do not know maybe you need to get together to figure it out.

The above excerpt indicates two different perspectives about the use of the tools. One supported by So is the recognition of tools as a means to "demonstrate" different representations to the students and the other, supported by Do and Ta, is to allow students to explore mathematical ideas independently. A third consideration, mentioned by A, is the collaborative dimension of learning that seemed to be reflected in Do's position. In this part of the discussion the teachers' attention shifts from management issues about the role of tools to other pedagogical issues closer to students' learning. However, arguments were not grounded in specific reasons related to learning but reflected the teachers' overall attitudes about teaching and learning.

Linking the Tools to Students' Mathematical Processes: The Tools as Tools for Thought

In the debriefing session after the first implementation the discussion about the appropriateness of tools again is prominent. Here, the teachers themselves address this issue. The teacher (Je) who implemented the lesson expressed concern about the way that the tools operated in his classroom.

Je: - what I noticed today was that the tools that we used, to some of them, became more of a distraction than of help. I tried to guide as much as possible without telling them what tools to use and what not to use. So far as getting that lesson out in the future, I do not know if I am going to maintain the same tools"

The arguments that Je offered were based on his observation that only a small portion of the class actually used the non-conventional tools such as squared transparencies. Choosing what tool was appropriate seemed to Je too difficult for the students. Several teachers responded to his impressions with justifications and arguments supported by different examples. In a number of cases, teachers interpreted students' difficulties as lack of familiarity and time for exploration.

El: I would just like to say that as problem solving strategies take time, tools take time also and I think, as you said, this was the first time maybe that they've seen some of them… I think they need to go through that trial and error almost with them to really begin to appreciate the value of the tools and how to then apply them…A calculator is not always the most useful tool and at the very beginning they think that this is what I need, this is what is going to solve my problem. So, we may need to give kids more exposure time to the tools.

Ma : *The more chance you give them to use tools in the classroom and I don't as much as I would like to, probably because of the time factor, that is what we have to get through and plus the … there is just so much to do that wouldn't it be nice to have the whole morning just to relax and slowly do this problem, you know what I mean.*

Other teachers support the effectiveness of tools to connect mathematics with problem solving in real life, one's power to select and reject particular tools, and how those tools might facilitate the learners development of flexible mathematical thinking.

Va: We need to allow for students to connect to real world mathematics and that problem solving is done with tools. A plumber has his tools, a chef has his tools, a mathematician has tools, too, and in the macro scheme of things everything becomes a tool. There was a subset that could envision the toolbox as a problem-solving tool. I just wanted to report back how important it is and you (Je) definitely need the connections for the students.

Do: I just think is so valuable for them even to use the wrong tool, to find our that there is a path that's going nowhere and then come back. This is a very valuable lesson to me. They try something, they take the risk, it doesn't work and they have to come back and start over.

Ta: At the beginning it was like: 'Why do we need this clear sheet?' I said: ' I do not know…. Use it as you please.' But five minutes later it was just like every one of them was using them. The two people there (with the Puerto Rico flag) were using the ruler and the other two on the other side (with the Japan flag) were using the grid and counted. You know, it was an easier task just to count. What was more difficult was to use the ruler to find area. And so the circle kids (the Japan flag) … I was surprised. Nobody else used the formula sheet, but they did because they had a circle, so how do you count it on the grid, so they put the grid aside. So we do not have to tell these girls to use the formula sheet, they just found it, take the flag, measure the diameter, take half, do the radius… But as I say nobody told them to do that, that's problem solving!

Transferring Experiences Across Different Tools: Mathematical Formulas as Tools

The teachers' observations that many students "*jumped into the formulas*" without understanding during the first two implementations, when each pair of students were provided a list of formulas, led them to hypothesize that the availability of mathematical formulas as a tool might prevent students from exploring mathematical ideas and constructing their own mathematical meaning. In the debriefing of the second lesson the teachers began to see another dimension in the role of tools, that of building connections among different representations and strategies for solving a problem.

Ta: They need to learn how to transfer from using the grid to pick up what formula to use. We are actually doing areas and perimeters now (in school) and I give them the grid and they count the squares and I give them the problem without the grid and obviously they do not know what to do.

El (referring to a recent workshop held as a part of the lesson study project): *Go back to what Makoto (Yoshida) said… We go back to basic shapes and how to manipulate the trapezoid to get something that it is meaningful for you and then you can still do the counting*."

In the final two implementations, the teachers excluded the formula sheet from the "toolbox" given to the class. In those sessions the students used a number of strategies that demonstrated their understanding of the main mathematical ideas of the task. In Va's 7th grade class, several pairs of students calculated the ratio of the red part to the whole by dissecting the flag into its various shapes, transforming these shapes to rectangles and calculating the areas. Two students, J and G, first measured the dimensions of the Antigua flag attempting to determine the fractional area of its two red triangles. Then, noting the symmetry, G reached for the paper copy of the flag and with his pencil divided it vertically into four parts and explained to J.

G: There is 1,2,3,4. If this (the triangle on the left side) *is one-fourth, this* (the triangle on the right side) *is one-fourth, it's equal to one-half!*

El, in describing the approaches used in her 6th grade bilingual class, made the following observation:

El: What fascinated me is that I never saw anybody writing a formula and I never saw anybody seem to think about the formula for some reason. Most of these students look like they went to some sort of dissection and using grid and using the idea of shape more than looking length times the width or looking at the area of the circle, even the circle.

During this discussion the teachers related these observations to their previous discussion about the use of formulas as they reflected on the students' overall mathematical experiences in the classroom. In the final debriefing meeting the tools are related to the development of students' mathematical strategies but not in a cause effect perspective. The teachers, based on their final implementations of the activity, their analysis of the work of the students, and their own reflection, appear to be much more aware of the complexity of that relationship.

Conclusions

The teachers' participation in the project over time did affect their actual practice. As individual teachers made decisions about the implementation for their students, they discussed and enacted changes both in the kinds of tools given to the students and in the way they were provided. They also moved towards more systemic conceptualizations of the tools grounded both in mathematical and pedagogical perspectives. For example, they considered the role of tools as an ongoing part of learning, based on their analysis of students' activity and on their developing understanding of mathematical learning. They also made connections between their own ways of reasoning and thosse of their students, questioning practices that they had followed in the past and showing increased awareness of the potential of alternative teaching approaches. These teachers, during the second half of the year, participated in another Lesson Study cycle. Preliminary analysis of those sessions provides evidence of their continuing attention to the issues described in this report and leads to the important question of whether and in what ways this close attention by teachers to the way students select and use tools in solving open-ended problems can be sustained over time.

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REFLECTION GONE AWRY

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This six year case study of a novice secondary mathematics teacher provides insights into how a teacher's drive to become a more reflective teacher impeded her teaching and her professional development until she left the profession and took the time to contemplate teaching from different perspectives. The findings suggest that teacher education, like education in general, must adapt to teachers' individual needs. Constant or skewed reflection on one's teaching practices can be debilitating and may be an inappropriate goal for some teachers.

Reflective teaching (Schön, 1983) has become a mainstay of teacher education programs and is well-aligned with the NCTM Principles and Standards as well as many state professional development standards for teachers. Articles and books too numerous to mention suggest reflective teaching as an integral dimension of teaching. According to Dewey (1933), reflection or reflective thought has the power to not only influence action but to change one's way of thinking. Therefore many teacher educators, including the ones in this research project, use reflective teaching as a framework to guide novice teachers as they develop pedagogical content knowledge (Shulman, 1986). Yet as mathematics teacher educators we must realize that too much reflection can impede progress and cause undue anguish. With overwhelming attention to the positive nature of reflection, educators may neglect to attend to negative repercussions of reflective thought or in other words, reflective thought that has gone awry. Findings from this six-year case study suggest that helping novices build from their embedded traditions (Arvold, 1998) promotes professional development, but that too little attention to ongoing and unresolved personal issues together with a push toward becoming more reflective can take a teacher over the edge.

This study was part of a large DOE-funded partnership project[1] that supported teacher education curricular design that promoted a greater understanding of how to teach diverse student populations in high-need schools. Within this large project, participating mathematics education researchers used the idea of Embedded Tradition as Springboard (Arvold, 1998; MacIntyre, 1981) to design programs and mentoring opportunities to prepare and sustain novice teachers' work with at-risk students. Research participants in a secondary mathematics cohort of fifteen mathematics majors studied together for two years and then four of them chose to accept positions in a nearby high-need district. The novices became determined to help students become empowered in mathematics. Rose, one of those teachers, provided us with unexpected insights into possible repercussions of the reflective process.

Rose (pseudonym) became a target participant during her second year in a two-year teacher education program. She remained in the study during the following two years as a high school mathematics teacher. During that time, she also participated in a monthly research/support group

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composed of the other three program graduates who were working in the district and the six-member research/support team. She continued in the study during the following year, her first year away from teaching and also the year after as she began full time graduate studies in mathematics education.

Research Design

Structural symbolic interactionism (Blumer, 1969) framed the study and supported the case study goal of better understanding Rose's sense-making as she began taking, playing, and making (Arvold, 2003; Mead, 1934) the role of secondary mathematics teacher for herself. Using the constant comparative approach and member checks, our research team analyzed data, coded and recoded data, and, during weekly research meeting shared ideas and participated in decision-making. Our focus was to understand the embedded traditions of our participants and help these novices springboard from them to construct their identities as mathematics teachers. Our searches for disconfirming evidence were continuous. The data included 6 years of electronic mail messages, journal entries, observation fieldnotes and pre- and post-interviews during Rose's two years as a university student and her two years as a high school mathematics teacher, monthly group interviews during the teaching years, and at least 3 additional individual interviews throughout each of the 6 years of the study. A natural flow from semi-structured interviews to participant-directed and participant-initiated interviews reinforced the research emphasis on teacher voice.

The Story of Rose

As true of most in her preservice teacher education cohort, Rose had a firm grounding in mathematics, polished leadership skills, and a dedication to teaching. In comparison to her classmates, she had a better feel for *reading* students and coming to understand how they made sense of mathematics. Personal struggles after the loss of her mother and renewed confidence on discovering herself as a mathematician during community college studies provided her with great insights and strengthened her desire and ability to help others become empowered in mathematics. Her burgeoning spiritual life was also contributing to her intense desire to help others.

She enjoyed working individually and collaboratively within the research-based and NCTM Standards-driven teacher education program, a program that encouraged students' personal professional growth through reflective practices and goal setting. The program also encouraged a healthy skepticism of research and of the multitude of standards that were entering the field of teacher education. She was recognized by her classmates as a most reflective student and provided plenty of evidence that supported this notion. Her final university project included reflective soliloquies both before and after the required classroom reenactment. In a journal entry she noted that she could identify with fellow classmates who had struggled with a conic section activity and added a comment representative of her typical style of self reflection.

> I'm not worried about being able to teach it – I think once I look at it in the book it will be easy to brush up on what I forgot. It just made me think a lot, realizing that most of us are very good at learning from traditional methods, yet still forgot much of what we learned that way, at least in terms of conics We should not be surprised at students who have the same problem.

Rose also shared more philosophical ideas. After a class discussion of parental and teacher responsibilities toward educating students, Rose's thoughts extended to how we describe people.

> The conversation about parents today was very interesting. It got me thinking a lot about the perspective of the "average" parent. I found that very difficult to understand, quite possibly because such a person is a mythical creation. I am coming to the conclusion that there is really no such thing as an average student or an average parent. It is a creation we have made to make people easy to compartmentalize, but I don't think they really exist.

Rose was also very down to earth and learned all she could about classroom organization and management during her student teaching. During my first observation of Rose as professional teacher, I observed students accessing file folders to deposit or pick up their work. The room was freshly painted (yes, she had painted it herself) and all was in perfect order. The classroom rules were quite clear to the students and most students followed them. Rose felt she was to blame for the few that did not.

Rose truly engaged students in mathematics. During a second year teaching observation, we noticed that students who had participated minimally in mathematics classes in the past were actually investigating strategies and performing rigorous mathematics during class. Yet Rose was disappointed for they did not progress as she had expected. Her caring disposition was obvious in the classroom but she was almost too calculating. As an observer I could imagine her conversation with herself as she taught. Students may have sensed her deep level of concentration and reflection as well and, especially during her first year as teacher, a few continually acted out in protest. In a post-observation discussion, she shared how relieved she was to hear my positive comments about her teaching actions. She had been thinking that she was not teaching well at all. She invited us to return often but at least a month usually passed before we returned to her classroom again and she was disappointed.

During a visit during Rose's second year teaching, the class flowed smoothly and the students were attentive and most responsive. She had developed a routine and stuck to it religiously. Rather than recite answers during class, students explained multiple ways of interpreting and solving mathematics problems. They looked back in their notes to find hints. They seemed to be trained well but displayed no passion for mathematics. Rose was disappointed. The students were not becoming empowered in mathematics even though they were now behaving properly and responding to her directives. She felt that she was doing something wrong and her frustration grew.

The support group meetings did little to allay Rose's frustrations. In fact, during one meeting, she actually broke down into tears after interpreting comments of a team mentor/researcher as evidence that this mentor/researcher totally missed her point. The discussion centered on ways to encourage students to take their homework seriously. She felt misunderstood and misguided. Fortunately she viewed my relationship to her as safe refuge and she shared with me her frustrations and a bit of family history that helped explain her reaction. She felt violated by a member of her own support group, and memories of earlier personal experiences had welled up inside her. Fortunately she and I worked things out and she continued to invite and accept our help with goal setting and planning but in retrospect, we believe we failed to provide the support she dearly needed. We did not understand the depth of her frustration. The school administrators were also oblivious to the turmoil she was experiencing. The principal held her in such high respect that he provided her opportunities to be a leader in the department. She led teachers in designing programs for state test reviews and she also developed, directed, and taught in the new high school study skill program. To observers, she was a high quality teacher and her students' progress was outstanding but to Rose, things were seen differently.

Near the end of her second year of teaching, she became convinced that something major was wrong with her teaching. Her teaching frustrations spilled over into all aspects of her life and the lives of her loved ones. She decided to take a break from full time teaching to assess the situation and reevaluate her professional goals. Although she resigned from her position, she had no intention of leaving teaching aside forever and she became quite upset when hearing others say that she had quit teaching. Teaching was not only her dream but also her passion. Only later did our understanding of her frustrations become more clear.

During the first year away from classroom teaching, Rose pursued another great interest of hers, teaching English as a second language. She taught international university students, but soon decided that she was more interested in teaching mathematics students. She remained a mathematics teacher as well but through a different venue. The following email request was acted upon and it also provided our research team with insights into her frustrations.

> I was thinking about something on my way home yesterday. Surprise, surprise! I was wondering if you need any help mentoring any of the student teachers. I obviously don't know everything there is to know about teaching, by a long shot, but I thought if you knew of a student who just really needed some TLC and encouragement during student teaching that could be something I could do to help them not feel so overwhelmed.

She did indeed assist both university students and high school students during this year away from the demands of full time teaching. We, as researchers, finally realized that her depth and breadth of understanding teaching and learning mathematics was overwhelming her as she reflected in and on actions in the classroom. She had had little time to relax and simply enjoy her teaching.

After a year of assessment of the situation, she applied for graduate studies in mathematics education. She stated her purpose quite clearly in her letter of application, "I have three main purposes for pursuing the M.S. in education: to grow as a teacher, as a leader of teachers and as a researcher." She realized that she needed to learn more to be all she wanted to be. She was searching for the missing pieces of the puzzle and she wanted to help others do so through her leadership and research. But most telling was her epiphany shortly after she began her studies.

> I just read the most amazing article…it is radically shaking my world!! It's in line with what you were telling me this afternoon, and this whole idea that the standards might not work for everyone, or in the same way for everyone is really shaking me up, in a good way! Thinking back, I realize how much I believed that the standards really were the way to teach everything, and if only I could just implement them sufficiently, all my students would excel. It never crossed my mind to challenge them [the standards], to think there were contexts in which they might not work… It's like a person with a broken computer who just keeps turning it on again, certain that if they press the button in just the right way it will work. I felt so bad that I could never implement them fully enough, which to me was evidenced by the fact that my students weren't learning. If I was implementing, then my students would be succeeding. It never occurred to me that there could be another factor at work.

Discussion

Yes, reflection is a positive attribute and one seemingly necessary for professional growth to take place, but too much of a good thing can be detrimental. Teachers like Rose may bound their reflectivity within an unproductive framework. They may be blinded by their sense of urgency as they try to teach at-risk students. Also in play with Rose were unresolved issues that led her to be overly critical of herself before entry into the program. As her instructor I had not realized how general instruction about reflective teaching had heightened her frustrations. I had supported her throughout the years by recognizing her great reflectivity but she had responded by trying to be even more reflective. I did not realize that I was ignoring personal/professional needs that influenced who she was as a teacher. We had succeeded in helping her springboard from her embedded tradition, but we had neglected to attend appropriately to the historical, the ongoing, and the visionary aspects of a teacher in the process of becoming a mathematics teacher.

Descriptions and types of reflection dot the landscape of teacher education research yet few if any speak to the challenges that arise if reflection is skewed in ways that paralyze a teacher. We as mathematics teacher educators/ researchers might consider how teachers' use of reflection can hinder the synergy that unfolds as quality teachers try to construct and sustain their roles as teachers.

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THE ISSUE OF FLEXIBILITY IN ONE STUDENT TEACHER'S USE OF MANDATED CURRICULUM MATERIALS AND OTHER INSTRUCTIONAL RESOURCES

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This report describes one student teacher's experiences teaching elementary mathematics with and without the support of mandated curriculum materials. Heather, the student teacher, planned and taught the majority of lessons utilizing the mandated materials, but also had the opportunity to create one particular lesson using a variety of alternative instructional resources. This alternative lesson creation appeared to offer Heather a chance to be more creative and flexible in her planning and lesson enactment. This experience not only afforded Heather an opportunity to personalize her mathematics instruction, but also shed light on and provided potential solutions to problems inherent in her more frequent use of the mandated curriculum. Implications for teacher education include providing opportunities for student teachers to analyze the materials they are asked to teach with and also encouraging lesson creation, at times, outside of the mandated materials.

Reform-based mathematics curriculum materials, developed in response to the National Council of Teachers of Mathematics [NCTM] *Standards* documents (NCTM, 1989, 2000) have been in use, to varying degrees, for over 10 years. Studies of teachers' use of such curriculum materials have offered important information about ways mathematics instruction emerges through interactions between teachers and texts (Lloyd, 1999; Remillard, 2000). This report considers issues surrounding the use of reform-based curriculum materials by student teachers. In contrast to more experienced teachers, for whom reform-based curriculum implementation commonly involves making the transition from traditional instruction and textbooks to more reform-based visions, many beginning teachers' use of reform-based curriculum materials represents their first experience teaching with instructional materials of any sort. At different points in time, it has been popular among teachers and mathematics educators to downplay the role of mathematics textbooks in planning and lesson enactment—for example, Ball and Feiman-Nemser (1988) found that student teachers emerged from their teacher education programs with the impression that good teachers avoided following textbooks and relying on teacher's guides. However, recent research has suggested that many beginning teachers greatly appreciate the guidance of published curricula (Kauffman, 2002; Manouchehri & Goodman, 1998; Remillard & Bryans, 2004). Given that most teachers generally view their student teaching internship as the most valuable and beneficial part of their preparation (Feiman-Nemser, 1983; Guyton & McIntyre, 1990), as student teachers are placed in school systems utilizing reform-based mathematics curriculum materials, it becomes increasingly relevant to examine their experiences utilizing such resources.

The study described in this report focuses on one student teacher, Heather, and her experiences teaching mathematics with and without the support of mandated curriculum materials. We were particularly interested in how Heather's development of mathematics instruction was impacted by different experiences with curriculum. We share Heather's planning routines, issues surrounding lesson enactment, and overall reflections in order to highlight those differences. In this study, we assume a bi-directional view of curriculum use. Called "curriculum use as participation with the text" by Remillard (in press) and "curriculum enactment" by Snyder, Bolin, and Zumwalt (1992), we focus not only on the ways in which Heather engaged with, interpreted, and adapted curriculum, but also on the ways in which Heather experienced and learned from her different uses of curriculum materials.

Methods

The Student Teacher and the School

Heather participated in student teaching during her final year in a 5-year teacher education program at a large, state university in the southeastern part of the United States. As an undergraduate student, Heather participated in one course focused on the particular mathematics content needed in elementary classrooms. Heather also participated in a mathematics methods course the summer before her fifth year in the program that included examination of the reform-based *Everyday Mathematics* (UCSMP, 1997) curriculum she used throughout her student teaching experience. For student teaching, Heather was placed in a first grade classroom at Clayton Elementary, a local suburban K-5 elementary school. The school utilized the county-mandated, reform-based *Everyday Mathematics (EM)* curriculum for mathematics teaching in all grades. Heather taught the 13 students in Ms. Greene's classroom for 12 weeks.

The Data

Most data were collected through classroom observations and semi-structured interviews. We visited Heather's classroom 15 times throughout the semester, with most visits taking place in 2-3 day consecutive blocks. When we visited, we interacted minimally with the students and with Heather and maintained a primarily non-participatory role. No member of the research team served as a university supervisor for Heather during the semester.

Heather was interviewed seven times during student teaching, both before and after weekly observations. Each of the first five interviews were focused on what Heather had planned and anticipated for upcoming lessons, or on reflecting and talking about lessons already taught. The final two interviews focused on Heather's experiences teaching mathematics in general throughout her student teaching experience. Artifacts and documents that included Heather's lesson plans and student teaching journals were also collected.

Analysis

All fieldnotes were typed up within 48 hours of each classroom observation, and analytic notes and memos were written at the end of each file. All interviews were transcribed before subsequent interviews were conducted in order to clarify any questions we might have and also to generate new questions related to areas of interest. During the more intensive data analysis phase of the study, electronic files were created to combine individual classroom visits with corresponding interview excerpts from before and after lesson enactment. As particular themes emerged from the larger data set, lesson segments and individual discussions that appeared to highlight the multiple facets of Heather's experience were selected for inclusion in this paper.

Findings

Throughout Heather's 12-week student teaching internship, her experiences with mathematics instruction centered primarily on the *EM* curriculum. Given the extensive amount of data obtained regarding Heather's experiences using the *EM* materials compared to much less data collected in regard to her experience creating a lesson without the use of the mandated materials, the two findings sections that follow are not meant to provide a comparison. Instead, we first present Heather's experiences as she taught with the *EM* materials in an effort to highlight her use broadly across student teaching, and then discuss in greater detail the less common experience Heather had creating a lesson without the *EM* materials.

Experiences with the Everyday Mathematics Curriculum

Heather worked very closely with the *EM* teacher's guide in her planning for individual mathematics lessons throughout her student teaching. Each weekend Heather prepared for the upcoming week's mathematics activities using a copy of the *EM* teacher's guide to help organize lesson structures. Heather explained, "I just read through this and make notes about the topics or kind of just generally. These aren't really detailed [points to her hand written planning book], and when I'm teaching, I have the teacher's manual up there with me" (Int. 1, 3/8/04). Although Heather reported reading through all of the material in the teacher's guide, she commented frequently that there was just too much material for her to cover in one 50-minute mathematics session: "There's two other sections that [the authors of the curriculum] think you should get to apparently in a math lesson and we're lucky if we get to the second one. That's just—that's been an issue" (Int. 2, 3/10/04).

Although Heather did regularly omit the last section of the *EM* lessons, she utilized particular questions, wording and explanations as provided in the main section of the teacher's guide. In order to have access to these suggestions, and to overall ordering of lesson components, during most lessons Heather had the teacher's guide with her in the front of the room, so detailed notes about what she was planning seemed to her to be unnecessary. Heather talked frequently about the scripted nature of the teacher's guide in relation to her desire to have the guide with her:

> I feel like it's a script, so I always have this book with me 'cause I never have it memorized. I'm never really like, 'okay, I know to go from here to here to here.' I always have it here so I can remember, 'okay, this is what I wanted to do next.' A lot of times, I just feel like if I miss a paragraph in the book then maybe that will throw the lesson off. So a lot of times it's too much information. (Int. 2, 3/10/04)

As Heather taught with the *EM* materials her continual focus was on lesson pacing. Heather commented frequently that working through the activities as posed in the *EM* curriculum got her "all messed up in the end" (Int. 1, 3/8/04) as she scrambled to get through each lesson on time. Heather described this lack of time in regard to planning and explained,

> I get so stressed out at the end of the lesson…You've just got so much to do in the last five or ten minutes. But that's just everyday, so that's nothing new. When I'm teaching other things that I've planned myself, I know what can be cut out. I know, "okay, I can start wrapping up from this point." I can just time it better. And I still run over and I still run out of time, and do the same things in other subjects, but it's definitely worse in math. It's definitely the toughest subject. I think it has a lot to do with the book because I'm like, "if it's all here I want to get it all in," and it's just not possible. (Int. 2, 3/10/04)

As student teaching came to a close, Heather summed up what she learned about teaching with the *EM* materials as she offered mathematics teaching advice to other student teachers who may be placed in the same classroom as she was placed this year:

> I would probably tell them the same thing that Ms. Greene told me—don't worry so much about sticking to the script of that day's lesson. It's okay to throw in your own questions and

> to pull away from what the teacher's guide says you should be saying and asking…cause that was very hard for me to get used to doing. When it's there, it's easy to think, well, "there it is," you know, I don't really need to be creative or worry about anything else—it's going to tell you what to do. I would just say to not worry so much about exactly what the book says—to kind of use it more as like a guide then a script. (Int. 6, 4/27/04)

In some sense, these words of advice echo what Heather feels took her an entire semester to discover. As Heather pointed out, before the start of student teaching, she was very excited to find that she was placed in a school system utilizing *EM*. Heather quickly came to realize, however, that curriculum use was much more complicated than she had anticipated:

> When I started out I thought, "I'm going Clayton County for student teaching…Great, math will be planned. I don't have to worry about it. I'm just going to have to be working on all the other lesson plans that aren't laid out for me." And, it's just been ironic because I think I've struggled the most with the scripted program. (Int. 2, 3/10/04)

Experience Creating and Teaching a Symmetry Lesson

Toward the end of her student teaching experience, Heather was given the opportunity to create one lesson without the use of the *EM* curriculum materials. Because Heather's cooperating teacher, Ms. Greene, had experienced a variety of difficulties with the particular symmetry lesson offered by the *EM* materials, she suggested that Heather create her own introductory lesson on symmetry. Heather expressed that she was "really excited cause I got to write my own lesson!" (Int. 3, 3/22/04) and explained that her cooperating teacher had brought in several old textbooks and teachers guides. Heather explained, "I looked through them, and one of them had an idea that I liked, so I kind of went with that one—but I sort of changed it a little" (Int. 3, 3/22/04). Heather's biggest change was taking the whole class activity as posed in the old textbook she utilized and converting it into small group activities. As Heather explained:

> Each group is going to get a sheet of paper and it's going to be divided into two columns and ones going to be symmetrical and ones going to be not symmetrical. I'm going to give each group a bag of shapes. Then, as groups, they glue them on the right column and then we'll just talk about it and talk about what symmetry—actually I think we'll start off talking about what symmetry is. If there is time, which I think there probably will be—and I haven't decided if I'm going to start with this or let it be an ending activity to make sure that there's time for it—I will give them paper and scissors and say, "fold it and cut and make something that is symmetrical and make something that isn't symmetrical." It'll be a very hands-on day. I'm excited. (Int. 3, 3/22/04)

When we spoke with her several hours after teaching the lesson, Heather exclaimed, "I thought it went great. I had fun with it and I was so relaxed. It was probably the best math lesson I've had this semester. I kept checking the clock and I'm like, "we're okay on time, everything's great!" (Int. 4, 3/24/04).

The decisions Heather made throughout this particular lesson seemed to be based primarily on student understanding. As she wrote in a formal reflection after teaching the lesson:

> I felt like I had rushed through the "introductory" part of the lesson, which usually takes up a good amount of the lesson. I was conscious of this as I was teaching, and for a minute I thought maybe I should slow down but then I quickly decided that I wanted them to get the bulk of their knowledge in this lesson from the group activity…. That was one of the best impulse decisions I have ever made! The students were excited to begin their group

> work…and I watched with pleasure as they justified their reasoning with each other and came to some sort of an agreement. (journal entry 2)

In addition to Heather's mid-lesson decision to move quickly through her introductory activity, she also added a new component to the lesson as she circulated to each group and listened to the struggles and discoveries being made. Heather commented:

> I just decided as I was up there to have [the students] come up and talk about [their symmetry posters]. And there were different things in each group that were unique to that group. So, I was like, well, we'll just let them share that. And we had time for it. And then we *still* had time to do the worksheet. (Int. 4, 3/24/04)

One group and one student's comments in particular seemed to have influenced Heather to incorporate group sharing as the lesson progressed. Heather explained:

> A lot of [the groups] put the shamrock on the symmetrical side and I looked at it yesterday and it's not [symmetrical] because of the way the stem curves out. There was one group who pointed that out and I asked them, "why did you put that there?" Roger was like, "well, because the stem curves, you can't do it." I was like, "oh my gosh, that was very observant of him." I wanted him to be able to share that. That was such a moment for me, cause there've been problems and just concerns there, and that was just—I loved it. (Int. 4, 3/24/04)

Overall, Heather felt that this was "one of the best lessons, especially the best math lesson," (journal entry 2) that she taught all semester. Because the symmetry lesson fell just over a week before the end of student teaching, Heather knew she would probably not have an opportunity to create another lesson utilizing alternative resources, although she expressed interest in doing so:

> I would say that I would definitely look more into doing my own things, or maybe taking some of the Everyday Math stuff out and putting more of my stuff in even if we kind of followed along with this [pointing to the *EM* guide]. It is definitely something I would want to do cause I feel like it went a lot better than some other lessons. (Int. 6, 4/27/04)

As the semester came to an end, and as Heather talked more about her expectations for the future, she discussed mathematics textbooks in general. Heather felt as though

> something in the middle is best. There [would be] enough information there to chose from, but it's not like you have to do this, and then go to this, and then go to this, and try to get to here. But then you're not just like, "where do I start, where do I get to?" either. I just think something like that, a little less than *Everyday Math* would be ideal. (Int. 6, 4/27/04)

Discussion

During her student teaching internship, Heather was afforded extensive opportunities to use and think about curriculum materials. Throughout her 12-week teaching experience, Heather's mathematics instruction was structured, almost exclusively, around the *EM* curriculum. Given the school-wide culture of acceptance and adherence to these materials, it is not surprising that Heather relied heavily on all the resources provided. Similar to the student teachers and beginning teachers from Kauffman's (2002) and Manouchehri and Goodman's (1998) studies, Heather drew heavily upon the teacher's guide for planning, instructional decisions and assessment. However, while Heather appreciated the planned mathematics lessons and relied on the details in order to pragmatically spend smaller amounts of time planning than in other subjects, Heather felt constrained and at times overwhelmed by the amount of detail and structure provided in the *EM* materials. Because Heather's experience teaching mathematics was so closely tied to using the *EM* curriculum as mandated, she spent much of her time learning how to use the resources—and the *EM* teacher's guide in particular—most effectively and efficiently.

In teaching with these materials, Heather also focused much of her attention, both in planning and lesson enactment, and even in her discussions about lessons after enactment, on lesson pacing and overall concern with time. Heather was met consistently with the issue of how to fit into one fifty-minute mathematics session everything she had planned to do from the *EM* materials and her decision-making appeared to be dominated by how to get students through each part of the lesson efficiently. Heather's experience teaching mathematics was focused on becoming more and more adept at delivering the curriculum.

Heather's experience planning and enacting a lesson based on a variety of resources, however, provided her an opportunity to be more creative and flexible in both her design and delivery of a mathematics lesson. As Lampert (2001) points out, "Lesson preparation involves figuring out how to connect particular students with particular mathematics…[and on] how to engage *this* class, with *its* particular variation of skills and understanding, in the study of ideas surrounding *this* piece of mathematics" (p. 117). As Heather was able to be more flexible in her lesson planning and focus on what she knew worked well for her particular students—namely small group instruction—her planning for the symmetry lesson was much more focused on the needs of her students. Additionally, Heather's mid-lesson decisions were much more focused on her particular students developing understandings. Her lack of apparent difficulty with pacing as the symmetry lesson progressed and her mid-lesson decisions to modify her plans as the students explored offered a markedly different teaching experience for Heather—an experience focused much more on teaching mathematics to her particular group of students than on teaching mathematics from a particular set of curriculum materials.

These findings are somewhat different from Ball and Feiman-Nemser's (1988) study focused on student teachers use of and opinions about mathematics texts. Heather's situation was initially different from the student teachers in that study—she emerged from her teacher education program with positive opinions of currently used mathematics curriculum materials—yet similar to those student teachers, Heather had a difficult time figuring out how to use a mandated curriculum successfully. And while Heather did not necessarily leave with the opinion that good teachers create their own lessons, contrary to the student teachers in Ball and Feiman-Nemser's (1988) study she did encounter much success in developing her own lesson. What we find is that although these two studies were focused on student teachers' use of and thoughts about very different types of mathematics texts, one conclusion is the same—student teachers need help figuring out how to use *and* learn from published texts and teacher's guides. But as Heather's case highlights, new issues arise as student teachers are asked to utilize more reform-based curriculum materials as they teach for the first time. As Lloyd (1999) has suggested elsewhere,

> When a reform minded teacher uses traditional materials in the classroom, he or she may be afforded more room for personalization because the goals of the materials are so different from his or her own goals. Because reform-oriented curriculum designers accomplish much of the alteration of mathematical content and activity in their production of materials, teachers with strong and innovative visions may experience a profound loss of previously held opportunities to personalize their instruction. (p. 246)

When student teachers are required to use reform-based curriculum materials, they may focus much more on figuring out how to successfully use that specific curriculum—especially when the curriculum closely aligns with their philosophies about teaching and learning—than on how to personalize instruction for their particular students. For Heather, an opportunity to create a lesson utilizing a variety of resources was not only successful, but also shed light on problems inherent in her use of a particular mathematics curriculum. And although student teaching was almost over, it also provided an opportunity for Heather to consider how she might use the mandated curriculum in a more flexible and creative way in the future.

Recommendations and Implications

As teacher educators we need to help preservice teachers learn how to work more creatively with the reform-based mathematics curriculum materials they may be asked to teach with. If we want student teaching internships to be focused on more than just curriculum delivery, we need to prepare and challenge preservice teachers to question the materials they work with and the curriculum they are asked to follow. If we ask student teachers to analyze the curriculum they use—a recommendation made by Zeichner & Liston (1987) over 15 years ago—they may be better able to focus on *both* curriculum delivery in particular *and* on learning about mathematics teaching more broadly. As Heather's case highlights, it may be beneficial to encourage lesson creation, at times, outside of the mandated curriculum. This would afford student teachers an opportunity to not only personalize their instruction and attend closely to their particular students, but may also push them to work flexibly within lessons as posed in a mandated curriculum. As student teaching remains a widely accepted component of teacher education, studies focused on its improvement remain critical.

Acknowledgement

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INVESTIGATING THE COMPLEXITIES OF MATHEMATICS TEACHING: THE ROLE OF BEGINNING TEACHERS' BELIEFS IN SHAPING PRACTICE

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This research investigates the factors that contribute to secondary mathematics teachers' choices during teaching. Given the current climate of reform in mathematics education, it is particularly important we explore which influences and experiences are important as beginning teachers begin developing their practices. Two case studies were conducted with beginning (first three years) secondary mathematics teachers. Literature in the area of cognition and teachers' beliefs provided a lens for interpreting interviews and observations. Results suggest that teachers' beliefs about mathematics and its teaching and learning, knowledge, and context form an interconnected, complex reality that critically influences what happens in practice. This study raises important questions concerning how to best prepare teachers to teach in ways that support ideals of current mathematics education reform, as well as suggests directions for future research in mathematics education.

In this paper I examine the relationship between beginning (first three years) mathematics teachers' beliefs and practices as they develop within the context of their subject, classrooms and school. Teachers' beliefs is an important area of study for mathematics teacher education (Thompson, 1992). Teachers' beliefs provide a window to understanding their actions, experiences, and how they interpret events, and can therefore help us to understand teaching and the process of learning to teach. More research is needed that explores the complex relationship between teachers' beliefs, knowledge, and the realities of the classroom (Putnam & Borko, 2000; Wilson & Cooney, 2002).

To explore these issues, I conducted two in-depth case studies using an interpretive case study design (Merriam, 1988) of beginning high school mathematics teachers, Laurie and James, during the 1999-2000 fall semester (September–January). The following questions guided my initial data collection and analysis: (1) What are beginning mathematics teachers' beliefs about mathematics and the teaching and learning of mathematics, as they develop within the context of their classrooms and schools? What appears to be the relationship between beliefs and beginning mathematics teachers' practice? (2) Which aspects of the school setting and what factors in addition to beginning mathematics teachers' beliefs seem to be most important in their pedagogical choices and teaching?

Theoretical Framework

Within mathematics education research a constructivist perspective has influenced research on teaching (Putnam & Borko, 1997). Researchers view teachers as thoughtful and reflective individuals, whose decisions influence all aspects of classroom instruction and learning. Research suggests that mathematics teachers' knowledge is contained within a context of beliefs about what constitutes mathematics and what teachers perceive to be their role as teachers of mathematics (Cooney, 1994; Thompson, 1992). Teachers' knowledge and beliefs are thus critical in understanding mathematics teaching and learning. I define the term *beliefs* to encompass

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elements of evaluation and judgment. What teachers know and believe acts as a lens through which they interpret teaching and learning, thus significantly impacting what they choose to do in practice and how they interact with students (Cochran-Smith & Lytle, 1992).

Although using the lens of teachers' beliefs can provide insight into practice, it cannot completely explain the complex nature of teaching. For this reason, teachers' actual practice, as well as other contextual factors, must be investigated simultaneously to understand the role beliefs play, as well as what other factors influence choices made in practice. As Wilson and Cooney (2002) conclude in their review of the impact of beliefs on teacher change, "It seems that both observing and interviewing teachers are necessary if one is interested in comprehending how teachers make sense of their worlds," (p. 144). With the two case studies in this study, I generated detailed stories based on conversations and observations that can shed light on the role beliefs play in shaping practice of beginning secondary mathematics teachers. Additionally, I explore other factors that contributed to the decisions these teachers were making about their teaching, thereby capturing the complexities of practice and setting.

Particularly at the secondary level, more research is necessary to better understand the interaction between beliefs and practice. Often, there is a disparity between espoused beliefs and what a teacher actually does in the classroom (Benken & Wilson, 1996; Wilson & Cooney, 2002). Furthermore, teachers often have seemingly conflicting beliefs, which are dependent upon specific teaching contexts. It has been suggested that more research that examines teachers' beliefs as they develop within practice would be helpful to better understand this relationship (Ball, Lubienski, & Mewborn, 2001). Knowledge of subject matter and how students learn, opportunity for professional support, a teacher's level of consciousness of her own beliefs, and social context also play a role in shaping practice (e.g., Cooney, 1994; Wood, Cobb, & Yackel, 1991).

Although some researchers are now looking at the context within which teachers' practices are developing, more research is needed to understand the complex relationship between teachers' beliefs, knowledge, and the realities of the classroom. Especially during early teaching experiences, existing personal theories of teaching and learning are reconstructed, thereby allowing for refinement of old beliefs and creation of new beliefs (Tobin, Tippins, & Hook, 1992). Beginning teachers' practice is an important and unique window to understanding teaching and the process of learning to teach.

Methodology and Analysis

Primary data sources collected over one semester included: interviews (12) and classroom observations (15) with each participant, and interviews with principals at participants' schools. Data were audio taped and transcribed. Laurie was in her third year of teaching in a large, suburban district located nearby a large, metropolitan city. James was in his first year of teaching in a small, rural town. Main selection criteria for participants included years of teaching experience, educational background, and willingness to participate.

Analysis of data took place in three stages: (1) First stage occurred simultaneously with data collection using direct interpretation (Stake, 1995) to identify important themes in participants' conceptions and experiences. To determine level of importance of themes frequency (stated and observed), level of emphasis articulated by participants, and responses provided by participants when specifically asked about developing interpretations were considered; (2) Second stage involved rigorously reviewing, reflecting upon, and comparing across all data following data collection; and (3) In stage three, all themes and assertions were tested and clarified by further reviewing data and checking for confirming and disconfirming evidence. Analysis occurred within and across the individual case studies.

Results and Discussion

The findings suggest that all theorized factors (e.g., teachers' beliefs about mathematics, and mathematics teaching and learning, teachers' content knowledge, and teachers' perceptions of context–school and classroom) are related in complex ways and played a role in shaping these beginning teachers' decision-making and practices. Although these factors influenced both teachers' practice, *how* they manifested in practice was individual to each teacher.

One central finding was that *aspects related to mathematics* (beliefs about mathematics and mathematics learning, depth of knowledge of mathematics, and perceptions of mathematical ability) were pivotal in shaping these teachers' practice. For example, Laurie's primarily traditional views about mathematics and its learning, interwoven with her conceptually limited content knowledge and perception that she could not teach high level mathematics, manifested in her having students follow teacher-led discussions, utilize memorized procedures encompassing networks of memorized facts, and practice content she considered basic.

On the other hand, James' strong and flexible content understandings and confidence in his ability to learn mathematics appeared, to some extent, to define his belief that mathematics involved making connections and thinking through problems using multiple approaches. James' pedagogical choices involved having students articulate their thinking about concepts in writing and during whole-class and small group discussions, generate new problems based on explored concepts, and identify and construct multiple approaches to problems.

Another central finding was that these teachers' affective beliefs related to their students, intertwined with and often supported by their beliefs about mathematics teaching and learning, also played a meaningful role in their decision-making and pedagogical choices. Most notably, a teacher's vision of her/his role as a teacher of mathematics and perceptions of her/his students' affect, which are both influenced by context and intimately connect to how teachers believe students learn, frame a teacher's *purpose*. This purpose can then guide teaching practice. Within the cases in this study, this cognitive and contextual frame of purpose influenced, along with the beliefs and knowledge discussed in the previous section, what mathematical content was emphasized, as well as how lessons were structured and how these teachers practiced caring.

For example, embedded in the choices Laurie made was a genuine concern for her students and their success. She considered it her role as a teacher to prepare them for future classes, as well as career/university. However, because of her beliefs related to what her students needed due to their low socio-economic position in society, she struggled to find a balance between covering material at an acceptable level of mastery and maintaining a positive, caring environment. Laurie's concern for students often interfered with her ability to remain focused on mathematics. Her formulaic approach to learning and beliefs related to student affect facilitated practices that appeared rigid compared to the ideas consistent with reform and usually did not support preparedness in terms of conceptually-based mathematical understandings. Although Laurie articulated that she focused on content understandings as well as basic skills, much of her observed practice involved spending time on tasks not centered on intended mathematical concepts, but rather on supporting students' mathematical self-esteem and motivating them extrinsically. This finding corroborates other research that suggests teachers' pedagogical choices are often connected to their perceptions of students' lives and needs according to socioeconomic variables (e.g., Sztajn, 2003).

Contrastingly, most of the affective issues to which James devoted time were centered on issues of student interest–in mathematics, learning mathematics, and in his classes. James believed that his students' perceptions of his passion for his subject and teaching would peak students' interest and motivate them to learn mathematics. Due to this belief, James readily shared his love of math with his students. He encouraged students to bring in math games to share and play at the end of class, as well as work through extra problems with him at the board when they were done with their assigned work. Most of James' pedagogical choices centered on their content understandings and attempts to connect the mathematics to their lives. To James, having students perceive him as excited about mathematics and teaching was paramount. He believed this goal to be the first step in students' openness to learning and feeling positive about the learning environment, as he saw students as only caring about that which they perceived the teacher to care. James enjoyed mathematics and believed that through content-focused, applied, and cooperative explorations, he could instill a regard for mathematics in his students as well. James showed caring toward his students by applying consistent expectations for all students, providing choice in how they learned mathematics, and creating an environment that centered on an enjoyment of mathematics and learning mathematics. As a result, he designed lessons that he believed would be meaningful to his students–meaningful in terms of cognitive-need based instruction and connections/application.

Conclusions

For this study, I sought to understand and describe beginning secondary mathematics teachers' beliefs about mathematics, and the teaching and learning of mathematics, and investigate the relationship between beliefs and practices as they developed during one semester of teaching. In order to create a complete picture of this complicated relationship, I also carefully explored the context within which these teachers' beliefs and practices were developing, as well as identified other factors (e.g., content knowledge) that seemed to be important to how their ideas manifested in action. Information learned from these case studies sheds light on how these teachers negotiated and thought about their experiences, and provides compelling evidence that beginning mathematics teachers' content knowledge and beliefs must be addressed and expanded in order to support them in teaching in ways that focus on conceptual understanding and incorporate ideals of the reform. Implications from these case studies are twofold: (1) They raise questions related to assumptions we make and current practice in mathematics teacher education, and (2) The conceptual underpinnings of the results can aid the field in developing a model that synthesizes the many factors that shape beginning mathematics teachers' practice.

While both teachers expressed genuine goals of improving students' conceptual understandings of mathematics, and when possible in ways that were aligned with existing reform ideals (e.g., NCTM, 2000), they readily communicated that other factors often took precedence. They did not always have time, support, or perspective needed to accomplish what they ideally wanted in their teaching. However, both teachers were perceived as good teachers, and in Laurie's case, a model for others to follow. These findings therefore raise important questions: What is the basis by which our teachers are being judged within their setting? How can we as teacher educators and mathematics education researchers help to expand the existing vision of accomplished mathematics teaching? Furthermore, how can we as teacher educators/researchers help teachers integrate what they often see as foreign practice with their existing vision of successful mathematics teaching? Perhaps we need to generate a middle ground, or simpler pedagogy, that will not be so difficult for teachers to align with existing knowledge, beliefs, other obviously important teaching goals, and the realities of beginning teaching.

Finally, we need to consider what kinds of support we provide beginning teachers. The cases of James and Laurie suggest that beginning teachers need models of teaching that challenge and ideally expand their existing visions of good mathematics teaching. Particularly in the case of first year teachers, they need to also receive on-going assistance and advice related to the daily realities of teaching (Gold, 1996). Additionally, in spite of large percentages of young teachers in many K-12 buildings, beginning teachers should not be assigned too many responsibilities outside of class, allowed to work in isolation, or considered experts, who after their first year are encouraged to mentor many new faculty.

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LEARNING THE LANGUAGE OF MATHEMATICS TEACHING: SITUATING THE EDUCATIVE EXPERIENCES OF PROSPECTIVE TEACHERS IN THE DOMAIN OF DIVERSE LEARNERS

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The problem addressed in our design experiment is how the educative experiences that teacher educators provide to prospective teachers promote learning the language for mathematics teaching. Over several years of educating prospective middle grades and high school teachers, we have noted the limited collections of instructional representations and activities that prospective teachers bring to a first methods class (Berenson et al., 2001). Their "language" collections are initially limited to symbols and numbers with few words. Here we identify several critical events within a larger design experiment that appeared to move 15 prospective teachers toward a perspective that required them to develop their language collections in a purposeful way. Our findings reveal and explain why many prospective teachers continue employing traditional approaches, always beginning their lessons with formalized learning. The researcher-instructors provided learning activities over two weeks that promoted image making, image having, and property noticing (Pirie & Kieren, 1994a). Most of the prospective teachers were then able to formalize these learning experiences within the domain of learner diversity, growing in their understanding of how to use their collections to increase their students' learning of mathematics.

Focus and Frameworks

The problem addressed in our design experiment is how the educative experiences that teacher educators provide to prospective teachers promote learning the language of mathematics teaching. Here "language" is defined broadly in the spirit of Vygotsky (Thought and Language) to reflect the tools and symbols that communicate mathematical ideas to students. Traditionally researchers framed representations within the domain of mathematical thinking. Goldin and Shteingold (2001), among others, summarized "representation" as either external or internal within the learning process. They explained that external representations are composed of different systems of representations; for example, numerical representations or metaphorical representations or graphical representations. Internal representations are often referred to as the learner's mental models and can only be inferred by his or her external representations. We trace many of these ideas back to Davis and Maher (1990) who considered the roles of mathematical representations inclusively across systems and recursively between internal and external representations. Some researchers use the term strictly to conform to mathematical convention accessing only one or two representational systems. Powell, Francisco, and Maher (2003) combine a number of different representational systems in advocating for the examination of gestures, speech, or written work when researching mathematical thinking.

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Our problem is focused on learning the representations used to communicate mathematical ideas within a teaching context. Together with *word*, *number*, and *symbol*; we consider graphs, tables, manipulatives, and figures as other examples of instructional representations that present or re-present mathematical ideas. Ball, Lubienski, and Mewborn (2001) classify this knowledge as knowledge *for* teaching mathematics. Contexts that situate the learning of ideas with problems, tasks, or activities are included within the language of mathematics teaching. Over several years of educating prospective middle grades and high school teachers, we have noted the limited collections of instructional representations and activities that prospective teachers bring to a first methods class (Berenson, et al. 2001). Their "language" collections are initially limited to symbols and numbers with few words. In this paper, we examine several critical events within a larger design experiment that appeared to move these prospective teachers toward a perspective of teaching that required them to develop their language collections in a purposeful way so as to increase their effectiveness as teachers.

We draw upon the work of Pirie and Kieren (1994a), Droujkova (2004), and Powell, Francisco, and Maher (2003) for the frameworks that undergird this experiment. Mapping the growth of understanding, the Pirie/Kieren model defines specific levels of activities. Here we are particularly concerned with the activities of *primitive knowledge, image making, image having, property noticing* and *formalizing*. Primitive knowledge is defined as what is known before and brought to the learning experience, while image making and image having are instances of learners building or revising models or representations of the idea before noticing properties of those images. These properties promote formalizing one's understanding with generalizations. The movement between these levels, explained by Droujkova (2004), is a metaphorical process at the image making/having and property noticing levels. In determining the events leading to understanding, we employ Powell and his colleagues' frameworks for classroom research, examining the learning contexts for critical events and then retrospectively analyzing the data.

Method

The purpose of a design experiment, is to develop and test theories over time within instructional settings (Cobb et al, 2003). The researchers actively participated as instructors or assistants and their tasks were to:

- Conduct experiment over 16 weeks of first methods course (15 prospective teachers)
- Teach and observe class collaboratively (four authors/researchers)
- Follow multiple strands of the experiment including knowledge of mathematics, instructional representations and activities to teach key concepts in algebra, geometry, and pre-calculus
- Meet after each class to examine prospective teachers' learning, problem solve, analyze events, and plan next lessons accordingly
- Generate and test theories related to prospective teachers' learning.

The 15 subjects of study represented a variety of groups in terms of race, age, and gender. Some were young undergraduates pursuing a degree in secondary mathematics education. Others were only interested in obtaining high school or middle grades licensure as second career options. The course met once a week for two hours over 14 weeks and then the pre-service teachers were assigned a 4-hour lab out in the schools as interns. The four researchers under the guidelines of a design experiment taught the class cooperatively. One of the aims of the class was to increase the communication between and among the prospective teachers through collaborative learning groups and whole group reporting, sharing multiple views and ideas of teaching mathematics.

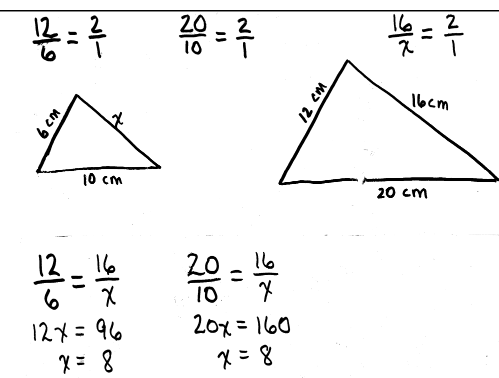
Sources of data were field notes and artifacts developed cooperatively and individually in and outside of class. Data were analyzed to identify critical events and within these critical events we determined the strands for study over time. The coding of these data were accomplished using five levels of the Pirie/Kieren model of understanding; primitive knowledge, image making, image having, property noticing, and formalizing. In this case, we expected to see changes in behaviors within the strands over time. Here we identify several critical events in relations to the pre-service teachers' selection of instructional representations for particular mathematical topics. We report our findings of two experimental strands; the decisions prospective teachers make concerning instructional representations when planning to teach a new high school topic and the decisions of teacher educators in their teaching of a first methods course to bring about growth of understanding instructional representations *for* teaching.

A Play in Two Acts

Setting the Stage

Noting our indirect, student-centered teaching philosophies, as instructors we chose neither to engage in direct instruction nor to judge prospective teachers' specific decisions about their choices of instructional representations. Instead we created and orchestrated experiences for prospective teachers to defend, reflect, and grow in their understanding of how to teach mathematics. Two learning experiences that were identified as critical events are described here. One experience in the fourth week of class asked prospective teachers to create instructional representations to teach the connections between scaling and ratio, proportion, and slope. Figure 1. presents a comparison of the instructional representations two groups of pre-service teachers.

Group 1



Group 2

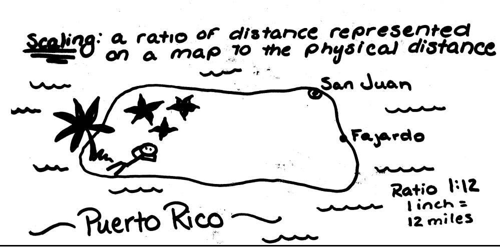


Figure 1. Two transparencies of instructional representations chosen by groups of pre-service teachers to communicate to students the connections between scaling and ratio and proportion

Act One

Viewing the groups' representations of scaling and ratio in Figure 1, one instructor asked the prospective teachers which one they would use with students. Their answers expressed intense emotion that is somewhat unusual behavior for undergraduates who tend to support one another's efforts. Some chose the formal representation of similar triangles and others a sketch of a map of Puerto Rico. This was the first time that the class had openly disagreed with one another, rolling their eyes or grimacing and groaning at their peers' choices. One strongly stated that *symbols and formulas are simpler,* another that *first you learn formulas*, and a third that *I need to give them the formula and show them how to use it*. Still others refuted this perspective saying that the real-world aspect of the map would make teaching more relevant and that students would learn more with the map representation.

Intermission

The researchers discussed this critical event of disagreement after class, noting that prospective teachers who favored beginning their lessons with formulas, equations, or definitions intended their students to begin learning at formal levels (Pirie & Kieren, 1994a). Those who favored the map of Puerto Rico intended for their students to fold back to their *primitive knowledge* to *make* new *images* relating scales of maps to ratio. We planned to have another class activity in the fifth week to follow up on the tendency of many prospective teachers to always begin their lessons with formal learning with no opportunities for their students to fold back to make images or note properties. The following methods activity illustrated in Figure 2 set the stage for the second critical event

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Abstract and Concrete; Introductions and Applications; Similarity: A Group Discussion

1) Come up with two "mini-sequences" or short lesson pieces about similarity.
   a) In the first mini-sequence, introduce students (high school or middle school) to similarity using a concrete representation, and then invite them to *apply* their understanding to an abstract representation.
   b) In the second mini-sequence, introduce your students to similarity using an abstract representation then invite them to apply their understanding to a concrete representation.
2) Discuss: What type of learners will benefit more from the first mini-sequence. Use yourself, your students, your friends and family as a source of examples.
3) Prepare a transparency with these "learner profiles" for the whole group.

Figure 2. Class assignment given to challenge "abstract first" advocates while introducing the INTASC standard of diversity.

Act Two

As shown in Figure 2, prospective teachers were asked to plan two different representational scenarios and then consider the characteristics of learners that would prefer one or the other. While we planned this as a group activity, the pre-service teachers preferred to, in some cases, present individual ideas on the set of instructions. An example of one student's notes is shown in Figure 3. This activity proved to be another opportunity for rich discussions, and the division of opinions expressed in the previous class was once again present. One prospective teacher stated that she preferred the abstract approach first because then *I can know if I'm right* (See her responses in Figure 3). Others nodded their heads or commented in agreement with these sentiments. Another rebutted this notion stating that he learned best *from applications that are familiar to me*. When voting on each approach one-third of the prospective teachers were in the abstract camp, one-third in the concrete, and one-third voted for both methods. It was our intent to acknowledge all opinions while providing thought-provoking questions to refocus the prospective teachers' on their students rather than their own preferences. Then we used their differences within the methods class to illustrate that not all students have the same learning preferences. They readily agreed that it would make sense to begin some lessons using concrete representations and other times to use abstract representations. One of the instructors asked if they, the pre-service teachers, would prefer to learn to teach only abstractly. There was no disagreement among the 15 students as to how they preferred to learn to teach. The acknowledgement among the whole class concerning the diversity of their students' learning preferences was the culmination of this second critical event.

These ideas of the learning diversity among their students were sustained throughout the course; in lesson plans, in descriptions of three INTASC standards, and in statements of educational philosophies. We anticipate that the robustness of the ideas of diversity of learners necessitating different approaches to teaching will have long lasting effects in these prospective teachers' formalized knowledge of teaching.

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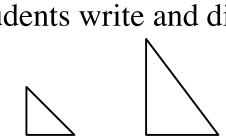
1) <u>Concrete</u> Introduction: students have sets of geometric shapes (1 small set and 1 large set. They sit back to back. One student describes a shape. The other student tries to pick out shape being described.

<u>Abstract</u>: Students write and discuss criteria for being similar

2) <u>Abstract</u>: 

<u>Concrete</u>: Start with a small photo with grid drawn on it– give students larger grid with no picture. Have students transfer picture

3) (concrete) explorative, hands on prefer, involved, visual learner

(abstract) organization, want rule first, logical structure, how's and why's

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Figure 3. Representations of concrete and abstract of pre-service teacher who prefers abstract first so she knows she is *right*.

Act Three

Portfolios are required of every education undergraduate and are built over the four methods courses and student teaching experiences following the INTASC Standards. In this first methods course, the portfolios consisted of an autobiography, philosophy of teaching, and three standards: content pedagogy, planning, and diversity. These were collected in the 16th week of the course, almost 3 months after the critical events lessons. Excerpts from eight pre-service teachers were chosen to show growth in their understanding of knowledge *for* teaching. PT2's portfolio was the only example of no evident impact from the two critical events.

Portfolio Excerpts from Abstract First Pre-Service Teachers

PT 1: White Female, Licensure Only, Age >30 (Diverse Learners). *Teachers need to be able to explain the lesson in several different ways, taking into account the different learning styles, strengths, and needs of the students in the class. … To be sure that I reach every student, I can then prepare an additional lesson using representations or enrichment to fit their learning styles. A visual or hands on activity may help some students " see" the concept.*

PT2: White Female, Licensure Only. Age 22 (Philosophy of Education). *First, I want to focus on theory, not just applications. Examples and practice of applications are important to solidify the theory; it is my opinion that when students have a strong foundation in theory they will be able to look at a higher level problem and know where to start ….*

PT3: Black Female Undergraduate, Age 19 (Content Pedagogy). *While using an appropriate instructional representation, teachers have the option of not only numerical and symbolic representations of ideas, but context real world situations where the mathematical idea is found. The use of charts, graphs and pictures are valuable representations for learners that rely on visuals to concretize learning. Basic definitions in your own words that define the idea or concept can be enhanced with manipulatives such as counters, unifix cubes, fraction bars or dice to reach a broader range of diverse learners.*

Portfolio Excerpts from Abstract OR Concrete First Pre-Service Teachers

PT4: White Female Undergraduate, Age 19, (Philosophy of Education). *Facilitating students' learning is a challenging, yet priceless, endeavor. Through research over the years, we have discovered that each student is unique in his or her way of mastering material, and I think that is a teacher's goal to develop different ways of teaching the same idea, so that all students will have an equal opportunity to learn and the concept will be more firmly grounded in each pupil's mind.*

PT5: Black Male Undergraduate, Age >25, (Philosophy of Education). *Upon creating a dedicated student, it is vital that the teacher understand the unique learning style of her pupil. Too often, students lag behind in math class because the material presented is too abstract. Teachers should use teaching methods that appeal to an audience of diverse learning styles.*

PT6: Black Female Licensure Only, Age >50, (Diverse Learners). *The 'active' learner will want to employ some type of mathematical formula to solve the problem right away. ON the other hand, the 'reflective' learner will want to thoroughly understand the problem before trying to solve it. A balance is needed so that students don't rush too quickly to a solution and perhaps veer completely off the paths to solving the problem and they don't also spend a lot of time thinking and discussing without actually trying to model solutions to the problems.*

Portfolio Excerpts from Concrete First Pre-Service Teachers

PT7: White Male Licensure Only, Age >23 (Pedagogy of Teaching). *Looking back at how I was taught in high school, how I am being taught in college, and how Mrs. Green teaches her* *class helped me to decide which kind of representations to use and which not to use. … I built a parallelogram out of plywood with bolts at the corners (so that I could change the angles) and hooks to connect the midpoints as a way of visualizing the quadrilateral midpoint theorem.*

PT8: White Male Undergraduate, Age >20 (Pedagogy of Teaching). *An effective instructional representation provides a means of assisting the learner in connecting the abstract mathematical constructs to concrete ideas and real-life problems.* He writes a textbook illustration of right triangle trig functions and the mnemonic, SOHCAHTOA, and comments that it is not a good representation because … *it does not relate the mnemonic device to the graphic in any fashion, a student cannot reasonably be expected to gather any information on understanding from this representation alone.*

The Reviews

Prospective teachers often choose abstract or formalized representations when planning and teaching mathematics. With this study we begin to understand why this is such a prevalent and ingrained practice. Viewing these teaching practices through the lens of Pirie/Kieren, it is evident that prospective teachers may be unable to fold back to make new images, and notice properties of the mathematics they will be teaching. As Pirie and Kieren (1994b) state, the growth of mathematical understanding is dependent on learning activities at inner levels to move with understanding to formalizing and other outer levels. Opportunities for the growth of understanding are lost when prospective teachers only provide their students with formalized representations such as formulas, definitions, and rules. Also, opportunities to engage in thinking metaphorically are lost since at the formalized level no metaphors are present in one's thinking. Without images and metaphors, it is difficult to reconstruct one's formalized mathematical understanding at a later time (Droujkova, 2004), as evidence by the difficulties these prospective teachers experienced in creating concrete activities and instructional representations for teaching.

Implications for Teacher Educators

Throughout our analysis of the results, another theme provides additional insights concerning teaching prospective teachers. The role of the teacher educator is evident in the activities and instructional representations that promoted the growth of understanding in how to teach mathematics. We avoided transmitting our formal knowledge while providing ample opportunities for prospective teachers to make images and develop metaphors about *teaching* mathematics. It is through these images and metaphors that opportunities are provided to revise images and notice properties before formalizing generalizations. In this story, the prospective teachers moved as a group after image making (*my own preference in learning mathematics*) to notice properties (*some of my students may prefer one way, others another way*) to formalizing a rule about the diversity of learners. By situating their generalizations in the domain of student diversity, the prospective teachers were able to apply this rule across many of the methods applications throughout the course. We never had to tell them that this was THE way to teach.

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EDUCATIONAL POLICY AS A VEHICLE FOR TEACHER LEARNING

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Educational policies aimed at improving school mathematics education have proliferated in recent years. Investigating how teachers make sense of these policies is imperative, for teachers ultimately decide what mathematics students learn and how they learn it. This research followed a group of 14 middle school mathematics teachers as they studied a particular policy document, Principles and Standards for School Mathematics (NCTM, 2000), tracing the ideas teachers developed about the policy and the impact on their beliefs, priorities, and practice. Analyses revealed teachers developed 5 different perspectives on the document, and these perspectives were aligned with particular features of teachers' local contexts. Results suggest documents like Principles and Standards can be generative – they can stimulate productive conversations among teachers, and such conversations can serve as fruitful sites for teacher learning.

The past two decades have witnessed an unprecedented increase in efforts to develop and disseminate educational policies aimed at improving school mathematics education. This policy proliferation is exemplified by the release of "standards" for school mathematics by the National Council of Teachers of Mathematics (e.g., NCTM, 1989, 2000). What impact these efforts might have hinges on if and how various constituents decide to implement the policy recommendations, and these decisions depend largely on constituents' interpretations and understandings of the policy (McLaughlin, 1987; Spillane & Callahan, 2000). Understanding how classroom teachers take up and make sense of instructional policy recommendations is particularly crucial, for teachers are the "final brokers" (Spillane & Callahan, 2000, p. 401) of instructional reform. Regardless of the various forces that aim to influence instruction, teachers are the ones who ultimately decide what mathematics students learn, and how they learn it.

Researchers have investigated the ideas that district policy makers construct from state science standards (Spillane & Callahan, 2000), the attempts of school sites to align mathematics instruction with the NCTM *Standards* (Ferrini-Mundy & Schram, 1997), and the efforts of an elementary mathematics curriculum committee to translate state standards into a district curriculum framework (Hill, 2001). However, there have been few efforts to understand how teachers gain access to and make sense of policy documents like the NCTM's *Standards*. This research followed a group of 14 middle school mathematics teachers as they studied one of the NCTM's most recent policy documents, *Principles and Standards for School Mathematics* (2000). In particular, this study investigated the following research questions: "What ideas do teachers develop about the policy – its purposes, messages, and perspectives?" and "How does engaging in the work of making sense of policy influence teachers' beliefs, knowledge, priorities, and practice?" This report will present selected results from this investigation.

Theoretical Perspectives

This study is situated within a broader framework conceptualized by the National Research Council (NRC, 2002) for investigating the influence of documents like *Principles and Standards*. This work focuses on a particular channel of influence identified in the framework – teacher

Lloyd, G. M., Wilson, M., Wilkins, J. L. M., & Behm, S. L. (Eds.). (2005). *Proceedings of the 27th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*.

development. In particular, this research takes up two guiding questions posed by the NRC: "Among teachers who have been exposed to nationally developed standards – How have they received and interpreted those standards? What actions have they taken in response?" (p. 35).

This research draws upon both cognitive psychological (Borko & Putnam, 1995) and situated perspectives (Putnam & Borko, 2000) on teacher learning and professional development. Teachers' knowledge, beliefs, values, and past experiences shape how they perceive and what they learn, and these in turn influence how they act (teach). Cognition also has a social component and is situated in particular contexts. Teachers' learning is influenced not only by their personal orientations, but also by their interactions within various social communities.

Methods

A professional development project was designed to provide middle school mathematics teachers with an opportunity to study *Principles and Standards*. Interested teachers were invited to apply with a partner from their school. From the pool of applicants, 14 middle school mathematics teachers – 6 school-based pairs and 2 individual teachers who agreed to team up – were selected to form a study group to read, discuss, and analyze *Principles and Standards*. Teachers were chosen so as to maximize diversity in terms of years of teaching experience, certification level, mathematics curriculum, and incoming familiarity with the NCTM *Standards*. Teachers met for a total of 16 sessions over 6 months, with each session lasting 3 to 6 hours. The central activity of each study group session was analysis of selected readings from *Principles and Standards*. To situate their study of the document in the activities of teaching, the group also worked on mathematics tasks, read teaching cases, viewed videos of mathematics teaching, and analyzed student work. Each teacher also maintained a journal in which they responded to prompts aimed to support them in reflecting on the readings and clarifying their ideas.

Data on teachers' developing ideas about *Principles and Standards* and the impact of teachers' study of the document on their beliefs, knowledge, and practice were collected from multiple sources. The primary data source was teachers' discussions of the policy document during the study group sessions. All 16 study group sessions were audio-taped and videotaped, and transcribed. Other sources of data included teachers' journal entries, interviews with teachers, teachers' discussions on an electronic listserv, observations of each teacher's classroom, and entry and exit belief surveys. In addition, all documents produced or shared during the study group sessions were collected and photocopied.

Analyses of the study group discussions were conducted in two phases. The first phase consisted of a turn-by-turn analysis of each teacher's individual contributions to the conversations. In this stage of the analysis, a conversational *turn* was the unit of analysis, and the goal was to trace the ideas that individual teachers were developing over the course of the project. The second phase consisted of a more global analysis of the discussions in which each study group transcript was chunked into distinct conversational *episodes* characterized by a shift in the topic under discussion. In this stage, an episode was the unit of analysis, and the goal was to characterize the nature and content of the group's discussions. Using principles from grounded theory (Strauss & Corbin, 1998), a coding scheme was developed to capture the ideas teachers were developing about the policy document and the major themes that emerged in the study group conversations. Two categories of codes emerged through this process: codes that captured the variety of ways teachers came to view the document and its purposes, and codes that captured the key issues that teachers were identifying in the document and grappling with in their discussions. Case studies of individual teachers were then developed to investigate the influence of each teacher's study of the document on their beliefs, knowledge, priorities, and practice.

Results

Analyses revealed teachers developed five different perspectives on the nature and purposes of *Principles and Standards*, and individual teachers were capable of assuming multiple perspectives. The ways in which teachers came to view the document were aligned with particular features of their local contexts. Case studies of individual teachers captured the ways in which their efforts to make sense of the policy influenced their beliefs, priorities, and practice.

Teachers' Views of the Policy

Teachers came to view *Principles and Standards* from five different lenses – as a warrant for their current beliefs or practices, as a lever for effecting change, as a vehicle for their own learning, as a springboard for rich discussions with colleagues, and as a tool for analyzing mathematics curricula. The discussion below illustrates three of these perspectives.

Policy as warrant. One of the most common perspectives was that of the document as a *warrant*. From this viewpoint, teachers envisioned using the document to defend their current beliefs or practices. For example, many teachers saw themselves using the document's stance on technology to defend the ways in which they used calculators in their mathematics instruction.

> Brian: Ok. I'll risk it, I'll say it. I liked on p. 32, "When teachers are working with students on developing computational algorithms, the calculator should be set aside."
>
> Janelle: I agree with that, too.
>
> DB: Where is that, Brian?
>
> Brian: It's on page 32, at the very bottom, the last sentence. I guess I saw it as *giving permission* [italics added] to do some back-to-basics type of instruction, or what would be seen as back-to-basics by the parents. I know that there's always a sigh of relief at open house when I say, "And sometimes this year your kids will not be allowed to use calculators. Please look out for that and don't let them." (5th Study Group Session)

Brian sees the document as authorizing him to put the calculators away when he is developing computational algorithms, a practice already in place in his classroom. Mimi also foresaw herself using the document to defend her use of technology.

> I do know that in reading the selections, I was certainly encouraged, especially in regard to the calculator use. I can use segments with this parent (and others) to clarify the issues, knowing that I have, in this document, the support of a large body of educated mathematicians and educators. . . . The document did repeatedly speak to the use of calculators and computers as tools that are a reality in the home and the workplace, and we cannot be considered responsible and simultaneously not use these tools in our classrooms. (3rd Journal Entry)

In this passage, Mimi anticipates how she can use the document to make a case to parents and "others" (identified later as school administrators and fellow mathematics teachers) for the importance of calculator use in mathematics instruction. In addition to their use of calculators, teachers foresaw the document as a sanction for other instructional decisions. For example, Mimi described referring to the document during a recent phone conversation with a parent to defend her practice of having students collaborate on mathematics problems.

> And that's what I'm working on with a couple of parents who have bright children. They don't want [their children] to share their knowledge. "My kid figured it out and I don't think

> it's her job to pull this other student along." That's a seventh grade parent calling me today in fact. Well, I tell this parent that Communication is one of the standards, too. And to say that that one is not as important as algebra, I don't think is fair. (6th Study Group Session)

Policy as lever for change. A second perspective assumed by teachers was that of the document as a *lever for change*. In contrast to viewing the document as a warrant for their existing beliefs and practices, teachers who viewed the document as a lever for change saw the document as a tool for brokering for new practices or resources. For example, some teachers envisioned using the document with administrators to lobby for more professional development opportunities. "They [the Writers] really pushed the issue of professional development. I thought, 'Oh this is a part that should be photocopied and given to my principal!'" (Dara, 3rd Study Group Session). Other teachers saw potential in using the document as a lever for catalyzing other teachers in their district to improve their mathematics instruction.

> Brian: So, that was kind of nice to be able to go on and actually read some of that stuff [the Standards for Grades 3-5] about what would be a high expectation of them [elementary grades]. I think that our district is feeling a lot more of the pressure to push the elementary school more, so it's nice to have a little more evidence of that…. I plan to have a meeting with our assistant superintendent and tell him what we [the study group] are doing and provide him with additional ammunition on this to push the elementary schools. (Interview 1)

Recognizing that his district is planning to raise expectations of its elementary school teachers, Brian sees *Principles and Standards* in terms of its potential to be used as "ammunition" for compelling those teachers to respond. Similar notions of the document's potential as a lever for change were explored by Tim:

> I also hope to obtain enough evidence from these sources to help back up my claims about how students in our building should be educated. As it stands now, I have had a difficult time convincing the other math teachers in my district about the things they should and should not be doing in their classrooms. Having a respected source to refer back to should help me greatly. (2nd Journal Entry)

In this entry, images of the document as a warrant and as a lever for change are intertwined. Tim foresees using the document to support his current views about how and what students should be taught; he also sees the document as a crucial tool in convincing other teachers of his ideas.

Policy as vehicle for teacher learning. A third perspective exhibited by teachers was that of the document as *a vehicle for their own learning*. From this perspective, teachers viewed themselves as learning not only *about* the document, but also *from* the document. For example, consider Joyce's reflections on her study of the document at the end of the project:

> I think that the document will have a different impact on each reader depending upon their particular situation and needs. Reading it again next year will probably affect me in different ways than it has this past year. Even as I reviewed sections of it in preparation for our presentation, I found new understanding (and questions) that I did not find the first time I read it. I view it as a learning tool that can be a continual resource for teachers who are interested in growing professionally. I hope that we can encourage other teachers to invest their time in its reading. (9th Journal Entry)

In addition to learning about the document's recommendations and perspectives, Joyce and other teachers saw themselves learning new ideas about mathematics teaching and learning. They also saw the document as an instrument for analyzing their practice and becoming more explicit about their beliefs and values. These views of the document as a "learning tool" emerged during several study group conversations. For example, during the beginning of the second session, teachers shared their overall reactions to their first readings from the document:

> Monique: It seems like it [the document] is common sense and nothing was so technical that you read something and were like, "Huh?" I mean, it seemed like it was kind of reconfirming what you already do or what you already thought.
>
> Kayla: I found myself almost reading it too fast because of that, and not thinking about it as hard as I should. Because I read through it and I thought, "Oh yeah, oh yeah, I do that. Oh yeah, I've thought about that." And then when I went back and read it again, I stopped to think about it and talked to another teacher in my building about it. I don't necessarily do what I thought I was doing. I don't know how to say it. The first time I read it through I thought, "Oh yeah, this isn't [anything new]." And then when I went back through it again and eventually, this is about my fifth time reading it since I got it last April, I thought there was a lot more here for me to think about. (2nd Study Group Session)

This passage reveals two very different perspectives on *Principles and Standards*. Monique reports that the document's recommendations align with and confirm what she already thinks and does, reflecting a notion of the document as warrant. Kayla admits to having the same initial reaction, but in reading the document more carefully and discussing it with a colleague, she finds the document leading her to reexamine her practice. In doing so, she realizes that, "I don't necessarily do what I thought I was doing". Studying the document helps Kayla become more aware of and explicit about her beliefs and practice.

Teachers also saw the document's inclusion of controversial recommendations and non-traditional perspectives as a catalyst to confronting their own ideas about school mathematics. For example, the teachers took up a conversation of what "algebra" should be for middle school.

> And that's why I thought reading this [the document], it shakes up what algebra is. If algebra at the middle school isn't just x, y, and equations, then what is it and what do we need to be doing? I thought this was really pushing at us to think about algebra differently for sixth, seventh, and eighth graders. (Kayla, 3rd Study Group Session)

Kayla characterizes the document's stance on algebra as provocative, as "shaking up" what algebra is. By taking a more radical stance, she feels that the document is "pushing at" teachers to reexamine their ideas about algebra for middle school. Teachers revisited this notion of the document confronting their thinking later during their discussion. The passage below opens with Brian inquiring about the document's discussion of the various meanings and uses of variable.

> Brian: It just felt much more explicit than some of the other stuff I read in the chapter. It takes up quite a bit of the chapter. . . . When I read it I also thought it was *trying to push people outside of their little box* [italics added], because most people would just think of, you know, a variable stands for a number that you find the answer for. And this was a way to push people a little bit further.
>
> Kayla: I think it's even more staggering if you have a chance to go to the 3-5 [grade band] and look at how they talk about the use of variable and starting it in grades three through five and looking at the different ways variables are used in a situation. I think it would make things so much smoother, flow better, but I don't know that . . . if all elementary teachers have the math background to be able to see that. . . .
>
> Janelle: Maybe that's part of this too, is that this document is a teaching tool as well. It's not just setting up standards, but teaching. Because I really put myself in a learner situation and I actually learned a number of things that I didn't know before, that I hadn't thought about in this way. So, I kind of like that, I absorb it. (3rd Study Group Session)

Brian and Kayla see the document as attempting to broaden teachers' understanding of variable to meanings other than variable as placeholder or missing value. Janelle builds on this discussion by suggesting that the document has been developed to do much more than just disseminate standards – it has been designed to serve as a catalyst for teachers' learning.

Alignment of Teachers' Views and Contextual Features

Results suggest that the ways in which individual teachers came to view the document were aligned with particular features of their local contexts. Important contextual features included the school or district's norms for teaching and for professional growth, the level of support for innovation from administrators and community, and the physical, financial, and human resources available to teachers. For example, teachers like Mimi who taught in districts where their mathematics curriculum was highly contested by parents and community members tended to see the document as a warrant for defending their instructional decisions. In contrast, teachers like Janelle who worked in more supportive, less contentious districts where they felt respected and trusted by parents and administrators were less likely to view the document as a warrant. These teachers felt free to experiment with new instructional approaches and were more likely to envision the document as a vehicle for their own learning.

Policy's Impact on Teachers

Case studies of individual teachers portray the educative potential of engaging teachers in the work of making sense of *Principles and Standards*. For Brian, reading and discussing the document pushed him to reexamine his beliefs about the use of technology in mathematics instruction. Before participating in the study group, Brian had experimented with various instructional technologies in his classroom but was not a strong advocate of them. In early study group discussions and journal entries, he shared feelings of frustration toward the demands posed by technology use. Over the course of the project, Brian's perspective on the role of technology began to shift. He became more explicit about his practice and dissatisfied with his limited use of technology. Gradually, his beliefs about technology became more balanced, and he began to recognize and espouse the advantages afforded by instructional technologies.

> I would like to think my classroom is fairly well aligned with the PSSM, but I have come to recognize several areas where I come up short: 1.) Use of technology: Although we use the graphing calculator, it is our only regular application of modern technologies available to us. . . . I hope to use more technology during the year. I am hoping to incorporate some of the e-examples from the NCTM website, as well as the Geometer's Sketchpad. (6th Journal Entry)

Consequently, Brian took steps to incorporate more technology into his classroom. He formed a partnership with his school's computer specialist to develop and implement a series of web-based lesson plans, including one designed around an electronic example in the document. Even after the professional development project ended, Brian continued to design, implement, and revise mathematics lessons incorporating technology, posting messages to the electronic listserv informing the other study group members of his progress.

For Janelle, reading and discussing the document helped her identify and reexamine a particular feature of her practice, namely her tendency to immediately provide students with an algorithm for solving a given problem. Study group discussions of two of the document's claims– mathematics concepts should be developed before procedures, and effective teachers are able to support students without doing the thinking for them – encouraged Janelle to reconsider her approach. "One of the biggest changes in my thinking is about giving the algorithm then solving problems, rather than thinking about things first without the rule or formula. I have to work at this a bit because I've learned it the other way" (4th Journal Entry).

Gradually, Janelle began to experiment with adopting this new "explore first" approach in her classroom.

> How has new NCTM knowledge changed my teaching? The #1 biggest way has to be in the "non-algorithm first" way. "Here, I'll show you how – then practice!" It will take me some time to reverse the engines, but I am committed to the change. One of the boys in my class who was doing sixth grade for the second time said that he now loved math and had been very unsuccessful before. It moves me to get lots of kids to that place. (6th Journal Entry)

As Janelle allowed her students to solve problems in their own ways, their attitudes toward mathematics began to improve. Collecting this type of evidence was compelling to Janelle, and strengthened her commitment to the approach. Indeed, toward the end of the project, she cited it as the most important thing she had learned through studying the document with the study group.

> "This one is the most important to me. I learned that the algorithm first then practicing is not the best way. Research supports the notion that doing the conceptual understanding and exploration first is more powerful and leads to deeper learning" (9th Journal Entry).

Concluding Remarks

Results from this investigation suggest documents like *Principles and Standards* can be generative – they can stimulate productive conversations among teachers, and such conversations can serve as fruitful sites for teacher learning. Further research is needed to shed light on how to effectively engage teachers in making sense of policy, and what ideas they develop by engaging in this work. Such research could inform the development of future educational policy.

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ACCOUNTING FOR AGENCY IN TEACHING MATHEMATICS

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This paper provides analysis of modified teaching sets conducted with three fifth-grade teachers. The teaching sets were designed to provide information on teachers' instructional practice in mathematics including the role of text resources in supporting teachers' reconceptualization of their practice. This sense making on our part involved understanding teachers' instructional reality and the role it played in supporting or constraining teachers' use of reform curricula. In our analysis we highlight the importance of accounting for agency in the process of supporting teacher change.

Introduction

This paper provides analysis of modified teaching sets (cf. Simon & Tzur, 1999) designed to provide information on teachers' instructional practice in mathematics including the role of text resources in supporting and/or constraining teacher change. Traditional views of professional development often assume that we can train teachers to enact instructional texts and strategies with fidelity and that this *fidelity to the curriculum* will lead to increased student achievement. This conception of professional development gives agency to text resources and places fidelity to the curriculum as the endpoint of professional development engagements. This approach stands in stark contrast to the goals of professional development engagements characterized by a design research approach which builds from teachers' understandings and contributions (cf. Brown, 1992; Cobb, Confrey, diSessa, Lehrer, & Schauble, 2004; McClain, 2004). When the emphasis of the professional development engagement is building from teachers' current understandings, the engagement is necessarily responsive and cannot be scripted. This type of ongoing engagement cannot be reduced to manuals, text resources or guides. This sentiment is captured by Carpenter and colleagues when they claim, "teaching is complex, and complex practices cannot, in principle, be simply codified and then handed over to others with the expectation that they will be enacted or replicated as intended" (Carpenter, Blanton, Cobb, Franke, Kaput, & McClain, 2004). This makes the notion of codifying professional development and handing it over to others as an image of travel untenable.

Our analysis addresses this conundrum by analyzing modified teaching sets collected from three fifth-grade mathematics teachers who are engaged in an ongoing collaboration with university partners[1]. The teaching sets contain a video-recorded pre-interview, a video recording of the teacher teaching a lesson, and a follow-up video-recorded interview. The sequence of interviews and observations were designed to find a way to characterize the teachers' stance toward instruction or what Zhao, Visnovska and McClain (2004) define as *instructional reality*. We find this term useful because it gives us a language for talking about the factors that influence teachers' decision-making processes such as perceived institutional demands,

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constraints and affordances. A central principle that guided our analysis was to assume that teacher' perspectives of teaching and learning and specific instructional practices they develop in their classrooms are always reasonable and coherent in the context of their instructional reality. Operating with this assumption enabled us to avoid a deficit view when examining the data and instead highlighted the necessity of generating reasonable interpretations of the teachers' instructional reality against which their practices can be understood. In the analysis, we therefore situate the role of agency as it relates to text resources against the teachers' instructional reality.

Theoretical Perspective

We incorporate two theoretical perspectives into our analysis in order to make sense of the complex dynamics involved in teaching and the institutional contexts through which it is enacted. First, we view teaching as a *social practice*. That is, we see the relationship between *social structures* (e.g. institutional settings – the classroom within the school and the school within the district), and local *events* (i.e. teachers' enactment of current instructional decisions within the context of the classroom) as mediated by the social practice of *teaching* (Fairclough, 2004). Second, we view teaching as a *distributed activity* and therefore situate teachers' instructional practices within the institutional settings of the school and school district.

We know from both first-hand experience and from a number of more formal investigations that teachers' instructional practices are profoundly influenced by the institutional constraints that they attempt to satisfy, the formal and informal sources of assistance on which they draw, and the materials and resources that they use in their classroom practice (Ball, 1993; Brown, Stein, & Forman, 1996; Cobb, McClain, Lamberg, & Dean, 2003; Feiman-Nemser & Remillard, 1996; Nelson, 1999; Senger, 1999; Stein & Brown, 1997). However, the approach provides a challenge in that we must coordinate the teachers' instructional reality in the context of professional development and their classrooms with analyses of the institutional setting. This is necessary in order to capture the intertwined system involving teachers' instructional practices and perspectives together with their experiences as they are trying to accomplish certain instructional goals within the institutional setting in which they work. Such experiences highlight the immediate challenges that teachers encounter, the frustrations they go through and the valuations they hold towards specific aspects of their instructional reality. We will therefore situate the analysis in an analysis of the institutional context by drawing on the analytic approach proposed by Cobb, McClain, Lamberg and Dean (2003). This approach builds from Wenger's (1998) work by delineating communities of practice within a school or district and analyzing three types of interconnections between them that are based on boundary encounters, brokers, and boundary objects. Boundary objects are especially significant since curriculum guides and texts are typically seen as carriers of meaning and authority.

Data

Data consist of modified teaching sets composed of a pre-interview, subsequent classroom observation, and follow-up interview on six middle-grades mathematics teachers involved in professional development collaborations. The data were collected in September of 2004. All interviews and observations were video-recorded. In addition, videotape of an interview with the principal was used to corroborate our conjectures about the demands and constraints within the school and district that were voiced by the teachers. There is also video from each of the bi-monthly professional development work sessions and more extended initial interviews with the teachers. However, for the purposes of this paper, we are restricting our analysis to the interview with the principal and the teaching sets with the three fifth-grade teachers.

Analysis of the Setting

The school in which we are working is a newly designated mathematics and science magnet school we call Iris Hill. Iris Hill, which accommodates students in grades five through eight, is located in a large, urban district in the southeast United States. The student population of the school is predominately minority. The school district, like many in the United States, is involved in a high-stakes accountability-testing program. Teachers and principals are judged by students' test scores. This use of test scores as an evaluation tool has created a climate in which preparation for the test dominates instruction time. Scores on tests for the 2003-2004 ranked Iris Hill fourth in the district. The teachers therefore viewed their test preparation and instruction as productive and effective, instilling confidence in the teachers and reinforcing their approach.

The principal was a former mathematics teacher in the district[2]. He had a strong commitment to the teachers and the students. He welcomed the collaboration with the research team and, according to the teachers, "strongly encouraged" them to participate in the professional development work sessions. The principal also advocated the use of the reform curricula at all grades and told the teachers not to worry about test scores, as that was his responsibility[3]. However, the teachers' identities were very much linked to their students' scores, creating a tension for the teachers.

A preliminary analysis of the school indicates that there are several informal communities in place; however, there are no formal ones. As an example, no designated content chairs, specialists, or grade level chairs have been structured within the institution. This lack of organizing structure enables the development of informal groups or communities based on common interests and needs. As an example, the three fifth-grade teachers have formed a tightly knit community. We are therefore particularly interested in the fifth-grade teachers since they are the only group of teachers in the building who collaborate with peers concerning their mathematics instruction. This collaboration is primarily based on creating efficient ways to organize classroom instruction. For example, each of the three teachers is responsible for a content area, creating copies of all worksheets and distributing them to the other two teachers. The teachers have common planning time and talk regularly about pace, ensuring that they are all covering the material at the same rate. Although there is no designated leader in the group, the most experienced teacher acts as a broker between the teachers and the Principal. Her seniority within the school and district is recognized by the other two teachers in the group as an asset when trying to accomplish tasks. There are no other apparent brokers among other mathematics teachers. They, in fact, are isolated and only meet during the scheduled professional development sessions. We therefore wanted to understand the interplay of the professional development engagements with the new curricula in the context of the fifth-grade teachers' interactions.

It is also important to note that the District Course of Study [DCS] and the original text resources served as boundary objects in that they were carries of the instructional demands and expectations from the District Office to the principals and teachers. The principals and teachers

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2 This work is part of a larger design experiment being conducted by Rich Lehrer, Leona Schauble, Pat Thompson, and Rogers Hall.

3 In February 2005, amidst internal turmoil and controversy in the school, the Principal was forced to resign. It is inappropriate for us to discuss any further details related to this event.

know the DCS is tightly correlated with the high-stakes test and their original text resources were correlated with the DCS. Teachers were confident that fidelity to the text would therefore ensure that students were well prepared for the standardized test. This was reinforced by the students' level of achievement on the test. For this reason, the practices of the teachers at the onset of the collaboration could be described as preparation for the test via the DCS and aligned text resources. Therefore, the change in texts created lack of confidence on the part of teachers that they were adequately preparing their students for the test. However, this lack of confidence was not evident in teachers' *use* of the new materials. They were genuinely attempting to engage students in mathematical investigations in the manner in which *they understood* the text. However, data from the pre- and post-interviews indicate that all three fifth-grade teachers were concerned about the rate at which material was introduced and what sense the students were making of the investigations. These concerns stemmed from the teachers' perception of the lack of fit between the new text resources and the demands for mastery on the test.

Against this background, we used analysis of the teaching sets to articulate a reasonable characterization of the teachers' instructional reality. From the interviews we learned that all three fifth-grade teachers praised the *new way of teaching* for giving them access to students' thinking. However, each praise was countered by a concern about a drop in test scores. The teachers therefore perceived the school (e.g. the Principal) and district as having conflicting goals. Their instructional practices during the previous academic year produced impressive results on the high-stakes test. Now the teachers viewed themselves as being asked to ignore those practices and embrace a new way of teaching that was counter to their prior approach. In addition, they lacked confidence in the Principal's ability to buffer them from the ramifications of lower test scores. Further, their identities were linked to the fact that they taught mathematics in a math/science magnet school. The expectation was that the students would perform well. The teachers therefore administered practice tests to determine what students were learning with the new curriculum. They expressed concern over how little the students were learning as indicated by the tests, and the amount of time the new curriculum was taking. This in tandem with dramatic changes in students' ability to talk about their mathematics placed the teachers on unfamiliar terrain. Their information was therefore contradictory — a negative result on the test but a positive result in discourse practices. This created a strained instructional reality, leaving the teachers in what they viewed as a precarious situation.

Analysis of Teaching Sets

In conducting the follow-up interviews, we were initially confused by the teachers' talk of the "new way of teaching" with the reform curricula that focuses on students' strategies. What we observed in all three fifth-grade classrooms were discussions in which careful attention was paid to allowing all students to share their "strategy" on a particular task. These strategies were simply the procedure the students employed to solve the task. Rarely did we observe the teacher engaging students in conversations about the actual mathematics involved in the strategies. As an example, we observed classes in which the students were investigating factors and multiples. The particular lessons focused on factors of 1,000 and 1,100. In one class students employed a variety of procedures including making modified factor trees, dividing by every number beginning with one until it "started repeating," and repeatedly multiplying two random numbers to see if the product was 1,100. The whole-class discussion entailed each group of students sharing their strategy. The discussions therefore took on characteristics of a show-and-tell where each contribution was equally valued. Students were praised for being able to explain their strategy and, in the subsequent interview teachers noted how pleased they were with the students' abilities to talk through the process.

In these settings, the teachers viewed their role as stepping back and watching the math unfold. They were committed to trying what they called, "this new approach" and all claimed that even if the text resources changed, they "won't go back to the old way" of teaching. However, their patterns of interaction with students were not focused on the concepts involved in the *mathematics tasks*. They were instead focused on *process*, meaning they were concerned more with whether or not students contributed their ideas to the classroom discussion rather than the content involved in those contributions. The goal was to get students to articulate their strategy both in the context of whole-class discussion and in homework assignments. The teachers viewed the importance of process as the primary goal of the new text resources. For example, in a post-interview, when Karen was asked to describe her class session on factors of 1,000 and 1,100, she stated, "Well what I like was working on our strategies, how to write them up. And everybody did a really good job of explaining what they were thinking" For Karen, students' explanations of their reasoning constitute good classroom conversations around important mathematical ideas, in this case factors and multiples. However, the process involved students' simply explaining their ideas, rather than the concepts embedded in the explanation. This is typical of Karen's responses throughout the interview when she spoke of her students thinking. The classroom observation of Karen's practice further corroborates this claim. During whole-class discussions nearly all forms of reasoning became acceptable contributions without much conflict. For example, the strategy of dividing 1,000 and 1,100 by every number until it "started repeating" was a time-consuming process for uncovering factors, although appropriate for discussing the patterns that might arise. However, this way of reasoning was praised as a good strategy without any question, either to the individual students or the entire classroom. If a student simply stated what he or she thought, whatever the intentions, it was a sufficient piece of evidence for student thinking and learning from Karen's perspective.

In response to the general question regarding the use of the curriculum materials, Karen mentioned, "Oh yes, um I like it [Investigations] cause it is not like a mindless type of thing. Like to me math used to be very step-by-step. I, I felt like I had to tell the students what to do all the time. And this, I don't feel like I have to tell what to do all the time. Um, they're doing more thinking for themselves." This statement is characteristic of the teachers' beliefs about the effectiveness of the new curriculum. Like Karen, the other teachers have expressed similar ideas about the mathematics involved in the new curriculum and its relationship to their instructional practices. The teachers seem to share the belief that following the guidelines and steps in the Investigations material implies a new orientation to teaching mathematical concepts (i.e. "a new way of teaching"). This new way, from the teachers' perspective, seems to be founded on the premise of getting students to think and reason in complex ways. But it is unclear what constitutes an effective strategy or evidence of complex reasoning.

The curriculum itself is the necessary tool for teaching mathematics in a new and effective manner. This is important for the teachers because they view this new way teaching as eventually translating into increased student achievement and successful completion of standardized tests. Therefore, the teachers perceive the new instructional materials as instantiating improved practice, which allows students to think differently about the mathematics involved. As an example, when asked about the new curriculum and its effectiveness, Gloria stated:

> Um, well first of all I like it I, I like the fact that somebody's thought through it and its good that somebody's thought through it for me. If somebody said 'okay, I want you to just stop and teach kids to think' ya know its not so easy. I, I thought I had always done that. I always asked a lot of questions I always said why, I always said, but, but there's a whole program established around that. Um, I like, its hands on, it gives kids more opportunity to solve things in different ways as opposed to just being computation based. Um, I just its, its excited the kids. I like it, I like, it takes a lot of planning. Even though its all script it's all ya know pretty much written out for you.

One interpretation of Gloria's perception of the curriculum is that it allows her to teach the students how to think about and solve problems differently than before. Like Karen, Gloria's classroom practice suggests that each strategy is a mathematically meaningful contribution. According to Gloria the students are thinking differently and contributing more to the classroom discourse because of the new curriculum. She also mentioned that the new text alleviated major time constraints because it has been "thought out" for her. It is important to note, however, that this does not imply a lackadaisical approach to teaching mathematics; rather, she perceived the curriculum as an affordance because it reduced the time-consuming planning required by the previous curriculum. Gloria viewed the curriculum as containing already prepared lessons that must be enacted following the steps provided by the text in each lesson, therefore giving the text a source of agency. This is interesting for us because it shows that the teachers rely on the text in the course of their daily instructional practices to not only bring out student strategies, but also to reduce the amount time spent preparing for lessons. As a result, the teachers rarely discussed instruction in tandem with the curriculum since the text is viewed as the authority on effective teaching practices.

This sentiment is echoed by Claire (the other fifth grade teacher), "It's kinda hard to follow to a tee because it kinda tells you what to say and what to write on the board. Some of them are easy to write on the board but I find myself kinda going off on another way." Although Claire agreed with Gloria about the step-by-step process the curriculum affords, she also noted that she diverged from the text at certain times. She later mentioned that this divergence depends upon the activity. This is interesting because she did not feel that diverging from the lessons presented in the book was appropriate. This hesitancy to construct her own meaning from how to teach the mathematics is evidence that she also viewed the text as the authority on how best to teach the students. Although struggling in this first year, the teachers believe that the information provided by the text is something to be learned. As Gloria's noted, "If we can just get through this year, we'll know what is in the book." This coming to understand the book is evidence of their placing authority with the text and adhering to a fidelity approach in its implementation.

Concluding Remarks

Analysis indicates that the teachers' instructional reality is currently creating a tension for them since they perceive their job as pursing conflicting agendas. Although the teachers praised the new text resources for giving them access to students' strategies, they also expressed concern over jeopardizing test scores. In the past, the texts were boundary objects in that they conveyed the objectives of the District Office. Our analysis of the teaching sets, along with the institutional setting, suggests that the teachers gave agency to the text resources as a means to ensure compliance of the objectives. Positive performance by their students on the high-stakes tests reinforced their placement of agency. Teachers were, in effect, de-professionalized in an attempt to control their performance as judged by the test outcomes of their students. Because the text resources concurred with the mandated objectives from the District Office, the teachers seemed reasonably confident that the curriculum would bring about a new and effective way of teaching mathematics. However, in order for the teachers to capitalize on the affordances of the reform curriculum, they must develop a sense of agency for their own practice. This may be loosely interpreted as finding a middle ground between the disciplinary resources provided and the teachers' perception of how to present the mathematics in a meaningful and coherent manner. Therefore, this analysis suggests that reform cannot be mandated and the textual resources provided through reform are not the primary carriers of meaning, much less of reform. However, the interplay of teacher, student and resources can support teachers' reconceptualization of their practice if they come to view the resources as a tool in this process (Meira, 1995, 1998). Viewing curriculum as a tool places primary agency with the teacher, who engages in what Boaler (2003) calls a "dance of agency" between the disciplinary domain of mathematics *and* teachers' beliefs and views of the instructional practices.

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CONSTRUCTING NOTIONS OF EQUITY THROUGH CURRICULAR REDESIGN: A DEPARTMENT'S APPROACH

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This research report examines notions of equity in the context of a high school mathematics department's curricular redesign efforts aimed at lowering the failure rates of students in low level mathematics classes disproportionately populated by low SES students and students of color. Twelve of the thirteen teachers in the department participated. Data sources included a focus survey, interviews, and field notes from observations of three key informants' classes and department meetings. Data analysis included constant comparison analysis, triangulation of the data, and a search for disconfirming evidence. I discuss the ramifications of the department's conceptualizations of equity on the curricular redesign process and the students' resulting access to advanced mathematics courses.

Mathematics is often viewed as a means to economic advancement and a subject that opens doors to those who are successful in advanced courses. Yet, mathematics has been employed to filter students out of educational opportunities (National Research Council (NRC), 1989). This claim is well-documented for students of color and for students from low socio-economic backgrounds who typically experience lower achievement than their White, middle class counterparts (Oakes, 1990; Lubienski, 2002; Secada, 1992; Tate, 1997. In the past few decades, equity issues in mathematics education have received increased attention. For example, the National Council of Teachers of Mathematics (NCTM), a leader in incorporating equity issues into mainstream discussions, has increased its attention to equity describing it as a societal need (NCTM, 1989) and later as a guiding principle in mathematics education (NCTM, 2000). In 2005, the NCTM Research Committee (Gutstein, Middleton, Fey, Larson, Heid, Dougherty, DeLoach-Johnson, & Tunis, 2005) discussed ways that research on equity issues could inform the mathematics education research community. Despite the increased attention, there have been few studies that examine either what characterize equitable mathematics education or teachers' understanding of equitable mathematics education (Gutierrez, 2002; Rousseau & Tate, 2003; Secada, 1991). This area of inquiry is critical as we plan instruction, develop curricula, and evaluate programs.

Purposes

Utilizing a case study design, I examine the conceptualizations of equity held by teachers in a high school mathematics department and how meanings of equity were negotiated and instituted as the department redesigned courses and restructured course offerings in order to respond to high failure rates in low level courses disproportionately populated by students of color. Gaining insight into these areas is critical as researchers continue to design and conduct research aimed at promoting equitable mathematics education. Findings from this research add to our understanding of how teachers, who are increasingly called upon to reevaluate and adapt their instruction to promote both equity and excellence, negotiate an understanding of equity as they undertake these complexities. That this research examines changes that impact access to

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advanced mathematics is significant since research links mathematics course taking with increased achievement. While access is central to this inquiry, in the next section of this paper, I argue that access is but one component of equity as I review conceptualizations in the field and later, examine conceptualizations held by teachers in the high school mathematics department at the center of this study.

Conceptualizations of Equity in Mathematics Education

Equity has taken on fairly traditional meanings in mathematics education although recent scholarship embodies a notion of social justice that includes non-traditional concepts. Classical meanings include emphases on equality, access, and outcomes. In this section, I review conceptualizations of equity and argue that given the context of department level reform, particular components of equity are critical to examine in order to create a more equitable learning environment. The purpose of this inquiry, however, is not to construct an exclusive definition or conceptualization of equity that is to be adopted by the field. Apple (1995) described equity as a "sliding signifier" without "an essential meaning," but rather with a meaning that is "defined by … use in real social situations with real relations to power" (p. 335). I argue that equity is a complex construct that is dependent upon particular circumstances (Apple, 1995; Gutierrez, 2002; Secada, 1989) and involves human judgments about fairness and justice (Gutierrez, 2002; Gutstein, 2003; Secada 1989; 1991). The existence of these judgments "appeal to the spirit…of the law" (Secada, 1991, p. 18) and prohibit an essential meaning.

In reviewing conceptualizations of equity, I examine three characteristics and three components of equity identified in the mathematics education literature. First, equity is described as a dynamic construct or a moving target (Apple, 1995; Gutierrez, 2002; Secada, 1989). This notion acknowledges that what is deemed equitable is contingent upon the particular circumstances being examined. Second, equity is a qualitative construct (Secada, 1989), and what is deemed equitable is a matter of judgment. The judgments are value-ladden (Secada, 1989; Gutierrez, 2002), involve matters of power (Apple, 1995), attend to matters of justice and fairness, and appeal to the spirit of the law (Secada, 1989). Third, equity is comparative. It is at this point that equity is often conflated with equality (Gutierrez, 2002; Secada, 1989). Equality connotes sameness, a lack of difference, or parity among groups (Secada, 1989). The conflation of equity and equality in education has been described as problematic (Secada, 1989; Tate, Ladson-Billings, & Grant, 1993) since equality or sameness (e.g. with respect to inputs) can be achieved and inequities still exist. Equity, however, does allow for comparisons since they are, at the least, necessary to identify the existence of inequities. Yet, equity goes beyond comparisons of sameness and includes an examination of fairness.

Equity can also be thought of as consisting of components that are evaluated in order to gauge equity as a whole. First, one must consider what is being assessed. Traditionally, one looks at inputs, processes, or outcomes. A restrictive (process-oriented) or expansive (outcome-oriented) view of equity may be adopted (Rousseau & Tate, 2003; Tate, Ladson-Billings, & Grant, 1993). An examination of inputs includes access to educational treatments. Large scale studies (e.g. *Multiplying Inequalities*) and reports (e.g. *Everybody Counts*) have examined inputs such as access to courses (Gutierrez, 1996; Lee, Croninger, & Smith, 1997; Oakes, 1990). Many smaller scale studies often assume a process-oriented perspective of equity. Noted examples include examinations of culturally relevant teaching (Tate; 1995; Gutstein, Lipman, Hernandez, & de los Reyes; 1997); the design, implementation, and research of programs such as Cognitively Guided Instruction (Carey, Fennema, Carpenter, & Franke, 1995) and the QUASAR

Project (Silver & Stein, 1996); and department level practices (Gutierrez, 1996; Gutierrez, 1999). Student outcomes such as achievement on national exams (Lubienski, 2002; Secada, 1992; Tate, 1997) or linking achievement with background factors (Moore & Smith, 1987; Rech & Stevens, 1996) have also been analyzed to gauge equity.

Secondly, one must consider the level and timeframe at which the unit of analysis is assessed. Gutierrez (2002) argues that assessments should be taken at various time frames and at various levels. Holding that a component of equity includes the examination of outcomes of the aggregate, equity can be gauged at various levels – classroom, department, school, or district. Patterns are informative, and we should conduct evaluations regularly in order to appraise efforts and achieve short- and long-term goals.

Finally, one must consider the ends for which students' mathematics education prepares them. Mainstream arguments for equity are economic-based (NCTM, 1989; NCTM, 2000; NRC, 1989) asserting that the goal is to prepare students to get jobs in a technological society. Democratic participation is articulated as a goal for mathematics education so that students are able to analyze and critique issues they confront and their place in society (D'Ambrosio, 1990; Tate, 1995).While democratic participation includes the ability to understand and employ mathematical arguments required to make decisions, mathematics education that promotes social justice is the intended end for researchers who argue that mathematics should also prepare students to employ their learning to change (as opposed to simply participating in) society on a personal (Frankenstein, 1995), local (Gutstein, 2003), and global level (Gutierrez, 2002). Mathematical content knowledge, from this perspective, is not to be diminished (Gutstein, 2003); rather, it should be viewed as a vehicle employed in the journey to achieve social justice.

Research has not focused on examining teachers' conceptualizations of equity in curricula design and policy construction. The aspects of equity discussed above provide a framework for thinking about equity in the context of mathematics education. In the following section, I describe a research study that I conducted in a high school mathematics department aimed at promoting equity through the redesign of several low level mathematics courses that were disproportionately populated by students of color. It is this context which serves as the basis for the following discussion of and suggestions for curricular redesign that zeroed in on more equitable mathematics education.

Methods

This case study was conducted in one high school mathematics department located in a small, mid-western, town. The high school, Rolling Meadow High School (Rolling Meadow), had a student enrollment of approximately 1500 students. The average class size was approximately 17. The racial demography of the school was 65% White, 28% African American, 5% Hispanic, and 2% Asian/Pacific Islander. Approximately 50% of the students in the school district qualified for free or reduced lunch.

Despite expansive mathematics course offerings ranging from pre-algebra and other remedial courses to both a traditional version and Advanced Placement Calculus, mathematics achievement at Rolling Meadow was low for all students and was especially low for students of color. Statistics for students who met or exceeded mathematics requirements on the state standardized exam were: 39% for all students, 44% White, 19% African American, 42% Hispanic, and 71% Asian/Pacific Islander. Four mathematics credits (semesters) were required for graduation, and students received elective credits for courses less advanced than pre-algebra. Additionally, honors versions of courses were offered for students enrolled in special interest academies (e.g. medical related fields). A cautionary view of the broad course offerings is warranted as research suggests that such promotes lateral movement (Gutierrez, 1996), and students learn more from a narrowed curriculum (Lee, Croninger, & Smith, 1997). Initial placement into mathematics courses was determined from an eighth grade placement exam. The graduation rate was 68% for the entire student body, 75% for Whites, 55% for African Americans, 41% for Hispanic, and 67% for Asian/Pacific Islanders.

Twelve of the thirteen teachers in the department participated including the department chair and the district's mathematics curriculum coordinator. Two teachers had joint appointments, the curriculum coordinator and a teacher jointly appointed in the applied technology department. All of the teachers were White; eight were female, and four were male. The average years of K-12 teaching experience was over 10 years. Four teachers had less than three years of K-12 teaching experience, and three teachers had previously taught at the postsecondary level. Three teachers, who were recommended by the principal, were key informants. The principal's recommendations were informed by two criteria: the key informants taught targeted courses and had varying levels of experience. The principal and one guidance counselor also participated in the study.

I collected data for eight months. Data included a survey, audio-taped interviews, and field notes from observations of five sections of targeted courses and department meetings. I observed each of the five classes at least twice a week for six months. I attended eight department meetings and two sub-department meetings which consisted of all teachers who taught a section of a particular course. Each teacher completed a focus survey administered prior to a 45-minute interview. The survey provided foundational information on the teacher's educational background, the teacher's views on the nature of mathematics, and the teacher's expectations of their students' ability to do advanced level mathematics. The key-informants participated in two additional 45-minute interviews. The principal and guidance counselor were interviewed once for 30 minutes.

Data analysis included constant comparison analysis (Strauss, 1987), triangulation of data, and a search for disconfirming evidence. Transcripts of each interview and field notes were coded utilizing an initial list of codes constructed from the research literature. This list was revised based on ongoing analysis. After analyzing data from each data source, I then analyzed data from combinations of sources to strengthen the inferences. I identified emergent themes and revised the list of codes collapsing multiple codes, adding new codes, and eliminating codes deemed inappropriate. The revised list of codes informed future data collection and my search for disconfirming evidence.

Findings

In this section, I describe the conceptualization of equity that guided the department's efforts and decisions. Interwoven into this discussion, I identify department and school level factors that significantly influenced the department's implementation of curricular changes, thereby, shaping policies and impacting the degree to which equitable mathematics education would result. I conclude with recommendations for departments that would engage in similar curricular redesign in order to lessen the likelihood of unanticipated and unintentional outcomes that perpetuate inequities.

I consider three components of equity: (1) what is being assessed, (2) the level and timeframe, and (3) the ends of the curricular redesign. Teachers in the Rolling Meadow mathematics department viewed equity narrowly – primarily as an issue of access. Among the changes implemented by the department was the noteworthy elimination of a four-semester sequence of algebra for which students fulfilled the mathematics graduation requirement upon completion. This version of algebra was replaced by Modified Algebra, a two-credit course. For many students in the targeted courses, this change meant that students would have to take another mathematics course after completing Modified Algebra. Consequently, the department designed an additional geometry course, Modified Geometry, and increased access. However, the curricular redesign had not addressed a key question – "Is that which is being distributed worth having?" (Secada, 1989, p. 74). The Modified Geometry course lacked formal proofs eliminating key opportunities for deductive reasoning and future mathematics course taking. Also, the pedagogy (or processes) of the modified courses had not been reconsidered. Rather, instruction focused, almost exclusively, on procedural knowledge. Finally, the department did not outline outcomes that would indicate success and relied on anecdotes to frame their assessment of the curricular changes.

The department neglected to articulate its vision for equitable mathematics education and did not identify measurable outcomes to gauge equity. This precluded future evaluations of the curricular changes as well as not systematically collecting data on enrollment or success in targeted courses. The teachers gave vague responses that could not be corroborated with data on what would count as successful changes.

A goal of the curricular redesign was to get students in targeted courses to take more advanced mathematics courses. The department asserted that the large number of students who graduated having only completed algebra contributed to their low scores on the state exam which included geometry content. Thus, the primary end was to increase achievement on standardized exams, although some additional focus was to prepare students to acquire jobs. Democratic participation or social justice was not an end goal. Moreover, the department's view of mathematics as a set of procedures went unchallenged and did not include mathematics as a tool for critiquing one's world or as processes such as reasoning, problem solving, and thinking critically.

The department viewed equity in terms of access to courses or content. It had not considered pedagogy or how different pedagogical approaches alter what mathematics students understand. The department had not focused on outcomes, and the expansive/restrictive dichotomy is, therefore, less helpful in examining the department's conception of equity. Moreover, each of the features of what is to be assessed – inputs, processes, and outcomes – is critical in creating equitable mathematics education since a focus on either one at the exclusion of the others permits the perpetuation of inequities.

An absence of critical perspectives characterized the departmental culture which fostered changes that resulted in restricted access. While some teachers disagreed with the design of the modified courses, their objections were not strong or framed in ways that would allow the dispute to be settled utilizing evidence. Rather, anecdotes were recounted, and a close examination of features of the curricular redesign was thwarted. Unexpectedly, a group of four teachers, each with less than four years of experience, were leaders in the curricular redesign. They went to conferences and were eager to implement new projects in their instruction and complete administrative tasks that allowed the department to redesign courses quickly and with little planning. The impact of their efficiency was mitigated by their lack of historical understanding of previous course redesigns in the department and their hastiness. The department could also be characterized as impulsive. It conceptualized, planned, and implemented new courses within the same academic year. Planning for a remedial course began during the fall semester and was taught by a first year teacher the next semester. The department's rush to implement change could not simply be viewed in terms of their willingness to identify and respond to inequities since additional planning would have likely permitted critical discussions about proposals and more thoughtful course designs.

A second factor that contributed to the negative, unintended outcomes of the curricular redesign was the department's means of addressing concerns related to guidance office practices. The department and guidance office had a strained relationship centered about opposing views on whether students would remain in courses after having failed the first semester of a two-semester course. Teachers evoked the sequential nature of mathematics and stated that they were the experts and knew what students needed to know in order to be successful in subsequent instruction. Consequently, the department constructed rigid placement policies which limited students' access to certain courses and created more remedial level courses to respond to the guidance office and principal's desire for students to remain in mathematics courses when they have failed the first semester of a course. The alternative would have been to place students in a study hall until they could reenroll in the course the following fall semester.

Conclusion

The department had increased access to geometry content. However, access was restricted, and students who were placed in pre-algebra in ninth grade were still unlikely to take a geometry course. This small gain in access was overshadowed by the creation of a new track, defined by the modified courses, and the neglect of attention to pedagogy. The outcomes which perpetuated inequities would have likely been diminished had the department taken more time to plan the courses and outline desired outcomes. The department's narrow view of equity went unchallenged and highlights the need for opportunities to critically examine proposals, provide opposing viewpoints, and negotiate a more complex conceptualization of equitable mathematics education.

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THE DIFFICULTIES OF THE TAXI PROBLEM: EXAMPLE OF NON-CONGRUENCY BETWEEN SEMIOTIC REGISTERS IN PROBABILITY

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The problem of the taxi is part of the classic literature of probability didactics and the psychology of the reasoning related with the use of heuristics. In particular, it was used to prove the existence of the base-rate neglect as an intrinsic to the human reasoning. In this article, adopting a didactic perspective, we present the results of a research that allowed us to identify some of the difficulties that this problem involves Using Duval theory of the semiotic registers of representation, we show that the difficulties are the same ones independently of the register that is assumed for its solution since they are associated to the non-congruency of the register of the problem in natural language with any other representation.

Introduction

In 1972, Kahneman and Tversky, in their searching for explanations for reasoning under uncertainty, proposed the following taxi problem for the first time that "together with its variants, has perhaps been the single most thoroughly investigated task in all the research on judgment under uncertainty " (Shaughnessy, 1992, p. 471):

> A cab was involved in a hit and run accident at night. There are two cab companies that operate in the city, a Blue Cab company, and a Green Cab Company. You are given the following data:
> (a) 85% of the cabs in the city are Green and 15% are Blue.
> (b) A witness at the scene identified the cab involved in the accident as a Blue Cab. This witness was tested under similar visibility conditions and made correct color identifications in 80% of the trial instances.
> What is the probability that the cab involved in the accident was a Blue Cab rather than a Green one? (Kahneman y Tversky, 1972)

Two data are presented, the prior probabilities, also called "base rate", given by the taxi composition in the city, and the likelihood given by the reliability of the witness, named "diagnostic information", and the posteriori probability is asked. Although the answer using Bayes theorem is 0.41, the people interviewed by Kahneman and Tversky answered in their majority 0.80. Since this value agrees with the diagnostic information (80% of correct identifications) the authors concluded that subjects neglect the base-rate (15% Blue cabs). "In the 1970s base rate neglect was considered a well-established fact, and was presented together with other "biases" and "errors" as evidence for the fallibility of human reasoning" (Gigerenzer and cols. 1989, p. 219).

Kahneman and Tversky's thesis was that people who are statistically naive make estimates for the likelihood of events by using certain judgmental heuristics. In this case the heuristic that explains the neglect of base rate is the *representativeness*. According to this heuristic, people estimate likelihoods for events based on how well an outcome represents some aspects of its parent population (Kahneman y Tversky, 1972)

In this case, people may feel that the single instance of the accident should be representative of the witness' 80% reliability data.

The Taxi Problem has been partially responsible for the current research by researchers for alternatives to representativeness to explain people's flawed probability estimates. Among these explanations *causality* and "*outcome approach"* stand out.

The explanation of causality given among others, by Kahneman and Tversky (1982) explains the attention given to the base rate in function of the causal relation that may exist between the rate and the objective event. When Kahneman and Tversky (1982, p.157), instead of the information of composition of taxis in the city gave the following information: *Although the two companies are quite equal in size, 85% of the accidents of taxis in the city are produced by green taxis and 15% by blue taxis*, the answers stopped ignoring the rate of base. The median of the answers, in this case, was 0.60 which is an intermediate value between the witness reliability (0.80) and the correct answer (0.41).

In order to explain why most people answered 0.8. Konold (1989) claims that some people perceive the taxi scene like a singular, unique fact, and therefore the purpose is to decide the result of the experiment as correct as possible. This form of assuming the random experiments is identified by Konold as *outcome approach*. Outcome-oriented people may believe their task is to correctly decide for certain what the next outcome will be, rather than to estimate what is likely to occur.

From a mathematics education point of view, the Taxi Problem is a difficult problem, even for students who have studied probability. "It is difficult for students to interpret exactly what they are being asked to do" says Shaughnessy (p. 471), and it's also difficult to interpret exactly what is given to them, we may add.

We present in this article some reasons that explain the difficulties people have with the taxi problem that makes them neglect the base rate. Our approach is didactic and is directed to the identification of difficulties that students, who have the enough resources to solve it, experience. The explanation that we give, extracted from the results of a research, use the point of view of Duval theory of semiotic registers of representation, that we explain briefly in the following section.

Theoretical Framework

Duval theory attributes an essential role in the learning processes of the mathematical concepts to semiotic representations. The understanding of mathematic objects depends on the availability and use of diverse systems of semiotic representation, of the transformations that on them take place and of the conversions that among them are made. Although the knowledge act is a cognitive act and the result of that act remains in the mind in the form of mental representations, the set of images and concepts that the individual has about an object and that give account of that knowledge are created based on the perception of external semiotic representations, which leads Duval to think that the function of these representations cannot only be the communication of the mental images, but that comprises an active part of this construction.

For Duval a system of signs is a Semiotic Registers of Representation if the signs form a system that characterizes them and that defines its form to construct them, if in addition they can be put under transformations under clear rules to obtain information that in his original form they do not have, and if they have their homologous in other representations. These characteristics are expressed by Duval in terms of the actions that a subject can exert on them and he denominates them cognitive activities of *formation, treatment and conversion*. Duval raises the necessity to know several representations of the mathematical objects in several semiotic systems of representation and to reach their coordination as a requirement for their understanding: "the understanding in mathematics requires the coordination of at least two registers of semiotic representation" (Duval, 2001, p. 3).

In summary, we could say that in Duval the understanding of a mathematical object is associate with the capacity to represent it in several semiotic registers, to make suitable treatments in them and of knowing how to find their equivalences. Creation of representations, treatments and conversions generate *integrating understanding*, that Duval mentions, and that supposes coordination between registers (Duval, 1995).

The accomplishment of conversions between semiotic registers is often a less simple activity of which it tends to be believed. Essentially the conversion means to establish a correspondence between the significant units in each register, which some times is not direct but that demands a reorganization of the information or the accomplishment of a treatment in one or both registers or, even the use of another different representation from the ones considered, before being able to make the conversion.

Duval referring to this phenomenon, classifies the representations in congruent and non-congruent depending on the three following criteria of congruency:

1. The possibility of a "semantic" correspondence of the significant elements: to each simple significant unit of one of the representations, a significant unit from the other register can be associated.
2. *The"semantic" terminal univocity*: to each elemental significant unit of the primary representation, correspond only one elemental significant unit in the register of arrival representation.
3. The organization of the significant units. The respective organizations of the significant units of the two compared representations, leads to the fact that the units in semantic correspondence are apprehended in the same order in the two representations (Duval, 1995).

An example of non-congruency due to the second criterion is the one that happens with the taxi problem and that we will analyze in the following sections.

The Research

The content of this article comprises the results of a research that tries to know the effects of the use of computational simulation on the understanding of random sequences, probability and conditional probability (Yáñez, 2003). The research was performed with six students of first year of engineering that still had not taken the basic course of statistics that is given in their curriculum, to whom an instruction in the basic probability concepts was given in 13 weeks with two weekly sessions of 2 hours each. Fathom (Finzer, 2000) was used to make the simulations.

Results and Discussion

The first time that the students had contact with the problem of the taxi was in the diagnostic questionnaire at the beginning of the course. Of the six students, five showed the neglect of base rate, answering 80%. The answer of one of the students clarifies the reason that justifies this forgetfulness: "*80% given that the witness identifies the blue taxi and that is what really matters and not the amount of green and blue taxis that exist in the city* ". Four months After the end of the instruction, the taxi problem was given to them again in a posteriori questionnaire with the desire to know the form in which they now confront it after being in contact with the simulation, the frequency tables, and the tree diagrams. The problem was shown with changes of percentages of the taxis to facilitate the programming of the simulation to 75% of green taxis and 25% of blue taxis.

The work session was divided into two moments: in the first, students worked individually and later, in the second, they worked collectively.

The dialogue that we show next took place at the moment of the group work. It provides us with explanations with respect to the form the students interpreted the information that the problem gives.

Calls our attention that the strategy they assumed to solve the problem in groups, is based on the formation of a table of frequencies similar to the one that the computer produces when activating the Summary Table command. The total of cases that the students assumed was 100 taxis.

Daniel made the following table of frequencies:

	G	**B**	
YES			
NO			
	75	25	

Researcher (R)**:** How do you express the witness' reliability?

Daniel: $P(Yes|G)$=80%

R: What does being right mean? If it was green, like in this case and it's right, what color did he say?

Daniel**:** I suppose it would be blue since it's the one we are referring to.

R: But, doesn't it say that he is right? It was green…

Daniel**:** No, it's that he identified the car as blue. Corrects and writes: $P\ (Yes|B)$= 80%.

R: Is that table right? ... What does the YES mean in the table?

Daniel: That he is right, he recognizes the color of the car.

Daniel began assuming the witness reliability as one conditional probability where the role of the conditioner event is assumed by the real color of the taxi, and the role of conditional event is played by the attribute to guess right with its possibilities of guessed right and did not guess right.

The difficulty that was generated can be explained by the double role of guessing right, since it is assumed as a conditional event, one of the attributes of his table, and in addition as a conditional probability. Moreover, Daniel used the outcome approach when he only thinks in blue because "it is the one we are talking about ".

R: Does guessing right come alone or does it depend on something?

Carlos: He guessed green right or blue right.

R: Then it's conditional, because if the car is green and it's right, it's because he said green, if he did not get it right it's because he said blue.

Daniel: Oh, ok, ok!

R: What are the marginal of Yes and No?

Jorge: Considering that it's 100 times the times that he had to put in order to see if he is right or not, then since they give us the percentage which is the probability of being right and not being right, it would leave us with 80 times right and 20 not right.

Daniel: But it's conditional. I mean $P(N|B)$ = 20%

Jorge: He guessed right a number of times and not right a number of times, there is no condition

Daniel: Let's look at the question: the tests were given under the same conditions of that night…

Jorge: Those are data, the tests were given guessing right a number of times and not guessing right the rest.

Daniel: what was given to guess right? A blue car? Why? Because the car was supposed to be blue.

Jorge: Right, but there is no condition.

Daniel: Because if it's like that, the table would not be able to be filled

Jorge: Yes it can be based on the general formula of conditional probability.

Carlos: Then there is a condition…. What is the condition?

The repeated Jorge interventions are in order to insist that the reliability of the witness is not conditional but that it deals with a marginal. In order to weaken Jorge's argument, Daniel uses a functional argument: "Because if it's like that, the table would not be able to be filled". This strategy works because it is Jorge himself who said: "Based on the general formula of the conditional probability", allowing the conclusive intervention of Carlos: "Then yes there is a condition.... what is the condition?"

Carlos: Ok, explain it to me, I don't understand that condition because you say that it was a blue car and he was right, if it was a green one and was right, then is the same condition. Because if it was blue and he was right is 80%, and if it was green and was right, also was right…

Jorge: It's part of 80% ...

It is from this moment where the difficulties to condense in a single one value what corresponds to two different conditionals are specified. From Duval's point of view, what is represented is the non-congruency between the representation in natural language and the algebraic representation since the semantic terminal univocity criterion is missing. The students start noticing this non-congruency when Carlos says "…and if it was green and was right, also was right".

R: Change Yes and No for Green and Blue, call the rows *G1* and *B1*. If *G* and *B* are the real colors of the taxis, what are *G1* and *B1* Carlos?

Carlos: After the test that was given to the witness

R: Ok, do something to see if it works now

Daniel: 80% by 75...

Jorge (writes on the board)

	G	**B**	
YES	60	20	
NO			
	75	25	

We have 75 green cars in the city, which are green in true and ...

Laura: *...he's wrong*.

Jorge: *It's wrong, it's 15* (completes the table)

	V	A	
V1	60	20	80
A1	15	5	20
	75	25	100

Carlos: Wait, wait, wait, where did the 20 come from?
Jorge: From the same place as the 60, here...
Carlos: meaning 80% of 25.
Jorge: 25 ___________ 100%

x ___________ 80% 20%

Laura: That's it! ...He was given 75 green cars and in 60 he said that they were green and in 15 he said they were blue. He was given 25 blue ones and in 20 he said they were green and in 5 he said they were blue.
Daniel: It's wrong, it's as I told you (goes to the board, erases and does the following table):

	B	G	
G1	20	15	35
B1	5	60	65
	75	25	100

Daniel: What I was telling you, I mean, since the witness said it was blue, then, 80% is taken positive for the, oh, blue. Then the conditional $P(V1\backslash A)$ which is true and blue. To find the values in the table it would be 80% x 25%, which is the probability of blue, gives us a value of 20. (Erases and corrects the table that he had done previously):

	B	G	
G1	5	60	65
B1	20	15	35
	25	75	100

$P(A1|A)=80\%$ (.80) x (.25) = 20
(.80) x (.75) = 60
$P(V1|A) = 20\%$ (20) (75) = 15

R: Laura, now that the table is right, solve the problem.
$P(B|B1) = 20/35$

After the suggested change by the investigator was made (to change Yes and No by the colors Green and Blue to talk about the color the witness says when a car in the same conditions as the one in the accident is shown to him), Jorge partially adopts the fact of conditionality of the witness reliability and makes correctly the calculations of the intersections associated to the green color. Then, retaking his idea that the marginal values of the new colors continue being the values of the reliability completes the table erroneously. The table is corrected when again the researcher takes part to say that the table is not well constructed, allowing Daniel to adopt the product rule and constructs the table correctly.

Conclusions

The results show several facts that call the attention and that explain the difficulty that this problem causes. Not only the simulation programming done by the students but also the use of the frequency table associated with the problem, demonstrates that the fundamental problem is given by the non-congruency that exists between the information shown and the representations used.

The non-congruency is due to the duality of values that involves the witness reliability: the value 80% (as well as the 20% complement) makes reference to two different conditional probabilities, $P(B1|B)$ *and* $P(G1|G)$ that refer to the probabilities of success related with the color of the taxis ($P(G1|B)$, $P(B1|G)$ (with equal values of 20% for the failure). Additionally, it is necessary to highlight that the use of the representations by Fathom simulation and frequency tables involve in a natural way the base rate, what avoided that the students fall again in neglect of the base rate.

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MAKING DECISIONS ABOUT DISCOURSE: CASE STUDIES OF THREE ELEMENTARY-LEVEL TEACHERS WITH VARIOUS BACKGROUNDS AND YEARS OF TEACHING EXPERIENCE

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Communication, or discourse, has been a central component of recent national reform efforts. The reforms call for a discourse that has students actively engaged as the teacher manages intellectual activity and monitors student understanding. Although the teacher's role is pivotal in determining how discourse manifests itself in the classroom, the role of the teacher in promoting discourse has not been given significant attention. This qualitative study investigated how three teachers made decisions regarding the use of discourse. A conceptual framework was developed to investigate this phenomenon using Shulman's model of the teacher knowledge base and teaching decisions outlined as pedagogical reasoning and action processes. Case studies of a novice and two veteran teachers (one with a relatively strong background in mathematics education) are presented.

National mathematical organizations recently have issued calls for reform in the teaching and learning of mathematics at all levels (MAA, 1990; 1991; NCTM, 1989; 1991; 1995; 2000), and communication has been a central component of these initiatives (Silver & Smith, 1996). The National Council of Teachers of Mathematics (NCTM) has been on the forefront of these efforts to incorporate more meaningful classroom dialogue at the PreK-12 level (1991; 2000). Such classrooms have the teacher managing intellectual activity and monitoring understanding while students take a more active role in discussions (NCTM, 1991, Silver & Smith, 1996). These discussions, referred to as *discourse*, include "the ways of representing, thinking, talking, agreeing and disagreeing" (NCTM, 1991, p. 34) about mathematics.

Although many factors contribute to classroom discourse, the teacher's role is fundamental in establishing the workings of the discussions (Chazan & Ball, 1999; Clark, 1997; Davis, 1994; NCTM, 1991), including the purposes of the communication and how ideas are exchanged. For instance, a univocal exchange is carried out to transmit knowledge whereas a dialogic exchange is completed to generate meaning (Brendefur & Frykholm, 2000; Lotman, 1988; Nassaji & Wells, 2000; Wertsch, 1991). In recent years, the NCTM (1991; 2000) has promoted discourse that represents more dialogic-type exchanges, increasing the level of complexity in the teaching of mathematics. Although the significance of communication continues to be stressed after more than a decade of reform efforts, many of the changes with respect to the implementation of discourse have not been fully realized in the classroom (NCTM, 2000). Since teachers hold the responsibility of managing mathematical discussions, it is important to better understand how they make decisions regarding classroom discourse in an effort to further promote and support meaningful classroom communication. Therefore, the intent of this study was to extend the literature by exploring the nature of teacher decision-making with respect to discourse in the teaching of mathematics.

Theoretical Framework

Research has indicated that effective classroom discourse is a function of many interconnected factors that support student understanding (Hiebert, et al, 1997), namely the nature of the task (Knuth & Peressini, 2001; Lappan, 1997; NCTM, 1991; 2000; Whitin & Whitin, 1999), norms of the classroom environment (Kazemi, 1998; Mewborn & Huberty, 1999; Yackel & Cobb, 1996; Zack & Graves, 2001), and the role of the teacher (Chazan & Ball, 1999). These factors and the interaction of these factors contribute to the purpose of the discourse. The purpose of discourse evoking univocal exchanges is to transmit exact ideas, while dialogic exchanges are conducted for the purpose of generating meaning (Knuth & Peressini, 2001; Lotman, 1988). Univocal exchanges tend to manifest themselves from tasks that are straightforward; are supported by environmental norms where the teacher is the mathematical authority figure; and has the teacher explain the mathematics while students listen. Dialogic discourse exchanges tend to develop from tasks that are open-ended, complex (Chazan & Ball, 1999), and require students to use strategies not previously learned (Rowan & Bourne, 2001); take place in an environment that promotes teachers and students sharing the authority for determining the mathematical validity of ideas (Ball & Chapin, 1997; Knuth & Peressini, 2001; Mikusa & Lewellen, 1999; NCTM, 1991; Prevost, 1996); and has the teacher facilitate discourse that is a medium for student thinking.

The research literature also indicates that teachers acquire a knowledge base of teaching, and they draw upon it to help make instructional decisions on a daily basis (Ball, 1988; Carter & Doyle, 1987; Shulman, 1987), which include decisions pertaining to classroom discourse. A framework that describes what teachers profess, understand, and do describes the teacher knowledge base, and Shulman (1987) has defined it as one that at a minimum would include knowledge of the content, general pedagogy, curriculum, learners, educational contexts, educational purposes and values, as well as subject-specific pedagogy. Teachers enact this knowledge base when making decisions through a process called pedagogical reasoning and action (Shulman, 1987), which includes “a cycle through the activities of comprehension, transformation, instruction, evaluation, …reflection” (p. 14) and then possibly new comprehensions. Questions still remain regarding how a teacher makes decisions concerning the use of discourse as they engage in pedagogical reasoning and action processes as they plan for, carry out, and look back on mathematical instruction. Therefore, this study looked at how teachers make decisions with respect to discourse as they engaged in the pedagogical reasoning and action processes in the teaching of mathematics.

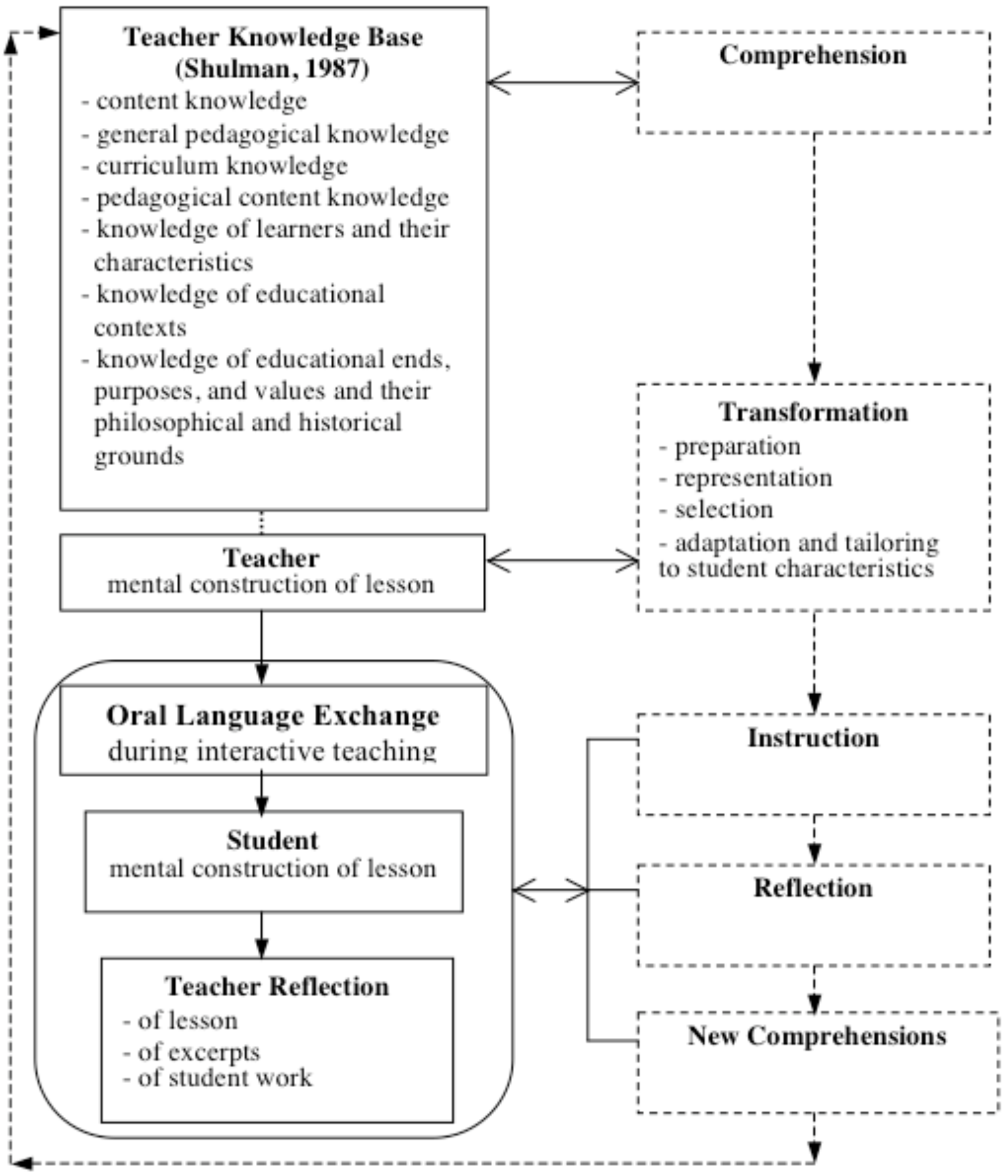
Modes of Inquiry

The purpose of this research was to uncover the complexities of the decisions pertaining to discourse in the teaching of mathematics, so the understanding of particular cases was appropriate (Rossman & Rallis, 1998; Stake, 2000). The nature of the research called for qualitative data collection and analysis techniques which would provide for an in-depth examination (Hamel, Dufour, & Fortin, 1993) and a thick description of each case (Lincoln & Guba, 1985; Stake, 2000).

A conceptual framework (see Figure 1) was developed to coincide with Shulman’s (1987) teacher knowledge base (TKB) and pedagogical reasoning and action processes. The framework proposes the cyclical nature of the TKB and pedagogical reasoning and action processes and the relationship between these processes and the teaching-learning cycle with respect to discourse.

The conceptual framework helped inform selection of participants, collection of data, and data analysis.

Figure 1. Conceptual framework



It was assumed that teachers with different levels of experience and mathematics backgrounds would provide a more holistic understanding of the decision-making processes with respect to discourse. One novice (teaching no more than three years) and two experienced (teaching for at least five years) elementary teachers, with at least one having a strong mathematics and/or mathematics education background were sought. Potential candidates from the southern New England area were identified based on the recommendation of state-level department of education officials, superintendents, principals, teachers, and other mathematics educators. The first three teachers to meet the above criteria and grant consent were selected.

Data Collection and Analysis

Data collection techniques relied on qualitative methods and were developed with the conceptual framework (see Figure 1) as a guide. Specifically, interviews and observations were designed to examine each participant's teacher knowledge base (TKB) and decisions prior to (the comprehension and transformation processes), during (the instruction process), and after (the reflection and new comprehension processes) teaching corresponding with their pedagogical reasoning and action processes. Audio recordings and notes were taken during all interviews and observations, and videotape also helped capture the happenings of the classroom during observations and stimulate participants' recall of the lesson. All interviews were open-ended and were developed by the researcher to coincide with the TKB and pedagogical reasoning and action processes frameworks. The interviews were semi-structured to allow the researcher to further investigate participants' thought processes. Participant interviews lasted between 1 and 1 1/2 hours; student interviews took approximately 20 minutes. Data collection took place at each participant's school across four or five school days in late Spring 2003. Table 1 summarizes the data collection in the order it took place and offers sample interview questions.

Table 1. Data Collection and Sample Interview Questions

Data collection	**Purpose**	**Sample interview questions**
Teacher knowledge base interview: Part 1	To gather data regarding each participant's TKB (except content knowledge, collected in Part 2)	What do you believe is/are the purpose(s) of discussions in math classes?
Pre-classroom observation interview: Lesson 1	To examine the comprehension and transformation processes that took place prior to teaching	How do you plan on going about teaching this lesson?
Classroom observation: Lesson 1	To examine the instruction process and identify unique discourse exchanges (e.g., change in direction of the dialogue or a revelation of a student's misconception)	
Student interviews: Lesson 1	To uncover three students' understanding (ranging from low to high in their general understanding of mathematics) of the lesson to later assist participants in discussing their reflection process	Students talked about what they had learned in math class, then explained how they solved problem(s) similar to ones posed in class.
Post-classroom observation interview: Lesson 1	To examine the reflection process, which included participants reviewing and commenting on the data collected during the student interviews	I am interested in understanding your thoughts at this point in the lesson (after viewing teaching segment of unique discourse exchange).
Pre-classroom observation interview, classroom observation, student interviews, and post-classroom observation interview: Lesson 2	To further understand participants' comprehension, transformation, instruction, and reflection processes and clarify data gathered for Lesson 1	
New comprehensions interview	To investigate and uncover any new comprehensions that may have occurred as a result of teaching, viewing their video excerpts, and reviewing student transcripts and work from Lesson 1 and 2	If you were to teach this lesson again, what specific aspects of the discussions would you incorporate/adjust into the new lesson and why?
Teacher knowledge base interview: Part 2	To further examine participants' TKB, specifically their content knowledge	Describe your background in mathematics.

Data were categorized as either discourse-specific (any response that specifically addressed discourse) or non-discourse-specific (all other data), and a variety of analysis techniques were utilized for both categories of data. The conceptual framework (see Figure 1) served as a guide for the analysis. Discourse-specific data collected from the participant interviews were analyzed using thematic analysis that incorporated open and axial coding to help identify patterns in the data. The data were chunked into complete thoughts and coded according to the TKB (from the Teacher Knowledge Base Interview) or pedagogical reasoning and action processes (from all other participant interviews) categories, then regrouped into these categories. The data from the classroom observations were transcribed and coded line-by-line using discourse analysis techniques adapted from Nassaji and Wells (2000). Specifically, each line was coded according to the type of exchange (e.g., a nuclear exchange contributes to the achievement of an activity or goal), move (i.e., a statement that is either an initiation, response, or follow-up to the response), and function of the move (e.g., 'disagree' indicated opposing an idea without being directly prompted). Non-discourse-specific data were scrutinized using thematic analysis techniques to capture specific types of knowledge or pedagogical reasoning and action processes. After being transcribed, the data were chunked into smaller cohesive units then coded according to the TKB and pedagogical reasoning and action processes categories. The data then were regrouped. The results were presented as three case studies with each case reporting the decision-making processes with respect to discourse prior to (i.e., the comprehension and transformation processes), during (i.e., the instruction process), and after (i.e., the reflection and new comprehension processes) teaching.

Results and Conclusions

Case 1: Nina, a Veteran Teacher with a Strong Background in Mathematics Education

Nina (all names are pseudonyms to protect participants' identities) was an experienced teacher who had been teaching for five years and her background included indicators of expertise in mathematics education. She had majored in elementary education as an undergraduate, had a master's in mathematics education, and was in the process of taking mathematics courses to become certified as a middle school mathematics teacher. She was an elementary-level mathematics specialist (she taught in a fourth-grade classroom for the purposes of this study) and had taught for five years. During the first four years of her career, Nina had received extensive training and provided professional development for teachers for how to use discourse to teach mathematics. She had presented at local, state, and national mathematics education conferences and co-written a book on using discourse in the mathematics classroom.

Nina's statements during the pre- and post-classroom observation interviews indicated that she consciously planned to use discourse to develop students' understanding of arithmetic and place-value concepts and problem-solving strategies as they solved number puzzles. She intended to ask students to compare their strategies and identify their advantages and disadvantages. Not only did she plan to have students share their ideas and defend them, she elected to bring out their misconceptions about the concepts and strategies. Nina was observed utilizing various methods to facilitate the discourse during instruction. She identified and addressed misconceptions that had come about or she had anticipated; encouraged students to contribute their ideas; ensured the class had a clear understanding of what was being discussed; positioned students to defend and justify their own and each others' ideas; and rehearsed whole-class discussions with students in small groups before addressing the entire class.

Overall, Nina had her students play a major role in contributing to the discourse, which helped dictate its course. They were asked to determine the validity of others' ideas based on their mathematical reasoning, clarify their thoughts, rephrase their peers' views, and offer their analysis of different problem-solving strategies. The decisions Nina made prior to, during, and after teaching her lessons indicate that she understands the significant role that discourse can play with regard to the teaching and learning of mathematics. As a result of these decisions, the characteristics of the classroom discourse described above suggest that it was dialogic in nature.

Case 2: Jane, a Novice Teacher

Jane was a novice teacher who had taught fifth grade for two years. She had majored in elementary education while pursuing her master's degree in a five-year teacher preparation program and had taken the general education mathematics requirement. Jane had attended regional NCTM and middle school conferences, targeting sessions that addressed major topics in her curriculum.

During the pre-, and post-classroom observation interviews, Jane shared that, as a beginning teacher, she had decided to follow the text closely, considered to be a reform curriculum, and ask the questions that were provided in the student workbooks. As a result of district and state expectations and her belief that students are social in nature, she felt it was necessary for her to incorporate discourse in her teaching. The decisions Jane shared during the pre- and post-classroom and new comprehension interviews indicated that she viewed discourse as a tool for individual students to share their ideas. Also, since she did not feel confident in her own understanding of spatial concepts, she believed having students share their solutions would give her the opportunity to learn from some of her own students (she held the view that some people had a natural ability to see figures spatially). During instruction, she had students discuss problems with each other then share their thoughts with the class. Although students contributed to the class discussion, they were not encouraged to completely defend or justify their ideas. Instead, after students presented their ideas, Jane led them to her understanding of the material as she closed in on the lesson objectives by revealing her own thinking processes and observations. At times she made observations about structures the students had built and also offered her own justification for students' observations.

In the end, although students contributed to the discourse, inevitably they did not alter its course. Instead, Jane's interpretation was at the forefront of the discussions. Consequently, it appeared that the decisions that Jane made in her planning, teaching, and looking back on her instruction helped establish a discourse that was more involved than univocal exchanges since she did more than transmit information to students. However, because it seemed as though she was primarily positioning herself, not the students, to make sense of the mathematics, it appeared that the discourse did not fully embody dialogic characteristics.

Case 3: Matt, a Veteran Teacher

'Matt' was a third-grade teacher in his twentieth year of teaching. He had received his undergraduate degree in elementary education, and had participated in few quality professional development opportunities in mathematics education. He was recognized as his districts' "Teacher of the Year" soon after data collection was completed.

The decisions Matt revealed during the pre- and post-classroom observation and new comprehension interviews indicated that he believed that students learn as a result of receiving information, being exposed to the concept repeatedly, and practicing the problems numerous times. Although he thought that he lectured to students too much, he felt that he had to either continuously describe procedures to students or find a different manner to explain the procedures based on how accurately students were able to describe the procedures themselves. These decisions appeared to manifest themselves during the classroom observations. During instruction, Matt demonstrated problems and presented information to students, eventually directing them to particular answers and procedures. To maximize learning, he believed students needed to connect the material to more concrete concepts. Thus, Matt presented his material by using objects, concepts, and real-life representations he felt were familiar to students.

The decisions Matt made with respect to discourse in the planning, teaching, and reflection of his lessons appeared to foster a univocal model of discourse since primarily used the discourse to transmit information and assess whether students had received his exact information.

Conclusions

Effective classroom communication is an essential component in improving mathematics education (NCTM, 1991; 2000). The results of this study support the literature in suggesting that a teacher's knowledgebase in conjunction with his or her pedagogical and reasoning processes influence the discourse that is carried out in the classroom. Nina had a relatively strong background in mathematics and appeared to make decisions that engaged students in dialogic discourse, while Jane and Matt both had a lower level of mathematics preparation and appeared not to engage students in discourse embodying dialogic characteristics. Questions still remain, however, regarding what factors or combination of factors have a greater influence on the decisions teachers make regarding discourse. For example, although Jane was a novice teacher, she was able to attain a discourse that was closer to the dialogic level than Matt, who was a veteran teacher. One could argue that it could be expected that teachers with more experience would have a greater facility with orchestrating the classroom discourse because they have had time to work on facets of teaching such as classroom management, scope and sequencing of the curriculum, among others. Also, although Jane used a reform curriculum, she did not utilize it in a fashion that allowed students to extensively engage verbally in higher-level processes characteristic of dialogic exchanges. Nevertheless, it appeared that making use of a reform curriculum provided a foundation from which Jane could base decisions to have students engage in discourse tending towards the dialogic level. This finding supports the research that indicates that the task influences but does not determine the eventual discourse, and, not surprisingly, training teachers on how to implement discourse using such materials may prove to be indispensable. Finally, the participants viewed the purpose for employing discourse differently, which appeared to be influenced by their beliefs about how students learn. Nina structured her instruction around discourse to encourage students to make sense of the material; Jane, who mainly implemented discourse because of district and state expectations, envisioned discourse as a way for students who naturally had great facility with the concepts to share their ideas; and Matt perceived discourse as a means to transmit his understanding to students because he believed students needed to repeat ideas in order to master them. This finding suggests that it is important for mathematics teacher educators to make teachers aware of the role and significance of discourse in classroom instruction and how it can be a catalyst for supporting student learning of higher-level mathematical processes.

Relationship of Paper to PME Goals

This research study embodies PME's goals in several ways. The literature base is comprised of works not only from the field of mathematics education, but also from linguistics. In addition, a more holistic understanding of the teaching of mathematics at the elementary level was achieved with the cooperation of practicing mathematics teachers. Implications of this research could serve to further support teachers implementing the type of discourse promoted in the latest mathematics education reform efforts.

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USING A MATHEMATICAL LENS TO INVESTIGATE A PROSPECTIVE MIDDLE SCHOOL TEACHER'S UNDERSTANDING OF SLOPE

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In this paper, we consider the mathematical activity of one prospective middle school math teacher as he considered and planned a response to an algebra student's question about slope. To do this, we recast a previously developed framework, drawing primarily on a mathematical lens, to interpret the prospective teacher's ideas about slope. In particular, we analyzed the tasks that the prospective teacher developed to address the algebra student's question to characterize the mathematics he 'called up' to support students' understanding of slope. We then overlay this analysis with an analysis of the regular classroom teacher's reflections about the lesson. By doing so, we explore the possibility of developing an understanding of the prospective teacher's understanding of how to teach in relation to how we, as teacher educators, might continue to support the learning of beginning teachers.

Teaching mathematics well requires the selection of tasks that invite learners to wrestle with important mathematical ideas, build mathematical connections and engage in meaningful discourse (National Council of Teachers of Mathematics, 1991; Hiebert et al., 1997; Stein, Grover & Henningson, 1996). A critical component of this practice is the teacher's knowledge about the content (Ma, 1999). Though we are not certain about *how* and *what* specific content knowledge plays out in teaching mathematics well, it is certain that knowing and understanding mathematics is essential to effectively anticipate and interpret students' ideas, a fundamental skill in orchestrating meaningful classroom discourse (Ball, Lubienski, & Mewborn, 2001; Whitenack & Knipping, 2002). Consequently, it is critical for prospective teachers to wrestle with important mathematical ideas to develop their own mathematical understandings for the purpose of teaching well.

Ma's (1999) work provides insight into the types of content knowledge necessary for teaching well. Ma's analysis of mathematical knowledge for teaching suggests that teachers who are able to teach well exhibit an understanding of the basic mathematical ideas, multiple perspectives and connections among those mathematical ideas, and how those mathematical ideas 'fit' within the larger mathematics curriculum. Acting upon such understandings in the classroom is what Ma refers to as having a "profound understanding of fundamental mathematics (PUFM)" (p. 124). In other words, Ma describes the mathematical knowledge that teachers must attend to 'in practice', not what the teacher knows per se. Interestingly, Ma's analysis of experienced teachers from the US and China revealed that careful lesson planning, involving in-depth analysis of school mathematics and careful task design, are critical links in the development of PUFM.

As teacher educators, we ask, "How might we engage prospective teachers in meaningful learning of mathematics for the purpose of teaching well?" and, more specifically, "What topics and/or tasks might be used to initiate the development of PUFM among prospective teachers?" Previous studies have demonstrated some of the potential learning gains associated with engaging prospective teachers in lesson study activities (c.f. Cavey & Berenson, in press).

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However, such studies were limited to some extent because prospective teachers were not able to explore issues that arose during actual classroom events. Although prospective teachers engaged in rich explorations and had opportunities to explore mathematical ideas and so on, the lessons that were developed were neither 'tested' in a classroom nor developed in response to actual classroom activities.

The current study addresses these concerns directly as we consider one prospective middle school math teacher's mathematics teaching understanding—the mathematical knowledge he enacted as he considered and planned a response to an algebra student's question about slope. We then overlay this analysis with an analysis of the regular classroom teacher's reflections about the lesson. By doing so, we explore the possibility of developing an understanding of the prospective teacher's mathematics teaching understanding in relation to how we, as teacher educators, might support the prospective teacher's learning. This study is embedded within the context of a longitudinal study designed to explore factors that contribute to the meaningful learning of mathematics for the purpose of teaching well.

Framework

Previously, we conducted a retrospective analysis via a thought experiment to identify a range of mathematical ideas a teacher might 'call up' to support her students' understanding of slope (Cavey, Whitenack & Lovin, in review). As a consequence of this work, we developed an interpretive framework that coordinated a mathematical lens with an instructional design lens to better understand the mathematics that teachers might draw on to teach slope. Here we recast our framework drawing primarily on the mathematical lens to interpret one prospective teacher's ideas about slope. We accomplish this task by considering the types of tasks that he developed to address a question posed by one of the algebra students, Kara (pseudonym).

The Student Question

Kara's question was particularly important because the teacher's response to it might make it possible for her students to explore a range of mathematical ideas. Kara's question was posed during the first lesson on slope after she had used her geoboard to find the slope of a 'ski slope' (line segment). She determined the slope by calculating the ratio of the 'rise' to the 'run', which she determined by counting spaces between pegs on her geoboard. Kara had originally created a segment that started at the top of the far left edge of her geoboard and stretched to the middle of the far right column. Subsequently, her line segment did not intersect with the bottom row of pegs, which had previously been used by the algebra students as the 'x-axis'. (See Figure 1.)

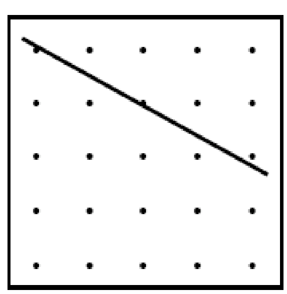

Figure 1: Kara's line segment that does not reach the bottom row of pegs (the 'x-axis').

To compute the slope, Kara wanted to count from the top left point down to the bottom left point (the 'origin') to determine the rise, and then from the 'origin' over to the line to determine the run. To accomplish this, Kara borrowed a geoboard from a classmate and put the second geoboard up against the right side of hers so that she could extend her segment down to the 'x-axis'. She then computed a slope of four eighths. Afterwards, she asked, "Can you just do like, do you have to start from the origin or can you just like go one down and see how many across it was?"

Researcher-Developed Trajectories

Kara's question relates in part to the idea that one can use any two points on a line to determine its slope. It is this aspect that we used to develop three possible hypothetical trajectories to address the following question: "What kinds of evidence might convince someone of the fact that any two points on a line can be used to calculate the slope value?" We hypothesized three main trajectories: 1) an equivalence class of ratios, 2) an equivalence class of similar triangles and 3) an equivalence class of parallel lines. These trajectories outline three mathematical 'paths' a teacher might follow to address Kara's question.

Although the concepts addressed by each of these trajectories are interrelated (and in some ways subsumed within each other), each revealed important differences. For example, if the teacher responded by asking students to generate and compare a finite set of equivalent ratios that originate from multiple pairs of points on a given line, we could infer that she acts on her understanding of slope as an equivalence class of ratios, fully aware that no matter which points are chosen from a given line, an equivalent ratio results. On the other hand, the teacher may decide to focus the discussion around similar triangles. If she did so, she might ask students to represent 'rise' and 'run' amounts for multiple pairs of points on a given line to generate a finite set of similar triangles. In this case, we might infer that the teacher draws on her understanding of slope as an equivalence class of ratios as well as the properties of similar triangles. Alternatively, the teacher may decide to invite students to construct different lines that are parallel to Kara's line to explore the class of lines with equivalent slopes. As such, we might infer that the teacher draws on her understanding of slope as an equivalence class of ratios that determine the direction of a line in the coordinate plane.

To analyze the prospective teacher's ideas, we found it useful to use constructs associated with the hypothetical trajectory lens. In particular, we analyzed the tasks that the prospective teacher developed to address Kara's question to characterize the mathematics he 'called up' to support students' understanding of slope. From this analysis, specific ideas related to two of the trajectories emerged. Before sharing the results, we discuss the methods and context of this research.

Methodology and Context

The prospective teacher who participated in this study, Ryan (pseudonym), attended a small university in the Eastern U.S. and was enrolled in a mathematics methods course for prospective middle and secondary teachers. Ryan was a career-switcher. He worked in the business world for approximately 15 years before deciding to teach math and was quite intent on bringing a real life perspective to the classroom. Ryan completed all components of this study as part of the methods course activities in conjunction with participants of the longitudinal study. The in-class components included viewing the videotaped lesson on slope up until Kara posed her question and working in groups of 3 to consider the mathematics involved in Kara's question to develop ideas about potential teacher responses. Outside of class, Ryan met with one of the researchers for an informal interview to discuss the mathematics of Kara's question and ideas for a potential teacher response. Subsequently, he prepared his own plan for a possible trajectory as a written assignment.

The videotaped lesson was generated via a previous study, where we collaborated with, Ms. Lowe (pseudonym), a middle school teacher, to explore her mathematics teaching understanding —the mathematically related knowledge she enacted as she made decisions and implemented lessons related to slopes of lines. The classes we observed were part of an Algebra I course for students in grade 7. Ms. Lowe had many years of experience, was interested in studying her own teaching, and seemed to have a deep understanding of how students best learn mathematics. Ms. Lowe's instructional practice aligned with recommendations made by the National Council of Teachers of Mathematics (2000, 1991). She valued her students' ideas, often highlighted and built on students' ideas during instruction, and more generally, frequently engaged her students in problem solving to explore mathematical ideas.

The data corpus includes video recordings of the secondary methods and algebra class activities and interviews with Ryan and Mrs. Lowe, Ryan's notes and written assignment, transcriptions of the algebra lessons, field notes, and an audio taped retrospective interview recently conducted with Mrs. Lowe. Here we refer to Ryan's written assignment to infer the mathematical ideas he 'called up' to plan a response to Kara's question. To accomplish this task, we analyzed each question/problem he planned to pose and the mathematical ideas inherent in each. As we did so, we identified aspects of the researchers hypothetical trajectories that are reflected in Ryan's plan. Subsequently, we rely on excerpts from the algebra class and recent retrospective interview with Mrs. Lowe to develop a sense of how we might continue to support Ryan's understanding of how to teach well.

Results

Ryan's Planned Response

In response to Kara's question, Ryan planned to continue using the ski slope context that Mrs. Lowe established at the beginning of the lesson. He also planned to ask students to "re-examine their geoboards with a fresh look, as if the concepts of coordinate plane and origin do not exist." Ryan indicated that he wanted students to, "consider the fact that the coordinate plane (and the concept of origin) is something that is imposed for specific purposes, but it really does not exist in the real world." To accomplish this, Ryan planned to initiate a whole-class discussion about why it makes sense to use a coordinate plane.

After the discussion about the coordinate plane, Ryan planned to ask students to work in pairs to solve 3 exercises. Exercise 1 is included below.

> *Assume Dragon's Breath is the steepest ski trail at Fun Time Ski Area and is approximately 1/2 mile long (measured on the surface of the ski run). Assume that it has a 1,200-foot vertical drop and has the same steepness throughout. 1) Draw a picture of the ski trail on the coordinate plane using the origin as the point opposite the ski trail and then calculate the slope. 2) Calculate the slope.*

Finding a solution to exercise 1 involves finding the slope of a line segment oriented in the same way that Kara and her classmates had been working with the geoboards. However, in this case, Ryan provided students with instructions to create a specific line segment in the coordinate plane. In addition, a conversion between miles and feet is required to obtain a consistent unit of measurement and the Pythagorean theorem is needed to determine the length of the 'run'.

Exercise 2 involved shifting the 'ski trail' (i.e. line segment) in the coordinate plane vertically by 2,000 feet and horizontally by 1 mile. In the written description, students are instructed to 1) draw a picture of the ski trail, and 2) calculate the slope. Ryan indicated in parenthesis that, "the answer should be the same as the answer to the first exercise even though different ordered pairs were used to represent points on the ski trail diagram." In solving this exercise, Ryan expected students to make another unit conversion before computing the ratio of rise over run.

In Exercise 3 Ryan wrote, "Assume the chairlift for Dragon's Breath ski trail has a midway station so that skiers can get dropped off halfway up the ski trail." Ryan then instructed students to 1) label the point for the midway station on the line segment drawn in exercise 1 and then 2) "calculate the slope from the midway station to the end of the ski trail." Determining the slope in this exercise involves calculating the slope of one portion of the original ski trail.

Ryan's Work & The Hypothetical Trajectories

Ryan's planned response shows evidence of his attention to two main hypothetical trajectories. By asking students to reposition a line in the coordinate plane and then recalculate the slope (exercise 2), we infer that he drew on his knowledge of equivalent ratios and parallel lines. By highlighting the issue of parallelism we infer that Ryan acted on his understanding of slope as an equivalence class of ratios that define the direction or the angle the line makes with any given horizontal. As Ryan wrote in his notes, "Apart from the location of the picture, or triangle, on the coordinate plane, it should be identical to the first picture." Further attention to slope as an equivalence class of ratios is illustrated by asking students to calculate slope for two pairs of points on the same line (exercises 1 & 3). By embedding an application of the Pythagorean Theorem within exercise 1, Ryan shows evidence of attending to properties of triangles, albeit not necessarily similar triangles.

Removing the Coordinate Plane

Ryan's emphasis on the coordinate plane as a means to represent the path of a line points to the need for such understanding to effectively implement instructional tasks related to slope. After all, Kara's question about whether or not one needs to start at the origin could legitimately be answered from this perspective. As Ryan wrote, "imagine that the coordinate plane, x & y axis, and origin have just vanished." By asking students to determine slope without the aid of a superimposed coordinate plane, we infer that Ryan drew upon his understanding of slope as a rate of change of vertical to horizontal distance. We might also infer that he drew on his knowledge of the coordinate plane as a fundamental tool for two-dimensional measurement and that his plans for a whole-class discussion reflect his intent to enact such knowledge in the classroom.

Mrs. Lowe's Actual Response

When Mrs. Lowe was faced with Kara's question during class, she had to quickly interpret the essence of the question and decide how to respond. In response, Mrs. Lowe invited students to hypothesize the answer to Kara's question. After asking Kara to repeat her question, Mrs. Lowe asked, "How many think yes? How many think no? How many think maybe?" Ms. Lowe then replaced an overhead geoboard with a coordinate grid (showing all four quadrants) on the overhead projector so that it was visible to all of the students. She then drew a line that passed through the points (0,4) and (5,0) and labeled the two intercepts. Mrs. Lowe and the students then pursued a discussion that included using different pairs of points on the line containing the points (0,4) and (5,0) to determine the slope. In their discussion, they noted that any two pairs of points seemed to yield an equivalent ratio of rise to run.

Like Ryan, Mrs. Lowe's response invited students to consider a specific line in the coordinate plane and to determine the slope ratio for several pairs of points on that given line. During Mrs. Lowe's introductory lesson on slope, she did not purposely lead them to considering parallel lines or towards making the connection to similar triangles. However, that is not to say that Mrs. Lowe had not been thinking about such connections during the lesson and in particular while thinking about a plausible response to Kara's question. Rather, Mrs. Lowe knew that parallel lines was a topic to be addressed in a few weeks and so made the decision to hold off on pursuing that connection. While reflecting on the lesson, Mrs. Lowe indicated that she thought several students were on the verge of making the connection to similar triangles, and in particular congruent triangles, during the lesson. However, that was not her primary objective for that day. As she noted after the lesson, she was most interested in students building an understanding of slope as a measure of steepness. As such, she did not 'push' students to pursue the connection to similar triangles. Mrs. Lowe also noted that making the connection between slope and similar triangles is a primary objective in her 8th grade algebra class, as she knows that generally, those students are ready to make such a connection.

Discussion and Implications

By examining Mrs. Lowe's response, we are not attempting to identify 'the right' response to Kara's question, but rather we aim to understand how teachers might put to use what they know about the mathematics they teach. In addition, we aim to explore how the beginning teacher might learn to organize his mathematical knowledge for the purpose of teaching well, putting into action that which will enhance the learning experiences of his students. Recall that Ma's (1999) work does not yield a prescription for exactly what teachers should know to teach well. Rather, she described the types of mathematical knowledge that are likely to come into being in the expert teacher's classroom. We get a glimpse of what such knowledge in action looks like when examining Mrs. Lowe's practice. In response to Kara's question, she acted upon her knowledge of slope as an equivalence class of ratios and relied on the coordinate plane as a tool for examining such measures. In addition, she managed to push aside her knowledge about slope in relation to parallel lines and similar triangles because of what she knew about the students' prior knowledge and the upcoming topics in the curriculum. Indeed, her practice exhibited elements of PUFM within the context of teaching slope.

The preliminary analysis of Ryan's proposed response illustrates that he 'called up' several constructs related to slope to address Kara's question. His ideas to compare slopes of parallel lines and to compare slopes of two portions of the same line related to two of the hypothetical trajectories (1& 3). In addition, his idea to 'remove' the coordinate plane may be a critical part of stripping away barriers to understanding slope as a rate of change. Indeed, he reminds us that the coordinate plane is the measuring tool for quantifying two-dimensional relationships and in particular, two-dimensional rates of change. In short, it seems that the task of creating a planned response to Kara's question, enabled him to call up important conceptual links in understanding slope of a line.

So, how might such information be used to support Ryan's continued development as a mathematics teacher? We suspect that just knowing how Mrs. Lowe responded is not sufficient to help Ryan become more aware of the complexities involved in making such daily decisions. Should we bring Mrs. Lowe in for a discussion in the methods class so that Ryan and other prospective teachers might have an opportunity to learn more about her thinking? Might we suggest that Ryan consider a curriculum guide for Algebra I to help him become better informed about where slope 'fits' in the curriculum? These among many other possibilities are some of the decision points we are faced with as teacher educators. As we continue our work with Mrs. Lowe to develop our joint understandings of how she uses what she knows, we are certain that other possibilities for supporting the learning of prospective teachers will present themselves.

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EPISTEMOLOGICAL DEVELOPMENT AND MATHEMATICS TEACHER LEARNING

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This paper reports the findings from the first semester of a longitudinal study that was designed to document the epistemological development of pre-service secondary mathematic teachers at a large urban southwestern university of the United States. Three pre-service secondary mathematics teachers (2 seniors and a junior) completed two constructed-response surveys and participated in an interview to discuss the surveys and other learning experiences. The analysis revealed that all three participants showed indications of Baxter Magolda's (1992) transitional knowing. Implications for secondary mathematics teacher education are addressed.

Rationale and Objectives

The reform movement in mathematics education advocates learner-centered mathematics teaching that emphasizes conceptual understanding, multiple representations, connections, communication, and problem solving (National Council of Teachers of Mathematics (NCTM), 2000). To implement reform-minded curricula as intended, mathematics teachers must view mathematics as an evolving, tentative field of knowledge, and mathematics teaching as contexts where children are provided opportunities to construct their own understandings of mathematical ideas (see e.g., Ernest, 1989; Thompson, 1992). Consequently, instructors within mathematics teacher preparation programs frequently encounter the challenge of initiating philosophical changes in what it means to know, teach, and learn mathematics.

An individual's epistemology is one's philosophy about the nature and justification of knowledge. In general, positions within models about epistemological development are characterized by evolving epistemic assumptions regarding certainty, sources of, and evidence for knowledge. Knowledge is first viewed as certain and free of context and human values and evolves to a view that knowledge is tentative, based in context and subject to the values and accepted reasoning norms of those involved (see e.g., Baxter Magolda, 1992; Belenky, Clinchy, Goldberger & Tarule, 1986; King & Kitchener, 1994; Perry, 1970/1999). A pre-service teacher's epistemology not only influences views of future teaching practices; it also influences how the pre-service teacher perceives the learning experience and what he or she learns in our teacher education courses. Put simply, the experiences we provide our students in our classrooms are filtered through epistemic assumptions regarding knowledge.

For example, individuals who adhere to the positions of dualism or multiplistic pre-legitimate (Perry), received knowing (Belenky et al), or absolute knowing (Baxter Magolda) turn toward perceived authorities for knowledge. They assume that this knowledge is legitimate based entirely on the authority's say-so. Knowledge is viewed as certain and unchanging, with the assumption that there is only one right answer. Within mathematics teacher education, a prospective teacher who adheres to such a view may be inclined to look toward instructors of mathematics or methods courses for *the* way to do mathematics or *the* way to teach mathematics (see e.g., Mewborn's (1999) participants in Stage 1). He or she may prefer learning environments in which discussions of multiple perspectives on a topic are not discussed. Or, when multiple

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perspectives are discussed, the prospective teacher will seek one right perspective. Furthermore, despite intentions of reform-minded mathematics education programs to purposely place the prospective teachers in unfamiliar contexts as a means to stimulate new thinking, this learner assimilates rather than accommodates (von Glasersfeld, 1991) ideas espoused by the program (see, e.g., the cases of Harriet (Arvold & Albright, 1995; Cooney & Wilson, 1995), and Henry and Nancy (Cooney, Shealy, & Arvold, 1998)).

Subjective knowing (Belenky et al.), multiplicity correlate (Perry), and independent knowing (Baxter Magolda) are positions that focus on internal sources of knowledge. The term internal is intended to imply a focus on self and intuition or a focus on others as intuitive individuals. Certain, single answers exist *for oneself* based on personal experiences, beliefs, and intuition. The same assumption is made about others. Certain, single answers exist *for others* and are based on personal experiences, beliefs and intuition. Multiple perspectives are considered legitimate (i.e., peers can be sources of knowledge), and knowledge is considered uncertain because of one's inability to fully understand the experiences of others. A prospective teacher from this perspective may appreciate whole and small-group class discussions in mathematics education courses but might be unwilling to evaluate others' views because he or she believes that everyone has the right to his or her own opinion or that everyone has his or her own interpretation of an experience (see e.g., Brenda in Chauvot, 2001, or see Wilson & Goldenberg's (1998) discussion of *extreme relativism*).

A third more common example for college students involve the positions of relativism subordinate (Perry) and transitional knowing (Baxter Magolda) in which some areas of knowledge are viewed as certain while other areas of knowledge are viewed as uncertain. Sources of and evidence for knowledge are dependent upon knowledge domain. Within the uncertain knowledge domains, multiple perspectives are epistemologically legitimate. As a consequence, a learner may expect environments that promote multiple, legitimate perspectives in some contexts but not in others. For example, a prospective teacher may expect a multiplistic environment in a methods course but not in a mathematics course.

This paper will report on the epistemologies of three pre-service secondary mathematics teachers during the first semester of a longitudinal study at a large urban southwestern university of the United States. 12 more participants will be obtained in the following semester. The eventual goal is to integrate results from the longitudinal study with the existing literature about epistemological development to better understand the role of epistemic assumptions in teacher learning and the kinds of experiences that might move pre-service secondary mathematics teachers to more complex ways of thinking.

Methodology and Data Analysis

A qualitative research design informed by Baxter Magolda's (1992) work was used for this study. Epistemic assumptions for each participant were inferred from two written surveys of constructed-response items and one digitally-recorded interview.

Participant Selection

Opportunistic sampling strategies (Patton, 1990) recruited 3 pre-service secondary mathematics teachers in multiple sections of a required reading in the content areas course during the first summer session of 2005. Students are required to have completed this course before they are permitted to enroll in the methods course and subsequent student-teaching semester.

Participants received a small stipend for participation in the study and agreed to participate for at least the summer and fall semesters.

Data Collection and Analysis

Data sources for this study were Baxter Magolda's (1992) Measure of Epistemological Reflection (MER), a class log writing assignment, and a semi-structured interview. The *MER* is a 7-page constructed-response questionnaire designed to gain an understanding of the participant's perspective on learning in college. It addresses each of the five strands of Baxter Magolda's (1992) Epistemological Reflection Model (learner, instructors, peers, evaluation and nature of knowledge). The *class log writing assignment* had constructed-response items that was designed to gain specific information about the participant's perspective regarding the roles of learner, instructor and peers in each of the courses the participant was taking that semester. This data helped to triangulate the information gained from the MER. The purpose of the *interview* was to allow for clarification on what was written on the MER and the class log writing assignment as well as to provide opportunity to discuss her learning experiences beyond the scope of the two written data sources. The MER was completed at the beginning of the semester, the class log was completed up to 3 weeks later, and the interview was conducted at the end of the semester. The three participants agreed to continue their participation in the fall semester where they will again complete the class log for each course and participate in an interview. 12 more participants will be obtained in the fall semester. This report represents a base understanding of three pre-service secondary mathematics teachers' epistemological perspective at the beginning of the study.

The analysis of the MER followed Baxter Magolda's (2001) recommendations for interpreting participants' responses. Each MER was read at least twice. Central reasons for each strand of Baxter Magolda's model were identified; the class log writing assignment and interview provided refined interpretations of the participants' perspectives. Interviews were listened to several times, and excerpts of the transcriptions were coded in terms of the inferred central reasons and characteristics of ways of knowing within Baxter Magolda's model as well as other models of epistemological development. Researcher biases were monitored through journaling and discussions of analysis with colleagues.

Findings

The Participants

All three participants were mathematics majors. Cindy (all names are pseudonyms), 21, was a senior but had two semesters of coursework and one semester of student teaching to complete. Her parents worked in the fishing industry. In addition to the reading in the content areas course, Cindy was taking advanced linear algebra. Ann (21) was a junior. Her father worked in retail and her mother was a childcare provider. Beth was 22 and also a senior. She was taking an educational psychology course along with the reading in the content areas course. Her parents worked in the space industry. At the time of this writing, an interview with Beth had not been conducted and is therefore not included in this analysis.

Epistemologies of Cindy, Ann, and Beth

All three participants showed indications of transitional knowing (Baxter Magolda, 1992) in that they each took on a learner's role to understand material rather than memorize material.

Also, each perceived some knowledge domains as uncertain; however, such occurrences were not prevalent in the data, particularly in Cindy's case.

Cindy preferred classes that focused on factual information (rather than ideas and concepts) because it was easier for her to remember facts than theory. Examples of courses that represented factual information were Calculus, where one learns and applies facts and formulas, and history, where one learns about what events happened when. An example of an idea and concept course was advanced linear algebra which involved theory and proof.

Cindy described positive learning environments as ones which involved instructors who repeatedly and patiently explained material until it was understood by students. Students should have opportunities to ask questions to clarify the instructor's explanations. For math and science courses, Cindy preferred to listen, take notes, practice with homework problems, and participate in study groups outside of class where students could "switch notes" and teach one another. Within other courses (reading in the content areas course), frequent small-group work was appropriate for sharing ideas gained from reading assignments and projects. She felt that she learned a great deal from her peers in this context. However, she was disturbed at times when the instructor did not provide closure to small group discussions to help students see how they would apply what was shared in discussions. Cindy described this course as different. She had never taken a course that used small-group work as a strategy for teaching the material but felt that this strategy made sense for the content that was addressed. This course was possibly an example of an uncertain knowledge domain for Cindy where her peers were viable sources of knowledge. However, she continued to rely on the instructor to help her make sense of the information.

Cindy was not conflicted with the notion that different instructors might provide different explanations of the same idea. She provided the example of two instructors who explained the concept of continuous function in different ways but "I still got the main idea in the end." In other words, different explanations did not change the facts. In the reading for content areas course, the different views were supported by the different experiences and background of students.

Ann's discussions about positive learning environments focused on "copious note-taking" and "understanding rather than memorization." She felt that she learned better in courses that focused on ideas and concepts (rather than factual information) because she liked to reason and think logically. Courses that focused on ideas and concepts were considered more difficult because of the need to understand rather than "memorize and regurgitate." For Ann, *understand* meant being able to apply the content in new contexts. Ann preferred instructors who gave lectures accompanied by "sequentially-written" notes that highlighted the important concepts with examples. Instructors should also invite questions from students throughout the lecture.

Ann claimed that in some cases, she preferred classroom environments where her peers did not talk. She qualified her position by stating "it depends on what type of class, but for math classes I really do prefer students don't do a lot of talking. … unless it's a problem-solving class where you discuss different ways of solving the problems." Philosophy courses and writing courses were environments in which small group discussions were appropriate and mostly opinion-based. Reconciling conflicting perspectives in these contexts was resolved through what one believed. These knowledge domains represented uncertain knowledge domains for Ann. The reading in the content areas course was also a context for small-group work and exchanges of ideas, but the content of this course represented certain knowledge that could be applied in multiple contexts. Like Cindy, Ann appreciated the instructor's role of helping students make sense of small-group discussions:

> I was able to grasp new points of views from my group discussion and reflect back to my own knowledge of what I have learned from the project. The group discussion was interesting. I was engaged because I was able to compare and contrast my findings with my classmates. The professor's input on the discussion made me realize new things I haven't thought of.

In some contexts, Ann was not distracted by instructors who "explained the same thing differently." In fact, she commended a friend who took two chemistry classes, taught by two instructors, in order to gain a better understanding. However, she struggled with the notion of choosing one explanation over another:

> Everybody is always sure on what they believe, but I believe you can never be sure unless you do a proof of it and it came out to be true. Even so, some people would still refuse to believe you. This is a tough question.

Finally, Ann felt that evaluating student work should be left for the TA (with monitoring by the professor), based on a "rubric of what is to be expected." The example she provided related back to her math courses where she wanted clear expectations of how much work needed to be shown to receive full credit. Evaluation was about demonstrating the level of understanding that had been taught in class.

Beth's written discussions about her learning emphasized the importance of understanding and applying knowledge as well as indicated an awareness of uncertain knowledge domains. In fact, her descriptions consistently delineated between certain and uncertain knowledge domains. Recall that to date, an interview with Beth had not been conducted to refine the following analysis.

Beth felt she learned best in courses that focused on factual information because the content was more clearly defined, specific, and less vague and ambiguous. In Beth's words, in these courses "questions will have yes/no answers. Subjects can be discussed in terms of definitions and properties, instead of, say, opinions. It seems to me that information can be relayed easier in this fashion." However, she cautioned, after the class is over, students would probably not remember very much of the material.

Beth preferred instructors who gave what she called interactive lectures. Such lectures occurred in courses that addressed ideas and concepts. The lecture format was described as less chaotic and distracting which allowed her to focus on the information. Interaction was a way to obtain student input through questions or comments. Beth explained, "when done 'right', it seems like more of a conversation between the teacher and students." Beth felt that interactive classes allowed for students to voice their opinions, give comments, and ask questions easily. This was "good for the learning process [because] you can hear and consider other people's ideas. Most people probably learn more when they are engaged in the subject." Beth's educational psychology instructor modeled effective interactive lectures that were "lively" and abundant with real-life examples that were helpful for making connections with the material.

In terms of multiple and possibly conflicting explanations about a concept, Beth argued that there would be times where one could be sure of an explanation, and there would be times where one might never be sure. DNA testing was an example she provided that could "rule out competing explanations" and some historical events were examples where it may be impossible to determine "why" because of the inability to understand individual thoughts and motives.

Finally, for Beth, evaluation of student work should not be based solely on the student's ability to demonstrate specific learned skills. Beth contended that evaluation of student work should also include the student's ability to follow directions and the thoroughness of the work. Following directions was important because "usually an instructor asks for work done a certain way because it facilitates a strategy for learning." Thoroughness allowed for grading for effort as well as "how well a student understood the material." She felt that the instructor, not an assistant, should be involved in the evaluation because the instructor would better understand a student's thinking in light of what was taught in class.

Discussion and Implications for Mathematics Teacher Education

Cindy, Ann and Beth exhibited transitional knowing, but in a limited way – there were not many knowledge domains that were perceived as uncertain. All three appeared naive in terms of their ways of knowing, yet, they are close to becoming practicing mathematics teachers. In what ways are they prepared to employ reform –minded instructional practices if they have not yet progressed to more complex ways of thinking? Although these findings cannot be generalized, the findings are consistent with previous research with pre-service secondary mathematics teachers in their junior and senior years (e.g., Chauvot, 2001 and Cooney et al., 1998) and research about epistemological development with college students (Baxter Magolda, 1992). How can we apply these findings to secondary mathematics teacher education? Two recommendations are proposed.

Early Intervention

It is unfortunate that the structure of many secondary mathematics teacher education programs delay our contact with students until the junior or senior year. Many of our mathematics education courses are designed to challenge their epistemic assumptions about mathematics. Such encounters earlier in the college career have the potential for initiating and sustaining change just as Ann's philosophy and writing courses challenged her thinking. A collaborative, interdisciplinary approach to secondary mathematics teacher education will help prospective teachers advance to more complex ways of thinking in *all* knowledge domains. Epistemological change is a slow process and should be treated as such.

Reform-minded Instructional Practices with Purpose

Many teacher educators already model reform-minded instructional practices in their classrooms. Our pre-service teachers are noticing, and research supports that students who experience for example, active, cooperative environments develop more epistemologically sophisticated beliefs than those in traditional classrooms (Hofer, 1999). It is important to note however that in the context of small group work, both Cindy and Ann discussed the importance of the instructor bringing closure to the discussion. Environments that are too multiplistic, from the learner's perspective, may hinder growth. We need to be mindful of the epistemological views held by the students in our classrooms and provide a balance of challenge and support to connect students from "where they are" to consider more complex ways of thinking (Baxter Magolda, 1992). As college educators, we need to not only teach the content of our courses, we must also fulfill our role as one in which we help our students develop more advanced ways of thinking.

This study will continue to follow the epistemological development of Cindy, Ann and Beth and 12 others throughout the rest of their college education into their beginning years of teaching. This documentation will provide a comprehensive understanding of experiences within mathematics teacher education that support and hinder changes in epistemic assumptions.

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HOW TEACHERS' PRACTICES MEDIATE CHARACTERISTICS OF REFORM TASKS IN RELATION TO STUDENT PARTICIPATION IN THE CLASSROOM DISCOURSE

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In this study, I consider how teachers' practices mediate the generativity and structure of reform tasks in providing opportunities for students to participate in the classroom discourse. I use a situated perspective to suggest that learning environments should be interactive in order for students to develop competency. I investigated the discourse practices of two seventh-grade teachers implementing the Variables and Patterns unit in the CMP curriculum. Three-fourths of the subtasks were rated as being at least moderately generative. The results suggest that teachers' practices especially mediated the moderately generative tasks. Implications are that implementations of curricula such as CMP are particularly sensitive to teachers' discourse practices.

In the current version of the mathematics reform characterized by the NCTM Standards (2000), much attention has been paid to promoting student participation in the classroom discourse. Forman, McCormick & Donato (1998) state that "new forms of instruction include more active participation of students in providing explanations, conducting arguments, and reflecting on and clarifying their thinking" (p.313-314). Student participation in the classroom discourse provides students opportunities to get feedback on their ideas and to evaluate their peers' ideas (NCTM), creating a more interactive learning environment.

The research on discourse in mathematics classrooms suggests that most classrooms, even most reform classrooms, are not very interactive. A few accounts exist in which students' ideas play a central role in the development of mathematical ideas (Hufferd-Ackles, Fuson, & Sherin, 2004; Leinhardt & Steele, 2005; O'Connor & Michaels, 1996): these accounts describe highly engaged students who make substantive mathematical claims which then become the focus of discussion by their peers. This study adds to those accounts by investigating how teachers' discourse practices mediate characteristics of reform tasks in the course of an implementation of a reform curriculum.

Reform curricula are designed to provide opportunities for students to collaboratively explore mathematical concepts. They do this by emphasizing connections between representations, setting problems in context, emphasizing problem solving, eliminating sets of repetitive tasks, and generally requiring that students attempt to make sense of the mathematics. The lack of repetitiveness in reform tasks and the more open-ended nature of the tasks provide opportunities for students to generate and offer insights into the mathematical concepts that are the focus for the instructional unit. The manner in which students use these opportunities to provide and react to claims during classroom discourse is mediated by the teacher's instructional practices, particularly their discourse practices. This paper describes two cases that illuminate how teachers' actions mediate the opportunities provided by the curriculum for students to participate in the classroom discourse.

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Generativity And Structure In Curricula

The classic tension in American education between promoting dialogue and establishing conventional knowledge (Palinscar, Brown, & Campione, 1993) can be found in the design of curricula. On the one hand, curricula facilitate variation in students' responses: this variation provides a foundation for dialogue as differences in students' responses become the basis for productive discussions. On the other hand, curricula establish common knowledge that serves to build understandings of conventional representations and concepts: this requires that curricula narrow possibilities for how students interpret and respond to tasks. The balance between these two influences is especially prominent in reform curricula, which tend to emphasize collaboration and communication.

I call these two competing design principles generativity (Stroup, Ares, & Hurford, in review) and structure. Generativity refers to the amount of variation of relevant responses a particular task affords. Structure refers to the way a task focuses students on particular uses of representations and formulations of concepts.

I adapt a task from Lampert (2001, p. 11) to illustrate generativity and structure. The task reads as follows. *A car is going 55 mph. Make a diagram to show where it will be after 15 minutes*. This task is highly generative because in Lampert's class there was no established diagram or procedure to approach the problem. Lampert's students could choose to focus on a strictly numeric approach or they could use invented representations to solve the problem. This led to a wide variety of responses which Lampert productively *made the focus of student discussion*. This example illustrates that generativity is applied to a task in relation to its place in the curriculum. Had the students solved five similar problems the day before and had established as a community a specific representation and set of procedures to approach this problem, the task would instead have been highly structured. The established representations and procedures would have served to structure the discourse toward specific ways of discussing rate problems, presumably guided by the teacher towards conventional formulations.

Generativity and structure generally have a reciprocal relationship. A task is highly generative in part because there are few established ways of speaking about or approaching a particular problem. This has the dual effect of increasing the variation in students' approaches but also makes it more difficult for students to explain their thinking in ways comprehensible to the classroom community. As students become more familiar with a type of problem or concept, they are able to draw on prior use of representations and ways of speaking about that type of problem. This has the dual effect of permitting them to explain their thinking in ways understood by the classroom community but also constrains the variation in which they would approach or discuss the problem. Discourse in this case is highly structured.

Theoretical Perspective

I use a situated cognition perspective which suggests that learning is mediated by the specific forms of activity in which a learner engages (Wertsch, 1991). In this view, learning and knowledge are not considered to primarily consist of universal qualities. This has specific implications for mathematics, the learning of which has been primarily characterized in individualistic psychological terms.

In order to build competency in a domain, learners need to experience a degree of interactivity. This entails that learners' ways of thinking impact the environment in which their practice is situated and vice versa. As learners react to and participate in the development of ideas within a learning environment, they begin to develop competency in a given domain (Gee, 2003; Lave & Wenger, 1991).

I am not suggesting, as others have (Greeno & MMAP, 1997), that there are prescribed practices which students must come to understand and master. Instead, I borrow from situated cognition scholars who suggest that thinking, learning, and acting are mediated by the specific context in which they take place (Gee, 2003; Wertsch, 1991).

In terms of mathematics classrooms, I interpret this to mean that, in order to develop competency as mathematical thinkers, students need to be involved in the development of ideas and practices in their classroom community. According to the situated view, students' opportunities to be involved in the classroom discourse are mediated by the curriculum and by teachers' practices. The nature of the tasks and the teachers' role in enacting tasks afford or constrain opportunities for students to generate and react to mathematical ideas.

Research questions

This study addressed the following research questions:

- How are generativity and structure manifested in the Connected Mathematics Project (CMP) (Lappan et al., 1998) curriculum?
- How are the generativity and structure of the CMP curriculum manifested in students' discourse?
- How do teachers' discourse practices mediate the design of a curriculum to impact student participation in the classroom discourse?

Methods

Participants

I selected two seventh-grade teachers based on the recommendation of the district-wide mathematics curriculum coordinator. These two teachers were among eight recommended by the district-wide curriculum coordinator as being effective teachers of the CMP curriculum. I initially contacted all eight teachers and ultimately selected these two based on factors of convenience and willingness to participate in the study. I wanted to recruit teachers recommended by the citywide coordinator to increase my chances of observing teachers who followed CMP in terms of task set up and sequencing.

Context

The two teachers, Lisa Robinson and Sven Johannsen, taught 7^{th} grade in two different schools in a mid-sized Midwestern city. The school district had mandated the use of CMP in all middle schools. Both schools had been using CMP for at least three years. This was Robinson's second time teaching the 7^{th} grade CMP units and it was Johannsen's first time. Both teachers had regular and intensive professional development in the use of CMP and both stated their firm support of the curriculum.

Task Analysis

I analyzed each task by describing: (1) similarities and differences from prior graphs in terms of what representations were provided, what students were required to do, and what quantities were involved in the problem; and (2) the choices available to students in working on and solving the problem. From this, I established a rating based on the two criteria of novelty and choice.

Table 1 – Rating descriptions

Rating	1	2	3
Novelty	The task has no new features. Prior tasks contain representations or concepts similar to current task.	There is at least one significant difference from prior tasks.	Task is different in more than one important way from prior tasks.
Choice	The task requires a highly prescribed output. There are no important decisions to be made in creating artifact or providing an explanation.	The task entails one decision in creating artifact or providing an explanation related to mathematical concept central to the instructional unit.	Creation of artifact involves several important choices related to mathematical concepts central to the instructional unit.

Table 2 – Examples of tasks for each category

Task description	Cho Rtg	Rationale	Nov Rtg	Rationale
Students are asked to find the mean number of bags.	1	There is an established procedure for finding mean.	1	Students are familiar with finding the mean.
Students are asked to find the intervals on a graph when the data rose and fell the greatest amounts.	2	Students may calculate and compare differences between values in the table or they may find the steepest positive and negative segments.	2	This is the first time students have been asked to find intervals which represent the greatest change. Previously they have had to only find differences.
Students must intepret data in two forms to decide which bike shop to rent from.	3	Students must compare data in two representations, which entails estimating data from the graph and determining when the data cross.	3	Students have not had to compare data in two representations nor have they had to consider two trends at once.

Classroom Data

I transcribed a total of 24 lessons between the two teachers. I selected for analysis only those parts of each lesson related to tasks described in the CMP curriculum. Within each CMP task, I divided the task into subtasks as presented in the textbook. Each sub-task was related to at least one episode and as many as eight. I defined an episode similarly to a Topically Related Set (Mehan, 1979), which consists of a series of exchanges around a topic, in this case primarily a subtask. In the event that an entire task was completed within a rapid series of IRE exchanges that I characterized as a single episode, I analyzed the task equivalently to a subtask. Overall, I identified 59 subtasks for analysis, 33 for Robinson and 26 for Johannsen.

Discourse Analysis

I focused my analysis of classroom discourse on student claims. I defined a claim as a mathematical explanation related to the task at hand. An explanation contains a complete thought and could be in a variety of forms, some of which are: a causal (if-then or because) statement that relates two ideas, steps used to arrive at an answer, or a description of characteristics of a representation or strategy. I focused on claims because these represented the clearest expressions of student thinking and consequently the most obvious sources of interactive discussions. In order to make a claim, I conjectured that a student had to make a choice or decision related to mathematics relevant to the task.

Subtask Analysis

For each of the 59 subtasks, I characterized their enactment as didactic or non-didactic. A didactic enactment consisted of a series of IRE exchanges with no opportunities for students to react to each other's claims. The vast majority of student turns in didactic enactments were in the form of short responses. For non-didactic enactments, the discourse took on a more conversational tone, with either the teacher or the students reacting to a student response. In the non-didactic enactments students were much more likely to provide a claim. For the analysis for this paper, I used the frequency which with students produced claims as a measure of opportunity for students to participate in the classroom discourse.

Results

Generativity and Structure in the CMP Curriculum

In this section, I present results from analyses on the subtasks which formed the basis of my discourse analysis. Because a number of the subtasks were enacted in both classrooms, the results I present below concern only 49 unique subtasks. In both categories, roughly two-thirds of the tasks were rated a 2. This is not surprising considering that reform curricula emphasize inquiry and de-emphasize repetitive tasks. The majority of subtasks provided some latitude in terms of how students could approach or respond to a task. Similarly, a majority of tasks had a novel aspect, whether, for example, it was considering new quantities or familiar quantities in a new representation.

Table 3 - Ratings for subtasks in both categories

Category	Rating of 1	Rating of 2	Rating of 3
Choice	14 (29%)	30 (61%)	5(10%)
Novelty	11 (22%)	35 (71%)	3(6%)

Teachers' Mediation of Task Characteristics

There were noticeable differences in the discourse patterns in the two classrooms. In general, Miss Robinson was interested in fostering discussion and often attempted to open up the discourse to student contributions. Two-thirds of the subtask enactments in her class were rated as non-didactic. By contrast, over two-thirds of the subtask enactments in Johannsen's class were rated as didactic. This ratio was particularly evident in tasks rated as 2's in either category. By contrast, the majority of subtasks rated as 1's were coded an being didactically enacted by both teachers while the reverse held true for the few tasks rated as 3's.

In general, Johannsen controlled the flow of the discussion and often provided rapid evaluations of students' explanations. His reformulation of students' responses was much more likely to close down discussions and often served as implicit evaluations. Robinson, on the other hand, was much more likely to ask questions of students after claims and to seek students' reactions to claims. As a result of Robinson's emphasis on encouraging student participation, her students were more likely to spontaneously provide explanations and reactions without having to be called upon.

A major difference across the two classrooms related to the correlation between the overall sub-task rating (the average of the choice and novelty ratings) and the number of claims made by students. In Johannsen's class, there was little difference in the correlations between the group of didactic and the group of non-didactic tasks. On the other hand, there was a distinct interaction in Robinson's class, as illustrated in the table below.

Table 4 – Correlations of claims with task rating

	Robinson	Johannsen
Didactic enactments	-0.56 (11)	0.04* (17)
Non-didactic enactments	0.62 (22)	0.11 (8)

The numbers in parentheses represent the number of subtasks in each category.
* Correlation with an outlying subtask removed.

The correlations indicate that in the task enactments rated as non-didactic in Robinson's class, the number of claims increases with an increase in the average of the choice and novelty ratings. The reverse holds true for tasks rated as non-didactic. Below, I interpret these findings as well as the differences between Johannsen and Robinson.

The discourse patterns in Johannsen's class were remarkably consistent across tasks. Johannsen's implementation of these tasks was fast-paced and geared towards completion: Johannsen expressed concern about the number of units he was expected to cover. The effect of this was to even out differences across subtask enactments. In addition, Johannsen often called on students for responses immediately after posing a question. This effect of this was to provide a steady stream of claims across tasks but this practice also diminished students' spontaneity.

Robinson, on the other hand, focused on student participation in discussions. She encouraged students to provide explanations and to challenge each other's explanations. The result of her focus on discussions was greater variation in discourse patterns in her classroom and a more frequent occurrence of mathematical discussions. The patterns evidenced in the correlations listed in table 4 reflect both the advantages and disadvantages of more generative tasks in relation to this discourse pattern. In cases where Robinson encouraged or allowed discussions to develop, the generativity of the task afforded greater participation. This participation took the form of students presenting various solutions or students disagreeing with a peer's explanation. When students disagreed, the novelty and openness of the task presented opportunities for multiple interpretations or for misinterpretations. In tasks rated as didactical in Robinson's class, it was apparent that she was seeking a particular answer or interpretation. In these cases, the ambiguous nature of more generative tasks restrained student participation because the answer was not as obvious as it would have been in a level 1 problem.

Discussion and Implications

The data in my study illustrate how teachers' discourse practices afford or constrain the opportunities provided by the tasks in the Variables and Patterns unit of the CMP curriculum. Not surprisingly, didactical discourse patterns constrained the generative aspects of tasks while more discursive practices enhanced the generative aspects of tasks. This in part argues for designing generative tasks and in part argues for more discursive practices if we wish to facilitate interactive classroom environments. The most substantive discussions involved level 3 tasks. The generativity of these tasks afforded greater variation in students' responses which served to spark claims and reactions to claims.

The results suggest sensitivity in terms of the impact of teachers' practices on how typical reform tasks get enacted. CMP contains a preponderance of moderately generative subtasks and there was a dramatic difference across the two classes in terms of how moderately generative subtasks were enacted. With increasing pressure to cover content, there is a strong temptation to teach in a didactical manner. This likely will constrain the interactive potential provided by reform tasks' generative characteristics.

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USING MULTIPLE-MISSING-VALUES PROBLEMS TO PROMOTE THE DEVELOPMENT OF MIDDLE-SCHOOL STUDENTS' PROPORTIONAL REASONING

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For this study, sixth and seventh graders worked two missing-value proportion problems that involve a scale factor of 2.5. The first problem required the students to calculate the value for only one unknown amount of an ingredient in a recipe, but the second problem required them to calculate the value for multiple ingredients. Overall, building up and between strategies were most common for both problems. The number of students who used the crossmultiplication strategy declined from the single-missing-value problem to the multiple-missing-values problem, and the number of students who used an incorrect additive strategy increased. The success rate declined sharply from the single-missing-value problem to the multiple-missing-values problem, which suggests that students who are successful at solving single-missing-value problems may lack an understanding of the underlying multiplicative relationship.

Introduction

A common concern of researchers (for example, Kaput & West, 1994; Lo & Watanabe, 1997) who focus on middle-school students' understanding of ratio and proportion is that students learn algorithms, particularly the crossmultiplication algorithm, before they have a sufficient opportunity to construct a conceptual foundation for solving missing-value problems. Ben-Chaim et al. (1998) found that students were more successful at solving missing-value problems than other types of proportion problems, such as numerical comparison problems, for which the students could not rely on an algorithm.

As part of a previous qualitative study (Clark, 2003), I gave students a cookie recipe that required four eggs and asked them in an interview to determine the amounts of the ingredients needed to bake a larger batch of cookies with ten eggs instead of four. Only one of the eight students approached the problem by determining the scale factor of 2.5 and multiplying the amounts in the original recipe by 2.5 to compute the amounts for the new recipe. None of the three students who used the crossmultiplication algorithm to solve a previous problem with a single missing value used the algorithm to try to solve the recipe problem with multiple missing values. One of these three students, Sheila— who was entering the eighth grade and was an excellent math student—struggled for 19 minutes on the multiple-missing-values problem and, after considering how to double and triple the original recipe, was able to formulate the following strategy: "If we use 10 eggs, then it's 4 times 2.5 equals 10. So that's the 10, and then we need to do everything else by 2.5."

From this experience, I made the conjecture that multiple-missing-values problems show promise in encouraging students to think deeply about the multiplicative structure and to develop meaningful strategies because these problems seem unfamiliar to them and because they do not apply the crossmultiplication algorithm. The purpose of this study is to investigate on a large scale the differences in success rates and strategies between a problem with a single missing

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value and a problem with multiple missing values and to use that information to set the agenda for follow-up qualitative studies.

Conceptual Framework

In general, the framework for the study is a constructivist perspective with the assumption that middle-school students can develop the conceptual foundation for solving ratio and proportion problems if given time and the exposure to a variety of situations (Lachance & Confrey, 2002). Specifically, the following were assumptions that guided the research questions and the design of the study:

- that a reliance on crossmultiplication to solve missing-value problems in the early stages of the development of proportional reasoning can limit the student's ability to establish a conceptual understanding of ratio relationships (Lo & Watanabe, 1997)
- that students should be encouraged to move from *building up*, or repeated addition, to more sophisticated forms of proportional reasoning based on *between* and *within* relationships (Karplus, Pulos, & Stage, 1983) as they progress through middle school and prepare for algebra courses
- that developing proportional reasoning—the "cornerstone of all that is to follow" (Lesh, Post, & Behr, 1988)—is critical to students' success in algebra, geometry, and beyond as they learn about topics such as linear functions, similar triangles, and trigonometry.

According to Karplus, Pulos, and Stage's (1983) classification system for numeric approaches to missing-values problems, a *between* strategy involves setting up two ratios, determining the multiplicative relationship between the ratios for the numbers of the units given in both ratios, and applying that multiplier to the remaining given number to determine the missing value in the other ratio. For example, if I gave a student the rate of riding a bike 5 miles in 20 minutes and asked how far would the cyclist travel in 60 minutes, a student using a between strategy would compare the 20 minutes and the 60 minutes, determine the multiplier of 3, and apply it to the 5 miles to get an answer of 15 miles. However, a *within* strategy involves setting up two ratios, determining the multiplicative relationship within the ratio for which both numbers are given, and applying that multiplier to the given number in the other ratio to determine the missing value. In the bicycle example, a student who recognized that the number of minutes in the given rate is four times the number of miles and then divided 60 by 4 would be using a within strategy.

Participants

The participants in this study were 252 middle-school students—124 sixth graders and 128 seventh graders—at a school for educational research in a medium-sized city (population between 100,000 and 250,000) in the southeastern United States. The admissions policy at the school ensures that the student body's demographic characteristics match those of the general community. Students from urban, suburban, and rural areas in six counties attend the school.

All the students worked two problems on paper, a single-missing-value problem the first week and a multiple-missing-values problem the second week, and were asked to explain in writing how they solved the problems. Students were randomly assigned to one of three groups. All the students worked the same initial problem, but the version of the second problem was different for each of the three groups. There was no instructional intervention, and none of the students were studying topics related to these problems in their math class. The purpose of the week in between giving the students the single-missing-value problem and the multiple-missing-values problem was to reduce the likelihood of a student's strategy on the first problem influencing the student's strategy on the second problem.

Data Collection

Problems were given to the students individually over two days in April and May of 2004. On the first day all the students were given the same problem, the punch problem, on paper and asked to respond by showing their work and then writing about how they solved the problem. The punch problem is as follows:

> *Bob made a pitcher of fruit punch by using 4 glasses of water and a scoop of drink-mix powder. He liked the taste and decided to make another pitcher. For the second pitcher, he used 10 glasses of water. How many scoops of drink-mix powder should he use to make the punch taste the same as the first pitcher?*

This problem was given to the students to determine their strategy for a single-missing-value problem that involves scaling by a factor of 2.5.

The following week the students were asked to work another problem, the cookie problem; however, the ingredients varied depending on which of the three groups the student had been assigned to. Group A received the easiest question:

> *Sally has a recipe for cookies that she likes. The recipe calls for*
> - *4 eggs*
> - *1 cup of sugar*
> - *2 cups of flour.*
>
> *She is preparing for a party and notices that she has 10 eggs in the refrigerator. So she decides to make a big batch of these cookies using all 10 eggs. How much of the other ingredients should she use so that the cookies taste the way she likes them?*

This version of the cookie problem requires the student to compute the amount of an ingredient for two missing values. Students in Group B were given the same cookie problem with the added ingredient of 1/2 pound of butter in the original recipe, and students in Group C were given the problem with the added ingredient of 2/3 teaspoon of vanilla in the original recipe (but not the 1/2 pound of butter). The multiple-missing-values problem for Group A contained all integer amounts and two missing values; the problem for the other two groups contained one fractional amount and three missing values.

Coding Scheme

Each student's strategy on a problem was classified according to the written solution and the student's written explanation of how the student solved the problem. The categories used are *between*, *within*, *building up* (Tourniaire & Pulos, 1985), *crossmultiplication*, and *additive*. In the case of no response, a response that did not fit any type of reasoning commonly used for missing-value proportion problems, or a response with conflicting evidence about the student's reasoning, the response was coded as *insufficient evidence*. The category of *building up* includes both arithmetic and pictorial approaches. The simple scale factor of 2.5 for these problems made it difficult sometimes to distinguish between building up and a between strategy because it was not always clear from the student's written solution whether the solution was based on adding an amount to itself and then adding a half of it or multiplying by 2.5. An additive strategy refers only to an incorrect assumption that adding, or subtracting, the same number from all amounts will result in the correct amounts for the new recipe.

Summary Statistics

Overall, there was a remarkable similarity between the success rates of the sixth and seventh graders on the problems. The students were more successful on the punch problem, the problem with only one missing value, than on the cookie problem, which had multiple missing values. Table 1 shows the number of students who answered each problem correctly (Yes) and the number who did not (No). The numbers in parentheses for the results for Group B and Group C are the adjusted numbers after giving students credit for being correct on all the missing values except for the final one. Because those in Group B and Group C had to compute an additional missing value, this adjustment makes comparisons with those in Group A more fair.

Table 1: *Summary of Students' Success on the Missing-Value Problems*

Grade	Punch Problem		Group A's Cookie Problem		Group B's Cookie Problem		Group C's Cookie Problem	
	Yes	No	Yes	No	Yes	No	Yes	No
6th	102	22	19	22	9 (18)	33 (24)	6 (12)	35 (29)
7th	108	20	16	27	11 (15)	34 (30)	7 (12)	33 (28)
Total	210	42	35	49	20 (33)	67 (54)	13 (24)	68 (57)

The overall success rate on the punch problem was 83.3 percent. Of the students in Group A, 66 out of 84 (78.6 percent) answered the punch problem correctly but only 35 (41.7 percent) answered the cookie problem correctly. Even when success on the cookie problem is defined only by whether the student computed 2.5 cups of sugar for the new recipe, the number of those successful in Group A was only 49 (58.3 percent). In other words, if you ignore whether students successfully computed the second missing value for the cups of flour in the cookie problem, the success rate drops from 78.6 to 58.3 percent from the punch problem to the cookie problem for those in Group A. This suggests that the mere presence of a second missing value makes a simple problem more difficult.

Table 2 shows the strategies that students in each group used to solve the punch problem and the version of the cookie problem that they were given.

Table 2: *Summary of Students' Strategies on the Missing-Value Problems*

Strategy	Group A (N=84)		Group B (N=87)		Group C (N=81)	
	Punch	Cookie	Punch	Cookie	Punch	Cookie
Building Up	35	32	35	29	25	32
Between	21	20	21	15	29	14
Within	5	2	6	4	7	1
Cross multiplication	12	4	13	1	6	2
Additive	7	14	4	15	5	8
Insufficient Evidence	4	12	8	23	9	24

By far, the two most common strategies were building up and a between strategy, most likely because of the simple scale factor of 2.5 in both the problems. There appears to be no impact from increasing the number of missing values on encouraging students to switch from building up to a between strategy on a problem with a scale factor of 2.5. As expected from the earlier study (Clark, 2003), the number of those who used cross multiplication dropped sharply, from 31 to 7, from the single-missing-value to the multiple-missing-values problem; however, what was somewhat surprising is that only 31 of the 252 students (12.3 percent) used cross multiplication on the initial problem. The number of students who used an additive strategy jumped from 16 on the first problem to 37 on the second problem. Therefore, the use of multiple-missing-values problems could expose a weakness in students' multiplicative reasoning related to proportions. The number of students using a within strategy fell from 18 for the first problem to 7 for the second.

Analysis

When building up, students establish how much of one of the units goes with a given amount of the other unit and iterate until reaching the target amount. For the punch problem, Dustin, a sixth grader, demonstrated this by writing the following:

$$\begin{array}{r}
4 \text{ glasses} = 1 \text{ scoop} \\
4 \text{ glasses} = 1 \text{ scoop} \\
+\ 2 \text{ glasses} = \tfrac{1}{2} \text{ scoop} \\
\hline
10 \text{ glasses} = 2\tfrac{1}{2} \text{ scoop}
\end{array}$$

He explained the build-up of the fractional portion by writing, "2 more glasses = a 1/2 since it is half of 4." The build up of the number of glasses is 4 + 4 + 2, which corresponds to 1 + 1+ 1/2 for the number of scoops. In general, a student building up establishes a corresponding x amount of the first unit and a y amount of the second unit and builds up to the target of the first unit by adding

$$x + x + \ldots + x + bx$$

where x is added a times, an integer, and b is the fraction of the x amount needed as the final piece of the build-up. Because Dustin needed a bx of 2 and was using an x of 4, he knew that b should be 1/2. Once a and b are determined, the student can compute the missing value by computing

$$y + y + \ldots + y + by$$

where y is added the same number of times as x.

Building up was also a common strategy for solving the cookie problem, but one error was common. In going from a recipe of four eggs, one cup of sugar, and two cups of flour to a new recipe with ten eggs, Rami determined the following new amounts: 10 eggs, 2 1/2 cups of sugar, and 4 1/2 cups of flour. She was able to determine a, the number of integer times to build up, and also determine b correctly, which is 1/2 in this problem, but she used additive instead of multiplicative reasoning for the fractional portion. Her solution can be represented by

$$y + y + \ldots + y + b$$

as opposed to by as the final term. Instead of building up the y amount of two cups of flour two times and then an additional one half of that amount, she reached four cups of flour and added 1/2.

A student who uses between reasoning instead of building up divides the target by the given amount to determine a and b combined as a mixed fraction or an integer and decimal portion and uses that one number as the scale factor. But students who do not fully understand the multiplicative relationship often use a hybrid approach by reasoning multiplicatively about the integer portion and additively about the fractional portion. For example, Tonya scaled her cookie recipe from the original recipe with four eggs to the new recipe with ten eggs by multiplying by two and adding two more eggs. She went from one cup of sugar in the original recipe by multiplying by two and adding two to get four cups of sugar in the new recipe, and she went from two cups of flour to six cups by the same method. It seems that a problem with a large fractional portion relative to the integer portion might help students make the transition to reasoning multiplicatively. For example, if an original recipe involves a 100:1 ratio and a new recipe contains 250 units of the first ingredient, it seems that students would be more likely to experience disequilibrium and investigate further if they used the hybrid approach to compute 2(1) + 50, or 52 units of the second amount, especially if they were using manipulatives.

Our goal should be to help students make the transition from building up to being able to recognize and use the scale factor. Because students do not rely on the crossmultiplication algorithm for multiple-missing-values problems, these problems seem appropriate to use in trying to achieve this goal. The results of this study indicate that multiple-missing-values problems can be designed for a wide range of difficulty depending on the number of missing values and the amounts of the ingredients involved. The easiest problems are those with an integer scale factor and only two missing values, but even with a scale factor as simple as 2.5 the cookie problem was difficult for most of the students in this study. As the scale factor becomes bigger and includes a more difficult fractional portion, it should become clear to students why a between strategy is more efficient than building up.

Researcher's Reflection

When I met the sixth graders the first day I could tell that they were excited to have a visitor. Because of human-subjects requirements, they had been given permission slips for their parents to sign, and therefore their teacher had been reminding them for several days that I was coming and that they needed to bring in their signed permission slip. Because of this, the students seemed to be anticipating a fun activity and were disappointed when they realized that all I was doing that day was giving them the punch problem, which I knew would be easy for most of them. Many students wrote how easy the problem was for them. One student, Pete, wrote his opinion in a way that probably expressed the feelings of most of the students:

> *I thought it was too easy. It was worded easy. In 3rd grade I could have answered this problem. Make harder ones next time. We are in the sixth grade almost in seventh. That was a third grade problem. Give us a sixth grade problem.*

As planned, I returned the following week and gave the students the cookie problem, which I thought was a "sixth grade problem" that would be challenging but that the students should be able to solve. Like most of his classmates, Pete did not answer the cookie problem correctly. The overall success rates were 42 percent for Group A, 23 percent for Group B, and only 16 percent for Group C. I was surprised by how difficult the cookie problem was for these students, especially the seventh graders.

On the first day that I visited the seventh-grade classes, the students were being taught the difference between combinations and permutations—a topic that I probably didn't encounter until my second year as a math major in college. As I helped small groups with their problem solving, I noticed that most students were struggling with this topic, but because this lesson seemed so much more advanced than the problems I was giving them to solve, I was worried that my multiple-missing-value problems would be too simple for these students. However, I was surprised at how poorly the seventh graders performed, in some cases slightly worse than the sixth graders. One possible explanation for this is that the sixth graders had worked traditional missing-value problems as part of an instructional unit in their classes about a month before my visit. As I watched the seventh graders, I wondered whether learning procedures for computing combinations and permutations would be of any benefit to them in algebra. From my limited interaction with them, I concluded that the majority of these seventh graders needed more experience and problem-solving opportunities with ratios and proportions to help prepare them to study linear relationships, functions, and graphing in algebra. As I watched them struggle with combinations and permutations, the often-repeated criticism "a mile wide and an inch deep" of middle-school mathematics curriculum came to mind. I believe that giving students opportunities to solve multiple-missing-value problems would help them develop a deeper and more meaningful understanding of multiplicative relationships, which they will need for success in algebra and beyond.

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TEACHERS DOING MATHEMATICS AS EXPERIENTIAL THERAPY

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It is not uncommon for elementary teachers to hold narrow or negative views of mathematics. Changing such views requires, at least in part, novel and rich experiences with mathematics that cause teachers to see mathematics and mathematics teaching in a new light. We conceptualize the purpose of teachers doing mathematics not in terms of gaining content knowledge but as experiential therapy. In this paper we report on (a) the development and (b) the research of a Mathematics Course for elementary pre-service teachers.

It's not surprising to meet people who wear their view that "I'm not a math person" as a badge of honor. It is concerning that this is also the case among prospective mathematics teachers. Given this mathematics education predicament, it is perhaps not too far-fetched to suggest that there is a need for 'math therapy', which, we will argue, needs to involve new and different experiences with mathematics.

In this paper we report on (a) the development and (b) the research of a Mathematics Course for elementary pre-service teachers whose goal is to immerse pre-service teachers in working with mathematics problems, where they experience rich mathematical activities, attend to school mathematics content deeply and more often experience the pleasure of mathematical insight (Gadanidis 2004). We conceptualize the purpose of teachers doing mathematics not in terms of gaining content knowledge but as experiential therapy. Ball (2003) suggests that the "goal is not to produce teachers who know more mathematics."

Development of the Mathematics Course

Prior to 2001, the core of our mathematics education program for elementary pre-service teachers consisted of 8 two-hour workshops (approximately 28 pre-service teachers per class) and 9 hours of lectures (approximately 440 pre-service teachers in a large auditorium). In 2001 we replaced the large lectures with 9 online modules accompanied by a structured online discussion. It has been difficult to add more time to our mathematics program, given monetary and timetabling constraints. We especially wanted to add a component where a space was created for pre-service teachers to (re-)experience school mathematics. In 2003, we added an elective Mathematics Course to experiment with what such a course might be like. The elective Mathematics Course was an 18 hour elective, with nine 2-hour classes, and it was offered to 25 pre-service teachers.

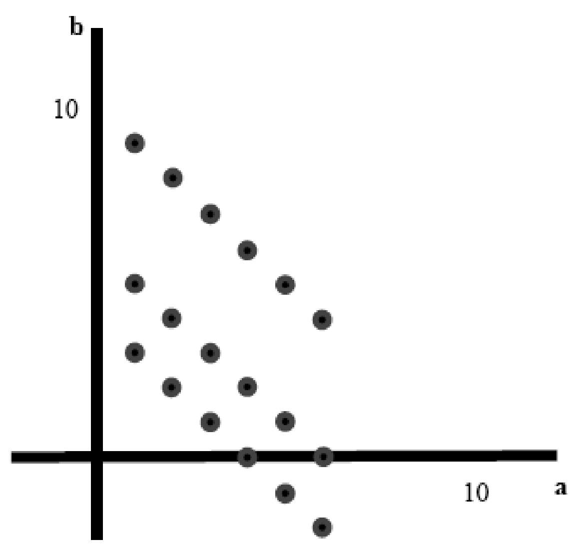

Figure 1. a + b = 10, a + b = 6 and a + b = 4

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Doing mathematics became the starting point for the course. Most elementary teachers have narrow views of what mathematics is and what it means to do mathematics (Fosnot & Dolk 2001; McGowen & Davis 2001). Fosnot & Dolk (2001, 159) suggest that "teachers need to see themselves as mathematicians," and towards this end we need to foster environments where they engage with mathematics and construct mathematical meaning. Mathematics experiences were designed to be interesting and challenging enough to capture their interest and imagination and to offer the potential for mathematical insight and surprise (Gadanidis, 2004). Classroom activities and assignments fostered reflection on and attentiveness to school mathematics subject matter it self. Two of the assignments (one at the beginning and one at the end of the course) asked students to describe their view of mathematics and to discuss an appropriate metaphor for their personal experience with mathematics. Also, in the last 5 minutes of each class, pre-service teachers took the time to write about what they learned and what they felt during the class. Their ideas were compiled into a single document, under the headings of 'learned' and 'felt' (anonymously), and this was distributed and discussed at the beginning of each class. The individual reflection and collective discussion opened up a space for attending to mathematics subject matter as well as to adequate ways of doing and warm experiences of mathematics.

A wide variety of mathematics problems and situations were explored in the Mathematics Course. What was common about the mathematical activities was that they had to be problem solving tasks that were non-routine and motivating to the teachers as well as had the potential to generate mathematical insight. For example, one of the problems in the first class explored the equation __ + __ = 10. Pre-service teachers rolled a die to get the first number and then calculated the second number. They wrote the pairs of numbers in table and in ordered pair form, and plotted the ordered pairs on a grid. We repeated this for __ + __ = 6 and __ + __ = 4. Some pre-service teachers expressed surprise that the ordered pairs lined up (see Figure 1). "I had the 'aha' feeling when I saw the diagonal line pattern on the graph. That was my favorite part." Pre-service teachers also noticed that the graph of __ + __ = 4 could be used as a visual proof of 6 + -2 = 4 and 5 + -1 = 4. That is, (6,-2) and (5,-1) line up with (4,0), (3,1), (2,2) and (1,3). They also explored equations whose graphs were not parallel to the ones above and whose graphs were not straight lines. Such mathematical connections appeared to be pleasing to pre-service teachers. "I loved the adding/graphing we did and how you should take problems and branch out … it really makes something in my mind click."

In 2004, we added 9 more hours of contact time with elementary pre-service teachers, in the form of 9 large group auditorium sessions, where they worked on small group problem activities. We selected 9 interesting problems to form the basis of each of the 9 Mathematics Sessions, respectively. We purchased enough materials to allow for 440 pre-service teachers to work in groups of four. In the last 5 minutes of each session, pre-service teachers completed and handed in a sheet outlining what they had learned and what they had felt during the session (this also served as a way of tracking attendance). These reflections were summarized and then shared and discussed at the beginning of the next session. A website was created to provide extensions and interactive explorations of problems. A new assessment component was added, where in the last workshop of the course, pre-service teachers would have thirty minutes to write an 'essay' on one of the problems dealt with in the Mathematics Sessions.

In 2005, we are eliminating the large auditorium sessions and replacing them with fully online mathematics activities and discussions. We're taking on this challenge in part because we want to see what is possible in terms of doing mathematics online. To help us in this endeavor, we're developing an online discussion tool, called Idea Construction Zone (ICZ), that allows for both (rich) text and graphical communication (diagrams can be created, posted, and edited by others). ICZ also allows for editable postings, where participants can co-author postings. We're also developing more interactive mathematical simulations, and adding video annotations which demonstrate activities, raise questions to explore, inject student commentary, offer expert teacher reflections, and so forth. Another reason for going fully online is the logistical difficulty of scheduling the large group sessions in an already full timetable.

In 2005, we are also offering incoming preservice teachers the option of taking a further mathematics course during the month of August. This course will focus on mathematics topics not covered in the 2005-2006 regular program. It is a fully online, quarter credit course, which appears on preservice teachers' transcript as a post-graduate certificate course. At the time of writing, more than enough preservice teachers have already registered for this course to make it a viable offering.

Research of the 2003 Mathematics Course

For the 2003 elective Mathematics Course, a multi-layered, structured content analysis was conducted of the reflections of 25 pre-service teacher (written at the end of classes 1-8) and of their two beliefs assignments. The content analysis adhered to the "stage model of qualitative content analysis" defined by Berg (2004, 286). A similar analysis will be conducted for the 2004 Mathematics Sessions, using data from the learned/felt reflections, the online essay writing collaboration, interactive explorations and discussions, the essay assessment, and from interviews of a small subset (20) of the pre-service teachers. The analysis of the data from the 2004 elective Mathematics Course revealed a number of themes. These themes are briefly discussed below.

Frustration

Many of the mathematics activities were novel experiences for all the pre-service teachers. Unlike typical mathematics teaching where the teacher seeks to *make learning easier* by organizing classroom experiences in small, bite-sized pieces, the Mathematics Course sought to create opportunities for pre-service teachers to *think harder*. However, thinking harder in a mathematics setting, especially for people with little experience doing so, can be a frustrating experience. Pre-service teachers expressed a level of frustration, especially with early course experiences, as reflected in the comments below.

> I'm still frustrated by my inability to see the conclusion or the point. I can't seem to push my thinking beyond the exercise to the solution, on my own.

As the course progressed, pre-service teachers' frustration seemed to dissipate.

> My feelings of frustration are gradually turning into curiosity as I begin to think about new ways of approaching math.

Attention and Insight

As pre-service teachers continued to engage with the mathematics problems in class, they began experiencing moments of insight. These were pleasurable moments that served as 'remuneration' for the effort and attention that they gave to the problems.

> I loved the adding/graphing we did and how you should take problems and branch out … it really makes something in my mind click.

> I feel so much better about this class. These math concepts are getting to be very exciting. I had a lot of moments where things just popped!

Collaboration

Although at times pre-service teachers were asked to think about a problem individually before attempting it as a group, the pervasive atmosphere of the Mathematics Course was one of collaboration. Being able to work with others helped reduce stress with doing mathematics and exposed pre-service teachers to the mathematical thinking of others.

> I have always feared math because I am not confident in my abilities. What helped me a lot was being able to work in a group. By discussing math problems in a group and then hearing from the whole class, I was able to look at problems in different ways.
>
> I felt really comfortable working in my group. It is easy to experiment with different things with other people vs working alone, and more ideas seem to come out.

Time

Pre-service teachers were given ample class time to work on problems in a low stress atmosphere. In fact, most of the problem solving was done in class, and take-home assignments mostly consisted of course readings rather than mathematics problems.

> It has been through being in a math class where I knew my mathematical abilities were not "on trial" that I feel I have learned the most about math. Knowing that I would not be penalized for not knowing the correct answer allowed me to take risks and experiment with math problems.
>
> I liked that we were asked what other methods can we come up with to test right-handedness/left etc. Then we were given <u>time</u> in class to go through and actually try ideas – it's been so long since I've had an experience like that in school. It was relaxing.

The Complexity of Mathematics

Most elementary mathematics teachers view mathematics as a subject of procedures for getting correct answers. As the Math Therapy course progressed, pre-service teachers started expressing more elaborate views of mathematics.

> Mathematics is a very complex subject that can be looked at from various angles and approaches.
>
> Math problems can be multi-faceted. One problem can be linked to another to another … There are different ways to approach a math problem.
>
> <u>Infinity</u> can be negative? What? I have never heard of a discussion of infinity or anything else "mysterious" in a math class (except this one). I guess an approach like this one (this class or Kamii etc) is just in the beginning stages of what math in school could be like.

Mathematics as a Human Activity

Pre-service teachers also associated the personal methods they 'discovered' for solving problems, as well as the process of doing such discovery, with natural human activity. They noted that their personal and informal ways of doing mathematics felt natural.

> I remember coming up with my own way of adding and subtracting (like the kids in the [Kamii] video). I tried to explain to my grade 4 teacher how I arrived at my answers and her only reply was "That's weird". I've never forgotten that. Class today showed me I wasn't weird at all, in fact I was progressing normally.
>
> Things have recently all come together for me regarding math. I see connections in and to math everywhere. Math has started to consume my thoughts.

Teaching Mathematics

At the beginning of the Math Therapy course pre-service teachers expressed apprehensions about mathematics and a lack of desire to teach it. As the course progressed, pre-service teachers started expressing an excitement for teaching mathematics.

> It was so interesting to see how attitudes towards teaching mathematics and mathematics in general changed from the first class to the end of the term. Initially, there were numerous concerns about one's ability to teach math and fears about adequacy. However, many of these concerns often changed to feelings of excitement. For me, personally, I found great pleasure in seeing how far everyone in this class has come, especially myself, regarding how comfortable we feel in our ability to teach math to students in an interesting an effective way. I would say that the most pleasurable experience from this course would be that I faced my 'math demons' and actually grew to enjoy a subject I thought would be my nemesis forever.
>
> In every area I study I seek to make personal meaning. I never thought I could do this with Math. Math has become one of my favorite things to teach.
>
> I feel much more confident in entering a math classroom now. I know that I can relax, face each topic with interest, and explore along with my students the intricacies and the beauty found there. For me, the beauty of math lies in being released from my phobia into this world of possibilities.

Beliefs and Practice

Although pre-service teachers' expressed perceptions of mathematics and of teaching and learning mathematics changed significantly as the course progressed, these new perceptions were also accompanied with apprehensions that they would be able to put them into practice.

> This class has completely shattered my understanding of math and how to teach math. It makes me feel that teaching math is going to be difficult – or at least more challenging than I previously thought. There are so many ideas – I feel overwhelmed.
>
> While I did really enjoy this course, I do feel more overwhelmed about teaching math than I did before the course. In many ways, I feel that the standard of what a good math teacher is has been raised.
>
> Pushing my creativity beyond simple math concepts is still foreign to me. I still feel like the only way I'll be able to teach math is the boring worksheet method with the teacher's answer book close by. However, I feel like I would be sacrificing another generation of students.

Research of the 2004 Mathematics Course

We have now collected and are in the process of analyzing data from the 2004 Mathematics Course (math essays, learned/felt reflections, and online discussions of 440 preservice teachers). We look forward to sharing our findings at the 2005 PMENA meeting. We will also be able to report on the new summer Mathematics Course for preservice teachers, as well as the first few weeks of the 2005 regular program Mathematics Course.

Math Therapy

Pre-service teachers in the elective Mathematics Course overwhelmingly expressed that experiencing the course helped change their view of mathematics and of teaching and learning mathematics. However, a single course experience cannot create comprehensive or permanent changes in teachers' perceptions of mathematics and mathematics teaching. Neither can we assume that such an experience will significantly affect teachers' classroom practice; teaching is also greatly affected by accepted teaching practices in the wider school community (Buzeika, 1999; Ensor, 1998) and by conflicting priorities (Skott, 1999). However, such insightful and pleasurable experiences of mathematics are important starting points for change in teachers' perceptions and classroom practice.

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EXPLORING ELEMENTARY TEACHERS' USE OF A NEW MATHEMATICS CURRICULUM

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This study focuses on elementary teachers' first-time use of a unit from one of the National Science Foundation (NSF) funded mathematics curricula: Investigations in Number, Data, and Space (Investigations). *The teachers were interviewed prior to beginning the unit, were observed teaching two to six times, and were interviewed at the completion of the unit. Specific emphasis was placed on the ways the teachers utilized the new curriculum materials to prepare for and implement instruction, and the exploration of factors that seemed to impact this utilization.*

Theoretical Framework

In the 1990's, NSF-funded curriculum materials were developed partly to support teachers to better understand the mathematics they were teaching and to utilize teaching approaches that support student understanding. These materials provide students with the opportunity to explore mathematics (Robinson, Robinson, & Maceli, 2000), and teachers with detailed information such as suggested questions and samples of student work. However, while adopting such materials helps support teacher change, the ways in which and the extent to which teaching practices change vary considerably. Remillard and Bryans (2004) suggest that teachers' orientations toward curriculum materials as well as their philosophy of mathematics teaching influence the ways in which they use the materials. According to Ball and Feiman-Nemser (1988), "Using [curriculum] materials thoughtfully requires an understanding of the meaning and possible consequences of the way they are designed and what they include" (p. 420). The way teachers interact with the curriculum materials also influences how they make instructional decisions. Examining teachers' use of a traditional textbook, Remillard (2000) argues that how teachers read curriculum materials impacts their teaching.

For this study, we conceptualized curriculum use as the ways in which teachers participate with curriculum materials to enact lessons. Thus we looked for evidence of teachers' use of the curriculum in their teaching. In particular, we focused on their use of questioning to elicit student thinking, and then the use of that thinking to propel the lesson. We did not assume that there was only one correct way to enact these materials; however, we did view the materials as embodying a particular philosophy for how learning occurs and as providing support for how to engage students in mathematics through the use of designated activities.

Methodology

Setting

This study took place in an urban school district, in a building that had recently become a magnet school with exploration as its theme. To meet this theme, teachers had begun writing

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their own curricula in some areas, and were considering the use of *Investigations* for mathematics. Thus, the six members of the committee decided to each pilot a unit from the *Investigations* curriculum and participate in the study of that pilot. Two data units were chosen for the pilot as they required little up-front preparation in terms of manipulatives and other instructional materials, and they provided a context conducive for exploration. The grades 1, 2, and 3 teachers used the unit *How Many Pockets? How Many Teeth?* (Economopoulos & Wright, 1998) and the grades 4, 5, and 6 teachers used *The Shape of the Data (*Russell, Corwin, Rubin, & Akers, 1998). This was done rather than having the teachers use different units from each grade level to allow collaboration and conversation among the teachers as they implemented the units. There was one day of professional development before the pilot, during which teachers worked on a data project in order to experience the various steps in the data collection and analysis process, analyzed a videotape of elementary students analyzing data in order to think about the teacher's role in encouraging students to think about data, and were given a unit overview.

Data

All of the teachers were interviewed before the data unit began. The interview was divided into four sections: 1) background and approach to teaching; 2) their magnet school approach; 3) what it means to understand mathematics, in general, and data, in particular; and 4) describing and analyzing data sets as adults and then analyzing elementary students' writing about the same data sets. Following the interview, one teacher withdrew from the study. The remaining five were observed 2-6 times during the unit and interviewed at the conclusion of the unit. Specific lessons were chosen for observations based on their potential to provide a forum for rich whole-group discussion. The intent of the observations was to capture teacher-student interactions during whole group discussions to analyze the extent to which student thinking and justification of their thinking were encouraged. The final interview revisited some of the ideas from the original interview, along with asking teachers how their approach to teaching and understanding of student thinking was impacted, and the ways in which the curriculum materials themselves were utilized to support their implementation. All interviews were audiotaped and transcribed. All classroom observations were either audiotaped or videotaped and at least two researchers took extensive field notes during whole-group discussions.

Analysis

The research team members individually analyzed the interview transcriptions and field notes and then met to discuss and refine the analyses. For the pre-interview, each member wrote about general reactions to the entire interview and then identified key findings from each of the four sections. The team then discussed any discrepancies in findings in order to reach consensus. For the observation field notes, individual analyses focused on identifying patterns in practice and characterizations of the ways in which student thinking was elicited. This was followed by an analysis of support provided in the text materials relative to the specific observed lessons. The team then discussed these patterns as well as initial thoughts about how the teachers' practice was tied to findings from the pre-interview. Finally, the post-interviews were analyzed for the extent to which the ideas expressed by the teachers changed from the pre-interview and whether they resonated with what was observed in their classrooms.

Results and Discussion

The main characteristic that the six participating teachers shared was that they were all members of the school's mathematics curriculum committee. As such, they all shared a commitment to thoroughly pilot these materials in order to make a recommendation to their peers about adopting them. Although this shared purpose provided a common context for all of the teachers, they were by no means a homogeneous group. Their teaching experience ranged from 2 to 20 years, and experience at this particular school ranged from 1 to 11 years. The three upper elementary teachers considered themselves "math people," while the lower elementary teachers did not. Two case studies are reported here, one lower and one upper elementary teacher, in order to highlight the impact of different characteristics on teachers' use of curriculum materials.

The Case of Carly

Carly (all names are pseudonyms) has been a teacher for 20 years, the last 12 years at third grade and 11 years in this particular elementary school. She does not consider herself a mathematics person and does not feel particularly strong in mathematics. Carly is committed to teaching mathematics with understanding, but is having a difficult time enacting this in the classroom. In the pre-interview, she stated:

> *I don't know if I'm teaching it to the kids so that…they're really getting a good understanding of math … I want to make it as concrete to those kids as I can so that they have a good understanding of it. 'Cause sometimes I feel like I'm teaching them the processes, but I don't know if they're really understanding what I'm teaching.*

Carly saw curriculum materials as being critical to supporting her efforts to effectively get at understanding. Finally, Carly indicated a strong belief that mistakes are part of the learning process and that students should expect to make mistakes and feel okay about it.

With respect to teaching data at the third grade, her comments clearly indicate a focus on the processes of creating and reading particular types of representations (e.g., pictographs, bar graphs, tables, and circle graphs). In considering other aspects of data analysis, like interpretation, prediction, and comparison, Carly had little experience and thus wasn't sure what to reasonably expect from her third grade students. In both the pre- and post-interviews, Carly expressed concerns about her own understanding of the mathematics of data. She had difficulty going beyond descriptions of single data points to searching for general trends in data or even using broader descriptions like measures of center. Carly was, however, able to recognize when analyzing student work that their responses dealt mainly with less substantive descriptions and did not involve interpretations of why the data may have looked the way it did.

Enacted Curriculum. In terms of her teaching, Carly made great strides in eliciting thinking from her students. Carly's whole group discussions typically began with relatively general open-ended questions, such as *What do you notice about the data? What does this information tell you?* Depending on the students' response to this question, Carly continued the conversation by taking one of three tacks: probing this idea further (e.g., *What do you mean?)*, restating the general question (e.g., *What else do you notice about the data?*), or asking a more specific question (e.g., *What is the lowest/highest age? How many people ate one pizza?*). Another characteristic of Carly's whole group conversations was her explicit efforts to let students know that she was asking them to expand on their thinking so that "we can all understand it."

Carly's explicit tone-setting, her open-ended questions, and her probing of students' responses all indicated a desire to elicit student thinking. In this unit on data analysis, the kind of thinking Carly worked to elicit focused on: reading and understanding a representation, describing and comparing representations, making predictions about data, and interpreting data. In general, Carly's efforts to elicit student thinking were more successful when she focused on goals other than simply reading a representation. The following conversation, which took place on the tenth day of the unit, is representative of her efforts to elicit thinking.

S1: How many teeth are there in the mouth?

T: How many teeth are there? 32.

S1: Somebody said they had lost 36 teeth.

T: This one is third grade, somebody in third grade said they lost 36 teeth. Do you think that could be possible?

Ss: Yes/No.

T: Wait a minute. Hold on. We have to- One at a time. Why do you think that might be? I am not saying that you are wrong, I am just saying explain what you think…

S2: …..teeth pulled…. [inaudible].

T: So, they could've lost their baby teeth and then they could've lost some adult teeth? [*Likely repeating student's comment*] Okay. That's possible. That's a possibility.

S2: Maybe…. [inaudible]

T: Maybe. Because you can loose adult teeth, only thing is that you can't get them ack. [*Likely repeating student's comment*] …

S3: You've wisdom tooth, too. That's how … people get their wisdom teeth pulled.

S4: But, you don't get wisdom teeth when you're like in your twenties.

T: Could somebody get wisdom teeth earlier?

S4: ….

T: It's probably going to be unusual. [*Likely repeating student's comment*]

Until that excerpt, most of the questions Carly had posed were about the mode and the range of the data the students had represented. After one of the students spontaneously asked Carly *How many teeth are there in the mouth?* and followed this with sharing an unusual aspect of the data, Carly pursued this with the question of *Do you think that could be possible?* It appeared that this question encouraged students to both offer their own thinking on how/if someone could have lost 36 teeth, and to consider, and potentially build off of, other students' thinking. After that lesson, Carly informally shared her thoughts on her facilitation of whole-group discussions with the researchers. In general, Carly felt that she was not asking questions that led to rich discussions, that she was focusing too much on descriptions and did not know what else she needed to ask about representations. In particular, she was shocked at the students' rationales for how one might have lost 36 teeth. According to Carly, her intention in focusing the class on this data value was to have students consider that this data was most likely "made up" by a student. It is significant that despite her intention, Carly encouraged the students' possible interpretations of this unusual data value.

<u>The Support of Curriculum Materials.</u> In her pre-interview, Carly indicated that the curriculum materials were necessary to inform her practice; in her post-interview, she noted that the lessons were quite helpful in providing her with ideas about how to specifically structure certain activities, what questions to ask, and what mathematical ideas to focus on. Despite this, Carly still expressed difficulties with both questioning and knowing what "understanding" should look like, particularly in the area of data analysis. In analyzing the videotapes, it appeared that seemingly simple alterations in wording of questions altered the focus of the discussion just enough to cause her more difficulties. For example, in one lesson the suggested prompt was *What have you learned about the group of students you collected data from?*, whereas the prompt Carly used was *Tell me about the data*. While both questions are equally open-ended, they have different foci: the question Carly used seemed to lead students to focus

on describing decontextualized data, while the suggested prompt may have helped students both describe and interpret the data in the context in which it was presented.

The Case of Danielle

The second teacher, Danielle, was also an experienced teacher. She has taught 16 years, all of them in fifth grade and the last four years at this elementary school. Danielle loved mathematics and was confident in both her own mathematical ability and her teaching of mathematics. It was important to Danielle that her students "own the math."

> *What I mean by that is: Do they understand why what they're doing works, not just that it works, but why it works, and can they apply it to something else?*

Danielle stated that her approach to teaching is inquiry and believed the *Investigations* curriculum to be a good match with her beliefs about teaching and the school's focus on exploration and inquiry. In the pre-interview, Danielle demonstrated solid understanding of data analysis. Danielle believed that student inquiry and discussion were critical to learning with understanding, but recognized the difficulty of questioning.

<u>Enacted Curriculum.</u> While Danielle also worked to ask more open-ended questions, her teaching contrasted with Carly's teaching in several ways. First, one immediately noticed the enthusiastic and fast pace of Danielle's teaching. Second, Danielle seemed to have a clear command of the important mathematical ideas, and a good sense of the path students needed to travel to get at those ideas. Despite this confidence, Danielle still struggled to ask questions that elicited student thinking. This issue is demonstrated in the excerpt described below from day 3 of the unit, where Danielle's class is discussing the typical number of brothers and sisters.

T: The question bottom line is "How many brothers and sisters does a typical student in [my] room this year have?" Listen again to the question. "How many brothers and sisters does a typical student in [my] class have?" Now, help me out with that "typical" word again. We introduced the word yesterday. Typical. Let me give you an example of a statistic with typical. "A typical 5th grader should be able to run a mile in less than 10 minutes." Listen to my statement again. "A typical 5th grader should be able to run a mile in less than 10 minutes." Now, the question is, what do I mean when I say that? What is typical? What is typical? Guys, one more time. My statement that I said was, to give you an example of typical, is "A typical 5th grader should be able to run a mile in less than 10 minutes." Okay. What do we mean by typical? Matt, what's typical?

S: Average.

T: Average. Explain that. Like, in this room, for example, how many people would be able to do that, do you think? Do you have a number in your head or amount of people? If a typical fifth grader should be able to do it, in our room, are most of the people going to be able to? Some of them? A couple? What do you think if it's typical?

…

T: So when I ask you the question about this data—What is the typical number of brothers and sisters of a student in [my] room?—could you answer that by looking at our data? What kind of number would you say? What kind of a range—would it be a range? Would it be a number?

S: A range.

T: Like what? What kind of a range?

S: Like, I don't know that—like a range for, from 1 to 10 or something like that?

T: Okay, it does range from 0-10, doesn't it? From 0 brothers and sisters up to 10, and we are pretty spread out. Guys, notice some other things about the patterns; it's hard. We probably have to talk about some more specifics. Some of these specifics you already mentioned. How many 3's there were; that there were no 9's.

Danielle's desire to help her students "own" their mathematical understanding was revealed when she asked the open-ended questions, such as *How many brothers and sisters does a typical student in [my] room this year have?* This was a question created by Danielle and was similar to those previously suggested about typical family size. However, the real-time dilemma of creating additional questions around the topic of typical surfaced through the asking of multiple follow-up questions, some of which veered away from a focus on the typical number of brothers and sisters in Danielle's classroom. For example, the decontextualized question, *What's typical?* can be interpreted to be a general question. Additionally, questions such as *Do you have a number in your head or amount of people?* and talk of "specifics" lean toward particular features of the data. Therefore, her questions fluctuated between the open-ended and the specific; that is, between questions about the general shape of the data and questions on specific features. This fluctuation may be characteristic of a tension for first-time users of *Investigations* between questioning to elicit student thinking and questioning to get a particular, known answer.

As shown in the excerpt, one noticeable characteristic of Danielle's whole group facilitation was her tendency to ask a series of questions. Each of these questions could have resulted in discussion; however, due to insufficient wait time, these potential discussions did not take place. Danielle recognized the importance of questioning and expressed that she struggled with asking appropriate questions. In her post-unit interview, Danielle talked about the difficulty of asking questions without leading students' thinking, saying:

> *The key is, when you ask the question, not to lead too much 'cause there's a fine line there. You can ask a question you think is an open-ended, but you're leading it so I have to be careful with that one. I noticed that in my thing and as I'm thinking about what I'd ask [a student], I think um, don't be too leading. Don't assume he already knows. Got to stay neutral which is really hard for me to do.*

<u>The Support of Curriculum Materials</u>. Regarding her use of the curriculum, Danielle tended to use the curriculum materials as a guide. She stated in the post-unit interview that to prepare to teach, she would condense the plans to what she thought she should remember. Observations revealed that Danielle followed the lessons as intended, including relevant tasks while sometimes choosing to revise tasks. Thus, her reading of the lesson seemed to focus on synthesizing the goals, and the general ideas of the activities. Her confidence in her knowledge of mathematics and the philosophy of the program may have contributed to her use of the curriculum materials as a guide, not seeming to recognize the support embedded in the curriculum for creating the kind of environment she desired.

Upon considering the curriculum materials from the perspective of a first-time user, it was not surprising that teachers might face challenges when leading discussions of difficult concepts like typical. Although the unit includes information for the teacher about the concept of typical, some general questions and sample student responses, it seemed this was insufficient to support Danielle's efforts to facilitate a classroom discussion around this topic. This was compounded by her expansion of some activities beyond their original intent. For example, there were two connected lessons that Danielle almost completely covered in one extended math class. On the following day (the day from which the excerpt above came), she took the remaining activity (intended as an introduction to a homework assignment), and chose to make an entire lesson out of it. In doing so, the original guiding questions (*How many brothers and sisters do you have? Do you think there will be more or fewer brothers and sisters than there were people in our families? Why?*) were clearly insufficient for such a major adjustment.

Discussion

For both teachers there was some indication that they needed more support in understanding differences between the focus of teaching data in their previous, more skill-based curriculum and the focus of the *Investigations* unit. In Carly's case, she was used to focusing heavily on particular descriptors and their terminology (e.g., mode, range), and the creation and naming of particular types of representations (e.g., pictograph, bar graph). Consider an *Investigations* activity she implemented in which students were comparing data on the same topic (*How many teeth have you lost?*), but with different groups of students (Sally's grade 4 class, Bob's grade 4 class). The written lesson suggests that students make some easy comparisons by looking at the range, "most common number of teeth," and unusual data points. Considering her past focus on terminology, it is not surprising that Carly added a focus on the use of the term "mode" and seemed to define understanding of that term as being able to say that mode means the "most common number." This resulted in shifting the focus to terminology, rather than using certain descriptors in the service of making comparisons between data sets. In Danielle's case, the concept of typical was a new one. Unlike the statistical measures she was familiar with (e.g., mode, median, mean), typical was introduced in *Investigations* as a way to get students to begin to consider how to capture the "essence of the data" or what is "usual." While one might consider the mode as representing a typical value in one data set, a range of values might be useful in another case. This flexibility would allow students to focus on describing a particular data set based on its characteristics, rather than simply "listing" statistics like mode and mean.

Conclusions

Given the circumstances of this pilot with little preparation time, both Carly and Danielle did a remarkable job using this new curriculum, and embraced areas of data analysis that had not been a focus of their prior instruction. Not only did these teachers express a desire and intent to continue to grow as they moved forward, but they specifically identified questioning as a key ingredient in using these materials well, and recognized the challenge in doing so. Differences in the way these teachers handled the challenge of using questioning to support student learning seemed to be impacted by their prior experiences with learning mathematics. In spite of Danielle's confidence in her knowledge and teaching of mathematics, she was challenged by the task of asking questions so that her students would own their understanding. Carly's difficulties with learning mathematics may have impacted her ability to be more patient with the learning process. This difference may explain Carly's heavier reliance on the curriculum to create a classroom culture where student thinking was primary. While the curriculum materials clearly supported both teachers in their enactment, we hypothesize that these teachers would have benefited from the explicit contrasts between the goals of the prior curricula they had used, and the *Investigations* unit they were implementing.

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AN ONGOING INVESTIGATION OF THE MATHEMATICS THAT TEACHERS (NEED TO) KNOW: THE CASE OF DATA MANAGEMENT

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Using complexity thinking as a framework for interpretation, in this article we offer a theoretical discussion of teachers' mathematics-for-teaching. We illustrate the discussion with some teachers' interactions around mathematics that arose in the context of an in-service session developed around the topic of data management. We use the events from that session to illustrate four intertwining aspects of teachers' mathematics-for-teaching. We label these aspects "mathematical objects," "curriculum structures," "classroom collectivity," and "subjective understanding." Drawing on complexity thinking, we argue these phenomena are nested in one another and obey similar dynamics, albeit on very different time scales. We conjecture that these aspects might serve as appropriate emphases for courses in mathematics intended for teachers.

Studying Teachers' Tacit Knowledge

The issue of teachers' knowledge of mathematics has been a prominent one for several decades now (see, e.g., CBMS, 2001). However, little progress has been made toward a consensus on the question of what teachers need to know. In our home institution, for example, teacher candidates in the secondary route (Grades 7–12) require a minimum of twelve 3-credit courses in mathematics. Most of these courses are drawn from 'stock' listings that include introductory calculus, linear algebra, discrete mathematics, geometry, and introductory statistics. Mathematics requirements in other pre-service teacher education programs at most English Canadian universities are similar.

These practices seem to be held in place by an assumption that courses in formal mathematics are vital to effective teaching. Unfortunately, as has been thoroughly developed, this belief is not easily substantiated. With regard to generic courses, for example, Begle (1979) and Monk (1994) demonstrated there to be at best a weak relationship between the courses taken by teachers and their students' performances on standardized examinations. Such results have prompted a pervasive belief that an emphasis on *more* mathematics in teacher education programs may be inappropriate. It might be that teachers require *more nuanced understandings* of the topics in a conventional curriculum (Freudenthal, 1973).

In an effort to inform discussions of the sorts of mathematical competencies that are necessary for effective teaching, we have undertaken a six-year investigation of the mathematics that teachers enact in their classrooms (see Davis & Simmt, in press). Currently at the beginning of the fourth year, this research is anchored in the assumption that most experienced mathematics teachers have a wealth of mathematical knowledge, although much of this know-how may never have been an explicit aspect of their educations. Indeed, it may not be popularly recognized as part of the formal disciplinary body of knowledge.

The work is informed by Ball and Bass's (2003) central emphasis on "job analysis," which entails close examinations of the skill requirements of teaching. These skills include, for example, interpreting concepts for learners, making sense of learners' varied understandings of topics, selecting suitable questions, and recognizing relevant associations among ideas. Such competencies require knowledge of how mathematical topics are connected, how ideas anticipate others, what constitutes a valid argument, and so on. The assumption is that the subject matter knowledge needed for teaching is not a watered down version of formal mathematics, but a serious and demanding area of mathematical work. We concur.

Complexity Thinking and Mathematics-for-Teaching

As with previous reports (Davis & Simmt, 2003, in press; Simmt & Davis, 2004), this writing is framed by complexity science, an attitude that has prompted us to interrogate several issues around teacher knowledge that are often taken for granted. We develop five here.

First, discussions of teachers' mathematics knowledge tend to be framed by an assumption that the individual is the sole site of learning and understanding. In our work, we deliberately take a more collectivist approach. Our rationale is that teaching always occurs in group settings—that is, in spaces that not only invite, but that compel negotiation of meanings and understandings. For the most part, we find that it is impossible to identify particular aspects of the final 'answer' with specific individuals within such collective settings.

Second, discussions of teachers' mathematics tend to focus on what might be called the 'front end' of knowledge—on established processes and conclusions—rather than on the images, analogies, metaphors, gestures, and applications that give shape to concepts. So pervasive is the emphasis on formal and explicit knowledge that the teachers often do not see examinations of figurative devices as mathematics, even though they readily acknowledge immediate and profound influences on their understandings and pedagogy.

Third, investigations of teacher knowledge tend to be level-specific, focused on the grade or grades that teachers are or will be expected to teach. While we would not dispute the importance of this emphasis, we would also note the value of discussions that span grades. In particular, the commonplace separation of pre-service and in-service programs into elementary and secondary strands can present some significant problems. For this reason, grades Kindergarten through 12 are represented in the cohort. As well, for each topic studied, a central task is to explore how it is developed through the K–12 curriculum.

Fourth, investigations of teachers' mathematics have tended to ignore the dynamic character of teacher knowledge—and, in particular, the manner in which efforts to study mathematics-for-teaching can affect the mathematics teachers know. It could be further argued that there is a tendency for investigations of mathematics-for-teaching to be organized around lists of the sorts of topics that others believe teachers should know, thus giving rise to reports of what teachers *should*, but often *don't* know. Instead of this emphasis on static deficiency, we regard knowledge in terms of dynamic sufficiency. We assume that knowing is constantly changing, in part because of our efforts to study such knowing. Further, we feel it to be important to be attending to what teachers *do* know—for the time being, at least, until we have a much better sense of the sorts of mathematical knowledge that are enacted when teaching.

Fifth, we note that discussions of teachers' mathematics tend to be organized around a radical break between 'objective knowledge' and 'subjective understanding.' This issue is no doubt anchored in the ancient assumption that pure knowledge exists on a metaphysical plane and that humans must thus use their wits and experiences to bridge the gap that separates personal construals of the world from actual truths about the world (see fig. 1).

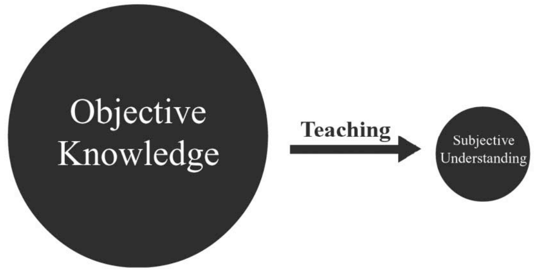

Figure 1. A depiction of the visual metaphor that underpins commonsense interpretations of knowledge, understanding, and teaching

Contemporary theories of knowing and knowledge depart from such conceptions, tending to invoke notions of nested rather than discrete realms (see fig. 2). In terms of the discourses that are now most prominent in discussions of mathematics pedagogy—namely radical constructivism, which focuses on individual sense-making, and various social constructionisms, which are more concerned with the emergence of bodies of knowledge—understanding and knowledge are seen to be enfolded in and to unfold from one another.

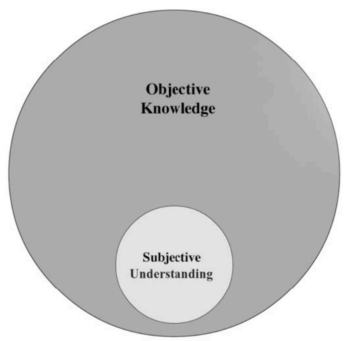

Figure 2. An image better fitted to constructionist and constructivist conceptions of knowledge and understanding

While this move does represent an important development in discussions of mathematics education, it also serves to perpetuate some troublesome separations. For instance, despite the fact that mathematics has been acknowledged to be subject to revision and elaboration—that is, mathematics is seen to evolve—in the context of teachers' knowledge of mathematics, it is still treated as a fixed backdrop. By contrast, subjective understanding is seen as inherently volatile. Hence, despite the epistemological shift witnessed in the field over the last quarter-century, there has been little or no reorganization of teacher education programs to address the separation of courses on established knowledge (i.e., on formal mathematics) and courses on how mathematics is established (i.e., on learning).

We worry that the maintenance of such separations contributes to a certain blindness to other complex, emergent forms that arise in the interactions of individual knowers and that are nested in bodies of collective knowledge. In particular, with regard to discussions of teachers' mathematics, relatively little attention tends to be given their knowledge of the dynamic forms of curricula and classrooms (see fig. 3).

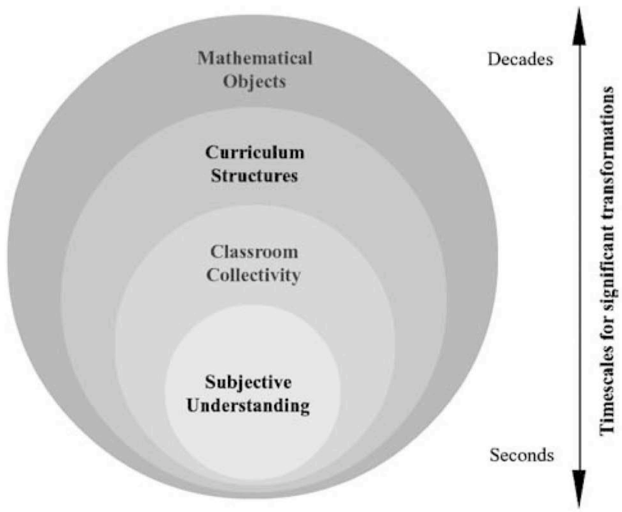

Figure 3. An image intended to foreground other levels of complex emergence that are of significance to mathematics educators

In other words, rather than framing discussions of teacher expertise in terms of the categories of objective knowledge and subjective understanding, we organize our investigations around a model in which these categories are two aspects in a grander continuum. When other levels of nested organization are examined, the tidy distinction that tends to be enacted between issues of objectivity and subjectivity becomes problematic. For example, matters of established mathematics come to be entangled in concerns for how curricula are constituted, which must be brought to bear on how classrooms are organized and how, in turn, the subject matter comes to be understood. The self-similar dynamics that operate at the various levels—albeit on very different time scales—also emerge as a vital issue when making sense of mathematics pedagogy.

The point here is mathematics-for-teaching is likely considerably more complex than has generally been assumed. In particular, it seems that it cannot be considered in terms of readily defined concepts or competencies that can be organized into courses designed especially for teachers. Rather, we would argue, such courses must give special attention to the dynamics of knowledge production—and we would offer our work with practicing teachers as an example. If the hope is that teachers might be capable of organizing rich and meaningful knowledge-producing experiences for their students, it would seem sensible to argue that they must not only be involved in similar sorts of experiences in their mathematics courses, but that they have opportunity to examine closely the structural dynamics at work in such settings.

Methodology

To repeat, in this work we aim to render explicit some of the usually tacit aspects of teachers' mathematics-for-teaching. As well, in the process, we explore the possibilities for knitting newly represented understandings into more sophisticated possibilities.

The cohort of teachers is a diverse one. Among the 24 participants, grades from Kindergarten through high school represented. In terms of professional experience, a few of the participants are at the beginning of their careers, several have taught for decades, and the rest fall somewhere in between. Most of the teachers are generalists, but two are mathematics specialists. Some teach in small urban centers, some teach in rural locations.

The cohort gathers for daylong meetings every few months. For their own part, the teachers see these meetings as "in-service sessions"; their principal reasons for taking part revolve around their professional desires to be more effective mathematics teachers. In contrast, for us, these events are "research sessions." We are explicit in the fact that we are there to try to make sense of their knowledge of mathematics and how that knowledge might play out in their teaching. The common ground, as we develop below, arises in the joint production of new insights into mathematics and teaching. Topics have ranged from general issues (e.g., problem-solving) to specific curriculum topics (as in the case of data management, developed here).

Informed by our understandings of complexity science, the meetings are organized around extended, group-based engagements with seemingly narrow activities. These tasks are developed around mathematical topics that are selected by the teachers themselves and they are designed in ways that allow us, as researchers, to map out some of the contours of their mathematical knowledge while also attending to the ways in which that knowledge is elaborated when is it made the explicit topic of inquiry. For example, the activity that is the focus of the session of data management was structured around one year's worth of newspaper weather reports that had been copied and assembled into binders. These reports contain information of average and forecast conditions, sunrise and sunset, moonrise and moonset, and temperatures from selected sites around the world.

A Brief Account of Key Topics of Discussion

As typically happens, participants organized themselves in self-selected groups of 3 or 4. Each group was each given a binder and instructed to "Organize, make sense of, and represent some data of a phenomenon." Grid paper of varying sizes, graphing calculators, and colored pens were made available. As might be expected, over the hour scheduled for the initial activity, the groups generated a range of graphs and summary statistics. The products included:

- a plot of moonrise times through the month of December (fig 4a);
- a line graph comparing the daily highs in four different cities;
- a line graph comparing daily temperatures and wind speeds in July;
- a curve of best fit of average daily highs for each month (fig. 4b);
- a line graph illustrating average daily temperatures and wind speeds for July (fig. 4c).

Of course, the 'shapes' of these products varied dramatically, and the differences were most pronounced in the contrasts between relatively smooth and very jagged curves.

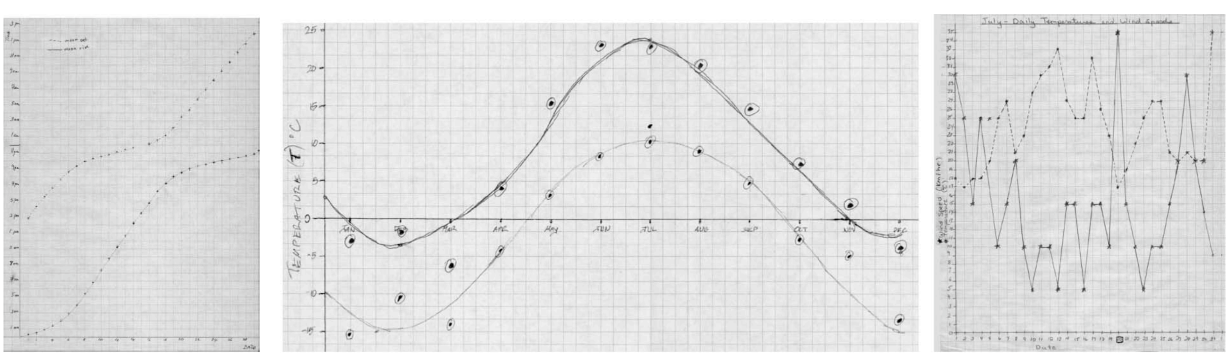

Figure 4. Sample graphs produced by the teachers

As the products were displayed and compared, it was evident to us that the teachers understood and acted on a distinction between descriptive and inferential statistics. In particular, the groups that focused on displaying data knew that they were doing something different from the groups that sought to identify patterns and relationships among graphs—framing this contrast with phrases such as "displaying data" (in ref. to fig. 4a) and "looking for a connection" (in ref. to fig. 4c). Even so, when we offered the terms *descriptive* and *inferential*, only a few could recall the formal distinction. This sort of observation underscores for us the importance of studying teachers' knowledge in action. That definitions of a few formal terms were not known by the teachers did not prevent them from acting on the distinctions.

More interesting to us was the attention given to a topic that simply is not addressed in local data management curricula: the differences in the sorts of phenomena that underlie the data. The plot of moonrise times (fig. 4a), for instance, resulted in a smooth predictable curve. By contrast, the graph of temperatures and wind speeds over a month-long period (fig. 4c) gave rise to seemingly random plots. Somewhere between these two extremes, graphs of long-term averages (of, e.g., temperatures, fig. 4b), gave smoother curves, but ones that still had anomalous points. In other words, the exercise of representing different data sets prompted an issue that none of the teachers had previously considered—namely, that different categories of phenomena were being represented, ranging from the easily predicted to the seemingly random.

The ensuing discussion was organized around a topic out of complexity science—namely the issue of three different types of phenomena: *simple*, *complicated*, and *complex* (see Davis & Simmt, 2003). Simple phenomena are those that arise in the mechanical interactions of a few parts, such as the moon in orbit around the earth. The behaviors of these sorts of systems, as the example of moonrises (fig. 4a) demonstrates, can be precisely calculated using straightforward laws, and the results generally give smooth curves when plotted. Complicated phenomena are also mechanical, but usually involve many interacting parts. Their behaviors are patterned, but not directly calculable and can appear quite volatile over the short term. However, using various tools of descriptive statistics, the long-term behaviors of these systems can be 'tamed' to give reasonably smooth curves (e.g., fig. 4b).

Complex phenomena, comprising the third category, are not mechanical. These sorts of phenomena might be described as 'learning systems' because two very similar systems can respond very differently to virtually identical circumstances, depending in large part on their particular histories. As such, complex phenomena may not generate patterned averages, even over the long term. There is always a potential for an entirely new profile to emerge that cannot be anticipated by the existing data. A hallmark of such systems is the scale-independent graph. For example, representations of a stock market's performance over a day, a week, a month, a year, and a decade can look remarkably similar—at least in terms of bumpiness of detail.

The issue of different categories of phenomena had been a topic in a previous discussion, and so that information served as a backdrop to the discussion of the different representations that had been generated by the teachers. The theme of that discussion was more toward "What gets obscured in the current curriculum emphases on representation, measures of central tendency, and related topics?" On that count, all present noted that their teaching of topics in data management, to that point, had never taken into consideration the very different phenomena that are represented with the current emphases on data display and low-level inference.

Conclusions and Implications

We believe that this example provides a useful illustration of the difference between teachers' mathematics-for-teaching and the sorts of subject matter competencies that their students will be expected to demonstrate. The distinctions between descriptive and inferential statistics and among simple, complicated, and complex phenomena are not necessarily ones that need to be introduced to learners, but they are valuable to understanding the sorts of mathematical knowledge that teachers can and do bring when jointly engaged in mathematics.

Further, this sort of event serves to highlight the complex, nested relationships among the body of mathematical knowledge, the formal curriculum, classroom collectivity, and individual understanding. As we argue in detail elsewhere (Davis & Simmt, in press), a critical aspect of this image is that each of the layers, while pointing to a discernible and coherent phenomenon, is entangled in the unfolding of all the other layers. Following an insight of complexity science, each of the layers could be argued to obey the same sort of fitness-oriented evolutionary dynamics. However, these occur on radically different time scales, ranging from fractions of a second for significant accommodations to individual understandings to decades and centuries for analogous changes to formal mathematics.

When teachers are engaged in activities that foreground the emergence of new mathematical understandings (such as the mathematical thinking that is associated with the three categories of phenomena discussed), the assumption of a fixed body of knowledge and a static curriculum give way to a realization that, as educators, they are participating meaningfully in the emergence of our culture's mathematics. For us, the weather map exercise with the teachers is valuable in that activities of this nature might support a realization that mathematical sensibilities evolve, which in turn might be useful to foreground the teacher's role in the participation in the evolution of Western mathematized sensibilities. In other words, we believe that experiences such as this one might be construed as a demonstration of how teachers, indeed, are participating in the production of mathematical knowledge—a claim that, of course, compels a definition of mathematics that includes all who are entangled in it (versus, for example, a restrictive and exclusionary definition such as "what mathematicians do").

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ALGEBRAIC REASONING OF YOUNG STUDENTS AND PRESERVICE ELEMENTARY TEACHERS

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The National Council of Teachers of Mathematics (2000) stresses the importance of the development of algebraic reasoning PreK-12. Our research studies revealed that 1-6 graders demonstrated sophisticated algebraic reasoning when solving problems with several unknowns. These findings led us to examine abilities of pre-service teachers to solve equations with two and three variables themselves, to predict students' difficulties in solving the problems, and create guiding questions for students. The study demonstrated that future teachers successfully applied algebraic solution methods to solve problems. However, they underestimated students' abilities to reason algebraically and suggested that students use arithmetic strategies. The paper presents results of three research studies conducted with elementary school students and pre-service teachers.

Introduction

Knowledge of algebra is essential for learning mathematics and other disciplines. One of the main strengths of algebra is that it is a tool for generalizing and solving variety of problems. Algebra is a symbolic language that enables users to describe and analyze relationships between quantities. By simplifying and representing problem situations in condensed ways, algebra allows students to see problem structures and become better problem solvers (Krutetskii, 1976; Schoenfeld, 1992).

The National Council of Teachers of Mathematics (NCTM, 2000) identifies the importance of algebra and stresses the significance of the development of algebraic reasoning beginning at the Pre-kindergarten level. The report from the National Research Council, *Adding It Up: Helping Children Learn Mathematics* (2001), points out that algebra stems from arithmetic and therefore, young students should develop understanding and have experience with algebraic ideas early in their schooling. Thus, the transition into formal courses of algebra will be eased.

The reformers of mathematics education call for algebra for everyone and recommend that students, as early as the pre-kindergarten level, solve algebraic problems (Davis, 1985; NCTM, 2000; Usiskin, 1997). Educational researchers concur. They claim that if students in grades K-8 are introduced to algebraic concepts gradually, they are able to build meaning of such abstract concepts including variable and equation (Kieran, 1992; Usiskin, 1997).

To address this goal, we need to take a closer look at students' algebraic reasoning abilities, at teachers' knowledge and understanding of algebra, and of teachers' readiness to facilitate students' algebraic reasoning.

Studies of Young Students' Algebraic Reasoning -- Theoretical Framework

Until recently, the study of algebra was reserved for middle and high school students who had matured enough to deal with the abstractness characterized by the use of symbols to represent relations among quantities (Kieran, 1992; Kieran & Chalouh, 1993; Lodholz, 1990). It

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is precisely this delay in the introduction of algebra that may explain the difficulty students experience in their transition from arithmetic to algebra (Dobrynina, 2001; Kieran & Chalouh, 1993; Tsankova, 2003), that connecting algebraic concepts to arithmetic concepts will facilitate students' abilities to reason both arithmetically and algebraically, and will produce favorable dispositions for the study of more complex mathematics. (Davis, 1985; Dobrynina, 2001; National Research Council, 2001; Tsankova, 2003; Usiskin, 1997)

Comparative studies have shown that in other countries formal instruction in algebra begins in grade 1 (Usiskin, 1997). Mathematics educators speculate that this is one of the explanations for the superior performance of students from the Pacific Rim and Eastern Europe on the Third International Mathematics and Science Study (1996).

Although many researchers believe that algebraic topics should be introduced to young students, there is a paucity of research data indicating what youngsters are capable of doing algebraically. Much of the research that does exist, focuses on methods for teaching children to solve missing addend problems. Page (1964) taught elementary students to solve for the unknowns in the "frame" equations, as he called them, by using a "guess, check, and revise" strategy. Peck and Jencks (1988), in a study with fifth grade students showed that using frame equations is a successful instructional approach to introduce students to solving equations. In a study involving students from grade 1 and 2, Carpenter and Levi (2000) showed, that young children, grades 1 and 2, could understand and solve "open sentences" of the form $\square + 6 = 9$.

Research with Students Grades 1 through 6 Solving Systems of Equations

Until recently, there has been paucity of studies that have examined the abilities of young children in grades 1 through 6 to solve systems of linear equations with two variables prior to instruction. In her study of students' algebraic reasoning abilities, Dobrynina (2001) investigated how 360 students in grades 4, 5, and 6 solved systems of equations with two and three unknowns prior to formal instruction in algebra. The students were asked to solve and describe their methods of solution for six problems, three two-variable and three three-variable problems. Dobrynina's results showed that without prior instruction in algebra, 11% of grade 4 students, 17% of grade 5 students, and 30% of grade 6 students employed algebraic methods to solve systems of equations with two or three variables. Furthermore, in all grades, solution success was higher for students who used the method of substitution (90% success rate) than for those who employed an arithmetic method such as guess-check-revise (70% success rate). Dobrynina discovered that without prior instruction in algebraic methods, 11% of her grade 4 students, 17% of her grade 5 students, and 30% of the grade 6 students employed algebraic methods to solve systems of equations with two or three variables. She also discovered that in all grades, solution success was higher for students who used algebraic methods than for those who employed arithmetic methods.

The results revealed that many children demonstrated deep algebraic reasoning prior to any formal instruction in algebra. Dobrynina (2001) identified two algebraic methods that were used by students to solve the systems of equations. They are: 1) the method of substitution, in which the value of a variable in one equation is determined and then that value is substituted in another equation; and 2) Simplifying and solving equations by adding and subtracting like quantities from both sides of an equation. Students who employed algebraic solution methods were more successful than those who used arithmetic. The table below shows the statistics for grade 4-6 students for a sample of 360 students:

Table 1: Solution Success by Strategy

Problem Type	Rate of success using Arithmetic method	Rate of success using Algebraic method
Two-variable problems	70%	93%
Three-variable problems	38%	89%

The purpose of Tsankova's study (2003) was to investigate not only whether students in grades 1, 2, and 3 can solve systems of equations with two variables independently, but also, how much children can accomplish with different degrees of prompting (hints) from adults.

Sixty students were interviewed, 20 students per grade level. During the test, if a student was not able to solve a problem independently, the researcher presented a preset sequence of solution hints, which varied in terms of the degree of assistance provided. The results revealed that students in grades 1 through 3 were able to solve systems of equations with two variables independently or with very little help. Without assistance, 70% of students successfully solved the problems using algebraic strategy in comparison to 5% of the students who were successful using arithmetic strategy. For students who had difficulties in solving the problems, problem-specific probing questions and hints were provided leading them to an algebraic strategy.

These findings led the authors to examine abilities of preservice teachers to solve problems with several unknowns, their readiness to anticipate students' reasoning and difficulties while solving the problems, and create problem-specific guiding questions to facilitate students' algebraic thinking.

Research Study with Pre-service Teachers

Research has shown a direct correlation between teacher's content knowledge and students' achievement in learning and understanding of mathematics. (Ma, 1999) To successfully guide students through development of their algebraic reasoning abilities, teachers themselves need to have an in-depth knowledge of the mathematics involved. The need of research of teachers' content knowledge and abilities to facilitate young students' reasoning is apparent.

The purpose of the study was to discover whether pre-service teachers can solve problems with two and three unknowns and what solutions strategies they use, and to uncover what types of assistance future teachers plan to provide for their students.

The problems used in the study were selected from the studies with students grades 1 through 6 discussed above. There were three different types of systems of equations shown below, where *x*, *y*, and *z* are unknowns and *a, b, c, d, e, f,* and *g,* are known values less than 30.

1) $x + y = a$
$x + y + y = b$
$x = ?\quad y = ?$

2) $x + x + x + y = c$
$x + x + y = d$
$x = ?\qquad y = ?$

3) $x + x + y + z = e$
$x + y + z = f$
$x + x + y = g$
$x = ?\quad y = ?\quad z = ?$

There were six problems used. The variables in the first problems were represented as geometric shapes, and the variables in the last three problems were represented using letters. An example of the two different variable representations follows:

Problem 2:

◯ + ◯ + ◯ + ⬡ = 16

◯ + ◯ + ⬡ = 13

◯ = ____

⬡ = ____

Problem 5:

d + d + d + f = 26
d + d + f = 19

d = ____

f = ____

To address the goals and check the methodology, a pilot study was conducted with 15 preservice teachers. The results revealed that only 5 out of 15 pre-service teachers both used and suggested algebraic solution methods. Future teachers underestimated students' abilities to reason algebraically and suggested the use of concrete objects and arithmetic strategies (guess and check) for elementary school students. Further more, pre-service teachers were "ready to overhelp" and facilitate too much too soon, and most of them felt that the use of letters as representing unknowns might be "confusing" for young students.

There were two major limitations of the pilot study. The first was that pre-service teachers were asked to complete the assignment at home. And the second limitation was the size of the sample. Therefore, a new study focused on a larger sample of 79 preservice teachers attending Methods of Teaching Elementary Mathematics courses. Future teachers were asked to solve problems involving systems of equations with two and three variables themselves twice -- in class and then as a homework assignment. In addition to solving the problems at home, future teachers were asked to predict possible pitfalls children can encounter in solving these problems, and finally create and improve guiding questions that would help children to overcome solution difficulties.

Results

A. Solution Success

Some pre-service teachers had difficulty solving the problems as shown in the table below.

Table 2: Lack of Solution Success

Problem Type	In class (N =79)	At home (N = 79)
2 variable frames	4.5%	1.5%
2 variable letters	6%	3%
3 variable frames	47%	3%
3 variable letters	20%	14%

As it can be observed from the table, the 3-variable problems were more challenging for the pre-service teachers to solve in class. Forty-seven percent (47%) of the pre-service teachers were not able to solve the 3-variable frames problem and 20% the 3-variable letters problem. This can be explained by the fact that when students encountered the frames 3-variable problem in class, they first tried to use arithmetic methods. Later, for the 3-variable problem (Problem 6), where the variables were represented as letters, some pre-service teachers realized that a more efficient strategy was needed.

In comparison, more students had solution success at home. The reason could be that students had more time to think about a better strategy or that they remembered the algebraic strategy they used in class at the end of the problem set. It can be speculated that solution success increased as more pre-service teachers used algebraic methods. The percentage of students who used arithmetic and algebraic methods are shown in Table 3.

B. Solution Strategy

Table 3: Solution Strategy per Problem

Problem/ Strategy	Arithmetic Strategy (N=79)		Algebraic Strategy (N=79)	
	In Class	At Home	In Class	At Home
Problem 1 2-variable frames	87%	52%	13%	48%
Problem 2 2-variable frames	77%	45%	23%	55%
Problem 3 3-variable frames	67%	34%	33%	66%
Problem 4 2-variable letters	63%	49%	37%	51%
Problem 5 2-variable letters	53%	38%	47%	62%
Problem 6 3-variable letters	60%	45%	40%	55%

As observed in Table 3, the use of algebraic strategy in class increased as students progressed through the set of problems. The first encounter with 3-variable problems (Problem 3), forced more students to apply algebraic strategies rather than arithmetic. It could be inferred that using more challenging problems leads to more sophisticated strategies. As soon as students discovered a more efficient algebraic strategy, they applied it more often.

Similar observations can be made about the solutions strategies used at home, namely, the use of algebraic strategy increased as students progressed through the set of problems. Also, it should be noted that more students used algebraic strategies at home than in class. Thirty-two percent (32%) of the students who used arithmetic strategies to solve frame equations switched to algebraic strategies at home, and 15% of the students who used arithmetic strategies in class to solve letter equations switched to algebraic solution at home.

Overall, in class, more students used algebraic strategies when solving problems where the variables were represented with letters than with frames. It can be speculated that since using letters to represent variables is traditionally more often used in algebra courses, pre-service teachers might have been more eager to apply more often algebraic strategies for problems 4 through 6. It could also be argued that since the letter equations came in second in the problem set, the participants in the study became more aware of a more efficient strategy. It is interesting to point out that in the study with students in Grades 1 through 3 (Tsankova, 2003), there were no significant differences in solution strategy because of variable representation (frames vs. letters).

C. Pre-service teachers' Recommendations for Students in Elementary School

Despite the fact that more students used algebraic strategies at home, and as a result, more students had solution success, 68% of the pre-service teachers did not consider the possibility that elementary students are able to understand and solve the problems using algebraic methods. To illustrate the discrepancy between what pre-service teachers recommended and what young students are able to do, examples from each follow:

Problem 3 (in the study with pre-service teachers):

What number belongs in each shape?
- Same shapes have same numbers.
- Different shapes have different numbers.

A	○ + ○ + △ + □ = 18	□	= ______
B	○ + △ + □ = 16	○	= ______
C	○ + ○ + △ = 17	△	= ______

Grade 4 student:

I looked at A, B and C. I noticed that in B compare to A, there was a circle missing and also two points missing on the equal sign. That tells me that the circle is two points. Then I compared C to A. The square is missing and one point is missing. That tells me that the square is one. Then I added two circles, got four, and found what to put into the triangle [13] to get to 17.

Pre-service teacher:

I would begin by having students look at the bottom number sentence. Could they think of any number strings involving doubles plus a number that equal 17? I would have them write down those possibilities. I would do the same for the second problem but working with variations of three numbers adding up to 16. I would have them look for intersection with their first list for sentence C. I would finally have them look at part A and have them think of doubles plus two numbers that could intersect with parts B and C but add up to eighteen. Trial and error would be the best way to start students working through this difficult algebraic problem.

It is clear, that the recommendation provided by the pre-service teacher is much more inefficient and time-consuming. It involves considering multiple cases and sophisticated skills at data organization. It should be noted, that most of the pre-service teachers who recommended arithmetic strategies, limited their guidance to the students to simply "guess and check." Other pre-service teachers' facilitation consisted of step-by-step instructions of how students should conduct the solution. Step-by-step instructions for using guess-check-and revise strategy for each equation separately does not allow for the students to perceive a problem as a system of equations and generalize a solution strategy.

Those 42% of the pre-service teachers, who were willing to facilitate students' algebraic thinking, were not able to provide meaningful guiding questions, hints, or suggest strategies to students. Example follows:

Pre-service teacher:

> Start by comparing all three [equations]. Figure out what they have in common and what is missing. Next, figure out which two [equations] are different only by one object. If only one object is missing, you should be able to find the number through subtracting. Once you found the value, plug it in, and the rest will snowball into place.

Conclusion

Our findings pose a concern about the readiness of preservice teachers to guide elementary students to use algebraic reasoning. That concern leads us to the question of developing better models for pre-service teacher education. During their program of study pre-service teachers need to be engaged in solving challenging problems themselves. To be successful, they need to be supported throughout their reasoning and in their reflection on the methods used all through the solution process.

Pre-service teachers need to examine results of past and current research on what young students are able to do and consider the implications for teaching. It is important that pre-service teaches create problem-specific protocols of guiding questions in order to develop abilities for facilitating children's algebraic reasoning in particular and mathematical reasoning as a whole.

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MATHEMATICS AND LITERACY: AN INTERDISCIPLINARY PERSPECTIVE ON TEACHING WITH REFORM-BASED CURRICULA IN URBAN MIDDLE SCHOOLS

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Reform-based mathematics curricula in the United States place new demands on both teachers and students for mathematizing situations that need to be interpreted through talk, texts, stories, pictures, charts, and diagrams. In urban middle schools, where many students are struggling with reading and writing, meeting these demands is especially urgent. In this study, we examine how teachers learned to recognize and to address the literacy demands in conceptually rich, but contextually complex curricular materials. The teachers' perspectives on the materials have implications for the design of curricula materials. This study also has implications for theorizing about teachers' learning in practice.

Over the past 15 years, the curriculum standards (NCTM, 1989, 2000) and several NSF-funded curriculum projects have resulted in an important shift in the contexts in which mathematics teaching and learning takes place in schools. One way of characterizing that shift is to describe these contexts as providing significantly more opportunities for students and teachers to mathematize situations that need to be interpreted through talk, texts, stories, pictures, charts and diagrams. Over this same time frame, many urban schools have been faced with increased cultural diversity among their students, including tremendous variation in primary languages and in experiences they bring to school. Many of these students, especially those who struggle with academic literacy, are among those at greatest risk for not learning from experiences with these more challenging materials (Schoenbach, Greenleaf, Cziko, & Hurwitz, 1999). Teachers who have adopted reform-based curriculum are faced with new challenges as they need to learn to provide opportunities for learning mathematics in ways that are significantly different from those involving use of traditional textbook materials. In particular, teachers need to learn how to support the development of students' mathematical communication, including their ability to read the texts of their curricula and to generate appropriate written responses. In this paper, we report the results of an interdisciplinary investigation on how secondary mathematics teachers learned to recognize and to address the literacy demands in conceptually rich, but contextually complex, curricular materials.

Theoretical Perspectives

This interdisciplinary work is informed by sociocultural research in mathematics and literacy education. While many studies of literacy in the 1970s and 1980s were concerned primarily with understanding learners' cognitive processes and teachers' instructional approaches in a variety of subject-area classes (Alvermann & Moore, 1991), more recent research has attended to the complex intersections of adolescent learners, texts, and contexts (Moje, Dillon, & O'Brien, 2000). Literacy has come to be seen as multifaceted, involving reading, writing, speaking, listening, and other performative acts—all taking place in certain social settings for certain purposes (Hicks, 1995/1996). Like other domains of study, secondary mathematics classes require teachers and students to use various kinds of literacies and to participate in various discourse communities specific to the domain, where certain kinds of literacy practices count more than others (Borasi, Siegel, Fonzi, & Smith, 1995; Siegel & Fonzi, 1995). For example, learning to write mathematically means learning to use everyday language along with the symbolic structures and the precision of language that are central to the discipline of mathematics. Learning to read mathematically means learning to identify and interpret mathematical quantities and understand the relationships among them. Understanding how teachers learn to teach in ways that engage students in communicative practices is the overarching goal of this research project. Our specific research questions were: (1) What are urban teachers' perspectives toward the literacy demands of reform-based curricular materials? and (2) How do urban teachers' practices change so as to support the development of students' mathematical communication?

Methodology, Data Sources and Analysis

Our methodological approach is the multi-tiered teaching experiment (Lesh & Kelly, 1999) which allows us to collect and interpret data at the researcher level, the teacher level, and the student level (see Figure 1). This multi-level approach is intended to generate and refine principles, programmatic properties (such as interventions with teachers), and products (such as shareable tools and artifacts of practice) that are increasingly useful to both the researchers and the teachers.

Tier #3 Researchers	Researchers, working with teachers, develop models to make sense of teachers' and students' learning and to re-interpret and extend their own theories.
Tier # 2 Teachers	Teachers work with teachers and researchers to describe, explain, and make sense of student learning.
Tier #1 Students	With teachers' support, teams of students investigate mathematical tasks in which they construct, revise and refine their interpretations of the problem situation.

Figure 1. The multi-tiered teaching experiment (adapted from Lesh & Kelly).

Central to our analytic approach is the notion that as researchers we examine the teachers' descriptions, interpretations, and analyses of artifacts of practice that were developed, examined and refined during our collaborative work. At Tier #1, the set of artifacts includes the writing that students generated in response to various text-based, mathematical tasks from the reform curricula, *Connected Mathematics Project (CMP)*, (Lappan, Fey, Fitzgerald, Friel, & Phillips, 1998). The artifacts located at Tier #2 consist of the literacy scaffolding tools that the teachers and researchers collaboratively developed to support students in interpreting and responding to the tasks. The artifacts generated and analyzed at Tier #1 and #2 generated the findings at Tier #3 that are reported in this paper.

The research was carried out by a team of university-based researchers in mathematics education and literacy education, working in concert with mathematics teachers in a mid-sized urban district that had recently adopted the CMP materials. The teachers were from one high school and its three feeder schools (two middle schools and a pre-K to 8th grade school). The results reported here are drawn largely from the work at the pre-K to 8th grade school. That school has approximately 860 students and 45 teachers and support staff. The school population is quite diverse with approximately 31% African-American, 21% Asian, 35% Caucasian, and 11% Latino/a students. Approximately 20% of these students are English language learners, and 25% have special needs. About 80% of the students qualify for free or reduced lunch. The teachers (n=5) were volunteers and were all but one of the mathematics teachers in grades 6 through 8. This was the second year using the *CMP* materials for most of the teachers. Four of the teachers were very experienced; one was in her first year teaching as we began this project.

During the first two years of the project, the work with the teachers consisted primarily of four on-going activities: week-long summer retreats, quarterly project meetings with teams from other project schools, bi-weekly team meetings, and "lesson cycles" (described more fully below). The week-long summer retreats provided an introduction to the project for the teachers and began our collaborative analysis of the supports and challenges to be found in using the reform-based curricular materials as text. The quarterly project meetings and the bi-weekly team meetings provided forums for the continued discussion of instructional approaches that might be used to support students' learning with the curricular materials and for developing more specific instructional goals and plans related to reading, writing and speaking mathematically.

The five teachers taught at three different grade levels (six, seven, and eight); they rarely shared planning times during the day and seldom were teaching the same the lesson at the same time, even when teaching at the same grade level. While a shared approach to examining common lessons was not feasible, the teachers did share a common focus on the need for their students to become better mathematical writers. This was in part driven by the high-stakes testing that takes place at the end of the eighth grade, where their students are asked to explain their reasoning or solution strategies in writing. All the teachers in grades 6 through 8 felt a school-level shared responsibility for preparing students for this exam. The focus on writing was also driven in part by the curricular materials that included many tasks that asked students to explain their reasoning or solution strategies. In addition, the curricular materials also included tasks that the teachers saw as higher-level thinking tasks, requiring more elaborated descriptions and explanations. Finally, the focus on writing reflected the teachers' concerns for many of their students who were not at grade level in reading or writing for reasons of second language learning, learning disabilities or special needs. This shared interest in student writing then became the focus of the discussions at the bi-weekly team meeting and of the "lesson cycles." It also drove the collection of student work to be analyzed and annotated. This “library” of student work was the principal artifact from Tier #1 (the student level) that we (both teachers and researchers) examined as we worked together.

Since our primary research questions concerned changes in the teacher perspectives and in their practice, we used "lesson cycles" to work jointly on planning, implementing, and debriefing lessons for supporting literacy opportunities for students. The lesson cycles began halfway through the first year of the project and continued through the second year of the project. Each teacher participated in a lesson cycle approximately once every three weeks with a member of the research team. Each lesson cycle consisted of three elements: (1) A planning session that followed the overall CMP guidelines for the investigations, but asked specifically the question "what are the literacy opportunities in this lesson?" In planning with this focus, the teacher discussed her ideas for reading the text, described opportunities for students to speak with each other, and identified prompts for student writing that would be used in the lesson. (2) The implementation of the lesson, where a member of the research team would observe the lesson, take extensive field notes, and generate questions for discussion that arose during the observation related to the literacy opportunities in the lesson. (3) A de-briefing session with the teacher, where the intent of the session was to collaboratively gain insight into the teachers' thinking about the literacy opportunities of the lesson and to collect shareable artifacts from the lesson, such as insights gained or tools used to support students' learning. The de-briefing session often centered on a discussion of the students' written work and how that might be used to inform subsequent lessons. The planning and debriefing sessions were audio-taped and later transcribed. Brief memos were written based on notes taken and the artifacts of the session. The lesson cycles and the bi-weekly team meetings were the primary sources of the artifacts and data that were examined at Tier #2 (the teacher level).

Our data sources included field notes from the summer work sessions, the quarterly project-wide meetings, and the bi-weekly school-level team meetings and the fieldnotes and transcripts from the lesson cycles. In addition, we examined the artifacts that were produced by teachers as they worked with the research team in examining student work and in developing instructional approaches for supporting students' written communication. Our analyses of these notes and artifacts of practice have been on-going and used to inform our continued work with the teachers. Hence, the findings reported here are emergent and draw on multiple data sources.

Results

Our analysis yielded three results. First, these middle school teachers perceived that reform-based mathematics curricula demand literacy in ways that are sometimes helpful and sometimes not helpful in teaching mathematics. In the first summer retreat, we asked the teachers to examine the introduction and the first investigation in the 7th grade CMP book, *Variables and Patterns*, in terms of the supports and challenges to comprehension it might present to a 12 year old literacy learner. We asked the teachers to attend to such literacy dimensions as the prior knowledge required to make sense of the text, the text structure, the language and vocabulary used, and the layout and organization of the materials. The teachers identified several strengths to reform-based mathematics curricula with reference to literacy development: the new materials built awareness among students that mathematics involves reading, writing and explaining, not just doing problems. The teachers believed that students interacting with CMP materials had a greater understanding of the need to use reading strategies in mathematics class than the students had with traditional curricula. They reported that students were learning to help each other construct meaning from text in small groups. Finally, the teachers saw the new curricular materials as holding students' interest and helping them make connections between mathematics and their lives outside the classroom.

Although teachers were enthusiastic about these trends, they also reported that the reform-based curricula presented significant literacy challenges for students. For example, they reported that students struggled to elaborate ideas in writing when faced with the sorts of extended response questions typical of these curricula. Teachers believed that students had difficulty with the type of problem solving required because they were used to problems being more “clearly defined.” The teachers reported that the curricular materials also presented challenges in the areas of readability and vocabulary and felt the curriculum developers appeared to assume a mastery of basic computational skills that many of their students lacked. Finally, they argued that while some of the scenarios in the materials were relevant to adolescents' lives and interests, others assumed experiences and/or cultural capital that their students simply did not have. Another teacher voiced the concerns of all when she noted that the vocabulary and the format of the reading and the questioning were "ahead of" where she found her students.

Second, we found that the curricular materials provided little by way of useful guidance for teachers in making instructional decisions to support students in reading their texts or in generating appropriate written responses. For example, in the investigation referred to above, the teachers' materials suggest that teachers "Read through the three opening questions and the first paragraph with your students," and "Read through the paragraph on variables with your students." These unelaborated instructions are in sharp contrast to the rich and detailed ways that the curricular materials provided guidance as to how students might approach the mathematics of the investigation. To address this gap, the teachers and researchers began to develop a set of literacy scaffolding tools and a library of student work.

The literacy scaffolding tools came in two broad categories: 1) annotations from teachers' planning related to the opportunities for reading, writing, speaking, and representing that they identified for particular mathematics lessons, and 2) heuristics to support literacy skills and strategies that could be shared directly with students to raise achievement. The teachers' planning for reading changed dramatically over the two years of the project. At the beginning of the project, the teachers described their current practices as either "ignore the reading and literacy part, and just deal with the math" or reading to and interpreting for the students. As one teacher described it, "many of us were just reading it [*CMP* text] to the students. They couldn't read it? I read it to them! And then I got really good at being able to summarize and interpret it for them!" During the first year and a half of the project, the teachers came to recognize that rather than look at the reading as a barrier, "we need to look at CMP, and the literacy element as an opportunity." The teachers recognized that they needed to develop strategies to support their students and began to explicitly plan for reading in their mathematics lessons in ways that included: (1) guided notes, (2) mentioning unfamiliar vocabulary in the launch of the lesson, (3) setting a purpose for the reading, (4) paired reading with sharing, (5) chunking the text, (6) photocopying the text so that it can be highlighted or written on, (7) using a large group debrief to help interpret the reading, (8) teacher modeling of her own reading strategies, (9) using paired coaching, and (10) using peer editing for interpreting the text. The critical shift that occurred for the teachers was seeing the literacy elements of the curriculum as affording an opportunity for learning rather than as a perceived barrier to the mathematics. They realized that they needed to guide students in developing literacy strategies in mathematics, rather than simply reducing the literacy demands in order to focus attention on mathematics concepts and skills.

A similar shift occurred in the teachers' writing practices as we began to look at students' written work. The initial library of student work consisted of three categories of students' writing: 1) the description of algorithms that were developed through the investigations; 2) the interpretations of the specific contexts for problem-solving; and 3) the writing about a central concept (such as similarity) multiple times over the course of a unit of instruction. In each case, the teachers selected samples of students' work that represented the range of responses that they found in their classrooms. They documented the context and purposes where the writing occurred and generated potential instructional strategies for subsequent lessons. This shared analysis made increasingly visible to the teachers how students' written expression of their mathematical thinking develops across specific content and contexts. This, in turn, led to changes in the teachers' practices.

The first change in practice occurred as the teachers began to use "quick writes," short, informal writing guided by an open-ended prompt, to drive their daily instructional practices. As with reading, at the beginning of the project, the teachers described their practices as including little or no writing. As one teacher described her practice, "little or no writing was going on in my math class. The [suggested] reflections [in the student materials] looked pretty, but I didn't have time to do them. I just left them. If I did do writing, I took out a state assessment question, and I said, okay kids, here's the question." This teacher would demonstrate how she would go about writing, but "there was no discussion at all about writing, what makes it good, what makes it acceptable, and what makes it mathematically correct." As we planned together in the lesson cycles, all of the teachers started using "quick writes" at the end of their lessons to see what the students understood at the end of a lesson or series of lesson. The prompts almost always came from the *CMP* materials, either directly or with minor modification. For example, one teacher used the following writing prompt: "When you add a positive and negative integer, you sometimes get a positive result; you sometimes get a negative result. Show that this is true." The student work from this prompt led the teacher to an analysis of the writing that reflected her increasing awareness of the needs of her English language learners as well as the need to continue working on the mathematical content before moving on to the next investigation. The initial use of quick writes soon gave way to a more focused instructional effort that led the teachers to construct "writing plans" at the unit level. These writing plans allowed them to look at student writing more systematically. The central question for the teachers became "How can we get kids to be better writers of mathematics?" Their classroom instruction had shifted from writing as non-existent to sporadic to systematic.

Third, the teachers articulated a need for a shift toward increased student independence in engaging with mathematics. The teachers wanted their students to be more independent in interpreting the text-based problems. Several of the teachers recognized the dominance of their role in interpreting the texts of the curricular materials for the students and began to change their teaching practices. One teacher expressed this shift by saying "Before this project, I felt that I needed to read, explain, and talk about every aspect of mathematics with my students. I was, in a sense, allowing my students to become too dependent on me for every assignment they completed." According to several of the teachers, what was more challenging, however, was determining how to reduce the scaffolding support systematically over time so that students would be able to resolve reading challenges more independently at the end of the school year than they had at the beginning.

The teachers saw a need for students to discuss their ideas about the reading with peers and to be able to express their ideas in writing. However, for students to have more opportunities to speak and write necessitated a concomitant change in the teacher's role. The critical insight for several of the teachers was that they needed to shift the role of engaging in the interpretative and communicative tasks from exclusively shouldered by themselves to a shared role with their students. As one teacher said, "We're trying to find ways to help them be better readers, and understand what they read. ... Kids need to be able to read this, too, and be able to understand. And are we really doing justice if we're taking all this language and interpreting it and summarizing it for them? Where's their responsibility? Where is their independence? How are they going to build on it?" The teachers noted that the manner in which tasks were laid out in the curricular materials did not easily allow them to foster gradual independence with literacy or mathematics, but, rather, group investigations with peer and teacher support gave way too quickly to expected independent performance. Moreover, the teachers needed to support such student independence in a setting where many students are English language learners or have special education needs or have had very limited school-based experiences in communicating their mathematical thinking.

Discussion and Conclusions

We have attempted to capture the teachers' perspectives toward what it means to teach mathematical concepts when using curricular materials that place new demands on students for reading, writing, listening and speaking. While the teachers saw strengths in the reform-based materials in terms of literacy development, they also found that these materials presented significant literacy challenges for their students. Many of their students struggled with generating elaborated written responses to tasks; the readability and vocabulary were difficult for students. Many investigations assumed computational fluency that many students had not yet acquired. Initially the teachers responded to these challenges as barriers. They would routinely read and interpret the problem texts for the students and skip the writing tasks in the curricular materials. As we noted earlier, and want to emphasize, the development of the mathematical concepts in the curricular materials was regarded as rich, engaging and coherently developed across investigations. Moreover, the teachers' guides provided carefully thought out strategies that teachers could use to support the development of students' mathematical ideas and suggested clear connections across investigations and units to other mathematical concepts. However, the teachers' guides lacked similar support for the related literacy demands we faced in an urban setting.

The critical change that occurred for the teachers was the shift from seeing the literacy demands as barriers to students' mathematical learning to seeing the demands as affording opportunities to support the development of students' abilities to communicate mathematically. This came about as we collectively examined the literacy focus of reform-based lesson and began to develop strategies to explicitly and systematically support students in reading and writing mathematically. Hence we would argue that this literacy focus became a power setting for teachers to learn in and from their own developing practices as they engaged in making sense of these materials with a diverse student population.

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JUSTIFICATION AS A SUPPORT FOR GENERALIZING: STUDENTS' REASONING WITH LINEAR RELATIONSHIPS

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This study presents an empirically-grounded framework describing the types of generalizations students constructed when reasoning with linear functions. Seven 7th-grade pre-algebra students participated in a 15-day teaching experiment in which they explored linear growth in the context of real-world problems. Qualitative analysis of the data led to the development of a taxonomy describing multiple categories and levels of generalization. The taxonomy distinguishes between students' activity as they generalize, or "generalizing actions", and students' final statements of generalization, or "reflection generalizations". Results suggest that a major factor influencing students' ability to generalize in more sophisticated ways over time was engagement in acts of proof and justification. Three ways in which justifying supported the development of students' generalizations are identified and discussed.

Objective

Notions of school mathematics have recently expanded to emphasize the development of students' abilities to create powerful generalizations (Kilpatrick, Swafford, & Findell, 2001). For example, researchers have defined algebraic reasoning in terms of generalizing knowledge into systems for representing and operating on mathematical ideas (Blanton & Kaput, 2002; Steffe & Izsak, 2002). By focusing on generalization, educators can help students lift their reasoning to a level at which they can focus on the patterns, structures, and relations across cases. While this expanded view of mathematics merits investigation into the role that generalization plays in furthering students' understanding, studies investigating students' abilities to generalize suggest that they experience severe difficulty in creating and using appropriate generalizations (English &Warren, 1995; Lee & Wheeler, 1987, Stacey, 1989). Although the field has thoroughly documented students' errors, less is known about what students do understand to be general or how teachers can more effectively promote correct generalization. Therefore, the purpose of this study was twofold, consisting of: a) an examination of the types of generalizations students developed when investigating linear relationships, and b) an identification of the factors that supported students' abilities to generalize productively.

Theoretical Framework

Lobato (2003) developed a framework for transfer called the "actor-oriented transfer perspective", in which the researcher shifts from an observer's (or expert's) viewpoint to an actor's (learner's) viewpoint by seeking to understand the processes by which students generate their own relations of similarity between problems. Under the traditional view (described by Singley & Anderson, 1989), researchers predetermine what knowledge will transfer, rather than making that knowledge an object of investigation. This narrow definition restricts transfer by requiring that students demonstrate improved performance between tasks, which may prevent researchers from capturing instances in which students construe situations as similar but do not demonstrate increased performance. In contrast, the actor-oriented perspective allows the researcher to focus on what is salient for students. Attending to students' perceptions of

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similarity, regardless of their correctness, can aid in the attempt to identify supports for generalizing.

The study reported here extended the actor-oriented perspective to account for the different types of generalizations students created. Instances of generalization were sought by looking for evidence of students a) identifying commonality across cases, b) extending their reasoning beyond the range in which it originated, and c) deriving broader results from particular cases. Evidence for generalization was not predetermined, but instead was found by exploring how students extended their reasoning, inquiring into what types of common features students appeared to perceive across cases, and examining what sense students made about their own general statements.

Methods and Data Sources

The study was situated at a public middle school located near a large southwestern city. Seven 7th-grade pre-algebra students were selected to participate in a 15-day teaching experiment, in which the students met for 1.5 hours each day. All sessions were taught by the author, and were videotaped and transcribed. One purpose of the teaching experiment was to explore the nature and development of students' generalizations as they emerged in the context of realistic problems about linear growth. The sessions therefore centered around real-world situations involving gear ratios and constant speed. Gender-preserving pseudonyms were used for each student.

Analysis of the data followed the interpretive technique in which the categories of types of generalizations were induced from the data (Strauss & Corbin, 1990). Transcripts were coded via open coding, in which instances of generalization were initially identified as they fit the definition described above. Evidence was then sought to determine the apparent meaning of the generalization to the student, the basis for each generalization, and the factors supporting the development of each generalization. Emergent categories of types of generalizations were ultimately tested, modified, and solidified through multiple passes of the data set.

Results

Types of Generalizations

Tables 1 and 2 identify the types of generalizations students developed while reasoning with linear functions. The taxonomy accounts for multiple levels of generalizing and distinguishes between students' activity as they generalize, or "generalizing actions", and students' final statements of generalization, or "reflection generalizations."

Table 1: Generalizing Actions

<table>
<tr><td rowspan="4">Type I: Relating</td><td rowspan="2">Relating Situations: Forming an association between situations.</td><td>Connecting Back: Making a connection between a current and a prior situation. (i.e. realizing that "This speed problem is just like the gears problem!")</td></tr>
<tr><td>Creating New: Inventing a new situation considered similar to an existing one. "He walks 5 cm every 2 s. It'd be like a heart beating 5 beats every 2 s."</td></tr>
<tr><td rowspan="2">Relating Objects: Forming an associating between present objects.</td><td>Property: Associating objects by focusing on a similar property. "These two equations both show the ratio between x and y."</td></tr>
<tr><td>Form: Associating objects by focusing on their similar form. "Those equations both have division in them."</td></tr>
</table>

Type II: Searching	*Same Relationship*: Repeating an action to detect a stable relationship. (i.e., dividing y by x for each ordered pair in a table to determine if the ratio is stable.)
	Same Procedure: Repeating a procedure to test validity across cases. (i.e., dividing y by x without understanding what relationship is revealed by division; dividing as an arithmetic procedure to determine whether the resulting answer is the same.)
	Same Pattern: Identifying a pattern that occurs across all cases. (i.e., given a table of ordered pairs, noticing that the y-value increases by 5 down the table.)
	Same Result: Determining if an outcome is identical each time. (i.e., given $y = 2x$, substituting multiple integers for x and noticing that y is always even.)
Type III: Extending	*Expanding the Range*: Applying a phenomenon to a larger range of cases. (i.e., having discovered that the ratio of differences is constant in $y = \mathrm{m}x$ cases, applying this idea to $y = \mathrm{m}x + \mathrm{b}$ cases.)
	Removing Particulars: Removing contextual details to develop a global case. (i.e., describing linearity as constant rates of change rather than as constant speed.)
	Operating: Operating upon a mathematical object to generate new cases. (i.e., if y increases 5 units for every unit increase for x, halving the 1:5 ratio to create a new ordered pair.)
	Continuing: Repeating an existing pattern to generate new cases. (i.e., if y increases 5 units for every unit increase for x, continuing the 1:5 ratio to create a new ordered pair.)

Table 2: Reflection Generalizations

Identification or Statement	*Continuing Phenomenon*: Identification of a dynamic property. "For every second, he walks 3 cm." Or "Every time x goes up 2, y goes up 6."	
	Sameness: Statement of commonality or similarity about properties, objects, or situations.	*Common Property:* Identification of the property common to objects or situations. "For each pair in the table, x is a third of y."
		Objects or Representations: Identification of objects as similar. "Those equations both relate distance and time."
		Situations: Identification of similar situations. "This speed problem is just like the gears!"
	General Principle: A statement of a general phenomenon; a description of a rule, pattern, procedure, or global meaning.	*Rule:* Description of a general formula or fact. (i.e., writing $2s = b$ or explaining "you multiply s by 2 to get b.")
		Pattern: Identification of a general pattern. "Down the x column it increases by 5 and down the y column it increases by 7."
		Strategy or Procedure: Description of a method extending beyond a specific case. "To find out if each pair represents the same speed, divide miles by hours and see if you get the same ratio."
		Global Rule: Statement of the meaning of a mathematical idea. "If the rate of change is constant, the data are linear."
Definition	*Class of Objects*: Development of a class of mathematical objects all satisfying a given relationship, pattern, or other phenomenon. "Any two gears with a 2:3 ratio of teeth will also have a 2:3 ratio of revolutions.'	
Influence	*Prior Idea or Strategy*: Implementation of an existing generalization. "What worked with the speed problem will work on this gears problem. Just divide y by x and see if the ratio is the same."	
	Modified Idea or Strategy: Adaptation of an existing generalization. "Dividing y by x doesn't work on this other speed problem, but you could divide the increase in y by the increase in x instead."	

Students' generalizing actions fell into three major categories. When *relating*, students formed an association between two or more problems, ideas, or mathematical objects such as equations. When *searching*, students repeated a mathematical action, such as calculating a ratio or checking a pattern, in order to locate an element of similarity. Finally, students who *extended* expanded a pattern, relationship, or rule into a more general structure. When extending, one widens his or her reasoning beyond the problem, situation, or case in which it originated.

Reflection generalizations were students' final statements of generalization; they represented either a verbal or written statement, or the use of a prior generalization. Reflection generalizations took the form of *identifications or statements* of general patterns, properties, rules, or common elements, or *definitions* of classes of objects. Cases in which students implemented previously-developed generalizations in new problems or contexts were also categorized as reflection generalizations under the third sub-category, *influence*.

Many reflection generalizations mirrored generalizing actions. For example, *statements of sameness* often accompanied the generalizing action of *relating*, and *statements of general principles* often accompanied the generalizing action of *searching*. The actions of noticing similarity, generating analogous situations, or searching for similarity resulted in declarations of sameness or articulations of rules and principles. These declarations were the reflection generalizations, while the acts that led to them were the generalizing actions.

The development of the taxonomy provided a way to distinguish between students' generalizations on multiple dimensions, as well as discern how those types were related to one another. Results indicated that students engaged in cycles of generalizing, developing more sophisticated, powerful general statements over time. One of the major factors influencing the progression of students' generalizations appeared to be engagement in acts of justification and proof.

Justification as a Support for Generalizing

Students' engagement in acts of justifying, particularly in the form of explaining why phenomena occurred, appeared to support the development of more productive generalizing in three ways: a) students' focus shifted from attending to number patterns to attending to quantities and quantitative relationships, b) students began to generalize about relationships rather than patterns, and c) students developed new general rules and principles. A data episode is presented to illustrates these three modes of support.

The episode begins with the following situation: Gear A, which has 5 teeth, spins 6 times by itself before it is joined to Gear B, which has 8 teeth. The two gears then spin together. Students were asked to generate tables of pairs of rotations for A and B. Two students, Julie and Mandy, first determined that when A increases by 8, B increases by 5 – thus communicating the reflection generalization of the *identification of a general pattern*. This action then sparked a connection for Julie: "Just like the, the other one where the, remember when we go up by one-half over here and one-third over here? You can also go up by four, two and a half over here or five, eight over here." Julie engaged in the generalizing action of *connecting back* to a prior table that she had created, in which she had noticed that while the values in her table increased by one-half and one-third, the values in everybody else's tables had increased by one and two-thirds. This had been a powerful experience for Julie because she realized that her table was correct, but different, and that one can choose different-sized increments for increases down the columns.

Now Julie realizes that one can make a table in which the increases are either (8, 5) or (4, 2.5), and both would be valid.

The students ultimately compiled their individual and group results into a large class table:

Table 3: Class-produced $y = mx + b$ table of gear rotations

A	B
1	0
2	0
3	0
4	0
5	0
6	0
7	5/8
8	1 1/4
9	1 7/8
10	2 1/2
11	3 1/8
12	3 3/4
13	4 3/8
14	5

The students then engaged in the generalizing action of *extending by continuing* the pattern of increasing eight for A while increasing five for B to create the pair (22,10). This caused Julie to then *extend by operating* on this 8:5 pattern, explaining "You could also do 16 and 10, I think." By this remark, Julie meant that it would also be possible to increase 16 for A while B increases 10. When asked what she was doing to generate new pairs, Julie said "Doubling them." Students were then pressed to explain why doubling the "increase 8 : increase 5" ratio worked to generate new pairs:

Teacher: Now here is the 64,000 dollar question. Why does doubling work?

Julie: Because it does.

Timothy: Because it's going up by a trend on either one. When one column goes up by a certain trend, the other is going up by another trend.

At this stage, the students struggled to produce coherent, deductive arguments. In addition, their generalization and argument focused solely on the patterns in the table: "it's going up by a trend on either one". The students had ceased to think about the pairs in the table as representing gear rotations, and instead focused on the patterns alone. However, this failed to produce satisfactory arguments, and the students continued to try to explain their reasoning:

Teacher: Why does that work? Larissa.

Larissa: Because, because, if you spin A and it goes eight turns then B'll go through five turns. So if you spin A, if A goes through 16 turns B *has* to go through 10 turns.

Teacher: Why?

Larissa: Because both of them…because of the number of gears, since B is… because the teeth have to go off each other. But in one rotation, in two rotations, A will go 16 times and B will go 10 times.

The pressure to explain why doubling worked pushed Larissa to shift her attention from the patterns in the table to the quantities of gear rotations, and the relationship between the rotations of the two gears. Larissa appeared to understand that turning the first gear another eight times to make 16 total turns would force the second gear to turn another five times, making a total of 10 turns. However, she struggled with articulating this idea clearly. Pushed to explain one more time, Larissa said:

Larissa: Since the…because the rotations, when every time A goes through eight teeth, no it spins – eight times, B spins five times.

Timothy: Because you, because as long as you're subtracting the six, you have to remember that. It's always going to equal, B is 5/8 of A. Every single time.

Through her attempts to explain why, Larissa created a new generalization: every time A spins eight times, B spins five times. This was a generalization that attended to the relationship between the rotations of the gears, rather than a generalization about the pattern in the table. Now Larissa could convey an image of continuation, noting what happened "every time"; thus her explanation contained the reflection generalization of a *continuing phenomenon*.

The students' struggle to explain also sparked an idea for Timothy. He produced another reflection generalization that had not previously been stated, a *general rule* that B will always be 5/8 of A once six is subtracted. The students' attempts to explain why doubling the increases worked, combined with their subsequent shift to thinking about the quantities of gear rotations, could have encouraged Timothy to focus on the direct multiplicative relationship between the rotations of the gears. He now made a statement about the ratio of A to B, when before the students had not focused on this ratio because it was obscured in the table.

This brief excerpt illustrates a larger result identified through the course of the teaching experiment. Specifically, engaging in acts of justification, particularly those that made use of deductive reasoning, was connected to the subsequent occurrence of two major types of reflection generalizations: statements of general principles such as algebraic or global rules, and statements of continuing phenomena. Furthermore, these relationships did not appear to be uni-directional. Students did not frequently produce a generalization, justify it, and then move on. Instead, the act of justifying itself appeared to push students' reasoning in ways that encouraged further generalizing.

Discussion

While the generalization taxonomy developed from this study represents only what occurred in a setting with 7 middle-school students studying linear functions, it sheds light on students' mental actions in a way that is not limited to describing specific strategies bound to specific types of problems. Instead, the taxonomy constitutes a framework that enables researchers to examine the origins of a student's generalization, what a generalization is about, how one focuses his or her attention while generalizing, and what type of actions one engages in while generalizing. More importantly, the study's results move beyond casting generalization as an activity at which students either fail or succeed in order to focus on what learners see as general. This move away from a success/failure model towards a more nuanced view of generalizing could support researchers' attempts to better understand students' learning processes as they generalize.

By categorizing generalizing actions and reflection generalizations into different types, it was possible to identify and track the changes in students' generalizing over time. As students engaged in the production of increasingly sophisticated generalizations, one activity that repeatedly preceded a shift in the nature of students' generalizing was the attempt to justify their results. By engaging in justification, particularly through the act of explaining why, the teaching-experiment students were able to subsequently generate more powerful generalizations. This suggests that the role of proof could be considered a support for more productive generalizing.

Current instructional recommendations encourage teachers to set up situations in which students can detect patterns from data and then generalize those patterns in increasingly formal ways (NCTM, 2000). Such recommendations embody an assumption that these types of generalizing activities will constitute a sufficient support for producing appropriate proofs. However, the research examining students' abilities to appreciate and produce proofs demonstrate that students struggle to appropriately justify their generalizations (Koedinger, 1998; Knuth et al., 2002; Usiskin, 1987). Results from this study suggest that a more productive approach may actually reverse the generalization/proof sequence. Students may initially begin generalizing in fairly shallow or unproductive ways as they struggle to make sense of quantitatively-rich situations. Rather than pressing students to produce correct and formal generalizations before moving onto proof, teachers might consider incorporating justification into the instructional sequence at this stage. One could capitalize on the role of justification as a support for more productive generalizing, which in turn can encourage the development of appropriately deductive arguments.

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COMPOSITION OF FUNCTIONS: PRECALCULUS STUDENTS' UNDERSTANDINGS

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The concept of function is central to most college algebra and precalculus courses; however, past studies reveal that even students who successfully complete these courses possess a weak function conception (Carlson, 1998; Dubinsky & Harel, 1992; Monk, 1992; NCTM, 2000; Sfard, 1992; Vinner & Dreyfus, 1989). This study provides new insights about how students understand and think about function composition at the completion of a course in precalculus. The process of validating and administering the Precalculus Concept Assessment (PCA) instrument provided a rich source of data. These data revealed specific function conceptions that were associated with flexibility in responding to function composition problems utilizing multiple representations.

Background

The concept of function has been studied extensively (Carlson, 1998; Dubinsky & Harel, 1992; Even, 1992; Monk, 1992; Sfard, 1992; Vidakovic, 1996; Vinner & Dreyfus, 1989); however, there has been little focus on what students understand about function composition. Breidenbach, Dubinsky, Hawks, and Nichols (1992) and Dubinsky and Harel (1992) proposed four levels of understanding for the concept of function: pre-action (little, if any, understanding of the concept of function), action (restricted to actual physical or mental operations on specific numerical values), process (involving a view of an entire transformation of quantities independent of any procedure, including coordinating multiple functions as required in composition and an ability to reverse the function process), and object (whereby a function is seen as a concrete entity on which other actions may be performed). Sfard (1992) reported that students have difficulty solving composition problems in which explicit formulas are not given.

Since a process view of functions involves both reversibility and coordination, composition is closely related to inverse functions developmentally. Vidakovic (1996) claimed that

> Subjects with schemas for composition of functions and inverse function can coordinate them (coordination of a higher order) to obtain a new process. For example, this coordination explains why the inverse of the composition of two functions is the composition of the inverses of two functions in reverse order. (p. 310)

While many authors have noted that composition and inverse problems are challenging for students and these problems are likely tied to a weak function conception, student understanding of function composition has not been a primary focus of study.

Methods

The 25-item *Precalculus Concept Assessment* (PCA) instrument was developed from a pool of 78 items, each grounded in and developed from the research literature (Carlson, 1998; Dubinsky & Harel, 1992; Monk, 1992; Sfard, 1992; Vinner & Dreyfus, 1989). Each item in the pool has been systematically tested and validated to assure that students who select a specific distracter (i.e., answer choice) have consistently provided similar justifications when explaining why they chose a particular distracter. We achieved this by alternating multiple revisions with clinical interviews of students who had previously completed each PCA item in writing. In our

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first round of interviews we administered open-ended items to students. We then classified the common errors and reasoning abilities that led to the various answers to construct five distracters for each item. In subsequent interviews, students were prompted to explain the reasoning and procedures that led them to their answer for each item. The items and corresponding distracters were revised until all wording was interpreted consistently with the design intent and all distracters were viable responses. Validation of the PCA items consistently revealed that students who select the correct answer were able to provide a valid justification for their response. The interview data collected during the instrument validation generated new distracter choices for the PCA items, while also providing a rich source of data for investigating the common understandings, errors, and reasoning abilities of students relative to specific PCA items.

The instrument was then administered to large numbers of students at the completion of a precalculus or college algebra course (Version G, n=1196, and Version H, n=652). Analysis of this data revealed patterns in students' responses on both individual items and item clusters. Follow-up interviews with students on the composition and inverse items provided insights into the reasoning patterns that supported various student responses. Select data from this analysis is reported.

Results

The mean score on version H of PCA, which was administered to 652 precalculus students at the end of their course, was 9.6 (out of 25). In general, student performance on the questions that address the concepts of composition and inverse was poor. Our results are summarized in the following table:

Precalculus Data		**Composition (% Correct)**					**Inverses (% Correct)**			
Version	N	Algebraic Formula	Graph	Table	Square Context	Circle Context	Algebraic Formula	Graph	Table	Definition
G	379	91%	45%	50%	20%	9%	8%	34%	13%	15%
H	652	94%	43%	41%	25%	17%	20%	35%	15%	16%

When students were given the algebraic expressions $g(x) = x^2$ and $h(x) = 3x - 1$ and asked to evaluate $g(h(2))$, 94% of the students responded correctly. The interview data supports that these students were able to execute the standard "plug and chug" approach to composition and function evaluation in general; that is, they were able to apply an algorithm to solve the problem with a limited understanding of what they were doing. Two additional questions asked students to evaluate a composite function using a table or graph. In these instances, 43% of the students answered correctly using the graph and 41% answered correctly using the table. Only 27% of students answered both correctly and 43% answered both incorrectly. The interview data supported that students who responded incorrectly were more likely to use language that suggests an action view of function while students who answered correctly were more likely to use language that suggests a process view of function (Dubinsky & Harel, 1992).

Most students, including those with only a mechanical, algorithmic understanding of functions, correctly answered the algebraic formula problem presented above. The following student's response was typical:

Oliver: So, I put the x value of 2 for $h(x)$, so $h(2) = 5$. Then you put 5 in for $g(x)$, which is 25 and that's the answer.

For tasks that involved composing functions defined by a table or graph, this level of understanding appeared to be insufficient. Example 1 shows how students interpret function composition given functions in tabular form, and Example 2 shows how students interpret function composition given functions in graphical form.

Example 1: Given the table to the right, determine $f(g(3))$.

a) 4 [41%]
b) -1 [27%]
c) 0 [13%]
d) 1 [16%]
e) 5 [2%]

x	$f(x)$	$g(x)$
-2	0	5
-1	6	3
0	4	2
1	-1	1
2	3	-1
3	-2	0

Some students found the input value in the $f(x)$ or $g(x)$ column and then found the output in the x column; they were reading the table backwards. The transcripts below illustrate the justifications three students gave for determining that $g(3) = -1$ and $f(-1) = 1$ for their selection of d as their answer.

Carrie: First, I need to find *g(3)*, oh, sorry, *g(3)* and then the x value is -1, so I plug in -1 for *f(x)*, then I get 1, so d is my answer.

Diana: Yeah, I went down to g of 3 [points to the 3 *g(x)* column], plugged it in and got x equal to one, to minus one [points to the -1 in the x column]. So then I had minus one in place of the g of 3, and then I said f of minus one and found it equal to one [runs her finger down *f(x)* column to -1 and then over to the 1 in the x column].

The value 1 was not among the original distracters. Due to the interviews, we replaced a distracter that was chosen by at most 10% of students with this new distracter. Subsequent tests resulted in 16% of students choosing 1 as their answer, and we further validated this distracter through interviews to ensure students who chose it were reasoning as described above.

When provided with the table, several students actively looked for a formula to describe the function. In the excerpt below, Gina initially appeared to search for a formula, but when her attempt failed, she reinterpreted the composition as multiplication.

Gina: I was trying to look for a formula to see what *f(x)* was but it's not working so I'm doing it wrong. That's not the right way to go about it… *g(3)* is 0. f of g of 3. Now I'm tempted to say it's f of g times *g(3)*, but I know that's totally wrong, but if I did do that, -2 times 0 would be 0. And I know that *g(3)* would be 0, so no matter what my answer is 0, even if I do it wrong. I don't know what, yeah, I'm just gonna say 0 because…

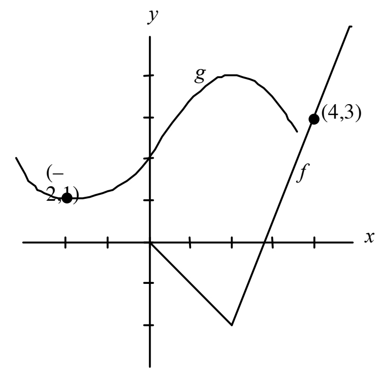
While Gina correctly read input and output from the table, she struggled with the notation of function composition. It was not uncommon for students to interpret the composition notation as multiplication. When students correctly answered the question in Example 1, they used language that suggested they had a process view of function. This is illustrated in the following excerpt:

Eddie: And for this one, I found where 3 was [pointing to 3 in x column], found g [pointed to 0 in g column] then plugged that into f [pointed to 0 in x column, slid over to 4 in f column] and found 4.

When faced with the graphical version of this question, students had two primary difficulties: 1) interpreting the input and output from the given graph, and 2) coordinating the output of one function as the input to another function. This item appeared on Version B (an early version) as

Example 2: Evaluate $g(f(2))$.

a) -2 [8%]
b) 1 [24%]
c) 0 [9%]
d) 2 [19%]
e) Not defined [27%]



The data suggested that 0 and -2 were not good distracters since only 8-9% of students chose them, and 13% of students did not answer the question. It was also noted that in some interviews, students were attempting to locate the maximum of *g* or the minimum of *f*. This led to our labeling these points on the graph. An additional distracter of 4 emerged from our interviews. Students often found $g(f(2)) = 4$ by finding $f(2)$ on the graph then moving their finger up until it intersected the graph of *g*, noting that the y-coordinate at this point is 4. Alan was interviewed after taking Version B:

Alan: Ok, using the graph, answer items 29 and 30, evaluate *g* of *f* of 2. Knowing what we do now, I selected negative 2 as the answer, *f* of 2 is, *f* of 2 is negative 2, *g* of negative 2, would be e, not defined, is that right? no. Can you tell I'm guessing? … Ok, so, *f* of 2 is negative 2, that we know, so what's *g* of negative 2? Negative 2, unless I'm reading this backwards, which would mean, what's the *g* of *f* of 2, no, that's not the answer because 4 isn't an option. I don't think I am reading it backwards.

This led to our replacing 0 (which had been chosen by 9% of students) by 4 (subsequently chosen by 28% of students). In the next round of interviews, Fred and Gina provided statements that confirmed the new distracter is more representative of students' thinking.

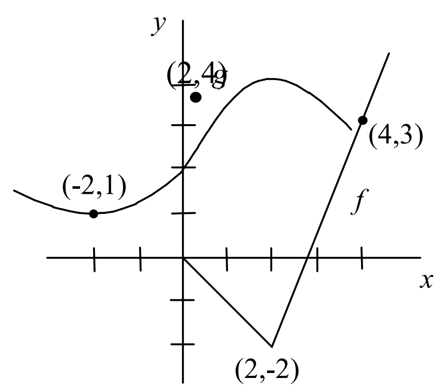
Fred: Ok, *f* of *x* is 2, and when *g* of *x* is 2, *y* equals 4.

Gina: ….[draws a line from (2,-2) up to *g* at (2,4), then over to y-axis] *g(f(2))*, 4. You look at the graph and you see it there.

Thus, Fred and Gina's responses supported that Alan's response of 4 is a common response. The item now appears on the current version of PCA as

Example 2: Use the graphs of *f* and *g* to evaluate $g(f(2))$.

a) -2 [10%]
b) 1 [43%]
c) 3 [6%]
d) 4 [28%]
e) Not defined [13%]



When students answered this item correctly, interviews confirmed that they were able to coordinate the input and output of the functions while recognizing that the x-axis represents the set of inputs and the y-axis represents the set of outputs. The following students' responses represented typical reasoning:

Eddie: I found *f* of 2 and it was kinda hard to see that on the graph, and so the output was -2, the y-value was -2, and then I found where *g* was negative -2 and that was 1.

James: Um it's asking for *g* of *f* of 2, so first you find *f* of 2 which happens to be negative 2, then you find *g* of negative 2, which is 1

Keith: So I'm going to go 1,2 and my *f* of 2 is going to be negative 2 ok and uh now my *g* of negative 2 is going to be um 1, yeah

Eddie nicely articulated that the y-value was the same thing as the output. For these students, it is apparent that they have a process view of function composition that allows them to coordinate the two functions using the output of *f* as the input for *g*.

There are also two questions that imbed composition in the context of the problem. One asks students to solve for the area of a square in terms of its perimeter (the square problem), and the other asks students to solve for the area of a circle in terms of an increasing radius expressed as function of time (the circle problem). Students performed significantly worse on these problems compared to the previously discussed composition problems. Of these students, 25% could answer the square problem correctly and 17% could answer the circle problem correctly. This decrease in success when solving contextual composition problems suggests that students do not have a fully developed process view of function (Dubinsky & Harel, 1992). Interviews revealed that students could not see one function as providing an input to the other function.

To further investigate the students' understandings, consider the circle problem:

Example 3: A ball is thrown into a lake creating a circular ripple that travels outward at a speed of 5 cm per second. Express the area, *A*, of the circle in terms of the number of seconds, *s*, that have passed since the ball hit the lake.

a) $A = 25\pi s$	[21%]
b) $A = \pi r^2$	[7%]
c) $A = 25\pi s^2$	[17%]
d) $A = 5\pi s^2$	[34%]
e) None of the above	[20%]

Distracter d was chosen by 34% of students and distracter e was chosen by 20% of students. The most common reason given for choosing d is that there has to be a 5 in the answer because that number was given in the problem. The following transcripts reveal typical student reasoning:

Alan: Um, radius is gonna be expanding, we can't use *r* because it's expanding, so we have to plug *s* in there, um, because we want it as a function of the number of seconds, *s*, which eliminates, *r* squared is the area of the circle, but the area is expanding, 5 cm per second so we need to have the 5 multiplier in there.

Ben: I'm going to guess it's d because pi *r* squared is the area of the circle, but I'm not going to choose that because there's a 5 they gave me for some reason, and I'm going to want to use it. So, I'm going to guess d, not e. D's the only one with a 5 in it. The other options I would have gone with were either b [now distracter a] or c, but those have 25 instead of 5. [later] I don't see where they got 25 from, but they could have squared 5. I don't know why they'd do that. So, I'm just going to guess d.

There were a small number of students who had set up the proper equation but neglected to square the 5 in the quantity $(5s)^2$ when computing their answers. This was not the usual reasoning that was given by students who chose d as is evidenced by the above statements. The distracter e was created as a "catch all" for the numerous solution methods students applied to obtain a variety of other answers.

Additional evidence of students being restricted by an action view of function comes from questions pertaining to inverses. From the developmental point of view, understanding of composition of functions and inverses are closely related. When students have a fully developed process view of function, they think of something that accepts inputs and produces outputs, but also realize that the process may be reversed. They are able to interpret function notation involving inverses, such as $f\left(f^{-1}(x)\right)=x$ and $f^{-1}\left(f(x)\right)=x$, in terms of "doing" and "undoing." The PCA has three questions that directly address understanding of function inverse. In one, students are asked to compute an inverse at a specific value from a table of data. The second question asks students to determine the inverse of an exponential function. The third requires being able to recognize the definition in a written statement. There is also a question that asks students to compute the inverse indirectly by asking them to solve for x given $f(x)=-3$ and a graph. Students had the least difficulty computing the inverse on this item; 35% of students answered correctly. On the items that used inverse notation, students did not perform as well. When using the table, 15% of students answered correctly, 20% answered the exponential question correctly, and 16% answered the definition question correctly. Only 1% of students correctly answered all three of the questions that used inverse notation. It is not surprising that the question the students performed best on was the one in which they could apply an algorithm to find the inverse function: Switch the x and y, solve for y, and label with $f^{-1}(x)$.

The most common error in all of these questions was for students to view the negative one in $f^{-1}(x)$ as an exponent and thus invert the expression or value multiplicatively. Let us now consider Example 4.

Example 4: Which of the following best describes the effect of f^{-1}, given f is a one-to-one function and $f(d)=c$?

a) f^{-1} inverts f, so $f^{-1}(d)=\frac{1}{f(d)}$ [11%]

b) f^{-1} inverts the input to f, so. $f^{-1}(d)=\frac{1}{d}$ [8%]

c) f^{-1} inverts the output to f, so $f^{-1}(d)=\frac{1}{c}$ [31%]

d) f^{-1} inverts f, so $f^{-1}(f(d))=d$ [16%]

e) a and c [33%]

Interview data indicated that students confused the inverse of a function with the reciprocal. How well a student understood input and output of a function often determined whether they saw distracters a and c as the same thing. John is an A student in calculus who chose distracter c. The following transcript from John's interview shows that he not only interpreted the negative one to mean reciprocal, but he also missed that distracters a and c were saying the same thing. John also could not distinguish between distracters b and c.

John: I don't remember um but that would go with, that would go, that would reflect, that would invert *f* so that *f* inverse of *d* would be 1 over *d*, I would think, I'm not sure.

Int: Does this relate at all to the other 2 inverse problems you've already done?

John: I don't know, maybe um if we put *f* of *c* is equal to *d* (mumbles something) well if I treat it the exact same way I treated the other 2 inverse problems it would be b or c.

Interview data supports the interpretation that this notational confusion is nearly universal for students who cannot reverse the function process or do not understand the roles of the input and output of a function.

The above composition and inverse problems illustrate that students have difficulties moving between function representations and interpreting function notation. Most precalculus students appear to have different procedures in place for solving each type of problem rather than a general coordination of inputs and outputs for the concept of function composition. The language and notational aspects of the negative exponent and function inverse appear to be very difficult for students to conceptually differentiate.

Conclusion

The PCA instrument provided new information about student function conceptions and difficulties when confronted with function composition problems. Analysis of students' scores on individual items revealed that most precalculus students at the end of the course possess an action view of function and are not adept at moving between function representations. This conclusion is supported by the fact that students were able to evaluate functions at specific points, but were generally unable to provide a correct response when confronted with questions that required a process conceptualization of function (a view of function as something that accepts inputs and produces outputs).

The PCA data revealed that composition problems which require students to determine the value of the composition of two functions at a point, given functions that are defined by algebraic formulas, required only an action view of function. It is noteworthy that more than 90% of the students in our study provided a correct answer to this question while less than 50% of the students provided a correct response to function composition problems in other representations. Interview data supported that only when students were asked to evaluate a composition problem utilizing a given table, graph, or context were they required to engage in process-level function thinking. Students with only an action view of function had difficulty distinguishing between the input and output of a function when the data was presented in a tabular form. This was revealed by their strong tendency to interchange the input and output when computing the composition of two functions for a specific input value, given that the functions were defined by a table. While these students had no difficulty using a graph to evaluate the output of the function for specific input values, their impoverished function view did not appear to support the reasoning patterns needed to compose two functions when given two function graphs. These students were also unable to reverse the process to evaluate a basic inverse question in which they were asked to determine the input when given a specific output and a graph.

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DESIGN OF DYNAMIC VISUAL SITUATIONS IN A COMPUTATIONAL ENVIRONMENT AS A SETTING TO PROMOTE THE LEARNING OF FUNDAMENTAL CALCULUS CONCEPTS

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The impact of technology on the teaching and learning of mathematics has challenged the principles and paradigms we used traditionally to design educational activities in mathematics. In this context, it is relevant to ask, how can dynamic environments help students overcome recurrent calculus difficulties that emerge in traditional or static instruction? What kind of dynamic environment activities should we devise to help students overcome such difficulties and also enhance their learning of fundamental calculus concepts? I investigated these questions by designing a set of dynamic situations in a computer simulation which involved basic concepts of calculus (function, rate of change, first derivative, maximum and minimum points, second derivative, concavity changes and inflexion points).

Introduction

Considering that the computing aspect does not represent difficulties anymore, it is pertinent to ask what kind of learning should we pay attention to. The National Council of Teacher of Mathematics (NCTM, 2000) points out that "When technological tool are available, students can focus on decision making, reflection, reasoning, and problem solving…Technology should not be used as a replacement for basic understanding and intuitions; rather, it can and should be used to foster those understanding and intuition" (page 24-25). In this context, the research work addressed to study the following type of issues: What is the role of dynamic environments to favor the above mentioned cognitive processes? At what level do these activities in virtual environments favor learning of fundamental calculus concepts?

Conceptual Framework

Kaput et al. (2002) state that technology is a representational infrastructure which develops new ways of thinking and reasoning. In the past, mathematical learning was carried out only by static or inert representations (paper and pencil, blocks, etc...). One essential characteristic of these media is that it does not react to the students' actions on the representations of the mathematical objects. In contrast, in an interactive medium, the students receive "answers" to their actions. According to Balacheff and Kaput (1994), this intrinsic property of computational media has deeply modified learning in mathematics.

In its origins the aim of calculus was to study the variation or change of natural phenomena; however, it was learned by means of static representations: graphic and / or algebraic symbolism, which did not highlight the dynamics that physical phenomena studied. Nowadays, modern technology has created the conditions to dynamically visualize such essential characteristic, to be able to manipulate the mathematical objects and, at the same time, see the effects on these actions in the dynamical representations shown on the computer screen. This framework was the basis to design a series of activities with computational media that simulate different settings in which a tank is filled and / or drained. The tasks were designed by a computing program that will be described later on. In order to evaluate the students' learning process, during their interaction

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with the activities, I focused on the degree and the strength of connections that students established for several concepts and the way they used these connections to interpret different settings.

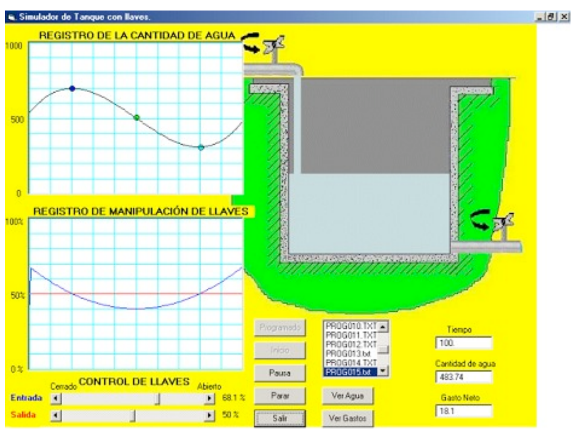
Methodology and Procedures

During the study, we videotaped and observed in deep the processes of a couple engineering students working together. Because our main interest was to observe the qualities of the behaviors that the students showed in the resolving of these tasks, we adopted a methodological approach qualitative in nature. The main objective was to observe the behaviors and difficulties shown by students when they interacted with the dynamic settings. The students were selected out from a group of ten who had already taken a Calculus course by means of a diagnostic test. This one contained the concepts involved in the tasks that were used in the research study. To illustrates the types of questions included in the test. We provided an example in which the students were asked to explain their answers:

"If **y = f(x)** is a decreasing function, then the concavity of its graph goes down. Is this right or wrong?"

We created computational software that simulates different dynamical situations related with in a tank in which we fill and/or drain water. Figure 1 shows the principal parts of this *simulator* software. Using the *mouse*, the tank is filled or drained by means of two scroll bars. One of them controls the inflow and the other one the outflow. The simulator includes twenty one programs in the *List Box* (see fig.1). Each one of them simulates different dynamical settings of the filling and /or drain water in the tank.

Figure 1



Now I will describe how the simulator works. When the inflow tap is more open than the outflow tap, we can see how the volume of the tank increases, while it decreases if the inflow tap is less open than the outflow tap. While we manipulate the scroll bars, two graphs are generated on the screen. In the top window we see a graph that represents the volume of water in the tank; in the lower one, there appears the graph which represents the handling of the taps (rate change).

When the graph is positive, it means that the inflow tap is greater than the outflow, when the graph is zero, it means that both taps are in the same position in the scroll bars and therefore, the graphs reach a minimum or maximum point. If the graph is negative, the inflow tap is more shut than the outflow. When the first derivative reaches a minim point, it means that inflow tap passes from being shut to being to being opened (or vice versa), which represents an inflexion point in the graph of the volume.

Before the students began to solve the tasks, they were given a time period to "play" with the simulator in order to become familiar with the features of the program. We designed twelve situations, which the students were exposed to. To illustrate one type of activities used, we present the following situation: “Suppose that at one moment the inflow is less than the outflow; then, for a certain period, the inflow tap starts to be opened at a constant rate until it is greater than the outflow tap. Later on, the inflow tap is again shut at a constant rate until it returns to the original position”.

The students were asked to select a program from the *List Box*. The selected one only showed the movement of water in the tank, but not the two graphs. The students' task was to make an initial verbal description and to sketch a graph that represented the behavior of the volume. Then, they had to answer the following questions:

What happened to the volume of water while the inflow tap was being moved?

What happened to the volume of water when both taps were in the same position?

What happened with the speed of the volume of water at the moment when the inflow tap passed from being opened to being closed?

What happened with the speed of the volume at the moment when the level of water was increased?

When the students finished this task, they were asked to select another program in the *List Box* which showed the same situation, but now showing the two graphs .One representing the volume and the other one the handling of the taps (first derivative). This allowed students to verify their answers. This interactive aspect, served as a feedback for students. Based on these two graphs, the students were requested to state what relations they could identify between the two graphs. We wanted to observe, for example, if they noticed the maximum or minimum points of the first derivative and if they associated the inflexion points of volume of the graph and what happens with the handling of the taps, before and after these points. We must point out that the students worked alone without the participation of the investigator, because we wanted to notice the behaviors with not external influence

Presenting the Results

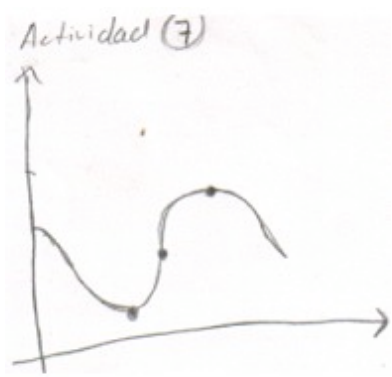
The students' behaviors of the students in each of the twelve situations were analyzed in order to identify the most outstanding tendencies, difficulties or misconceptions showed by the students in the learning process. In the initial activities, the students showed the following behaviors: In a situation in which the taps were supposed to be open and fixed, but, at the same time, the inflow tap was more opened, we asked the students “What happens with the volume of water in the tank?” The students replied that “it was filling”. It was a quite general answer and it seemingly did not catch their eyes that the taps were fixed, which is interpreted as the volume increasing constantly.

In the next task, in which it was assumed that the two taps were open but not fixed, and the inflow tap is more open than the outflow tap, but now the volume, got into the tank more quickly, the students exhibit the same pattern: “The volume is increasing”. In general terms, they were right, but they still had not noticed the changing characteristics, that is, the students didn’t pay attention to the changing volume rate. When the students created the graph, they drew a quite steep growing straight line. Therefore, at this initial stage, they did not realize that, at the same time, there is a constant of water that will affect the graphic form.

Another tendency exhibited in this initial stage was when students interpreted that the volume was constant at the maximum points or at the minimum or inflection points. For example, “there will be a moment [inflection point] in which it will be constant. However, as the process advanced, the students’ behaviors began to show advances in the conceptual understanding and to overcome some difficulties. In order to support this affirmation, we presented evidences of this progress. We were based on an intermediate activity that was previously used as an illustration.

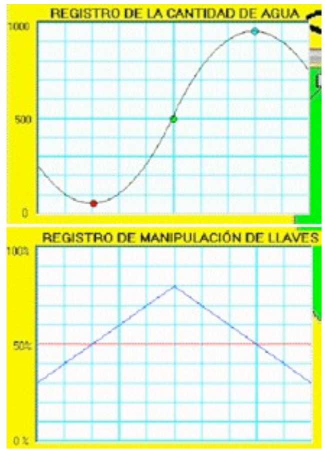
This task demanded to identify the maximum, minimum and inflexion points, as well as the changes of concavity. In addition, the situation demanded to pay attention to the change in the rate volume. When the students ran the list box in the program that simulates a given situation, but without showing the graphs, the student A says: "First, the volume decreases at a constant speed; there is a point in which it stopped [a maximum point] and then it increased, but at a growing speed". Please note that the students pays attention to how the volume changed, fact that was not perceived in the initial activities. When the students began to approach the vicinity or inflexion zone point, difficulties to interpret what happens there emerged. For example, their explanations became hesitant: Student B: "It also begins to increase uniformly”. Student A also expressed the same insecurity with an incomplete idea: "It begins to close uniformly and then...” In this episode, this student exhibited a false conception: "Since [the volume] keeps uniform, then the speed remains uniform, doesn’t it? Nevertheless, student B shows his disagreement with this interpretation: "Speed does not remain uniform; that is why the speed increases faster and faster". It is during these dialogues that the student realizes what was happening to the behavior of the volume at this point: "Because there must be an inflexion point… it is when the tap is being shut again .." and emphasizes: "here you are opening it and here you are shutting it again”.

Figure 2



Once the students provided the verbal description, they drew a very acceptable graph representing what was happening with the volume in the tank (see figure 2). This graph shows the essential features of the dynamic situation, for example, underlining the minimum, maximum and inflexion points as well as the concavity changes. But the most important finding was that the aspects of the graph were associated with the manipulations of the taps. The students ran the program and saw the graphs shown in figure 3.

Figure 3



Based on in these graphs, the students established the following connections: "we noticed that the blue line [change rate] is an indication of the slope of the curve [volume graph]; because of this, both the blue line and the slope increased; the blue line reached the zero value, started to increase and again reached zero... After this, it decreased" (student B).

On the other hand, we must point out, that both students showed a limited concept of the *change* idea. For example, when the students saw a graph that represented the volume of water, in general, they could say in which parts of the graph the volume increased, decreased; that is, on these parts they conceived the variation of the volume like a *fluent*.

However, when they were asked what happened at specific points (minimum, maximum or inflexion points) their point of view about the *change* became *static*: “the volume is constant or there is a point in which they ‘stop’ ”

Conclusions

As a result of the above evidences, we think that the designed set of computational media activities has the potential to promote students' understanding of basic calculus concepts. However, we cannot affirm categorically that the learning advancements shown by students are due only to one factor: The dynamic situations in the computer program. We think that the students’ progress is a combination of various interactions (verbal, written and dynamic representations) which aided the students' learning process, in spite of the misunderstandings that emerged here.

Although the students' learning misconceptions arose, we can see that further study is needed to see if the researcher’s direct intervention might help the students overcome obstacles by themselves.

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THOSE ALREADY LEFT BEHIND: LOW ACHIEVING HIGH SCHOOL MATHEMATICS STUDENTS GET ACTIVE

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This study investigated the implementation of a meaningful and challenging curriculum in conjunction with increased personal teacher-student interaction for teaching mathematics to low achieving mathematics students, focusing on changes in their educational achievements and motivation. The study addressed factors that contributed to change in educational achievements and motivation. Achievement was influenced by collaborative work, opportunities to communicate, and assessments that valued student effort. Increased achievement was noted by investigating perseverance, retention rates, placement in a higher mathematics class in the next year. Factors that contributed to a change in motivation included the use of real-world activities that built on students' interests, and students' enjoyment of mathematics.

Purpose of the Study

On January 8, 2002, President George W. Bush signed the No Child Left Behind Act. This legislation was designed "to ensure that all children have a fair, equal, and significant opportunity to obtain a high-quality education and reach, at a minimum, proficiency on challenging state academic achievement standards and state assessments" ("No Child Left Behind," 2001). This goal for "fair, equal and significant" learning opportunities is championed by the National Council of Teachers of Mathematics (NCTM) in their Equity Principal in the *Principles and Standards for School Mathematics* (2000): "All students need access each year they are in school to a coherent, challenging mathematics curriculum that is taught by competent and well-supported mathematics teachers. The unfortunate reality, however, is that many students, by the time they entire high school, are already behind their peers in their learning of mathematics. These students are often grouped into low-level mathematics classes, causing a "dead-end effect" (Gamoran et al., 1997). Furthermore, the teachers of these classes often believe that these students are incapable of higher order thinking, thus the teachers tend to ask fewer questions requiring students to develop those skills (Zohar, Degani, and Vaakin, 2001). Rather, these classes are dominated by "low quality teaching…characterized by teachers' low expectations…valuable time spent on managing students' behavior, and most class time devoted to paperwork, drill and practice" (Linchevski and Kutcher, 1998).

The study addressed in this paper investigated low achieving high school mathematics students in a specially designed class entitled "Explorations in Algebra" in a mid-western city. The class was designated for students who experienced "great difficulty" in their 8th grade mathematics class and "was not intended for students who plan to attend either a community college or a four-year college" by the district course description. Straying from the structure of the previous years, the students were given no worksheets, were encouraged to interact, and were graded heavily on their participation and effort. For the 2003-2004 school year, the design of the course was one that was rooted in activities, real world applications, and cooperative learning. The class incorporated a meaningful and challenging mathematics curriculum that was designed by the researcher and the teacher of the class. The curriculum emphasized real-world activities

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and cooperative learning. In addition, the study included increased personal interaction between the teacher and the students. Every student in the school was randomly assigned to a homeroom for fifteen minutes each day over the lunch hour. Each of the students in "Explorations" was assigned to the same homeroom, which was headed by the teacher of the class. This time was largely unstructured and allowed the teacher to interact informally with the students daily.

This paper addresses the issue of low achieving high school mathematics students and their educational achievements and motivation by addressing two research questions:

1. What factors contributed to a change or a lack of change in the educational achievements of low achieving students during the implementation of a meaningful, challenging mathematics curriculum in conjunction with increased teacher-student interaction?
2. What factors contributed to a change or a lack of change in the motivation of low achieving students during the implementation of a meaningful, challenging mathematics curriculum in conjunction with increased teacher-student interaction?

Theoretical Framework

Because this study focused on the individual experiences of the students as well as the overall classroom culture, the underlying framework was the emergent perspective. This framework "involves the explicit coordination of interactionism and psychological construction" (Cobb & Yackel, 1995, p. 6). This allowed for a lens to interpret both the psychological and the social phenomena in the study at the classroom level.

In addition, the study extensively used the expectancy times value motivational theory (Wigfield & Eccles, 2001). This motivational theory suggests that motivation is the product of two factors: whether a student expects that they can accomplish a task, and whether the student values that task. If either aspect is missing, the overall product is zero. The expectancy times value model incorporates the wide variety of factors that influence achievement motivation, including past experiences, beliefs, aptitude, the influence of others and their beliefs, self concepts, and goals. This theory was used to interpret student motivation, but also influenced lesson design.

By using these two complementary theoretical perspectives, the study could address the classroom as a whole, the students' perspectives within the classroom, and the students' participation and reasons for that participation in the classroom. Using the emergent perspective, the examination and interpretation of the classroom from both the social and psychological aspects, investigating the students and teacher as a collective group and as individuals was possible. The expectancy times value theory provided a framework for interpreting why students would choose to participate in class or not. Using these two perspectives in conjunction with each other allowed for a comprehensive view of the classroom, the development of the classroom environment, and the students themselves.

Method of Inquiry

Stake (2000) describes that in an instrumental case study, the case is studied "to provide insight into an issue or redraw a generalization. The case is of secondary interest, it plays a supportive role, and it facilitates our understanding of something else" (p. 427). This study was an instrumental case study that investigated the experiences of the teacher and students in a course designated for low achieving mathematics students. The goal of the case study was to come to a better understanding of low achieving students, the teachers of these students, and a course design that may be beneficial for them.

The researcher observed the class and accompanying homeroom nearly daily from September 2003 until March 2004, keeping extensive field notes. An external observer also viewed several classes over the course of the study, and kept field notes and a reflective journal. The course began with 13 students. By the end of the first semester, one student had transferred to an alternative school. 8 new students were assigned to the class at the beginning of the second semester. Each of the students in the class completed an initial survey to identify their levels of expectancy and value of mathematics as either high (H), moderate (M), or low (L). Based on the outcome of this survey, five students were selected to be interviewed three times each throughout the course of the study: Danny (H/H), Diego (M/M), Steve (L/M), Jerry (H/H), and TJ (H/M). In addition, the teacher of the course (Mrs. Jensen) kept a daily reflective journal was interviewed four times throughout the study. The interviews were transcribed and analyzed using constant comparative analysis, as were the field notes and journals. Throughout the analysis, themes emerged regarding student achievement (perseverance, cooperative work, and communication), expectancy (students' self-perception, and future class expectations), value (career goals, future needs of mathematics and utility), enjoyment (specific activities, design of the class, and working collaboratively), and student beliefs (this class is easy, this class is dumb, the content is review, the benefit of activities).

Results and Conclusions

Students showed dramatic changes in their educational achievements and motivation. Factors that influenced changes in achievement were the use of collaborative work, opportunities to communicate, and assessments that emphasized effort. It should be noted that the study did not make use of pre and post-test scores, factors commonly associated with achievement. Rather, the study investigated other indicators of achievement, such as perseverance in problem solving, class attendance and retention, and the promotion to a higher mathematics class. For this study, these indicators are collectively referred to as educational achievements. A variety of factors influenced these changes. Factors that influenced a change in motivation were the use of activities, focusing lessons on student interests, and student enjoyment.

Collaborative Work

In the beginning of the study, students were unfamiliar with cooperative learning. Mrs. Jensen described it "like pulling teeth" in the first week of the course (Teacher Journal, 8/25). Students were often intimidated to tell their group members when they did not understand a concept. Even after a month of cooperative learning, students still criticized group members when they did not understand: "Danny was so confused about the previous day's work that he could not express his questions His group mates began to tease him about the math. Then, they started to tease him about his size. As a result, Danny completely shut down and quit working on the day's lesson" (field notes, 9/26). In addition, when students did understand the material, they were often reticent to help their group mates. Student would often simply share answers with no explanations: " Diego 'helps' his group mates by giving them his work. When asked by the teacher to explain what he did, he said 'I only said I was a mathematician, I never said I was a teacher" (field notes, 9/26).

By October, students were beginning to hone their collaborative skills. Mrs. Jensen described in her journal: "Students finished their similar houses. Ross and Elvira came to a near impasse over how to make the roof. They took a group vote. The learned to work together!" (journal entry, 10/24). Students in their final interviews expressed positive insights about collaborative learning: "Just because I like working with other people, and stuff. If you've got more people, you can talk and make sure you are doing it right. Just because you have more people there. Because there is more people there that you are working with and they're trying to get it done as well as you are. And you are trying not to let down the people you are working with" (Steve, Final Interview). "We have done it probably more than half the time we have been in class, working in groups and stuff like that. I have come to like it a little more than I did...Because I can talk to somebody, like, she tells us how to do it, and then I can go into a group, like, say they don't understand something, then I can try to explain it a little more. And maybe help them learn too" (Jerry, interview 2).

Communication

In the first three months of the study, only two students ever volunteered an answer without being called upon by Mrs. Jensen. When students were called on, they would only offer a final answer. Even when they were pressed to give their reasoning, students often had little to contribute (field notes, August through October). By the second semester of the course, students' responses became more intricate, with students sometimes asking to move to the front of the classroom to give their explanations. During a probability lesson, in order to explain the sample space of tossing three coins, Steve said, "I am going to need to use the overhead. (After proceeding to the overhead projector and listing the possible outcomes, he continued.) So player one only had a 2/8 probability of scoring, and player two and three had a 3/8 probability of scoring. Players two and three and the same chance, and player one had a worse chance" (field notes, 1/14).

A similar experience happened when Maria asked to finish teaching a lesson. Prior to this day, Maria had never volunteered an answer, and had in previous lessons become so frustrated with either the content or her group members that she had stormed from the room. Mrs. Jensen described the event in her journal: "Success day! Maria saw what was happening with player three. She said to me, ' I know what's happening and I know what we are going to do the rest of the hour.' I said, 'what?' She proceeded to explain that we would ask which player had the advantage. Then we'd look at selected student scores, then the whole class. I said, 'wow, you could teach the class today!' and she agreed to do it! She faltered at first, but her confidence was evident as she walked the class through the process of finding the experimental probability. She responded well to the wise cracks from the students. I couldn't believe that she persevered through the entire lecture. I saw a totally different Maria today. She was sweet, happy, and obviously having fun. She enjoys center stage. I really think the homeroom has helped shape the mood towards this class and its members."

Assessing Effort

A third factor that influenced student achievement was emphasizing effort in assessment rather than assessing only from scores. Tests and grades did little to influence student performance because the students in the class simply cared little about their grades or were satisfied with mediocre grades. In September, after giving a test, Mrs. Jensen asked students to make corrections to their errors to earn back half of the missed points. One student who had attained a 60% asked, "What if we are happy with our grades?" When students' grades were based on tests and not on daily activities, students seemed to care about neither. As a result, negative behaviors were prevalent in the daily class activities.

Throughout the first semester, Mrs. Jensen gradually began the place the assessment emphasis on student participation, and replaced tests with large projects. This change encouraged students to engage in the classroom activities and help reduce behavior problems. Mrs. Jensen reflected in an interview, "I think that once they realized that they were going to be assessed on the amount that they tried that their attitudes started changing, and they realized they weren't going to be beaten down by a grade if they were slow, or didn't get it. That they could ask, and try to understand and that we would respect that learning process. I think that they opened up and their attitudes towards math and towards thinking changed."

Evidence of Increased Educational Achievements

Changes in achievement are difficult to quantify, and it is even more difficult to attribute causation. Hence, it is important to note that the claims of changes in educational achievements were merely concomitant with this study. Evidence of educational achievements was identified as perseverance in problem solving, attendance and retention, and promotion to a higher mathematics class.

In the first semester, students would frequently have temper tantrums and outbursts at the beginning of a new activity. Within minutes, they would curse, quit, and sometimes physically leave the classroom. "At the beginning of class, the students were given their assignment and I overheard them saying how hard it was going to be. Maria even said that she was too dumb to do something this hard" (external observer journal). As the study progressed, students' perseverance not only increased, but the students became aware of these changes. When asked to identify a change in himself over the school year, Jerry replied, "Probably my self discipline because I wouldn't just go into a class and do an activity. It would be fine with me, but I would have to have discipline to sit there and actually do the things, and interact with some kids that should be doing it but aren't." Diego commented about similar changes in himself, commenting that he believed these changes were attributable to the Explorations class: " I don't think it would have happened [without this class]. I would have just given up. Just give up."

The retention and attendance rates also indicate success. The last time the Explorations course was taught, fourteen students had enrolled in the course, and only six finished it. Furthermore, only one student was recommended to proceed on to the next level mathematics course. In the Explorations course observed in this study, thirteen students were enrolled in the first semester, and another seven enrolled in the second semester. Over the course of the year, only two students did not finish the class, and both of those students left the school completely. Furthermore, six students were recommended to proceed on to the next higher mathematics course (an algebra class taught over two years).

Factors Contributing To Changes In Motivation

Motivation was defined as the product of student expectation for success and the value they attributed to the content. Two factors contributed to students' improved motivation: the use of activities with real world connections, and enjoyment.

Virtually every day, the Explorations class maintained the same pattern. Students were presented with a beginning problem that either reflected the lesson of the day before or introduced the new lesson of the day. Students would work and discuss this problem for five minutes, and then would be given an activity that would allow students to explore a new mathematical concept. In the last ten minutes, students would discuss the outcome as a whole group and recap the mathematical concepts of the day. Students came to see the value of these activities, as demonstrated by a comment from Diego's first interview: "because they make you see things. They make them [mathematical concepts] stand out" (Interview 1). Jerry described how the use of activities improved his expectation of success: "Yeah, I have more confidence, because a couple of our activities have been, like, probability and percentages and stuff. And I never really knew how to do that, and now I know how to do that. And it is better for me knowing that I know how to do it" (final interview). In addition, Diego commented that the daily activities prevented him from 'ditching' class: "No. I haven't done it [ditched] once this year. Because I know that when I go in there that this is the fun part of the day. It is the only thing fun in the day. In every other class, you are like (leans back in chair, crosses hands on his chest, tilts head up towards the ceiling, and stares off into the distance).

Another contributing factor to motivation, which was not addressed in the Expectancy Times Value model, was pure enjoyment. Each of the interviewed students commented on the enjoyment level of the Explorations class, and how that enjoyment motivated them to learn. Jerry stated, "To me, I think of math as a little more fun instead of just, 'oh, it's math, it's going to be boring again.' To me, going into class, I am thinking, 'we are going to do another project.' It's cool, it's not like we are going to sit here and do nothing, or do a worksheet or whatever." Each of the interviewed students also commented on former mathematics classes being teacher centered and using worksheets frequently. The contrast of the active classroom was noticeable to each of the students. Jerry summarized it, stating, "If we are doing a project and it is fun for me, I actually want to do it instead of just sitting there and writing down answers to be done with it. I would actually want to do it, and have fun doing it."

Implications

The main implication of the study was that low achieving mathematics students, who are typically not given the opportunity to interact and engage in their mathematics classes, benefit in both educational achievements as well as in motivation as the result of an activity-based, meaningful and challenging curriculum. In the context of the NCTM Standards and the No Child Left Behind Act, finding ways to improve the mathematical experience for previously low performing students is increasingly important. This study gives evidence that a dramatic change in the classroom structure for low achieving high school mathematics students may yield positive results.

Literature in the field of special education (Carnine, Jones, & Dixon, 1994; Jones & Wilson, 1997) suggests that using multiple approaches to solving a problem is confusing for low achievers and that students may not persevere through the process. Jones and Wilson charge, "the premise that secondary students with LD [Learning Disabilities] will construct their own knowledge about important mathematical concepts, skills, and relationships, or that in the absence of specific instruction or prompting they will learn how or when to apply what they have learned, is indefensible, illogical, and unsupported by empirical investigations" (p. 149). Statements like this only contribute to the idea that low achieving mathematics students should continue to be subjected to mathematics classes that do not challenge them to become autonomous thinkers, that do not challenge them to work cooperatively or to investigate multiple approaches to problem solving, and that do not encourage them to learn to communicate mathematically. The study described in this paper directly contradicts this idea by suggesting that an open, active environment that encourages cooperation, communication, and multiple approaches and deemphasizes a teacher-centered classroom is exactly the kind of environment that will help low achieving students improve their educational achievements and motivation.

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PROCESSES OF COMPREHENSION SHOWN BY HIGH SCHOOL TEACHERS IN SOLVING OPTIMIZATION PROBLEMS WITH THE USE OF TECHNOLOGY

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What aspects of mathematical practice are enhanced when problems that involve change or variation are approached through the use of technological tools? How does the use of particular tool (dynamic software, excel or graphing calculator) help the problem solver to represent and solve those problems? How the problem approaches that appear when using distinct tools are connected or can be complemented? These were the research questions used in this study to document the work exhibited by high school teachers when solving a set of optimization problems with the use of technology. Results indicate that the use of technology became important for teachers to represent the problem, formulate conjectures, search for relations, generalize results, and make connections.

Introduction

To what extent the use of technological tools (dynamic software, excel or graphing calculators) help high school teachers explore and solve problems that involve change or variation? What kinds of representations are favored with the use of the different technological tools? What kind of conjectures and observations do high school teachers exhibit in their problem solving approaches that involve the use of technology? In general, what type of reasoning do teachers develop in their mathematical experiences with the use of particular technological tools? These are fundamental questions that are part of the research agenda in mathematics education. Hence, the use of each tool may provide different environment and conditions for the problem solver to represent and deal with information to solve mathematical problems. As a consequence, it becomes important to document types of reasoning that problem solvers show as a result of using a particular tool. Santos (2004) states that the systematic use of dynamic software, excel, or calculators helps students represent and explore mathematical properties embedded in the problem or situation from graphic, table or algebraic representations.

> The electronic technologies – calculators and computers-are essential tools for teaching, learning, and doing mathematics. They furnish visual images of mathematical ideas, they facilitate organizing and analyzing data, and they compute efficiently and accurately. They can support investigation by students in every area of mathematics, including geometry, statistics, algebra, measurement, and number. When technological tools are available, students can focus on decision making, reflection, reasoning, and problem solving. (NCTM, 2000, p. 24)

In this study, we are interested in analyzing high school teachers« approaches to solve series of optimization problems found in calculus textbooks. In particular, the extent to which they utilize technology to explore distinct types of representations that appear during their problem solving approaches. The research questions that directed the study were: What aspects of mathematical practices do high school teachers show while working the problems with the use of technological tools? How does the use of particular tool (dynamic software, excel or graphing

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calculator) help the problem solver to represent and solve those problems? How problem approaches that appear when using distinct tools are connected or can be complemented?

Components of a Conceptual Frame

Problem-solving has been recognized as a central activity for students to learn mathematics. In this perspective, the use of technology in problem solving environment is seen as an opportunity for students to enhance their mathematical processes that involve looking for patterns, formulating conjectures, searching for arguments and communicating results. For example, the use of dynamic software may become a powerful tool for students to represent geometrically and graphically variation phenomena without using algebraic procedures; while the use of excel may offer students the possibility of examining relevant properties of the problem through the use of a systematic list or table. Both types of representations emerge as a result of examining properties of the situation or problem from two related ways: the construction of a dynamic representation of the problem and the exploration of particular cases via the algebraic model of the situation. An integrating principle in problem–solving approaches is that students have the opportunity to pose questions around the problem that lead them to recognize relevant information needed to comprehend and explore meaning associated with concepts (Postman & Weingartner, 1969). In this context, students conceptualize their learning as a continuous activity in which they constantly formulate questions, use distinct representations, look for patterns, present conjectures, support and communicate results (Thurston, 1994).

The NTCM (2000) points out that the use technology can help students understand mathematics, but it shouldn't be used only as a means to carry out basic operations. Rather, through the use of calculators and computers students can construct more representations of the tasks and focus on examining examples or particular cases to generate conjectures and eventually pose their own problems.

> The graphic power of technological tools affords access to visual models that are powerful but that many students are unable or unwilling to generate independently. The computational capacity of technological tools extends the range of problems accessible to students and also enables them to execute routine procedures quickly and accurately, thus allowing more time for conceptualizing and modeling. (NCTM, 2000, p. 25)

According to Williamson and Kaput (1999), an important consequence of the use of technology in mathematics instruction is that it can provide conditions for students to explore relationships inductively. Thus, students tend to perceive mathematics in an experimental way (by interacting with technology) and explore relationships that eventually they need to formalize or justify. That is, mathematical ideas emerge from the processes of examining relationships in geometric, numerical and algebraic contexts. By seeing and dealing with different representations of problems, students can visualize relationships and reflect on mathematical properties that are important to solve those problems (NTCM, 2000)

The Participants, the Design, and the Procedures

Twelve high school teachers, who were taking a graduate course in mathematics education, participated in the study. The course took place during one semester and included two weekly sessions of three hours each. An important objective of the course was to work on series of tasks with the use of dynamic software, excel and calculators. It is important to mention that most of the participants in the study had never used the Cabri Géométre dynamic software or the TI 92 calculators; while most of them knew Excel or they had already worked with it.

A total of eight activities were implemented during the development of the sessions. The main topic in each activity was to solve "typical" optimization problems that appear in most of calculus textbooks, but now solved them with the help of the Cabri Géométre dynamic software, the TI 92 calculator, or Excel. Each session followed a structure that included:

i) The instructor introduced the activity to the teachers and explained to them ways to report their approaches to the problem.
ii) The participants worked on activity either individually or in pairs, using the available technological tools.
iii) The participants handed in a written report with the corresponding computer files showing their approaches to the problems, comments and extensions of the original problem. Cabri Géométre offers an option of keeping records of the problem solving processes and this information became important to analyze the teachers' work.
iv) Some participants were interviewed at the end of the course. They were asked to work on one problem and were asked to reflect on the meaning of fundamental concepts that appeared during the solution of the task.

In general, the instructional principle around this problem-solving approach is that teachers conceptualize learning as an inquiry process in which they need to formulate questions, reveal and contrast their own ideas, and present distinct arguments to communicate their results. To describe what emerged or transpired during the development of the problem sessions, we decided to focus on presenting mathematical features that we identified as crucial during teachers' interaction with one problem. Since we are not offering a detailed analysis of the teachers' performance, we decided to first identify distinct types of representations that were present during the solution of the activity. Secondly, we comment on mathematical properties that became transparent in using dynamic, numeric, and algebraic (including the use of derivative techniques) approaches to deal with the problem. And thirdly, we recognize the need and importance for teachers to examine those representations of the problems achieved through the use of technology and those that usually appear with the use of paper and pencil. That is, rather than privileging one particular approach to the problem, we take the position that it is important for teachers to move, in terms of meaning, across all those representations. In this context, all the participants have an opportunity to contrast mathematical properties that appear in paper and pencil approaches with those relationships that emerge from representing the problems through the use of technological tools. Indeed, looking at the problem from distinct perspectives seems to be an important habit that teachers can develop by using technological artifacts.

After solving the activities with the use of technology, teachers also had the opportunity of working with the same activity with the only use of a pencil and a sheet of paper. Some questions that helped organize and structure the analysis include: What problem-solving approaches do teachers exhibit while working on the optimization problems? What types of representations do they use when they work with the different technological tools, and how do they interpret them? What type of information do they identify as relevant to solve the problem? To what extent the use of different tools favors the search for connection or extensions of the problem? What types of advantages or limitations appear in representing and analyzing the task with the use of each tool?

Presentation of Results

To present mathematical features that distinguish each approach to solve the problem, we identify distinct problem solving episodes to look at mathematical properties associated with the type of representations of the problem that appear during the process of solving the task. We take only one example to illustrate the type of reasoning that emerges during the process of posing questions and ways to respond them with the use of the tools; however, the task is representative of a family of problems that high school teachers use in their calculus courses. In order to present the results we identified problem-solving episodes that include understanding the problem, construction of a model, and making sense of results. The problem used to present the results is:

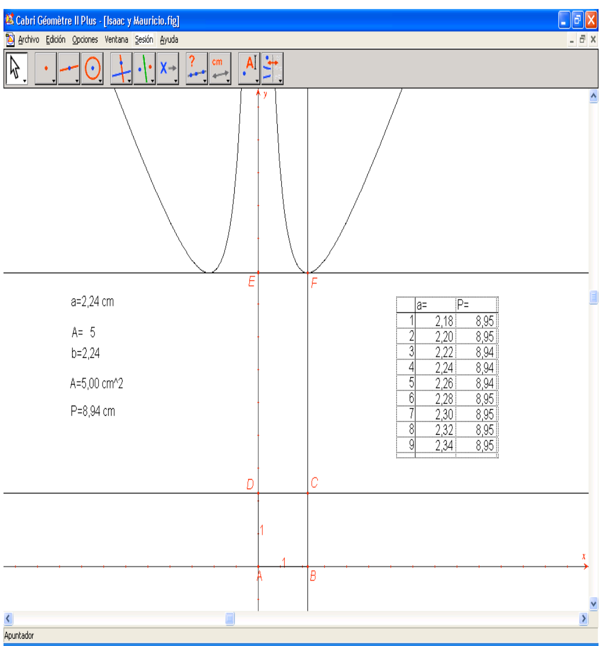
From all rectangles that have a fixed area A, find the one that has the minimum perimeter.

First Episode: Understanding the Problem

What does it mean to have rectangles with a fixed area? How can we measure the perimeter of rectangles with a fixed area? How can we represent geometrically the relationship between the rectangles with fixed area and their corresponding perimeters? How can I trace the perimeter variation of those rectangles? These types of questions were initially discussed among the participants and led them to identify relevant information to construct a dynamic representation of the problem.

Second Episode: Construction of a Model

Three distinct ways to represent dynamically the problem appeared during the teachers« interaction with the problem. Mauricio and Isaac used Cabri Géométre to represent and analyze the case in which fixed area of the rectangles was 5 cm^2 (*Figure 1).*



	a=	P=
1	2,18	8,95
2	2,20	8,95
3	2,22	8,94
4	2,24	8,94
5	2,26	8,94
6	2,28	8,95
7	2,30	8,95
8	2,32	8,95
9	2,34	8,95

Figure 1: Dynamic approach used by Isaac and Mauricio in order to approximate the solution of a particular case.

Procedure:

1. They drew the Cartesian axes and located the origin *A*.
2. On the *x- axis* they drew segment *AB*.
3. They measured segment *AB*, which they denoted as *a (*one side of the rectangle*).*
4. They drew a perpendicular line to the *x-axis* passing by point *B*.
5. Selected 5 as the given area and calculate — to determine *b* (the other side of the rectangle).
6. They transferred value *b* on the *y- axis*, that is, *AD* = — .
7. They drew perpendicular line, to *y-axis* passing by through point *D*. This line intersected the perpendicular to *x-axis* that passes by *B* at point *C*.
8. They drew rectangle *ABCD* and calculated (using the software) its area and perimeter.
9. They transferred the perimeter's magnitude of the *ABCD* rectangle on the *y axis*, and located segment *AE* as that measure.
10. They traced a perpendicular line to the *y-axis*, through the *E* point. This straight line cut the perpendicular line to *x-axis* that passes by *B* at *F*.
11. They drew locus of point *F* point when point *B* is moved along the *x- axis*.
12. They tabulated several values of *a*, and the respective perimeter of the *ABCD* rectangle.

Third episode: Making Sense of the Results

The locus shown includes positive and negative values of *b*, what does this mean? Isaac and Mauricio argued that it was enough to pay attention to the positive values to analyze the behavior of the perimeter, since *a* and *b* represent the side of the rectangle. Any point $P(a, p)$ on the locus of *F* has the coordinates one side of the rectangle and its corresponding perimeter. When Isaac and Mauricio moved point *B* along the *x-axis*, they observed that the point on the locus that is nearest to the *x-axis* represents the point with coordinates the values of the one side of the rectangle and its perimeter. That is, the locus of point *F* helps to identify the rectangle with minimum perimeter.

When these teachers constructed a table with some values around what they identified visually as the rectangle with minimum perimeter, they realized that there were various values of the side of the rectangle that were associated with that minimum value. Here, they realized that the software was a tool to approximate the dimensions of the rectangle with perimeter minimum. In this case, when the fixed area of the rectangle was 5 squared units, they noticed that the side for the rectangle with minimum perimeter was around 2.24 units. Based on this, they made a conjecture that the minimum perimeter is reached when the rectangle becomes a square. This conjecture was later proved by using an algebraic argument.

Another pair of teachers (Marco and Pablo) relied on other type of representation, using Cabri Géométre to approach the problem (*Figure 2)*.

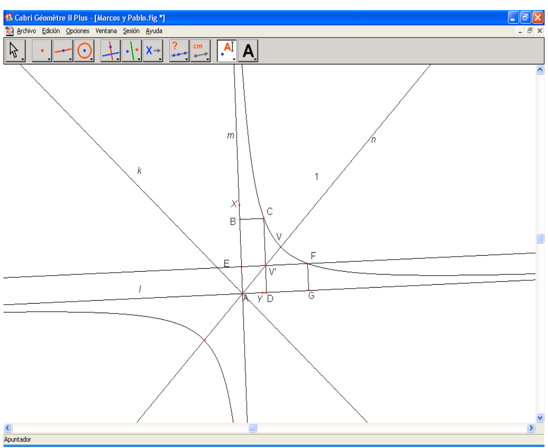


Figure 2: Dynamic construction constructed by Marcos and Pablo to deal with a particular case.

Procedure:

1. They drew line l and selected point *A* on it.
2. They drew a perpendicular line *m* to line l passing by point *A*. On line *m* they selected point *X*.
3. They drew segment *AX*.
4. They took point *E* on segment AX.
5. They drew segment *AE*.
6. They constructed a circumference *s* circumference with center in *A* and radius 1 unit radio. This circumference cut line l at *Y* point.
7. They drew line EY.
8. They drew a parallel line to line EY passing by *X* point. This line intersected the line l at G.
9. They constructed a perpendicular line to l passing by point *G* and a perpendicular line to *m* passing by point *E*. These two lines get intersected at point *F*.
10. They drew rectangle *AGFE.*
11. They traced the locus of point *F* when point *E* is moved along *AX,* that seems to be a hyperbola.
12. They drew the bisector *n* of angle EAY.
13. They identified the locus of point *F* as the hyperbola.
14. They noticed that line *n* cut the hyperbola at point V.
15. They constructed a circumference with center in *A* and radius AV.
16. They drew a perpendicular line *k* to line *n* passing by point *A*.
17. They located point *C* as the symmetric point of *F* with respect to line *n* and similarly the symmetric point *B* of point *G* with respect to line *n*.

18. They drew a perpendicular line to line *l* passing by point *C*. This line cut line *l* at point *D* .

19. They drew the rectangle *ABCD* and located V^1 as the intersection point of *CD*, *EF*, and line *n*.

The length of segment *AX* represents the area (*A*) of the rectangles. Angle *XAG* is a right angle and *AV* is its bisector, triangles AEV^1 and ADV^1 are right triangles and isosceles. Since polygon $DAEV^1$ is a parallelogram then this polygon is also a square.

How Did They Identify the Solution? Triangles *EAY* and *XAG* are similar, that is, it is fulfilled that $\Delta EAY : \Delta XAG$, which means that $= \dfrac{XA}{EA} = \dfrac{AG}{AY}$.

Now, since *XA* represented the *A* area of the rectangles and *AY* measures 1 centimeter, then $A = AG \bullet EA$. Here, they used this information to construct all rectangles *EAGF* with fixed area *A*. When Marcos and Pablo moved point E noticed that:

> When the perimeter of rectangle ABCD increases, then the perimeter of square $EADV^1$ decreases, that is, there is an inverse relationship between the perimeters of those figures. Thus, when the perimeter of the square reaches its maximum value, then the perimeter of the rectangle reaches its minimum value. This happens when the vertex of the square coincides with the vertex of the hyperbola".

Yet another approach shown by Isaac and Mauricio to solve this problem involved the use of Excel *(Figure 3)*. By taking 5 squared units as the fixed area and selecting different values for the sides of the rectangles, they calculated the corresponding values of the perimeters.

Àrea	Base	Altura	Perimetro
5	1	5	12
5	2	2,5	9
5	3	1,66666667	9,33333333
5	4	1,25	10,5
5	5	1	12
5	6	0,83333333	13,6666667
5	7	0,71428571	15,4285714
5	8	0,625	17,25
5	9	0,55555556	19,1111111
5	10	0,5	21

Àrea	Base	Altura	Perimetro
5	2	2,5	9
5	2,1	2,38095238	8,96190476
5	2,2	2,27272727	8,94545455
5	2,3	2,17391304	8,94782609
5	2,4	2,08333333	8,96666667
5	2,5	2	9
5	2,6	1,92307692	9,04615385
5	2,7	1,85185185	9,1037037
5	2,8	1,78571429	9,17142857

Área	Base	Altura	Perimetro
5	0,5	10	21
5	1	5	12
5	1,5	3,33333333	9,66666667
5	2	2,5	9
5	2,5	2	9
5	3	1,66666667	9,33333333
5	3,5	1,42857143	9,85714286
5	4	1,25	10,5
5	4,5	1,11111111	11,2222222

Área	Base	Altura	Perimetro
5	2,2	2,27272727	8,94545455
5	2,21	2,26244344	8,94488688
5	2,22	2,25225225	8,9445045
5	2,23	2,24215247	8,94430493
5	**2,24**	**2,23214286**	**8,94428571**
5	2,25	2,22222222	8,94444444
5	2,26	2,21238938	8,94477876
5	2,27	2,20264317	8,94528634
5	2,28	2,19298246	8,94596491

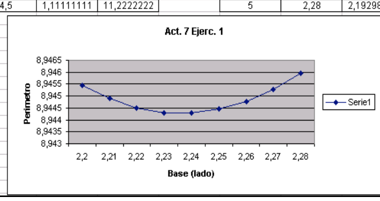


Figure 3: Discreet approach shown by Isaac and Mauricio to approach the problem.

Figure 3 shows the process of refining values assigned to one side of the rectangles. The values included variations of one unit, 0.5 units, 0.1, and 0.01 units.

These teachers observed that while varying one side of the rectangle in 0.5 units the solution reminded on the interval (2,3) and continued refining the partition until they identified 2.24 and 2.23. Here, they represented graphically this information and visualized the solution to the problem.

During the development of the session, the participants also had opportunities to approach the problem by using algebraic approaches. Here, they confirmed results that had gotten when using the dynamic software or excel and discussed differences among those approaches.

Discussion of Results

The approaches carried out by the teachers show that the use of distinct technological tools offers the possibility of constructing various types of representations of problems that involve variation or change. Each representation can be analyzed in terms of mathematical properties that are important to comprehend concepts associated with this type of problem. For example, when using dynamic software, teachers« first goal was to think of the problem geometrically.

Segments, perpendicular lines, right triangles, symmetry and similar triangles were some basic ingredient to construct a representation that modeled the problem to observe and quantify variation dynamically. Teachers realized that they were able to construct the dynamic representation of the problems without relying on algebraic procedures. Here, they identified the potential of discussing this type of problems even with students who have not developed algebraic competences. The use of Excel became important to analyze the problem in terms of refining values of one side of the family of rectangle with a fixed area. Here, teachers recognized that the process of refining a partition is an important aspect to grasp the concept of limit. The use of algebraic procedures was seen by teachers as an opportunity to deal with the general case and as a way to verify results that emerged from using dynamic software and excel.

There is evidence that most of the teachers showed changes in using the dynamic software to represent the problem throughout the development of the sessions. Initially, they mainly focused on using commands to calculate lengths of segments, areas or perimeters of figures that appeared in the problem representation. Later, it became evident that they were also interested in comparing ratios of particular figures (lengths, perimeters, areas) and determining loci of points or figures. In addition, they became interested in providing arguments to support conjectures that emerged when moving particular elements of the problem representation. The analysis of particular cases through the use of Cabri Géométre or Excel allowed them to observe relations, patterns, results, and eventually to propose general cases.

Teachers recognized that the TI 92 calculator became a useful tool to deal with operations that involve calculating derivatives, solving equation, and graphing the function that represents the problem. Here, they were concern about the meaning associated with those operations and tried to explain connections between what they did using excel or the geometric software and their calculator work. What does it mean the derivative of function perimeter? Why does it mean to solve *f«= 0*? These were the types of questions that teachers discussed during the sessions. In this context, teachers recognized that the use of distinct tools to deal with the same problem offered them the possibility of examining fundamental concepts embedded in this type of problems from distinct angles and as a consequence to achieve a robust understanding of those concepts.

Acknowledgement

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PROOF AS LITERATE MATHEMATICAL DISCOURSE IN PAST AND PRESENT: PERSPECTIVE ON STUDENTS' WORK

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The objective of this study is to reconsider proof as literate mathematical discourse for communicating mathematical knowledge and with this perspective, reconsider students' difficulties in creating proofs. For this purpose I examine different kinds of proofs that are drawn from the work of Euclid, from work of a contemporary skillful mathematics instructor, and from proofs provided by university students. In the analysis of the different discourses that are used in these proofs, I consider four dimensions along which, according to Sfard, literate mathematical discourses can be distinguished from other types of communication (1) the mathematical vocabulary, (2) the mathematical visual mediators, (3) specific discursive routines, and (4) endorsed narratives. The analysis provides an insight into the students' cognitive obstacles to creating a proof.

Introduction

Historically, Thales (600 B.C.) was the first mathematician who saw the necessity of proving a general geometric proposition (Davis & Hersh, 1980), and Euclid (300 B.C.) was the first one who organized the geometry and number theory of his time into a coherent logical framework, whereby each result could be deduced from those preceding it, starting with a small number of postulate regarded as self-evident (Hartshorne, 2000). Since then the idea of the proof, established itself as the characteristic aspect of mathematics.

The notion of proof has evolved in mathematics through the centuries to the point where proofs nowadays, look very different from proofs of the past. Unfortunately, the evolution of proof has not been matched by an evolution of understanding of proof. Proof is now one of the most misunderstood notions of the mathematics curriculum (Schoenfeld, 1994), and therefore one of the greatest challenges for students, mathematics educators, and researches alike.

Over the past decades, many of those involved with mathematics education have also turned their attention to proof as a social construct and product of mathematical discourse (Hanna, 1983; Hersh, 1993; Knuth, 2002). As Manin (1977) stated, "a proof becomes a proof after the social act of accepting it as a proof" (p. 48). In addition to the social nature embodied in the process accepting an argument as a proof, the product of such process (i.e. a proof itself) also provides a means for communicating mathematical knowledge with others (Schoenfeld, 1994). However, the social nature of proof traditionally has not been reflected in the proving practices of school mathematics (Knuth, 2002).

Considering the idea that the language of proof can also be used to communicate and to debate, this study proceed to the social aspect of proof. I consider several theorems or claims and examine different kinds of proofs for these claims using the lens of the communicational framework developed by Sfard (2001a). The different kinds of proofs are drawn from the work of Euclid, from work of a contemporary skilful mathematics instructor, and from proofs provided by university students. The analysis of different discourses that are used in these proofs, according to the aforementioned framework, provides insight into the students' cognitive

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obstacles on the pathway to creating a valid proof. As such, this framework may serve as a useful tool for the diagnosis and possible remediation of students' difficulties.

Theoretical Framework

In this study I will use the communicational framework developed by Sfard (2001a). The communicational approach to cognition is based on the learning-as-participation metaphor, and conceptualization of thinking as an instance of communication. That is, as one's communication with oneself. She believes thinking like conversation between two people involves asking questions and giving answers. In this approach learning mathematics is an initiation to a certain type of discourse. In this communication framework Sfard and Cole (2002) acknowledge two kinds of mathematical discourses: colloquial and literate. The colloquial discourses are known as everyday or spontaneous. Unlike, literate mathematical discourse, which is the objective of school learning and required deliberate teaching.

Sfard (2001a) considers four dimensions along which literate mathematical discourses can be distinguished from other types of communication (1) the special mathematical vocabulary, (2) their special mediating tools, in the form of symbolic artifacts that have been created specifically for the purpose of communicating about quantities, (3) their discursive routines with which the participants implement well defined type of tasks, and (4) their particular endorsed narratives, such as definitions, postulates and theorems, produced through the discursive activity.

It seems this theoretical framework of mathematical discourse would offer a new approach to the analysis of mathematical proof and students' learning of proof. Since, itemized components of literate mathematical discourse not only detail what should be considered in generating proof and communicating through that, but also provide a tool for the diagnosis of possible obstacles in generating such proof. Furthermore, it would offer a framework for the examination of proof as an evolving discourse, for in the work of Euclid although crude by modern standards, there is available the first illustration of what present-day mathematicians would call a mathematical discourse (Newsom, 1964).

Study

For this study I adopted three propositions from *Euclid's Elements* (2002), which can be also found in the most of the number theory textbooks.

- *Proposition 24, from book seven*:
  If two numbers be prime to any number, their product also will be prime to the same.
- *Proposition 30, from book seven*:
  If two numbers by multiplying one another make some number, and any prime number measure the product, it will also measure one of the original numbers.
- *Proposition 29, from book nine*:
  If an odd number by multiplying an odd number, make some number, the product will be odd.

I adjusted the language of these propositions and presented them to 110 pre-service elementary school teachers. They were invited to determine whether the statement was true or false, and then to prove it or provide counterexample respectively. Students' proofs for these propositions, in addition to Euclid's proofs and proofs presented by a contemporary math instructor, provided good variety of discourses on mathematics.

I had close look at the role and the form of the characteristics of the literate mathematical discourses, according to the aforementioned framework, in each of the proofs. Although in different forms, the observations verified the inevitable presence of the components of the literate mathematical discourses in Euclid's and contemporary proofs. Nonattendance or wrong attendance of those components in students' works, however, can be seen as the factors that impede students' active participation in creating a proof, and communication through proof.

In this paper, I will focus on the proofs for the second proposition, proposition 30 from book seven of *Elements* (Euclid, 2002, p.177).

Proposition:

If a prime number p is a factor of $a \times b$**,** then p is a factor of a*,* or p is a factor of b**.**

Proof:

According to the fundamental theorem of arithmetic we have

$$a = (p_1^{n_1})(p_2^{n_2}) \dots (p_k^{n_k})$$

$$b = (q_1^{m_1})(q_2^{m_2}) \dots (q_t^{m_t})$$

Where p_i for $1 \le i \le k$, and q_j for $1 \le j \le t$ are prime factors. And, n_i for $1 \le i \le k$, and m_j for $1 \le j \le t$ are natural numbers.

p is a factor of $a \times b$, then $a \times b = p \times s$ for some whole number s. Or,

$$(p_1^{n_1})(p_2^{n_2}) \dots (p_k^{n_k})(q_1^{m_1})(q_2^{m_2}) \dots (q_t^{m_t}) = p \times s$$

Since based on the fundamental theorem of arithmetic each composite number can be expressed as the product of primes in exactly one way, therefore there exist $1 \le i \le k$, or $1 \le j \le t$, where $p = p_i$ or $p = q_j$.

If $p = p_i$ then $p|a$,

If $p = q_j$ .then $p|b$.

Euclid's proof:

For let two numbers A, B by multiplying
one another make C, and let any prime
number D measure C;
I say that D measures one of the
numbers A, B.
For let it not measure A.
Now D is prime;

A _____
B ________
C _________________
D ___
E _________

Therefore A, D are prime to one another. [VII. 29][1]
And, as many times as D measures C, so many units let there be in E.
Since then D measures C according to the unit in E,
Therefore D by multiplying E has made C. [VII. Def. 15][2]
Further, A by multiplying B has also made C;
Therefore the product of D, E is equal to the product of A, B.
Therefore, as D is to A, so is B to E. [VII. 19][3]

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[1] [VII. 29] Any prime number is prime to any number, which it does not measure.
[2] [VII. 29] Any prime number is prime to any number, which it does not measure.

But D, A are prime to one another,

Prime are also least, [VII. 21][4]

and the least measure the numbers which have the same ratio the same number of times, the greater the greater and the less the less, that is, the antecedent the antecedent and the consequent the consequent; [VII. 20][5]

Therefore D measures B.

Similarly we can also show that, if D do not measure B, it will measure A.

Therefore D measure one of the numbers A, B.

Q.E.D.

Mathematical vocabulary: The hallmark of Euclid's proof is the precise use of well-defined words for concepts, numbers, operations, and the rules of inference. However, the same precision can be seen in a contemporary proof, there can be also found an equivalent sign or symbol for most of the technical words, which is the distinguished character of the contemporary mathematical discourse.

Analysis of students' works revealed that familiarity with the colloquial uses of words might have given them means for an ad hoc interpretation. Let me start with the term proof. Students live in a world in which the term proof may mean different things in different contexts. As such, students often do not recognize the need for a mathematical proof for given statements. Indeed, their lack of understanding of concepts usually leads them to use technical words in an inappropriate manner or using inappropriate words for a technical purpose.

In this study, almost half of the students justified their decision with an inductive argument, as exemplified below.

> If a prime # "p" is a factor of "a" × "b"
> then "p" is a factor of "a" or "p" is a factor of "b"
> ie, a = 5 b = 4 $a \times b = 5 \times 4 = 20$ (1,20,2,10,4,5)
> p = 2 (p is a factor of 4 (b)
> ie, a = 6 b = 8 $6 \times 8 = 48$ (1,48,2, 24, 3, 16 12, 4, ..
> p = 3 → 3 is a factor of 6 (a)
> True: because the prime number must be a factor of either one of the numbers you are multiplying for it to be present in the product.

The results confirm the findings of prior research (Harel & Sowder, 1998; Hoyles, 1997; Martin & Harel, 1989; Fischbein & Kedem, 1982) that suggests a strong reliance on empirical proof schemes. However, the study deals with the root of this tendency, which might be in the use of the common word in students' colloquial language and mathematical discourses.

Mediators: "Communication either inter-personal or self-oriented (thinking) would not be possible without symbolic tools, with language being the most prominent among them" (Sfard, 2001b, p. 28). The strong use of language in presenting a mathematical idea is the salient aspect

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[3] [VII. 19] If four numbers be proportional, the number produced from the first and fourth will be equal to the number produced from the second and third; and, if the number produced from the first and fourth be equal to that produced from the second and third, the four numbers will be proportional.

[4] [VII. 21] Numbers prime to one another are the least of those, which have the same ratio with them.

[5] [VII. 20] The least numbers of those which have the same ratio with them measure those which have the same ratio the same number of times, the greater the greater and the less the less.

of Euclid's proofs. The only non-lingual mediator in his proof is the line segments that he used as an icon or pictorial means for representing numbers. In a contemporary proof, algebraic symbols are the main mediator tools created specially for the sake of literate mathematical discourse. Surprisingly, in students works' there was not a high tendency to use algebraic symbols. The majority of students used numerical examples, as a visual mediator for understanding and showing the validity of the proposition.

Routines: Discursive routines are patterned discursive sequences that the participants use to produce in response to a certain familiar type of utterance expressing a well-defined type of request, question, task or problem (Sfard & Cole, 2002). In the case of mathematical discourses, the routines in question are those that can be observed whenever a person performs such typical mathematical tasks as calculation, explanation (defining), justification (proving), exemplification, etc. The routines with which interlocutors react to the given type of request (e.g. "justify") may vary considerably from one mathematical discourse to another, whereas one of the special characteristics of the literate mathematical discourse is that its routines are particularly strict and rigorous (Ben-Yehuda et al, 2005).

In Euclid's work, Aristotelian logic rules are the set of meta-rules or the routine that specify the 'how' of the proof, the rules such as "If it is true that statement A entails B and statement B entails statement C, then statement A entails C."

Considering the major role of the algebraic notations and symbols in contemporary proofs, the routine (beside the logic rules) is the rules for manipulating the algebraic notations and symbols. However, generalization based on limited number of numerical examples was the dominated routine that was used by the majority of the students. Results also reveal that the mostly invisible rules that guide the general course of students' communicational activities is under influence of their everyday discourse. For example, we know "or" is always used inclusively in mathematics, while "or" in everyday discourse is used both exclusively and inclusively. In following example of students' works we can see how such colloquial understanding might mislead the whole process of a proof.

$P = 7 \Rightarrow a \times b = 9 \times 14 = 126 \qquad 7 | 14$

Let $a = 4$ factors (1,2,4)
Let $b = 6$ factors (1,2,3,6)
$4 \times 6 = 24$ factors (1,2,3,4,6,8,12,24)
2 is a factor of 24
$\Rightarrow$ 2 is a factor of 4 (a) and is a factor of 6 (b)
$\therefore$ the above statement is false and it should state :
p is a factor of a or p is a factor of b or p is a factor of a and b.

Endorsed Narratives: Endorsed narratives, in the communicational framework, are the narratives that are accepted by mathematical communities and are labeled as true (Sfard, 2001a). They include such discursive constructs as definitions, postulates, proofs, and theorems. However, in general, they are produced throughout the discursive activities. Indeed, as a result of mathematical routines, according to a set of well-defined rules, the new endorsed narratives will be constructed from previous endorsed narratives. This chain of constructing endorsed narratives from previous ones is the main characteristic of Euclid's proofs. As seen in the above-mentioned proof, he constructed the proof of the given proposition based on four pre-proved propositions and one definition.

In the contemporary proof of the proposition, we can see the key role of the Fundamental Theorem of Arithmetic, as an endorsed narrative that supports and guides the whole process of the proof. The students' works, however, were mainly based on their intuition. In some of students' works it can be seen that they had good understanding of the proposition but they did not know how to present it in the form of a mathematical proof. Consider for example the following response

$p \mid a \times b$

Then $p \mid$ (prime factors of a) (prime factors of b)

$p \mid$ (prime factors of a) or $p \mid$ (prime factors of b)

p must be a prime factor of a or b.

The given argument shows the student understands the proposition and the proof, however, she did not mention any statement to support her argument. The majority of the conclusions were based on experimental arguments.

Discussion And Conclusion

A well-structured deductive proof offers humans the purest form of reasoning to establish certainty. In the work of Euclid's and modern texts however, based on totally different discourses, the persuasion aspect of proofs is salient. This aspect was the weakness of students' arguments. The results show the students' proofs are very subjective and are based on their intuition, which is in contrast with the social nature of proof. It seems that they did not consider social aspect of the proof; that a proof should be convincing for a third person who reads it. Indeed, they were satisfied with an argument that was convincing enough for them.

Examining the components of literate mathematical discourses in different discourses in past and present provides an opportunity to see their importance. However, an interesting contribution of my study deals with the gaps that the lack of each of these communicational means can cause. Indeed, these gaps may impede students' active participation in creating a proof, and communication through proof.

Results show that the students' arguments were highly under impression of their colloquial discourse. It might remind us that we cannot teach or evaluate students in isolation and the fact that we cannot disjoin them from their everyday experiences. The important point that the work never starts from zero brings up this question: how can we use and lead students' background and experiences to introduce and develop new concepts?

Future research will determine the scope of applicability of a communicational framework for analyzing issues related to teaching and learning proof.

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USE OF EXAMPLES AND COUNTEREXAMPLES IN UNIVERSITY TEACHING: THE IMPROPER INTEGRAL

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This paper addresses some results on the use of examples and counterexamples by University students. The work was developed in the frame of a Didactic Engineering, a methodology which allows both research and an improvement of practice. Some activities using counterexamples are showed and we comment the results of the students. Our results, although preliminary, are encouraging.

Introduction

The work presented in this paper is a part of a broader research project which addresses the question of the teaching and learning of the Improper Integral concept at University. After analysing how students learn this concept (and some obstacles, difficulties and errors that this learning generates. See González-Martín, 2002 and González-Martín & Camacho, 2004a to see some results), a Didactic Engineering was designed in order to improve this learning and to get a better coordination between the graphic and algebraic registers in tasks related to this concept. In our opinion, the design of Didactic Engineerings is a useful way of observing how the students learn, in addition to their potential to improve learning.

The main contributions of our proposal are:

1. The systematic use of examples and counterexamples;
2. The coordination of the graphic and algebraic registers;
3. The explicit use of knowledge about series and definite integrals;
4. The changes in the *didactic contract* to give the student more responsibility in his learning process;
5. The use of the CAS *Maple V*.

This paper focuses the first of the previous points. We take into account Artigue's statement (1992) that some kind of validation must be done when the graphic register is used and we think that some particular examples and counterexamples really serve for this purpose. However, some reluctances (not only on the part of the students, but also on the part of some teachers. See Eisenberg & Dreyfus, 1991) must be overcome:

- Cognitive: visual aspects are more difficult.
- Sociologic: visual aspects are more difficult to teach.
- Beliefs about the nature of Mathematics: visual aspects are not mathematical.

In this sense, our work using examples and counterexamples tries to show that sometimes visual reasoning helps to avoid long proofs, giving this way a mathematical validity to the graphic register.

Some of the questions that guide the design of our proposal are usual in research in University Teaching (see Selden & Selden, 2001):

- How can be the culture of the classrooms be changed in order to make students perceive Mathematics not as a knowledge which is received in a passive way, but as a knowledge built in an active way?,

Lloyd, G. M., Wilson, M., Wilkins, J. L. M., & Behm, S. L. (Eds.). (2005).
Proceedings of the 27th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.

- Which are the effects of several strategies of cooperative learning in the students' learning?,
- Would visualization help First Year Calculus students to build proofs?

This last question is explicitly addressed in our research. Moreover, some of the objectives of our research which are addressed in this paper are:

- To generate a teaching sequence for the improper integral concept which incorporates both the graphic and algebraic registers.
- To study to what extent the modifications in the usual *didactic contract* change the students' attitude towards improper integrals.
- To analyse whether the active use of examples and counterexamples in teaching, together with the use of the graphic register as a valid working register, may produce improvements in the student's learning.

Theoretical Framework

The construction of our Didactic Engineering has been guided by Brousseau's Theory of the Didactic Situations (Brousseau, 1988, 1998).

According to Brousseau (1998), two fundamental ideas play the role of engine to design good *situations*:

- Knowledge appears essentially as a control instrument of the situations.
- The student learns by adapting himself to a *milieu* which is a factor of contradictions, difficulties and imbalances, as it happens within human society. This knowledge, fruit of the student's adaptation, becomes apparent through new answers which are the proof of his learning.

The *milieu*, or environment, is another concept of great importance in Brousseau's theory. It appears as a system which reacts to the students' actions, both in a collaborative and in an antagonist way.

In our design we give a privileged role to the *adidactic milieu*, constructed to produce retroactions with the students. The students are in the middle of a problematic situation which needs of an adaptation to look for optimal strategies (later, institutionalisation will produce knowledge about the Improper Integral). Moreover, our situations try to produce *optimal* answers using the graphic register.

Taking the concept of *milieu* into account to the production of *didactic situations* implies some changes in the *didactic contract*. The student cannot adopt a passive attitude and the teacher cannot any longer be just a lecturer. If the student is supposed to adapt himself, to look for optimal strategies, the teacher must give him a new responsibility, the student must feel that he is an important part of the process of learning.

The design of our situations and the activities we have used has taken into account Duval's theory (1993) of the registers of semiotic representation. Duval emphasises that mathematical objects are not directly accessible and we only can apprehend them using their representations. To differentiate a semiotic representation and a mental representation, Duval distinguishes three fundamental activities which characterise a register of representation:

- They allow the **formation** of a representation,
- They allow the **treatment** of a representation within the register where it was formed,
- They allow the **conversion** of a representation into another one in a different register (conserving the whole or a part of the initial representation).

If we take into account that every representation shows a part of the mathematical concept it represents, the interaction between different representations becomes an essential task in the understanding of mathematical concepts.

Moreover, we take into account some other authors' contributions (as Hitt, 2000), who believe that not only are important the transformation tasks within a register and the conversion tasks between registers, but also the confrontation with examples and counterexamples. The search of examples and counterexamples is not algorithmic nor procedural and it requires of a more flexible and dynamic mathematical thinking than the one which is usually taught at University (Selden & Selden, 1998).

Methodology

Our Didactic Engineering was developed through eight sessions in the classroom, at the end of the academic year 2002-2003 and it was completed with two sessions in the computer lab. Approximately, 25 students inscribed in the First Year of Mathematics Degree took part regularly in our Engineering. The sessions were first videotaped and later analysed (for the subsequent *a posteriori* analysis).

As a consequence of the use of the Theory of Didactic Situations, the students were given more responsibility. We organised some small working groups to allow the students discuss some questions and some debates were set (we tried to adapt the classical *scientific debate*, see Legrand, 2001).

During the development of the sessions, the students were asked to fill three working sheets in groups (regarding questions to be immediately introduced, facing them to new situations that allowed them to operationalise their new knowledge) and they were also asked to prepare individually a *table of convergence* for the most usual functions and to solve three problems.

At the end of our Engineering they were given a Test of opinion, a Test about the use of examples and counterexamples and, finally, a Test of contents. The results we show in this paper come from the written material the students filled.

Some Results

During the classroom sessions the use of the graphic register[1] was introduced little by little to, later, illustrate the construction of examples and counterexamples in this register.

We tried to reproduce to some extent the historical development of the theory of improper integration. To do this, we gave great importance to the geometrical aspects (particularly the interpretation of the integral as an area). On the other hand, we first developed some intuitions and later the theory and the results were developed. This way, the students could already try to anticipate some results using their intuitions and their mathematical knowledge about series and definite integrals.

[1] More information about the use of the graphic register can be found in González-Martín & Camacho, 2004b.

The first significant counterexample that was built is the following one:

$$\nexists \lim_{x\to\infty} f(x) \nRightarrow \int_a^{\infty} f(x).dx = \infty$$

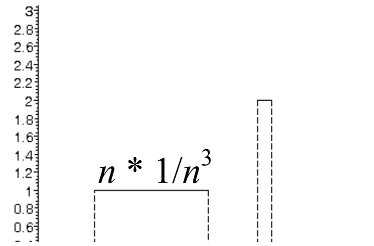


Figure 1

which contradicts everyday' intuition. Students normally only have a repertoire of prototypical functions (almost always continuous), so their *image* of a function without a limit at infinity matches a sinusoidal function. However, if we use their knowledge about series (this foreseen knowledge enriches the *milieu* created for the activity) it is easy to create a function which forms a rectangle with area $1/n^2$ above every positive integer n, as we show in Figure 1.

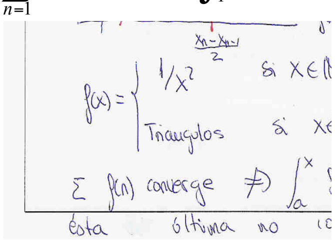
On the other hand, this function may also be used as a counterexample to the following statement:

$$f(x) \text{ is not bounded } \nRightarrow \int_a^{\infty} f(x).dx = \infty$$

avoiding in this way some obstacles that a traditional teaching generates.

Later, the students themselves have to create their own examples and counterexamples to the different questions that appear. Although some students still try to "prove" a statement using an example (see Alcock, 2004 and Selden & Selden, 1998), some other students show that they are able to create their own counterexamples to questions as the ones we show in Figures 2 and 3[2].

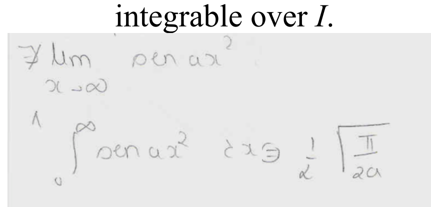
$$\sum_{n=1}^{\infty} f(n) < \infty \nRightarrow \int_1^{\infty} f(x)dx$$



They create "triangles" joining the points $(n, 1/n^2)$, $(n + \_, a)$ and $(n + 1, 1/(n + 1)^2)$.

Figure 2

Consider $I = [a, +\infty)$ and let f be continuous in I. Prove that $\lim_{x\to\infty} f(x) = 0$ is not a necessary condition for f to be integrable over I.



The student found the function $sin(ax^2)$ in a book for engineers.

Figure 3

[2] The first example was produced by three students and the second one is one of the questions in the problems the students had to solve.

Not only were the students, in general, able to create "easy" counterexamples, but they also had good results in the Test of contents. It consisted of 10 non-routine questions where the graphic register was privileged and were coordination between registers were necessary. 20 students filled the Test and 11 had an average greater than 5 (over 10). Moreover, 5 students had their average over 8.

Question 6 explicitly asked for examples and counterexamples:

> We say that a function $f(x)$ is locally integrable in an interval $[a, b)$ when…
>
> Give an example of a function $f(x)$ and an interval $[a, b)$ so that f is not locally integrable in $[a, b)$.
>
> If $f(x)$ is locally integrable in $[a, b)$, can we state that the improper integral $\int_a^b f(x)dx$ is convergent?

Seven students gave both correct examples and counterexamples to the last two questions. In fact, all the students who could complete the definition were able to answer the next questions.

Regarding the students' attitude towards the use of examples and counterexamples, 20 students filled the corresponding test. Among them, 11 stated that they feel comfortable using counterexamples and 9 said they do not. The most repeated reasons were: "*They are difficult to find / It is difficult for me to find a suitable one*" (7), "*They are useful to prove something false / They show 'easily' that something is false*" (8) and "*They are an easy way to learn a given property / They help me to remember-understand the theory*" (5).

In the question regarding whether they consider the method of learning using counterexamples to be effective, 19 students answered affirmatively and only 1 answered 'no'. The most repeated reasons were: "*They have a visual – fast character / They are easy to understand, clear*" (5), "*They help to learn – understand / They are a complement to theoretical questions*" (4) and "*If you find one, it means that the property is not true*" (5).

The question in the Test of attitude concerning their opinion about the methodology used had the following possible answers:

0 – Very little adequate;
1 – Little adequate;
2 – Adequate;
3 – Very adequate

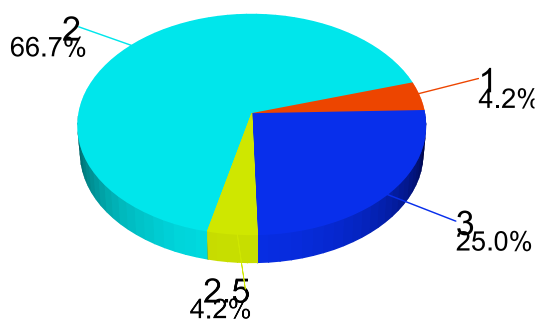


Figure 2

Twenty-four students filled this test and their answers are distributed as we show in Figure 4[3]. A fourth of the students considered it as "very adequate" and two thirds of the students think it is "adequate".

Some Conclusions

The analysis of all the material we collected (video recordings, written materials, analysis) during our sessions let us state that it is possible to generate a teaching sequence for the Improper Integral concept which takes into account both the graphic and the algebraic registers. Moreover, changes are made in the usual *didactic contract* and the students have accepted their new responsibility (in this sense, some *myths* of University lecturing have been avoided. See Alsina, 2001).

Benbachir & Zaki (2001) state that the construction of examples and counterexamples is an activity that allows a richer learning than the one achieved with traditional teaching and we agree with them. The students have shown to be able to answer to non-routine questions and even to give examples and counterexamples to some questions (and in cases of not being able to construct one, to find it in books). Also, the results of the Tests let us think that an improvement of the understanding of the Improper Integral concept has occurred (if we compare the results to those shown in González-Martín, 2002 and González-Martín & Camacho, 2004a).

The data we have shown in this paper, together with the analysis of the students' working sheets and the problems they had to solve allow us to state that it is possible to make University students see the use of examples and counterexamples as an useful activity for their learning and to make them learn how to create their own ones. Although the construction of adequate counterexamples needs of some time and experience, we can see how students accept its use (giving this way a validity to the graphic register which is unusual at University teaching) and they acknowledge that it provides an effective learning method which mobilises their previously acquired knowledge.

One of the questions to tackle after our research will be the possibility of designing a whole semester taking more into account the use of the graphic register, the construction of examples and counterexamples and the cooperative work. The local results of our experience encourage this task.

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[3] One student ticked the answers 2 and 3, so we have assigned him a 2.5.

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WHY REFLECTIVE ABSTRACTION REMAINS RELEVANT IN MATHEMATICS EDUCATION RESEARCH

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The paper defines the Piagetian notion of reflective abstraction as it has been applied in mathematics education research for over twenty years. The history of reflective abstraction in the field is briefly summarized and recent definitions of reflective abstraction are described with the intent to address the criticisms of reflective abstraction from the situated cognition community. Additionally, the utility of new approaches to reflective abstraction for curriculum development and classroom teaching is discussed. In particular, this paper intends to underscore recent original contributions in reflective abstraction, such as those represented in the 2005 American Educational Research Association (AERA) symposium, Abstraction in Mathematics Learning: Comparing Alternative Emerging Conceptions, and other recent work in the field.

Significance of the Issue

A central issue in mathematics education is improving our ability to accurately describe characteristics of students' conceptual development and features of classroom culture that support that development. Researchers have used a variety of theoretical constructs to explain what they believe to be occurring during students' learning experiences; basing these concepts on written artifacts and analyses of classroom activities and individual interviews. One of the more robust of these theoretical constructs is reflective abstraction. This paper defines and provides a brief history of reflective abstraction in mathematics education, describes fundamental problems posed from the situated cognition perspective, and describes several recent approaches to reflective abstraction that attempt to address these concerns. The paper also describes how modern applications of reflective abstraction are employed successfully: (1) from a curriculum-development perspective and (2) for suggesting effective classroom teaching practices.

A Brief History of Reflective Abstraction in the Mathematics Education Literature

Piagetian Foundations

Reflective abstraction may explain the way that students construct conceptual knowledge. The notion has a long history as noted by von Glasersfeld (1991) who quoted Locke (1690) in describing reflection. "…so I call this *Reflection,* the ideas it affords being such only as the mind gets by reflecting on its own operations within itself" (pp. 45-46). Piaget (1970, 1985) concerned himself with the issues involving a subject's interactions with external objects and the subject's internal mental operations. Von Glasersfeld (1991, 1995) in works describing his theory of radical constructivism, summarized three types of reflective abstraction discussed by Piaget: (1) reflective abstraction; (2) reflected abstraction; and (3) pseudo-empirical abstraction. The first type of abstraction refers to the subject's ability to project onto a new level and reorganize a structure created from the subject's own activities and interpretations. The second type, called reflected abstraction, is important for understanding higher-level cognitive development. The distinguishing characteristic of reflected abstraction is that not only is the subject able to project the structures created by his activities onto a new level and reorganize them; the subject is also *consciously aware* of what has been abstracted. Reflected abstraction refers to the metacognitive awareness of the subject concerning his or her activities and the organization of cognitive structures. The difference between the third type, pseudo-empirical abstraction, and the other two is that the subject's reflective activity must take place in the context of sensorimotor objects or materials. Dubinsky (1991) argued for the use of Piaget's reflective abstraction concepts as a tool in helping students develop advanced mathematical thinking and advocated an instructional approach to "induce students to make specific reflective abstractions" (p.123). In order to induce these reflective abstractions successfully, one has to place careful emphasis on developing appropriate mathematical learning tasks.

Piaget and von Glasersfeld's ideas concerning reflective abstraction were applied by Cifarelli (1988) and Goodson-Espy (1998) in studies where specific, observable problem-solving actions were used to define levels of reflective abstraction. Cifarelli defined levels of reflective abstraction to describe a learning process and described levels attained by college students while solving algebra word problems. Goodson-Espy used these levels in conjunction with Sfard and Linchevski's (1994) theory of reification to describe college students' transitions from using arithmetic to algebra. The levels of reflective abstraction defined were: *Recognition, Representation, Structural Abstraction,* and *Structural Awareness*. The differences between these levels are described in terms of whether a solver can:

- recognize having solved a similar problem before;
- re-use previous solution methods on a problem;
- develop novel strategies for a problem that the solver has not used previously;
- anticipate sources of difficulty and promise during the solution process when using a previously applied method;
- anticipate sources of difficulty and promise during the solution process when using a new solution method;
- mentally run-through methods used previously;
- mentally run-through potential solution methods;
- demonstrate conscious awareness of problem solving activities and decisions.

At the highest level, structural awareness, the problem structure created by the solver has become an object of reflection. The student is able to consider such structures as objects and is able to make judgments about them without resorting to physically or mentally representing solution methods. As solvers attain the higher levels of reflective abstraction, they become increasingly flexible in their thinking. An important feature of Cifarelli's levels is that they are a step toward describing whether a solver is conscious or unconscious of certain concepts during their problem solving activity and help identify whether a solver is using previously applied solution methods or if she is using novel problem solving methods. Cifarelli and Cai (in press) are continuing to examine the significance of students' problem solving decisions in their conceptual development. The next study we will summarize is that of Simon, Tzur, Heinz, and Kinzel (2004) and their efforts to describe an elaboration of Piaget's (2001) reflective abstraction as a mechanism for mathematics conceptual learning.

Reflection on Activity-Effect Relationships (Simon, Tzur, Heinz and Kinzel)

In their 2004 article in the Journal for Research in Mathematics Education, Simon et al. explain the difficulties inherent in applying the ideas of radical constructivism in an instructional setting, specifically defining the problems associated with engendering cognitive conflict. They discuss the *learning paradox* (Pascual-Leone, 1976) and the responses of Piagetian researchers. In brief, Simon et al. define a learning mechanism as an elaboration of reflective abstraction, i.e. "…attribute development of a new conception to a process involving learners' goal-directed activity and natural processes of reflection" (p. 318). The article is significant because it exemplifies how this approach to reflective abstraction can be directly and effectively applied to mathematics instruction. The mechanism for conceptual learning that Simon et al. define involves mental activity, activity sequences, learners' goals, and effects. Of critical importance is their statement that "…learners enact available activities in service of their goal" (p.320). The learner's activity is described as being constructive, rather than inductive, resulting from the learner reflecting on a pattern in the activity-effect relationship instead of reflection on a pattern in the outcomes. The work clearly describes a case demonstrating how Piaget's ideas of assimilation and accommodation work in conjunction with reflective abstraction to define a learning mechanism. The paper also specifies how this information can be used for lesson design and defines a four-step process: (1) specifying student's current knowledge; (2) specifying the pedagogical goal; (3) identifying an activity sequence; and (4) selecting a task. While the work of Simon et al. demonstrates one approach to modernizing Piagetian notions of reflective abstraction, other researchers are working to improve the construct using different ideas. Before describing these new approaches, it is useful to describe some of the problems associated with the earlier versions that have been raised by the situated cognition community.

Resolving Theoretical Questions: New Approaches to Reflective Abstraction

Abstraction has often been rejected by researchers (Fuchs, et al. 2003) using situated theories of cognition due to objections involving decontextualization. Abstraction is described as a decontextualization process in which one gradually moves away from the empirical aspects of a situation where something was originally learned. Fuchs et al. (2003) describe abstractions as deleting details across exemplars and avoiding contextual specificity in order to realize generalization (p. 294). Noss, Hoyles, and Pozzi (2002) note that:

> *…an abstraction is deemed to be "apart" from, even above, the situation of its genesis. Abstractions are therefore, by definition, not about situations. Rather they involve expressions—which, although they may derive from specific situations, are meant to shift away from that situation. (p. 206)*

A fundamental problem is then that abstraction is seen as requiring the process of decontextualization for generalized learning to take place or for knowledge to be "transferred" in the classic sense from one situation to the next and yet decontextualization is deemed unacceptable because it separates knowledge from concrete experience (Beach, 2003). Cobb (2003, 2004, 2005) has argued in three recent AERA symposia that situated theorists should not dismiss abstraction on the basis of decontextualization, rather they should engage in adapting and developing alternative approaches to abstraction. In a 2005 AERA symposium, Cobb and other researchers engaged in describing such alternatives, four of which are briefly described below.

American Educational Research Association 2005 Symposium—Abstraction in Mathematics Learning: Comparing Alternative Emerging Conceptions

Four alternative conceptions of abstraction, described as being consistent with a situated cognition perspective, were discussed in this symposium. These conceptions were defined as:
(a) collective abstraction; (b) situated abstraction; (c) abstraction in context; and (d) actor-oriented abstraction.

Collective Abstraction as a Shift in the Social Situation of Learning (Cobb).

Collective abstraction was defined by Cobb's research team during a series of classroom design experiments targeted at understanding the interactions in the immediate social situation of students' mathematical learning (Cobb, 2005). Collective abstraction intends to account for the social dimensions of students' learning which also provides a context for interpreting students' individual activities. Cobb describes the social structure as a classroom microculture and analyzes students' mathematical learning in the context of the class norms and practices. Cobb described collective abstraction as occurring when members of a community, such as a class group, collectively use prior group experiences or the results of such activity, as an explicit object for class discourse. This means that the groups' previous discussion and activity becomes an object of reflection. Cobb contends that protocols of action, which he refers to as inscriptions of prior activity, are necessary for this process. Individual student mathematical learning is viewed as activity within the classroom culture, involving processes of reorganizing activity and the use of tools and symbols.

Situated Abstraction (Noss and Hoyles).

Noss and Hoyles define situated abstraction to explain the mathematical understandings that develop as individuals interact in a particular setting; using a given set of tools and a common community discourse. The ideas developed as a result of their studies involving children's use of computational systems (Noss & Hoyles, 1996) and later via studies (Noss, Hoyles, & Pozzi, 2002) based on various workplaces. They contend:

> *In both classes of settings, the process of generating and expressing mathematical meanings with the symbolic tools available tends to produce individual and collective understandings that are divergent from standard mathematics. Thus unlike official mathematical discourse, situated abstractions are the result of using the available web of tools and are articulations (for oneself and for others in the community) that depend on the artifacts and discourse of the available representational system. (Noss & Hoyles, 2005)*

The situated abstraction approach has been used in a series of studies by Mitchelmore and White (2000) in which they describe a process by which children angle concepts through progressive abstraction and generalization. In the work of Noss and Hoyles, and that of Mitchelmore and White, they contend that individual and collective understanding develops in concert with specific environments, tools, and communities of discourse.

Abstraction in Context (Hershkowiz, Schwarz, and Dreyfus).

Hershkowitz et al. (2001) described an abstraction model, based on individual student case studies from a large empirical study of a constructivist curriculum project. They described a model for the genesis of abstraction—paying particular attention to the process aspects of abstraction. They defined three dynamically nested epistemic actions within this model: constructing, recognizing, and building-with. They contend, "to study abstraction is to identify these epistemic actions of students participating in an activity of abstraction" (p.1). The researchers consider abstraction as "…vertically reorganizing previously constructed mathematics into a new mathematical structure" (p.2). This vertical reorganization activity "…indicates that that abstraction is a process with a history; it may capitalize on tools and other artifacts, and it occurs in a particular social setting" (p.2). Thus, they approach abstraction from a sociocultural point of view and base their definition of abstraction on five principles: (1) abstraction is an activity in the sense of activity theory—a chain of actions undertaken by an individual or a group and driven by a motive that is specific to a context; (2) context is a personal and social construct that includes the student's social and personal history, conceptions, artifacts, and social interaction; (3) abstraction requires theoretical thought and may also include elements of empirical thought; (4) a process of abstraction leads from initial, unrefined abstract entities to a novel structure; and (5) the novel structure comes into existence through reorganization and the establishment of new internal links within the initial entities and external links among them (p.14). The fifth principle is crucial to explaining the genesis of abstraction as a learning process and the authors clearly specify how this principle plays out in practice and were among the first to do so.

The Actor-Oriented Abstraction Approach: Coordinating Individual and Social Levels of Abstraction (Lobato).

The actor-oriented approach emerged through research directed at describing an alternative interpretation of educational transfer, called, *actor-oriented transfer* (Lobato, 2003). Transfer is seen as, "...the personal creation of relations of similarity, or how "actors" (rather than "observers" or experts) see situations as similar" (Lobato, 2005). The actor-oriented approach seeks to describe how features of instructional environments, including curricular materials, social-cultural norms, tool use (such as graphing calculators), and classroom discourse interact to affect the conceptual attributes to which students pay attention. Actor-oriented abstraction includes modifications to the Piagetian construct of reflective abstraction to address criticisms from the situation cognition perspective. The first criticism addressed is that reflective abstraction is solely an individual psychological construct and does not account for the contribution of the wider environment (including all facets of the learning situation), with the second criticism addressed involving the decontextualization problem. The actor-oriented abstraction approach seeks to describe both individual and social levels of abstraction and uses the device of *attention focusing* to do so. Reflective abstraction is used at the individual level to identify "…regularities in mental records of actions by focusing on and isolating certain properties while suppressing others" (Lobato, 2005). The social level of abstraction is addressed by identifying *records of focus* which are the mathematical ideas on which students seem to have focused their attention while interacting with particular representations in the classroom environment (Lobato, Ellis, & Mu–oz, 2003). Lobato, et al. seek to understand why these records of focus exist and account for them by what they refer to as *focusing phenomena* in the classroom including the teacher's actions, classroom discourse, curricular materials, and available tools. Decontextualization is addressed in this approach by broadening reflective abstraction to include what is called *dynamic actor-created contextualization* which is defined as "…the dynamic creation of multiple personal contexts from the student's perspective. Each context is populated by different mathematical objects and/or different relationships among objects" (Lobato, 2005). While viewpoint (novice/expert) is often central in problem-solving research, and plays a role in other researcher's approaches to reflective abstraction, Lobato's work specifies in significant detail how careful consideration of viewpoint addresses fundamental, problematic issues.

Conclusions

While earlier work in reflective abstraction provided useful tools for describing the conceptual growth of individual students, its weakness was that it was difficult to make recommendations concerning how these ideas could be used to develop curriculum or to influence classroom instruction. As the previous paragraphs reveal, the current projects dealing with abstraction are intimately involved with both curriculum development and classroom methodology. Earlier approaches to reflective abstraction focused on the creation of learning tasks designed to induce reflective abstraction and conceptual learning, and focused on the learning activities of the individual student when faced with these tasks, while the scope of more recent approaches have broadened. The learning "situation" is considered not only to be the specific learning tasks, but also the individual classroom culture. Classroom discourse is credited with its proper role as being crucial to the concepts that individual students develop, and the limitations of curricular materials alone to spur cognitive dissonance is recognized. Comparing the different approaches to reflective abstraction that have been discussed, one can summarize the approaches as encompassing the following components:

- Describing an individual student's current knowledge of a concept;
- Describing an individual student's problem solving actions;
- Designing learning tasks likely to spur reflective abstraction in individual or group settings;
- Describing the mechanism of reflective abstraction in the context of tools and language;
- Describing the mechanism of reflective abstraction in the context of the classroom environment and culture including the discourse;
- Describing reflective abstraction from the perspective of the actor or the observer.

It is exciting that the new approaches to abstraction being described are not incompatible. Indeed, it seems that they are all describing important and different facets for what happens in the process of learning to understand a concept.

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UNDERGRADUATES' ERRORS IN USING AND INTERPRETING VARIABLES: A COMPARATIVE STUDY

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This study examined undergraduate basic algebra, college algebra, and calculus students' abilities to use and interpret algebraic variables. Participants completed an algebra test to identify four hierarchical levels of variable use, from the most basic, such as ignoring or evaluating variables, to the more complex uses of variables as generalized numbers or varying quantities. Students' levels of variable use were determined, and types of common errors were tabulated and compared across the three courses. Results showed that levels of variable use generally increased with course difficulty. The levels were related to the mean course grades for college algebra and calculus students. Considerable numbers of students in all three courses exhibited high error rates on test items requiring the interpretation and use of variables as generalized numbers or functionally related quantities to represent word problems. Common errors included using variables as labels, making strictly literal translations, and failing to extract meaning from variable expressions.

Overview

Research has documented some of the difficulties that students have in using and interpreting algebraic variables, such as using variables inappropriately as labels and representing word problems incorrectly in equation form (Gray, 2002; MacGregor & Stacey, 1993, 1997; Sokolowski, 1997; Trigueros & Ursini, 2003). Difficulties with using and interpreting variables are likely to contribute to poor performance in undergraduate mathematics courses, including calculus (Gray, Loud, & Sokolowski, 2005; White & Mitchelmore, 1996). It is generally expected that students entering calculus will be able to appropriately use and interpret algebraic variables as generalized numbers and functionally related quantities. Students who need further development in this area typically take one or more prerequisite algebra courses.

This research examined college students' uses and interpretations of algebraic variables when they entered basic algebra, college algebra, or calculus courses. This study documents these students' levels of variable use (Hart, Brown, Kerslake, Küchemann, & Ruddock, 1985; Küchemann, 1981) and compares these levels to course grades. The study further identifies errors that students made when attempting to use and interpret variables in expressions and equations that required the understanding of variables as generalized numbers and as functionally varying quantities. Common errors are compared across the three courses, and explanations for these errors are posited.

Theoretical Framework

It has been suggested that one underlying cause for the learning difficulties associated with the notion of variable is the number of differing uses and interpretations of algebraic variables (Schoenfeld & Arcavi, 1988: Usiskin, 1988). Among these are the use of variables as specific unknowns, as generalized numbers, or as varying quantities (Küchemann, 1981; Schoenfeld & Arcavi, 1988; Usiskin, 1988). Küchemann (1981) developed an algebra test that identifies four

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hierarchical levels of variable interpretation according to differing uses. Students whose scores on this test indicate that they are at Levels 1 or 2 are primarily ignoring or evaluating variables or using them as labels, three extremely elementary uses of variables. Students whose scores indicate that they are at Level 3 can appropriately use variables as specific unknowns and generalized numbers, while those who can interpret variables appropriately in functional relationships fall into Level 4.

Formulating and interpreting expressions and equations that represent relationships found in word problems is well-recognized as a major difficulty for students in secondary and undergraduate mathematics (Kieran, 1992). This study provides evidence to support the notion that a weak understanding of the algebraic variable contributes to this difficulty.

Method

The present study employs an adaptation of Küchemann's algebra test (Sokolowski, 1997) to determine undergraduates' levels of variable use and makes comparisons in the ways these students use and interpret variables in basic algebra, college algebra, and calculus courses. The test was administered on the first day of class to 346 students at two private Liberal Arts colleges in New England. Students' levels of variable use as identified by this test were determined according to Küchemann's criteria (Hart et al., 1985). Percentages of students at each level were compared across the three courses, and mean grades for students at each level were calculated for each course. Finally, responses to four test questions requiring the use or interpretation of variables as generalized numbers and as functionally related quantities were compared across the three courses.

Results and Discussion

Levels of Variable Use

In general, students' levels of variable use increased with course difficulty. Percentages of students who tested into each level in each course are shown below in Table 1.

Table 1. Percentages of Students in Each Level by Course

Level	Basic Algebra n=75	College Algebra n=94	Calculus n=177
1	25.3%	22.3%	2.3%
2	38.7%	29.8%	7.3%
3	32.0%	35.1%	31.1%
4	4.0%	12.8%	59.3%

Although students in basic algebra and college algebra were similarly distributed among Levels 1, 2, and 3, the greatest percentage of basic algebra students tested into Level 2, and the greatest percentage of college algebra students tested into Level 3. Importantly, more than one-half of entering college algebra students used variables at Levels 1 and 2. These students were not using variables correctly as generalized numbers and functionally-related quantities. Although a distinct majority of calculus students tested into Level 4, more than forty percent were using variables at Levels 1, 2, or 3, which are levels far below expectation for entry into this course.

The mean course grades, based on a four-point scale, for students testing at each level were computed at the end of the semester and are shown below in Table 2.

Table 2. Mean Grade in Each Level by Course

Level	Basic Algebra n=75	College Algebra n=94	Calculus n=177
1	1.89	1.06	.68
2	1.48	1.28	1.72
3	1.67	1.76	1.63
4	1.77	2.92	2.77
Overall	1.65	1.61	2.29

There appears to be no pattern in mean grades at each level for students in basic algebra, a course designed for students who typically have difficulty with algebra. However, mean grades generally increased according to level for students in college algebra and calculus, the more traditional courses for entering college students. These data indicate that an advanced level of variable use is an important factor for success in college mathematics.

Common Errors in Using and Interpreting Variables

For the purposes of this paper, four test questions were analyzed in detail. These questions required the interpretation and construction of algebraic expressions and equations. These questions appeared throughout the test and are renumbered here for the purpose of discussion.

Errors involving algebraic expressions. Two test questions required students to interpret given variable expressions involving the numbers and prices of two items. One of these also asked students a question that required them to formulate a variable expression as a response. These questions were:

1a. Small apples cost 8 cents each and small pears cost 6 cents each. If a stands for the **number** of apples bought and p stands for the **number** of pears bought, what does $8a + 6p$ stand for? Correct response: Total cost or price of fruit

1b. What is the total number of fruits bought? Correct response: $a + p$

2. Bagels cost b cents each and muffins cost m cents each. If I buy 4 bagels and 3 muffins, what does $4b + 3m$ stand for? Correct response: Total cost or price of bagels and muffins

Because this study investigated students' errors, percentages of *incorrect* responses for Questions 1a, 1b, and 2 are given below in Table 3.

Table 3. Percentages of Incorrect Responses by Course

Problem Number	Basic Algebra n=75	College Algebra n=94	Calculus n=177
1a	81%	76%	50%
1b	84%	85%	64%
2	63%	55%	24%

As would be expected, the percentages of incorrect responses for each question decreased as the course difficulty increased. Nevertheless, there were substantial numbers of students in all three courses who did not answer these questions correctly.

For Question 1a, which asked for an interpretation of the expression, $8a + 6p$, calculus and college algebra students showed similar types of errors. Their most common error was to substitute 8 for a and 6 for p, and then compute to arrive at an answer of 100 or $1.00. It appears that some students used the given price as the number of fruit. Others may have replaced the variables with the numbers given in the problem because they felt compelled to give a numerical rather than a descriptive answer. These responses suggest that students were using variables as labels: a for apples and p for pears (Hart et al., 1985; Herscovics, 1989; McGregor & Stacey, 1997). The second most common error among calculus and college algebra students was an answer that explicitly included the words, "8 apples and 6 pears," or, "the number of fruit." These responses reflect confusion in the meaning of the 8 and the 6, which represent respective prices in $8a + 6p$, again suggesting the use of variables as labels. The third most common error for the calculus and college algebra students took one of two forms: "8 cents for a apples plus 6 cents for p pears," or, "price x number." These responses are direct translations from the variable expression to words and show a recognition that the expression represents both the price and the number of fruit. This indicates an appropriate interpretation of the expression, but an incomplete understanding of the expression as a whole, i.e., the total cost.

Basic algebra students exhibited the same errors described above for Question 1a, but in different proportions. Their most common error was to respond with, "8 apples and 6 pears," or, "the number of fruit," the result of using the variables as labels for apples and pears.

Question 1b, which asked for the total number of fruits bought, showed the highest error rates of all questions analyzed for students in all three courses. By far, the most common error was to give the answer, "14," or, "14 fruit," reached by adding 8 (*apples*)+ 6 (*pears*). Even high percentages of calculus students used variables in this way as labels. Perhaps they did not consider giving an answer in the form of an algebraic expression (Kieran, 1992). The second most common error for students in all three courses was to leave this question blank or to say that there was not enough information, probably concluding that a variable expression did not provide information about the numbers of fruit.

Question 2 was similar to Question 1a in that it required the interpretation of a given variable expression that included prices and numbers of two different items. In contrast to Question 1a, the numbers in Question 2 referred to the numbers of items, and the variables referred to the prices. Students in all three courses were far more successful with Question 2 than they were with Question 1a. For calculus students, the most common error was to interpret the variable expression as representing the number of bagels and muffins rather than the cost of these. This error suggests the use of variables as labels. A second common error was to give a response in the form of a direct translation, "4 bagels at b cents plus (or +) 3 muffins at m cents." A third type of error was an answer referring to the number bought and cost. The second and third errors show recognition of the components of the variable expression and may be considered transitional, incomplete responses that do not reflect the level of abstraction required to give the correct answer, "total cost."

College algebra students' errors for Question 2 took the same forms as those described above for calculus students. However, the highest frequency of college algebra students' errors was in the "number bought and cost" category noted above.

Basic algebra students' most frequent incorrect response to Question 2 was to somehow refer to the total number of items bought or to write, "4 bagels plus 3 muffins." These errors indicate an interpretation of the variables as labels for the bagels and muffins.

Errors involving algebraic equations. Two test questions required students to construct algebraic equations. One of these also asked students to compute a value represented by the equation from the previous question. These questions are:

3a. Mary's basic wage is \$200 per week. She is also paid another \$7 for each hour of overtime that she works. If h stands for the number of hours of overtime that she works, and if W stands for her **total** weekly wages (in \$), write an equation connecting W and h. Correct response: $W = 200 + 7h$

3b. What would Mary's total wages be if she worked 4 hours of overtime? Correct response: \$228

4. Fine point black pens cost \$3 each and medium point red pens cost \$2 each. I went to Staples in Salem, New Hampshire, and bought some of each type of pen, spending a total of \$25. If b is the number of black pens, and if r is the number of red pens bought, what can you write about b and r? Correct response: $3b + 2r = 25$

Percentages of *incorrect* responses for Questions 3a, 3b, and 4 are given below in Table 4.

Table 4. Percentages of Incorrect Responses by Course

Problem Number	Basic Algebra n=75	College Algebra n=94	Calculus n=177
3a	77%	78%	47%
3b	17%	16%	12%
4	60%	41%	24%

Basic algebra and college algebra students performed similarly on Questions 3a and 3b and showed greater percentages of incorrect responses than did calculus students. For Question 4, the percentages of incorrect responses decreased considerably as course difficulty increased.

Question 3a required students to construct an equation relating two variables, W and h. Common errors included expressions or equations taking these forms: $W + 7h$, or $W + 7h$ = total; $200W + 7h$, or $200W + 7h$ = total; $w + h$, or $w + h$ = total. The first of these was the most common error among calculus students, while the first and third were the most common errors among both college algebra and basic algebra students. All of these errors seem to indicate reluctance to follow the instruction in the problem to let W represent the *total* weekly wages. Instead, students used W to represent *basic* weekly wages and h to represent hourly wages. They incorrectly used the variables as labels in order to write a formula to calculate total wages (Kieran, 1992; Stacey & MacGregor, 1999). Some students may have been making a direct translation from the wording of the problem to a symbolic form (Kieran, 1992).

Question 3b required students to compute the total weekly wages for a given number of overtime hours. All students were markedly better at computing the result than they were at representing the situation with an equation. This finding suggests that most students understood the situation, but were not able to express it symbolically, a common result that has been reported by other researchers (Stacey & MacGregor, 1999; Trigueros & Ursini, 2003).

Question 4 provided prices for black and red pens and a total amount spent. The variables, *b* and *r*, were identified as the number of each type of pen bought, and students were asked, "What can you write about *b* and *r*?" More than three-quarters of the calculus students answered this question correctly. Calculus students' most common incorrect response was to leave a blank, possibly because this question appeared at the end of the test and was between two difficult questions. The most common error for college algebra and basic algebra students was, "$b + r = 25$," a response that results from using variables as labels for sets of black and red pens (Herscovics, 1989). A less common error, shown by students in all three courses, was, "$b = 5$, $r = 5$." This response is actually one pair of solutions to the question of how many pens at each price could be purchased for $25. It demonstrates the tendency of undergraduate students to evaluate variables using arithmetic methods, as has been reported for high school students (Hart et al., 1985; Kieran, 1992; Stacey & MacGregor, 1999).

Questions 3a and 4 were the only questions analyzed that required equations as correct answers. Students in all three courses performed considerably better on Question 4 than they did on Question 3a. It is possible that students wrote the correct equation for Question 4 while incorrectly using variables as labels, which may have contributed to the high success rate for this question. Students could have been writing $3b + 2r = 25$ to represent either, "3-dollar black pens and 2-dollar red pens equal $25," or, "3 black pens plus 2 red pens equal 25." Alternatively, students could have performed better with Question 4 because it resembles a typical textbook word problem. They could simply lift the numbers and letters out of the text in the order they were given and use them to construct an equation. Also, in contrast to Question 3a, Question 4 gives a numerical total, which appeared to assist students in writing an equation.

Conclusions

This study shows that college algebra and calculus students' mean course grades increased along with levels of variable use, as determined by KŸchemann's algebra test (Hart et al., 1985). As would be expected, basic algebra students used and interpreted variables at the lower levels of Küchemann's (1981) hierarchy. Error analysis showed that these students exhibited the greatest frequency and variety of errors. Although more college algebra and calculus students tested into Levels 3 and 4, respectively, they had difficulty using variables as generalized numbers and as functionally-related quantities. Frequently observed errors for all students included using variables as labels and making strictly literal translations between algebraic expressions or equations and their verbal interpretations. Students also exhibited the inability to attach meaning to algebraic expressions or to use expressions as generalized answers. An unexpected finding was the frequency with which calculus students made many of the same errors as did college algebra and basic algebra students. These types of errors are similar to those found in previous studies involving college age and younger students. (Trigueros & Ursini, 2003; MacGregor & Stacey, 1993, 1997). In particular, the tendency to incorrectly use variables as labels when the variables were defined as the first letters of the objects or quantities they were intended to represent was pervasive among students in all three of the mathematics courses examined.

This study found high numbers of students entering college algebra and calculus who, despite their previous study of mathematics, did not have the ability to use and interpret algebraic variables at Küchemann's (1981) more abstract levels. It would be useful for college mathematics instructors to have a sense of their students' abilities to use and interpret variables at the beginning of the semester. They can then explicitly focus on developing students' understandings of variables as generalized numbers and varying quantities in courses from basic algebra through calculus.

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A NUMERICAL APPROACH AS A PROBLEM SOLVING STRATEGY IN ALGEBRA

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Students' difficulties in constructing and assigning meaning to algebraic equations have been at the center of numerous research studies. We have found that while most students know how to manipulate formulas, simplify algebraic expressions and solve some typical textbook problems, many fail to interpret and construct basic algebraic equations when problems are presented verbally. In this study, our goal was to analyze the causes for difficulties of a college student in constructing and interpreting algebraic equations, and investigate the results of the numerical approach on his solution strategies. In advanced mathematics classes, the use of methods more advanced than these numerical ones for solving problems becomes necessary. However, some students might not benefit from these advanced strategies. In this case, students' solution strategies are a result of their reasoning about the problem prior to attempting to construct an equation or make inferences from an equation. The results of this study indicate that the LCM method could be helpful for students as a problem-solving strategy to construct equations, make inferences from the equations as well as understand the relationship between the coefficients and the value of the variables.

Students' difficulties in constructing and assigning meaning to algebraic equations have been at the center of numerous research studies (e.g., Booth, 1984; Kieran, 1992). We have found that while most students know how to manipulate formulas, simplify algebraic expressions and solve some typical textbook problems, many fail to interpret and construct basic algebraic equations when problems are presented verbally. That is, students develop algorithmic strategies leading to solutions of algebra problems, but they may not understand the underlying structures of the word problems. Thus, students encounter two main dilemmas when confronted with problems presented in words: 1) difficulty in constructing equations that would lead to a solution to the problem, and 2) difficulty in creating meaning for the equations they are using. In this latter case, students are able to construct equations for problems similar to ones they have solved previously, but do not always understand the meaning behind these equations they are using.

Background

Some researchers (e.g., Booth 1984; Linchevski & Livneh, 1999) believe that students' difficulties in algebra stem from their lack of understanding of arithmetic and suggest that teachers start with students' knowledge of arithmetic to help students make the transition from arithmetic to algebra. Sfard (1991) has described two fundamentally different approaches to teaching abstract mathematical concepts. The first is the *structural* approach in which operations performed on algebraic expressions yield algebraic expressons. For example, a teacher may ask a student to simplify an expression such as $3x - x + 2x$. The second is *operational*, an approach which leads to a numerical result. Here, a teacher may ask a student to solve $2x + y$, when $x = 1$ and $y = 2$; this is computational and the result is numerical. Sfard has concluded that the operational approach should precede the structural, because the structural appoach is the more advanced stage of concept development. The terms *operational* and *numerical* have been used

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interchangably to describe this last approach; in this paper we will use the term, numerical, to describe the approach.

Several similar numerical methods have been suggested. In order to help students construct equations, Soloway, Lochhead and Clement (1982) utilized a computer program with problems such as the following: "Write an equation using the variables S and P to represent the following statement: "There are six times as many students as professors at this university." Use S for the number of students and P for the number of professors." This enabled students to find the number of students, when the number of professors was entered. When students analyzed their self-generated data, they were able to construct the necessary equations without difficulty, and as a result, their students were successful with this enter-and-check numerical approach.

Similarly, Rubio (1990) asked students to use a trial-and-error method in order to analyze a problem and reach a solution. After several trials of different numbers, the process that students used led to an equation for the problem. Demana and Leitzel (1988) suggested an approach that depends on a calculator and problem solving in order to strengthen students' understanding of arithmetic. After students solved problems numerically by building tables and using guess-and-check procedures, they investigated the problems geometrically by making graphs of the relationships in the problems. Demana and Leitzel concluded "basic concepts of algebra are accessible to students in their arithmetic experiences through numerical computation and problem solving. In particular, the ideas of variable and function can be established, as a consequence, in later college preparatory courses" (p.68). Bernard & Cohen (1988) utilized an interactive computer program in order to help students to find a root of an equation. Students were asked to generate different values for testing. After generating and evaluating some values, students determined whether or not a root was larger and what a root could be. Then, they generated new values to find the root of the equation. Generate-and-evaluate methods helped students formulate their understanding of roots and the solutions process.

In these ways, we too have seen that students have been successful with numerical approaches in which students are asked to calculate one variable by entering the value of the other variable and checking the results. The use of such numerical approaches facilitated students' conceptual understanding of the relationships between the variables described in the problems. Consequently, students were more likely to create meaning for the equations.

The LCM Method

In advanced mathematics classes, the use of methods more advanced than these numerical ones for solving problems becomes necessary. However, some students might not benefit from these advanced strategies when they begin the study of algebra and have difficulty understanding symbolic representations. In this case, since a numerical approach makes the problem more visual and concrete, and numerical operations can be manipulated easily by students, we will describe a unique numerical approach for students'constructions of equations for some types of algebra problems. Students' solution strategies are a result of their reasoning about the problem prior to attempting to construct an equation or make inferences from an equation.

This method utilizes the least common multiple (LCM) of the coefficients in equations (hereafter called the LCM method). For example, students may experience difficulty when constructing an equation for the following task:

> Write an equation using the variables R and B to represent the following statement: At Vallapart Motors there are 4 blue cars produced for every 5 red cars." Use B for the number of blue cars and R for the number of red cars.

Students may write "$4B = 5R$" to represent the relationship between the number of blue cars and the number of red cars. Using the LCM method, the least common multiple of the coefficients in the equation (or any multiple of the least common multiple) can be used to calculate the values of the variables and to find out which variable has a greater value. The least common multiple of 4 and 5, which is 20, is used as the total number of blue or red cars. Then, the number of production runs a factory needs to produce 20 blue or 20 red cars is calculated. The factory needs 5 runs for the blue cars ($5 \cdot B = 5 \cdot 4 = 20$) and 4 runs for the red cars ($4 \cdot R = 4 \cdot 5 = 20$). In other words, the equation that represents the relationship statement in the question is "$5B = 4R$." Our work with students demonstrates that these values also help students understand the relationship between the coefficients and the value of the variables.

We tried this approach with a student over a period of several months. The results of the study are summarized in the following paragraphs. Our goal was to analyze the causes of difficulties of a calculus student in constructing and interpreting equations in algebra, and to investigate the results of the numerical approach on his solution strategies.

The Case of Eric

We worked with Eric, a calculus student who had been successful in his algebra classes. In this study, three audiotaped interviews were conducted in 9 months. The second interview was conducted three months after the first interview and the third interview was conducted 6 months after the second interview. The purpose of solving the problems after the first interview and asking the same questions in subsequent interviews was to investigate whether or not the participant would later use the same strategies that led to his misunderstanding in the first interview. We discussed the solutions to the problems during the first interiew, and three months after this meeting, we asked him to solve the same problems. Six months after the second interview, we again asked Eric to solve the problems. During the interviews, the following algebra word problems from previous studies were used:

<u>Task 1</u>: Write an equation using the variables S and P to represent the following statement: "There are six times as many students as professors at this university." Use S for the number of students and P for the number of professors. (Clement, Lochhead & Monk, 1981)

<u>Task 2</u>: At Vallapart Motors the equation $5B = 4R$ describes the relationship, which exists between B, the number of blue cars produced and R, the number of red cars produced. Next to each of the following statements place a T if the statement follows from the equation, an F if the statement contradicts the equation, and a U if there is no certain connection.

a) There are 5 blue cars produced for every 4 red cars
b) The ratio of red to blue cars is five to four.
c) More blue cars are produced than red cars (Kaput & Sims-Knight, 1983)

<u>Task 3</u>: Write an equation using the variables C and S to represent the following statement: At Mindy's restaurant, for every four people who order cheesecake, there are five people who order strudel. Let C represent the number of cheesecakes and S represent the number of strudels ordered. (Clement, Lochhead & Monk, 1981)

These questions allowed us to infer the students's understanding of variables and equations. We wanted to investigate whether using a numerical approach would help him construct equations and make inferences from equations as well as whether or not he would continue to understand the LCM Method.

Description of Eric's Work and Thinking

In the first interview, for the students-professor problem he wrote "$6S = P$". Suddenly, Eric realized that the equation was incorrect, because the numbers that he substituted for P and S did not satisfy the equation. He corrected the equation by using the method of guessing-and-substituting discussed above. The following illustrates his thinking: "As professors (he wrote $6S = P$), so for every student you multiply (he is checking the equation $6 \cdot S = P$) that would give you [long pause], that is incorrect. Students equal professors times 6."

When he was given Task 2, he solved the equation $5B = 4R$ for B and R correctly. He concluded that one blue car was equal to $\frac{4}{5}$ red car or one red car was equal to $\frac{5}{4}$ blue car. Then, he said there were one-quarter blue cars for every red car. At this point he did not realize that the number of red cars is more than the number of blue cars. He also did not understand the relationship between the unknowns and the numbers in the equation. As a result, he said that there were 5 blue cars produced for every 4 red cars.

It was easy for him to solve the equation for R and B and determine whether the statement "The ratio of red to blue cars is five to four" was true or not. However, he failed to interpret the ratios $\frac{4}{5}R$ and $\frac{5}{4}B$ correctly. He did not notice that the ratio of red to blue cars is $\frac{5}{4}$, whereas the ratio of blue to red cars is $\frac{4}{5}$ which is less than one, so more red cars are produced than blue cars. As a result of this, he answered that the statement "More blue cars are produced than red cars" is true.

Three months later, he was interviewed again to determine whether he would use the same strategies that led to his misunderstanding in constructing and interpreting equations in the first interview. He explained his understanding for Task 1 and constructed the equation correctly. Then he was given Task 2 and asked to determine whether the statements were true or not, and the same misconceptions surfaced. He correctly solved the equation for B and R and said that one blue car was equal to $\frac{4}{5}$ red car and one red car was equal to $\frac{5}{4}$ blue car. However, he did not know which number was bigger. He assumed that the number of blue cars was 5, because the coefficient of B was 5 and the number or red cars was 4, because the coefficient of R was 4. As a result of this misconception, he answered that the statements "There are 5 blue cars produced for every 4 red cars" and "More blue cars are produced than red cars" are true.

The task was rephrased as: At Vallapart Motors there are 4 blue cars produced for every 5 red cars. Use B for the number of blue cars and R for the number of red cars. At this time we asked him to use the LCM method, described above, in which the least common multiple of numbers is used to make sense of the equation. He calculated the least common multiple of 4 and 5 and used this as the total number of blue and red cars. Then he was asked how many production runs the factory needed to produce 20 blue and 20 red cars: he said 5 runs for blue and 4 runs for red and wrote the equation "$5B = 4R$". He realized that the bigger the coefficient the smaller the value of the variable and noticed how he misrepresented the equation for the first question.

As a follow-up, the professor-student problem was rephrased as:

> There are 9 students for every 2 professors at this university. Use S for the number of students and P for the number of professors.

We asked him to solve this problem by using the LCM method. He said the least common multiple of 2 and 9 is 18. He assumed that there are 9 students and 2 professors in every group of 11 people. Then, he said he needed 2 groups in order to have 18 students and 9 groups in order to have 18 professors. After this point, it was very easy for him to write the equation "$2S = 9P$". Then he wanted to solve the second question again. He set equation $5B = 4R$ equal to the least common multiple of 4 and 5, which is 20 and solved the equation for B and R. He assumed that the total number of cars was 20. If $5B = 20$, then the factory needed to produce 4 blue cars in every production. Similarly, if $4R = 20$, then the factory needed to produce 5 red cars in every production.

The third interview was conducted 6 months after the second interview. He was asked to solve Task 2. He had no difficulty finding the ratio of red to blue cars. In order to determine whether or not the statements were true, he substituted 4 for B to have a number that could be divisible by 4. At this point it was easy for him to determine that the statements (a) and (c) were false.

Then, he was asked to construct an equation for the statement in Task 3. First he rewrote the information as $4C$ and $5S$. Then he substituted 4 for C in order to have 20. He calculated the values of C and S that helped him construct the equation. The following is an excerpt from the interview that illustrates his thinking.

> Student: So for every 4 cheesecake, there are 5 people who ordered strudel. $4C$ and $5S$ now I'm going to check this.
>
> Teacher: Why did you write $4C$ and $5S$?
>
> Student: It is just because 4 and cheese are together, so I was not thinking mathematically. I was just rewriting the question. One way to check this equation I put in 4 there (substituting 4 for C in the equation $5C = 4S$) so I get 20. if I divide 20 by 4 solve for S and that would be 5.
>
> Teacher: Why did you substitute 4?
>
> Student: Because I know it (i.e., 20) would be multiple of 4. I like full numbers.

Clearly, as a result of using the LCM method, Eric has developed meaning for the relationship of the variables and their coefficients.

Results

The results of this study revealed that the student had little understanding of constructing equations for problems, although he took some advanced mathematics classes. He failed to construct the equation for the first task in the first interview. He did not make the same mistake in subsequent interviews and constructed the equations for Task 1 and Task 3 correctly. However, he had difficulties in interpreting the equation in Task 2 and considered the coefficients as the values of the variables. In addition, although he successfully solved the equation for B and R in the first and second interviews, he could not determine which number was bigger. It was observed that the numerical approach helped him understand the relationship between the coefficients and the variables. He realized that his way of using the coefficients as the values of the variables would lead him to misrepresent the values of the variables. In addition, the LCM method helped him understand the relationship between the value of the coefficient and the value of the variable. When he was confronted with the problems six months later, he used this numerical approach to verify and interpret the equations, and with this approach, he was able to create meaning for the equations given in the problems.

Conclusions

Although the relationship between students' knowledge of numbers and letters is too complicated to be described by any simple models (Wong, 1997), the results of this study indicate that the LCM method could be helpful for students as a problem-solving strategy to construct equations as well as to make inferences from the equations. As teachers, we should be aware of students' difficulties in interpreting and constructing equations, because there will always be students who have difficulties understanding these concepts. Using the numerical approach as an initial step may facilitate students' learning as long as we avoid meaningless manipulations within the number system (Linchevski & Livneh, 1999).

Although further study needs to be conducted, the results of this study have shown two things.

1. The student, who had difficulties in constructing equations and making inferences from equations, was not able to retain his understanding of his former, and generally more typical, strategies for solving problems three months after the first interview.
2. Using the LCM method, employing the least common multiple as a problem solving strategy, the student was able to retain his understanding and solve problems after six months.

This study has reported the case of a student who experienced success using the LCM method. Further research is warranted, particularly studies that test whether the student's success might be representative of other students.

Of course, not all your students will go on to study more advanced algebraic methods; however these students too can be successful in solving problems. But problems such as the following can lead students to the use of variables as placeholders in problems as teachers build on students' numerical solutions methods. The following problems, although ostensibly different, also may be solved using the LCM method.

<u>Problem 1:</u> An oil tank can be filled with oil in 10 hours by one pipe. Another pipe can fill it in 15 hours. If both pipes are used to fill the tank, how many hours will it take them?

<u>Problem 2:</u> John spent 2/5 of his monthly allowance for rent and 1/8 of his money for utilities. The total amount of money that he spent is \$630. How much money did he have at the beginning?

<u>Problem 3:</u> Tom deposits 1/3 of his monthly salary to his savings account every month. However, this month he had to lend 3/5 of this amount to his friend and deposited \$320 into his savings account. How much does he deposit into his savings account every month? How much is his monthly salary?

For example, the following also shows how the LCM method may be used for Problem 2:

We need to find the least common multiple of key numbers in the problem. The least common multiple of 5 and 8 is 40 and this quantity, since it can represent the amount, 40 "units" of money John had originally, can be used as the placeholder in the problem. Now the task is to find the value of one unit:

John spent $\frac{2}{5} \bullet 40$ units = 16 units for rent.

John spent $\frac{1}{8} \bullet 40$ units = 5 units for utilities.

Since he spent \$630 for rent and utilities, 16 units + 5 units = 21 units = \$630. Therefore, 1 unit is \$30. John had \$30•40 = \$1200 at the beginning. Let's substitute $\frac{2}{5} \bullet 40$ units for 16 units and $\frac{1}{8} \bullet 40$ units for 5 units in the following equation:

16 units + 5 units = \$630. This gives us: $\frac{2}{5} \bullet 40$ units + $\frac{1}{8} \bullet 40$ units = \$630.

We know that John had \$1200 at the beginning and $\frac{2}{5} \bullet \$1200 + \frac{1}{8} \bullet \$1200 = \$630$

After solving some similar problems, you may wish to introduce the concept of unknown for some of your students at this point and help students write the equation using *X* as a placeholder.

$\frac{2}{5} \bullet X + \frac{1}{8} \bullet X = \$630$, where *X* represents the money John had originally.

Thus students learn to appreciate the unknown as a placeholder.

By viewing algebra as a strand in the curriculum from pre-kindergarten through high school, as teachers we can help students build a solid foundation of understanding and experience as a preparation for more sophisticated work in algebra in the middle grades, high school, and beyond (NCTM, 2000). Students can learn to appreciate algebra as the language through which much of mathematics is communicated as they develop effective strategies for presenting algebra from conceptual, representational, and problem-solving perspectives. We invite you to share your experiences with the LCM method with us.

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MATHEMATICAL CARING RELATIONS AS A FRAMEWORK FOR SUPPORTING RESEARCH AND LEARNING

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Mathematical caring relations (MCRs), a framework for conceptualizing student-teacher interaction, was used in a year-long constructivist teaching experiment with 4 6th grade students. MCRs supported (1) the extension of previous research on how students construct improper fractions and (2) the learning of students and their teacher (the researcher). Establishing a MCR entails aiming for mathematical learning while attending to affective responses of both student and teacher. Although all students entered the experiment with the splitting operation deemed necessary for constructing improper fractions (Steffe, 2002), during the experiment 2 students did not construct improper fractions. One of these students is the focus of this paper. The current hypothesis is that splitting does not automatically engender the coordination of 3 levels of units that seems necessary to construct improper fractions. Analyzing MCRs in research is seen to facilitate interactions that can lead to learning and to validate the experiential difficulties of learning.

Purpose

Mathematical caring relations (MCRs) is a framework for conceptualizing student-teacher interaction that conjoins cognitive and affective realms (Hackenberg, 2005b). This paper explores how establishing MCRs in a year-long, constructivist teaching experiment with four sixth grade students supported the extension of previous research on how students construct improper fractions (Steffe, 2002). The paper addresses two central questions: What does describing and analyzing affective responses contribute to research on mathematics learning? What learning occurred (and did not occur) for one of the students and her teacher (the researcher and author) in the process of establishing a MCR?

Background

Interaction and Learning

Social interaction is basic to mathematics learning: Interactions among teachers and students form a major, though not comprehensive (cf. Confrey, 1995), impetus for the learning of both teachers and students. I contend that any social interaction can trigger depletion and stimulation for the people involved. I define *depletion* as a feeling of being taxed, usually accompanied by a decrease in energy or a diminishment of overall well-being. I define *stimulation* as a feeling of being excited or awake, usually accompanied by a boost in energy or a stronger sense of aliveness. In any social interaction these two feelings may be negligible and the dominant feeling may be “evenness” or neutrality. But in some social interactions the two feelings fluctuate more obviously or one greatly outweighs the other. In social interactions focused on learning, sustaining some level of depletion is often necessary for subsequent feelings of stimulation. However, feelings of depletion may dominate for a variety of reasons.

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Caring Relations

Sustained depletion may contribute to comments from both students and teachers that the other party doesn't care. Students who experience sustained depletion may say that their teachers don't care about students' lives or ways of thinking, which implies a lack of awareness about or valuation of them. These students don't feel *cared for* by their teachers. In turn, teachers may say that these students don't care about learning, a particular subject area like mathematics, or school. These teachers don't feel cared for *as teachers* because they don't experience their students' responsiveness to and engagement with the activities they orchestrate and the questions they pose. These feelings may not correspond with the other party's intentions—teachers may care very much about their students, and students may be interested in learning. But teachers and students often fail to develop *caring relations* (Noddings, 2002). Such failures can interrupt or stunt learning for both students and teachers.

In Noddings' (2002) notion of caring relations, a teacher (a carer) orchestrates experiences in which a student (a cared-for) feels that her or his need for care in student-teacher interactions is satisfied. For Noddings, establishing care as a teacher means being engrossed in students' ideas and worlds while working cooperatively with students to realize and expand their ideas and worlds. Receiving care as a student means experiencing renewed interest or activity, an increase in commitment or energy, or even a "glow of well-being" (p. 28). A caring relation is reciprocal because evidence of the reception of the teacher's care is what the teacher needs most to feel "cared for" by the student and to continue to care. Thus caring is not simply a feeling, or a virtue. Caring is an orientation to co-create and participate in social interactions that are responsive to the cognitive and affective states and needs of both teachers and students.

I conceive of establishing care in *mathematical* interaction as inseparable from engendering mathematical learning (Hackenberg, 2005b). Mathematics teachers may act as carers in general, but they start to act as *mathematical* carers when they hold their work of orchestrating mathematical learning for their students together with an orientation to monitor and balance feelings of stimulation and depletion that may accompany student-teacher interactions. From a student's perspective, participating in a MCR involves being open to the teacher's interventions in the student's mathematical activity and pursuing questions and ideas of interest.

Theoretical Framework

Mathematical Learning

Because MCRs cannot be established without aiming for mathematical learning, situating MCRs in a model of learning is essential. Following Piaget (1970) and von Glasersfeld (1995), I define mathematical learning as a process in which a person makes relatively permanent modifications or reorganizations in her or his ways of operating in response to perturbations (disturbances) brought about by her or his current ways of operating. The outcome of mathematical learning is a new way of operating adapted from a previous way of operating. An outcome is more or less permanent if, from the learner's point of view (but not necessarily at the level of awareness), it continues to be useful to the learner in on-going interaction. If the new way of operating solves situations not previously solved and can serve in further learning, then it can be considered more powerful than a previous way of operating.

Schemes and operations. I use the phrase "way of operating" to refer to a range of repeatable activity in which a person engages, such as a student regularly telling the teacher that his head hurts when a problem seems difficult. In contrast, I use the words "scheme" and "operation" more specifically. An operation is a mental action, such as repeating an item in imagination to create a plurality of items (an iterating operation). Operations are the components of schemes, goal-directed ways of operating that consist of a situation, an activity, and a result (Piaget, 1970; von Glasersfeld, 1995). To initiate the activity of a scheme, a situation must be perceived by a person as similar in some way to previous situations in which the person used the scheme. This perception or recognition is the result of assimilation, the basis for construction—and modification—of schemes. The perceived situation then triggers the activity of the scheme, which may be mental or physical or both. The person generally anticipates that the result of the scheme, an outgrowth of the activity, will be expected or satisfying in some way.

Accommodations. Modification or reorganization of a scheme—i.e., an accommodation—may occur when a person's current schemes produce an unexpected result: The person does not achieve her intended goal. This "disturbed" state of affairs is one example of a perturbation and is often accompanied by a sense of disappointment or surprise. As von Glasersfeld (1995) emphasizes, a person's "unobservable expectations" (p. 66) are instrumental in initiating a perturbation because what is crucial is the degree to which the unexpected result "matters" to the person at an intentional or unintentional level. This aspect of perturbations means they are not always consciously conflictive: An unexpected result may remain largely unnoticed by a person and yet have some impact on a person's subsequent activity—perhaps in a vague sense of unease or a somewhat heightened interest. Thus even perturbations that are mostly outside of immediate awareness involve an affective aspect. As a person (consciously or unconsciously) eliminates perturbations, or equilibrates, the perturbation has the potential to trigger an accommodation.

Affective aspects of perturbations. The affective aspect of a perturbation is a major point of connection between mathematical learning and mathematical caring. That is, experiencing a perturbation can be accompanied by feelings of both depletion and stimulation. Feelings of depletion may occur if a person senses that she or he does not know what to do to eliminate the perturbation, or that such activity will be particularly onerous. If a feeling of depletion is too great or extended for too long, a student may feel overwhelmed, which may impede engagement in mathematical activity either immediately or in the future. Perturbations can also provide stimulation in the form of a challenge, particularly if a person senses that she or he can meet that challenge, or that such activity itself will be enjoyable. If a feeling of stimulation is sufficient, the student's interest in or curiosity about a situation may prolong mathematical activity and open new opportunities for learning. If, over time, feelings of stimulation outweigh feelings of depletion, the student may feel mathematically cared for.

Enacting Mathematical Caring Relations

Teacher's activity. In establishing MCRs, teachers attempt to pose situations in which students may construct more powerful schemes while experiencing a balance between stimulation and depletion. Three activities help teachers to enact MCRs with their students. First, teachers pose situations that harmonize with students' mathematical ways of operating and affective responses to mathematical activity. Second, teachers pose situations that challenge students—that open opportunities for them to make accommodations and thereby expand their mathematical ways of operating. Third, teachers track students' affective responses to mathematical activity as indications of whether they feel mathematically cared for, and then teachers make adjustments in the situations that they pose to reinitiate harmonizing with and challenging students' ways of operating. In this process, teachers also monitor their own feelings of stimulation and depletion.

Student's activity. When a student's response to a teacher's mathematical care includes consideration of a new situation the teacher has posed, or a sense of interest or aliveness, the student participates in a MCR with the teacher. Students who do so are likely to feel that they are being listened to, that their ideas are valued, and, perhaps, that they are understood. As a result, these students may experience stimulation—may feel energized or stronger in some way. Students may also feel some depletion—for example, uncertainty or confusion—in response to provocations posed by the teacher. The provocations can be stimulating if students find that it is possible—or even satisfying—to resolve them (although some may not be resolved quickly!). Such experiences may help students sustain or increase their engagement in mathematical activity engendered by the teacher or by themselves.

Connections between engendering caring and learning. Decentering from one's own ways of operating as a teacher is required to orchestrate learning for students and to initiate MCRs with them. In decentering, teachers practice close listening to and observing of students' ways of operating. Teachers test out activities and pose problems that they conjecture students can solve with their current schemes and operations. The teacher's main goal in listening and observing is not to confirm her or his own mathematical thinking but to make images of and conjectures about the students' mathematical thinking. In effect, the teacher is trying to learn mathematical ways of operating from students. In this sense the teacher works to harmonize with students' ways of operating, a central aspect of establishing mathematical care.

Yet to engender student learning, teachers also must act. Based on their images of and conjectures about students' ways of operating, teachers pose situations to provoke perturbations that students may eliminate by making accommodations in their current schemes. Such situations can expand students' mathematical realities in ways that students have likely not envisioned. Doing so can be stimulating for students and accomplishes the aspect of challenging them in establishing mathematical care. As teachers continue to observe and reflect on the consequences of these interactions with students, teachers may feel stimulated by their sense that they are communicating with their students and providing mathematical care.

Methods of Inquiry

In a constructivist teaching experiment (Steffe & Thompson, 2000), researchers seek to understand and explain how students operate mathematically and how their ways of operating change in the context of teaching. Since researchers' mathematical knowledge may be insufficient to understand students' ways of operating, researchers aim to learn mathematics from students. Researchers also engage in on-going conceptual analysis of how students *might* operate in mathematical situations. Based on learning from students and conceptual analysis, researchers make conjectures and test them through posing tasks in teaching episodes. Teaching practices include presenting students with problem situations, analyzing students' responses, and determining new situations that might allow students to make accommodations. Thus this methodology is compatible with establishing MCRs because researchers harmonize with students' current schemes and open opportunities for students to construct new schemes.

Four sixth grade students from a rural middle school in Georgia participated in my teaching experiment from October of 2003 to May of 2004 (Hackenberg, 2005a). They were invited to participate after demonstrating during unrecorded selection interviews that they reasoned multiplicatively. I taught them biweekly in pairs for two to three weeks, followed by a week off.

Each teaching episode occurred during school hours, lasted 30 minutes, often involved the use of a computer software program called JavaBars (Biddlecomb & Olive, 2000), and was videotaped with two cameras. Two witness-researchers assisted in videotaping, offered feedback on the teaching activities, and provided triangulation of perspectives in data analysis. Intensive retrospective analysis of the videotapes began in the summer of 2004.

The Splitting Operation

In their selection interviews, all four of the participating students demonstrated that they had constructed the splitting operation (Steffe, 2002). This fundamental multiplicative operation is involved in solving a problem like the following: "Here's a picture of my stick, which is five times longer than yours. Can you make your stick?" Students who can solve this problem engage in partitioning and iterating nearly simultaneously. Students need to posit their bar, which stands in relationship to the given bar but is also separate from it: Their bar can be iterated five times to make the given bar, and at the same time their bar is formed from partitioning the given bar into five equal parts. So students who solve the problem are aware of at least one multiplicative relationship between their bar and the given bar, namely that their bar taken five times produces the given bar. They may not necessarily be aware that 1/5 of the given bar produces their bar, depending on the state of their fraction language and fraction schemes. Solving a splitting problem is more complex than solving the problem of making 1/5 of a candy bar. In this latter problem, making 1/5 is a result of a student's fraction scheme, whereas Steffe has commented that the concept of 1/5 is *input* to the splitting operation, even if the student does not name the part as 1/5 of a bar (Leslie P. Steffe, personal communication, February 11, 2005).

Steffe has posited that a student's construction of the splitting operation opens the way for the student to construct improper fractions. He states, "Upon the emergence of the splitting operation, I regard the partitive fractional scheme as an *iterative fractional scheme*" (2002, p. 299, italics in the original). Steffe places so much importance on splitting for the construction of improper fractions (and an iterative fractional scheme) because splitting involves positing a bar that is independent from a given bar and yet stands in relation to it. Students who have constructed only *partitive* fractional schemes can disembed 1/5 of a bar from a whole bar but cannot iterate it beyond the whole (i.e., 6/5 of a bar makes no sense to these students). Students who have constructed an *iterative* fractional scheme can disembed unit fractions and iterate them any number of times so that, for example, 17/5 means one-fifth iterated 17 times and simultaneously means three groups of five-fifths and two more one-fifths. In this way, 17/5 is a unit of 17 units, any of which can be iterated five times to make the whole (5/5), another unit embedded within the 17/5. Thus 17/5 can be seen as a structure involving three levels of units.

Results and Discussion

Bridget's Lacuna

Although both Deborah and Bridget had constructed a splitting operation at the start of the teaching experiment, they had not yet constructed an iterative fractional scheme. But on November 11th, Deborah made a bar 2/15 longer than a unit bar and called the result 17/15 of the unit bar. I infer that for her 17/15 was 1/15 iterated 17 times and simultaneously 2/15 more than a whole. My hypothesis, which I support by the discussion to follow, is that she operated with three levels of units: She viewed 17/15 as a unit of seventeen units, any of which could be iterated fifteen times to make the whole, 15/15, another unit. Seventeen-fifteenths was distinct from this whole (versus part of it) and yet still stood in relation to it. So I can attribute an iterative fractional scheme to Deborah at this point.

However, Bridget called the bar 17/17. She later agreed with Deborah's assessment of the bar and could often operate with "small" improper fractions (fractions just more than one) as Deborah did. But "large" improper fractions (fractions quite a bit greater than one) continued to be problematic for Bridget. For example, on February 9th we were imagining a long peppermint stick of length *P*. I asked them to imagine how they'd use *P* to make another stick that was 15/4 of *P*. Deborah said to make *P* three times and add on 3/4 of *P*. Bridget said, "you've got to keep going, you've got to go 14 more of the original one." My hypothesis is that Bridget was *not* operating with three levels of units: For Bridget, 15/4 was a unit of 15 units, but she did not form units of four of those units as equal to *P*.

Even at the end of the experiment, Bridget demonstrated this lacuna. On May 3rd I posed the following "apple" problem: "An apple costs 75 cents; how much does five-thirds of an apple cost?" Bridget exclaimed, with some annoyance in her voice, "Oh my gosh!" Then, while Deborah swiftly solved the problem, Bridget said, "Two and one-third?" For Deborah, five-thirds was one-third five times and she could take it as a given in operating (i.e., 5/3 of an apple costs five times $.25 or $1.25). Bridget did not seem to be able to operate similarly at that time, and appeared glum that Deborah could solve the problem so quickly.

Establishing a MCR with Bridget

Because early on November 11th I perceived that Bridget's activity with fractions larger than one was different from Deborah's activity, I posed this problem to the girls in that episode: "This cake is magic in that the cake fills right back in when you take out a piece. Can you make a cake that's 17/5 of that cake?" Such problems are designed to free students from feeling confined to the material of whole—i.e., the magic refilling of the cake allows them to use material that goes beyond the whole. So in posing a magic cake problem to Bridget I was harmonizing with my inference that she had a partitive fractional scheme but not yet an iterative fractional scheme. However, she was able to complete these problems with ease by disembedding and iterating a unit fraction beyond the whole, yet without eliminating the lacuna in her activity with fractions larger than one (as I have described in the previous section). So posing such problems did not sufficiently challenge her to make an accommodation in her fraction scheme and thereby expand her mathematical world, a central aspect of establishing a MCR.

In fact, throughout the experiment Bridget often made small, and sometimes large, improper fractions. But she did not seem to be able to take small or large improper fractions as a given in solving a problem. So improper fractions seemed to remain "odd" or not quite "legitimate" numbers for her. However, because she tended to be able to interpret Deborah's responses to problems involving improper fractions, I did not always take seriously enough the evidence that fractions larger than one bothered her and that not having constructed improper fractions may have prevented her from making accommodations similar to Deborah's during the experiment.

For example, in December the girls worked on problems like this one: "Make a 3/3-bar. Make that into a 9/8-bar without erasing the thirds marks." I infer that Deborah made an accommodation in her iterative fractional and multiplying schemes in order to solve this problem, but Bridget remained stumped by it. At the time I was aware of the depletion Bridget seemed to experience as we worked on these problems, but I was not sure what to do so that it might be alleviated through her operative activity. In retrospect, I can see that asking Bridget to make a 3/3-bar into an 8/8-bar (or something "easier" like a 4/4-bar) might have been one way to better mathematically care for her, provoking the coordination of two different fractions within the same bar while eliminating the "bothersome" improper fraction.

Similarly, in early May I posed a series of problems like the "apple" problem. Bridget appeared unable to operate and seemed emotionally shut down. I decentered enough to understand that she was experiencing significant depletion, and I endeavored to break down the problems so that she might act more independently, as well as feel more autonomous and in control. But because my interventions relied heavily on my own ways of structuring the problems, my suggestions were not very effective for her and indicated that I had not decentered enough cognitively. Thus my suggestions did not alleviate her depletion very well, and the longer she remained in a depleted state, the more depletion I felt!

My interactions with Bridget in early May triggered a perturbation for me that set off a rather fervent search for better ways to communicate with her mathematically. Out of that search came a gradual reestablishment of our MCR over the final three episodes of the teaching experiment. I began to take more seriously that large improper fractions were cognitively and affectively problematic for her. So I carefully planned a sequence of tasks for her in subsequent May episodes that avoided taking improper fractions as given. Bridget responded quite positively to the sequence, both cognitively and affectively (Hackenberg, 2005a). Thus I can conclude that we reestablished a MCR. But I cannot conclude that, at that time, Bridget made an accommodation in her partitive fractional scheme in order to construct an iterative fractional scheme.

Hypothesis Revision

While the splitting operation still seems to be instrumental in the construction of an iterative fractional scheme, it does not appear to be sufficient for it. Thus my current, revised hypothesis is that students can construct the splitting operation, in which units of the whole become iterable *within the whole*, without also constructing three levels of units that are necessary for constructing improper fractions. I conjecture that constructing a splitting operation *with fractional parts* is what allowed Bridget to iterate *beyond the whole* even though she did not yet operate with three levels of units. That is, Bridget could make the whole bar given 2/5 of the bar. So for her 2/5 was 1/5 two times, and she could use her splitting operation on 2/5 by partitioning it into two equal parts, either of which could be iterated five times to make the whole bar. Thus she could also iterate 1/5 to make, say, 17/5 of the whole bar, but 17/5 lost its relationship to the whole. Since another student in the teaching experiment operated similarly to Bridget, it is possible that many students may construct this way of operating with fractions larger than one.

Conclusions

Although Bridget did not learn to make improper fractions during the experiment, the depletion she demonstrated triggered significant learning for me, the researcher. My revision of the hypothesis about what is necessary for students to construct improper fractions opens the way to establishing better MCRs with students who operate like Bridget. Bringing forth the construction of three levels of units appears to be crucial, but such learning is not easy to engender! So, understanding that students like Bridget cannot yet operate with three levels of units may allow teachers to pose tasks that harmonize better with these students' current schemes—and therefore do not bring about the sustained depletion that Bridget seemed to experience. Thus the contribution of attending to affective responses in orchestrating and researching mathematical learning appears to involve (a) opening possibilities for making and refining interventions so that both teachers and students may learn more, and (b) validating the difficulties that both teachers and students may experience in eliminating perturbations.

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STUDENTS' REVERSIBILITY AND THEIR UNDERSTANDING OF THE LIMIT OF A SEQUENCE

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This research examined how college students apply their conception of limits to determine the convergence of a sequence. In the research, students were given a tool to visually measure how many terms of a sequence are or are not close to the limit within a given error bound. The results show students' ability to properly perceive the intuitively reverse relation between the given error bound and the index defining the sequence, which plays a crucial role in understanding the limit of a sequence. Another important result was the improvement in students' perception of such a reverse relation through the visual activity. These results imply that students' development of reversibility should be considered in teaching the limit of a sequence.

Introduction

Teaching and learning the concept of limit has long been an important and interesting research subject to mathematics educators. One body of work makes use of Piaget's theory of cognitive development and/or focuses on students' readiness for instruction related to limits (Brackett, 1991; Piaget & Inhelder, 1967; Taback, 1975). A second body of work focuses on the role of informal reasoning, such as concept images or metaphorical reasoning, which may influence understanding the concept of limit (Davis & Vinner, 1986; Lakoff & Núñez, 2000; Oehrtman, 2002; Tall & Vinner, 1981). A third body of work investigates instructional strategies to help students overcome the cognitive obstacles in learning the concept of limit (Cottrill et al., 1996; Kidron & Zehavi's, 2002; Mamona-Downs, 2001; Teo, 2002).

Unfortunately, students have experienced difficulties in conceptualization of limits (Szydlik, 2000; Tall, 1992; Tall & Vinner, 1981; Williams, 1991). Students' misconceptions about limits impact their understanding of other related topics (Benzuidnhout, 2001; Cornu, 1991; Merenluoto & Lehtinen, 2000; Sierpinska, 1987; Tall & Schwarzenberger, 1978). Such difficulties result in negative experiences in solving relevant mathematics problems. Repetition of such negative experiences will eventually deprive students of their desire to solve mathematical problems, and will lead them to not trust their own mathematical thinking in solving problems.

Theoretical Framework

In general, students who start to learn limits are led to conceptualize the idea of a limit in the same order as they read the limit symbol $\lim_{n\to\infty} a_n$. To be precise, they are taught the limit of a sequence as a certain value which, as the index goes to infinity, the sequence is approaching or getting close to. However, thinking of limits in this order tends not to be successful in precisely understanding the concept of limit. It should be noted that, in the $\varepsilon - N$ definition of limit, an index number N is properly chosen after the error ε is determined to control the error between the sequence and the limit. However, in contrast, students typically first choose an index number and next determine how close the term corresponding to the index is to the limit value.

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Comparing the process of students' thinking (for a given index *N*, see what the error ε is) with that of the mathematically rigorous definition (for a given error ε, find the corresponding index *N*), one see that the order of finding the error and the index is reversed. If we accept that the order of students' intuitive thinking is the same as in reading the limit symbol, then the order of students' intuitive thinking is natural, and the order required in the rigorous definition can be regarded as a reverse process. Courant and Robbins (1963) and Fischbein (1994) point to such a counter-intuitive nature of the reverse thinking process as a factor in students' difficulties in understanding limits.

In line with this viewpoint, in this paper, the process of thinking implied in the $\varepsilon - N$ definition of limits is called the reverse thinking process. To be precise, reverse thinking in the context of the limit of a sequence means reversing the intuitive order in which students think about N and ε in determining the convergence of a sequence. The research reported here addressed the relation of students' intuitive conception of the limit of a sequence to their ability to reverse the order between ε and *N* as required in the rigorous definition.

Research Design and Methodology

This study was conducted in a Midwestern university with a fairly diverse department of mathematics from September 2004 to November 2004. The subjects of this research were students who were acquainted with and had used the limit symbol, but had not had any experience with rigorous proofs using the $\varepsilon - N$ definition of a limit. Twenty one students in calculus courses for engineering majors and 12 students in a calculus course for biology majors voluntarily took a survey. Among them, 12 students were selected for a series of 1-hour semi-structured, task-based interviews once a week for 5 weeks; and 11 of these students completed the whole series of interviews.

The task-based interviews consisted of a pretest, six tasks, and a posttest. Each interview was designed to foster a conceptually rich environment to optimize the chances of development in intuitively understanding the limit of a sequence. The following five typical types of sequences were included in the interviews: Monotone bounded, unbounded, constant, oscillating convergent, or oscillating divergent sequences. Students represented each sequence numerically as well as graphically, and they determined convergence of the sequence. In addition, students used tools, called $\varepsilon -$strips, which were specially developed for this study to describe the $\varepsilon - N$ relation in the rigorous definition of limits. Each $\varepsilon -$strip was made of translucent paper so that students could observe the graph of the sequence through the $\varepsilon -$strip. In addition, each $\varepsilon -$strip had constant width, and in the middle of it, a red line was drawn so as to mark a possible limit point. As an example of such a representation of a sequence, Figure 1 illustrates the graph of a sequence $a_n = 1/n$ along with an $\varepsilon -$strip:

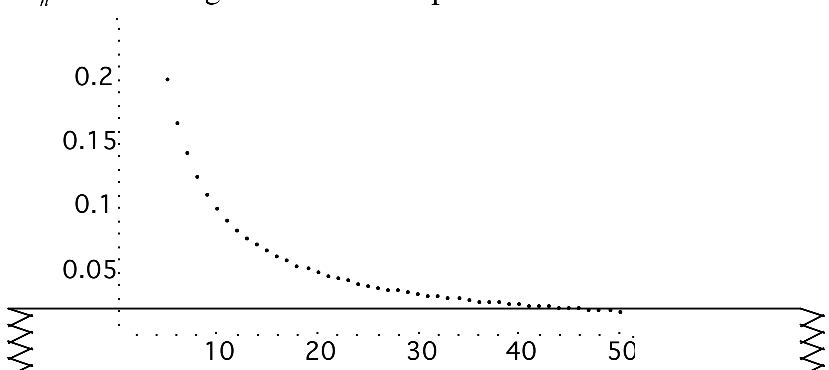

Figure 1. A graph of a sequence with an epsilon strip

These $\varepsilon-$strips were shown to students during Task 2 for the first time, and all through the tasks from Task 2 on, the student had opportunities to intuitively explore the $\varepsilon-N$ relation in defining limits through hands-on activities with the $\varepsilon-$strips. Finally, students were presented the following statements as though they were other students' ideas:

$\varepsilon-$Strip definition A:	*A certain value L is a limit of a sequence when infinitely many points on the graph of the sequence are covered by any epsilon strip as long as the epsilon strip covers L.*
$\varepsilon-$Strip definition B:	*A certain value L is a limit of a sequence when only finitely many points on the graph of the sequence are NOT covered by any epsilon strip as long as the epsilon strip covers L.*

Students evaluated the propriety of the above $\varepsilon-$strip definitions A and B as statements representing the limit of a sequence, and compared them with their own conception of the limit of a sequence.

The overall theoretical frame of the data analysis was based on building grounded theory from a collective case study, suggested by Lincoln & Guba (1985) and Stake (2000) as follows: (1) Find main themes emerging from the preliminary coding procedure; (2) triangulate the collected data to build the grounded theory around the main themes; (3) construct a model of the case studies with major attention to the necessity for specifying all the theoretical elements and their connections with each other; (4) afterwards, build in illustrative data, selected according to the required validity, reliability, and ethical issues.

Results

Students' Conception of the Limit of a Sequence

Students' major conceptions of the limit of a sequence appeared as follows: (1) Regarding asymptotes as limits; (2) regarding cluster points as limits; (3) regarding limit points as limits. Students who regarded asymptotes as limits considered a limit as a straight line such that the graph of a sequence approached arbitrarily close, but did not surpass or cross. They often described a convergent sequence as getting close to, but not being equal to such asymptotic lines. On the other hand, students who regarded cluster points as limits considered a limit as a value that infinitely many terms of a sequence were getting close to or being equal to. Students who regarded either asymptotes or cluster points as limits seemed to have a misconception about the uniqueness of the limit value, and believed a sequence could have multiple limits.

Students' Conception of the Reverse Relation Between ε and N

There were students who did not regard the $\varepsilon-$strip definitions as proper descriptions of limit. Several of them did not relate ε to N for any value of ε at all. These students had difficulties relating their own conception of limit to the $\varepsilon-$strip definitions.

On the other hand, there were several students who imagined the completion of the infinite process of increasing or decreasing the value of ε, with the value of ε ultimately ending at infinity or 0. For instance, EMMA, who regarded cluster points as limits, considered the case as ε was getting infinitely small and then eventually became 0. When $\varepsilon = 0$, it would not be possible for any terms to get within the error bound 0 of the limit. Therefore, according to $\varepsilon-$strip definition A, the limit of this sequence should not exist. EMMA thought that this was what $\varepsilon-$strip definition A implied. However, EMMA believed the limit of a sequence was 0 when applying her own conception of limit to the sequence. By experiencing such cognitive dissonance, EMMA concluded that $\varepsilon-$strip definition A did not properly represent the limit of a sequence.

<u>EMMA on Task 2</u>: $a_n = 1/n$

EMMA: Well [pause], I think saying that 'the strips are covering zero' is kind of [pause] misleading because, I mean, you can say that your strips [pause] get so thin that it's just like a line and then does not include anything other than zero.

There were also several students who saw the relation between ε and N only as a static feature. These students also experienced difficulty in understanding the $\varepsilon-$strip definitions. They even perceived that it was enough to check only one, or at most some, $\varepsilon-$strips rather than all $\varepsilon-$strips to see if the sequence was convergent or not. Without seeing the difference between using any $\varepsilon-$strip rather than some $\varepsilon-$strips, these students pointed out that $\varepsilon-$strip definition A might mislead them into determining 0.1 as the limit of the sequence $a_n = (-1)^n 1/n$ even though 0.1 was not the limit of that sequence.

<u>EMILY on Task 5</u>: $a_n = (-1)^n 1/n$

EMILY: Like, that [A] should, how it sounds like [pause], when infinitely many points on the graph of the sequence are covered by any epsilon strip as long as the epsilon strip covers L. So, yeah, if we have it on [pause] 0.1, if it [an epsilon strip] extends all the way to right, we will still have infinitely many, many [pause] points in this strip [pause], and the limit is also in this strip.

There were students who perceived just static images of the graphical representation of a sequence for any fixed value of ε. However, such static images still made it difficult for students to grasp the role of reverse thinking in the context of limit. They did not regard the value of ε as going to 0 to determine the limit of a sequence. Consequently, they did not understand that the $\varepsilon-$strip definitions could be used to reduce any error bound as much as was desired.

Students' Conception of the $\varepsilon-$Strip Definitions and Their Own Conception of Limit

It was found that students' conception of the $\varepsilon-$strip definitions was positively associated with their own conception of limit. Table 1 describes students' conceptions of limit and their responses to the propriety of $\varepsilon-$strip definitions A and B as descriptions of the limit of a sequence. Students who thoroughly understood the dynamic notion of the reverse relation between ε and N evaluated the $\varepsilon-$strip definitions as shown in Table 1.

Table 1. Students' responses to correctness of $\varepsilon-$strip definitions A and B according to their own conception of limit and of convergence

Conception of Limit	**Asymptote**	**Cluster point**	**Confusion between Cluster point & Limit point**	**Limit point**
A is Correct	No	Yes	Yes	No
B is Correct	No	No	Yes	Yes

Students who regarded asymptotes as limits of a sequence responded that neither $\varepsilon-$strip definition A nor B was proper to describe the limit of a sequence. The main reason was that neither $\varepsilon-$strip definition could properly identify asymptotic lines as the limit of a sequence. For instance, ELISA determined an oscillating divergent sequence $a_n = (-1)^n(1+1/n)$ as having two limit values, 1 and -1, due to the fact that this sequence had two asymptotic lines. However, ELISA had to determine this sequence as divergent according to $\varepsilon-$strip definition B because there was an $\varepsilon-$strip where there were infinitely many points on the graph of the sequence outside the $\varepsilon-$strip. ELISA did not agree with $\varepsilon-$strip definition A as a proper description for a limit of a sequence, either. ELISA pointed out that according to $\varepsilon-$strip definition A, even a sequence which was getting close to and eventually equaled a certain value should be determined as convergent.

<u>ELISA on Posttest</u>: $a_n = \begin{cases} 1/n & \text{if } n \le 10, \\ 1/10 & \text{if } n > 10. \end{cases}$

I: Does this sequence have a limit?

ELISA: [pause] Umm, no.

I: How can you tell this?

ELISA: Because umm [pause] hmm [pause] I guess by that definition [A], this would be, the limit would be one tenth,

I: Can you explain if you follow student A's definition, why one tenth should be the limit of the sequence?

ELISA: Umm, because the, when you place the epsilon strips [placing an epsilon strip] over one tenth, umm there are infinitely many points [pause] umm going out, because the rest of points are all gonna equal to one tenth. So I guess the limit to this would be one tenth.

I: What do you think?

ELISA: I just always thought that a sequence like went out to a certain value, like it kept going out but never actually reach it. And so, if that's the case, then this sequence wouldn't have a limit because it actually reaches it. Then that [A] would not be [pause] right, the right definition.

As seen in the above dialogue with ELISA, such a contradiction between her own conception of limit and the result from the $\varepsilon-$strip definitions seemed to make it hard for her to regard either $\varepsilon-$strip definition as the proper descriptions for the limit of a sequence.

On the other hand, students who regarded cluster points as limits of a sequence responded that $\varepsilon-$strip definition A was correct but not $\varepsilon-$strip definition B. This was because there were some examples of sequences, such as $a_n = (-1)^n(1+1/n)$, that have multiple cluster points, all of which were limits for these students. On the contrary, they had to call such a sequence divergent if they applied $\varepsilon-$strip definition B.

<u>EMMA on Task 6</u>: $a_n = (-1)^n(1+1/n)$

I: Can you explain it [$\varepsilon-$strip definition B] one more time?

EMMA: Umm [pause]. A certain value *L* is a limit of a sequence when infinitely many points on the graph of the sequence are covered by any epsilon strip as long as it covers *L*. And that's true there are infinitely many points and this is on the sequence. And here [pause] this [B] is not true umm there are infinitely many points outside the epsilon strip [pause] while the epsilon strip is in fact umm a limit of the sequence. So this [B] isn't true here. [pause] Because the student [B]'s saying that for a value to be a limit of a sequence finitely many, only finitely many points [pause] should not be covered by any epsilon strip. And [pause] here, by this argument, 1 would not be a limit of the sequence. So, well, student B is wrong.

I: But 1 is a limit of the sequence?

EMMA: Right.

Several students did not recognize the difference between cluster points and limit points. These students responded both $\varepsilon -$strip definitions A and B were correct to describe the limit of a sequence.

BECKY on Task 5: $a_n = (-1)^n 1/n$

BECKY: They both explain it, umm [pause], they both explain it, like, what we were saying because when umm [pause] you have a strip covering the limit which is zero. There is [pause] like they are both true. There is an infinite amount of umm points in it because they are getting closer to it. [pause] So, and if it extends out, all of them are gonna be [pause] like covered by it. And then also for B, there is only a finite number outside because like there are only gonna be outside but [pause] up until the point where it starts getting like small strip. And all would be [pause] inside the strip, so.

Only students who properly understood the limit of a sequence without confusing it with other concepts such as asymptotes or cluster points recognized the difference between $\varepsilon -$strip definition A and B, and responded $\varepsilon -$strip definition B but not $\varepsilon -$strip definition A as correct to describe the limit of a sequence.

Concluding Remarks and Implication for Teaching and Learning

In this research, $\varepsilon -$strips were given to students to examine students' reversibility, which means the ability to think of the infinite process in defining the limit intuitively in terms of the index and to reverse the process simultaneously by finding an appropriate index N for an arbitrarily chosen error bound ε. The results of this study show that students' understanding of the rigorous definition of the limit of a sequence is associated with not only their conception of limit but also their reversibility. The better they recognized the dynamic feature of "any ε" appearing in the $\varepsilon -$strip definitions , the better they understood the $\varepsilon - N$ defition of limit.

In addition, it should be noted that there was improvement in students' reversibility through the $\varepsilon -$strip activity even though there was no procedure for indicating students' errors, correcting students' misconceptions about limit, or confirming the propriety of the $\varepsilon -$strip definitions to students during the interviews. It seems that the $\varepsilon -$strip activity can be regarded as an effective instructional method in teaching the limit of a sequence. Therefore, this study implies that reverse thinking should be considered in developing instructional strategies for the limit of a sequence.

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SUSANNA'S TOOLS TO PERCEIVE THE DERIVATIVE

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The acquisition of an advanced mathematical concept can also begin from the perception of an object. Perceptions are not apart from thinking and an individual's perceptions depend on his/her representations. In this case study, a grade 11 student, Susanna, attended a task-based interview. It was analyzed how Susanna could perceive the derivative of a function from the graph of the function and what kind of representations she used for this. Susanna could perceive, for example, the sign and the maximum point of the derivative. She used representations of the increase and the steepness of the graph for this and supported thinking with gestures. This case shows that with appropriate representations students can perceive essential aspects of the derivative from the graph of the function, and that students can consider the derivative as an object at the very beginning of the acquisition process.

Introduction

Nowadays it is widely recognized that cognition is not only inside individuals' heads but also distributed to the social-cultural environment that includes tools (cf. Sfard & McClain, 2002; Rasmussen & al., 2004). Also the role of perceptions in learning mathematics is acknowledged (e.g. Tall 2004). In this paper it is studied how one student, Susanna, could perceive the derivative from the graph of the function and what kind of representations, including internal and external sides, she used for this.

Following Piagetian distinction, there are at least two ways how concept acquisition can begin. One way is to perform a symbolic action on an object and from this action to construct a new concept. The other way, according to Gray and Tall (2001), is that the concept acquisition begins from the perception of an object. Gray and Tall call this kind of perceived object an embodied object. The embodied objects are mental constructs of the perceived reality, and through reflection and discourse they can become more abstract constructs, which do not anymore refer to specific objects in the real world (ibid.). In Tall's (2004) theory of three worlds of mathematics the conceptual-embodied world includes real-world objects and internal conceptions that involve visuo-spatial imagery. The other two worlds are the proceptual-symbolic world in which symbols allow people to swift from processes to do mathematics to concepts to think about, and the formal-axiomatic world which consists of deductive reasoning (ibid.).

There are many studies on students reasoning in the conceptual-embodied world. Berry and Nyman (2003) found that university students moved from the instrumental understanding of the calculus towards relational understanding when engaged in tasks where they sketched the graph of the function from the graph of the derived function, and then created the corresponding movement and compared that to the graph given by the motion detector. They recommend that before entering the formal symbolic calculus, students should understand the underlying concepts which can be enhanced with tasks like those in their study (ibid.). In a study of Schorr (2003), middle school students explored speed and motion especially with graphic representations and computer software, and according to the results, students build powerful ideas of related concepts. So she concludes that meaningful mathematical experiences in the mathematics of motion are possible even at grades 7 and 8 (ibid.).

Also gesturing as part of students' reasoning has gained much attention in the literature. McNeill (1992) has argued that gestures together with speech are an essential part of thought processes. According to McNeill, there are different kinds of gestures, of which deictic, iconic, and metaphoric gestures are discussed here. Deictic gestures indicate something, iconic gestures resemble something and metaphoric gestures represent abstract ideas (McNeill, 1992, p. 12-18). In a study of Radford et al. (2003), gestures seemed to help the student to focus attention on particular aspects of graphs, and according to Rasmussen et al. (2004, p. 319), gestures are part of expressing, communicating and reorganizing one's thinking.

Characterization of Representation Concept

Traditionally, a representation is conceived as something which stands for something else, and representations are divided into internal and external ones (cf. Janvier, 1987; Goldin & Kaput, 1996). Internal representation refers to mental construction of an individual and external representation to physically embodied, observable configuration (Goldin & Kaput, 1996, p. 399-400). Goldin and Kaput (1996, p. 401) emphasize that it is always an individual who does the representing.

The traditional view of representation has been criticized lately. For example, there is a danger that representations may be thought to be mere representations of some objects and separated from meaning (Sfard, 2000, p. 43-44). According to Sfard (2000, p. 44), this position implies that objects and meanings are more important than representations and that meanings should be learnt before signs. The traditional view of representation implies that representations are only used to store information (Sfard & McClain, 2002, p. 155) and that the role of signs and symbolic tools is only to support and aid students (Sfard & McClain, 2002, p. 154; Radford, 2000, p. 239; Meira, 1998). Thus, this view does not use all the potential power of tools. Also the dichotomy of internal versus external representations has been found artificial. Traditional views often take a standpoint that external representations reflect the mental structures of an individual (e.g. Radford, 2000, p. 239) and that learning is the growth of mental structures (Sfard & McClain 2002, p. 154). Even when the decisive role of the student is acknowledged, representations are often analyzed from the expert's point of view as if external representations would include meanings (Cobb & al., 1992; Meira, 1998). Thus, these analyses do not address the uses or constructions of the representations.

Meira (1998) has emphasized that the focus of studies in representations should move towards students' use and construction of representations. This move towards conceiving representations as tools has been made, for example, by Radford (2000). In his study there is "a theoretical shift from what signs *represent* to what they *enable* us to do" (Radford, 2000, p. 241). Sfard (2000, p. 48) has argued that representations are not born as such and that they *may* become to stand for something else later. Several authors have emphasized that meanings are constructed through the use of signs (Sfard, 2000; Radford 2000, p. 241).

I believe that we should not abandon the concept of the representation but take account the criticism and new directions and thus build on previous significant theories of the representation. For example, Goldin's (1998) theory of representational systems is suitable for this study when stressing the point that characters in one such system do not necessarily symbolize something else. Thus, building on the mentioned theories and views, the representation is characterized in this study as follows:

> *A representation is a tool to think of something which is constructed through the use of the tool. The representation has the potential to become to stand for something else but this is not necessary. The representation consists of external and internal sides which are its equally important parts and do not necessarily stand for each other but are inseparable. The external side is visible to other humans through the senses but the internal side is not.*

For example, a student may use the steepness of a graph of a function as a representation of a derivative of the function (see the results). This means that the steepness-tool allows the student to perceive some aspects of the derivative, for example, the maximum point of the derivative. Now, the student's conception of the derivative may have been constructed through the use of steepness and other representation-tools. It may be that on some occasions the student may use steepness to represent a derivative, but this is not a case in all situations where the steepness is used. There is also an external side of steepness, for example, the mere graph on the paper, speech or some gestures. Obviously, there must be some internal side because for some people the graph would not allow to perceive the derivative. It is not the case that the external side only reflects the internal side, but it is the interplay between them that allows the student to use this tool efficiently.

Methodology

This case study focuses on Susanna's use of representations when perceiving the derivative at a task based interview. Susanna was selected to this study because her success in mathematics was thought to be defective and it would be interesting to investigate how she is reasoning when compared to more advanced students (cf. Hähkiöniemi, 2004, 2005). In a pretest, Susanna could read values from a distance-time graph and state when the distance is at its greatest. On the contrary, she could not state when the velocity is positive, negative or zero. She also calculated the instant velocity at a particular point as a distance over time at the point which gives an average velocity. When the algebraic expression and the graph of a function were given, she reasoned the maximum value and the domain of negative values of the function incorrectly and did not notice that in the graph these do not make sense. She also could not draw a tangent to the graph.

This study is not a teaching experiment, but a specific teaching period was designed to give students experiences with several representations so that there would be more to investigate than their applications of symbolic procedures. I taught the five-lesson teaching period in the autumn of 2003 as a part of a Finnish grade 11 course. The teaching period consisted of the first five lessons on the subject of the derivative. Different representations and open problem solving were emphasized in teaching. The teaching period began by examining motion graphs and by perceiving the rate of change of a function from its graph. Moving a hand along the curve, placing a pencil as a tangent, looking how steep the graph is and the local straightness of the graph were used as representations. It was discussed how the abovementioned representations can be used to see the sign and the magnitude of the rate of change. The problem of the value of the instant rate of change was discussed and the derivative was defined through the solution of this problem.

After the teaching period, Susanna attended a 60-minute task-based interview. In this paper, only the task where she was asked to make observations of the derivative of a function from the graph of the function (Fig. 1) is discussed. Inductive analysis was based on the original video and transcript. From the whole interview, the situations where Susanna used some representation were located. From each of the situations it was analyzed how she used these representations and what they allowed her to do. Then all the situations were compared to each other. This way, an analysis of one representation was reflected against the analysis of other representations. Susanna's uses of representations were also compared to the uses of the other four students. This allowed noticing common and distinct features in the students' use of representations and seeing some aspects of Susanna's behaviour in a new light.

The excerpts from the interview are translated from Finnish, and "- -" in the students' transcripts means that the text is snipped. Gestures are explained between brackets () and the points which indicate the use of a specific representation are underlined.

Susanna's Perceptions of the Derivative

It was found that Susanna had constructed imagistic representations of the derivative which allowed her to perceive the derivative from the graph of a function. Susanna also used gestures along her thinking processes. It seems that her cognitions were distributed and that her thinking can be characterized as an interplay between internal and external sides of representations. The following excerpts show aspects of her thinking:

Interviewer: The graph of the function *f* is given in the figure (Fig. 1). What kind of observations can you make of the derivative of a function *f* at different points?

Susanna: Well, if you start to look from here *(points to the graph at -3)*, then here the increase speeds up, we go upward *(traces the graph with finger from -3 to -1.6)*. Then at the top *(points to the graph at -1.5)* it is zero, it goes horizontally *(draws a horizontal line in the air)*. Then again it slows down here *(turns pencil a bit, traces the graph with pencil from -1.5 to 0.8)* and here again it is zero when it goes horizontally *(points to the graph at 0.8)*. And then again upward from here the rate increases *(traces the graph with pencil from 0.8 to 2)*. - - Here the decrease is constant *(traces the graph with pencil from 2 to 4)*.

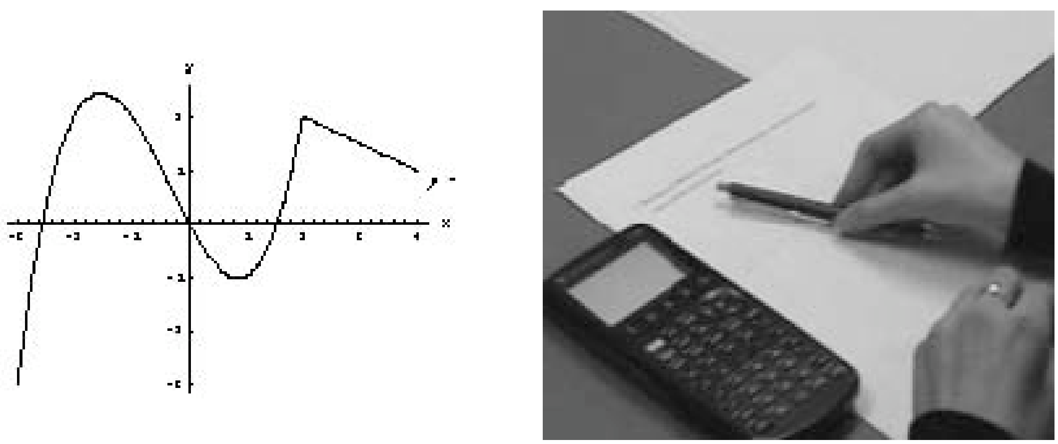

Figure 1: The graph of a function at the task 2 and Susanna's gesture with pen to see the steepness of the graph at the point 0.7

Above, Susanna seemed to speak of the increase or rate when she mentioned "the increase speeds up", "it is zero", "it slows down", "it is zero", "rate increases" and "decrease is constant". The first and partly the third utterances are incorrect. Thus, Susanna probably confused the change of the function with the change of the derivative. After this Susanna wondered what the derivative would be at the point 2:

Interviewer: What would the derivative be at that point *(points to the graph at 2)*? At about two.

Susanna: It can't really be zero, because it doesn't actually go horizontally at any point. Hmm. Or in a way it can be zero, for example, if you look at it with the pen *(moves pen as a tangent from ascending to descending at the point 2)*, then at some moment it be horizontal. - - [Pause.] If it is at point two, then if you look, then in a way it would have to be one. - -

Interviewer: On what grounds?

Susanna: Mm. Well. Ex. *(Holds pen above the notion $y = f(x)$ in Fig. 1)*. If you take the derivative at the point two, then [pause]. *(Writes $y'(2) =$)*. There's only the *x (points to the notion $y = f(x)$ in Fig. 1*), so then it would be one.

First Susanna used a pencil as a tangent to look at the derivative and after that she differentiated *x* from the symbolic expression $y = f(x)$ concluding that the derivative would be one. After that she used the increase and the steepness of the graph to see the sign and maximum and minimum points of the derivative:

Interviewer: When would the derivative be positive in general at the whole graph and when negative?

Susanna: It would be positive approximately from here to somewhere there *(points to the graph at -2.6 and -1.5)*, when the graph rises upward *(moves pencil upward)*. And then from somewhere there to there *(points to the graph at 0.8 and 2)*. - - Negative from somewhere there to about there *(points to the graph at -1.5 and 0.8)*. And from that on *(points to the x-axis from 2 to 4)*. - -

Interviewer: Could it be said when the derivative is at its greatest and when at its smallest?

Susanna: At its greatest it is when the graph rises most steepest upward *(moves pen upward)*. It could be *(puts pen as a tangent to the graph at points -3 and 1.9)*. Hmm. Somewhere there *(points to the graph at [-3, -2.6])*. Or here *(points to the graph at 1.9)*. Where it falls most steepest downward then, hmm *(holds pen as a tangent above the graph at about point 0)* somewhere hmm. It's a bit hard to see, but somewhere there *(points to the graph at 0.7)*. - -

Interviewer: How did you look that it goes most steepest downward there?

Susanna: Mm. Here *(puts pen as a tangent to the graph at [2, 4])* it clearly goes not as steep as there *(puts pen as a tangent to the graph at 0.7, the pen's position is steeper than at [2, 4] and it is steeper than it should be, see Fig. 1)*.

Interviewer: Ok. What about there (*points to the graph at -0.6*) compared to that point (*points to the graph at 0.7*)?

Susanna: Yes. *(Puts pen as a tangent to the graph at -0.6)*. It falls quite slowly. It won't quite go. (*Puts ruler as a tangent to the graph at -0.6, its position is steeper than that of the pen*). It could be also there the steepest.

Susanna seemed to use the increase of the function as a tool to perceive how the function changes at some interval. For example, the sign of the derivative was easy to perceive when the derivative was represented as the increase of the function. Along the representation of the increase she used the steepness of the graph. Steepness seemed to represent the magnitude of the change of the function. So it was an especially good tool for perceiving local properties of the derivative, such as the maximum point. Yet Susanna perceived the minimum point of the derivative incorrectly. It seems that Susanna placed the pen steeper than it should be at 0.7 and this misled her. Susanna made iconic gestures of moving hand along the graph and moving her hand upward or downward in the air while considering the increase. These gestures, as well as the graph itself and her speech, seemed to be external sides of her representation of the increase. She also made a metaphoric gesture of placing a pencil as a tangent to the graph. This gesture seems to be an external side of the steepness representation. Susanna also used some deictic gestures to indicate some points in the graph.

Discussion and Conclusions

It was shown how Susanna could perceive and reason many essential properties of the derivative: the sign of the derivative, the zero point and the maximum point of the derivative and the interval when the derivative is constant. She also mixed up a little: she confused the increase of the function with the increase of the derivative, she used the differentiation rule inappropriately and she determined the minimum point of the derivative incorrectly.

I have tried to emphasize the distributed nature of knowledge and break the classical external–internal dichotomy by focusing on external and internal sides of the representation. Susanna's case shows that with appropriate representations students can perceive essential aspects of the derivative from the graph of the function. Especially, the representations of the increase and steepness seemed to embody important relations between a function and its derivative. Gestures of imitating the graph, tracing or pointing to the graph, and placing the pencil tangent-like to the graph as external sides of representations served to externalize thinking and to support perceptions. Also appropriate internal sides of representations are needed, which is evident by Susanna's difficulties when perceiving the negative steepness.

Susanna's case supports arguments that gestures are important to thinking (McNeill, 1992; Radford & al., 2003) and part of expressing, communicating and reorganizing one's thinking (Rasmussen & al., 2004, p. 319). For an advanced person in mathematics these gestures may seem meaningless or useless. But for a novice, like Susanna, they may be in great aid and help to focus attention on particular aspects, such as like increase and steepness.

Susanna's perceptions of the derivative focused largely on the graph as a physical object while she recognized some aspects of the derivative. So in her way to learn to see the derivative in the mathematical way she was still at the beginning. For example, she noticed such things as the graph going upward and the steepness of the graph, but she did not perceive how the rate of change of the derivative was changing. The latter would have required perceiving aspects that expect a more disciplined way of seeing. She also used physical objects to see these aspects and had problems with "negative" steepness. This proves that she is still focusing on very concrete aspects. Referring to the concept of transparency (Meira, 1998), Susanna did not seem to use her representations of increase and steepness very transparently because she focused more on these tools than to the derivative which can be seen through them. Nevertheless, we can consider Susanna's perceptual activity as a powerful stage of learning the derivative because she could do some reasoning, and when compared to reasoning in a formal symbolic context, this reasoning was more mathematical.

This study suggests that with special kind of representations students can consider the derivative as an object that has some properties such as sign and magnitude. This is possible even at a very early stage in the learning process and even for students like Susanna, whose previous success in mathematics is not excellent. Although Susanna considered the derivative as an object, this object did not probably have much inner structure because in the previous analysis it was found that Susanna had a poor procedural and conceptual knowledge of the limit of the difference quotient (Hähkiöniemi, 2005). Thus, Susanna considered the derivative as an object and made some good perceptions but did them more or less intuitively without knowing why they can be made. According to the theory of Sfard (1992, p. 75-77), a conception like this is called pseudostructural, and it is unsatisfactory and may be harmful. Contrary to this, it seems that for Susanna such a conception was helpful. Also according to the theories of Tall (2004) and Haapasalo and Kadijevich (2000), a starting point like this could be very fruitful for learning. As for Tall's theory, Susanna's representations were embodied objects, and in the theory of Haapasalo and Kadijevich, Susanna's use of the mentioned representations corresponds most likely to spontaneous procedures.

With regard to Tall's (2004) theory of the three worlds of mathematics, Susanna worked in the embodied world and in the symbolic world. At many points of the interview the symbolic world seemed rather procedural and confusing to her. Instead, in the embodied world she demonstrated some conceptual knowledge connecting some features of the graph of the function to its derivative. Still she could do much better also in the embodied world and connect this world to the symbolic world.

It may be that students like Susanna would benefit if they had opportunities to work with graphs when learning concepts of the calculus (cf. Berry & Nyman, 2003). This may be possible even before high school (cf. Schorr, 2003; Radford, 2003). Although this research is not a teaching experiment, the results suggest that teaching may have a positive influence on students' abilities to perceive mathematical aspects. Rasmussen et al. (2004, p. 305) emphasize, that the meanings associated with gestures are both individually and socially constructed. Thus, teachers should pay attention to gesturing and encourage students to it.

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MAKING AND MANAGING DISTINCTIONS— THE CASE OF SAMPLING DISTRIBUTIONS

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The concept of sampling distribution is one of the most complex topics encountered in an introductory statistics course. The goal of our research is to examine prospective secondary mathematics teachers' understandings of sampling distribution. Six students' responses to two open-ended interview tasks were analyzed in order to characterize their understandings of concepts related to sampling distribution. In the course of our analysis, a persistent theme arose in the data - students' thinking seemed fundamentally influenced by their treatment of important mathematical distinctions. The variation in the extents to which prospective secondary mathematics teachers with considerable mathematics background make and manage distinctions suggests that this feature of their thinking may help characterize the nature of mathematical thinking at more advanced but still developing levels.

The Case of Sampling Distributions

Inferential statistics is founded on reasoning about the characteristics of a population on the basis of information obtained in a sample. Reasoning effectively requires understanding that while the composition of individual samples from a population may vary tremendously, the means of all possible samples of a given size will follow a predictable pattern as described by the Central Limit Theorem. The Central Limit Theorem describes the pattern of distribution of all possible means from samples of a given size, the sampling distribution of sample means. Since this collection of the means of all possible samples is commonly inordinately large, we typically work with a subcollection. (For linguistic clarity, we use the phrase "a 'proper' distribution of sample means" to make a distinction between the sampling distribution and the distribution of sample means that is not generated from all possible samples.) As data analysis assumes an increasingly important role in the secondary mathematics curriculum (NCTM, 2000), it is especially important for secondary mathematics teachers to develop a strong and flexible understanding of sampling distribution. This study focuses on the understanding of sampling distribution by prospective secondary mathematics teachers.

To understand sampling distribution one needs to build on understandings of distribution (including center, spread, and shape), population, samples, and the act of sampling. Adding to the complexity, the statistical language associated with sampling distributions requires students to distinguish among "sample," "sample distribution," "distribution of sample means," and "sampling distribution of sample means." In addition, depending on whether one is reasoning about a population or a sample, there is a need to account for the fact that some properties are true under any conditions and some properties need to be probabilistically conditioned. For example, consider the property of a mean of a distribution being equal to the mean of the

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population. While this property applies to the sampling distribution, which includes sample means from the set of all possible samples of a given size, it does not apply to proper distributions of sample means. Although we cannot predict the mean of the proper distribution of sample means, we can calculate the probability of that mean falling within a given interval about the population mean. That is, we can make a probabilistically conditioned statement, not a deterministic one about the relationship between the means of the population and the mean of the proper distribution of sample means.

The importance and difficulty of learning about sampling distributions is underscored in current literature (Saldanha & Thompson, 2002; Lipson, 2003; Chance, delMas, & Garfield, 2004). Lipson (2003) provided evidence that the understanding of sampling distribution affects the quality of understanding of inferential statistics. Chance and colleagues (2004) observed a complication that can arise when reasoning about a sampling distribution – based on the need to distinguish between the distribution of a single sample and the distribution of several sample means. Because of the foundational role of sampling distribution in understanding statistical inference, the layered complexity of the concept of sampling distribution, and the increasing importance of teachers developing solid understanding of statistical ideas, we chose to investigate how prospective secondary mathematics teachers understand sampling distribution. We believed that the mathematical understandings of these advanced mathematics students would go beyond the process understanding described in much of the literature on advanced mathematical thinking (Asiala, et al., 1996) but would fall short of object-level understanding. In the course of this investigation, we observed in students' understanding a characteristic that seemed to distinguish more advanced levels of understanding in ways that theories about process and object understanding (Asiala, et al., 1996) seem not to capture. That characteristic centered on the ways prospective secondary mathematics teachers did or did not make important mathematical and statistical distinctions in the context of a complex situation. Not only is the ability to make mathematical distinctions particularly important in understanding sampling distribution but it also affects the complex mathematical decisions involved in classroom teaching.

Making Distinctions

Understanding sampling distribution requires making many distinctions and offers a rich context for examining the nature of distinctions students may or may not make. The necessary distinctions include distinctions in language, in measures of center and spread, in sampling, and in probability. Distinctions in language are critical because similar language is used to describe different essential concepts: sample, sample distribution, distribution of sample means, and sampling distribution of sample means. Understanding of sampling distribution involves a nesting of concepts, and navigating this web of concepts requires making distinctions about the level of nesting about which one is thinking. The complexity of this nesting, and hence the intricacy of the distinctions students need to make, is revealed in the following consideration of what is involved in understanding sampling distribution. Understanding of measures of center and spread of a population is needed in order to be able to reason about the possible values for measures of center and spread of samples drawn from that population. Understanding possible values for the measures of center and spread for a sample leads to the need to understand the probabilistic nature of sample statistics. Understanding the probabilistic nature of statistical measures is central to the understanding of the set of means of all possible samples (of a given size). In order to understand sampling distribution, students need to have a sense of how probable a sample mean is rather than whether or not the specific mean is possible. Yet students also need to understand that some properties of the sampling distribution are absolutely true, such as the mean of the sampling distribution is equal to the mean of the population.

Students also need to be able to make distinctions as they reason about aspects of the sampling process. Students need to distinguish between the effects of sample size on the characteristics of the sampling distribution and effects of sample size on the characteristics of an individual sample. A clear distinction also must be made between the proper distribution of sample means that results from simulation and the actual sampling distribution. The proper distribution of sample means may or may not possess the same characteristics as that of the sampling distribution, and ignoring these distinctions may result in erroneous conclusions.

By examining student understanding of sampling distribution with an eye to making distinctions, we have begun to identify distinctions that seem crucial to that understanding or that seem problematic for students. We identified ways that students made distinctions in their wording and sought evidence to distinguish between students' genuine understanding and word choice. We looked for evidence of where some students made important distinctions that others failed to make, and we looked for students' recognizing distinctions in one setting while failing to recognize similar distinctions in a different setting.

Description of the Study

Setting and Subjects

The eighteen subjects in this study were prospective secondary mathematics teachers enrolled in a course designed to broaden their understandings of basic statistical concepts they might encounter in teaching some of the reformed curricula for high school mathematics. The class regularly used Fathom Dynamic Statistics™, and several of the activities on sampling distributions involved simulating the production of collections of means for samples of a given size. These students were arguably successful mathematics students (with a mean mathematics GPA of 3.26) and had completed an average of six mathematics courses at the upper undergraduate level. Although fifteen of the eighteen prospective teachers in this study had completed a senior-level course on the theory of probability, none had had a formal college course on statistics or data analysis.

Data and Data Analysis

The primary sources of data for this study were two one-on-one 60- to 90-minute problem-solving sessions per student designed to elicit the student's understanding of sampling distributions. During the second interview, students were asked questions related to distributions of a population, of a sample taken from the population, and of 500 sample means. The graphs for the three distributions are displayed in Figure 1. The graph on the left shows the income distribution from the 1990 U.S. Census for a Pittsburgh suburb. The graph in the middle is that of a distribution of one sample of size 10 from the given distribution of Pittsburgh-area incomes, and the graph on the right is that of the distribution of the means of 500 size-10 samples from the given distribution of Pittsburgh-area incomes. Students were asked probing questions about the relationships among these three distributions and to explain what would happen if various hypothetical modifications to the sampling process were made. These sessions were video-recorded, and recordings were transcribed and annotated. Data were supplemented with recordings of class sessions and copies of students' written work. The analysis was focused on interviews with six students who demonstrated a willingness to verbalize their thinking and who articulated a range of understandings about sampling distribution. Analysis sessions typically centered on a line-by-line interpretation of each student's overall understandings related to sampling distribution. Analyses included summaries of student reasoning about sampling distributions and related statistical concepts, identification and refinement of emerging themes, and identification of specific examples of those themes.

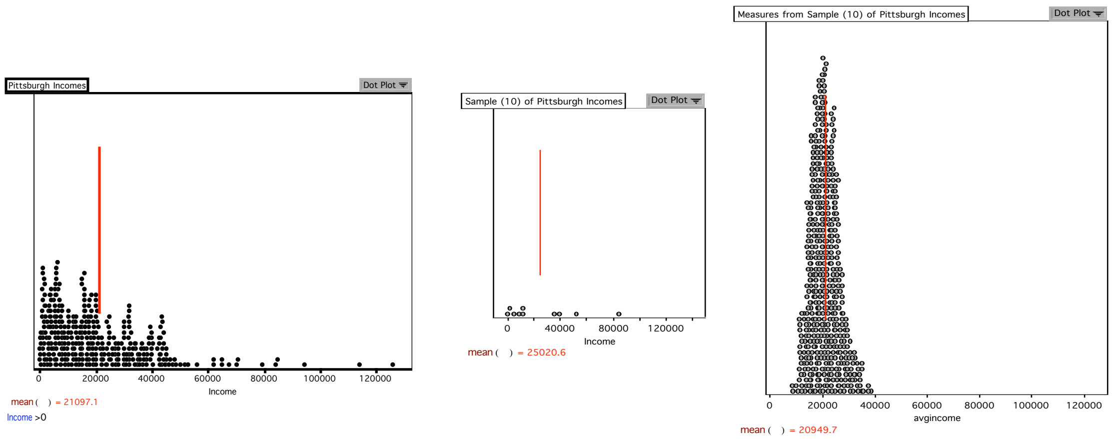

Figure 1

This report will illustrate the persistent overarching theme of making and managing distinctions that arose in the analysis of these data. We used the term "managing distinctions" to describe the ways in which students called on prior distinctions in reasoning about a complex situations. Having identified the making and managing of distinctions as an emerging theme, we analyzed each student's responses in the context of the Pittsburgh population situation to determine the ways in which students were making and managing distinctions. Although we focus our analysis in this paper on the Pittsburgh population situation, we tested our observations against student responses on a similar sampling distribution problem centered on data from a statewide school mathematics assessment.

Results

Making and Managing Distinctions

The nature of reasoning about sampling distributions is inherently complex, and these complexities can lead to difficulties in reasoning about sampling distributions. Our analysis revealed three primary areas in which important distinctions among statistical ideas are necessary. One needs to use precise language in order to distinguish between statistical terms with precise meaning and use in everyday language. One also needs to be able to formalize informal notions mathematically in order to make conceptual distinctions beyond the limitations of everyday language. Lastly, one must be able to make distinctions between closely related concepts in the complex setting of sampling distributions.

The difficulty in reasoning with statistical terms is compounded by the fact that many statistical terms have everyday meanings that are less precise than their statistical meanings. For example, when comparing the spread for different distributions of data, Tate appeared to equate the terms "spread" and "range" with an interval of values from minimum to maximum. He did not distinguish between the well-specified meaning of the mathematical entities he was describing and the common usage of the terms. In other cases, students used proper statistical terms but struggled with describing the corresponding concept or using the concept in their reasoning.

In addition to knowing and being able to use precise language for concepts, one must be able to make conceptual distinctions by recognizing the essential characteristics of the concept in order to mathematize the concept. In some cases, students recognized the essential characteristics of the concept, but were unable to express their ideas formally. For example, Eva identified the essential feature of an outlier – that of a value far removed from the majority of the data – but was only able to provide pieces of a formal definition of the term. She had difficulty mathematizing the concept and was unable to be more precise than what everyday language allowed.

Furthermore, the use of informal language impacts one's ability to make distinctions that would be available through the use of precise mathematical language. When Eva attempted to express mathematically the impact that an increase in sample size would have on a set of sample means from the same population, she used the phrase "more concentrated around the mean" to describe the behavior of means from samples of size 50 in comparison with means from samples of size 10. Similarly, Jennifer claimed that you are more likely to get a mean close to the population mean for larger samples. Jennifer's argument was based on the concept of "closeness" in spite of the fact that, like Eva, she failed to elaborate on the meaning of the terminology she introduced. In both cases, Eva and Jennifer failed to mathematize this concept of closeness. It was not clear whether they were conceiving of closeness to the mean as the same percentage of data values contained in a tighter interval about the mean or as a larger percentage of data values contained in the same interval about the mean or neither.

An essential feature of reasoning about sampling distributions is to recognize the inherent complexity of the concept along with related concepts and be able to distinguish among closely related concepts. For instance, Eva was asked about the mean of a size-10 sample from the given population of Pittsburgh-area incomes. In her response, she emphasized the possible compositions of the samples, without recognizing that some samples are more likely than others. She articulated that the number of outliers in a sample, the spread of a sample, and the shape of a sample could vary considerably, but she did not attach probabilities to any of these characteristics. When asked whether some means are more likely than others, Eva was able to talk about more likely compositions of samples, but was unable to extend her reasoning to talk about more likely means. Eva's consideration of the possibilities for the samples and means without the corresponding probabilities indicates that she did not fully grasp the complexity of the situation, and even when she acknowledged the probabilistic nature of the task, she was unable to incorporate this complexity into her reasoning.

In order to reason successfully about the task described in this study, one needs to not only recognize the complexity involved in reasoning about a population, a sample, a proper distribution of sample means, and a sampling distribution but also be able to reason about statistical concepts within that complexity. For example, Karl, Tate and Jennifer were all able to distinguish the graph of 500 means of size-10 samples from the sampling distribution, although they varied considerably in how they used this distinction in their reasoning. Karl clearly understood the difference in magnitude between 500 sample means and the sampling distribution and was able to convincingly argue about the maximum value for the sampling distribution versus the most likely maximum value for a distribution of 500 sample means. Additionally, Karl was able to reason that the mean of a given size-10 sample may or may not be represented on a given distribution of 500 sample means and was able to articulate the impact of the given sample mean on the distribution of 500 sample means. Karl was able to make distinctions between the sampling distribution and a proper distribution of sample means and to explain the differences at appropriate times.

In contrast, Tate acknowledged the distinction between the set of 500 sample means and the sampling distribution but did not consistently use this information in his reasoning. At times, he reasoned from the set of 500 sample means as if it were the sampling distribution. In one example, Tate used the graph of the distribution of 500 size-10 sample means to conclude that the range of 500 size-50 sample means would be smaller than the range of 500 size-10 samples. Although this reasoning is correct for the range of the sampling distribution, Tate incorrectly applied this reasoning to the sets of 500 sample means. Later Tate was asked to determine which was more likely – a single income more than $25,000 or a size-10 sample with a mean more than $25,000. Again, Tate reasoned from the set of 500 sample means as if it were the sampling distribution. Later, Tate indicated that his response might be different if he had all possible samples, but he was unclear about how to reason about any potential differences.

Jennifer also acknowledged the distinction between the set of 500 sample means and the sampling distribution but did not consistently use this distinction in her reasoning. She was asked to superimpose a sketch for the curve of a distribution of 500 size-50 sample means over the distribution of 500 size-10 sample means, Jennifer drew the curve as if she had been given the graph of the sampling distribution by using the range of the given graph as a guide for the range of the superimposed graph. In her next statement, however, she acknowledged that the two graphs had the same number of points. The closeness in time of Jennifer's recognition of the difference (between the distribution of 500 sample means and entire sampling distribution) to her treatment of them as the same suggests that she may not have viewed the distinction as relevant to the question at hand. Although Jennifer recognized the relevant distinctions, she failed to apply those distinctions in an appropriate setting.

Karl, Tate, and Jennifer provide three examples that indicate different ways students can incorporate the complexity of a situation and make associated distinctions in their reasoning. Karl recognized the distinctions among the mathematical entities and was able to use those distinctions consistently and appropriately in his reasoning. Tate and Jennifer recognized the same distinctions, but did not consistently apply those distinctions in their reasoning. They may have been unclear about how to proceed with a particular distinction, or they may not have seen a particular distinction as relevant.

Managing Distinctions within a Probabilistic Setting

Not only are there issues about making distinctions among similar concepts but there is also a more systemic issue of knowing what rules apply under given constraints. These systems of constraints consist of assumptions made and the conditions under which mathematical relationships hold. In statistical settings such as the study of sampling distribution, the probabilistic qualification of properties of entities or relationships between entities presents a distinctly different set of conditions under which certain properties hold than students typically encounter in school mathematics.

The most striking examples of the tendency of students to look for absolute truths rather than conditioned truths had to do with instances for which students reasoned deterministically when it was appropriate to reason probabilistically. That is, they reasoned as if the given conditions determined a single predictable result instead of a probabilistically conditioned result. At times, students refrained from making any statements about a relationship when they could not make a certain conclusion but could have made a probabilistically qualified statement. Eva, for example, concluded that no conclusion could be drawn about the shape or spread of a sample because of the wide variety of possibilities.

At other times, students made absolute statements when probabilistic qualification was in order. One such instance occurred when students were questioned about how the graph of 500 sample means would change if the number of samples increased. Jennifer and Eva both claimed that the more samples you take, the closer their mean is to the population mean without conditioning their responses with a statement about behavior in the limit or about the fact that while their conclusion is probable it is not certain (the larger number of samples could still have means that are outliers). A second instance occurred in a number of places at which students were considering the likely composition of a sample. Jennifer, for example, discussed the variability of the mean of a distribution of sample means without taking into account the fact that some means would be more likely than others. This phenomenon of accounting for the possibilities of a sample composition and not the associated probabilities was common although not universal among the students we interviewed. Students often lost track of when probabilistic qualifications were in order.

Discussion

Understanding of sampling distribution requires maneuvering carefully across a complex terrain of conceptual interrelationships and managing distinctions among those relationships. The variation in the extents to which prospective secondary mathematics teachers with considerable mathematics background make and manage distinctions suggests that this feature of their thinking may help characterize the nature of mathematical thinking at more advanced but still developing levels. The prospective secondary mathematics teachers in our study could accurately describe the construction of a sampling distribution, but as we listened to them talk about their understandings and to apply the concept of sampling distribution to real data, we grew to appreciate the complex nature of the concept. In a situation that required thinking about samples and distributions of both populations and sample means, the prospective teachers inconsistently made appropriate distinctions among the objects about which they were reasoning and among the rules that governed those objects. They reasoned deterministically when their reasoning should have been probabilistic. This lack of agility in their thinking suggests aspects of statistical reasoning that may need greater instructional attention. Moreover, it is our belief that this notion of the essential role of making and managing mathematical distinctions at appropriate times may account for difficulties in mathematical and statistical reasoning in a wide range of advanced mathematical settings. Our study is exploratory. It suggests the promise of investigating in a further study the role of making and managing distinctions in prospective teachers' understandings in other areas of mathematics.

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PURPOSES FOR MATHEMATICS CURRICULUM: PRE-SERVICE TEACHERS'PERSPECTIVES

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This study examined pre-service mathematics teachers' interpretations and perceptions of curriculum materials and texts. The students were asked in a mathematics methods course to compare and contrast reform curriculum with traditional curriculum with particular attention to the views of mathematics and problem solving implied by the materials. The assignment elicited students' beliefs about their role as mathematics teachers as well as how students should learn mathematics as well as the possibility of the integration of the approaches described by each text.

Objectives and Purposes

Mathematics teaching at the secondary level (middle and high school) is a complex endeavor. Teachers must maintain and develop knowledge about mathematics itself and about teaching and learning mathematics (Ball & Bass, 2000; Shulman, 1986). In addition to mathematics knowledge, prospective teachers bring a variety of prior experiences that impact their mathematics ability. Prospective teachers range from recent college graduates in their 20s to people working on a second or third career in their 40s and 50s. So, understanding the beliefs about mathematics, teaching, and learning is an important issue for understanding pre-service teachers and their learning to become teachers (Artzt & Armour-Thomas, 2002; Peressini, Borko, Romagnano, Knuth, & Willis, 2004). In addition to the complexity of mathematics teaching in different settings, the documentation and assessment of learning about becoming a mathematics teacher within the university setting is complex. This study presents one means of eliciting and documenting prospective teachers' understanding of mathematics teaching and learning by asking them to examine and critique curriculum materials.

A central part of mathematics teaching is the curriculum used in the classroom (Ball & Cohen, 1996). There has been a significant amount of debate related to the characteristics of effective curriculum materials and their implementation in the classroom which is beyond the scope of this paper. However, exactly how mathematics teachers interact with and use curriculum is still largely an open question (Remillard, 1999). In particular, Remillard points to distinctions between textbooks and the enacted curriculum in the classroom. Other investigations have examined materials use in the classroom (Ball & Feiman-Nemser, 1988; Sosniak & Stodolsky, 1993; Spillane, 2000). This study examines prospective mathematics teacher responses to a task requiring analysis and critique of two sets of curriculum materials: one a reform-oriented, standards-based series and the other a traditional set of materials. The two types of materials represent different views of mathematics, teaching and learning. Both types of materials were approved by the state for classroom use. The prospective teachers understand that textbooks will play a significant role in their teaching particularly as they work to cover state and district mathematics standards. Recent reports have called for the inclusion of teachers' perspectives in the evaluation of such materials (National Research Council, 2004) since the teachers' will be the primary users of the materials. In addition, pre-service teachers enter certification programs with preferences, beliefs and interpretations about the role of curriculum materials in teaching and learning which influence their later implementation of any set of materials. The study presented here includes two questions. First, what are pre-service teachers' impressions of the two types of curriculum? Second, what are pre-service teachers' interpretations of how those materials define mathematics and mathematics problem solving?

For the purposes of this study, the analysis will focus on students' comparison of traditional textbooks and reform textbooks and their role in teaching and learning mathematics. While students in the course also completed lesson planning, unit planning, field observations, and problem solving activities as well as readings related to mathematics teaching and learning, the textbook analysis assignment served a different purpose in the classroom. The assignment provoked a lengthy class discussion about the role of textbooks and, by extension, how students learn mathematics. The assignment also served to introduce students to the types of materials that might support the practices we had been discussing in class and in the assigned readings.

This assignment revealed significant differences in students' perceptions of textbooks and mathematics teaching. For instance, one criticism by students of the *Connected Mathematics*, *Math in Context* and *Math Connections* texts was that they were not sufficient as mathematical references (e.g., no definitions provided, very few formulas or algorithms). This raises the question of what role textbooks serve in the mathematics classroom. Do they primarily provide tasks to develop understanding or should they serve as reference materials? The students also varied in the perception of problem solving and what it meant to do problem solving tasks or exercises in different contexts.

Methods

Context for Data Collection and Analysis

Methods of Teaching Mathematics in Secondary School is the first of a two-semester sequence of mathematics teaching methods courses. Both courses are designed to provide a launch into a teaching career. There is a focus on mathematics learning from the secondary students' perspective and the relationship to instructional decisions by the teacher. As in most mathematics methods courses, the students complete a number of assignments related to teaching and learning including problem solving, lesson planning, examining classroom video tapes, field observations, and the investigation of different teaching practices. As will be described later, the students in the class have diverse backgrounds and mathematical experiences. Some of them are also practicing mathematics teachers or may be working in schools in other capacities. All of the students are working towards teacher licensure in secondary mathematics.

I taught the course in Fall 2004 and Spring 2005 for the first time at George Mason University. I re-designed the course and updated the assignments and readings based on similar courses offered at other universities. My goal in the course was to provide students the means to critically examine and think about mathematics teaching and learning. In particular, I wanted to expose them to the complexity of the role of the teacher and the perspective of learners in a mathematics classroom. As in mathematics teaching, I wanted to elicit the students' ways of thinking about a topic (in this case, via textbook analysis) and provide a space conducive to critical discussion and examination of different viewpoints. As with mathematics learning, the students were at different stages of learning to be teachers. Some students were teaching mathematics at the same time as they were completing licensure so could bring the day-to-day work of teaching into our class. The textbook analysis was one means of eliciting their thinking about a tool for mathematics teaching and made explicit what might otherwise be implicit assumptions and beliefs enacted in different types of textbooks. The assignment was given about halfway through the semester so they had some exposure to different ways of thinking about mathematics teaching and learning.

Since many mathematics teachers rely on textbooks and other learning materials for instruction and planning, the textbook analysis assignment was developed in order to help students unfamiliar with current innovations in textbooks learn about what might be available in their classroom. I asked them to compare and contrast reform textbooks with traditional mathematics textbooks. The goal was also to have students consider and reveal their expectations and beliefs about mathematics, problem solving, teaching, and textbooks. In particular, the students had assumptions and expectations about mathematics curriculum materials that may not have been fulfilled by the reform-oriented texts. The students also identified the qualitatively different approaches to teaching and mathematics in the various textbooks. Ironically, although the assignment did not ask the students to say which series they preferred, they usually included a statement of preference somewhere in their writing.

In contrast to the discussions in the course related to observing teaching (either that they had observed in schools or on video tape), a discussion and writing about a textbook was qualitatively different because the discussion was related to a tangible object rather than a person. While teaching practices or behaviors may be attributable to the personality of a teacher, years of experience, or the students in a classroom, a textbook discussion may have been more tangible because the students could relate to the need for textbooks as part of their own teaching (either future or present). Criticisms of a textbook could have been easier or less threatening because there was no individual person involved. The students could also identify themselves as the teacher using the textbook they examined. So, rather than thinking about someone else's classroom (which they may find idealized or impossible to realize as a novice teacher), they could envision themselves using the texts and consider different ways that mathematics learning and teaching might occur in the context of different materials. In addition, the students could consider different visions of teaching and learning as well as how their own teaching practice might look as they integrated different types of materials.

Participants

Participants in this study were graduate students enrolled in an introductory course for methods of secondary mathematics teaching. All students in the secondary teaching program at George Mason University are graduate students. Over the two semesters of data analyzed for this paper, 22 students responded to the textbook analysis assignment (14 men, 8 women). Due to critical teacher shortages in the local region, nine students were employed as mathematics teachers (five in public high school, one in private high school three in middle school). Since these nine were practicing teachers, some of their comments were related to a textbook they were currently using in their teaching. Other students were teaching other subjects, working as substitute teachers, employed in non-education fields, or staying home with children. Most had background either in mathematics or a mathematics-related field (e.g., engineering, business, or economics). Most were coming to teaching after a career in a different field. The students ranged in age from mid-20s to late-50s so represented a diverse set of educational experiences. Students were either working toward a master's degree in education or had completed a master's degree in another field. However, due to the diversity of their experiences and the range of ages in the course, they represented a broad range of mathematical experiences and knowledge.

Stages of Data Analysis

Data analysis included the collection of the written responses to the following questions related to comparing a reform-oriented textbook and a traditional mathematics textbook at a particular grade level.

1. Select a mathematical topic (e.g., addition of fractions) and compare the textbooks' treatment of the topic. How are they similar? How are they different?
2. How would you, as a teacher, structure your teaching differently with each type of textbook?
3. How do the textbooks view mathematics differently? How do the textbooks view mathematical problem solving differently?

The first question (mathematical topic) was selected in order to help focus the students' attention on the textbook and provide some means of comparison for two types of materials that are qualitatively different in approach. Students reported working out some of the examples and the exercises. They also reported on the organization and appearance of the pages in this section (e.g., colorful pages, distracting or extraneous information). The second question placed them in the position of a teacher. The assignment was placed about halfway through the semester so the students had observed other mathematics teaching in the schools or on videotape, and they had completed readings related to mathematics learning and teaching. A focus of many methods classes (including this one) is on lesson planning and the organization of instruction. The question was intended to help them think about the planning process using the different types of materials as well as how a mathematics class would be organized using the different materials. Their responses included how they would use small groups, the role of the teacher as either facilitator or leader, and the kinds of assignments and activities the students would engage in during the class. The final question related to mathematics and problem solving was intended to elicit their impressions of the content they were teaching and assumptions about that content embedded in the two sets of textbooks. The question also elicited responses about how students should or could learn mathematics in different classroom scenarios.

The students submitted their responses and participated in a class discussion about their impressions of the textbook series. The data analyzed here includes only the written responses to the questions; however the class discussion revealed distinctions made by the students that impacted the subsequent analysis and coding. The first stage of coding characterized their responses to questions 2 (teaching) and 3 (view of mathematics and problem solving). Responses to each question were aggregated and coded for content qualitatively (Miles & Huberman, 1994). Even with the small sample, there were some consistent patterns of responses to the questions which reflected the students' conceptions of mathematics teaching and learning.

Results

The written responses to the textbook analysis assignment did elicit student interpretation about the dichotomous philosophies regarding teaching and learning represented by the two textbooks series. A view into their perspectives as novice teachers gave insight into how curriculum materials may or may not be used in a mathematics classroom. In any analysis of an artifact, there is a level of interpretation on the part of the reader (even in my analysis of the textbook analysis assignment). In this case, the students' interpretations as novice teachers were based on their prior experiences as mathematics learners, their vision of future work as teachers, and any experience in classrooms in between.

Results at this stage of the analysis include the following issues. First, the students' perceptions of the role of textbooks in the classroom influenced their critique. Second, the students' views about what secondary students' need from mathematics instruction in order to learn and understand topics varied. For teaching, they often said that some combination of traditional practice and worked-out examples and methods using real-world problem solving would be ideal. In most cases, they advocated some blending of the two approaches or would have liked having both types of textbooks in the classroom. Table 1 shows the textbooks analyzed by the students.

Table 1: *Textbooks Analyzed*

Reform Textbook	Number	Traditional Textbook	Number
Connected Mathematics	13	Algebra	11
Math in Context	6	Pre-algebra	3
Math Connections	3	Geometry	2
		Other	6

A common critique of the standards-based materials was the lack of practice problems, the lack of worked out examples and the need for a textbook to function as a reference when students are working on mathematics problems and exercises. For instance, the students were looking for more algorithms, definitions, and formulas in the textbooks. In addition, they were also looking for more exercises for practice. From one students' perspective, the textbook should be a "toolbox of knowledge" available as students are working on problems. If the standards-based materials do not function as reference materials, what impact does that have on mathematics teaching? From the pre-service teachers' perspective, the textbook needed to function as a reference book. In terms of practice problems, the pre-service teachers at this stage still feel they are a necessary part of mathematics teaching and learning. For example one student states, "I feel this text could also use some supplementary exercises, because I, too, believe in the power of practice. I'm glad that this book lets the students develop the concepts out of necessity in problem solving, but without practicing applying these concepts, I fear retention of the main concepts will be limited."

A critique of the traditional materials was that they did not provide enough opportunity for students to develop conceptual understanding of mathematics topics. For instance, one student stated, "The *Connected Math* book focuses on the concepts and how they relate to other mathematical concepts. The primary focus is not on regurgitating a formula to find answers. I believe that the [traditional textbook] has a tendency to promote just finding the answers." The comment connects to other statements by students which described how the reform textbooks emphasized the process or the "journey" to the answer where the traditional textbooks focused on the answer itself. This is a fundamental difference between the two approaches with implications for teaching and learning. So, while the students critiqued the lack of practice problems in the standards-based materials, they also felt the traditional materials moved too quickly from providing formulas to practice problems or there was not enough focus on connecting and explaining concepts beyond giving a worked-out example. In contrast, for the reform textbooks, the students used words like "discovery" and "making generalizations" to describe the process of learning and teaching. They felt that the reform materials were more student-centered and conducive to small group learning.

Overall, even though there was no question about textbook preference, the students often implied a preference for one approach over the other. However, they also advocated a blend of the two approaches and saw the need for students to develop conceptual understanding and to complete real-world problem solving applications of mathematics without leaving behind the application of formulas and algorithms. One practicing teacher in the study described how she moved between the textbooks depending on the mathematical content and her perceptions of the coverage of the topic. She described accommodations for different students and the assumptions made by the reform textbooks about students' prior mathematical knowledge. Other students described possible accommodations and concerns about different kinds of learners using the materials. The students could see advantages and disadvantages of both methods and fell somewhere on a continuum between exclusive use of one or the other.

One challenge to the analysis was the students' use of vocabulary in unclear ways. For instance, "problem solving" was used in a number of ways to refer to one-answer exercises as well as longer investigations. Connecting a comment to the textbook to which it referred helped clarify the language. However, the use of mixed language has implications for the study of other types of reflections about teaching and learning by novice teachers. The use of vocabulary in unclear ways could also represent the novice teachers' emerging understanding of mathematics teaching and learning as they are still constructing meaning for ideas like problem solving. This could impact how other reflections about teaching are interpreted and highlights the importance of clarifying language.

Further analysis will include examining the relationships between the students' mathematics experiences (as presented in a mathematical autobiography) and their critique of the textbooks. Other extensions of the study include examining responses to other course assignments as well as documenting some aspects of classroom discussion. For instance, one student who originally responded negatively to the standards-based materials in his written response changed his mind after hearing evaluations by his peers in class.

Conclusion

The study supports and extends prior work investigating teachers' use of curriculum materials with an emphasis on pre-service teachers. Remillard (1999, 2000), Collopy (2003), Ball and Feiman-Nemser (1988) and others have examined how curriculum and textbooks play a role in teachers' learning and development. Teachers in those studies had a variety of reactions to curriculum and enacted different materials in different ways. Teachers' dynamic use and development of activities from a variety of sources is consistent with the findings of this study. The pre-service teachers' advocated using a blend of the reform methods and traditional practice problems. As Stodolsky and Grossman (1995) found, the mathematics department culture will likely play a large role in the students' use of curriculum as mathematics teachers whether that means they have the autonomy to use materials independently or must conform to the expectations and requirements of the department. In any case, the methods course was the beginning of their exposure to divergent views about curriculum materials and the role they might play in the classroom.

The investigation of pre-service teachers' perceptions of mathematics textbooks provided some insight about their beliefs about mathematics learning and teaching as well as their roles as teachers. They found advantages and disadvantages to both types of materials and could see how their role as teacher would be different using each set. However, most of the students determined they would need aspects of both curricula in order to be effective teachers. If that is the case, is there a middle-ground between the perceptions of mathematics learning in the curricula? Is the effectiveness of standards-based curricula undermined when teachers' add other types of materials to instruction? How is teaching impacted when teachers are given curriculum with which they have fundamental disagreement? In any case, if reform materials are adopted by a school, as stated by a student, "almost every teacher will have to be re-trained into this new method of teaching. If they are not, their lack of the new teaching style will be obvious to the students. This will not help the learning process at all." Many students also noted how different the reform textbooks were from the way they had learned mathematics. Training teachers to teach in a different way than they learned is a continuing challenge.

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PATTERNS OF SECONDARY MATHEMATICS STUDENTS' REPRESENTATIONAL ACTS AND TASK ENGAGEMENT IN A SMALL-GROUP TECHNOLOGY INTENSIVE CONTEXT

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This study investigates high school students' mathematical thinking in a technological context by focusing on representational actions and engagement in mathematical tasks when students have access to powerful technology tools. Two frameworks were employed in the analysis to illuminate the interplay between representational acts, types of tasks, and technology use in small-group and whole-class settings.

Purposes and Perspectives

When mathematics teaching and learning occurs in technology-intensive environments, students often engage with tasks and use resources that differ markedly from what might be available in non-technological contexts. Exploration, generalization, reasoning, and justification tasks are but a few of the processes that technology can facilitate. Because students' performance on various tasks may be influenced by the resources available to them in a technology-intensive environment, one step in understanding the potential influence can be to determine the nature of or patterns in students' use of resources, such as mathematical representations, as they engage in tasks when they have access to technology.

This study addressed the research question: What are patterns of students' engagement in representational acts and mathematics tasks in the context of a technology-intensive curriculum? The way in which the notion of "task" was defined in this study differed somewhat from the way "task" has been defined in other research literature (e.g., Friedman, 1976; Stein, Smith, Henningsen, & Silver, 2000). Our definition of task assumes that tasks are goal-driven, may be implicit or explicit, can be posed by the curriculum, teacher, researcher, or student, and as students are working on a large task, subtasks may be introduced. Categories of tasks were developed through a cyclical process involving coding and refinement.

As students engaged in different mathematical tasks while they had access to technology, they made use of mathematical representations. In addition to coding the task on which students were working, the research team also coded students' representational acts using the MAGICAL framework (Zbiek, 2002). Coding of representational acts involved the identification of the mathematical object represented (e.g., geometrical point, function, polygon), the type of representation (e.g., graph, table, symbolic rule), and action performed by students. The two-tiered coding process enabled the researchers to identify patterns of representational acts and task engagement as students solved mathematics problems using technology.

Methods and Data Sources

The study took place in a heterogeneously grouped class focused on intermediate algebra. The course was offered in block-scheduled format classes that were 85 minutes in length and met five days per week for one semester. The course used three modules from the *Technology-Intensive Secondary School Mathematics Curriculum* (TISSMC, Heid, Zbiek, Blume, Choate, &

Foletta, 2004) The teacher of this class attended three two-day workshops to prepare to use the materials in her class prior to implementation. During class and outside of class each student had access to a computer algebra system (TI-89 calculator), and pairs of students had access, in the classroom, to computers with *The Geometer's Sketchpad* (Jackiw, 1990) software.

The class consisted of 31 students from grades 10-12. Because seven of these students were classified as special needs students, there was a collaborator teacher in the classroom in addition to the regular teacher. The typical student in this course had taken two years of introductory algebra as well as Geometry. Data were collected from the small-group work of 12 students. The research team consisted of four members[1] who were present during data collection. One of the research team members served as the primary classroom teacher on the days when small-group data were collected. This research team member conducted class and facilitated small groups of non-target students. The regular classroom teacher and collaborator also assisted and observed non-target students. The remaining three members of the research team served as facilitators for the 12 target students in their small groups, asking questions, eliciting clarification of students' statements, and suggesting things for the group to explore. All sessions were videotaped and audiotaped to capture students' work on paper and with technology. Written artifacts were also collected. Verbatim transcripts were created from the videotapes and audiotapes and analyzed.

Four members of the research team read transcripts initially to identify tasks. A description of the task categories that were used in the analysis is included in Figure 1 and discussed further in Heid, Blume, Hollebrands and Piez (2002).

Task Category	Code	Description
Identify	Identify object (IO)	The task is to identify the *name* of an object when the characteristics of that object are presented.
Describe	Describe observation (DO)	The task is to state what is seen (visual to perception)
	Describe procedure (DP)	The task is to describe or identify a procedure that is already known or observable by the student.
Elaborate	Compare/Explain/Describe phenomenon (CED)	The task is to compare two different mathematical objects or representations,make sense of one thing in terms of something else, or describe a phenomenon or procedure that is not perceivable by the senses.
Produce Value	Produce a value or output given an input (PE)	The task is to provide output value(s) given particular input value(s).
	Produce an input value given an output value (PS)	The task is to provide one or a set of input value(s) given one (or a set of) particular output value(s).
	Produce a graph (PG)	The task is to produce a graph
Corroborate	Corroborate a result (CR), procedure (CP) or generalization (CG)	The task is to provide additional evidence that what is given or found is true. [Determine if something might be true or false]

[1] In addition to the authors, members of the research team included Glen Blume, The Pennsylvania State University, and Cynthia Piez, University of Idaho.

Predict	Predict (Pt)	The task is to describe what might happen under certain conditions in a novel situation. Students are asked to come up with a conjecture. There is no expectation that the students have enough information to deduce the answer.
Justify	Justify (J)	The task is to provide a logical argument for why something happens [Establish truth of finding *in the spirit of the field of mathematics rather than in terms of what convinces the student*.]
Generalize	Generalize (G)	*Generate* a relationship that holds for an entire class. The task is to generate a relationship from instances that are given or from logic.
Generate	Generate function specifics (GFS)	The task is to produce a function rule. Given a rule, graph, table, or situation.
	Generate a procedure (GP)	The task is to create a procedure that does not already exist.

Figure 1. Task types used in the analysis of the data

After the transcript was coded for types of tasks, a second team consisting of two to four researchers coded transcripts for representational acts using the MAGICAL framework (Zbiek, 2002). Types of actions described in the framework are: Manipulate, Ascribe, Generate, Interpret, Connect, Augment and Link. A description of each of these different types of representational acts is included in Figure 2 and discussed further in Zbiek (2002).

Code	Description
Manipulate (M)	Change one representation of a mathematical object to another of the same type.
Augment (Au)	Make prominent something that is already in a representation.
Generate (G)	This is the production or introduction of the first representation of that type.
Interpret (I)	This involves giving meaning to a representation by interpreting it in terms of a situation or in terms of an abstract concept.
Connect/compare (C)	Connect or compare two representations of the same type. They represent two different mathematical objects.
Ascribe (As)	Produce the first representation of a situation or of an abstract concept.
Link (L)	For one mathematical thing (equivalence class) a representation of one type is connected to a representation of another type.

Figure 2: Categories for coding representational acts (Zbiek, 2002)

The two sets of codes were then placed sequentially on graph paper using one horizontal line to represent one line of transcript (Figure 3). This allowed researchers to examine patterns within and relationships across the different coding schemes. This paper focuses on data from one of the three small groups (Hank, Jenny, Jim, and Rachel) during a two-day small group observation.

Results

A way to describe the flow of reasoning that occurs in whole class and small group settings is by analyzing patterns of representational act sequences. Several patterns were identified. For this paper, two patterns will be illustrated. One pattern seemed to reflect instances of student engagement in inductive reasoning and a second pattern seemed to characterize the refinement of a conjecture.

Instances of Inductive Reasoning Are Reflected in Representational Act Sequences

The pattern of inductive reasoning was generally characterized by a sequence of Manipulation representational acts followed by Connecting, Linking, or Interpreting representational acts (Figure 3). Closer examination of these inductive reasoning patterns revealed that there were differences in the types of representational acts performed by students when they were participants in a small group and when they were participants in a large group. Examples taken from the whole-class researcher/teacher led discussion and from small group work will be used to illustrate these differences.

Representational Act	Task		
M T: What is the floor of 0.25? Ss: Zero. Lines 312-314	**Produce Value (FE)** T: What is the floor of 0.25? Ss: Zero. Lines 312-314		
M T: What about the floor of 0.5? Ss: Zero. Lines 314-315	**Produce Value (FE)** T: What about the floor of 0.5? Ss: Zero. Lines 314-315	**T, Produce Graph** Lines 303-346	
M T: What is the floor of 0.75? Ss: Zero. Lines 318-319	**Produce Value (FE)** T: What is the floor of 0.75? Ss: Zero. Lines 318-319		
M T: One? Ss: One. Lines 320-321	**Produce Value (FE)** T: One? Ss: One. Lines 320-321		

Figure 3. Illustration of a coded transcript depicting a pattern of inductive reasoning

Whole-class setting.

The following description of the whole-class discussion shows how a sequence of representational acts that include Manipulation followed by Interpretation/Linking/Connecting may be indicative of inductive reasoning and focuses attention on the participant in these representational acts, which in this case is the teacher/researcher. After students had worked in small groups to complete an introductory task, the teacher/researcher convened students to discuss their work. The teacher had noted that in response to the question that requested students to sketch $y{=}ceiling(x)$ and $y{=}floor(x)$, most made errors by connecting all consecutive input-output values creating a graph that appeared "stair-like." To focus students' attention on why the graphs should not be connected everywhere, the teacher posed a sequence of Produce Value tasks (e.g., "what is the ceiling of 0.75?") and engaged students in a sequence of Manipulation representational acts (e.g., find ceiling (0.75)).

In response to each of the Produce Value tasks, the teacher plotted the ordered pair on a graph that was drawn on the front board. Students engaged in a sequence of Manipulation representational acts and it was the teacher who engaged in the Interpretation representational act when she used these values to arrive at her conclusion that the points (0.75,0) and (1,1) should not be connected. The occurrence of this change of task and representational action highlights the complexities involved in orchestrating whole-class discussions.

Small-group setting.

While the pattern of Manipulation acts followed by Interpretation, Connecting, or Linking acts that occur in the whole class setting is also reflective of students' engagement with inductive reasoning in a small group setting, the performer of the representational acts is different. In particular, we note that in a small group setting it is the students who perform the Interpreting, Connecting and Linking acts – acts that might be described as being at the crux of inductive reasoning. An example is provided that takes place as students are working on a slider graph that represents the function family $h(x)=a*(floor(b(x-c)))+d$ (Figure 4). In response to the curriculum Generalization task, students engage in a sequence of Manipulation acts as they drag the slider, d, to the right and left. As this occurs, students make comments about what is happening in the graph in response to a student-generated Describe Observation task. This sequence of Manipulation acts is followed by a sequence of Linking and Connecting acts as *students* generate a conjecture about how changing d affects the graph of the function family.

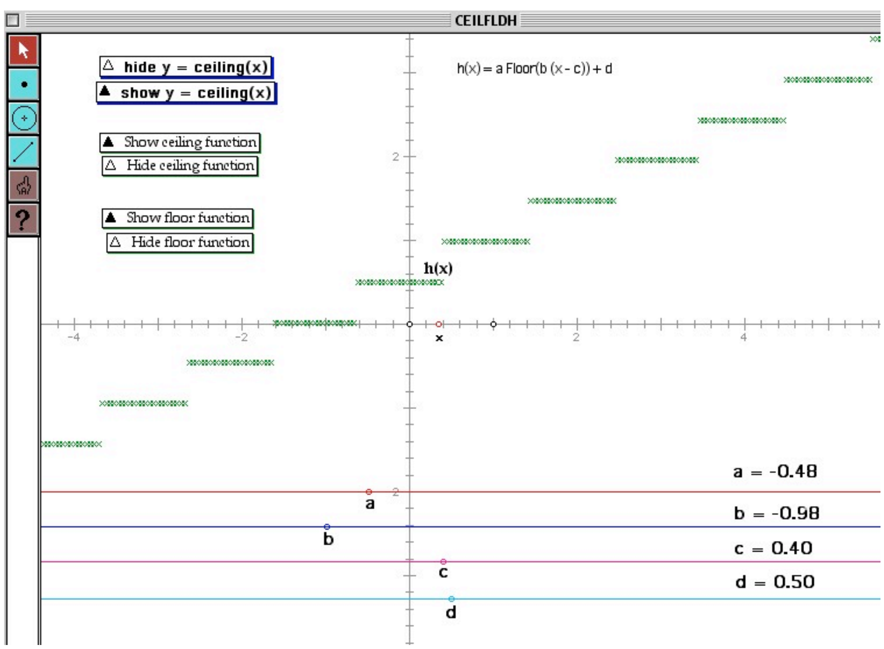

Figure 4. The graph of the function h with rule $h(x) = a\ [\ b(\ x- c)\ ] + d$.

While the sequence of representational acts in both cases follows the same pattern of Manipulation followed by Linking, Interpreting or Connecting, closer examination reveals that it is the teacher in the first example and students in the second example who are doing the Linking, Interpreting or Connecting. The MAGICAL coding enables us to attend to representational acts that are being performed and focuses our attention on who is performing them.

The task coding in each episode highlights how the original tasks (Produce Graph, Generalize) are transformed in response to students' work on each problem. In the first case, it is the teacher who poses a sequence of Produce Value tasks with the expectation that students will notice errors in the graphs they have produced. In the second episode, the Manipulation representational acts occur in the context of a Generalize task. These Manipulation acts produce graphs that allow students to describe changes that occur when the value of the parameter *d* is decreased and increased which seems to support a response to a student generated Describe Observation task. In both cases, it appears that the cognitive demands of the original tasks are reduced. While this is often considered an undesirable feature, in these episodes the joint coding allows us to conjecture that the reduction in the demands of the task might be a necessary step. Student responses to these easier tasks, particularly in the small-group setting, occurred in conjunction with the generation of representations that assisted students in their final Linking and Interpreting representational acts that are critical in reasoning inductively.

Students' Refinement of a Conjecture

A second pattern of representational acts was identified that seemed to reflect students' refinement of a conjecture that was made while they were working in a small group. Typically, one member of the group made a conjecture and other members of the group contributed additional, different information to further refine the first conjecture. This pattern was characterized by a sequence of Linking, Connecting, or Interpreting representational acts, interspersed with Manipulation representational acts and different members of the group often performed these different types of acts. To illustrate, an example is provided that takes place as students are working on a slider graph that represents the function family $h(x)=a*(floor(b(x-c)))+d$. Students are asked to describe the effects of changing the value of *a* on the graph of *h*.

After dragging the slider *a* to the left and right Rachel performed a Linking act when she presented the first conjecture about the effects of *a* to the small group: "When we move it [a] left –it's [the graph is] rotating" (519). This was followed by Jim's observation, "I think it [*a*] might change the distance between the two flat points [steps in the graph] because when you had it [*a*] at zero they [the steps] were a straight line" (524-527), that narrowed in on what Rachel might have been referring to when she noticed the graph was rotating. Rachel dragged the slider *a* to the left until it was -0.01. Jennie contributed to the refinement of the conjecture by focusing on the invariant features of the graph. She stated, "And that point [the y-intercept] stays in the same spot" (529). The initiation and refinement of the conjecture is reflected in a sequence of Linking and Interpreting acts that are interspersed with Manipulation acts (Figure 5).

Representational Acts		Task		
	L Rachel: When we move it left - it's rotating. Line 519			**C, Generalize**
M Rachel drags a toward zero Lines 525-526	**L** Jim: I think it might change the distance between the two flat points Lines 524-525		**S, Describe Observation** Lines 519-529	What are the effects of changing the value of a
	L Jim: because when you had it at zero they were a straight line. Lines 525-527	**S, Corroborate Generalization** Lines 525-578		on the graph of h(x) = a Floor(b (x - c)) + d?
M Rachel drags a to - 0.01 Line 528				Lines 517-587
	L Jennie: And that point stays in the same spot Line 529			

Figure 5. Illustration of a coded transcript depicting a pattern of refining a conjecture

There is evidence that the observation Jennie made while they were refining the conjecture was incorporated into the strategy Jim suggested to determine the effects of changing *a*. Jim noted "well if you know the one doesn't move [the y-intercept], it just stays there, if we put that on zero, we could probably try and measure if *a* does exactly change the distance [between the flat parts]" (538-540). We see the merging of Jennie and Jim's Linking acts to create a strategy for testing the conjecture that was built up by the group. Manipulation acts alone were not sufficient to address the tasks that were posed by the students and curriculum. Rather we note that the Linking acts were essential components in the development of the conjecture within this small group.

Many researchers have emphasized the importance of providing students with multiple-linked representations. The fine-grained analysis we have presented sheds light on the ways in which students in whole-class and small-group settings interact with representations as they solve mathematics tasks with technology. This analysis highlights the importance of encouraging students to engage in linking, connecting and interpreting representational acts. It also demonstrates that sometimes a reduction in the task level is necessary for students to make links and connections among different representation and use this information to attend to the original task. These insights were gleaned using analytical tools that show promise for better understanding the processes involved when students use representations as they work on mathematical tasks using technology.

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DISCOURSE THAT PROMOTES MATHEMATICAL REASONING: AN ANALYSIS OF AN EFFECTIVE ALGEBRA TEACHER

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Mathematical reasoning requires the effective use of representations, the formation of convincing justifications, and the communication of patterns and relationships through generalizations. Engagement in these practices may support students' understanding of mathematical ideas. In this study, I examine how an effective high school algebra teacher supported her students' mathematical reasoning through the organization of classroom discourse. I identify a discourse routine I call share, compare, and analyze (SC&A), examine relationships between the discursive practices of sharing, comparing, and analyzing and the mathematical practices of representing, justifying, and generalizing. I found a strong relationship between this set of discursive practices and students' intensive engagement in mathematical activities.

Introduction and Overview

Mathematical reasoning is supported by identifiable mathematical thinking practices. Specifically, mathematical reasoning involves the effective use of *representations*, the formation of convincing *justifications*, and the communication of patterns and relationships through *generalizations* (RAND Mathematics Study Panel, 2003). There is evidence that engagement in these practices supports the development of students' mathematical understanding (Greeno & Hall, 1997; Martino & Maher, 1999; Maher, Martino, & Alston, 1993; Moschokovich, Schoenfeld, & Arcavi, 1993). Fostering these practices in the classroom should therefore be a key goal of mathematics instruction.

But how can teachers foster such practices? This study closely examines videotapes of an effective high school algebra teacher, with the aim of better understanding the relationships between her classroom's discourse practices and students' engagement in the mathematical practices listed above.

The key finding is that the teacher established norms for students to *share, compare,* and *analyze* their own and each other's mathematical reasoning. She did this, in part, through the use of a discourse routine, which I have called *share, compare, and analyze* (SC&A; see also Horn, 2005). In this paper, I argue that sharing, comparing, and analyzing supported students' engagement in mathematical reasoning in several ways. First, consistent discourse routines allow students to attend to academic content rather than classroom procedure by minimizing management problems and transition times (Cazden, 2001). In addition, the expectation for students to share, compare, and analyze their thinking created space for cognitively-demanding questions (Martino & Maher, 1999) and structured norms that support high intellectual press (Kazemi & Stipek, 2001).

In the remainder of this paper, I will first connect this study to prior work on classroom discourse and students' 'mathematical thinking. Then I will explain the methods used for this analysis, followed by a report of the findings that emerged. In the end, I argue that the discourse

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in this classroom made mathematical thinking practices of representation, justification, and generalization necessary, transparent and accessible to students.

Theoretical Framework

Using a sociocultural framework on learning (Forman, 2003), this study examines the discourse of an effective high school algebra classroom. Classroom discourse is the site for investigation because it can be viewed as both a scaffold for students' learning, as well as a site for reconceptualizations of their thinking (Cazden, 2001; O'Connor & Michaels, 1993). Understanding the organization of classroom discourse and its relationship to mathematical learning are an aim of this paper.

The Structure of Classroom Discourse

Research on the discourse of mathematics classrooms has frequently examined three interrelated levels of talk. At the broadest level, studies have examined classroom *norms*, or regularities in classroom interactions that set participants' expectations (Yackel & Cobb, 1996; McClain & Cobb, 2001). One of the findings of this line of work is that classroom *routines* often organize norms by creating predictable sequences of classroom interaction.

Ostensibly similar routines may have different consequences for students' mathematical engagement, however, depending on the kinds of prototypical *exchanges* that constitute the routines. For instance, Kazemi and Stipek (2001) found that while teachers may share a routine of *eliciting multiple solutions* from students, the actual content of the exchanges within the routine varied significantly. They describe this difference as one between *low-press* and *high-press* exchanges, with the latter prompting the students to focus on mathematical connections among various solutions.

While norms, routines, and exchanges all have been used to describe the discourse of mathematics classrooms, they are highly interdependent. This study attempts to examine all three simultaneously by looking at a classroom discourse patterns alongside the kind of mathematical activity these patterns supported.

Mathematical Practices

Mathematical reasoning is comprised of (at least) three important mathematical practices. Research indicates that engagement in the practices of representing, justifying and generalizing helps students develop mathematical understandings. For example, students' understanding of mathematical objects can develop through the construction and interpretation of *representations* (Greeno & Hall, 1997). By understanding connections across multiple representations of a mathematical concept (such as "linearity"), students gain a deeper grasp of the concept (Moschokovich et al., 1993). Similarly, when students *justify* an asserted solution, they often reorganize their thinking about the problem in the process, often resulting in a more sophisticated understanding (Maher et al., 1993). Likewise, a press toward *generalization* may support students' understandings across classes of problem situations, helping them to make important connections (Jurow, 2004; Martino & Maher, 1999).

If students' engagement in these three mathematical practices assists their learning and understanding, then educators need a better grasp of how to support such engagement in our classrooms. By looking at students' engagement in these three practices alongside the organization of classroom discourse, this analysis seek to specify ways that teachers can help students learn mathematics by doing mathematics.

Research Procedures

Identifying an Effective Classroom

The data from this study come from a four-year longitudinal study of mathematics teaching and learning in three California high schools (Boaler, 2004). During the first year of the study, all ninth graders in college preparatory mathematics classes took a pre- and post-test of mathematical achievement. The students in Ms. Larimer's Algebra class at Railside High showed the most dramatic gains in their scores on these assessments out of all the 9th graders tested at three high schools, indicating the effectiveness of her teaching. In addition, in questionnaires and interviews about students' beliefs about and experiences of mathematics, students at Railside were more intrinsically interested in math and reported enjoying the subject more than other students in the study (Boaler & Staples, 2003; Boaler, forthcoming). This is particularly notable since the majority of Railside's urban students came from groups traditionally underrepresented in higher education.

Data and Methods

The primary data for the present analysis are six videotaped classroom sessions of videotape from Ms. Larimer's Algebra 1 class. The data were collected early in the school year,[1] where presumably the teacher would have a greater need to be explicit in her expectations about adequate mathematical responses. Using methods from conversation analysis and sociolinguistics, transcripts of the videotapes were created.[2]

After a period of open coding, the SC&A routine was found to be in use across the class sessions. SC&A episodes become the first unit of analysis investigated. We coded sequences of talk in which Ms. Larimer used a routine of guiding the students to *share* their responses to a problem, *compare* their solutions, and *analyze* similarities and differences to develop mathematics. There were variations to the routine — sometimes solutions were shared and made the object of discussion in front of the whole class, sometimes the solutions were discussed in small student groups.

At a finer-grain, we coded teacher-initiated exchanges in which (a) Ms. Larimer prompted students' engagement in the mathematical practices of *justifying, representing*, and *generalizing* and (b) these activities were visibly taken up by students.[3] (For more details on the coding scheme, see Horn, 2005.) Two class sessions were double coded to check for interrater reliability, which was initially 88%. After a discussion of the contested codes, we reached consensus on the appropriate coding decisions.

Our goal was to see how much overlap there was between the SC&A episodes, the practices of sharing, comparing, and analyzing, and the observable presence of these mathematical thinking practices. For our final phase of analysis, we went back to the videotapes and did a minute-by-minute coding of five of the six the class sessions to verify how frequently SC&A overlapped with students' engagement in mathematical practices. We created data displays of this coding and used those displays to calculate statistics about the classroom discourse and mathematical practices. To help us better understand the patterns in Ms. Larimer's classroom, we created a similar data display of a videotape of another well-regarded (but not as highly

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[1] I thank Megan Staples and Jo Boaler for collecting this data and sharing it with me.
[2] Ashley Watson and Lynee Lawson assisted in this phase of analysis.
[3] Lynee Lawson helped with this phase of analysis, supported by a Zesbaugh Scholarship.

effective) teacher's classroom. This second teacher (hereafter, *the comparison case*) taught in a similar setting as Ms. Larimer, had a similar amount of classroom experience, and also used a mathematically rich curriculum.

Results

Sharing, Comparing, and Analyzing were Normative in this Classroom

The discourse routines. The SC&A routine helped normalize the discursive practices of sharing, comparing, and analyzing. Sharing, comparing and analyzing were encapsulated in two SC&A routines, one for the whole class and one for the small group.

The whole class SC&A routine consisted of: (1) *a general compare prompt* (the teacher reminds students of their audience role to compare their solutions to the ones presented); (2) *a share prompt (*teacher invites a student or student group to present a solution); (3) a specific compare prompt (students are asked to compare their solution to the specific one presented); and (4) *a comparative analysis prompt* (teacher encourages the analysis of different answers).

The small group SC&A routine consisted of: (1) *signaling interdependence within student groups* (the teacher reminds class of their responsibility to work together and make sure everyone understands); (2) *a share prompt* (in the small group setting, the teacher redirects students' questions to their group members); (3) *a compare prompt* (the teacher solicits different students' thinking); and (4) *an analyze prompt* (the teacher may focus a small group on a key aspect of the problem through a clarifications or otherwise seed the conversation to direct them toward a key idea to support their reflection on their thinking).

Inherit in these routines is the valuation of mathematical difference (Yackel & Cobb, 1996), along with the interactional means to reconcile differences in a way that is mathematically productive (Ball & Bass, 2000). Ms. Larimer used these routines as a springboard for challenging questions, yet the discourse was structured enough that students began to anticipate her expectations for adequate solutions and explanations, an important support for their own internalizations of the mathematical thinking practices supported by SC&A.

The last part of the routine, wherein the teacher led students through an *analysis* of the differences among their solutions, was critical to supporting students' engagement in generalizing, representing, and justifying. Sometimes the analysis led to the development of publicly shared generalizations of the solutions overall. At other times, the class's collective analysis helped reconcile seemingly different solutions, allowing students to build connections across multiple representations. In both cases, working toward the connections across solutions required students to justify their thinking and their problem solving strategies.

Normalizing sharing, comparing, and analyzing. While each of the six class sessions included at least one SC&A episode, these discursive practices were not bounded by the routines. Analysis of the data displays showed that across five of Ms. Larimer's class sessions, *sharing* took up 54.4% of the class time (SD=14.0), *comparing* took up 23.4% (SD=13.5), and *analyzing* took up 37.6% (SD=19.2). These practices often co-occurred, as in the "comparative analysis" segments of talk. Nonetheless, over half of the students' time was taken up by these discursive practices, making them normative in this classroom.

Students in this Classroom Frequently Engaged in Mathematical Practices

In addition to spending the majority of their time engaged in sharing, comparing or analyzing their mathematical thinking, students engaged in the mathematical practices of generalizing, justifying, and representing with great frequency in this classroom (see Table 1).

When contrasting Ms. Larimer's data displays with those of the comparison case teacher, I found that the portion of class time spent engaged in *any* mathematical practice (first row of Table 1) did not differ significantly. However, the amount of time students engaged in *multiple mathematical practices* did vary significantly. In the comparison classroom, students spent only 17% of the time engaging in two of the practices simultaneously and *no* time engaging in three of the practices simultaneously. By comparison, Ms. Larimer's students spent on average 17% of their time engaged in two practices simultaneously and 20% of their time engaged in three practices, for a total of 37% of their time engaged in multiple mathematical practices. Also, comparing the ratio the first two rows of Table 1, there are class sessions in which as much as 94% of students' time engaging in mathematical practices involved the engagement in multiple mathematical practices.

I speculate that it is this *intensive engagement in mathematical activity* that accounts for the remarkable learning gains of students in this classroom. In instances where students are creating representations to justify their thinking, for example, they are required to do more intensive mathematical work than they would be in simply creating a representation. It may be that multipurposed mathematical activity supports a higher level of engagement with mathematical ideas and thus supports greater student attainment.

	9/13	**10/11**	**10/23**	**10/31**	**11/20**	**Mean**	**S.D.**
% time engaged in math practices	58	48	55	49	54	**52.8**	4.2
% time engaged in *multiple* math practices	50	35	52	24	23	**36.8**	13.8
Justifying	48	16	37	17	20	**27.6**	14.2
Representing	48	48	48	27	43	**42.8**	9.0
Generalizing	53	34	52	40	22	**40.2**	13.0

Table 1. Percentage of class time students visibly engaged in the three mathematical practices.

The Social Practices of Sharing, Comparing, and Analyzing Co-occurred with Students' Engagement in Multiple Mathematical Practices

The coexistence of the discursive practices of sharing, comparing, and analyzing with intensive engagement in mathematical activities obviously does not necessarily indicate a causal relationship. To further explore the nature of this relationship, I examined the frequency of students' engagement in the mathematical practices *alongside* the use of sharing, comparing, and analyzing. Figure 1 shows the frequency of each discursive practice alongside the frequency of engagement in mathematical activity across five of Ms. Larimer's class sessions. The graph shows that an increase in the frequency of mathematical activity co-occurred with an increase in the frequency of each discursive practice. Likewise, decreases in mathematical activity were accompanied by decreases in each discursive practice.

In addition, the minute-by-minute coding of the class sessions showed clusters of sharing, comparing, and analyzing accompanied by clusters of mathematical activity. Figure 2 is a representative eight-minute interval from one of the data displays that shows this co-occurrence at a finer-grained level.

Although this evidence still does not verify a causal relationship between the discursive practices and students' mathematical engagement, it suggests a robust relationship worthy of further examination.

Discussion

This study sought to understand relationships between the organization of discourse and students' mathematical activity in an effective high school math classroom. By investigating discourse alongside students' engagement in mathematical activity, this study suggests classroom possibilities for other teachers wanting to support their students' learning. Although there may be other factors that supported the positive learning outcomes in this classroom, this analysis may suggest something about a link between classroom discourse, mathematical activity, and student learning. The further exploration of this relationship supports both learning of mathematics and research about that learning.

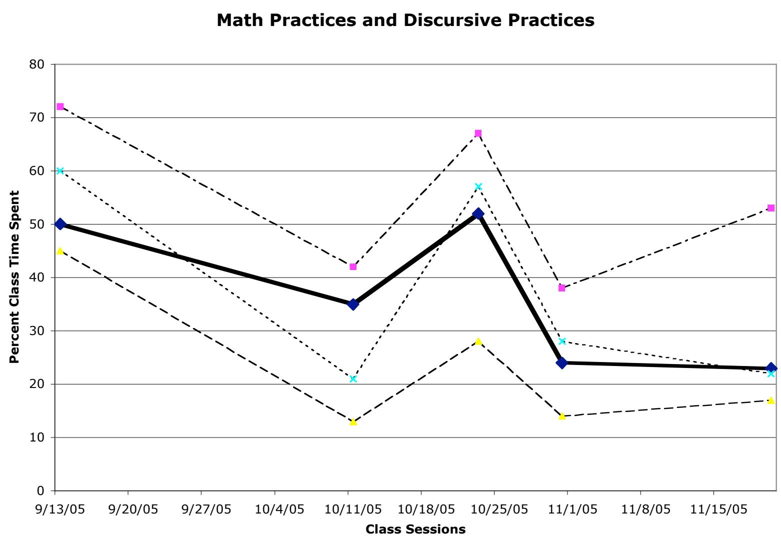

Figure 1. The top dashed line indicates the percentage of class time spent engaged in *sharing*. The second dashed line indicates time spent *analyzing*. The third dashed line indicates time spent *comparing*. The thick line indicates the percentage of time students engaged in multiple mathematical practices.

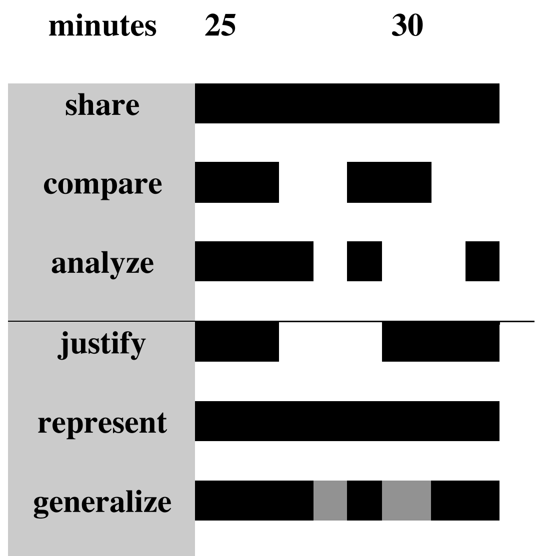

Figure 2. An eight-minute excerpt from the 10/11/2000 data display of the minute-by-minute coding of mathematical and discursive practices.

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COORDINATING TECHNOLOGICAL TOOLS IN LEARNING ANGLE AND TURN

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This paper presents an exploratory study performed with sixth grade children (11 years of age) concerning learning the notions of angle and turn. Observation was made of simple drawing tasks using Logo. The Cabri-II dynamic geometry software was also introduced, which permitted for observation of both technological tools' functionalities, and may be compared and/or complementary. The students performed exploratory and simulation tasks within Logo and Cabri-II, to finally determine both turns and measurement of the angles involved. Specifically, activities based on manipulating these learning environments permitted the students to overcome difficulties with the direction of turn and the measurement of angles greater than 180º.

Background and Theoretical Foundations

This research work takes up some results that Clements and Battista together with Sarama (2001) indicated for using Logo in learning geometry. These authors found that when K-6 students perform drawing tasks, the

> children… cannot alter the drawing procedure in any substantive manner (Karmiloff-Smith, 1990), much less consciously reflect on it. In creating a Logo procedure to draw the figure, however, students must analyze the visual aspects of the figure and their movements in drawing it, thus requiring them to reflect on how the components are put together. (Clements, Battista, and Sarama, 2001, p. 142)

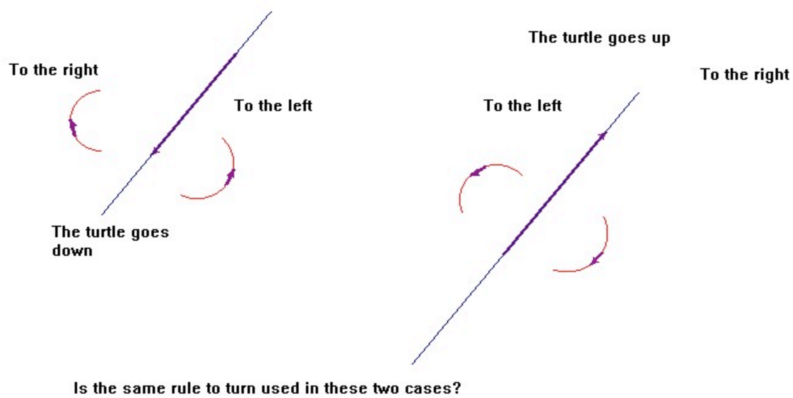
Their pre- and post-test model was repeated using the series of tasks and problems they considered to investigate children's knowledge of angle measurement. For example, "2.12. A boat is mailing on a lake, heading toward its home. It goes forward 60 yards, turns right 80º, goes forward 152 yards, turns right 160º, and goes forward 173 yards. It is "now back to its original position on the lake. How much does it have to turn to be facing toward its home again?" (Clements, Battista, and Sarama, 2001: 59).

Setting all these questions permitted diagnosis of the difficulties the subjects had in perceiving and measuring angles greater than 180º, as well as determining and calculating the direction of turn. These difficulties became more evident when pupils were involved in execution of simple drawing tasks using Logo, such as a house (Ocaña 2003) or a pine tree.

A suggestion put forward by Magina and Hoyles (1991) was also taken up: introducing a cardboard clock with a moveable hand as another possible context for the study of angle and turn. We considered this also pertinent insofar as mathematical turn rules are usually referred to clockwise.

Nonetheless, measurement tools provided by the protractor and the Logo computer program kept likely being insufficient to overcome difficulties simple drawing tasks present in measurement and determination of turn.

Figure 1. Rule to Turn



For example, Logo's turtle (Figure 1) can move forward in any direction, yet it does so lineally in such a way that it is not evident to children that the same turn procedure is being applied both forward and backward.

Further, children's performance with the protractor invited introduction of another technological tool adequate to surpassing the detected difficulties with measurement. Cabri-II was then presented in response. Its menu includes a toolbox (see Figure 2) that performs precise calculations and measurements.

In reality, introducing Cabri-II dynamic geometry software allowed for observation of how functionalities of the two technological tools (Cabri with Logo) may be compared and/or complementary.

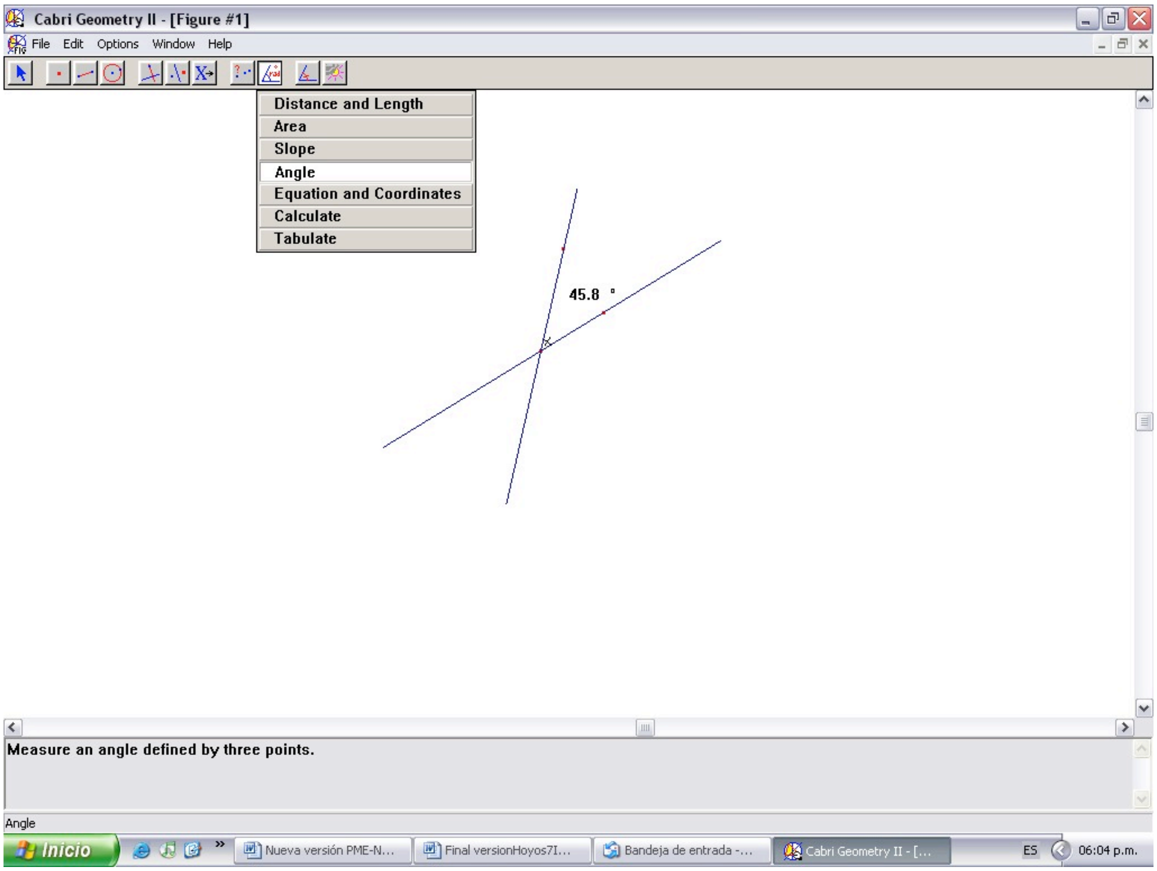
Verillon and Rabardel (1995) affirm the potential for technological tools to address specific learning difficulties concerning functionalities of the artifacts in play: "if cognition evolves, as genetic epistemology has shown, through interaction with the environment, then it can be expected, in the course of its genesis, to have to accommodate to the particular specific functional and structural features which characterize artifacts. Does this have an effect on cognitive development, on knowledge construction and processing, on the nature itself of the knowledge generated?"

This study aimed to incorporate both dynamic geometry software programs, with the idea of comparing the results from manipulating different learning environments (cf. Vincent et al., 2002; Hoyos, 2002). Specifically, the hypothesis was put forward that inclusion of an additional computer learning environment such as Cabri-II would provide specific, functional features that would enable students to leave behind the difficulties associated with determining turn and angle measurements greater than 180º, that were not solved by working with Logo and the protractor.

Aims, Methodology, and Some Results

This exploratory research began by confirming difficulties children have in solving tasks involving concepts of turn and angle. The questions drafted by Clements, Battista, and Sarama (2001: 56-60) were used for this purpose.

Figure 2. Angle measurement with Cabri-II



Then a series of activities were organized for pairs of pupils on computers in a lab equipped with 16 machines loaded with educational software, specifically Logo and Cabri-II.

For the total stages of exploration were held eight work sequences with the pupils, each lasting from 50 to 100 minutes, twice each week.

Worksheets were drafted to guide their activities consisting of a series of instructions to manipulate the software, drawing simple tasks to complete, and questions to answer. Activities basically consisted of exploring each of the two computer programs in the order presented, the drawing of simple figures such as a house (Ocaña 2003) or a pine tree, and measuring angles with the respective technological means Logo and Cabri-II provide. The tools alternated in the order shown in Figure 3.

All the worksheets were collected. The answers the students gave after activities and to the questions were analyzed. Of particular interest were the events during the final working sessions with Logo and Cabri-II, which were videotaped and transcribed.

The following are the general results from the exploration.

An observation that may be made on the sequences carried out in stage 2 is that it constituted a first approximation to using Logo, particularly drawing, an experience the students found interesting

Also, everything was fine while the students merely had to follow instructions. But it should be noted that at the end of sequence 3 the students were to make a pine tree in Logo, a task for which they did not receive measurements for sides or angles. They had to first obtain the sides and angles using a protractor and ruler, based in a drawing they had in a printed handout. In this task, the students' executions demonstrated they were incapable of manipulating an ordinary protractor to measure or obtain angles greater than 180º. This being the case, they thus could not fulfill the task of "telling" the turtle how far to advance and how much to turn.

Figure 3. Stages of Exploration

Activities de Detection, Observation, and Analysis
Stage 1. Application of pretest
Questionnaire 1: Diagnostic
Stage 2. Use of Logo
Sequence 1: The Logo program
Sequence 2: Teaching the turtle
Sequence 3: Distances and angles
Stage 3. First analysis of results
Analysis of first results and selection of students.
Stage 4. Use of concrete materials
Sequence 4: Comparison of angle measurement with concrete materials
Sequence 5: Cardboard clock with concrete materials
Stage 5. Use of Cabri-II
Sequence 6: Angle measurement with Cabri-II
Stage 6. Use of Logo and Cabri-II
Problem-solving using Logo and Cabri-II

In sum, they found this pine tree task quite arduous. Some pairs were unable to solve that problem within Logo, and others found it quite useful, because they could keep on drawing by trial and error. That is, they could "correct" the turtle's moves and back it up — although mistakes are not fully erased, leaving traces in the final product. They observed how the drawing came out, and "corrected" it when necessary.

Stage 3 included an evaluation of the results from the previous stages. The conclusion was to select only eight students to pursue the remaining stages. The main criterion of selection was for complete reports (worksheets filled in or solved) on the sequences performed.

Phase 4 planned for children to perform activities with concrete materials. As suggested in Magina and Hoyles (1991), children were provided a cardboard clock with a moveable hand, protractor, pencil, and eraser. The purpose for this phase with the clock activity was to discover how the students related Logo's right and left turns with clockwise and counterclockwise movements, respectively.

As a result of this activity, we obtained that only one fourth of the students adequately recognized right turns as related to clockwise movement, and still were unable to measure angles that exceeded 180º.

In Stage 5, Cabri-II was then introduced as a technological tool, because using the toolbox to measure and calculate (see Figure 2) was likely to allow the students to overcome the difficulty demonstrated in Stage 4.

The Cabri-II activities may be briefly characterized as simulating the two directions of clock movement by placing the clockwise in various positions and using the Cabri-II *Angle* command to measure the angle sizes determined by such configurations, especially those which measure was greater than 180°. The following were the main instructions for the activity set out within Cabri-II.

Measuring Angles within Cabri-II

Part I

1. Construct two lines through the same starting point. Verify that it is possible to make them turn as the clock hands do.
   - Describe how the hands move:

2. It would seem like we can use this computer program to draw our own clock with hands. Let's try. We can start by following these steps.
   a) Construct two straight lines through the same starting point O.
   b) Construct a circumference whatever size you like with "O" as the center.
   c) Select *Vector* from the third toolbox Construct two vectors over your two straight lines, one smaller than the other. These vectors will be your clock hands.
   d) Observe how it is possible to move the hands "to the right" and also "to the left." Quickly sketch two clocks: one that shows you moved the clock hands to the right and the other that you moved the clock hands to the left.

   - To the right:…

   - To the left:…

   e) After putting our little hand vertical, we will "hide" the corresponding straight line to lock our little hand in place. We do this by choosing *Hide/Show* in the last toolbox, and clicking on the geometric object we want to hide.
   f) Observe how it is possible to move the big hand to any position by holding the straight line we used to obtain it. Put the big hand in six distinct positions. Quickly make six sketches that show the six distinct positions you put the hands in:
3. Now let's use the software to measure the angles determined by the different positions we put the big hand in. To do this we will follow these steps.
   Click on Cabri's *Help*, and also click on the toolbox's second-to-last button, the one for calculations and measurement. Choose *Angle*. Create a point on the fixed hand, another in the center of the circumference, and another on the hand that is free to move. You will see how the measurement of the angle determined by the position of the fixed hand and the big hand appears. Move your big hand and observe how the angle measurement changes. Move the big hand until you obtain an angle of 0°, 90°, and 180°.

Part II

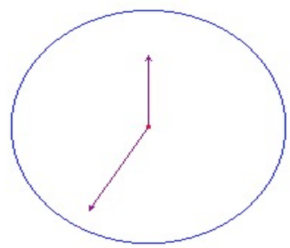
1. Make three quick sketches that show the position of the hands determinig these measurements.

   0º: 90º: 180º:

2. Using Cabri-II to measure the angles greater than 180º.

   a) Put the big hand so that the angle with the little hand is 180º. Make the second hand move further to the right and calculate the angle measurement in this position.

   b) Make a quick sketch that shows the angle configuration you used, and also finally you must measure that angle. The angle you should finally measure should be one angle like the following:



   c) What is the sketch of your clock hand positions?

   d) What is the measure of the angles greater than 180º you defined? What were the measurements of the angles you could use to calculate it, and finally (after making your calculations) report the following:

   Measurements of the auxiliary angles:____________________

   Measurement of the angles greater than 180º:_____________________

It is worth noting that Cabri's *Angle* command only works with angles less than or equal to 180_, requiring reflection and planning on how to calculate those of greater dimension. These may be the reasons why a general angle measuring competence is finally achieved.

Finally, in the last problem-solving stage, the students worked on the computer using Logo. One of the most important results from this stage may be that all of the students succeeded in solving the original boat problem, where the correct answer is 120º to the right.[1] The reader should keep in mind that the boat problem is notoriously difficult for children; in fact, pretest results showed that not a single one of the 92 students solved it at the outset of the study.

[1] It was quite interesting that, when asked how they arrived at their answers, the students explained they had followed the instructions correctly — even though it was they who had given the instructions to the turtle.

Some Preliminary Conclusions

Preliminary results obtained in this exploration indicate that, in effect, both technological tools fulfilled specific functions, distinct yet complementary, for learning the notions under study. Both dynamic geometry computer programs, Logo and Cabri-II, allowed the students to quickly overcome the difficulties initially detected.

Acknowledgements

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EFFECT OF A COMPUTER SIMULATION AND DYNAMIC STATISTICS ENVIRONMENT ON THE SAMPLING DISTRIBUTIONS' MEANINGS

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This research paper reports the results obtained from the effect that a computer simulation and dynamic statistics environment have on the meanings that undergraduate students attribute to sampling distributions. In different stages of simulation process were identified that the dynamic interplay of simultaneous representations and immediate visualization provided by software Fathom (Finzer, 2002) were crucial for some students to understand important properties of sampling distributions and an adequate interpretation of probabilities. The foremost observed difficulties in the simulation process were linked to the use of symbolic representations of software and mainly to the formulation of the population model.

Purposes and Background

Sampling distributions are fundamental to comprehend statistical inference. However, results of some research works have showed that sampling distributions constitute a difficult concept for students, because it demands integration and application of several statistical ideas such as sample, population, distribution, variability, and sampling (Chance, delMas & Garfield, 2004), and for formal approach based on random variables and distributions of probability that traditionally have prevailed on its teaching (Lipson, 2002).

As from incorporation of computer technology in education, it has looked to replace or complement such approach through an empirical focusing supported by the relative frequency approach to probability. Research works made by Lipson (2002), Saldanha and Thompson (2003) and Meletiou-Mavrotheris (2004), suggest that a dynamic and of multiple representations' environment, even not free from difficulties, can help students to develop an adequate comprehension of the concept of sampling distributions and to apply it in their reasoning about inference. However, research needs to be done in order to understand how students build their meanings about that concept in computer environments with such features.

Our research is inscribed in this framework. Specifically we have posed the question: What is the effect of a computer simulation and dynamics statistics environment on the meanings that undergraduate students attribute to sampling distributions?

In order to carry out the study, we selected Fathom (Finzer, 2002) as a representative software of dynamics statistics environment. This software allows to visualize all the process of building sampling distributions and to link the concepts involved through multiple simultaneous representations (graphs, tables, and formula). Thus, a change in data or parameters can be visualized immediately in the rest of representations.

Theoretical Framework

In this study, we use the meaning of a mathematical object as stated by Godino and Batanero (1994; 1998). In this model, the meaning is conceived as a practical system used by people to solve a field of problems where the concept arises. This system can be observed by means of

Lloyd, G. M., Wilson, M., Wilkins, J. L. M., & Behm, S. L. (Eds.). (2005).
Proceedings of the 27th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.

various elements involved in mathematical activity developed by people in solving problems. There are five elements or meaning components:

1. *Problem-situations (phenomenological elements).* Field of problems or situations where the concept under study arises.
2. *Language (representational elements).* Any verbal or written representation used to represent or refer to concepts and features involved in a problem.
3. *Actions (procedural elements).* Procedures or strategies used in solving problems.
4. *Concepts and properties (conceptual elements).* Concepts, properties and their relationships with another concepts involved when solving a problem.
5. *Argumentations (validative elements).* Argumentations or validities used to convince others of the validity of our solutions to the problems or the truth of the properties related to the concepts.

From this point of view, the comprehension of a mathematical object consists of a continuous and progressive process where students acquire and connect different elements involved in the meaning of concept. The model considers three levels to analyze the meanings: epistemological, cognitive and instructional. In this report, however, we only use the cognitive level to identify the meaning elements used by students in solving problems (personal meaning).

Methodology

The study was conformed by 8 activities that were developed along 16 sessions of 1.5 hours each in a computer laboratory with 11 computer science students (19-21 year olds) who were studying a statistics course at the National Polytechnic Institute (Mexico). The activities involved different population types and they were encouraged to vary the sample size in order to explore certain concepts such as sample variability, central limit theorem and to calculate probabilities of sample values. In each activity, the students were asked to establish the link with theoretical results obtained by means of paper and pencil. Data gathering was done by means of two questionnaires applied before and after the activities, also data sheets for each activity, diskettes with results of the activities; videotapes and final interviews with some of the participants.

Results and Discussion

Results of this research report describe the most representative meaning elements that students developed about sampling distributions. Furthermore, we present the progress shown by two students (Monica and Omar) about their meanings.

Phenomenological Elements

All posed situations belong to a field of sampling distributions in which the aim is to calculate probabilities of some sample values by means of a deductive process in which the population distribution and its parameters supposedly are known. This is a representative example of the situations treated:

In a bearing factory there is a misadjusted machine that produces 30% of defective bearings.

a) If a sample size 80 is taken. How many defective bearings do you expect in the sample?
b) What is the probability that in a sample size 80 would be 30 or more defective bearings?
c) What is the probability that in a sample size 80 would be 20 or less defective bearings?

Representative Elements

Along the simulation process, students established relationships among population, sample and sampling distribution through various graphical representations such as histograms, data tables, tables with descriptive measures, expresions to calculate sample statistics and proportions of sample results and they carried out adjustments in a theoretical distribution with an empirical distribution.

As an example, we see in Fig. 1 the use of different representations of population and samples made by Omar. He linked the different representations among them in order to explore the concept of sample variability and to compare the sample and population values.

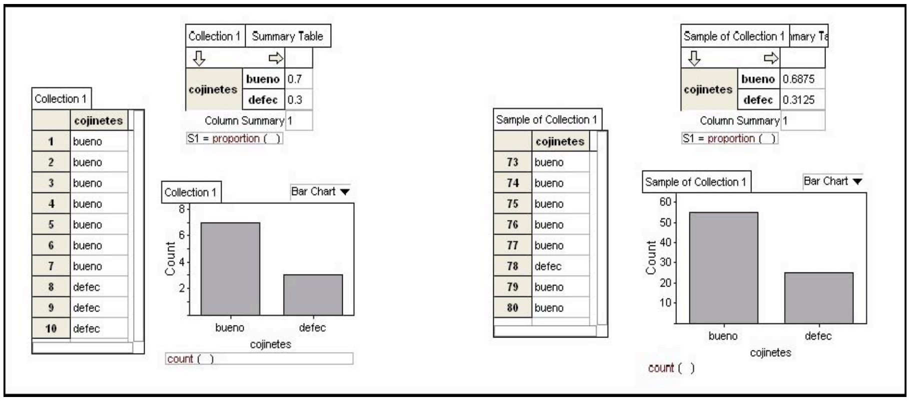

Fig. 1: Multi-representational scheme about sampling

The relationship between population and samples could be visualized dynamically and immediately while taking sample by sample because the multiple representations feature of Fathom. This influenced Omar to develop an adequate understanding of sample variability and to correct some misunderstandings shown in the diagnostic questionnaire. Similarly, as we'll see, in other stages of simulation process, Fathom representations were important for students to construct correct meanings of different involved concepts.

Procedural Elements

In the process of simulation of a sampling distribution with Fathom, we have distinguished three stages, as regards to the concepts involved and the activities needed to construct them:

1. To formulate the population model.
2. To construct the sampling distribution.
   a) To select a sample of a given size from the population and to define the statistics to calculate.
   b) To repeat many times the process of sample selection and to form a collection of calculated statistics in order to gather a sampling distribution.
3. To cut the sampling distribution to determine what proportion (empiric probability) of statistics is greater than a given value or is located between two given values.

Doubtlessly the hardest stage for students was the formulation of the population model because they needed to identify outstanding features of the population, such as the type of variable and the values of parameters. Usually their actions and strategies about formulation weren't well guided in most of the activities. As an example, one activity was about a binomial population in which there were a 30% of defective bearings. Most of the students considered both outcomes to have the same probability and only 2 students could establish the model without help.

The next stage of the process consisted in sampling the population and defining the statistics to calculate. This was particularly difficult in the initial activities because students weren't familiarized with the software. Later they repeated the sampling process, grouped the statistics and constructed the sampling distribution. In the third stage, they turned to generate a formula to calculate probabilities of sample values. The main difficulty was the inadequate operation of connectives "and" and "or" in defining the intervals.

The procedural elements used by students were qualitatively different to those used by theoretical approach by means of paper and pencil. For instance, some procedures that are central in theoretical approach, such as the standardization of sampling distribution and the use of probability tables, were not required in the simulation environment. On the other hand, some actions in which students usually aren't involved in a paper and pencil environment, such as the formulation of population and the construction of sample distribution —because they usually are given to students through formula—, were central in simulation environment and students played an important role in their development.

Conceptual Elements

Different concepts and properties were brought into play in each stage, such as sample variability, the effect of sample size in the behavior of sampling distributions and the calculation of probabilities of sample results. In the diagnostic examination, most of students showed incorrect ideas about these concepts. For instance, Monica argued that in 5 repetitions of 10 tosses should appear exactly 5 heads in each repetition. After doing the activities she modified her sense about the meaning of variability, accepting the fact that even though the probability is 1/2 the sample not necessarily had to have the same proportion. This was observed in the post-test. Furthermore, she extended the meaning of variability connecting it with the sample size. Let's see the answers she responded to the researcher about the variability and sample representativeness in the context of a uniform discrete population with a mean equal to 3.5.

R: What is happening with the sample results?

M: The results vary. Obviously mean and standard deviation vary.

R: What would you do to reduce variation?

M: To increase the sample size.

R: Well, now increase to 50.

M: Less variation of the mean is observed and standard deviation stays almost the same. Possibly including all decimals the results were 3.5 and 1.70 [the population values].

R: What conclusion do you get from that?

M: That the greater the sample, the nearer the sample values are to the population values.

As for Omar, who also showed difficulties in the comprehension of sample variability in the diagnostic examination, in an interview at the end of the activities and in the context of a binomial population with p=0.30, he responded:

R: Take a sample size 10, repeat the process and watch what happens with the proportion of defective articles.

O: It does not move away from 3.

R: How would you expect that proportion to vary?

O: Neither very greater that 3 nor very less than 3.

R: Give me certain limits between the results could be.

O: Almost always they will be between 2 and 4, more or less.

R: Now, watch how is the proportion of defective articles each time you take a sample and compare it with the proportion of defectives in the population. Is there any relationship between both of them?

O: In some way, because the proportion of the population is the real proportion and the sample is not exactly the same, because we are not taking all the population, it is just a sample. So, obviously that from the sample is going to be different to that from the population.

R: Yes, but, how different?

O: Not so much but, the greater the sample, the nearer to the real one it will be.

We can see in Omar's answers that he has constructed a correct idea about variability of proportion around the population parameter because he points out that most of the times the sample proportion will be between 2 and 4 defective articles. Furthermore, he considers that the greater the sample, the more representative from the population. This was, doubtlessly, a reflection of the situations in which he worked in the environment of simulation and dynamical statistics.

As for the properties of sampling distributions, students failed to recognize them in the diagnostic examination since only two of them recognized correctly the sample size and the variability in 5 distributions and only four students pointed out that the mean of sample distributions was equal to the mean of the population. Furthermore, in many cases the explanations in their answers weren't normative. In the post-test their answers were better and seven students understood the implications of sample size in the variability of sampling distributions and nine of them understood that a sample distribution is centered in the population parameter.

Let's see again Monica's case who, in the context of the same activity, arranged a graphical representation (see Fig. 2a) that was essential to understand the properties of sampling distributions as well as to interpret the effect of sample size on probabilities.

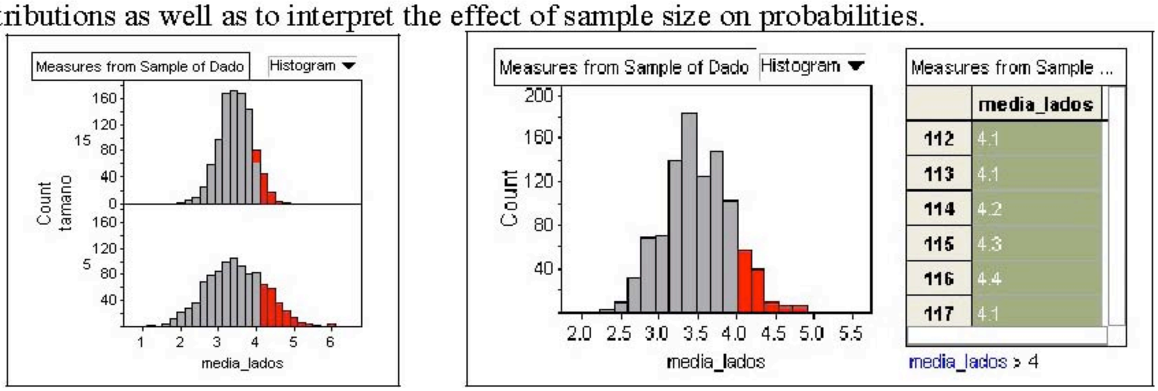

Fig. 2a: Sampling distributions for n=5 and n=10. Fig 2b: Sampling distribution for n=30

R: Do you think is there some relationship between the population and the sample distribution?

M: Well, here in the population are the values of the die and here are the means of the values of dice.

R: Then, isn't there any reason for them to be alike?

M: I think not. In the population the values are original and in the sampling distribution the values are already processed.

R: If there were a variable population like a swing instead of a uniform population like this, do you think that would there be repercussion on the form of the sample distribution?
M: No, usually the form of sampling distribution is normal.
R: Independently of the form of the population?
M: Yes.
R: What happens with the mean of the sampling distribution when compared to the mean of the population?
M: Usually the mean approaches to the population mean.
R: What happens with the standard deviation?
M: The standard deviation varies; it becomes smaller as sample size increases.
In calculating and interpreting probabilities, Monica linked a table with values of means with the histogram and used filter option to shade interesting results (see Fig. 2b).
R: In which of both distributions is more likely to appear a mean of 4 or more?
M: In that of size 5.
R: Why?
M: Because we have much more values toward the right side of 4.
R: If it were necessary to bet; to what values would you bet?
M: Between 3 and 4.
R: And if you want to increase your probability to win?
M: Then I would choose between 2 and 5.

As we see, in the dynamic environment created by means of technology, Monica could create fundamental notions concerning to sampling distributions theory, as sample variability, its dependence in relation with the sample size, the convergence of sampling distribution towards the normal distribution and the effect of sample size on the distributions variability and on probability of sample results. Some practices that support the meaning of those notions are associated to acts and consequences of the work in computer environment. These meanings are very far from practices consistent in searching a formula and execute algorithms, as is typical in a paper and pencil environment.

Validative Elements

One of the contributions of dynamics environments in statistics is to permit students to appropriate control mechanisms to correct their solutions. By instance, they use the formula inspector, which warns them when they make syntactical mistakes, the shading of areas to confirm that they are calculating the required proportion, the adjusting of empirical distribution to theoretical distribution (see Fig. 3). Furthermore, also they keep the comparison with theoretical results obtained by means of paper and pencil.

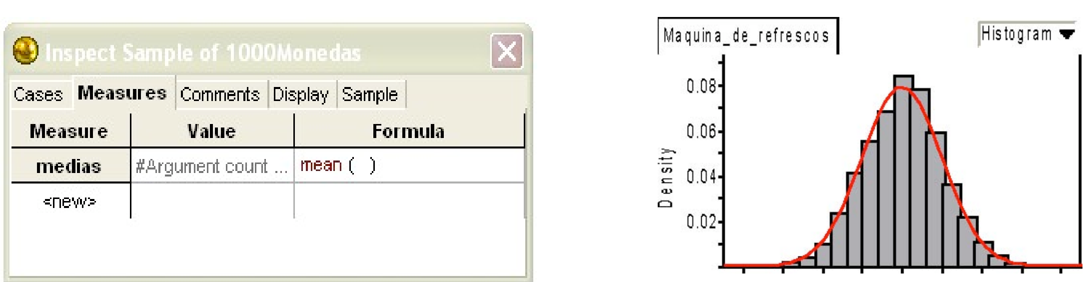

Fig. 3: Some resources to validate used in simulation process.

In a final interview, they were asked about the validity of the results obtained by means of simulation. All the students concluded that the obtained results were enough precise compared to those obtained by means of the application of formulae and probability tables.

Conclusions

The actions and strategies used by students to solve the problems were qualitatively different to those that use a theoretical approach as a consequence of computer environment. The different representations, the dynamical links among them and the immediate feedback offered by the software when some data or parameters are changed, had an important effect on the students' meanings. This was the case of Monica, who could construct fundamental notions concerning to sample distributions theory, such as sample variability, the convergence of sampling distribution towards a normal distribution and the influence of sample size on the variability and on the probability of sample results. Students got involved in the management of different concepts and their relationships, focusing the process as well as the final result, in contrast to paper and pencil focusing, which is usually the aim in probability. Finally, the resources provided by the software to validate constituted an important element for students to detect some mistakes while they developed the simulation process and permitted them to manage themselves independently of the teacher along some stages of the process. Students' meanings were evolving as the study developed, refining gradually their strategies and centering their attention on the conceptual aspects of problems. The post-test results inform of an increase in the answers, not only in quantitative terms but also in qualitative terms: they made statements more normative than in the diagnostic examination.

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POINTS OF INTEREST: TEACHERS' SELF-IDENTIFIED PROBLEMS OF PRACTICE

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In this study, four teachers were asked to identify classroom-teaching situations that they "wondered" about from their own lessons. Each teacher was using an inquiry-based, NSF funded curricula (Investigations in Data Number and Space or Connected Mathematics) to teach fractions. Results show that the teachers' predicaments centered on interactions in which they struggled to understand students' invented strategies. Teachers often perceived themselves as "stuck" rather then empowered because they did not have the "just-in-time" strategies to support children's novel strategies. Even so, these teachers strove to find ways to support student thinking and implement the instructional intentions of inquiry based mathematics practices rather than resorting to more didactic approaches.

Introduction

Researchers have made progress in the study of teaching as a problem-solving practice (Simon, 1999). Studies about teaching have shown that teachers have to engage in complex teaching problems that cannot be solved with ready-made solutions (Ball & Bass, 2000). Methods used to study these problems that pay attention to: (a) the type of teachers invited to participate, (b) the type of teaching problems studied, and (c) who chooses the problems that are chosen for analysis can lend support to the complexity of teaching, and at the same time provide a venue to make claims about teaching and the problems of practice that can narrow the gap between research and practice. While researchers are using methods that tap into the teacher's perspectives, the problems of practice central to the studies are usually selected by the researchers. This study results show what kinds of teaching problems are identified when teachers identify them from their own lessons.

Conceptual Framework

The NCTM standards define problem solving as "engaging in a task for which the solution method in not known in advance" (p. 52). We can apply this definition to the problems teachers solve. Problem solving can be considered the goal of all instruction (Ball & Bass 2000), and understanding the problems of teaching can be treated as we think about children's problem solving. Teachers construct their knowledge of teaching through problem solving just as children can construct solutions to problems in math (Carpenter 1988).

If teaching is problem solving, then what are the problems that teachers encounter? One way of conceptualizing the types of problems teachers encounter is to think of them in terms of "core activities" (Ball & Bass, 2000). Other projects implicitly identify situations dealing with children's thinking as learning places for teachers. When teachers make children's mathematical thinking central to their practices, they are able to translate what they know to novel situations and topics (Franke, Carpenter et al., 2001). However, listing particular problems of practice from the perspective of a researcher may miss or minimize the kinds of problems are most salient for teachers. Problem solving is the ability to recognize a problematic situation, so the teachers'

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cognitive sense of the problems not only frames how they are able to deal with them, but also helps us understand what problems are relevant.

The Study

This study places emphasis on the teacher's role in problem making by asking, *"What are the problems teachers identify as they teach children using inquiry approaches?"* And second, *"How do teachers reason about these problems?"*

The four teachers (3rd, 5th and 6th grades) who participated had a range of 2 to 5 years of experience with the curricula. This sample represented teachers who were willing to implement new practices, but who were not completely developed in using the approach. All the teachers were teaching a unit on fractions and had a high attendance rate for the after school professional development on fractions and children's thinking (at least 8 out of 11 or better). The teachers' lessons (about one hour each) were videotaped from 7 to 10 times. After each lesson they wrote their responses to the following question:

> "Did anything happen today during the lesson that caused you to stop and wonder what to do? Who was involved and what was it about? How did you deal with the situation? (Describe how ever many situations of this kind you experienced today.)"

These written responses were analyzed to look for themes, and three clips of these interactions were used in a post-unit interview with the teachers. The questions were designed to probe the teachers' thought processes during the interaction and to get a deeper description of the teachers' perceived problems of practice.

Results: What Did Teachers Wonder About?

The teacher-selected interactions were analyzed by their features using the written post-lesson reflections. Later the interactions from the videotaped lessons were studied and analyzed to describe the problems in more detail. Across all four teachers there were similar, situative features among the interactions. There were 44 total problems; seven involved the teacher's difficulty using the task as prescribed in the curriculum, and four involved issues on student discipline. The rest (33) involved student strategies at some level.

First, each interaction is categorized in terms of the *type of situation* which describes the activities that were prominent in during interaction. Most of the situations concerned children's work. Within *children's work*, issues of validity, and children's use of informal mathematics framed teachers' problems. A surprising number of situations had to do with children's use of area representations of fractions. Also, teachers sometimes wondered about the quality of their *responses* to children's work. The teacher sometimes directly referred to how they were able to respond as a problem (e.g., "I did not want to tell too much"). Teachers responded to students' strategies with supporting actions, reflecting a range of responsiveness to student thinking. Analyzing the situations associated with the response features added insight to why these situations may have been problematic for the teachers. In some cases, they wondered about their ability to respond, but in others the level responsiveness seemed to be a result of their predicament.

Types of Situations

Table 2.1 summarizes the types of situations, and Table 2.2 describes the features of student work:

Table 2.1 Types of Situations (n=44)

Student's attention	The teacher expresses difficulty with student's behavior	4
Implementing tasks	The teacher expresses difficulty with the task as suggested by the text.	7
Student work	The teacher expresses difficulty or her comments are associated with interaction with a student and the student's strategy	33

Table 2.2 Features of Student Work (n=33)

Valid teacher-introduced (alternative) approaches	Strategies introduced by the teacher, usually as a result of suggested activities in the curricula. None of these strategies were algorithms.	8
Valid student-created approaches	The strategy is introduced by the child, and leads to or could have lead to a correct solution to the math problem	9
Invalid approaches	These approaches could not have or do not lead to valid solutions.	16

Managing Students' Attention and Implementing Tasks

Teachers wondered what to do when their students did not pay attention. Four of the situations concerned management of students. Three of these situations concerned one particular child's inability to pay attention during math lessons. The teacher was particularly aware of this child, because she was having difficulty in math but often refused to participate in the lesson. In the fourth situation, the class was distracted and restless and so had they problems paying attention during whole-group discussion.

Seven of the situations concerned difficulties with implementing the task. Sometimes the teachers were frustrated when the materials were lost or hard to keep track of, and at other times they questioned productivity of the tasks suggested by the curriculum. In the Investigations curriculum lessons are often linked to particular student-created manipulatives that are used throughout the week. Teachers believed that having these manipulatives was crucial to being able to do the activities from day to day. If they felt the manipulatives were too cumbersome, they worried about the effectiveness of the lesson.

Student Work

The remaining 33 situations concerned students' work. In these cases, students used strategies to solve problems that were (a) valid, teacher-introduced alternative approaches: (b) valid, student-created approaches: and (c) invalid approaches.

Valid, alternative, teacher-introduced approaches. In an effort to discourage dependence on standard algorithms, the *Investigations* series and the *Connected Mathematics* curricula suggest alternative methods of solving problems presented in the lessons. These alternative methods are designed so children can build conceptually based understanding of the content. For instance, instead of teaching children to convert fractions to percents using division (i.e. divide denominator into numerator), fifth graders are encouraged to use what they know about common fractions and to use those relationships to find other fractions. In addition, manipulatives like the 100 grid and percent strips are used to make area representations of percents.

Both of the curricula used by the teachers in this study introduced alternative strategies. In these situations (7 out of 8) the students had at least a partial solution to the math problem when the teacher began to interact with them. Teachers often were helping children use the strategies they introduced, and often the teachers were faced with new facets of the strategy they had not considered. In contrast to student-created approaches, alternative teacher-introduced approaches were used by children as a result of instruction, so the teacher had a self-imposed expectation to understand the approach.

For example, Ms. Edwards (5th grade) encouraged an alternative approach suggested in the teacher's guide for finding percent equivalents for common fractions using a 100 grid. This entailed counting out the number of squares on a 100 grid equal to the denominator and then shading the amount indicated in the numerator. This is repeated until the squares are accounted for. Figure 2.3 below shows what the solution looked like when Mario and Diego used this method to solve the problem 3/8 = ?/100.

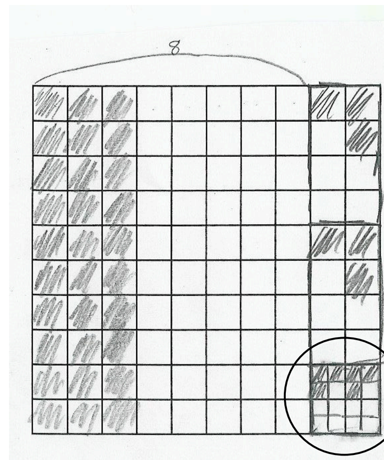

Figure 2.3. Mario's solution.

The children had already begun shading the fraction when Ms. Edwards checked in on their progress. During the interaction she watched as the students shaded in 3 out of 8 tens. Then they chose 8 squares, shaded in 3 and another 8 squares, and shaded in 3. Finally they had just 4 squares (100ths) left. They decided to divide each square into 4 parts. When the two students finished shading in all the parts representing 3 out of 8 as in Figure 2.3, the students along with Ms. Edwards struggled to name the last 4 partitioned, shaded squares (see the circled portion in Figure 2.3) as a fraction of 100. I believe the approach became problematic because the denominator was not a factor of 100, and to shade in 3 out of 8 in the last iteration, partial squares should be shaded, not 3 out of 8 whole squares.

Together, they decided that the shaded portion was 6/8 squares instead of 1 and 1/2 squares. The logic here is that they had partitioned 2 squares into 4ths to make 8 partitions, and then shaded 3 of 8 sections, twice. As a result, they incorrectly calculated the percent equivalent to 3/8 by adding 30 + 6 + 6/8 to get 36 6/8 (rather than 37 + 1/2). At that point Ms. Edwards questioned the reasonableness of the 6/8 part of the answer, asking, "Is this right? This can't be right." After the lesson she wrote what she wondered about: "Mario and Diego dividing 3/8—the last four squares were divided into 4ths each. We redrew and had a hard time thinking of the parts as not wholes". Her attention for this particular case was drawn to executing the strategy and finishing the solution. Rather than a focus on children's thinking, her comment indicated that she was focused on learning how to use the strategy for herself.

Valid, student-created approaches. Another method that decreases the necessity of teaching algorithms is to allow children to invent their own strategies. Nine predicaments involved valid, student-created approaches. For most of these situations the student had solved the problem and was explaining the strategy to the class or to the teacher. Teachers recognized that the students had correct answers but wondered about the child's thinking and puzzled about how a child could devise such strategies. Student-created approaches are a result of the child's construction, so the teacher's primary resource for understanding it is the child's thinking. Student-created approaches are strategies not directly introduced by the teacher, so the teacher is not expected to understand it from an execution standpoint. Instead, teachers can focus on how the student understood the strategy.

For example, Ricky had explained that two same-sized but different-shaped amounts were the same, saying, "But half of this one is half of this one," and pointing to sections shaded in his picture. Ms. Marks (3rd grade) wrote,

> A few times I wasn't understanding what a student [Ricky] was trying to describe and that made it frustrating because I found myself losing sight of the original question and what I was trying to get all the students see or understand. Ricky was explaining his thoughts and I got confused.

Later in a private conversation, she told me that she did not know that a third grader would be able to solve a problem without drawing and just could use his own logic like he did. In fact, she confessed that she had not thought of comparing the two areas as halves of halves, but now understood what Ricky might have been thinking.

Invalid approaches. Sixteen situations involved a student using a strategy that was invalid (see Table 2.2), and thus the teacher perceived the student as struggling. In some of these cases children were inventing a strategy that could not have led to a productive answer. In other cases, the children were applying a previously taught strategy incorrectly; sometimes these were standard algorithms, and at other times they were alternative approaches.

For example, Ms. McDonald approached Reggie, a sixth grader, who had solved a problem involving ordering fractions. The problem the children were solving was to put four basketball players' shooting ratios in order from worst shooter to best shooter: Naomi, 19 out of 25; Bobbie, 8 out of 10; Kate, 36 out of 50; and Olympia, 16 out of 20. Who has a better chance of making the next shot? The students could have used percentages to compare the four players' shooting accuracy, but Reggie compared the four ratios using fraction bars. This had not been suggested by the curriculum or by the teacher. However, his drawing lead to an inaccurate conclusion. Ms. McDonald recognized that the student's drawing would lead to an incorrect answer but had little success in leading him out of it. She wrote, "Reggie was trying to draw a comparison of 4 fractions and was not succeeding b/c they were not divided into equal parts." She said that the comparison did not work because the divisions of each bar were not even. After working with him for a while she understood that Reggie had made each ratio as a fraction of shots out of total attempts, but was unable to convince him why his comparison method was flawed. She finally told Reggie to go over to another student for help.

How Did the Situations Impact the Way Teachers Responded to Students?

Whether the student was using a student-created or alternative, teacher-introduced strategy or an algorithm, when the strategy was invalid, teachers expressed their view of the situation as one in which they could not understand where the confusion on the part of the child arose. Often, in the above category of invalid approaches, they resorted to telling the student more directly what to do or to abandon the strategy altogether. This was not true of the valid approaches.

Teacher's responses to children arose as an important feature of the situations that involved student work since it seemed that certain types of situations were associated with particular forms of support. Given that these teachers expressed beliefs that children could devise their own strategies and that understanding is as important as or more important than learning procedures, it is not unexpected that they would wonder about their ability to support children's thinking. For example, when the predicaments included a children's area representation as a part of their strategy, the teachers were able to be more responsive to children's thinking than for other types of mathematical representations. In fact, of the 18 interactions that featured a child's representation, in only three of these, teachers resorted to more directive responses.

Predicaments Not Dilemmas: Teachers' Reasoning

I was struck by the similar way teachers felt about these situations. Teachers expressed feelings of discomfort and confusion. They rarely talked or wrote about choosing between one possible solution or another; instead, their feelings indicated that they were in the middle of a predicament. For example, teachers wrote that they felt kind of "stuck," "confused," and "surprised" with their situation. The teachers' descriptions reflected feelings that were distinctly uncomfortable or unpleasant. They described the problem as feeling unsure of what to do next and as having "a hard time thinking."

In Heaton's (2000) self study during her first year of implementing a new curriculum, she describes her feelings of being uncertain or stuck, the trial-and-error based decision making, and the challenge of finding out how to support children's ideas. These were also the ways teachers in this study described their problematic situations. If the teachers had conceptualized their situations as dilemmas, they would have talked about competing choices. Instead, the nature of the situations and the way the teachers referred to them as not knowing what to do situate these problems as predicaments. It is important to make the distinction between predicament and dilemma; if it is a dilemma, one assumes the teacher has enough knowledge about teaching mathematics to consider viable options, but if the situation is a predicament, the teacher may have only a partial understanding of the situation and has to take some sort of action and hope it works.

Discussion

Two points can be made about these data that point to positive change. Given that the nature of schooling in the past 30 years, despite the push for compliance to the NCTM Standards (Fullan and Stiegelbauer 1991), has remained relatively and deeply unchanged and there exists an extreme pressure of accountability testing in most schools. However, these teachers persisted in looking for ways to teach for understanding through children's work. Secondly, this sample of teachers showed a heightened awareness of the problems of practice noted in teacher knowledge literature as places where teachers learn key ideas about inquiry practices of teaching mathematics, and promote student learning (Franke, et al,2001; Petersen et al, 1989).

As Burbles and Hansen (1997) claim, predicaments are inevitable in teaching. Even though they felt uncomfortable and were unsure, teachers appreciated their predicaments as learning opportunities. During the postunit interview, I asked Ms. Marks to compare her previous year's implementation of the Investigations curriculum to the present year's instruction. Just before this exchange, we had been discussing her predicament in which she was surprised to discover a students' particular way of thinking about a fraction problem.

Interviewer: Oh, so you didn't do this lesson last year? This specific one?

Ms Marks: Oh I did, but I didn't, it seemed like last year I only had 13 kids, and so we would do everything pretty much whole group on the carpet, and they would give ideas, and so the kid, if there was a child who didn't see that—

Interviewer: Ohh.

Ms Marks: I would not have picked up on it. As much as I would have this year. Because it is more in small group and with you guys [the University of Texas math group], I have been really—

Interviewer: Going around—

Ms. Marks: —Looking at them more closely.

Interviewer: So you are thinking that last year that someone might have done the same thing, but since you were on the carpet just throwing out ideas, you think you wouldn't have seen that?

Ms. Marks: Um hum. And then I would have just changed it [the child's ideas] already instead of [letting them] exploring it themselves.

Frameworks constructed to understand teaching mathematics through inquiry practices could begin with descriptions of the problems of practice as teachers see them. Research designed to examine the reality of teaching through inquiry approaches can illuminate accounts of developing teachers' problems of practice. Insights about these problems from the teachers' perspectives helps explain differences in teachers' learning experiences, impacts of curricular changes, and understanding of how teachers interpret, respond to and add to their knowledge base for teaching.

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TOWARDS CONCEPTUAL UNDERSTANDING IN THE PRE-SERVICE CLASSROOM: A STUDY OF EVOLVING KNOWLEDGE AND VALUES

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Mathematics education reform requires deeper and broader understanding of mathematics on the part of teachers than ever before. Such understanding must be conceptual in nature, yet many teachers have learned mathematics themselves in a purely procedural way. Challenges exist in overcoming beliefs grounded in previous experiences to enhance the valuing of conceptual learning by teachers. This study examines the procedural and conceptual knowledge and values of pre-service grade four to ten teachers during their regular mathematics methods course and examines how these characteristics change during their education program. This study is relevant as part of the broader effort to study the kinds of learning needed by teachers to teach well.

Introduction

Vast changes have taken place in many mathematics curricula in recent years. For example in Ontario, Canada the elementary mathematics curriculum includes the content strands of the Standards (NCTM, 2000), as well as embedding the process strands in more general statements (MOE, 1997). However, pre-service teachers enrolled in teacher education programs at this point in Ontario have not studied this curriculum themselves in school as elementary or secondary students, nor have the practicing teachers with whom they will be working in their school placements. Preparing teachers with such traditional experiences to teach in a reform based way thus becomes a challenge.

Theoretical Framework

The behavior of teachers in classrooms may be inhibited by a lack of knowledge and skill (Ross et al, 2003), yet taking a larger number of undergraduate mathematics courses does not seem to improve the general conception of teachers about appropriate mathematics teaching (Foss, 2000; Fennema and Franke, 1992). Teachers' content knowledge of mathematics is crucial for improving the quality of instruction in classrooms (Ambrose, 2004; Hill and Ball, 2004; Stipek at al, 2001; Ma, 1999). Hill and Ball (2004) also cite a lack of measures of teachers' content knowledge as a difficulty in determining what features of professional development contribute to teacher learning.

Details about the types of understandings required for "profound understanding of fundamental mathematics" (Ma, 1999) are as yet unclear. Such understanding may include both procedural and conceptual aspects. Competing theories have been proposed regarding the development of conceptual and procedural knowledge, which have often been pitted against one another (Rittle-Johnson and Koedinger, 2002). In fact it is possible that "teaching quality might not relate so much to performance on standard tests … as it does to whether teachers' knowledge is procedural or conceptual" (Hill and Ball, 2004, p. 332). Hiebert (1999) states that instructional programs that emphasize conceptual development, with the goal of developing students' understanding, can facilitate significant mathematics learning without sacrificing skill

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proficiency. It is timely and highly relevant to examine the role of teachers' subject matter and pedagogical content conceptions in helping them move from the traditional classroom to the reform classroom (Lloyd and Wilson, 1998). It is the premise of the current study that appropriate mathematical knowledge should include both conceptual and procedural understanding, and that such understandings should be complementary not adversarial.

However, even appropriate knowledge may not be sufficient for teachers to choose to teach differently from the ways in which they learned mathematics as children. Foss (2000) feels that majoring in mathematics does not necessarily broaden a preservice teacher's view of mathematics or develop the preservice teacher's conception of mathematics teaching and learning. Influencing teachers' beliefs and values may also be "essential to changing teachers' classroom practices" (Stipek et al, 2001, p. 213). Ambrose (2004) feels that while prospective elementary teachers may underestimate the importance of subject-matter knowledge in teaching, their beliefs about mathematics and teaching may also contribute to the problem. She feels that belief change may be incremental, and that teacher educators need to consider creating several "intense experiences" (p. 117) which influence students' existing belief systems throughout the teacher preparation program, in order to leave "vivid impressions" (p. 94).

Given the likelihood of teachers holding beliefs based on prior experiences, as well as the difficulty of building a conceptual understanding *after* a procedural approach has been encouraged (Hiebert, 1999), pre-service teachers may be faced with a great challenge in improving their conceptual understanding, especially if their own experiences of learning mathematics in school consisted mostly of memorizing procedures. Such past experiences likely influenced both the types of knowledge teachers initially have as well as their perceptions and beliefs about mathematics learning.

Many factors may underlie the issue of lack of conceptual understanding. As one example, Ball (1990) reports that prospective teachers' specific knowledge of division of fractions was founded mostly on memorization rather than conceptual understanding. Furthermore, such understandings are cumulative; it is unlikely that a conceptual model of division of fractions can be achieved if the division operation itself is not conceptually understood.

Hill and Ball (2004, p. 345) feel that "teachers *can* learn mathematics for elementary school teaching in the context of a single professional development program" and that an important feature of successful programs is to foreground mathematical content. They state the need to probe more carefully into effective content of professional development.

Ross at all (2003) show that beliefs found in teachers' self-report surveys do relate to subsequent student achievement, yet cite aspects of teacher practice that may be hard to access on self-reports, such as "whether teacher-student discourse probes deep conceptual understanding" (p.345). Both beliefs and knowledge do appear to play a role in effective teaching (Stipek at al, 2001, Cooney et al 1998; Foss, 2000), and thus the premise of the current study is that content knowledge, both procedural and conceptual, as well as beliefs, particularly beliefs about the importance of different types of mathematical understandings which will be referred to in the current context as *values*, may all be important factors to consider.

Methodology

The subjects in the study were 114 teacher education students in five different classes taking a one-year mathematics methods course for teachers of grades four to ten. Pre-test and post-test data related to both values and mathematical understanding were examined.

Participating students were asked initially to respond to two written surveys. They were asked to answer questions using a four point Likert scale pertaining to their beliefs about the need for procedural as well as conceptual understanding by both students and teachers. Such beliefs will henceforth be referred to as procedural and conceptual *values*. As well, they were asked to answer a second survey containing several mathematics questions similar to those used by Ma (1999) and to provide a "diagram or model with explanation or an example or story problem" to illustrate the meaning of the operation involved. Correctness of the method was scored as procedural knowledge, while provision of a diagram, explanation or model was scored as conceptual knowledge. These surveys were re-administered at the end of the course. Teachers were shown their initial survey scores midway through the course, and asked to write a brief reflection as to how well they felt the survey had characterized them initially. Scoring was done by three researchers working together until consistency was reached.

Treatment during the course consisted of a strong emphasis on the mathematics content, particularly at a conceptual level. For example, in studying fractions, participants were given fraction bars and challenged to work in groups constructing meaning for the various operations, which were then shared in detail as a larger group. No standard algorithms were allowed unless fully justified. Assessment also focused on conceptual understanding.

The scoring of the surveys yielded scores in four areas: procedural knowledge, conceptual knowledge, procedural values and conceptual values. Rather than framing conceptual and procedural knowledge and values as competing ideas, the premise is that *both* of these understandings are important for teachers. Hence teachers were presented with the four scores on a two dimensional graph, similar to a Cartesian plane but with four positive axis. Each of the four variables listed above was shown on one of the axiis. The participants joined the four points to create a quadrilateral. The location and size of the quadrilateral was meant to give teachers an overall visual sense of their current *Profile*. Roughly speaking, the width of the shape gave participants a sense of the breadth of their knowledge, the height a sense of the value they placed on different mathematical understandings, and the location a sense of their overall approach to mathematics teaching and learning. The mean scores on the pretest and posttest surveys are estimated graphically in Figure 1. The individual *Profiles* were shared with the participants.

One of the purposes of the study was to examine whether such a visual tool was useful to participants as a way to conceive their personal data. The graph was meant to convey to teachers that there is more than one aspect to teacher preparation, and that not everyone has the same skills, values and areas for growth. It seems likely based on the research described previously that aspects of each of the four variables shown are necessary for teachers to teach well. This graphical aid does not "throw out" the value of procedural proficiency in favor of a conceptual understanding as some critics of math reform might have argued has been done, nor does it pit any one variable against another. Rather it attempts to illustrate an evolving teacher as a combination of traits which may interact with one another in complex ways, and allows the teacher to see visually some possible components of their personal values and skills. The idea of the *Profile* is to encourage teachers to reflect on how they would personally like to develop.

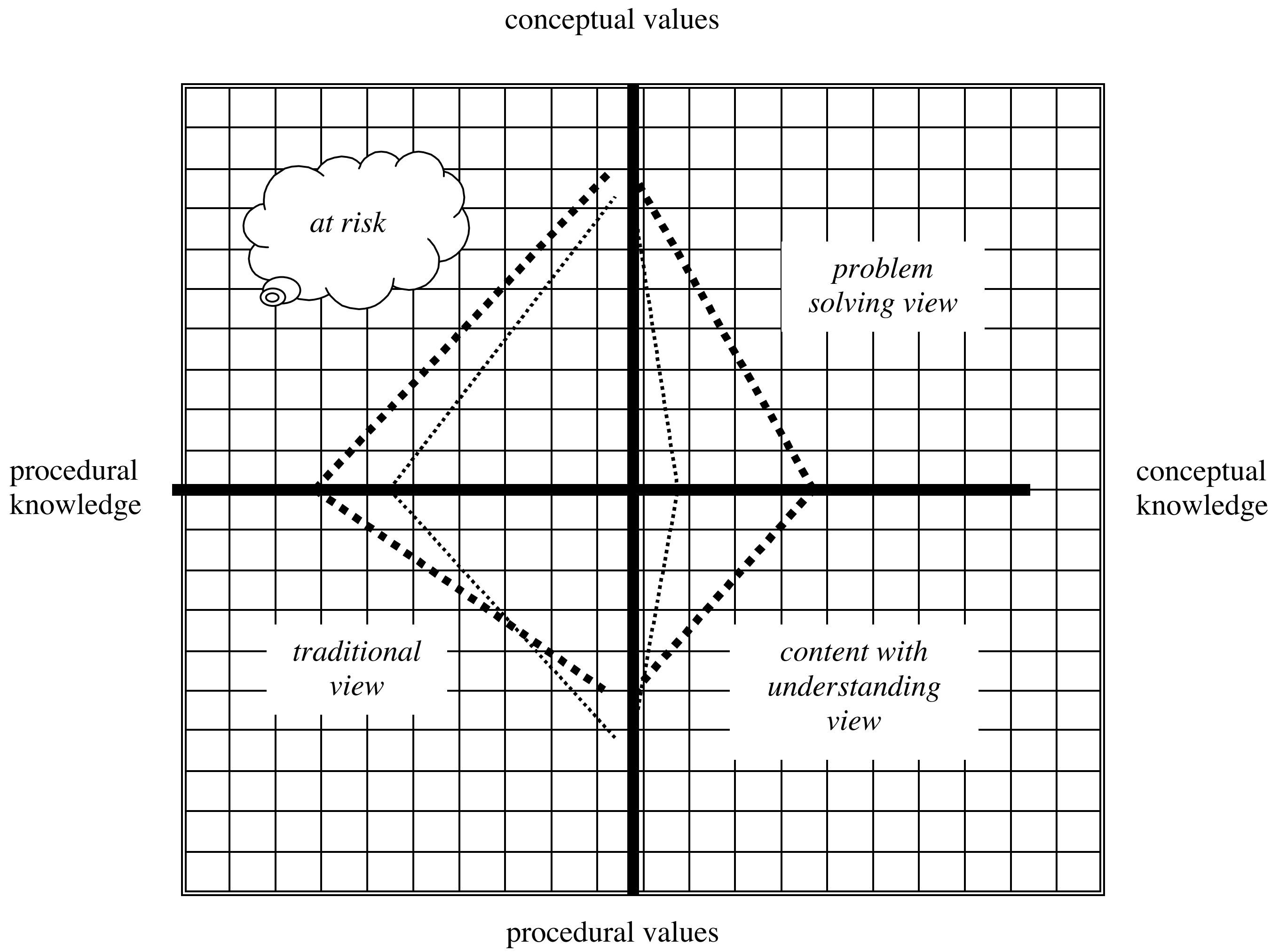

Figure 1. Profile graph illustrating typical participant scores using graphical estimations of the mean score of each variable, with posttest scores shown darker. (Maximum score is 10 on each variable, and one square represents one unit in each direction).

Results and Discussion

Pretest results show that initial teacher procedural knowledge appears to be moderate, while conceptual knowledge was extremely low (mean score of 1.1 out of 10). However, values related to both procedural and conceptual understanding were generally moderate to high. In their written feedback many students explained that their valuing of conceptual understanding had changed during the first few weeks of the course, which were devoted to learning through problem solving. (The survey was administered either before or after the third class in the course, depending on the section.) This accidental result might show that even during such a short time as a few weeks, conceptual values may have been influenced.

Participants saw reflected in their graph a visual representation of their strengths as well as in which areas they need to continue to evolve; for many this was in the area of their conceptual mathematics knowledge. It was suggested to participants that those whose shapes were predominantly in the top right quadrant might be more open to a problem solving focus in mathematics, while those in the lower left might be more traditional. The model might be extended to propose that students in the top left quadrant might be at risk to revert to a more traditional position if they were not able to deepen their conceptual knowledge. Many participants subsequently stated in their journal reflections the desire to work hard on their conceptual understanding, and some cited that seeing this need on their personal *Profile* graph as contributing to this resolve.

By the end of the course, (13 classes later), change was evident for many participants, particularly in the area of conceptual knowledge. Tables 1 and 2 show the summary data.

Table 1 - *Knowledge Pretest and Posttest Scores*

	Pretest		Posttest	
	Procedural	Conceptual	Procedural	Conceptual
Mean	5.71	1.10	7.10	3.93
Median	6.00	0.00	8.00	4.00
Standard Deviation	2.253	1.587	1.248	2.812
Minimum Score Out of 10	0	0	2.5	0
Maximum Score Out of 10	10.0	6.0	10.0	9.0
Valid N	109	109	114	114
Missing N	5	5	0	0

Table 2 - *Values Pretest and Posttest Scores*

	Pretest		Posttest	
	Procedural	Conceptual	Procedural	Conceptual
Mean	6.30	7.21	5.45	7.62
Median	6.30	7.30	5.70	7.85
Standard Deviation	1.327	1.251	1.498	1.365
Minimum Score Out of 10	3.0	4.0	1.7	3.3
Maximum Score Out of 10	10.0	9.3	9.7	10.0
Valid N	107	107	114	114
Missing N	7	7	0	0

An important feature of the course under discussion may have been the explicit discussion and examination of teacher values related to the use of conceptual understanding and problem solving in mathematics learning. Such awareness may have allowed participants to choose to focus on conceptual learning, and to attend to its inherent value. It is also possible that the emotional importance to the students of deeply understanding the mathematics, having come fact to face with the weakness of their own conceptual knowledge, created a strong motivation to learn. It is interesting that both conceptual as well as procedural knowledge improved, even though the focus was almost exclusively on the conceptual aspect. Interestingly, procedural values slightly declined, while conceptual values rose, over the course of the study.

Teachers' beliefs may have been influenced enough during the initial experiences of learning through problem solving in the earlier weeks of the course prior to the study to allow them to invest in significantly deepening their understanding, but the detailed relationship of these variables is as yet unclear. Further study is on-going to examine the relationship of the proposed variables and instruments to examples of teacher behavior in actual classroom practice. Beliefs alone may not sufficient to support reform-based teaching (Raymond, 1997). It is hypothesized that conceptual understanding, in cooperation and interaction with conceptual values, may be another important factor in teacher development.

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STUDENTS' UNDERSTANDING OF LOGARITHMIC FUNCTION NOTATION

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The purpose of this study is to investigate how college students interpret logarithmic notation and how these interpretations inform students' understandings of rules for working with logarithmic equations. The framework used for this study is the procept theory of Gray and Tall (1994), which proposes that the use of mathematical symbols enables mathematical concepts to be treated as both a process and an object at the same time, and that it is the ability to deal with this dual nature of notation that separates the less able math student from the more able. The results suggest that most students in the study lack a process-object understanding of logarithms.

Purpose and Background

Students struggle greatly with both the concept of logarithms as inverse functions and the processes and procedures needed for working with logarithmic equations. Much of this difficulty stems from trouble students have interpreting notation used to express logarithms. What is it that students are "seeing" when working with logarithmic and exponential functions? What meanings are they constructing about the symbols involved? To be successful, students must be able to interpret the symbols used as both an expression of the object of a logarithm and an indication of the process needed to work with the function (Gray & Tall, 1994; Kinzel, 1999; Sajka, 2003; Weber, 2002b). The term *process* here means a sequence of steps that a student might perform to execute a mathematical problem. This paper investigates the idea that notation used in logarithmic and exponential functions is often misunderstood by students and becomes a hindrance to their conceptual understanding of log functions (Stacey & MacGregor, 1997).

Little research in math education has looked specifically at students' understanding of logarithms (Weber, 2002a; Weber, 2002b). However, researchers such as Dubinsky & Harel (1992), Kinzel (1999), Sajka (2003), and Tall et al (2000) have taken a close look at students' understandings of general function notation. Researchers suggest that the dual nature of the symbols, serving as both an indication of a particular operation and an object upon which to be operated, can be difficult for students to interpret (Kinzel; Gray & Tall, 1994, Sajka). For example, an expression such as $f(x) = 3x+1$ can be interpreted as a rule for a procedure, or as an object that can be manipulated (Kinzel). Students must be able to understand the context clues in mathematical problems to decide which interpretation to follow, which can be a difficult task (Sajka). Both Stacey & MacGregor (1997) and Kinzel have observed that students have a limited understanding of algebraic symbols used in different contexts and the ways that they are used to communicate in mathematics.

According to Gray and Tall (1994), mathematicians cope with an object's dual nature as a process and an object by giving it the same notation. For example, they use *a/b* to represent both the process of division and the object of a fraction. This dual use of notation enables advanced mathematical thinkers to deal with the duality of object and process, but less-able learners are left unable to comprehend this duality and, instead, remain focused only on a process understanding of functions and their notation (Gray & Tall). They rely on memorized procedures evoked by certain notation. A *procedure* here means a specific algorithm used to

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implement a process (Gray & Tall), and *procedural thinking* is defined as strict attention to procedures and the algorithmic tools that support them (Tall et al, 2000). Procedural thinking on a problem may allow students to do specific computations correctly without understanding why their algorithms work; however, students will encounter difficulties as they try to build new ideas on their procedural understanding (Tall et al). This is particularly true when learning logarithmic functions, because until this point in their learning of functions, students have been presented with notation that gives a clear rule for what to do with an input value (Hurwitz, 1999). For example, $f(x) = 2x+3$ means to double the input and add three. Logarithmic functions cannot be considered in the same fashion. Understanding logarithmic functions relies on being able to interpret the notation and symbols involved. The definition of a logarithmic function in many textbooks is given as follows:

$$\log_a(x) = y \text{ if and only if } a^y = x.$$

Students must be able to understand this notation as showing both a process and an object to successfully work with the function (Weber, 2002b). The logarithmic notation is used here as both a referent to a specific logarithmic function, and as an indicator of the value needed to be used in an exponentiation process. The value x is both an input value for the logarithmic function and the product of y factors of a (Weber, 2002b). To solve both exponential and logarithmic equations, students must be able to understand connections between the logarithmic and exponential forms and be able to combine and reverse the processes involved in both forms (Dubinsky & Harel, 1992; Weber, 2002a). However, Hurwitz finds that logarithmic notation leaves students "bereft of a succinct way to verbalize the operation performed on the input" (p. 344), and that the change from the familiar $f(x)$ makes it difficult for students to interpret the logarithm as a function output. The question to explore, then, is what do students understand when they see logarithmic notation? Do students interpret logarithmic functions as representing both an object and a process? How does their interpretation guide their understandings of rules for manipulating logarithmic expressions and working with logarithmic functions and equations?

Theoretical Framework

The phenomenon that this study intends to explain is the way in which students understand the concept of logarithms and use these understandings to solve problems that involve logarithms. The intention is not to look specifically at the learning process, but rather the understanding and conceptions that have been formed as a result of learning. The framework used for this study is the procept theory formulated by Gray and Tall (1994), which proposes that the use of mathematical symbols enables students to consider mathematical concepts to be both a process and an object at the same time, and that it is the ability to deal with this dual nature of notation that separates the less able math student from the more able.

In this framework, symbols act as a pivot between being able to think of a symbol as a process and as an object (Tall et al, 2000). Gray and Tall (1994) refer to this notion of a symbol representing both a process and a concept resulting from that process as a *procept*. They define an elementary procept as consisting of three components: a process, an object produced by the process, and a symbol representing the process or the object. Procepts, then, consist of several elementary procepts that all have the same object. *Proceptual thinking* is defined as an "ability to compress stages in symbol manipulation to the point where symbols are viewed as objects that can be decomposed and recomposed in flexible ways" (Gray & Tall, p. 132). In other words, students who think proceptually have the flexibility to see symbols as an indicator for carrying out a procedure, or as a compact representation of an object that can be manipulated and acted upon. For logarithms, an example of an elementary procept might be the symbol log(x), in the equation $\log(x) + \log(x - 9) = 1$. The symbol represents an object that can be decomposed and recomposed into a new object according to the properties of logarithms, and also indicates the process of exponentiation. This study examines the extent to which students demonstrate proceptual thinking when interpreting notation to solve such problems. Under this framework, the researcher tries to determine if, as Gray and Tall suggest, less able students are not simply slower learners but are, in fact, developing different techniques for problem solving due to their interpretations of the symbols.

Methods

The subjects for this study were first-year college students in two different pre-calculus classes, MA-107 and MA-111, taught by the researcher at a university. Most of the students in both classes were between the ages of 17 and 20. Many of the students were weak in their understanding of basic algebra skills.

There were two phases of data collection for this study. Phase I involved a five-problem questionnaire administered to 59 pre-calculus students enrolled in MA-111. The questionnaire was voluntary, and no points were offered toward the class grade for participation. The purpose of the questionnaire was to give the researcher some understanding of the different ways in which students interpret and manipulate logarithmic functions. Two of the five questions were used in the analysis. Data gained from this questionnaire was used in the design of the second research tool. However, due to time constraints on the researcher and the students, no students from MA-111 were able to participate in the second part of the study.

Phase II of the data collection was a three part task-based interview conducted with two students from MA-107. These students had just finished a lesson on exponential and logarithmic functions and had been recently tested on the material. The two girls interviewed, Anna and Lynn, were chosen by the researcher because: 1) they were active learners who asked questions and discussed problem solving strategies when doing group work in class, and 2) they had used incorrect methods to solve the equation $\log(x) + \log(x+9) = 1$ on their tests. Questions were chosen by the researcher to try to determine what sense the students were making of different logarithmic forms and their interpretations of the notation.

Results

Phase I

The questionnaires provided some interesting insights into students' understanding of logarithms. The first question contained five pairs of logarithmic expressions in which students were asked to circle whether the pairs were equivalent or not and to explain their choice in a sentence (see Figure 1). Almost all students reasoned correctly that the expressions in question *a*

a.	$\log_3(2)$	= or ≠ (circle one)	$\log_4(2)$
b.	$\log_a(1)$	= or ≠ (circle one)	$\log_b(1)$
c.	$\log_3(x) + \log_3(x+1)$	= or ≠ (circle one)	$\log_5(x) + \log_5(x+1)$
d.	$\ln(x)$	= or ≠ (circle one)	$\log_{10}(x)$

Figure 1. Paired expressions problems from the questionnaire

were not equal because the bases were different, but 26 of the 59 students used the same argument to say that the expressions in *b* were not equal. The notation here did not seem to suggest a process or object, perhaps due to the generic values for the bases. Students saw no meaningful relationship in the log symbols that supported mathematical activity (Kinzel, 1999). A similar lack of procedural and conceptual understanding of logarithms was exhibited in question *c*. Here, 19 out of 59 students believed the expressions were equivalent, most reasoning that log was irrelevant because it could be "cancelled out". The logarithmic notation obviously had no meaning for these students.

Another surprising result was that 35 out of 59 students answered on question *d* that $\ln(x)$ was equivalent to $\log(x)$. In these classes, $\ln(x)$ is notation for log base e, and $\log(x)$ notates log base 10. The special notation was presented and reinforced in class; however, a majority of students misinterpreted it. According to Kinzel (1999), teachers must be explicit about what is being represented by mathematical symbols if they are to be understood and used properly. Perhaps the students' misconceptions came from insufficient explicit teaching of this concept, but it could also be that the change from *log* to *ln* caused major problems for students and prevented them from forming a process or concept view of $\ln(x)$.

The second problem on the questionnaire required students to solve for x given the equation $\log(x) + \log(x+9) = 1$. Students were asked to show all of the steps they would take to solve the problem and to write at least one sentence explaining their work. The results were that only 22 out of 59 students used a correct method to solve the problem. Twenty-one others solved by again "canceling" the log notation and solving the remaining linear equation, and 16 could not do the problem at all. This problem was explored in more detail in Phase II.

Phase II

In the interviews, the students were given three tasks and asked to "think out loud" and explain how and why they solved each problem. The subjects, Lynn and Anna, were guided to make some connections among their work on all of the problems. To help distinguish between the students' meaningful knowledge of facts and rote-learned facts, analysis included comparisons of each interviewee's solutions to the three tasks as well as some comparison between the two students (Gray & Tall, 1994).

For task one, students were asked to solve for x given the equation

$$\log_5(x) + \log_5(x+4) = 1. \qquad (1)$$

Both Anna and Lynn used the same incorrect methods to solve Equation 1 as they had used on a similar problem on their classroom test. For Lynn, the problem brought to mind a procedure, which was itself incorrect and provided no indication of a concept understanding of logarithms. Her method was simply to eliminate the logarithmic notation from the problem entirely and then solve the remaining linear equation. The following transcript describes her thinking:

Lynn: First set these [expressions of x] equal since logs are the same, so I'll get $2x$ plus 4 equals 1. Subtract the 4, divide by 2.

Interviewer: What about this problem allowed you to do that?

L: Well, um, the logs are the same, the fives, so when they're the same you can assume that they're equal so you can solve it like that.

I: Okay so what happens to log?

L: I guess they cancel out.

I: What does that mean to you?

L: They're both, we don't have to include them. They both cancel out and that gives you that there (pointing to her result). They're equal or, (pause) I guess it's like when you have a negative 1, or a 1 and a negative 1. That would give you zero, so that's how they cancel out. You don't include them.

Lynn interpreted the presence of two log functions with the same bases as things that negated one another, although her comparison to adding 1 and –1 is surprising and her reasoning here is unclear. When asked if in her first line of work that showed $x + x + 4 = 1$, she thought the x's "cancelled out" because they were the same, she said no, that that was just a property of logarithms. It is apparent that she not only lacked a conceptual understanding of log functions, but also a correct procedural understanding for this problem. Lynn did not see logarithms as objects to be manipulated, in fact, she saw no need for the logarithmic notation to remain with the problem, even when checking her answer of x= -3/2 in the original problem. The notation did not evoke any anticipation of a need to change the function to exponential form. It is most likely she was simply remembering from class work that the log notation "disappeared" on these types of problems, and was applying a method that she had identified as necessary for making this happen. Lynn's' solution method is an example of how, in the absence of a proceptual understanding of a logarithmic expression as both an indication of the exponentiation process and an object to be manipulated to be used in a procedure, students will make up their own rules in math to make their solution match what they remember from previous examples. This supports Gray and Tall's (1994) idea that less-able students are not learning correct techniques more slowly, but are instead developing their own techniques. In fact at the end of the interview, after the researcher helped Lynn recall properties of logs so that she could produce the correct solution to this problem, she admitted that she remembered doing the problem that way, but that her method "seemed so much easier." She made no acknowledgement of her method violating any laws of mathematics.

Anna, on the other hand, initially recognized the logarithms in Equation 1 as objects that could be manipulated and that needed to be converted into a new object before processes could be applied. However, she used the properties of logs incorrectly and dropped the logarithmic notation from the problem, producing $x(x+4) = 1$. She, like Lynn, did not possess any real understanding of the concept of logarithms or any anticipation for using exponentiation in the solving process. She simply solved the resulting quadratic equation and obtained two irrational solutions, which she did not check back in the original problem. Her work is described below:

I: So what happened to log? Why is log in this first line but not the second? Anna: They cancel out.

I: Can you explain?

A: I don't know. I think that's how I remember it is they cancel out.

I: What does canceling out mean to you?

A: They have a, I don't know if complementary is the right word, but another one was there...so that positive or negative or something like that. Two of something.

I: When you look here [pointing to x (x+1)] you think multiplication. When you look here [pointing to log5(x+4)] do you think multiplication?

A: No…this I see as something with like an exponent like 5 to the something equals x +4. Anna was relying primarily on memorized facts, justifying her thoughts by saying what she recalled from class. She also exhibited algebraic misconceptions about the concept of canceling.

Anna's last statement in the transcript is interesting because it shows that when given a single log expression, the notation suggested a process of exponentiation. Both Anna and Lynn further demonstrated procedural understanding on the other tasks. For example, students were shown twelve index cards containing logarithmic expressions or equations for task two, and were asked to talk about what came to their minds when they saw each card. On most of the index cards involving a single equation, both students consistently converted the log equations to exponential form and then solved the resulting equation for *x*. Similarly, both students were able to use the correct procedure to solve the problem

$$\log_7(2x+1) = 2 \qquad (2)$$

for task three. For these types of problems, it might be tempting to suggest that the students had some conceptual, or even proceptual understanding of the logarithms. However, their approaches to the first problem leads this researcher to believe that their abilities to solve problems like Equation 2 were based on rote-learned fact, and not an example of a meaningful knowledge of logarithms that could be used to derive new facts or solutions (Gray & Tall, 1994; Tall et al., 2000). The introduction of a second log term in the problem completely changed the students approach to the problem. This may be due to the students' fixation on the process aspect of these problems. According to Tall et al, "knowing a specific procedure allows the individual to do a specific computation" (p. 8). In other words, viewing Equation 2 as only a process of converting to exponential form limits students' ability to see the expression as an object available for further manipulation in a more complicated equation like Equation 1 (Gray & Tall, 1994; Kinzel, 1999). These students lacked the proceptual understanding needed to be able to interpret the log symbolism in a flexible way, which is essential for successful mathematical thinking (Gray & Tall, 1994; Tall et al, 2000). These results are not unique to the two interviewed students as indicated by the results on the questionnaires.

A few other results from task two are worth mentioning here. When considering the index cards, both students struggled with the expression

$$\log_4(-16). \qquad (3)$$

Both girls converted this to exponential form, and then proceeded to guess and check answers on their calculator. They thought that the answer should either be a negative number or a fraction, and even though they could not come up with a result, they both suggested that there probably would be one if they kept looking. The fact that they believed that four raised to a power could produce a negative result is an indication of a major lack in understanding of both a process and concept of exponentiation. According to Weber (2000a, 2000b), a process understanding of exponents is the starting ground for building an understanding of exponential and logarithmic functions, which could help explain the difficulty students had solving logarithmic equations.

A problem with natural log was witnessed again in the interviews where, when shown an index card with the expression log (10), both of the interviewed students were able to determine that the answer was 1 because 10 raised to the 1-power equals 10. When shown ln(*e*), they also answered that it was 1, but neither student could provide more explanation than it was a known fact from class that natural log and *e* "cancel each other out." They were relying on rote-learned facts because the notation did not call to mind any other available tools for solving this problem or verifying the known result.

It is also interesting to note, that when shown the index card log(*xy*), Lynn recognized this symbol as an object that could be decomposed by the laws of logarithms to log(*x*) + log(*y*), but that the expanded form in Equation 1 in the first task problem did not suggest the use of this law. This indicates that Lynn's understanding of the concepts of expanding and condensing log functions was based primarily on memorized facts, and her inability to recall these facts caused her to solve the problem incorrectly.

Conclusion

The results of this study indicate that the students did not, in general, have a proceptual understanding of logarithms. The presentation of single logarithmic forms evoked the procedural response of rewriting the problem in exponential form in almost all cases. However, the addition of a second log form to the equation no longer prompted students to anticipate a change to exponential form; their learned procedure no longer fit the given form of the equation. In fact, they failed to demonstrate any process or object understanding of logarithms, and instead invented their own solution methods for "getting rid of" the logarithmic notation and finding x. It does seem, as Gray and Tall (1994) suggest, that less successful students are not necessarily slower at learning correct methods and in need of more time to understand concepts, but are simply doing different mathematics than the more successful mathematical thinkers. Furthermore, it is important to note that the students examined in this study showed no indication of moving away from their different methods or recognizing their faults. This is exemplified by the fact that both Anna and Lynn had already been tested on logarithms and had received their tests back with corrections, yet they both answered Equation 1 using the exact same incorrect method they had employed on their tests. If students are not held accountable for using material learned in math classes except on a single test, or perhaps twice if the material is revisited on the final exam, then there is little incentive for them to learn from their mistakes.

The major problem that students seem to have with logarithms is making meaningful connections to the name logarithm and the notation used to represent it. For example they may be able to relate to the notation $f(x) = x^2 + 3$, and give it a literal interpretation such as f is the function that takes a number for x, squares it, and adds three, but logarithmic notation is much more ambiguous; it does not "tell" students what to do. Without a deep understanding of its function and use, students must rely on memorization, which usually proves unsuccessful in the long run. For proceptual thinking to exist, the concepts of logarithms and exponentials should be as closely linked as addition and subtraction, where exponentiation is understood simply as a flexible reorganization of logarithmic facts (Gray & Tall, 1994). Teachers need to find ways of doing more than just teaching the processes and properties involved with logarithmic functions and instead make logarithms objects to which students can relate. With a meaningful understanding, students may be able to think about functions such as logarithms as both processes and conceptual objects and become more successful advanced mathematical thinkers.

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STUDENTS' COLLOQUIAL AND MATHEMATICAL DISCOURSES ON INFINITY AND LIMIT

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The study presented in this paper, which serves as a pilot study for a future comprehensive project, is to investigate how students deal with the concepts of infinity and limit. Based on the communicational approach to cognition, according to which mathematics is a kind of discourse, we try to identify the characteristics of students' discourse on the topics. Four American and four Korean students were interviewed in English on limits and infinity and their discourse was scrutinized with an eye to common characteristics as well as culture, age, and education-related differences.

Introduction

In this study, students' thinking about infinity and limit is investigated based on the *communicational approach to cognition*, according to which mathematics is a kind of discourse. The reason for a comparison between American and Korean students is that the Korean mathematical words for *infinity* and *limit* in a mathematical context do not appear in colloquial Korean language. Therefore, while American students have experience with the colloquial use of the English words *infinity* and *limit*, Korean students have little experience with colloquial Korean use of the mathematical terms. There are several reasons why this kind of study may be important. First, there has been little research on the mathematical concepts of infinity and limit using discourse analysis as a methodology. Discourse analysis holds promise of answering some previously unanswered questions. Second, such investigations may lead to methods for helping students overcome their difficulties in understanding limit and infinity, and may have implications for K-16 curriculum. Third, this approach may have implications for investigating advanced mathematical thinking in other areas. Finally, applying the communicational method to culturally different groups of students may shed light on the impact of culture on how students understand advanced mathematical notions.

Theoretical Background

Epistemology and History of the Notions of Infinity and Limit

The histories of the mathematical concepts of *infinity* and *limit* have been interwoven since their beginning. The story of infinity begins with the ancient Greeks. For the Greeks, infinity did not exist in actuality, but rather as a potential construct. Although there was the notion of bounded processes, there was no concept of limit as a concrete bounding entity.

In the Middle Ages, Christianity came to value infinity as a divine property. With the developments of astronomy and dynamics in the 16th century, there was an urgent need to find methods for calculating the area, volume, and length of a curved figure. In the 17th century, to find the areas of fan-shaped figures and the volumes of solids such as apples, Kepler used infinitesimal methods (Boyer, 1949). Throughout the 18th century, calculus lacked firm

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conceptual foundations. At the end of the 18^{th} century, mathematicians became acutely aware of inconsistencies which plagued the theory of infinitesimal magnitudes.

Today's notion of limit emerged gradually in the 19^{th} century as a result of attempts to remedy the uncertainties existing within mathematical analysis at that time. Cauchy and Weierstrass were pioneers of the movement toward a rigorous calculus. At this time, limit turned into an arithmetical rather than geometrical concept, as it was before, in the context of infinitesimals. Infinity was now actual rather than potential. In order to complete Weierstrass' foundations of arithmetic, Dedekind and Cantor developed the theory of the infinite set.

In spite of the mutual interdependence of the concepts of limit and infinity, there has been little research to examine students' understandings and difficulties of both of them simultaneously.

Learning the Mathematical Notions of Infinity and Limit

Various aspects of the learning about infinity and limit have been investigated over the last few decades. Anchoring their research in the analysis of the *mathematical structure* of the notions, Cottrill et al. (1996) report that there are two reasons for student difficulties with limits. One reason is the need to mentally coordinate two processes: $x \to a$, and $f(x) \to L$. The other is the need for a good understanding of quantification related to ε and δ. Borasi (1985) suggests several alternative rules about how to compare infinities based on students' intuitive notions (within this tradition, see also Cornu, 1992; Tall, 1992).

Other research has focused on *misconceptions* and *cognitive obstacles* related to infinity and limit. Fischbein, Tirosh, & Hess (1979) and Tall (1992) emphasized the role of intuition. One source of difficulty with the notion of infinity is the belief that a part must be smaller than the whole. Other researchers (Cornu, 1992; Davis & Vinner, 1986) stress the influence of language. Students might have had many life experiences with boundaries, speed limits, minimum wages, etc. that involved the word "limit". These everyday linguistic uses interfere with students' mathematical understandings (Davis & Vinner, 1986). Przenioslo (2004) focuses on the key elements of students' *concept images* of the limits of functions. Still others have focused on *informal models* that act as cognitive obstacles (Fischbein, 2001; Williams; 2001). According to Williams, informal models based on the notion of actual infinity are a primary cognitive obstacle to students' learning.

Finally, some researchers address students' difficulties through the lens of the theory of actions, processes, objects, and schemas (APOS; see Weller, Brown, Dubinsky, McDonald, & Stenger, 2004). Weller et al. speak about the cognitive mechanisms of *interiorization*, *encapsulation*, and *thematization* that are used to build and connect actions, processes, objects, and schemas.

Conceptual Framework for this Research

The present study takes as an assumption that when students come to the classroom to learn the notions of infinity and limit, they already have a certain amount of knowledge that comes from daily experience. The use of a given concept in everyday language can be crucial for students' future learning. Therefore, for those who teach the subject it is important to find out how students use the notions of infinity and limit in colloquial discourse.

Most of the past research on learning limits and infinity was grounded in a neo-Piagetian, cognitivist framework which does not seem quite appropriate for this type of study as it underestimates not only the inherently social nature of student thinking, but also the role of discourse and communication in learning and in other intellectual activities. Our project is guided by a conceptual framework within which school learning is seen as aiming at a change in ways of communicating. In particular, learning mathematics is seen as tantamount to becoming more skilful in the discourse regarded as mathematical. The word *discourse* signifies any type of communicative activities, whether with others or with oneself, whether verbal or not. Four distinctive features of mathematical discourses are often considered whenever discourses are being analysed, compared, and watched for changes over time: *words and their use*, *discursive routines, endorsed narratives*, and *mediators and their use* (Ben-Yehuda et al., 2004). In our ongoing study, particular attention has been paid to the participants' uses of the keywords *limit* and *infinity* in colloquial and mathematical discourses; to *discursive routines*, that is, repetitive patterns of both these discourses; and to *endorsed narratives* about limits and infinity, that is, to propositions that the participants accepted as true.

Design of Study

Research Questions

Our interest in characterizing the mechanisms of students' thinking about infinity and limit led to the following research questions:

1. What are the leading characteristics (in terms of word use, endorsed narratives, and discursive routines) of students' colloquial and literate (school) discourse about infinity and limit?
2. Do the students' colloquial and literate discourses on infinity and limit change with age and education?
3. Are there any salient differences between the discourse of native English and Korean speakers on infinity and limit? Can these differences be accounted for in terms of the differences in the colloquial uses of these words in English and in Korean?

Methodology

Each ethnically distinctgroup included one elementary school student, one middle school student, one high school student, and one university undergraduate (to refer to groups' members, we use symbols such as A_5 for the American 5th grader, K_{10} for the Korean 10th grader, and A_U for the American undergraduate.) The four American students were English speakers from the United States and the four Korean students were non-native English speakers from South Korea whose first language is Korean. Because the interviews were to be conducted in English, the four Korean students who were selected had been living in the United States and attending US schools for more than 3 years.

The interview questionnaire consisted of 29 questions, organized into eight categories. The first two categories aimed at scrutinizing students' *colloquial* discourses on infinity and limit, whereas the rest were targeted at investigating students' *mathematical* discourses on the topic. Examples of the interview questions are shown in Figure 1.

The interviews lasted 30 to 40 minutes. The conversations were audio- and video-taped and then transcribed in their entirety.

Data were analyzed so as to identify and describe the three distinctive features of the respondents' discourses: word use, routines, and endorsed narratives. At the next stage, the analysis of the data was guided by the three research questions. In this paper, we present *word* *uses* and *endorsed narratives* of the respondents' discourses on *limit* and *limited*, and compare the results obtained in the two ethnic groups.

I. Create a sentence with the following word: (a) *Infinite,* (b) *Infinity*.
II. Say the same thing without using the underlined word.
(b) Eyeglasses are for people with limited eyesight.
III. Which is a greater amount and how do you know?
(d) A: odd numbers, B: Integers
IV. $\frac{1}{4} = 0.25, \frac{2}{8} = 0.25, \frac{3}{12} = 0.25, ...$
How many such equalities can you write?
V. What do you think will happen later in this table? How do you know?
VI. (a) What is the limit of the following $\frac{1}{x}$ when x goes to infinity?
VII. Read aloud: $\lim_{x\to\infty} \frac{x^2 + 3x}{2x^2} = \frac{1}{2}$. Explain what it says.
VIII. (a) What is infinity?

Figure 1: Representative samples of questions from each category

Selected Findings on the Use of the Words *Limit* and *Limited*

Students' responses to the two questions relevant to our present subject, (questions I and VIII - (b)) have been summarized in Table 1. Students' responses to question I may count as representative of the colloquial discourse on limit, whereas their answers to question VIII – (b) seem to be a result of the students' attempt to respond according to the rules of literate (school) discourse. This conjecture seems reasonable because of the fact that in this latter part, the interview was contextually framed as interview on the *mathematical* concepts of limit and infinity. The analysis that follows is guided by the three research questions listed above.

Question 1: What are the leading characteristics of students' colloquial and literate (school) discourse on limit?

Use of words. The use of the words *limit* and *limited* can be characterized according to two characteristics: their area of *application* and the measure of their *objectification* (this latter term is explained below)

- *Application*. Almost all students' responses refer the words *limit* and *limited* to *processes that are bounded* in a certain quantifiable aspect, mostly in time. In the colloquial language the processes are of watching TV([1]), playing ([5], [10]), eating ([2], [6]), driving ([7], [15]), spending money ([4], [8]), and admitting diners in a restaurant ([16]). In the more school-like discourse (see responses to question VIII – (b)), there is no mention of any concrete circumstances, but the observation that the words limit and limited refer to processes is evident from expressions that imply a change in time, such as "limit can go on" ([17]), "numbers can't go past" ([19]), "you … never reach the limit" ([20]; cf. [23]), "limit is the furthest something to go" ([22]).

- *Objectification*. The use of a noun counts as objectified if this noun is applied as if it referred to a self-sustained, discourse-independent entity. Evidence of such use can be seen only in the utterances of 10th graders and university students; in all such cases the entity is presented as a particular number ("the speed limit is 50" ([7]), "ten dollars" ([4]), eighteen (12]), "a certain number" ([23]), "special number" ([24]); the status of the utterance [18] is unwarranted, in the present context).

Endorsed (endorsable) narratives. The following responses seem to complete the expression "Limit is…" to a narrative that is endorsed as true by at least some of the students, in some of the contexts (the narratives did not have to be articulated in exactly this form; their endorsement was inferred from statements actually made by the students):

- *an upper boundary* – In the majority of statements *limit* is used to describe as something that limits the process from above; this is true whether one speaks about playing, eating, spending money, driving, or admitting diners. It is true in the more abstract context, as evidenced by expressions such as "limit means like the maximum" ([21]), and "limit is the farthest [for] something to go" ([22]).

Students		I. Create a sentence with the following word: (a) Limited	I. Create a sentence with the following word: (b) Limit	VIII. (b) What is limit?
American	A_5	**[1]** I am limited on my TV for one hour	**[5]** I have a limit on video games	**[17]** Limit can go on and will stop at some point.
American	A_7	**[2]** There are limited amount of cookies	**[6]** We must limit the amount of candy we eat	**[18]** The limit is the value of a number
American	A_{10}	**[3]** I am limited to my ability	**[7]** The speed limit is fifty	**[19]** Limit is something that numbers can't go past. They have to stop at a certain point.
American	A_U	**[4]** My spending money each month is limited to ten dollars	**[8]** There is a spending limit on my credit card	**[20]** As not in mathematics, limit is the absolute ending of something like no more. In a mathematical sense, it would always be the ending of…like an answer, but sense… numbers are infinite. You can get to the answer but never reach the limit because there's infinite numbers... kind of opposite in a way because there isn't… there is no limit in infinity.
Korean	K_4	**[9]** The time is limited	**[13]** There is a limit	**[21]** Limit means like the *maximum*.
Korean	K_7	**[10]** I have a limited time for playing outside	**[14]** Our teacher put the limit to how much time we have to test	**[22]** Limit is the farthest something to go… like where it stops…like boundary you cannot go on… I think, infinity is the opposite of the limit
Korean	K_{10}	**[11]** My ability is limited	**[15]** I can't drive such at the speed limit	**[23]** Limit is a certain number that you can't reach but you get very close to
Korean	K_U	**[12]** Let's limited to eighteen	**[16]** That restaurant limits people's number	**[24]** Limit is *a special number*…for instance if I say limit four, then there is one, two, three, and four. Not over four.

Table 1: The summary of answers to questions I and VIII (b)

- *finite* - It seems that some students use *limited* (*bounded*) as *having an end* (that is, stopping at some point in time) and thus as an opposite of *infinite*; see [20] and [22].
- *unreachable* – In the process usage, the "unreachable" meaning occurs when one approaches the limit but does not actually attain it. This narrative appears only in the responses of the university students; see [20] and [23]. (In [20], based on our reading of A_U's responses to other interview questions, we interpreted the words "the answer" as referring to the result one gets while calculating limits according to known mathematical algorithms.)

Question 2: Do the students' colloquial and literate discourse on limit change with age and education?

Based on the above descriptive findings, we can speak about the following age-related characteristics of the students' use of the word *limit* and related endorsed narratives.

Differences in the *use of the words.* Two such differences seem to appear with age. First, the *area of application* of the discourse on limits evolves hand in hand with shifts in the students' everyday experience: The younger students (grades 4, 5, and 7) speak about limits in the context of children's favorite pastime activities, such as watching TV, playing videogames, or eating sweets. The 10^{th} graders speak about driving and individual abilities, two issues clearly quite significant to young people of this age. Finally, the university students make references to limits and limitations in the context of spending money and visiting restaurants. Second, the discourse limit gradually undergoes *objectification* and the uses of the noun limit as signifying a self-sustained entity become more frequent.

Differences in *endorsed narratives*. There seems to be no change in the students' endorsement of the colloquial narrative of limit as the *upper bound*, where all the other values produced in the limited process must remain on one side of the limit. One salient education related difference is the endorsement of the narrative on the unreachability of the limit, which we saw only in the mathematical discourse of the older students (thus the conjecture that it was related to education). In older students the discourse on *limit* becomes problematized, as they start seeing a possible incompatibility between *finiteness* and *unreachability*.

Question 3: Are there any salient differences between the discourse of native English and Korean speakers on infinity and limit?

Unlike in the case of the discourse on infinity, no systematic differences were found in the use of the words *limit* and *limited* between native Korean and English speakers.

Discussion and Conclusions

In the part of our study reported previously (Kim et al., 2005), we observed a considerable difference between the American and Korean groups in both the context in which the words of *infinity* and *infinite* were mentioned and their application to numbers. Our present data, however, indicate almost no difference between the two groups in their discourses on *limit*. We ascribe this finding to the fact that unlike the word *infinity*, the words *limit* and *limited* are quite common in English colloquial discourses. The Korean students, in spite of the fact that English was their second language, might already have been sufficiently familiar with the colloquial uses of the English word *limit* to make it irrelevant that there was no comparable use for the Korean mathematical term *limit* in everyday Korean discourse. Thus, this pilot study implies that in a more extensive future study, native Korean students should be chosen who *have not studied mathematics* in English and should be interviewed about both colloquial and literate discourses on infinity and limit *in Korean*. Although the sample was too small to allow for generalizations, what we found in this study can serve as a basis for hypotheses to be tested in a future, more comprehensive project.

Based on our findings, we can conclude that the use of the words *limit* and *limited* coming from students' colloquial discourses conflicts with the one imposed by the formal mathematical definition of limit in at least two respects. First, the narrative on limit as *an upper boundary* implies that all the values produced in the process must be on the same side of the limit. Second, the story of the limit's *unreachability* precludes the possibility that some values produced in the "limiting" process will be equal to the limit itself. Clearly, neither of these interpretations follows from the mathematical definition of limit. In addition, the use of *limited* as *finite* seems in conflict with the narrative about limits as being a property of infinite processes (the possibility that a process with an infinite number of steps may still be finite in time is also difficult to conceive). Students' colloquial discourse of the notions may thus have an impact not only on the students' later use of the mathematical keywords, but also on other aspects of their mathematical discourse – a fact that the teachers should keep in mind while planning instruction.

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MATHEMATICS LESSON STUDY TEACHERS LEARN ABOUT REPRESENTATIONS TO PROMOTE COMMUNICATION: A CASE STUDY

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This paper investigates how mathematics teachers' thinking about the role of representations in mathematics teaching and learning developed and shifted as they planned, taught, and revised three lessons during lesson study. Five middle school mathematics teachers increased the use of representations to facilitate student communication, support mathematical reasoning, and reveal student thinking. Specific episodes during each lesson study phase are suggested as events that generated teacher change.

Introduction and Purpose

Lesson study has spread quickly since its introduction to the U.S. in 1999, but has been studied in few U.S. settings (Chokshi & Fernandez, 2004; Lewis, 2002a; Stigler & Hiebert, 1999; Yoshida, 1999). This paper presents a U.S. case of mathematics lesson study where five middle school teachers worked collaboratively during one school year on promoting student mathematical discourse through planning and teaching research lessons. As teachers experienced how mathematical representations facilitated mathematical conversations, they increased the use of representations in their research lessons. This study investigated how the teachers' thinking about the role of representations in mathematics education developed and shifted during lesson study.

Theoretical Framework

Lesson study is a form of teacher professional development that originated in Japan (Fernandez, 2002; Lewis, 2002a, b; Lewis and Tsuchida, 1998; Stigler and Hiebert, 1999; Yoshida, 1999). The lesson study process consists of an iterative design cycle of collaborative teacher activities: 1) formulating goals for student development, 2) studying existing instructional materials, 3) planning a research lesson designed to make the goals visible in the classroom, 4) having one team member teach the lesson while others observe and collect data, and 5) using the data to redesign instruction in response to student learning (Lewis, 2002b).

This lesson study process provides a foundation for teacher learning with multiple opportunities to collaboratively consider teaching and student learning. By investigating how teachers' discussions and research lessons evolved during the process we were able to trace the shifts in teachers' thinking about the role of mathematical representations in student discourse.

The use of representations, "physical objects, drawings, charts, graphs, and symbols that are used to create, compare and communicate one's thinking" (NCTM, 2000 p. 280), is fundamental and is one of the standards in the Principles and Standards for School Mathematics (NCTM, 2000). Mathematics educators emphasize the value of using representations and models as essential elements in supporting students' reasoning and understanding of mathematical concepts and relationships and in communicating mathematical approaches, arguments, and understandings to one's self and to the classroom community (Chapin, O'Connor & Anderson, 2003; Gravemeijer, Cobb, Bowers, & Whitenack, 2000; Monk, 2003; NCTM, 2000). Focusing

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on representations as a tool for facilitating communication and for generating mathematical meaning, the two questions that guided this study are: (1) How does teachers' thinking about mathematical representations develop and shift during their participation in lesson study and (2) What aspects of lesson study appear to generate such shifts in teachers' thinking?

This study is from dissertation research that is part of a larger NSF-funded research project on the emergence of lesson study at different educational sites.

Methods and Data

Five mathematics teachers (Mr. Doyle: eighth grade; Mr. Rodgers: seventh and eighth grades: Ms Hayes: seventh grade; Mr. Linn and Mr. Ray: sixth grade) and their students participated in the study at a middle school in California serving a diverse student population (note that all teachers' and students' names are pseudonyms). The teachers met approximately 30 hours during the 2003-2004 school year. This lesson study group completed three lesson study cycles teaching the research lesson to one class each in Grades 6, 7, and 8. The first author was a participant observer of the group who took field notes and videotaped all lesson study meetings, research lessons, and post-lesson discussions. Teachers were interviewed and audiotaped winter and spring. The first author offered teachers occasional support by suggesting and making available materials to augment their lesson study work (e.g., articles about student mathematical discourse, summaries of their goal-setting and post-lesson discussions, summarized student strategies, and suggestions about using representations to facilitate mathematical conversations).

The videotapes of the lesson study planning meetings, the three classroom research lessons, and the post-lesson discussions, in addition to the audiotapes of the teacher interviews, were transcribed. These transcriptions and the lesson plans were then examined, coded and analyzed for evidence of change in the teachers' thinking about representations and student mathematical conversations. Although other themes and categories also emerged in the larger study, this paper focuses on the development and shifts in teachers' thinking about mathematical representations.

Results and Discussion

At the beginning of the year, the teachers chose "promoting student mathematical conversations" as their lesson study goal. The focus of the research lesson was algebra with the lesson objective: "*Students will use patterns and knowledge of sequences to discover and discuss relationships between building height, building number, apartment number and apartment floor level. They will derive formulas for determining the set of apartment numbers that share a given floor level.*" The first lesson plan included a drawing of numbered apartment buildings given to students and shown on the overhead projector (see figure 1). Students first determined the apartment number when given its location: "Mr. Doyle moved into the 5th building, in the apartment just below the penthouse. What is the apartment number and how you know?" The

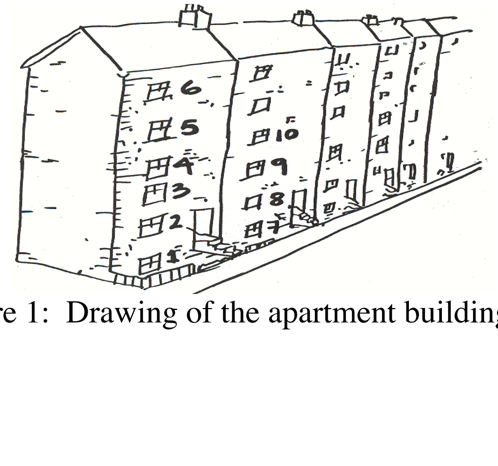

Figure 1: Drawing of the apartment buildings

class discussed students' strategies for determining different apartment numbers. Next, students were given their own apartment numbers and asked to determine their building and floor level. Students were asked to find someone in the class who is on the same floor and discuss their strategies. Then students put their numbers on the white board according to floor level only so multiples of 6 are not evident. The class discussed strategies for finding floor levels, the numbers on each floor level, and rules or formulas for finding the floor when given an apartment number.

Teachers' Thinking About Representations and Scaffolding While Planning Lesson 1

Mr. Doyle taught the lesson first and based the draft of the research lesson on a sample problem from the Math Assessment Collaborative (MAC). The original problem had a drawing of the apartment buildings (3 floors instead of 6), a worksheet, and a partially filled out table. Mr. Doyle told the teachers he extended the buildings to six floors and wanted to eliminate the "scaffolding" (worksheet and table) to allow for more student exploration. Mr. Rodgers, who taught the second lesson, expressed his concerns.

Mr. Rodgers: … if I don't give them a chart, if I don't give them a table, if I don't give them something then my kids are going to scurry around or be frustrated… This is interesting how…we've decided we've been over-scaffolding, …that scaffolding has a pejorative feel to it …

Mr. Doyle: I don't think that scaffolding is necessarily pejorative…but it's doing the thinking for them. …You're taking something away that is essential to becoming a better problem solver.

The teachers' discussions about scaffolding surfaced differing views. Some teachers felt they broke down mathematics problems too much so students no longer have to think to get answers, "I'm doing the math, the students just guess." Some types of scaffolding direct the students to solve the problem one way or gives students the "answer." Others felt scaffolding can clarify the problem and enable students to more clearly understand the situation so they may make meaning for themselves. Teachers agreed they need to carefully choose what kind of scaffolding to use, but they had divergent opinions about which kind and its use. Though Mr. Doyle mentioned anticipated student strategies, "They could use diagrams, repeated addition, a formula, an argument based on the relative position of this specific apartment to the one above it," the teachers did not discuss how students would explain their thinking and strategies to the class.

Lesson 1

Mr. Doyle explained the math problem and gave out and displayed the drawing on the overhead projector (Figure 1). Students were engaged as the class discussed how to find the apartment numbers. When students were given their own apartment numbers (between 45 and 100) and were asked to find the floor level, seven out of the 30 students extended the buildings on the drawing and wrote in all of the numbers up to their apartment number. Three students made a table filling in the numbers. Whole class discussions of the problems stayed on an abstract level (e.g., using numbers, equations, and mathematical terms). One student drew imaginary buildings in the air when attempting to show her conceptualization. Another student tried explaining his strategy of finding the nearest common multiple and adding up or subtracting down to the apartment number. Students had difficulty following his verbal explanations, so he pointed to a multiplication chart on the wall. His number, 78, was not on the chart, and the students could not easily determine the closest multiple or see that 78 is a multiple of 6. Consequently, the students' attention lagged.

During the post-lesson discussion observers noted difficulties students had communicating their ideas to the class. An outside observer who was experienced in lesson study referred to this and asked, "Are there representations that would enable students to see one another's work and catch onto explanations better…and would enable teachers to see students' work?"

Revising and Planning Lesson 2

A representation, a grid six boxes high and 10 boxes wide was introduced by the first author at a lesson study meeting, and the teachers considered how it could focus students' attention, facilitate explanations, and promote understanding. Again, differing views of the role of the grid representation emerged, as well as when to use it in the lesson. If introduced early, some teachers feared the grid would be too directive, stifling creativity. Other teachers asked that if the grid could stimulate conversation and promote understanding, why not give it to them earlier in the lesson? The grade six teachers, Mr. Linn and Mr. Ray, thought the grid could assist students in understanding the problem. Ms Hayes reminded teachers that students drew similar representations during lesson 1. Mr. Rodgers viewed its role only to assist students in articulating their thinking. The morning before the lesson he said, "…the representation comes at the very end, …people will be explaining what their rules are, rationalizing why they think they work, showing what works. I don't know if it will promote the conversation so much as help one person explain it."

Lesson 2

During the class discussion near the end of the lesson there was disagreement about which building number was correct. One student was trying to explain how she interpreted the quotient of her solution. She had divided the apartment number by 6 (each building had 6 floors) and the quotient gave her the building number. But, since she had a remainder she counted on to the next building and the remainder was the floor number. She was having a difficult time explaining it and again, the students' attention waned. Mr. Rodgers then put a large grid representation on the board and asked her to "show the class how it works." She attempted to make connections between her thinking and the representation, but with only five minutes left, the lesson ended.

At the post-lesson discussion some teachers felt the grid representation was not as successful as they had planned. Even so, Mr. Ray observed that the representation, "the chart," gave focus to the student discussion:

Mr. Ray: Then when you put the chart up, I sensed a change in the room. Suddenly people began to relax and start to focus…they had something to get their attention…

Mr. Rodgers: …But unfortunately the first person that went was Betty (pseudonym), a high-status person who barely used it, she just sort of repeated her algorithm…just pointed in that direction. She didn't really need it...

Other teachers observed students spontaneously creating similar representations, drawing out the buildings and filling in the numbers in order to access the problem. An outside observing teacher said, "One person was using the powerful drawing method to convince the other person who was using her formula that she had come up with."

Revising and Planning Lesson 3

As the teachers reflected on the first two research lessons, their thinking about mathematical representations was beginning to shift. They had observed situations where students were having difficulty expressing and understanding each other's ideas and were considering how the grid representation could have facilitated the communication between students. For example, in lesson 1:

Mr. Doyle: …then Bill got up at the board, …a bunch of people weren't listening…it made me anxious because I saw half the class had lost interest…

Mr. Rodgers: This was where we thought that a visual would promote, and everyone would stay more engaged because they would be able to follow. (Bill) had in his head what he was doing, but he wasn't able to convey it.

The teachers also reflected on lesson 2 when two students had argued about their differing solutions and never came to a mutual understanding. The teachers wondered if the grid had been available could those students have communicated their strategies more effectively to each other. They realized they had been rushing the students through the lesson, and decided to take two days for lesson 3 and allow more time for student exploration and discussion using the representation. The teachers decided to observe the lesson on day 2. Because the third lesson was taught in a sixth grade class, they decided to start with 4-level buildings on the first day instead of six. Seeing the potential of the grid representation, they chose to make additional tools available (desk copies of the grid representation, graph paper, and centimeter cube blocks). Mr. Linn wanted to have guiding questions written out for the students and made up a worksheet. He planned to use the grid representation earlier in the lesson saying, "…that representation makes explaining it easier for the kids… that tool will allow the kids who are counting to demonstrate that they are counting. And kids who are doing multiples can go 6, 12, 18…and if someone does the 6n - 1, they have an illustration to go along with their verbalization."

Lesson 3

On day one, Mr. Linn posted the grid representation on the board as students worked to solve the first problem. He demonstrated on the grid for students having difficulties. In the discussion following, Mr. Linn asked two students to use the grid to demonstrate their strategies, the counting and the multiples methods, making their thinking visible to the class. The class did the problem again with 6-story buildings and another student clearly articulated the rule saying, "…six times the building minus one gives you the apartment below the penthouse." On day 2, students were given their own apartment number (in a 6-story building) and worked on finding the floor level. Mr. Linn asked students to put their apartment numbers on the correct floor level on the grid representation, telling students to ignore the columns as representing the buildings and focus only on the rows. Students became confused wanted to place their numbers in the correct building also.

The teachers discussed this confusion during the post-lesson discussion. Mr. Rodgers said:

> … part of the problem there was that you had the grid up, the big poster up, and at first it was used as actual building number, building number one, 1 through 6, building 2, …7 through 12, and then when they were putting [their numbers] up, you used the same grid, but now this wasn't building number one any more, now they could put it anywhere.

Mr. Linn responded:

> I realized that part of the lesson was a bit confusing…kids were trying to put (their numbers) in the correct floor and the correct building, and I was trying to isolate the two… In retrospect, kids would have been ready to put it by floor and by building, eventually. And it wasn't necessary to isolate it, which was the idea I had after watching Mr. Rodgers teach it."

Before lesson 3, Mr. Rodgers had asked if the planned changes in the lesson would "amplify the differences between the educational haves and have-nots." At the post-lesson discussion Mr. Rodgers remarked that it was exciting when the higher-performing students started right out on the second day with "higher-level stuff," and yet the scaffolding changes "raised the floor…in terms of the 'comfort level' of the lower kids, and they were "more grounded and more able to function throughout the whole problem." Teachers commented on the lesson's effectiveness.

While one may question the usefulness of investigating a lesson taught in different grades, it was valuable to teachers in several ways. Teaching, observing, and analyzing the lesson at three grades allowed teachers to see the overlapping performance range of students in each grade, compare strategies at other grade levels possibly helping teachers understand strategies used by their own more (or less) advanced students, and begin to identify a trajectory across grade levels.

Summary and Implications

During lesson study teachers increased their use of representations to support student communication, student reasoning and understanding of mathematical concepts, and to help teachers see their students' thinking in new ways. The changes we found in teacher thinking about representations and the development of their new perspectives are summarized in table 1.

	Teachers shifted from thinking…	to thinking…
1	students only needed a rich problem to have rich mathematical conversations	rich mathematical conversations take careful planning.
2	students will be able to articulate their thinking and explain their strategies coherently without help	representations can give structure to discussions and assist students in clearly expressing their ideas.
3	students will understand each other's explanations without help	Representations can help students catch on to the other students' explanations.
4	students will follow and understand whole class discussions carried out on an abstract level with only words and equations written on the board	representations on the board can focus students' attention, increase engagement, and support student reasoning and understanding during class discussions.
5	tables or other representations would be too directive, limit student exploration, or do the thinking for students	well-chosen representations allow students to use multiple strategies, and higher-performing students can still move to a more abstract level.
6	representations should be generated by students only	teachers can generate representations that will help them show the connections between multiple strategies.
7	students would be able to easily switch their use and understanding of a representation	representations are powerful in conveying meaning and students readily internalize that meaning.
8	scaffolding plans would increase achievement gap or remove the richness of the lesson	scaffolding measures, such as representations, can support and stimulate all students
9	it is all right for teachers to rush students through the lesson, cut off class discussions, and/or skip the closing summary of the lesson if too many activities were planned for the class period	teachers should plan less activities and allow time for students to do in-depth exploration, share their strategies and solutions, and discuss the connections between the strategies.
10	representations have limited uses, or are only necessary to help struggling students explain their ideas	representations can be effective for supporting student problem solving.
11		representations can make students' thinking visible to the teacher.
12		representations can help students see the connections between multiple strategies.

Table 1: Shifts in Teachers' Thinking About Representations

These changes can be traced back to particular episodes in each lesson study phase that initiated, encouraged, and/or supported the changes. The teachers' perspectives in the left column were evident at the beginning of the year or emerged during lesson study planning discussions. Some of the shifts in thinking occurred early in the process. For example, shift #1 began for most teachers during lesson 1 as they observed students struggling to explain and understand each other's strategies. Then during the post-lesson discussion an outside observer suggested that teachers find a representation to facilitate communication between students. The teachers may have been particularly ready to pick up on this idea and begin making shifts #2 and #3 because they had just observed the students struggle. The later shifts in thinking occurred after teachers observed lesson 2 and/or lesson 3 and reflected on the students' conversations. In this way, research lessons provide a laboratory for teachers to take a big idea from research, that mathematical representations support mathematical reasoning and facilitate student discourse, and study it in the classroom. Another important feature in this process was the multiple opportunities teachers had to actively engage with their colleagues and examine their beliefs and knowledge of mathematics. It was the lesson study meetings where teachers' differing experiences and ideas about scaffolding and representations surfaced, where teachers wrestled with diverse opinions, analyzed student strategies, and developed their new ideas and perspectives. Each phase in the lesson study process provided different venues all contributing to the teachers' learning in important but different ways.

This research found evidence of shifts in teacher thinking about mathematical representations and their connections to particular episodes in the different phases of lesson study. Future research will investigate how these shifts in teacher thinking played out in the classrooms. Were teachers able to put into practice any of their new ideas about facilitating discussions and using representations? Examination of the videotapes of the teachers' regular lessons during the same school year and analysis of student artifacts may help determine if teachers were able to make significant changes in their lessons and pedagogy. Future investigation of this lesson study group may contribute to researchers' understanding of what happens when teachers participate in lesson study.

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MEASURING ALGEBRAIC SOPHISTICATION: INSTRUMENTATION AND RESULTS

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Because number sense "resists the precise forms of definition we have come to associate with the setting of specified objectives for schooling" (Resnick, 1989), it retains an ineffable quality that makes it difficult to observe and measure. Arcavi (1994) has extended number sense to the algebraic realm under the rubric "symbol sense." Building on Arcavi's work, this paper grapples with how to understand and measure students' algebraic sophistication (conceived in the spirit of, but somewhat more broadly than, symbol sense). We report on an instrument developed to measure algebraic sophistication for 131 Calculus II students at an elite private university, exploring the subconstructs identified for algebraic sophistication, and the degree of sophistication found for these students.

Increasingly, mathematics educators are turning away from skills and concepts as the only goals of mathematics instruction to embrace "sociomathematical norms" (Yackel & Cobb, 1996), "habits of mind" (Cuoco, Goldenberg, & Mark, 1996), or "mathematical dispositions" (Kirshner, 2002, 2004) as additional important accomplishments for students. Norms, habits of mind, and dispositions, in this sense, aren't limited to affective tendencies such as "enjoyment of math" or "persistence in problem solving," but include valued ways of thinking and of approaching mathematical activity.

"Number sense" is a wonderful exemplar of the attractions and challenges of such interests. For, while highly valued in our community, number sense is not reducible to a discrete listing of skills and concepts, and hence "resists the precise forms of definition we have come to associate with the setting of specified objectives for schooling"(Resnick, 1989, p. 37). In the algebraic sphere, Arcavi (1994) introduced "symbol sense" as a concomitant to "number sense" in the arithmetic realm. Arcavi has done an excellent job of setting forth concrete examples of some constituent aspects of symbol sense. However, as with number sense, symbol sense retains an ad hoc character making it difficult to address in curriculum, and difficult to measure.

Our study takes up the challenge of symbol sense under the rubric of algebraic sophistication–a label that signals a somewhat broadened interest from that pursued in Arcavi (1994). As with number sense and symbol sense, we assume that algebraic sophistication must retain a measure of the ineffable, the uncodifiable. Nevertheless, it is important for our community to continue to grapple with its many facets, to find ways to understand it more fully, and to learn how to identify its presence and absence with respect to particular programs of instruction.

We report, here, on an instrument used in a study to evaluate an innovative Calculus II curriculum at an elite private university in the Midwestern United States. The instrument was designed to provide a measure of students' algebraic and functions sophistication, reflecting our conjecture that mathematical accomplishment in such higher courses might depend on students' sophistication.

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The instrument was administered in January 2005 to 122 students (9 students were absent). Our purpose in this paper is to present the 6-item instrument, discuss the sophistication subconstruct each item addresses, and review students' performance on each. Some of these items were adapted from Arcavi (1994) or taken elsewhere from the existing literature; however, 3 of them were newly developed for this study. Dissemination and discussion of such instruments is vital to our efforts as a research community to support an educational focus on learning goals that, while difficult to measure, remain vitally important to a sound mathematical education.

Item 1: The Reification Subconstruct

The evolution of algebra through a sequence of process/object reifications has been carefully analyzed and presented by Sfard (1995) and others in relation to cognitive obstacles encountered by students in their development of algebraic understanding. For instance, algebra novices initially read $x + y$ as indicating the process of adding two numbers to get a result. However, process/object reification entails a dual understanding of $x + y$ as both the process of addition, and the result achieved through that process. This dual understanding is difficult to achieve. Many students simply adopt a "pseudostructural approach" (Sfard & Linchevski, 1994, p. 116), manipulating such expressions, but with no grounding in their structural relations.

Item 1 addresses the reification subconstruct through the use of 2 related subproblems that require the student to find all solutions for x in the interval $[-\pi/2, \pi/2]$.

1a. $(2x + 1/2)^2 - 2x + 3 = 2 1/4$, and
1b. $(2\sin x + 1/2)^2 - 2\sin x + 3 = 2 1/4$

Note that the 2 subproblems have similar form, except that x in the first problem has been replaced by $\sin x$ in the second. The student who has a reified understanding of variable will be able to exploit the solution from 1a to solve 1b. The student with a pseudostructural understanding will have to recompute the entire solution in 1b. (The instructions present the 2 problems together and urge the student to "be as efficient as possible" in their solution strategies.) This item provides a strong inducement (advantage) for a reified solution according to a metric of the advantage in such problem pairs developed by Wuensch and Kirshner (1997).

Following are the response options for the 2 subproblems.

1a. A. $-1, 1/2$ B. $\pm\sqrt{2}/2$ (or $\pm 1/\sqrt{2}$) C. $\pm\sqrt{3}/2$ D. $1, -1/2$ E. no solution in $[-\pi/2, \pi/2]$
1b. A. $-\pi/2, \pi/6$ B. $\pm\pi/4$ C. $\pm\pi/3$ D. $\pi/2, -\pi/6$ E. no solution in $[-\pi/2, \pi/2]$

Note that B is the correct solution for both problems. However, our scoring rubric for 1a investigates a range of possible performances:

For B score = 2 (correct symbol manipulation)
For B score = 1 (correct, but used quadratic formula unnecessarily–for $4x^2 - 2 = 0$)
For A, C, or E score = -1 (careless error, or weak algebra skills)
For D score = -3 (overgeneralized distributivity)*

*A student who begins the solution by expanding $(2x + 1/2)^2$ as $4x^2 + 1/4$ (overgeneralized distributivity) will obtain D as their answer, and so is deemed algebraically less sophisticated than a student who merely makes a careless error (A, C, or E).

The vast majority of students (79%) were able to successfully solve this equation. There was a wide range of errors represented in the remaining 21% of the sample, but only 2 of these students used overgeneralized distributivity to obtain D. Four of the 120 students responding to this item applied the quadratic formula.

The scoring rubric for 1b, requiring examination of the written work, is concerned mainly with whether the student has re-solved the equation, or shown a reified understanding by substituting the result from 1a:

For B	score = 3 (correct answer obtained through substitution–reification)
For B	score = 1 (correct answer obtained by re-solving whole equation)
For A, C, D, or E	score = 2 (incorrect answer obtained by substitution–reification)
For A, C, D, or E	score = -1 (incorrect answer obtained by re-solving equation)

(Additional points were deducted for unnecessary use of quadratic formula [deduct 1] or for overgeneralized distributivity [deduct 2] if done in 1b but not 1a.) The total score over the whole 6-item test could range from -16 to +17, making a score of 0 a neutral outcome.

Only 10 of the students (8%) used substitution to obtain a correct answer, and a further 2 students obtained an incorrect answer using that method. The majority of students (58%) got the correct answer by re-solving the equation. The rest of the students obtained incorrect answers, or else displayed unsophisticated methods like substituting the multiple choice options.

Item 2: Representational Flexibility and the Point-Set Interpretation of Function

The process/object distinction also can be examined in relation to students' flexibility in mediating between symbolic and graphical representations of functions (Moschkovich, Schoenfeld, & Arcavi, 1993). From a process view, students begin to graph functions point-by-point, each point an association between an x-value and a y-value derived from it by algebraic means. However, eventually one departs from the point-by-point graphing process by joining together points to create an object—a solid line. The challenge of reification is to maintain a dual understanding of the line as at once a whole that can be transformed and translated in lawful ways, while still retaining the point-set intuition of the line as consisting of infinitely many points. For many students these 2 aspects of the graph/function become dissociated from one another. They can perform routine tasks with graphs and functions but lack the sophisticated association between process and product.

One such routine task is the vertical translation of graphs in association with a changing constant term in the defining equation for the function. Given equations of the form $y = \mathrm{f}(x)$ and $y = \mathrm{f}(x) + C$, many students will recognize that the graph of the second equation is a vertical translation of the first. However, this capability may be dissociated from the point-set understanding of function. Item 2 presents such a system; however, the form of the equations masks their simple algebraic relationship. The student, therefore, cannot rely on the resemblance of the 2 equations to one another to discern the graphical implication of vertical translation. They must, instead, algebraically manipulate the system, and then grapple with the fact that all of the variables cancel out, yielding a numerical equation that is always false.

Item 2 asks the student to solve the system of equations and identify the true statements.

$$y = 3\cos^2 x + 6x\sin x + 5$$

$$y = 3\sin x(2x - \sin x)$$

i) The graphs are vertical translations of one another.
ii) The graphs intersect at 1 point.
iii) The graphs intersect at 2 or more points (but a finite number).
iv) These graphs cannot be drawn.
v) There are infinitely many solutions to the system of equations.
vi) There are no solutions to the system of equations.

The response options are: **A.** i and vi **B.** ii **C.** iv and vi **D.** iv and v **E.** iii

Forty percent (40%) of students scored 3 points on this item for their choice of A, displaying a point-set interpretation of functions and graphs. A further 15% of the students scored 1 point for choosing C. These students recognized that the system of equations has no solutions, but were unable to work out the graphical implications of that fact. Examining the work of a further 4 students (3%) (with varying answers: D, B, or E) showed they did recognize the system as inconsistent, but did not know how to interpret it. These students received a score of 0. A score of -2 was earned by the remaining 42% of the students who made computational errors, offered no work, or selected no answer.

Item 3: Functions Notation as a Window to Function Concepts

As with graphical representations, function notation can come to be dealt with at a pseudostructural level–manipulating notational forms without grasping the structural relations they are intended to represent (Vinner & Dreyfus, 1989). A telltale sign of such maladaptive learning is explicit misreading of functional symbols. For instance, if the simple signifier "f" represents a function (an association of domain and range elements), the more complex signifier "f(x)" might seem to the unsophisticated student to represent a more complex concept. Item 3 tested this proposal by asking students whether, for a function with domain and range the real numbers, f(x) represents i) function of x, ii) the y value corresponding to x, or iii) a number. Responses that included i) were deemed unsophisticated in relation to those that included ii) and/or iii).

Unfortunately, this item did not function as intended, actually producing a small negative correlation with students' total score on the instrument. Only 4 students of 122 respondents selected a response that excluded "function of x" as a valid way to read or interpret f(x). A poll of several mathematicians indicated a consensus opinion that though technically incorrect, informal parlance, even among mathematicians, is so loose (e.g., $f(x) = e^x$ is the exponential function) that a student's unsophisticated response might not always reflect an unsophisticated understanding. Thus, this item would need to be reworked to become an effective measure of algebraic sophistication.

Item 4: False Compensation and Systematic Versus Ad Hoc Procedures

This item, adapted from Arcavi (1994), provided a diagram of a rectangle with sides labeled L and W, and asked students to consider the effects on area that would result from increasing one dimension by 10% and decreasing the other by 10%. Many students might suspect that these offsetting operations cancel each other out, resulting in no change to the area. An unsophisticated student would select "no change" as their response without further investigation. About 4% of our sample did so, earning a score of -3. The most sophisticated response was to encode the situation algebraically and determine, correctly, that the new area would always be less than the original. About 29% of students earned a score of 2 for this approach. Somewhat less sophisticated an approach would be to check arithmetically by selecting specific values for L and W. While arithmetic tools are valuable, in this problem the results may lack generality, or require much more labor to confirm for many cases. 34% of students received a score of 1 for adopting this strategy. Most of the remaining 33% of the students made an arithmetic error and concluded either that the resulting area will always be larger than the original, or that the direction of change depends on the particular values selected for L and W. These students scored -2 on this item. A few students making these errors using algebraic means earned a score of -1.

Item 5: The Reversal Error

The famed “Students and Professors” problem (Clement 1982) was included in the instrument to gauge the sophistication of students’ use of variables. In writing an equation that relates the number of students and professors, at a certain university that has 6 students for every professor, students who non-reflectively match the word order in the problem statement, or who understand variables as labels (e.g., S is “students”) rather than as representing quantities that are balanced in an equation, will reverse the variables and write $6S = P$ instead of $6P = S$. Twenty eight percent (28%) of students making this error, and another 7% making various other errors scored -2 for this item. The 65% of students who provided the correct answer scored +2.

Item 6: Metacognitive Skill and Representational Flexible

Ever since problem solving was identified as the heart of the “Agenda for Action” (NCTM, 1980), the need for students to monitor their solution processes, recognize unproductive approaches, and seek alternative ways to construe or represent the problem has been emphasized in the research community (e.g., Schoenfeld, 1985). The final item, adapted from Arcavi (1994), presented in algebraic form a problem that also can be interpreted geometrically: For how many different values of a will exactly 3 solutions be found for this system of equations:

$$x^2 - y^2 = 0 \text{ and } (x - a)^2 + y^2 = 1$$

Approaching the system algebraically leads to complex equations that are nearly impossible to interpret. However, recognizing that the first equation represents graphically a pair of diagonal lines passing through the origin and that the second equation represents a circle of radius 1 centered at $(a,0)$ on the x-axis leads easily to the observation that there are 2 values of a satisfying these conditions.

This item proved to be too demanding for these Calculus II students. Only 3 of the 122 Calculus II students responding to this item adopted a geometric approach (and only 2 of these 3 got the correct answer). As a result, we recoded this item so that students received a score of 0 for an unsophisticated response, instead of a negative score.

We have developed an alternate, 2-part, version of this item that would provide more support for students’ display of representational flexibility in future studies: 6a. A pair of 45¡ diagonal lines cross a coordinate system at the origin (i.e., they form an "X" at the origin). How many different circles of radius 1 centered somewhere on the x-axis intersect with these diagonals at exactly 3 points? 6b. For how many values of a does the following system of equations have exactly 2 solutions:

$$x^2 - y^2 = 0 \text{ and } (x - a)^2 + y^2 = 1$$

Psychometric Properties

In the spring of 2005, 122 students at Washington University in St. Louis (WUSTL) enrolled in Calculus II were given this test near the start of the semester. WUSTL is a top ranked university with selective entry requirements. The mean math-SAT score for students in the course was 717, with a standard deviation of 51.04 (data available for 107 students). Our on-site ethnographic observations revealed a generally serious and hard-working student body.

Within a possible range of -16 to +17, the algebra quiz scores for these students ranged from -7 to 11, with a mean of 2.13 and standard deviation of 4.6. Thus, students tended, overall, to be rather neutral, or relatively unsophisticated, with respect to their algebra performance (recall that small positive scores could be earned for most items by merely getting the computation correct). Given the overall strength of the students in their mathematical preparation, the performance on the algebra sophistication instrument suggests a need to improve secondary school and university level curricula with respect to this important learning outcome.

As a first step toward quantitatively measuring algebraic sophistication, analysis of the test results revealed serious limitations to the current instrument. Pearson Product Correlations showed only weak correlations among the items (.15 or less, most not statistically significant at the .05 level). The Mokken scale coefficient H was calculated to measure the homogeneity of the questions in terms of measuring a single underlying trait or ability. The scale score of .11 (on a scale of 0-1) was very low, suggesting that the questions do not measure a single underlying latent construct (.3 is considered the minimum for even a "weak scale").

In addition to internal correlations among the items, the algebra sophistication scores were correlated with several scores external to the instrument: SAT and/or ACT scores; the final examination score in the Calculus II course; and scores on a Calculus Concepts quiz we developed to measure students' understanding of the calculus. In general, these correlations were weak. There was a statistically significant but weak linear positive association between the standardized SAT/ACT Math scores and the total algebra sophistication score (r =.280, p-value = .002). (Only 7% of the variability in the algebra quiz score is explained by the SAT/ACT Math score.) There was also a moderate statistically significant relationship between the algebra scores and the final exam scores (r = .323, p-value < .001); however, the relationship to the Calculus Concepts Quiz was not statistically significant (r = .136, p-value = .209).

There are two possible interpretations that can be given to the low internal correlations among the test items. The first is that the subconstructs of algebraic sophistication targeted by the various items are relatively independent of one another. Alternatively, perhaps students do not consistently display their algebraic sophistication in testing circumstances. Note that most of the problems could be solved using sophisticated or unsophisticated means. Perhaps students sometimes opt for a "safe" rather than a sophisticated strategy, depending on local circumstances like tiredness or mood that may change from minute to minute. Such considerations point to the subtlety of measuring this elusive construct.

The weak external correlations of the algebra sophistication scores are still more difficult to interpret. One possibility is that algebraic sophistication is a multidimensional construct, weakening our ability to see overall effects. Alternatively, it is possible that this highly capable and successful student population lacked the variability that would have been needed to discriminate outcomes. Finally, the low external correlations could reflect the fact that the potential for sophistication is stifled in standardized tests like SAT and ACT tests and on departmental examinations that tend to foster attention only to routine methods for solving problems.

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PROOF WRITING: THE MOVE FROM SEMANTIC TO SYNTACTIC REASONING

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Students at the collegiate level have a great deal of difficulty learning to prove. The literature suggests that both syntactic and semantic reasoning is necessary for students to reason through proving activities successfully. This paper describes a case study of students from a "learning to prove" course into their advanced calculus/real analysis course. The students' learning to prove process is examined with an emphasis placed on their reasoning while proving. The story is illustrated best by examining one student's transformation (Doug) from semantic to syntactic reasoning in contrast to the transformations of his peers (Dustin and Lynn). This paper suggests that not only do students need to be able to reason semantically and syntactically, but they must also have the strategic knowledge to determine when each type of reasoning will be helpful.

The purpose of this paper is to describe the strategic knowledge students need to reason both semantically and syntactically in the process of proof. In order to help students learn to prove, many universities include some type of learning to prove transition course from Calculus to the upper division courses in their mathematics major curriculum. Research within these courses indicates students have a great deal of difficulty learning to prove (e.g, Harel & Sowder, 1998; Moore, 1994). This paper looks at how student reasoning plays a part in the student's proving activities, by describing the journey of one student, Doug, from his learning to prove transition course into a beginning real analysis course. His journey highlights the need for students to possess the ability to reason both semantically and syntactically as well as the need for strategic knowledge to know when to use each type of reasoning.

Literature Review

Student difficulties with proof include content knowledge as well as logical reasoning and notation. Research has found students have a great deal of difficulty unpacking logical statements and following a logical chain of reasoning (Hoyles & Kuchemann, 2002; Selden & Selden, 1995). Other difficulties include the inability of students to see the global picture, the "overview" or big picture involved in a proof (Jones, 2000; Selden & Selden, 2003). Likewise students lack the language to communicate effectively in proof writing (Dreyfus, 1999). Finally Dreyfus remarks, "In most cases, they still lack the conceptual clarity to actively use the relevant concepts in a mathematical argument" (p. 91).

Weber (2001) found that students needed more than just the conceptual knowledge and deductive reasoning skills to successfully produce a proof. He reports the need for domain specific strategic knowledge, or heuristic guidelines, which help the student determine a successful course of action. Edwards and Ward (2004) found that real analysis students lacked this strategic knowledge in that they had difficulty understanding the role that definitions played in an analysis proof. Moore (1994) also found that students did not understand the role the definitions played and likewise often did not understand the definitions involved. The strategic knowledge needed in a beginning real analysis course includes the ability to understand and use

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domain specific definitions as well as counter examples and theorems, and to know when each will be useful (Weber, 2001).

Theoretical Perspective

Weber and Alcock (2004) distinguish two types of reasoning in the proof production process: semantic and syntactic. Semantic reasoning is when "the prover uses instantiation(s) of the mathematical object(s) to which the statement applies to suggest and guide the formal inferences that he or she draws" (Weber & Alcock, 2004, p. 210). Syntactic reasoning is when the students produce a proof solely by manipulating definitions and facts in a "logically permissible way." Weber and Alcock suggest that both semantic and syntactic reasoning are necessary, and that a proof produced with syntactic reasoning may not be as convincing to students.

Weber and Alcock's terms semantic and syntactic reasoning are consistent with Hart's (1994) terminology, which he used when describing a proof writing model developed in the theme of the van Hiele levels of geometry. Hart suggests four levels of student's proof writing. Those students who are unable to write proofs are considered level 0. Those students who prove theorems syntactically are at level 1. Hart distinguishes between a concrete semantic reasoning (level 2) and abstract semantic reasoning (level 3). Even though Hart's model is not recursive, it is suggestive that students can be expected to follow a path from syntactic reasoning to semantic reasoning. However, if a student reasons semantically this does not necessarily imply the student can reason syntactically. Together the findings of these two studies (Hart, 1994; Weber & Alcock, 2004) suggest a trajectory for students from syntactic reasoning to semantic reasoning and suggest that students be able to reason either way when needed.

Methods

Four students who had participated in a semester long teaching experiment in proof writing course and were currently enrolled in a one unit advanced calculus workshop were asked to volunteer for several one hour interviews throughout the semester. While the students were enrolled in different advanced calculus courses, the course syllabi were similar, covering topics in real analysis including set theory, limits, convergence, functions, differentiation and integration. Students were expected to write proofs in their advanced calculus courses on homework and exams. The one unit workshop met weekly for an hour or more to work on problems from real analysis in small groups. I will focus on the journey of one particular student from this larger study, while using comments from some of the other students to compare and contrast.

The interviews were conducted in two parts. Students were first asked to discuss their current struggles in their advanced calculus course. The second portion of the interview consisted of a series of content related questions and proof tasks. Students were asked to talk aloud as they worked through the problems. Video data of the interviews were first coded in a manner consistent with grounded theory (Strauss & Corbin, 1990). The data was then coded with an emphasis on semantic and syntactic reasoning. It was also noted when students were using the strategic knowledge noted by Weber (2001).

Results

At the beginning of the semester the students were asked to consider the following statement:

> *Problem 1 -- Let x and y be irrational numbers such that x-y is also irrational. Then define the following sets A={x+r : r is rational} and B={y+r : r is rational}. Show that the intersection of A and B is empty.*

Each of the students approached the problem in a different way. I will first describe their answers to this question and then describe their reasoning trajectory while proving across their semester of advanced calculus.

Doug was a vocal student in class. He worked very hard to perform well, but was not as successful in his coursework as he would like to be. He received a C in his transition to proof course and dropped the advanced calculus course the first time he tried it. He received a B in the advanced calculus course his second time through, which coincided with the period of this study. Doug read the problem and thought quietly.

> *Doug: Okay I am trying to find a way to relate, right now, A and B. I am thinking element chasing but it's kinda hard when (mumbles to himself) x-y is also irrational uh shoot…*

Doug was looking to find a meaningful relationship between the sets in the problem. His focus was semantic as he attempted to make inferences based on his instantiation of the sets. He did mention the idea of "element chasing" a type of set theory proof. This is a domain specific strategy similar to the strategic knowledge referenced by Weber (2001). The difference is that Weber suggests part of the knowledge is to recognize when this strategy would be useful. Doug did not recognize that an element chasing strategy is not likely to be helpful.

Initially Doug had difficulty with the set notation; he then switched tactics and started to look at A-B to find an element in the intersection. It took him several minutes to orient himself to the problem. Doug then articulated that he would like to prove that $x + r \neq y + r$ for all r, where r is a rational number. He struggled to prove the statement using an element strategy; thus Doug again regrouped.

> *Doug: I am trying to go back and think how to say it clearly. My basic idea is since x can't equal y at any point then you know that r+x is never going to be equal to r+y.*
>
> *Int: What if the "r's" were different? What if we change this to r1 and r2 cause they don't have to be the same in this right?*
>
> *Doug: No.*
>
> *Int: How do you know that even though x doesn't equal to y there isn't some r to get me to y. To get it so that $x + r_1 = y + r_2$?*
>
> *Doug: Alright now I want to go back to rational is a finite decimal…*

At this point, per Doug's request, Doug and the interviewer reviewed the definition of rational and irrational. When further questioned Doug was able to define an element of the set A to be some $a = x + r$, but he was not able to finish the problem. In this session Doug did not produce a proof. While Doug did investigate the definitions related to sets and irrational numbers, his work was primarily focused on relating the sets and looking for meaning in the statement. His reasoning seemed to be semantic, in that he was looking to understand and find meaning and relationships between the entities in the statement.

In contrast to Doug's work on problem 1, several of Doug's colleagues reason either syntactically or semantically, but were more successful in their proving. On problem 1, Doug's classmates Dustin and Lynn both produced proofs by contradiction. Lynn was a hard-working student who tended to perform well in her courses, but felt like she was struggling with the material. When asked to explain her method of the first question, Lynn recalled a heuristic from her transition to proof class.

> *Lynn: When you're trying to prove an intersection… like an empty set, isn't it easier to like, contradict it? And assume that there is something in that, than deal with the definition of an empty set?*

Lynn's strategy determined her proof would be syntactic. Her access of a heuristic was consistent with Weber's strategic knowledge (2001).

On the other hand Dustin tended to reason semantically. Dustin was a top student in the mathematics department. He received A's in both his transition to proof course and ultimately in his advanced calculus course. On problem 1 Dustin also produced a proof by contradiction but his initial thoughts were more semantic in nature than Lynn's.

> *Dustin: So like all of these are irrational. Both sets A and B both have all irrational numbers. Now, why? How come there's no elements that are in both in sets A and B. Hmmm… because a does not equal x and x does not equal y?*

Dustin's comments and approach to this problem at first were very similar to Doug's. The difference was that Dustin chose to follow a heuristic about using a proof by contradiction to prove that something is an empty set. Both students focused on the sets, A and B and looked for some reason for their intersection to be empty.

The Rest of Doug's Story

Doug continued to work at producing successful proofs throughout the semester. In the next interview, Doug was asked to prove that the limit of a sequence is unique. After some clarification of the statement, he began flipping through his textbook to find this theorem and its proof in the book. At this point Doug was trying to understand the meaning of the statement, but relying heavily on the textbook for definitions and proofs.

In the mid semester interview Doug was asked to prove that a uniformly continuous function maps Cauchy sequences to Cauchy sequences. He began by writing out the two definitions and then looked in his book to find a theorem or proof that looked helpful. He came across the proof of the statement "Let f be a uniformly continuous function and x_0 an accumulation point, then $f(x_0)$ has a limit if and only if the sequence x_n converges." Since the proof looked similar to the definition of Cauchy and used a uniformly continuous function, Doug decided to use it to model his proof of the statement at hand. Although the proof he produced was somewhat acceptable, he was unable to explain why he had used certain notation. He was able, when pressed, to correctly explain the key idea of the proof based in the meaning of the definition of uniform convergence.

> *Doug: Okay, since the sequence is Cauchy, okay, when you have the uniformly continuous, the numbers that you plug in if they are individually the difference between them is less than your delta then your transformation through the function is less than some epsilon. It's just saying that no matter what since it's uniformly continuous there is a uniform ray of transformations so basically any two points on the sequence on the domain aspect translates to a set difference on the range aspect and if that works with just x's then the entire two sequences have a certain amount of difference.*

Hence Doug was capable of reasoning out the underlying meaning of the proof, but did not choose to do so until pressed by the interviewer.

By the final interview of the semester, Doug's reliance on memorized definitions and theorems had become an issue. He was spending hours researching in the library to find a proof or a theorem that would help him complete his homework. When asked to prove that a continuous function maps a convergent sequence to a convergent sequence Doug described his method of proof.

Doug: How do you do it? You've got to state the definitions of both first and then you have to show that – and then repeat it and it's actually a fairly simple proof.

At this point Doug relied solely on manipulating the definitions, and was no longer focused on finding meaning in the situation. He produced a proof syntactically, but he missed the meaningful connection between the two definitions, specifically that the epsilon from one definition is the delta from the other.

Throughout the semester Doug went from looking for meaning in order to produce a proof to working only with memorized definitions, theorems and proofs. His journey was not a successful one as he never made the connection between his semantic reasoning and his syntactic reasoning. This data is consistent with Weber and Alcock's (2004) claims that both semantic and syntactic reasoning are necessary for proof production. Doug's situation seemed to indicate that being able to reason in both manners is not sufficient for successful proving, but that strategic knowledge about when to reason semantically and when to manipulate definitions syntactically is required.

Doug's transition from looking for meaning to memorizing definitions was not made without influences. The professor in Doug's advanced calculus course rewarded students who had memorized formal definitions. Most class periods began with a pop quiz. These quizzes predominately asked the students to produce the formal definition for a concept or a theorem regarding a concept. Thus it is likely that this influenced Doug's desire to know the definitions and use them whenever possible to answer a question.

Doug's Colleagues

Throughout the entire semester Lynn and Dustin were both successful proof writers in their own way. Lynn's proofs were predominately syntactic similar to Doug's later reasoning, while Dustin's proofs were predominately semantic similar to Doug's early reasoning. In either case Lynn and Dustin exhibited both types of reasoning. Likewise Lynn and Dustin often made comments which exhibited their domain specific strategic knowledge.

When asked to start a proof Lynn often focused on the necessary structure involved. For example, when asked to prove an "if and only if" statement Lynn said, "Well you have to show both ways with an if and only if statement. So that's like the first, which way are you going to prove first." Her response was consistent for a different type of problem. When asked to prove the uniqueness of a limit Lynn stated, "Well so I guess with uniqueness proofs usually they are by contradiction. You assume that a sequence has two separate limits and then assume they are equal and come to a contradiction."

As Lynn worked further on the uniqueness of a limit proof she reasoned syntactically. In fact in the beginning of the semester she made several comments consistent with syntactic reasoning.

Lynn: I am using the definition of the limit just writing out what that means and seeing if there is anything I can manipulate. I don't know if it's going to get me anywhere, but that's what I am going to try.

Like Doug's work at the end of the semester, Lynn's reasoning from the beginning of the semester seems based in the need to manipulate symbols or definitions to make it work. However, as the semester progressed Lynn began to look for connections that "make sense."

Lynn: I mean in order to get anywhere…But what I am trying to figure out is how uniformly continuous and the Cauchy input go together. I mean how they interact. [works for a *few minutes then clarifies] What I am trying to do is just fit f of the sequence, a function of a sequence into the definition of Cauchy. But would it just be still greater than or equal to m and n? I understand cause those are still determining your elements in the sequence so that makes sense.*

She was focused not solely on the manipulation of the definitions but the interaction of the two concepts and specifically that her instantiated elements still "make sense." This was certainly semantic reasoning. However, when stuck or frustrated, Lynn reverted to the syntactic reasoning. In the final interview of the semester she was frustrated with a proof involving absolute continuity.

Lynn: I have no clue how to relate those. We are looking at this definition and it has sums. And I am much better looking at a definition and like at least attempting to manipulate things look like each other than I am to look, I don't know.

Thus through the end of the semester Lynn preferred a syntactic reasoning approach to problems.

On the contrary Dustin predominately reasoned in a semantic way. His initial approach to a problem was to read it and then make sure he believed it. When he was not sure of where to start, Dustin looked for connections.

Dustin: Okay this goes for any proof in general. The first thing I would do before I would do a proof that I haven't done before is to convince myself the statement is true. Then sometimes, before it used to happen a lot more than it happens now, sometimes seeing why, like understanding this, like when you understand something is like…there is a reason why you say, "oh okay so that is true," and I think the key to proof is in the thing that tells you its true like there is a connection. There is a connection, when you read something and say its true you have reasons for it. Those reasons might not be so easy to write down but you know they exist. So those reasons, sometimes there is the answer about how you are going to go about proving it. I don't know if this is the case for this one.

Even though this is the self described approach that Dustin took, he also exhibited some comments consistent with looking for a heuristic as well as syntactic reasoning. When asked to prove the limit of a sequence is unique, Dustin's first response was "this one definitely sounds like a contradiction one." He proceeded to write out the definition of the limit and reasoning syntactically.

Discussion

It should be noted that the course work in a beginning real analysis course consists predominately of proving statements whose proofs are a symbolic manipulation of the definitions of the concepts they involve. Thus for these students syntactic reasoning played an important role in their success in the classroom. In fact, this curriculum can certainly focus students on the memorization of definitions and use of syntactic reasoning to produce a proof.

We see that using Doug's method of putting the definitions on the page and looking for some way to manipulate them was a successful method of proof for Lynn. Doug was also able to produce some correct proofs with the help of models from his textbook, his notes, or other sources found in the library. Likewise, we see Doug's early method of looking for meaning and searching for a connection between the entities in the statement was a successful method of proving for Dustin. Yet overall we see Lynn and Dustin were significantly more successful in their proving attempts than Doug.

Notice Doug transitioned through the semester from semantic reasoning to syntactic reasoning. This path is somewhat contrary to Hart's suggested model (1994). Likewise we see very little transition in Lynn and Dustin's reasoning throughout the semester. It is not clear if their transitions took place prior to entering the study or were not as drastic, since they combined both types of reasoning and developed the knowledge of when to use each or to use them in conjunction.

The difference between Doug and his colleagues is two-fold. First we see Doug depending on outside sources for the strategic knowledge referenced by Weber (2001). While both Lynn and Dustin recognized that a common strategy for proving something is an empty set was to use proof by contradiction, this never occurred to Doug. Likewise, Lynn and Dustin often referenced some heuristic when making decisions about the structure of the proof.

Second, we see Lynn and Dustin reason both semantically and syntactically at some point even though their preference for one type of reasoning may dominate. As noted by Weber and Alcock (2004), both semantic and syntactic reasoning are necessary for students to be successful in proving. However, for Doug, in his zeal to develop the syntactic reasoning skills that were encouraged by his professor, Doug discarded the semantic reasoning he had developed earlier. It seems that his lack of the strategic knowledge to know when each type of reasoning would be useful would have been helpful. It is this knowledge that is key to the success exhibited by Dustin and Lynn. Thus in addition to the strategic knowledge suggested by Weber (2001) and the semantic and syntactic reasoning, students need to know when to pursue a proof using syntactic reasoning and when it is more important to reason semantically. This knowledge is necessary for successful proving.

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"IT'S LIKE HEARING A FOREIGN LANGUAGE." MATHEMATICS DISCOURSE: A NEGOTIATION OF INTERFERENCE

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Frequently in my teaching of secondary school mathematics, I have heard students comment on how hearing the language used in mathematics is like hearing a "foreign language." Investigated in this study of a ninth grade mathematics classroom is whether teachers and students appropriate language in mathematics differently and to what extent the differences in appropriation contribute to this perception of "hearing a foreign language." One key finding from this study that may provide some explanatory power is that students experienced interference between natural language and ways in which natural language is used in mathematics.

Introduction

Mathematics makes requisite of the English language in ways that are inconsistent with more common and natural uses of the language. Students may be unaware of such differences in usages. Reported in this paper are findings from a study of a ninth grade mathematics classroom that suggest that differences between natural language and mathematical discourse, comprised of a *mathematical language register* (MLR), contribute to this perception of mathematics sounding like a foreign language. This in turn potentially creates interference for students as they attempt to negotiate meaning in mathematics.

Romaine (1989) says that interference is the extent to which users of more than one language are able to keep the two languages separate. There may also be positive or neutral interference; however, these other types of interference are not elaborated herein. Although an extensive discussion of interference exists in research literature pertaining to bilingual speech communities, research that considers registers within a particular discipline such as mathematics is limited. The challenge is then to conceptualize the literature from applied linguistics, second language learning, and sociolinguistics in the context of mathematics discourse in order to develop an understanding of how interference might negatively affect student outcomes.

Theoretical Framework

Mathematical Language Register (MLR)

MLR is language (including symbols) utilized uniquely but perhaps not exclusively (i.e. may be *borrowed* from natural language) in mathematics and is distinguished from that used natural language or in other areas of academic language (Dale & Cuevas, 1992; National Council of Mathematics Teachers [NCTM], 2000; Pimm, 1987; Winslow, 1998). Halliday (1978) explains further the notion of register and more specifically register in mathematics:

> A register is a set of meanings that is appropriate to a particular function of language, together with the words and structures which express these meanings. We can refer to 'mathematics register', in the sense of the meanings that belong to the language of

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> mathematics (the mathematical use of natural language, that is: not mathematics itself), and that a language must express if it is being used for mathematical purposes. (p. 195)

Pimm (1987) says this with respect to mathematics register:

> Registers have to do with the social usage of particular words and expressions, ways of talking but also ways of meaning. […] pupils at all levels must become aware that there are different registers and that the *grammar*, and *meanings*, and *uses* of the same terms and expressions vary within them and across them. (pp. 108-109)

As Pimm (1987) suggests increased awareness is necessary for both teachers and students on the differences in registers. He notes that a lack of awareness can lead to "register confusion" (p. 88).

In contrast, natural language consists of a register that is colloquial, common, familiar, and includes everyday conversational language (Chamot & O'Malley, 1994; Delpit, 1998; Heath, 1983; Orr, 1987; Peregoy & Boyle, 1997). Words such as cancel, if, and table, for example, must be relearned within the mathematics register (Pimm, 1987; Wagner, 2003). MLR is not highly functional or transferable to contexts outside the mathematics classroom.

Language Proficiency in Mathematics Education

Cummins (1984; 2000) theorizes that there are two different levels of language proficiency. Cummins' analysis is specific to ESL students but can be generalized to students of mathematics as well. The first he refers to as *basic interpersonal communicative skills* (BICS) or the skills one uses to communicate with others in everyday communications. The second is referred to as *cognitive/academic language proficiency* or CALP. CALP is "conceptualized in terms of language in de-contextualized academic situations" (p. 136). Cummins describes de-contextualized academic situations as being *context-reduced* as opposed to *context-embedded* (p. 138). Context-embedded language is language that is embedded within experiential learning while context-reduced language is the language that is required, symbolic or otherwise, in abstraction.

The development of CALP has strong implications for content area instruction such as mathematics. For example, Cummins (1984) observes that students are most likely to speak with each other in "peer appropriate ways" regardless of their second language proficiency (p. 136). This suggests that, even when opportunities arise for students to engage in mathematical dialogues with one another, they may do so using primarily natural language or BICS and may not further develop their CALP. A further consideration is that mathematics, which can require abstract reasoning, is dependent on CALP as the context of instruction tends to be context-reduced. Cummins cautions that students may appear as being significantly more advanced in their language proficiency than they are, but remain limited in their CALP. Through dialogue and classroom activities, educators may perceive students to have a good understanding of concepts; however, in situations requiring CALP, this perception often proves false.

Understanding Interference

Interference manifests itself in a variety of ways in mathematics education as students progress through the BICS/CALP continuum. The negative implications of interference are that students are at risk of not learning. Collins (1993) refers to this as the denial of "the pedagogic message" because of "a match or mismatch of codes" (p. 119). Following is an elaboration on two closely related perspectives (although possessing subtle differences) of interference: (1) macro-interference that arises from earlier and later acquired languages and (2) micro-interference that arises from linguistic codes within an acquired language or register. Both perspectives offer alternative ways of conceptualizing the relationship between natural language and MLR.

Macro-Interference

Macro-interference is interference that arises from earlier and later acquired languages. Kellerman and Sharwood Smith (1986) describes interference between earlier and later acquired languages in their construct of *crosslinguistic influence* as the "interplay between earlier and later acquired languages" (p. 1). Richards (1984) also introduces *error analysis* and *intralingual interference* as the juxtaposition of a second language (native language) on a target language. Richards asserts that some languages may be more challenging to acquire in that there exists a "universal hierarchy of difficulty" (p. 13) and that context and prior knowledge are significant factors influencing interference. Natural language can be conceived of as the earlier acquired or native language and MLR can be conceived of as the later acquired or target language.

Micro-Interference

Micro-interference describes interference that occurs between linguistic codes *within* an acquired language or register. The existence of MLR suggests that in a mathematics classroom there is the potential for two different registers to be used at any given point – MLR and natural language. Blom and Gumperz (1972) describe two types of interference conceptualized as micro-interference: *situational* and *metaphorical*. Situational interference occurs as a result of mismatched cues associated with class and social situations while metaphoric interference has as its antecedent's topic or subject matter. Blom and Gumperz's work emerges from investigations of social constraints and linguistic patterns that arise from a single communicative system. Communication in single communicative systems requires that the participants "agree on the meanings of words and on the social import or values attached to a choice of expression" (p. 417).

Bernstein (1972) offers another perspective related more closely to the relationship between types of codes (e.g. MLR and natural language) and represents the effects of the combination of both situational and metaphoric interference. Bernstein distinguishes between two types of performance codes: elaborated (formal) and restricted (informal). Bernstein suggests that the selection of a particular code is not arbitrary but rather dependent on context and social structure in that "different forms of social relations can generate very different speech systems or communication codes" (p. 473).

Methodology

The data for this study was collected over a 3-week period in a ninth grade mathematics classroom in 2003. There were 21 student participants, one teacher, and myself as participant-observer. Data collection techniques included: (a) participant observation/researcher memos, (b) participant interviews and artefacts (e.g. journals, class notes, etc.), and (c) transcriptions of tape recordings of classroom interactions. Five students were selected for student interviews on the basis of one or more of the following criteria: (a) classroom participation, (b) questionnaire responses, and (c) written journal responses. Two of the students were ESL. It is important to note that the responses of the students whose first language was not the language of instruction mirrored those of students who did indicate that English was their first language.

The teacher was also interviewed. The teacher was a veteran teacher who had taught mathematics previously on numerous occasions. The data detailed herein is drawn in particular from lessons pertaining to algebra (i.e. addition or subtraction of polynomials, distributive property, etc.). The interviews were conducted immediately following the conclusion of the unit of algebra. Students had access to their notes and text during the interviews and were encouraged to refer to these resources if they needed assistance in answering an interview question.

I adopted a grounded theory approach for this investigation. Drawing from Strauss and Corbin's (1998) grounded theory model, discourse data was coded, further classified, until theories, substantives or otherwise, emerge. The coding of the data was largely a content analysis (Berg, 2004) identifying lexicon belonging to the MLR. Context was therefore considered to make this determination. Key episodes were then identified for further examination through other data sources such as interviews and student journaling.

Results

There are two sources of interference observed that jeopardized students' progression along the BICS/CALP continuum. These are: (1) teacher-talk and (2) student-talk interference.

Teacher-Talk Interference

During my classroom observations, teacher-talk interference was most visible. Two factors contribute to teacher-talk interference. The first factor was that the teacher dominated classroom discourse and thus students "tune-out." The second factor is the use of a predominantly highly formalized register; that is, MLR. Observed were extended periods of time whereby the teacher dominated talk which increasingly minimized the students' attention and thus acted as a source of interference. This was most notable in instances where the teacher attempted to engage students in the learning and the students were completely disengaged and could not even follow where the teacher was in the particular lesson. Take the following classroom discourse excerpt:

1. Teacher: This is our last topic in <u>algebra</u> and it's actually not going to be terribly different from the stuff that you've already done. This is why it leads nicely into the review exercises which prepare you nicely for the test. That is my plan. Do this topic, do the review exercises and finish the morning with our review. <u>Adding</u> and <u>subtracting polynomials</u>. All right. Again, a lot of times I find people look at a question like that and they go home and say look at all those <u>terms</u>, look at all those <u>positives</u> and <u>negatives</u>, look at all those <u>exponents</u>. I can't do that, and throw up their hands in frustration. But, all it would take is for them to take two seconds and look at it and realize wait a minute, what's the operation that I'm being asked to <u>perform</u> here? What's the <u>operation</u> I'm being asked to <u>perform</u>? And how can I rely on prior knowledge? Watch. What's the <u>operation</u> here? [long pause, student is selected via a hand gesture by the teacher]
2. Student: <u>Division</u>. <u>Brackets</u>, basically <u>multiplication</u>.
3. Teacher: I don't think so.
4. Student: <u>Addition</u>? What was the question again?
5. Teacher: You've got 4 choices. What is the <u>operation</u> here?
6. Student: [shrugs]
7. Teacher: I don't think so, to look at it, you've got a <u>set</u> of <u>brackets</u>, but the important part is what's in between them. It's the <u>positive</u>, so you're being asked to <u>add</u>

> this polynomial, what kind of polynomial is this? A trinomial. With this trinomial right here. All right. Now before we do BEDMAS, according to BEDMAS, we have to do brackets first, right. And when we say bracket we mean everything inside the brackets. Well can you do anything with what's in the brackets right there? Are we done?

The underlined words are from MLR. Of importance is the extensive use of MLR in the absence of any qualification or elaboration. Also illustrated in this example is the extent to which teacher-talk dominated the discourse. From my memos following this episode of instruction, I noted that students continued to struggle during seat work and made comments that they still did not "get it." During the semi-structured interviews I asked students to describe, explain or show me a "polynomial." Only one of the five students put forth an adequate, although vague, explanation. I asked each of the five to show me a polynomial in their notebook and although 3 out of the 5 did so, they prefaced their selection by "I think" or "I'm not sure" or "Is that correct?"

Student-Talk Interference

Student-talk interference occurs through the use of primarily an informal register (BICS) or natural language. One problem that emerged in the study was that there was limited discussion amongst students or even between the teacher and the students. Therefore, to explore the potential for student-talk interference, questions were asked during the semi-structured interviews to simulate peer-to-peer discourse (i.e. assist a peer with a homework question). Students occasionally talk in what appears to be a higher level of language proficiency than is actually present somewhat mimicking authoritative discourse. This excerpt, however, demonstrates the extent to which meaning is restricted to natural language although MLR was used:

8. DK: Your advice was to "expand it" – what did you mean by this?
9. Student: Expand? Expand, expand, what does it mean; I don't know. Um, not really, expand. Expand, make it bigger, stretch it, expand it. That's probably what it means in math.
10. DK: What did you mean by simplify?
11. Student: Simplify in math, like find the answer and work it out.
12. DK: Now, what's the difference between simplify and evaluate?
13. Student: Evaluate is where, evaluate would be evaluate.
14. DK: Do you get an exact answer with simplify and evaluate?
15. Student: Not really.
16. DK: How are they different? Can you show me an example from your work?
17. Student: I don't know. I know difference, but … No, I don't know. Probably something is wrong or something and you never know.

My content analysis of classroom discourse revealed that students often failed to use MLR. The interview data suggests that even when MLR is used, a clear understanding of the intended meaning cannot be assumed.

Discussion

Teacher-talk interference emerges from the use of a highly formalized register which may be incompatible with the level of language proficiency of students and thus act as a source of interference from an assumed a commonality in language code which may be false. Teacher-talk interference may also arise from the dominance of classroom discourse as shown in the first example (lines 1-7); whereas, student talk emerges from the use of primarily informal or natural language. When students have gaps in understanding, their inappropriate use of MLR has the potential to create interference for their peers during peer-to-peer discourse. Indeed, students appeared to be unaware of differences in registers and used primarily BICS to communicate about mathematics. This last example (lines 8-17) shows how in de-contextualized situations students who are limited in CALP may be unable to access what they know.

Richards (1984) asserts that some languages may be more challenging to acquire and as such the students may be more inclined to resort to discourse that are more familiar – both in use and in meaning making. Bernstein (1972) suggests that the selection of a particular code is not arbitrary but rather dependent on context and social structure in that "different forms of social relations can generate very different speech systems or communication codes" (p. 473). It would then be reasonable and expected that during peer discourse informal code (i.e. natural language) may be predominant. As a consequence, Blom and Gumperz (1972) assertions of situational and metaphoric interference may also have explanatory potential in this study in that assumptions of a shared meaning between teacher and students and between students may be false.

Conclusions

This study concludes that students experience interference from the ways in which language is used in the mathematics classroom. This in turn contributes to the perception of "hearing a foreign language" that students report. The interference can be conceived of having two sources: (1) teacher and (2) student. In each case, the assumptions that frame language competency, language use, and generalizability of natural language to mathematical contexts forms the basis of the interference.

Addressing issues of interference in mathematics education creates multiple challenges. The first challenge is to minimize, or potentially utilize interference as an opportunity to facilitate learning. The second challenge is to be able to identify those students who are perhaps more at risk. The third challenge is to understand the ways in which MLR proficiency can be explicitly developed. Students' ability to appropriate and develop MLR and negotiate interference independently cannot be assumed. Further research is needed to consider how interferences can be minimized in the classroom such that optimal learning opportunities are created for all students.

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RE-LEARNING DIVISION BY FRACTION: AN EXPLORATORY STUDY OF PROSPECTIVE TEACHERS' LEARNING

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This paper reports an attempt to develop prospective elementary teachers' understanding of mathematics for teaching related to the topic of division by fraction. The pre-test data indicated that most of the participants could correctly perform this operation with fractions but could neither devise any alternatives to the traditional 'invert and multiply' algorithm nor explain why this procedure works. By the end of the teaching intervention, however, most prospective teachers could not only perform a division by fraction task in different ways, but could also provide various explanations for how and why this procedure works. This study suggests that it is possible to help prospective teachers re-learn mathematics content in ways that can be useful in the context of teaching.

This paper reports on a teaching intervention study aimed at developing prospective elementary teachers' understanding of division by fraction. Fractions, division, and division by fraction are familiar topics in the upper elementary and middle school mathematics curriculum. Yet students' and teachers' difficulties with these topics have been widely documented (e.g., Ball, 1988b; Ma, 1999; National Council of Teachers of Mathematics, 2000). The topic of division by fraction has been found to be particularly challenging for prospective and practicing teachers of elementary and secondary school mathematics (Borko et al., 1992; Simon, 1993; Tirosh, 2000).

These studies document that while most teachers and prospective teachers are able to perform the calculation and obtain the right answer when asked to solve exercises with division by fraction, they seldom are able to explain how or why this procedure works or to devise alternatives to the traditional 'invert and multiply' procedure. While much attention and progress has been made in identifying prospective teachers' difficulties with this particular topic, researchers have yet to systematically study how teacher educators might help prospective teachers re-learn this topic with understanding.

A common view in the United States is that elementary mathematics is "basic", an arbitrary collection of facts and rules in which doing mathematics means following set procedures step-by-step to arrive at answers (Ball, 1991). This perception of mathematics may be satisfactory for knowing mathematics for one self but when the case is knowing mathematics for teaching, a procedural orientation is not enough; teacher's own subject matter knowledge should move beyond mere memorization of rules and execution of procedures to a knowledge that is rich in representations, connections and explanations (Hiebert & Lefevre, 1986).

Considering the aforementioned research literature and following a "teaching experiment" approach (Tirosh, 2000) I set out to investigate ways of strengthening prospective teachers' understanding of division by fraction in ways that they could in turn be able to teach it to school students. In particular, my aim throughout this study was to answer the following questions: (1) How might prospective teachers' knowledge of division of fractions – before and after my teaching intervention – be characterized? and (2) What is the impact of my teaching intervention on prospective teachers' knowledge of division of fractions?

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Theoretical Framework

The design of the research and teaching intervention draws on the National Research Council's (2001) notion of mathematical proficiency and Graeber's (1999) forms of knowing mathematics. With respect to the former, learning mathematics is viewed by the NRC not as a one-dimensional trait but as a synthesis of five interwoven and interrelated strands: *conceptual understanding* – comprehension of mathematical concepts, operations and relations; *procedural fluency* – skill in carrying out procedures flexibly, accurately, efficiently, and appropriately; *strategic competence* -ability to formulate, represent, and solve mathematical problems; *adaptive reasoning* – capacity for logical thought, reflection, explanation, and justification; *productive disposition* – habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy.

With respect to how these forms of knowing come to bear on the teaching of mathematics, Graeber (1999), in examining forms of knowing mathematics, finds that it is important for preservice teachers to understand that executing an algorithm, or getting the right answer, does not imply conceptual understanding. Prospective teachers according to Graeber (1999) must understand that students who possess one form of knowledge do not necessarily possess other forms of knowledge. For instance, students may hold procedural knowledge of how to divide two fractions but have poor conceptual knowledge of either fractions or division. If preservice teachers enter the classroom without making the distinction between conceptual and procedural knowledge, they are apt to take existence of one type as evidence of existence of the other.

Data Sources and Analysis

Participants

Participants in the present study were 10 (8 female and 2 male) prospective elementary teachers within the age range 22-24 years old. The sample was selected randomly from a total number of 20 Caucasian senior preservice elementary teachers enrolled in a mathematics education methods course at a large Midwestern University. The course was open only to students admitted to the teacher certification program and its focus concentrated on three directions: (a) Examining teaching as enabling diverse learners to inquire into and construct mathematics-specific meanings, (b) Adapting mathematics to learner diversity and (c) Exploring multiple ways diverse learners make sense of the elementary mathematics curriculum. During the semester I visited the class on a weekly basis where I observed and participated in the class activities alongside the prospective teachers.

Research Instruments

Immediately prior and a week after the teaching intervention the participants were asked to respond to a written survey, which consisted of three tasks: [A] *People seem to have different approaches to solving problems involving division with fractions. Please show all of the ways you can think of to solve this:* $1\frac{3}{4} \div \frac{1}{2} = ?$ [B] *Imagine that you are teaching division by fractions. To make this meaningful for kids, something that many teachers try to do is relate mathematics to other things. Sometimes they try to come up with real-world situations or story-problems to show the application of some particular piece of content. What would you say would be a good story or model for* $1\frac{3}{4} \div \frac{1}{2} = ?$ (Kennedy, Ball & McDiarmid, 1993, p.49; Ma, 1999, p. 55) [C] Imagine that you are a teacher and one of your students says: "*I know that when I'm supposed to divide two fractions, I have to turn one of the numbers upside down and multiply, but I don't* *know why all of a sudden it gets changed to multiplication, I also forget which one to turn upside down and I get a bunch of the problems wrong*" (Borko et al., 1992, p. 202). How might you respond? Teaching Intervention. The teaching intervention was designed based on the participants' responses to the pre-test. The analysis of the pre-test responses showed that task A and C were the most troublesome to these participants. The teaching intervention was administered to the whole class of 20 students and it focused on strengthening the two mathematical proficiency strands that were found to be underdeveloped (conceptual understanding and adaptive reasoning).

The intervention consisted of two parts (60 minutes each). The first part of the teaching intervention focused on exploring 4 different ways of solving division by fraction. The second part focused on exploring 3 different explanations for why the standard 'invert and multiply' algorithm works. The focus of the first part was to answer the question: "How can we solve division by fractions without using the 'invert and multiply' algorithm?" The class examined in 'expert groups' four distinct ways to perform division by fractions; two of the ways were pictorially oriented [use of fractions strips (Figure 1) and area model (Figure 2)] whereas the remaining two were algorithmic ones. Then the groups were reconfigured into five groups with four different 'experts' in each and taught each other what they had learned in their respective 'expert' groups.

The second part of the intervention followed the same jigsaw method explained above and focused on clarifying the reasoning behind the standard division by fractions algorithm. Two of the explanations were algorithmic ones (Figure 3) whereas the third one (Figure 4) was pictorial. The importance of this jigsaw method lays on the fact that it promotes communication and mutual respect in the process of learning, and empowers the learners to figure out things on their own rather than rely on the teacher as the sole source of knowledge. Apart from the group discussions, pre-service teachers were also encouraged to exchange their views and reflections as a whole class.

Results

Both intervention sessions were videotaped and a rough transcription was created. Participants' responses were initially coded and, then, interpreted in the light of the first four strands of mathematical proficiency. It is important to clarify here that the strand of productive disposition was not examined in the current study. The diagnostic questionnaire/pre-test indicated that all pre-service teachers were able to solve correctly, efficiently, and appropriately the number sentence $1\frac{3}{4} \div \frac{1}{2}$ [Task A]. All 10 used the invert and multiply procedure. Six provided another way (pictorial) of solving the number sentence, and only one provided three different ways of solving. This indicated that the flexibility aspect of procedural fluency was not well developed in all the participants. The participants also appeared to have well developed the strand of strategic competence; 6 out of the 10 participants formulated (in Task B) a problem representing the measurement model of division whereas 1 prospective teacher developed a problem representing the partitive model. However, the dominancy of the "invert and multiply" procedure (exhibited in 10 out of the 17 correct responses) in Task A, coupled with the participants' inability to provide an explanation in Task C as to why this procedure works, revealed the weak status of the strands of conceptual understanding and adaptive reasoning.

The analysis of the post-test data showed a development of the participants' ability to represent division by fraction in multiple ways. There was an increase from 17 to 25 total correct responses to this task. The majority (8 out of 10 prospective teachers in contrast with 6 during the pre-test) was able to provide two to four representations for solving the number sentence $1\frac{3}{4} \div \frac{1}{2}$ [Task A]. It is interesting to note that among the 25 successful responses [Task A], 11 matched two of the new ways learned during the teaching intervention. Apart from the above, the impact of the intervention is evident if one considers the decrease of the standard "invert and multiply" algorithm from 10 out of 17 correct solutions in the pre-test to 10 out of 25 during the post-test.

The analysis of the post-test data also revealed development in the strand of adaptive reasoning. While no one provided a successful response to Task C during the pre-test, after the teaching intervention 6 out of the 10 participants provided an effective explanation. All six explanations were modeled after two of the explanations learned during the teaching intervention. These findings reveal the participants' capacity to think logically about the rationale underlying the standard division by fraction algorithm, and also suggest growth in their conceptual understanding.

In sum, this short teaching intervention, which focused on the generation of multiple strategies for solving a division of fraction problem, led increased teachers' subsequent use of multiple strategies, as well as an increased ability to provide explanations for why strategies worked.

Conclusions and Implications

"*The development of mathematical proficiency requires thoughtful planning, careful execution and continual improvement of instruction…*" (NRC, 2001, p. 424). This study represents an initial, small-scale attempt to promote the development of prospective elementary teachers' proficiency in division by fraction. Without underestimating the difficulty of the topic of division by fraction, the analysis of the present study's data has important implications for teacher education programs, particularly in terms of addressing deficits in undergraduate students' ability to understand and reason in mathematics in ways that is useful for teaching.

Preservice teachers need time to experience as learners activities like the ones described in this paper in order to be able to design mathematics lessons that promote more than procedural fluency. Teaching students traditional algorithms like the "invert and multiply", while not necessarily inappropriate, requires careful thought about students' potential for understanding. One consequence of teaching such computational formulas without a connection to understanding is that students learn to see mathematics as an arbitrary set of rules and see the teacher and textbook-taught algorithms as having more authority than their own fraction sense and reasoning abilities. Prospective teachers need to acknowledge alternative solutions to problems; otherwise, "students' reasoning may be undervalued or, more seriously, be declared incorrect if valid or correct when invalid" (Graeber, 1999, p. 203).

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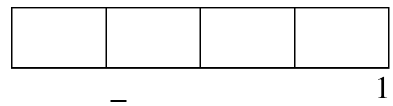
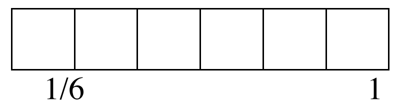
Appendices

Let's suppose we have to figure out the result of $\frac{3}{4} \div \frac{1}{6}$

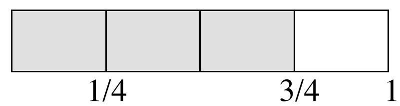
If we have a fraction strip representing fourths:

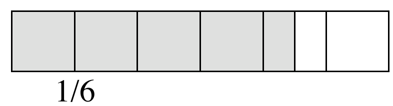


and one fraction strip representing sixths:



we can ask the question how many times does 1/6 fit into 3/4. Visually



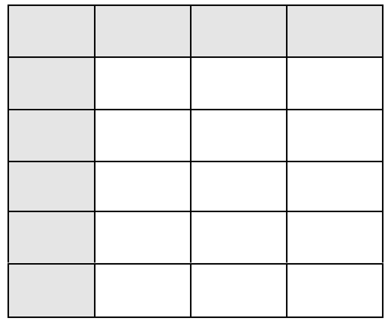


we can see that one sixth fits into three fourths $4\frac{1}{2}$ times.

Figure 1: Division by fraction using fraction strips

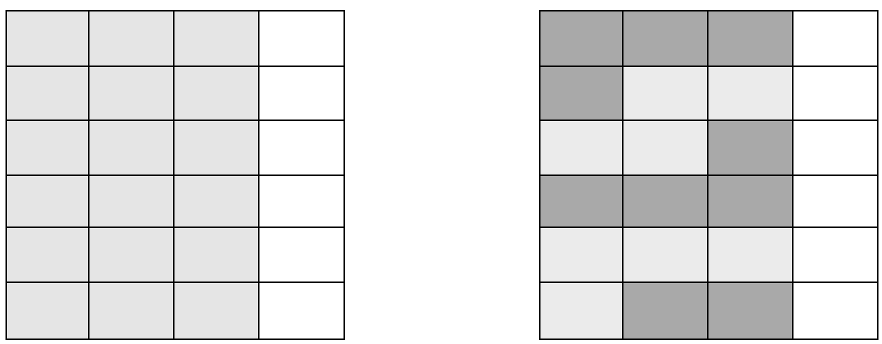
Let's suppose we have to find the result of $\frac{3}{4} \div \frac{1}{6}$

First we draw a square and divide the one side into fourths and the other one into sixths:



From the above drawing we can see that $\frac{1}{6}$ corresponds to 4 small pieces out of 24.

Then, we shade $\frac{3}{4}$ of the whole square, that is, 18 small pieces. Our question is how many times does $\frac{1}{6}$ (or 4 pieces) fit into $\frac{3}{4}$ (or 18 pieces).



In this way we can see that $\frac{1}{6}$ (or 4 pieces) fits into $\frac{3}{4}$ (or 18 pieces), 4 times and there are 2 pieces left, that is, $4\frac{2}{4}$.

Figure 2: Division by fraction using the area model

Let's suppose we have to figure out what is the result of $\frac{3}{4} \div \frac{5}{6}$

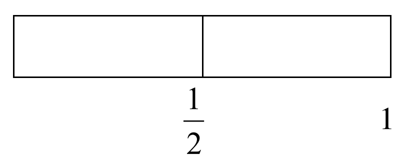
If we convert the fractions so that they have common denominators then:

$$\frac{3X6}{4X6} \div \frac{5X4}{4X6} = \frac{(3X6) \div (5X4)}{(4X6) \div (4X6)} = \frac{(3X6) \div (5X4)}{1} = \frac{3X6}{5X4} = \frac{3}{4} X \frac{6}{5}$$

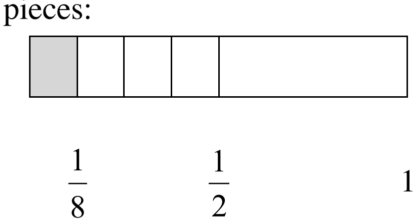
This shows that dividing $\frac{3}{4}$ by $\frac{5}{6}$ is basically the same as multiplying $\frac{3}{4}$ by the reciprocal of $\frac{5}{6}$

Figure 3: Explanation of the "invert and multiply" using common denominators

[a] To find $\frac{1}{2} \div 4$ visually, we draw one whole divided into halves:

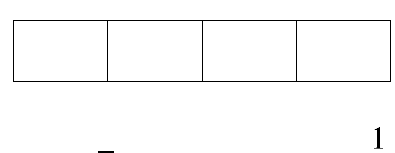


Then, we divide one half into 4 pieces:

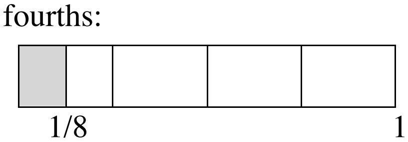


The shaded piece is $\frac{1}{8}$ of the whole. Therefore, $\frac{1}{2} \div 4 = \frac{1}{8}$

[b] To find $\frac{1}{2}$ x $\frac{1}{4}$ (one half of one fourth) visually, we draw a whole divided into fourths:



Then we draw half of one of the fourths:



The shaded piece is $\frac{1}{8}$ of the whole. Therefore, $\frac{1}{2}$ x $\frac{1}{4} = \frac{1}{8}$

From [a] and [b] we can conclude that: $\frac{1}{2} \div 4$ is the same as $\frac{1}{2}$ x $\frac{1}{4}$

Figure 4: Explanation of the "invert and multiply" algorithm using the area model

USING A SET OF FRAMEWORKS IN DESIGNING A COMPUTER ALGEBRA LEARNING ENVIRONMENT DEDICATED TO FUNCTIONS

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The paper starts from classroom situations where students learn about functions and algebra by experimenting and conjecturing using technological tools. Four theoretical frameworks help to consider the limitation of these situations in practices based on the use of non-symbolic software like dynamic geometry and spreadsheets. The paper focuses then on potentialities of classroom use of computer algebra packages that could help to go beyond this shortcoming. It looks at a contradiction: while symbolic calculation is a basic tool for mathematicians, curricula and teachers are very cautious regarding their use by students. The rest of the paper considers the design and experiment of a computer algebra environment Casyopée as means to contribute to an evolution of classroom practices.

Introduction

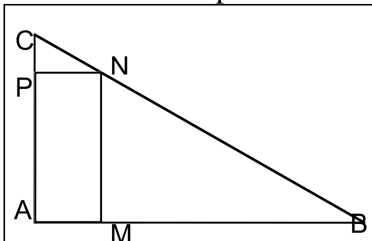
This paper considers classroom situations where students learn algebra by modeling a functional relationship and experimenting with help of technological tools. Recent curricula stress their potential contribution to students' learning and attitudes. For instance this is an extract from the French curriculum for upper level (Second, 10^{th} grade).

> *By solving problems, modeling situations and progressively learning to prove, students can begin to understand the nature of mathematical activity: identifying a problem, experimenting on examples, conjecturing a property, building an argumentation, writing out a solution, verifying the properties and evaluating their relevance regarding the problem. It is possible to study geometrical situations, the independent variable being a length and the dependant variable an area. The problem is then often to look for a maximum, a minimum or simply a value.*

According to the curriculum, technology is able to support this approach.

> *Computer tools can help a quasi-experimental approach.... It favours students' more active attitude and commitment. Possibilities for observing and manipulating are much wider.*

This is an example of a text of problem for students:



ABC is right-angled in A. AB=5. AC=3. AMNP is a rectangle, M, N and P being respectively on [BA], [BC] and [AC].

1. What values takes the distance AM?
2. Draw a table of values of the area of the rectangle for several values of the distance AM.
3. How does this area vary?
4. Is there a value of AM giving a maximum area?
5. Can you prove it?

Figure 1 displays screenshots from three software applications often proposed for students' experimentation. With dynamic geometry, a student is able to draw a figure, to animate it by dragging point M, to obtain values dynamically, and to draw a graph. With a spreadsheet, (s)he can get a table by entering a list of value for AM, then a formula for the area and copying this

Lloyd, G. M., Wilson, M., Wilkins, J. L. M., & Behm, S. L. (Eds.). (2005).
Proceedings of the 27th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.

formula in cells. A Computer Algebra System (CAS) can help him (her) to obtain expressions of the difference of a conjectured maximum area and the current area, especially for the proof.

How effective are these situations to learn algebra? This question is now raised in many classrooms, as, urged by social demand, curricula recommend the use of technology and stress its interest for experimentation or exploration. The approach of this paper will be first to discuss theses situations using a set of frameworks. Then, taking into account the limitations of existing software it will consider the Casyopée project that a team including the author is developing.

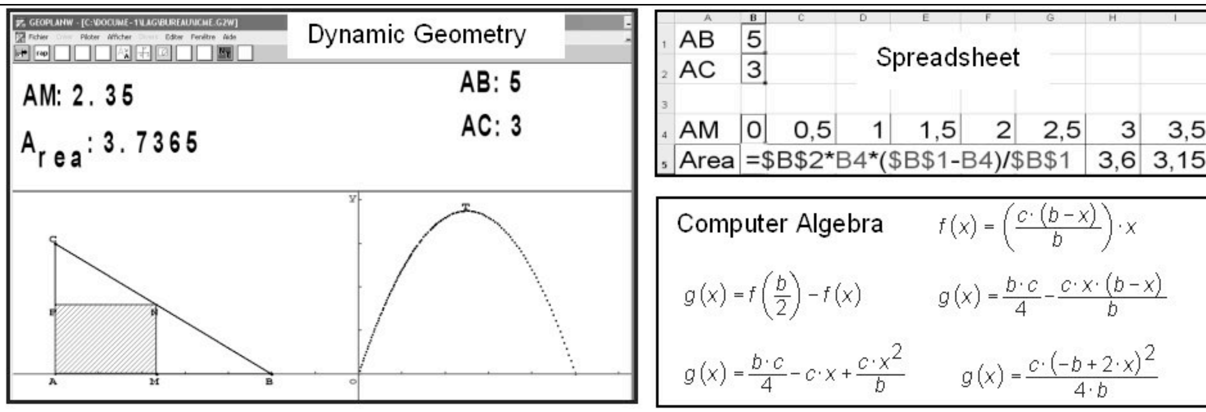


Figure 1: three technological aids for experimenting

Frameworks

Four theoretical frameworks help to think about this question[1]. First, experimental activity is not an end in itself and thus an epistemological study of the knowledge at stake should not be missed. It is not a so obvious issue, since curricula advocate experimentation and use of technology for general reasons –better pedagogy, imitation of mathematical research methods- rather than refer to specific contributions to knowledge. I will use Kieran's (2004, p.24) categories of algebraic activity –generational, meta-level, transformational- related to functions as a didactical and epistemological framework.

- *The* generational activities *of algebra involve the forming of the expressions and equations…*
- *The* transformational (rule-based) *activities includes, for instance, collecting like terms, factoring, expanding, substituting, working with equivalent expressions and equations (…)*
- The global / meta-level mathematical activities *include problem solving, modeling, noting structure, studying change, justifying, proving, and predicting (… they) cannot be separated from the other activities (…) otherwise the algebraic purpose is lost.*

Beyond epistemology, a key issue is how genuine experimental activity can exist in classroom. In mathematical research, experimentation and theorization are interweaved processes. Constraints of teaching/learning make this interweaving more difficult, and the result

[1] Tackling the complexity of classroom use of technology through complementary frameworks is a basic choice, but difficult to explain in seven pages. Lagrange (to appear) develops more these frameworks, especially the "anthropological approach", and their specific contribution to the issue of classroom computer aided experimental approaches in algebra and functions.

is often a ‘poor induction’: some experimentation is done as an entry to a new subject and then a theory is presented to the students as a model of the experimental situation. Technology is often presented as means to overcome this difficulty, because it should lighten the experimentation, easily providing data for observation and interpretation, but, in this domain, hopes were often deceived. This issue is "anthropological", in the sense of Chevallard (1999) since it involves transposition from mathematical research to schools and viability of approaches to knowledge.

Observing classroom use of CAS in the years nineties, we noticed that situations prepared by teachers did not succeed because they did not consider the students' relationship with the tool and its evolution. The instrumental approach (Guin, Touche 1998) was built to give account of the interweaved processes of learning mathematics and appropriating the tool. Experimenting with a tool, these processes certainly have important effects because gathering and interpreting data depends on the user's knowledge of the tool.

Software design is a dimension important to consider. Yerushalmy (1999, p. 169) reminds us that technological tool designers clearly worked towards providing means for a better approach of experimentation (or exploration) as a support for students’ conceptualization when they initiated ‘*this revolutionary view of software that meant to introduce new ways of building and reflecting on knowledge (that) was the essence of micro worlds…*’ She nevertheless stresses that, beyond the stimulating prospects that such new design offers, its insertion into a curriculum remains problematic. In her conclusion (ibid, p. 183-185) she notes that “*there is encouraging evidence about the impact of various specific software capabilities on exploration (and also) discouraging evidence about work with educational software that does not always act as the idea generator it was designed to be*”. What is at stake is the visibility of educational intentions in software design. Designers should “*realize and articulate their perhaps unconscious decisions and turn them into conscious design considerations*” and help “*teachers focus on the finer properties and messages of the tools they use in the classroom*”.

Interest and Limitations of Dynamic Geometry and Spreadsheet

In many curricula, dynamic geometry and spreadsheet are favored. The French curriculum stresses their interest while mentioning CAS only to warn teachers against misuses of calculators by students. In UK, CAS calculators are not allowed in A-level examinations and they are rarely used in teaching and learning. In contrast, using Dynamic Geometry or spreadsheet for the type of problem considered above is not uncommon in classrooms. By dragging the point M, (figure 1) students can look at the dependency of the area and the length. By defining a point whose coordinates are the length and the area and by tracing its position then can build a graph. Spreadsheets operate at a more symbolic level and a more general approach is possible. For instance in figure 1 the sides of the triangle are stored in cells and the formula refers to them by means of absolute references.

With reference to Kieran’s categories, dynamic geometry and spreadsheet contribute to the generational activities. The dynamic character of these applications helps to understand the dependency between position and area. At the beginning of the situation, using dynamic geometry, it is a pure geometrical dependency. By displaying numerical values of the dependant variable, dynamic geometry helps to pass from a geometric to a numerical relationship. ‘Tracing’ these values helps to express this relationship into the graphic register. Provided that students could master the specific formalism, using a spreadsheet could help to see the relevance of a *formula* to make the relationship works without a figure.

Global/meta activities are limited by the fixed numerical values of the lengths of the triangle's sides. The position of M for the maximum area at the middle of the sides does not appear as a general property. Experimentation, conjecture and -possibly- conceptualization occurs in the limited context of a fixed size triangle. Using a spreadsheet, the relative and absolute references in the formula provide means to express a family of functions and thus contribute to global/meta activities. The drawback is that these means are far from the variables and parameters used in everyday mathematics. Difficulties then result from a distance between objects in the experimental activity and 'ordinary' mathematical objects. To students, instrumented gestures and techniques evoke objects existing mainly in the 'enactive' interaction with the computer.

If technology is limited to dynamic geometry or spreadsheet, students have to use paper and pencil to carry on transformational activities for proving and there is no technology aided experimentation or problem solving in these activities. Students' difficulties for understanding and mastering the power of transformational activities and formalism are well known in the paper/pencil context. In is also difficult for teaching to design efficient learning situations. The result is a wide gap between 'enactive' functional activities and a real algebraic treatment. Dynamic geometry and spreadsheet practices cannot bridge this gap, because in these applications formula exist to be numerically evaluated and not to be transformed.

Given suitable conditions (preparation of the students in the use of software, students' previous knowledge…), dynamic geometry and spreadsheet certainly encourages an active approach of the problem and help to focus on functional properties (dependence, growth and decay, graphs and even formula…) But in my meaning, a real algebraic experimentation should go beyond this 'enactive' exploration of functional properties and link it with a work on expressions, aiming 'transformational' understanding as expressed by the curriculum's objectives:

> *Students should be able to recognize the form of an expression (sum, product, square…), to recognize various forms and to choose the most relevant form for a given work.*

The Case of CAS

Computer algebra was created to help mathematicians to go beyond mere numerical experimentation. In education, these tools could be really useful to overcome students' difficulties in algebraic manipulations in the transformational activities. Mastering algebra implies abilities for anticipating the effects of transformations and appreciating the relevance of a form for a given task, perhaps more than applying formal rules. CAS potentially contributes to help students concentrate on this 'sense of transformations' because in paper/pencil transformational activities, algebraic manipulations and skills are necessary to get a given form, possibly hiding the relevance of the outcome.

CAS also helps algebraic activities at global/meta level. As compared with spreadsheet's expression of a family of functions in the 'generalized' triangle problem, CAS offers expressive means much closer to ordinary mathematical notation.

In spite of the above potentialities, in many places, like we saw above, CAS has a bad name and there is no wide classroom use. Didactical research introduced the instrumental framework to make sense of classroom use of CAS. The instrumental concern is not limited to CAS, but students' genesis of CAS seems to be particularly intricate. Actually CAS' design does not take the user's knowledge and the task into account. Using CAS, a student is confronted with multiple capabilities offering new potentialities for action, but also with difficulties to take advantage of these. (S) he needs much time and specific situations to sort out these capabilities, integrating them as instrumented gestures into techniques and link these to his(er) mathematical knowledge (Lagrange, 1999).

This is consistent with Yerusalmy's (ibid. p. 172) observation that "*the design (of CAS) serves the agenda of the tool designers –reaching a result in the smoothest possible way*" and this design contradicts with an educational agenda that should give students control over experimentation and provide for support to the organization of the curriculum. I can see the lack of enthusiasm towards CAS in various places as recognition of difficulties in CAS' classroom implementation.

The Casyopée Project

In the design of Casyopée, the four theoretical frameworks presented above are involved. One goal is to provide students a technological aid in the three categories of algebraic activity. We aim also easier instrumentation and better curricular adaptation by creating symbolic capabilities easy to connect with usual secondary mathematical practices, and to encourage methods or techniques as a way to conceptualization.

Casyopéee's organization is designed to help students to keep clear of erratic behavior by concentrating on relevant objects in problem solving, to make sense of experimentation and to develop methods. As a difference with standard CAS, which operate mainly on symbols, each object has a clear status with regard to the curriculum: real number, function, parameter… Functions are defined on IR or on reunion of intervals. While standard CAS' window is just a memory of commands and feedback, Casyopée's interface displays the objects relevant for a problem. These objects are dynamically updated like in a spreadsheet when some change is done.

We wanted to facilitate graphic and numerical experimentation like with a grapher, while encouraging transition to global/meta activities by generalization using symbolic computation. We developed a special feature -instantiating and de- instantiating parameters. Together with the dynamic organization of objects, it is an aid to students' experimentation and problem solving involving families of functions. Dynamic instantiation –named '*pilotage*'- helps to study numeric cases and deinstantiation -'*dépilotage*'- corresponds to generalization.

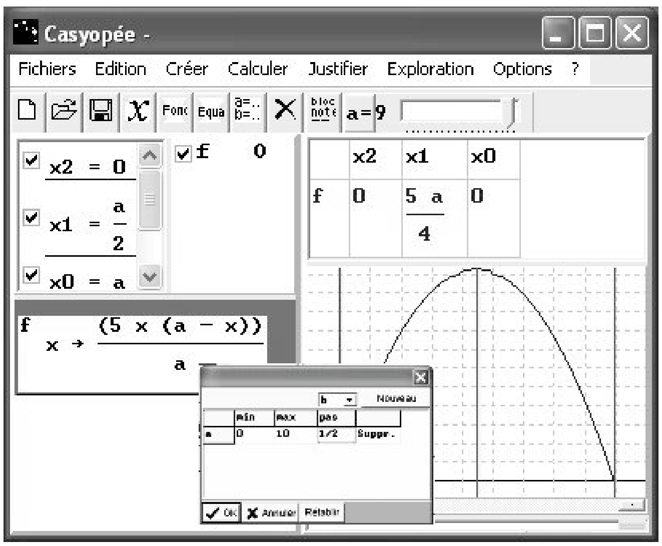
Students' development of meaningful techniques and transformational understanding implies classroom practices of experimentation in several registers (graphic, numerical, algebraic…), interweaved with algebraic reasoning and writing. Thus we wanted that students could go beyond merely reading properties on a graph or table, tackling situations with no direct numerical or graphic evidence. In these situations, experimenting should be articulated with building algebraic proofs as means to assess the validity of properties. To achieve this, in addition to providing graphs and numeric tables, Casyopée helps the student to use algebraic reasoning and to keep track of properties conjectured or proved. Theorems in secondary algebra and calculus, relative to properties of sign, variations and zeros of functions are implemented as elementary steps of proof that students can operate as consistently as possible with the pragmatic at this level. Steps of proof are based on premises found from a form of a function or already proved or conjectured.

Casyopée uses the algebraic knowledge of a CAS kernel (Mupad). The kernel is not directly visible to students. Its more obvious aid is to algebraic transformation and calculations, but it is also a support to the whole students algebraic activity, for instance solving equations or checking the definition of a function. The Mupad kernel is also much used by Casyopée in steps of proof.

An experimentation of the above problem was done in a classroom at the end of the year of Second with an early version of Casyopée. Students had worked before on problems similar to the above-discussed situation. The goal was to offer them a more general problem –with a parameter- and a more self-directed activity -no detailed guiding. We choose to have only one side of the triangle as a parameter. The general problem was presented at the beginning together with the capabilities of the environment. The worksheet asked students (Figure 2), to:

1. enter the function $x \to 5x(a-x)/a$, instantiate and animate the parameter **a**, do a graphic and numerical exploration for several values of **a** in order to conjecture the maximum,
2. **f**(α) being the maximal volume, define for each value of **a** the function $x \to f(\alpha) - f(x)$ do an algebraic study of these functions in order to find their signs and conjecture a general property regarding the maximum,
3. deinstanciate **a** and build a general proof using Casyopée.

At step 1, an interval and a step had to be chosen for the parameter in a dialog box (bottom). The graph and algebraic results were dynamically updated. The table above the graph displays exact values of the function.



At step 3, students had to build a proof of the sign of **g**. Using Casyopée, this proof in done like demonstrated below.

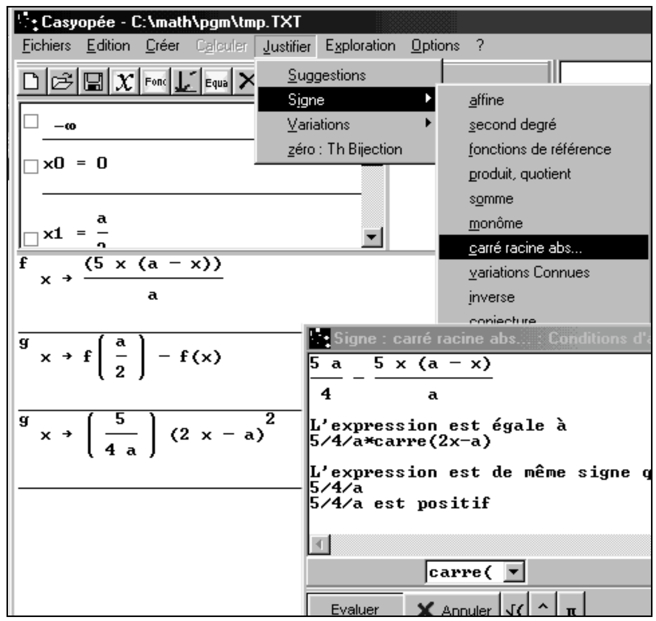


Figure 2

At step 2, students had to fill in a table whose lines were similar to the following:

a=	x in	α=	Expansion of **f(α)−f(x)**:	Factorisation of **f(α)−f(x)**:	Sign of **f(α)−f(x)**
$\frac{23}{2}$	$[0; \frac{23}{2}]$	$\frac{23}{4}$	$\frac{10x^2}{23} - 5x + 115/8$	$5/184\,(4x-23)^2$	Positive

We expected that, from this table, students could understand the connection between α value of **x** for **f(x)** maximum and the factorization of **f(α) − f(x)** as a square (times a positive constant), connect this also to a graph -a parabola tangent to the **x**-axis, and think of a proof.

At step 4, this proof could be written out like the following:

Let us define **g**: $x \to f\left(\frac{a}{2}\right) - f(x)$. *The factorized form is* $x \to \left(\frac{5}{4a}\right)(2x-a)^2$

It is the product of a positive number and a square. Thus it is positive. We proved that for all **x** *in the interval* **[0;a]** **f(a/2) ≥ f(x)** *and thus* **f(a/2)** *is the maximum volume.*

Observation and Evolution of Casyopée

The observations reported and discussed here are about students' understanding of instrumented gestures with Casyopé when working on the problem. Defining a function and its interval of definition, instantiating and de-instantiating a parameter, using commands for the values of functions and for algebraic transformations were relatively easily recognized by students as corresponding to usual mathematical gestures. Generalizing by means of deinstantiation was not a difficulty and clearly emphasized the general property and proof. Casyopée 's objects and instrumented gestures are indeed consistent with the curriculum.

Exploring efficiently the graphic window was more difficult. For instance, students confused the maximum of the function with the maximum value of the variable though they did a similar experimentation before when using other software or calculators. These instrumented gestures have no correspondence in the usual mathematics. In spite of previous practice with other software, students did not appropriate them enough to be able to do an easy transfer into a new environment.

Proving with Casyopée was supposed to encourage transformational activities and focus on methods. In the version of Casyopée developed at this time, students had to look into a menu for the relevant entry and fill two successive dialog boxes (figure 2, right). They perceived the corresponding gestures as more constraining than ordinary paper/pencil proof. Further developing the environment, we created bypasses making Casyopée do a part of the proof when requested by the student and allowed by the teacher. We also made Casyopée issue detailed messages about the proof, as close as possible to usual formulations, that students might insert more or less directly when writing out the proof. In consequence, proof is now less constraining and Casyopée offers students a valuable help in the writing of the proof.

In the maximum area rectangle problem, the geometrical framework is important for the generational activity, as a first step in modeling and for the interpretation. In the present development of Casyopée, the activity is entirely on an algebraic definition of the function. Thus, using dynamic geometry together with the environment would be valuable. To avoid the complexity of using two applications and to provide a transition from geometrical generation to algebraic modeling, we are thinking on the possibility of diversifying the way users can define a function: as an algebraic formula but also from a geometrical definition (tangent..), from a relationship between variable geometrical objects or as a regression from empirical data.

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HOW STUDENTS VIEW THE GENERAL NATURE OF THEIR ERRORS: IMPLICATIONS FOR INSTRUCTION

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This study examined how two students viewed the generality of their proportional reasoning errors as they attempted to generalize numeric situations. Using a teaching experiment methodology we studied the reasoning of two students over 18 instructional sessions. One student, Dallas, appeared to recognize that the proportional reasoning error applied to all cases of a particular problem situation and began to apply this reasoning across problems. The other student, Lloyd, exhibited difficulty seeing the generality of his mistaken use of proportional reasoning and regularly repeated this error during the study. We encourage further discussion with students about the generality of their errors at the problem level and across problems.

A marked shift has occurred in the way student errors are to be viewed by mathematics teachers. Early mathematics education researchers (e.g., Buswell & Judd, 1925) were cognizant of various student mathematical errors. As such, teachers were expected to recognize and attempt to eliminate student errors to reduce student confusion. As stated by Kerr and Lester (1976), the teacher "must learn to anticipate places where students may have trouble . . . Active involvement may be the easiest way of learning mathematics, but it increases the likelihood of floundering, misdirection, or misconception by the student" (p. 115). In contrast, mathematics educators in the late 1980s and 1990s, aided by an alternative perspective of how students learn, began to view errors differently—noting that student examination of errors could build understanding. Recommendations were made to use errors as "springboards" to deepen student understanding (Borasi, 1987) so they can understand the concepts underlying their errors (Hiebert et al. 1997).

However, most teachers are unfamiliar with how to use errors as springboards for discussion and little research exists about how students view and reconcile the errors they make. In our work, we have examined the challenges that students face as they recognize errors, attribute errors to various sources, and reconcile errors (Lannin, Townsend, & Barker, 2005). As we continued to analyze student errors, another important dimension emerged—student views of the generality of their errors. Here we describe our current theoretical framework for this dimension and provide examples of student thinking in relation to the whole-object strategy (Stacey, 1989), a mistaken application of proportional reasoning. Thus, our research question is: How do students view the generality of the errors they identify?

Theoretical Perspectives

The view of errors as "sites for learning" is essential to the classroom that builds on student sense-making (Hiebert et al., 1997). As students engage in problem solving—solving tasks for which they have no previous solution strategy (NCTM, 2000)—errors naturally occur. We view errors as Hiebert et al. do, that they are part of the process of "improving methods of solution . . . moving from methods that do not work as well, and may even be flawed, towards methods that

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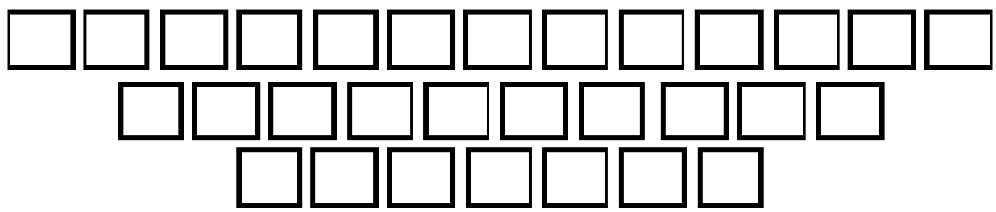
work better. Making mistakes is a natural part of the process; it even may be essential sometimes” (p. 48).

A common error that occurs as students attempt to construct generalizations is an incorrect application of proportional reasoning (Stacey, 1989; Swafford & Langrall, 2000). An example of this could occur as a student attempts to determine the number of seats in the tenth row for the Theater Seats Problem (see Figure 1). The student could use the fact that 19 seats are needed for the fifth row, doubling this amount to find the number of seats in the tenth row. In this case the strategy generates the incorrect number of seats for the tenth row.

In a theater there are 7 seats in the first row. The increase in the number of seats is the same from row to row. Below is a diagram of the first three rows in the theater.



How many seats are there in the 4th row of the theater? In the 5th row? In the 10th row? In the 23rd row? In the 38th row? In the 138th row of the theater? Write a rule that would allow you to calculate the number of seats in any row.

Figure 1. Theater Seats Problem

We use the mistaken application of proportional reasoning to illustrate the generality of errors that students appear to recognize. This strategy error could be applied to various types of linear and non-linear situations. Research (De Bock, Verschaffel, & Janssens, 2002) has demonstrated that this error is quite resilient.

Method, Data Sources, and Analysis

Eight students, varying in ability level, were purposefully selected from the fifth grade population of 80 students in a Midwestern elementary school. Students were paired in two high/medium and two low/medium groups throughout 18 instructional sessions, occurring over four months. The study utilized the teaching experiment methodology (Steffe & Thompson, 2000) in which a pair of students explored and attempted to generalize numeric tasks to test hypotheses about “students’ unanticipated ways and means of operating as well as their unexpected mistakes” (p. 277). The first author served the role of teacher and was assisted by the co-authors, who provided alternative perspectives on the students’ thinking. The students used a variety of strategies and were consistently prompted to explain their thinking. Students were encouraged to reflect on the strategies they utilized and to examine any inconsistencies that occurred when they shared their results. Each episode was captured on video with a separate camera focused on each participant. Further evidence included the researchers’ field notes, students’ written work, and video screen-capture of student’ computer spreadsheets.

Analysis of the data employed elements of grounded theory (Strauss & Corbin, 1998), allowing salient categories to emerge. We first identified all instances where proportional reasoning was used. Then we used constant-comparative analysis to develop, test, and revise categories related to students' views of the generalizability of their errors. This process led to four levels that we discuss further in the following section.

Data Interpretations

From the data we constructed a framework to describe the different dimensions for which students view the generality of their errors. This framework is composed of four levels:

Not An Error: The student does not see the strategy as an error.

Local-Level Errors: The student views the error as applicable only to a particular instance of the situation (e.g., the student views the error as applicable for a jump from $n = 4$ to $n = 8$, but not for a jump from $n = 5$ to $n = 10$).

Problem-Level Errors: The student views the error as incorrect for any application in a particular problem situation (e.g., the student recognizes that the use of proportional reasoning as an error for all cases within a particular problem situation).

Cross-Problem Level Errors: The student views the error as incorrect for a particular class of problems (e.g., the student recognizes that proportional reasoning will produce incorrect results for any linear situation that is not a direct variation situation).

We found that analyzing the error levels along a continuum (see Figure 2) was more productive than viewing them as discrete points. For example, some students appeared to recognize that erroneous proportional reasoning occurred for more than one particular case, but it was unclear whether they recognized that the error applied to all cases in a particular problem situation. In such situations we placed the students' reasoning along the continuum of local-level to problem-level. In addition, we found that some students recognized erroneous proportional reasoning in various problems, but these students were unable to clearly define the problem situations for which this error occurred. At this point the students' reasoning was placed between the problem-level and cross-problem level.

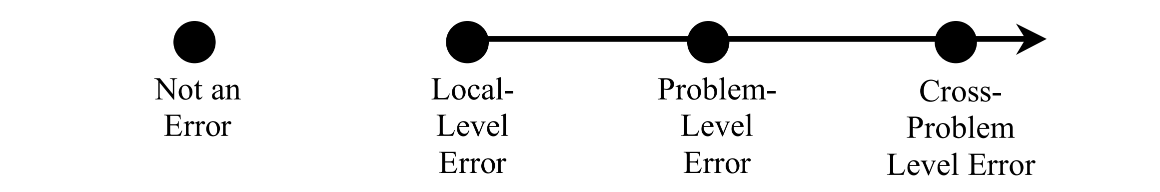


Figure 2. The Error Continuum

In the following paragraphs we describe the views of proportional reasoning demonstrated by Lloyd and Dallas for three instructional sessions. We detail the changes that they appeared to make in the way they viewed their use of proportional reasoning.

Session #6

In the Theater Seats Problem (Figure 1), both students correctly determined the number of seats in the fourth row (16) by adding 3 to the number of seats in the third row (13). When calculating the number of seats in the tenth row, Dallas incorrectly used proportional reasoning, doubling the correct result for row five (19 seats) to obtain 38 seats in the tenth row. Lloyd used a recursive strategy, adding 3 repeatedly to arrive at 31 seats, but left off a group of three seats. A dialogue began about their strategies for finding the number of seats in the tenth row.

Dallas: I wasn't going to keep adding all the way up to get to the 10th row. But I knew that 5 times 2 was 10, so I just multiplied 19 by 2, and got 38.
Teacher: Lloyd, how many seats do you think there are for the tenth row?
Lloyd: I think there are 31.
Dallas: I got 38. Nineteen times 2 is 38.
Lloyd: So how did you get the 19 times 2?
Dallas: Because in the fifth row there were 19. The fourth row had 16 (seats), so I added 3 again to get 19. Then I timesed that by 2 to get 38.
Lloyd: I entered 16 and pushed plus 3 once. [Displaying 19 on his calculator.]
Teacher: That's how many (seats) there are in . . .
Lloyd: In the fifth row. And I pushed "plus 3" six more times (starting from the fourth row). Cause I knew 6 plus 4 was 10, so I just (added 3) six more times. So the answer is 31.
Teacher: So we disagree on the answer for the tenth row.
Dallas: I think we're both wrong. I timesed my number by 2, but I have to be adding 3 every time. I don't think I could just times it by 2.

Dallas explained that he could take 19 and add 15 (3 times 5) because each row added 3 and there were five more rows. Meanwhile, Lloyd returned to using his calculator, carefully counting the number of threes on his fingers so that that he added six threes to 16 (the number in the fourth row), arriving at 34 seats. Both now agreed that 34 seats existed in the tenth row.

Dallas appeared to see that proportional reasoning did not provide the correct number of seats when moving from the tenth to the 20th row (Local-Level Error). At this point, he did not deeply explore why proportional reasoning would be incorrect and could have believed that the proportional reasoning provided incorrect values for other instances.

For the 23rd row Dallas calculated the number of seats by starting with 38 seats (the incorrect value he obtained for the tenth row) and added 39 (i.e., 13 times 3) for the extra 13 rows that were added to the tenth row, arriving at a value of 77. Lloyd started with 34 seats in the 10th row and doubled that value to arrive at 68 seats in the 20th row. For the remaining three rows, Lloyd added an additional nine seats for the remaining three rows with three additional seats each. Neither student appeared to question the other's strategy as both obtained the same, though incorrect, value of 77 seats in the 23rd row. When questioned about Dallas' previous use of proportional reasoning (row five to row ten), Lloyd said, "maybe you (Dallas) did it wrong." When calculating the number of seats for the 23rd row Lloyd appeared to believe that the error committed by Dallas was not a strategy error, but was a mistake with his calculations.

To determine the number of seats in the 50th row, Dallas returned to using proportional reasoning, doubling the number of seats in the 23rd row (77) in an attempt to obtain the number of seats in row 46. He then added 12 more seats for the four additional rows, arriving at 166 seats for the 50th row. Lloyd multiplied the number of seats in the tenth row (34) by 5 to obtain the result of 170 seats for the 50th row. When discussing their results, Dallas stated that both strategies should be correct, but was unsure why his answer differed from Lloyd's by four seats. At this point, both students appeared to believe that proportional reasoning was either not an error or was a local-level error that occurred for a previous particular instance.

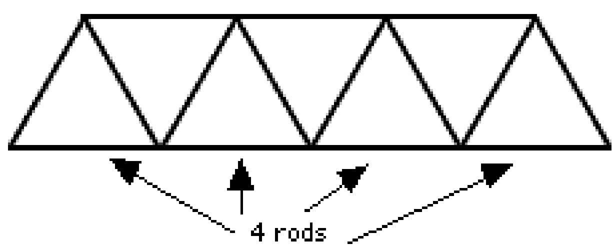
To continue encouraging Lloyd and Dallas to think further about their strategies, they were asked to explain how to find the number of seats in the 10th row using only addition. Dallas stated that the number in the tenth row could be calculated by adding nine 3s to the number of seats in the first row, resulting in 34 seats. Both students agreed that this would provide the correct value for the tenth row. When asked to determine the number of seats in row 20, Lloyd returned to using proportional reasoning, doubling the number of seats in row ten (34) to arrive at 68 seats for row 20. When questioned how to find the number of seats in row 20 starting with the first row, Lloyd said that he could add 19 more threes to the seven seats in the first row. After computing 7 + 3*19, Lloyd and Dallas found that there were 64 seats in the 20^{th} row. Lloyd stated that the answer must be 64, but was unsure why doubling did not provide the same result. He grabbed his calculator and added 3 repeatedly onto seven, again arriving at 64. At this point both students abandoned the use of proportional reasoning, but appeared unsure why this strategy did not provide correct answers for these instances. During the remainder of the class period Lloyd and Dallas used recursive or explicit rules.

At the end of the session Lloyd and Dallas appeared uncertain about their use of proportional reasoning. They either believed that proportional reasoning provided incorrect values for every instance of the problem (problem-level error) or that it created enough errors that it could be not reliably used in this problem (local level to problem-level error).

Session #8

Lloyd and Dallas began working on the Beam Design Problem (see Figure 3) by counting the number of rods for a beam of length four. Dallas then developed an explicit rule that involved multiplying the length of the beam by 4 and subtracting one rod. For a length-five beam Dallas used his explicit rule whereas Lloyd continued to draw and count the number of rods. Lloyd changed strategies for a length-ten beam employing a recursive strategy; he repeatedly added 4 to obtain the correct result of 39 rods. Below is an excerpt of their discussion as they attempted to find the number of rods for a length-20 beam.

Beams are designed as a support for various bridges. The number of rods used to build the beam bottom determines the beam length. Below is a length-4 beam.



How many rods are needed to make a beam of length 5? Of length 8? Of length 10? Of length 20? Of length 34? Of length 76? Of length 223? Write a rule for how you could find the number of rods needed to make a beam of any length.

Figure 3. The Beam Problem

Teacher: What would you do for a beam of length 20?

Lloyd: I think you just do 39 plus 39, see if that might do it. (Both beams) are (length ten), and I know 10 plus 10 is 20.

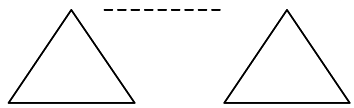
Teacher: Can you explain why you would take 39 plus 39 in terms of the beam?

Lloyd: Because it might be a quicker way.

Dallas: I think you would have to add one more on the top to go with the one (part of the beam) before it. I think you have to add one more after you times it.

Teacher: What do you mean you have to add one?

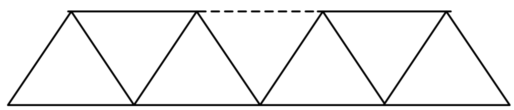
Dallas: Say you (want to make) a rod length of two; you have to add this one (a connector rod). I think you have to add one after you times it.



Teacher: Lloyd, what do you think about that?

Lloyd: I don't understand what he is saying.

[Dallas draws two beams of length-two and shows that connecting the two would require adding an additional rod between the length-two sections.]



Teacher: So you think it would be incorrect to just take 39 plus 39?

Dallas: Yes, because you would have to add one.

Teacher: Lloyd, what do you think?

Lloyd: I agree now.

Note that Dallas discussed the case of doubling a length-ten beam by providing examples of doubling a length-one beam and a length-two beam. Dallas appeared to view the error as a problem-level error, implying that you can examine any "doubling" case for this situation and the same error will occur. He applied similar reasoning when determining the number of rods for a length-40 beam, combining two length-20 pieces and adding one (i.e., 79+79+1).

When asked about a length-37 beam, Lloyd multiplied 39 by 3 and added 2 to connect the three groups, stating that he needed to count how many rods were required for a beam of length seven and add that on as well. Lloyd's thought process appears to illustrate that he moved to thinking about his use of proportional reasoning at the problem-level. However, he exhibited difficulty applying his thinking to a length-37 beam as he left off a beam tying the three length-ten beams to the length-seven beam. The examination of other cases provoked Lloyd to deal with erroneous proportional reasoning at the problem level though he was unable to successfully adjust his strategy.

During session #18, Lloyd returned to using the proportional reasoning for the Straw Problem (a problem similar in structure to the Beam problem in which students determined the number of straws needed to construct any number of joined squares). He applied this strategy mistakenly to situations involving 20 squares and 245 squares. Only when Lloyd was asked to draw a picture and count the number of straws for a 20-square figure did he begin to recognize that his use of proportional reasoning was incorrect. From his drawing of the 20-square case Lloyd noted that one straw must be subtracted to produce the correct result. However, he was unable to explain why he subtracting one was necessary when using the proportional strategy. Later in this session he was provided a problem for which he could appropriately apply proportional reasoning. However, he continued using the doubling and subtracting one strategy. Lloyd appeared to move from not viewing proportional reasoning as an error, to seeing it as a local-level error. He then mistakenly applied his calculational adjustment to this new problem situation, suggesting that he looked at the error as a cross-problem level error by applying an adjustment to his calculations rather than considering the underlying reasons for his error.

During Session #18 the teacher showed Dallas a fictitious student's thinking for the Straw Problem in which the student used a doubling strategy to determine the number of straws for a jump from 10 to 20. Dallas noted that this strategy provided incorrect results, stating that he needed to subtract a straw due to the overcounting that occurred when joining sections.

Discussion

Throughout the study, both students periodically used proportional reasoning in various settings. This demonstrates that both students initially believed that proportional reasoning could be applied across tasks. However, a problem arose when situations were provided that did not involve direct variation. In these situations the students needed to examine the use of proportional reasoning at the local, problem, or across problem levels. For instance, Lloyd initially used the proportional strategy incorrectly for the Beam Problem, because he was unaware that it produced incorrect results. Even when confronted with discrepant results, it appeared that he did not develop a deep understanding of why proportional reasoning would provide incorrect results. During the sessions, Lloyd at times recognized that the proportional strategy did not apply at a local level, but struggled to expand this to other instances. Due to Lloyd's lack of success with the proportional strategy beyond the local level, he continued to use this strategy repeatedly.

Dallas was able to link his reasoning to the context and representations of the problem and to the diagram to make sense of particular situations. After Dallas explained the adjustment for a particular case (local level), he tried to extend his reasoning to the problem level. Although Dallas may have initially believed that proportional reasoning could be applied across all problems, by deeply understanding his error for particular cases he was able to apply his understanding of the error to the problem and cross problem levels, changing his conception of his use of proportional reasoning.

Allowing students to make errors and learn from their errors can be a powerful instructional strategy in the mathematics classroom. However, teachers often wonder why students make the "same mistake" over and over again. It appears that students may not see their errors as "the same," instead perceiving their errors as local level errors that do not apply to other instances. Students need to be asked to consider how their errors apply to situations outside particular instances, exploring the domain for which a particular error occurs at the problem level, and moving towards the cross problem level.

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CONJECTURING AND PROVING AS PART OF THE PROCESS OF DEFINING

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Researchers and mathematicians have argued that students should be engaged in the activity of defining mathematical concepts. This report looks at the role of proving in students' defining activity. A preliminary framework is offered to account for the ways in which proving can contribute to the process of defining. Three categories of contribution (motivation, guidance, and assessment) are illustrated in the context of two classroom episodes (one from a geometry course and another from a group theory course) in which students are engaged in defining.

Introduction

Freudenthal (1973) noted that definitions are generally not preconceived but are just the finishing touches of the mathematical activity of defining. He argued that students should not be denied the opportunity participate in this activity. De Villiers (1998) also argued that students should be actively engaged in the process of defining in order to highlight the meaning of the content and to allow students to actively participate in the construction and development of the content. He further suggested it might be essential to engage students in the process of defining in order to increase their understanding of definitions and the concepts to which they relate.

De Villiers (1998) pointed out that defining is inherently a complex mathematical activity. Zandieh and Rasmussen (in preparation) take defining to include not just formulating a definition but also activities such as negotiating and revising a definition. These activities may involve generating conjectured definitions, creating examples to test the conjectures, and trying to prove whether or not a conjectured definition "works" in the sense of doing the job that the definition is being created to do. Zandieh and Rasmussen include these activities as part of what they mean by defining. They also note that the defining process includes negotiating both the way the definition should be formulated and the deeper issue of what the concept should mean.

While the research literature suggests the value of engaging students in the activity of defining, there is still much we need to know in order to support this kind of mathematical activity. De Villiers (1998), drawing on Freudenthal (1973), elaborated the activity of defining by describing the categories of constructive and descriptive defining. Descriptive defining involves singling out some properties of a well-known object while constructive defining involves creating new objects from familiar ones. Zandieh and Rasmussen (in preparation) added additional structure to the activity of defining by considering students' defining in terms of Gravemeijer's (1999) levels of mathematical activity (in the task setting, referential, general, and formal). Additionally, researchers have conducted teaching experiments in order to develop instructional sequences and instructional theories to support defining in particular content areas (e.g., De Villiers, 1998; Larsen, 2004, Rasmussen, Zandieh, King, & Teppo, 2005).

The goal of this paper is to further explore and elaborate the activity of defining by analyzing the role of proving as students develop definitions in the context of college geometry and group theory. Our focus on this aspect of defining activity is inspired by Lakatos' (1976) case studies that illustrated the historical development of mathematical concepts through a process of "proofs and refutations" – a process that featured movement back and forth between proving and

Lloyd, G. M., Wilson, M., Wilkins, J. L. M., & Behm, S. L. (Eds.). (2005).
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defining. We have observed this same phenomenon in our instructional design research at the undergraduate level. By analyzing the role of proving in students' defining activity in different content areas, we aim to develop a better theoretical understanding of the interaction of these important aspects of mathematical activity. In turn we expect this understanding to guide the development of instructional theory and practice in support of students' mathematical activity.

Research Methods

The data for this research comes from teaching experiments (Cobb, 2000) in college geometry and elementary group theory. While these teaching experiments varied in many respects, they both included the development of instructional approaches designed to support mathematical learning through the students' own mathematical activity. In particular, each of these teaching experiments involved engaging the students extensively in defining. Data consisted of videotapes of all class sessions and photocopies of the students' written work. The data analysis consisted of multiple phases of iterative analysis (Cobb & Whitenack, 1996).

In the following sections we present two episodes in which students were engaged in developing definitions for specific mathematical objects. The first episode is situated in a college geometry course. The students were engaged in defining a special class of triangles on the sphere (small triangles) for which the Side-Angle-Side (SAS) congruence theorem is valid. The second episode is situated in a group theory course. The students were engaged in developing an improved definition of subgroup. Each of these defining activities is an example of constructive defining because they involve the creation of new objects from familiar objects. Furthermore in each case there was an original definition for the students to work with as they developed the new definition. The geometry students had already developed a definition of triangle that makes sense on the sphere while the group theory students had agreed that a subgroup is a subset of a group that is also a group under the same operation. However, there were significant differences between these two defining activities as well. In the geometry episode, the students were developing a new class of spherical triangles by adding more restrictive conditions to the existing definition. The main criterion for evaluating this new definition was that SAS holds for the new class of triangles. In the group theory episode, the students were developing an alternative but equivalent definition of subgroup. The main criterion for evaluating this new definition was that it be equivalent but more efficient than the original definition.

Geometry Episode: Defining a Class of Triangles for which SAS is True

Background

The class had previously developed a definition for triangle on the surface of the sphere and had begun to explore properties of these triangles. During the class session discussed below, the students were working on proving SAS on the plane and determining whether or not SAS was true on the sphere. If SAS was not true for all triangles on the sphere (which it is not), the students were to prove that by coming up with a counterexample. The students were finally asked to find a subset of triangles on the sphere, labeled by the textbook (Henderson, 1996) as "small triangles," for which the theorem is true.

Proof as Motivation and Guidance for Defining

The students' experience the previous day with a wide variety of unexpected triangles on the sphere had given them a sense of what might be meant by a small, medium or large triangle --the small triangles being the ones that look most like planar triangles. So, student defining of "small triangle" was influenced in part by this preordained label. However, most of the student discussions of potential small triangle definitions were focused on issues that are related to proving: proving (finding a counterexample) that SAS is false in general for triangles on the sphere and proving that SAS is true for all triangles on the plane.

Early in the small group discussion the students in the group readily found a counterexample for SAS by noting that the endpoints of two given sides of a triangle could be connected on the sphere by two different line segments thus creating two non-congruent triangles with a side, included angle, and side in common.

Amy: So we've got side-angle-side. And one side would be going around there and the other one would be all the way around back like that.

Sam: But they're not congruent. So, it doesn't work. You have to limit it and say that it only works for small triangles. That you can't go all the way around the sphere.

Amy: And did we define small triangles?

Sam: No, but we probably need to.

Amy: Yeah. [pause] Small triangles is an area less than half the area of the sphere?

Note that Amy's definition will eliminate one of the two triangles in the counterexample, but it also may simply have been an attempt to limit the size or area of the small triangles. At this point the students were distracted and did not return to the issue of defining small triangle until after working on the proof that SAS is true on the plane. The proof encouraged by their textbook involved using symmetries and transformations to line up the side, angle, and side that are given to be congruent. At this point the students had to use a "fact" discussed in class that, on the plane, two points (in this case the two end points of the given congruent sides) determine one and only one line. This means there is only one segment that can be the third side of the triangle. Therefore, the triangles are congruent because one has been made to lie on top of the other.

Amy: So this proof doesn't work on a sphere?

Tom: No.

Sam: It must be because of the last part.

Amy: Right. So what do we do as far as proving it on the sphere?

Cindy: Because it doesn't work.

Amy: But it does work for small triangles right?

Jay: You've got to define small.

Sam: As long as you defined a straight line to be the shortest distance, not just any distance from B to C. That's why it falls apart in the sphere case because you can go outward from B and come inward on C. It could be the shortest distance. I mean maybe that's the definition of a small triangle is if you have points A, B, and C they're connected by a straight, shortest distance line.

Amy: Okay. Yeah. I like that too. I like that better. It's more concrete.

Sam described a way to define small triangle that has a direct connection to both eliminating the counterexample and replacing the step in the proof that is true only on the plane. The defining was motivated by the counterexample and the proof on the plane, but these proofs also directly guided the creation of Sam's definition. Note that Sam's definition is more closely related to both the counterexample and the proof on the plane than Amy's earlier definition.

Proof as a Way to Assess Defining

Immediately following the discussion above, the students began the process of assessing whether their definition would accomplish the purpose for which it had been created, i.e. assessing whether, with this definition, they could prove SAS was true for small triangles.

Amy: So how do we prove that it does work for small circles [sic] on the sphere if we can't use reflection symmetry? [pause] Rotation?

Tom: Couldn't we do it if we're working with small triangles? Can't we use reflection symmetry?

Cindy: We're gonna have to prove, like he said yesterday—We're going to have to take the image, rotate, reflect—

Jay: Doesn't this proof work for small triangles anyway?

Amy did not answer Jay directly but continued to push the group to confirm that the transformations and symmetries involved in the planar proof would work on the sphere as well. Then the teacher moved the class back to whole class discussion before the small group reached the point in the proof for which the small triangle definition is needed. Even though Sam had mentioned in the previous episode that the "last part" of the planar proof was probably the part that failed on the sphere, Amy made a point to assess whether each part of the proof would work on the sphere using the new small triangle definition.

Episode Wrap-up

On the next day of class a number of possible definitions for small triangle were discussed with respect to whether they made the SAS congruence theorem true. As the course continued, these discussions extended to whether or not the various definitions were equivalent and whether or not each would allow for the Angle-Side-Angle theorem to be true.

Group Theory Episode: Developing a More Useful Definition of Subgroup

Background

The class had previously developed a definition for group and agreed (after some discussion) that a subgroup would be a subset of a group that was a group under the same operation. In the class session described in the following episode, a group of three students was engaged in the process of developing a more useful definition for subgroup. *Proof as Motivation for Defining*

As the defining process began, the students had a working definition of subgroup as a subset that is a group under the same operation. Although this definition is quite natural, it is somewhat inconvenient to use in practice because in order to prove a subset is a subgroup it is necessary to verify that the subset satisfies all aspects of the group definition (closure, associativity, identity, inverses). Thus the motivation for developing an improved definition was the desire for a definition that made it easier to prove that a subset is a subgroup. The goal for the students was to determine the smallest number of properties that would need to be verified in order to determine whether a subset was a subgroup. As it turns out, it is only necessary to check that a subset is 1) non-empty, 2) closed under multiplication, and 3) includes the inverse of each of its elements. So, it is not necessary to verify associativity or the existence of an identity element.

The process began with Steve making a conjecture that it was only necessary to check that a subset is closed under the operation in order to show that it is a subgroup.

Phil: Closure, and after closure…

Steve: I think it's just closure.

Mike: You only need to check closure as long as you know it's a subset of a group.

Proof as a Way to Assess Defining

After a conjectured definition had be proposed, the students were able to assess it by attempting to prove that it was equivalent to the original definition. In the following excerpt, Phil outlines a proof that closure is sufficient to show a subset is a subgroup. Note that although Phil's attempt to take care of the infinite case was unsuccessful, the remainder of his argument works in the finite case. So, while this proof attempt verified that it is sufficient to check closure in the finite case, it did not allow the students to successfully evaluate the conjecture that closure is sufficient in general, because the students did not notice the subtle flaw in the proof. (While it is true that even in the infinite case each element appears exactly once in each row and column of a group's operation table, this property is not necessarily inherited by closed subsets.) However, the teacher was able to offer a counterexample to more fully assess this conjecture.

Phil: Closure means each element appears exactly once.

Mike: Closure says each element appears exactly once? Exactly once in what?

Phil: If something is closed and you have a finite set then basically every element of that set is in the row.

Mike: Not necessarily.

Teacher: So what if it's not a finite group?

Phil: Well, we already proved it for the infinite case that each element will appear exactly once in each row and column. So if we know it's going to appear exactly once in each row or column then we can make x the arbitrary element which means a times the arbitrary element still guarantees that a is somewhere in there. So if we solve for x then x would have to be I. And then if we know I is in the group then we can basically say a times some arbitrary element will still give me I in the group, and then if you solve for x…

Teacher: So what are you even trying to define or prove here?

Phil: I'm trying to prove you only need closure.

Teacher: So consider the following example: real numbers under addition. Is that a group?

Phil: Yeah.

Teacher: Now, consider the following subset: positive numbers under addition.

Proof as a Guide for Defining

The counterexample offered by the teacher was also useful as a guide for the ongoing development of the definition. In the discussion show below, the students analyzed this counterexample with the goal of improving their definition. The result was a new conjectured definition: a subgroup is a closed subset of a group that contains the inverse of each of its elements.

Phil: I forgot to say it has to have the same group operation.

Teacher: I didn't change the operation.

Mike: It's not closed.

Teacher: Are you sure?

Phil: Not a subgroup because don't have inverses.

Teacher: You didn't say I had to have inverses. You said I only had to be closed.

Steve: He's right.

Phil: Trying to think of a way around it.

Steve: So it's inverses and closure.

Episode Wrap-up

The discussion continued for a while longer. A number of conjectured definitions were offered up by the students and then rejected as they developed counterexamples. Eventually Phil offered a further refinement of Steve's original conjecture that closure would be sufficient, and Mike offered an improved version of the desired alternative definition of subgroup.

Phil's new conjecture: "If you're talking about an infinite group you were talking about finite groups before so maybe there's a couple different cases. If it's finite then you only need closure."

Mike's new conjecture: "But you can get identity from the inverse law I think. The inverse law says that for all a in S there exits an inverse in S such that it makes I, and that also has to be the inverse…and from that you can derive that $aI = a$ from this."

Following this small group discussion Mike's conjecture was discussed during a whole class discussion. The condition that the subset be non-empty was added in order to make a proof (that the new definition was equivalent to the original definition) work. Finally a subgroup was defined as a nonempty subset of a group having the properties that 1) it is closed under the operation and 2) the inverse of every element of the subset is also in the subset. Later in the course the students were asked to prove Phil's conjecture, that in the finite case it was sufficient to check closure, using this definition.

Conclusions

In the two episodes, the students' proving activity contributed in a variety of ways to their defining activity. As an initial step toward developing a framework for making sense of the role of proving in students' defining activity, we categorize some of these contributions as follows.

Proof as Motivation for Defining

Proving served a motivational role in both of the episodes. The purpose of creating a definition for small triangle is to make it possible to prove (on the sphere) a theorem that is true and extremely useful in the plane. The purpose of creating an improved definition of subgroup is to make it easier to prove that a given subset is a subgroup. From an educational perspective, the fact that proving can motivate defining should not be overlooked. This aspect of defining activity can allow students to see that "defining is more than describing, that it is a means of the deductive organization of the properties of an object." (Freudenthal, 1973, p. 417).

Proof as a Guide for Defining

Proving also served as a guide for defining in both episodes. In the geometry episode, the proof of SAS on the plane coupled with the proof (counterexample) that SAS is not true in general on the sphere suggested that a successful definition would not allow each pair of vertices to be connected by a side in two ways. In the group theory episode, the proof (counterexample) that closure was insufficient in general suggested that the existence of inverses might be particularly important. This particular contribution to students' defining activity has been particularly powerful in our developmental research. For example we have found that reflecting on the process of proving that two specific groups are isomorphic provides a great deal of guidance as students define isomorphism (Larsen, 2004; Weber & Larsen, in press).

Proof as Way to Assess Defining

Proving also provided a way for students to assess their defining as they went along and a way to evaluate their final definition. In the geometry episode, students were able to evaluate their definition by considering whether it made it possible to modify the planar proof of SAS to work on the sphere. In the group theory episode, students were able to evaluate their definitions by proving whether they were equivalent to the old definition.

Informally, these three categories of contribution could be restated as follows. The role of proof in defining is to 1) tell you what job the definition needs to do, 2) suggest what the definition ought to look like in order to do that job, and 3) to let you determine whether it actually does the job it is supposed to do.

Final Comments

The purpose of the framework of categories described in the previous section is to 1) add structure to our understanding of the process of defining by highlighting the important ways that proving can contribute to students' defining activity and 2) to inform instructional design and practice by identifying the ways that proving can be leveraged to support students' defining activity and concept development.

Finally, one of the goals of mathematics education research is to improve students' understanding of and ability to construct proofs. Much of students' difficulty with proof can be tied to their understanding and use of definitions. The research literature makes it clear that students' concept images are not well connected to definitions for concepts and that they struggle to use definitions in proofs (e.g., Edwards & Ward, 2004; Moore, 1994; Tall & Vinner, 1981). Evidence is emerging that suggests that engaging students in defining can help them overcome some of these difficulties (Weber & Larsen, in press).

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STUDENTS' REASONING WITH SMALL AND LARGE TRIALS IN PROBABILITY SIMULATIONS

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When in a technology environment, we need to consider the ways in which students create goals for their activity and reason about the usefulness of both small and large trials. In several prior and on-going studies with children ages 9-13, it has been observed that many students use similar approaches to choose the number of trials when conducting probability simulations and often make similar observations regarding the distribution of empirical results with a large number of trials. This paper is a theoretical reflection on the common themes observed across studies and students' struggle between choosing trial sizes when the goals of an activity are shifted from finding an exact empirical distribution to match an expected theoretical one to obtaining empirical results close enough to make an estimate of an unknown distribution.

When considering research on students' learning of probability, it useful to reflect on results from several studies where similar phenomena emerge from student's work with similar tools and commonly used tasks. Such cross-cutting analysis of students' work across studies may help researchers and teachers understand why students select a number of trials to use in a simulation and how they interpret their empirical results.

Conceptual Perspective

There is almost universal agreement that technology should play a predominant role in probability and statistics education (e.g., Ben-Zvi, 2000; National Council of Teachers of Mathematics, 2000; Parzysz, 2003; Scheaffer, Watkins, & Landwehr, 1998) but that research has only begun to scratch the surface on documenting students' understanding of probability when using such tools as simulation software (Jones, 2005). Technology allows students to generate a large amount of simulated data and represent the data in various ways. Furthermore, generating large sets of data allows students to quickly experience phenomena like the law of large numbers in a meaningful way. This law tells us that the *probability* of a large difference between the empirical probability and the theoretical probability limits to zero as more trials are collected. Thus, it is possible, although unlikely, to have an empirical probability substantially different from the theoretical probability, even after a large number of trials.

In many studies, researchers claim that students have a tendency to fall into thinking about the "law of small numbers" as a type of representativeness heuristic (Shaughnessy & Bergman, 1993; Tversky & Kahneman, 1971). Using this heuristic, "people believe that even small samples, perhaps a single outcome, should either reflect the distribution of the parent population or mirror the process by which random events are generated" (Shaughnessy & Bergman, 1993, p. 181-182). Much of the early research on students' reasoning about the effects of small and large trials were conducted in interview settings or with surveys and paper-and-pencil test instruments (e.g., Batanero, Serrano, & Garfield, 1996; Fischbein & Schnarch, 1997). More recent research has attended to students' reasoning about sampling and variation that provides a perspective on how little variation students often expect when sampling a small number of times (e.g., Shaughnessy & Ciancetta, 2002; Watson & Kelly, 2004). The results of many of these studies seem to support the notion that students can fall prey to reasoning with the "law of small numbers." However, most of these studies did not ask students to conduct a probability simulation (with real objects like dice or with technology) where they had to choose their own number of trials. When in a technology environment where students are given free choice to decide the number of trials to investigate, more research is needed to consider students' goals for their simulation activities, what they may anticipate as a result of their activities, and how they reason about the usefulness of both small and large trials to meet their goals.

Data Sources

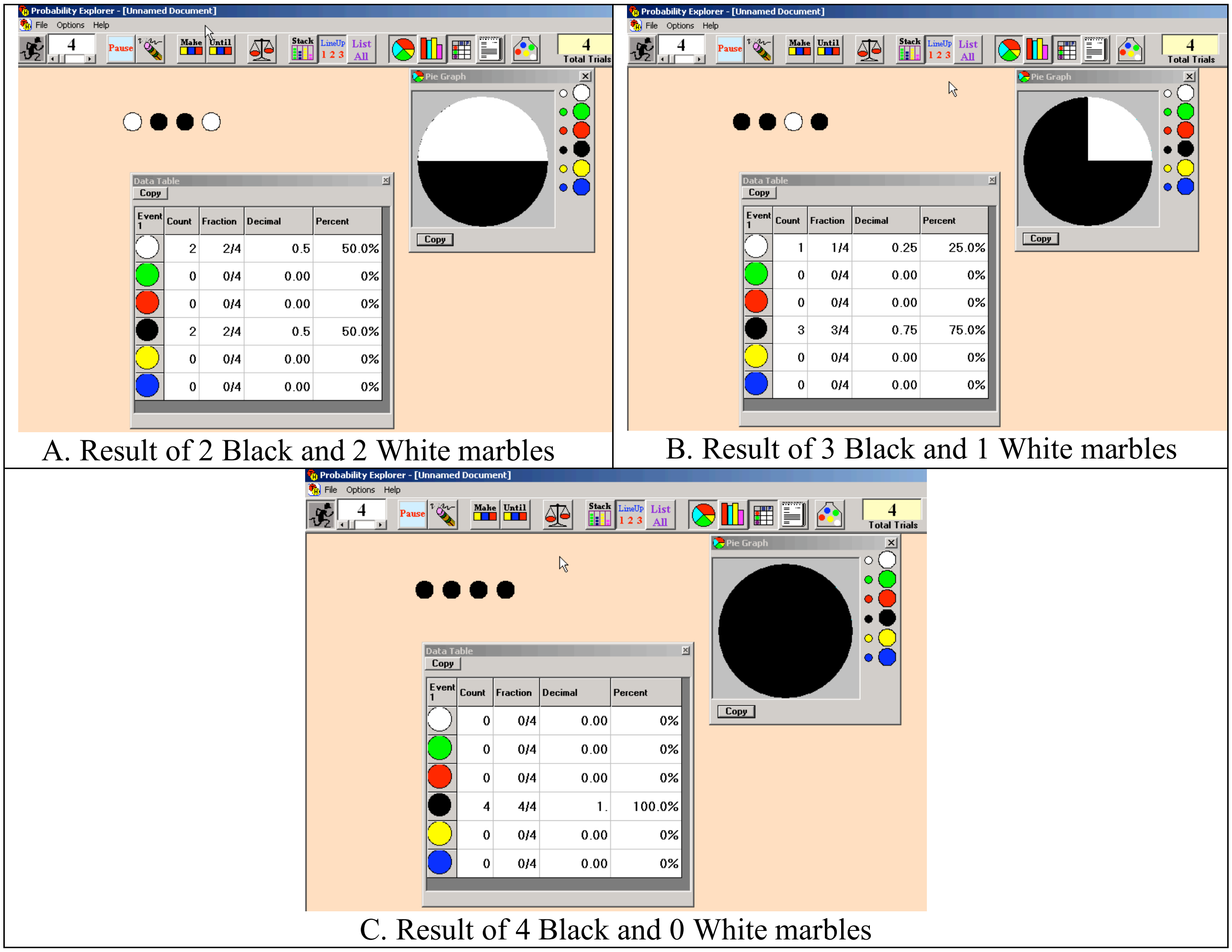
In several prior and on-going studies with children ages 9-13 (see Drier, 2000a; Drier 2000b; Stohl & Tarr 2002; Lee, Rider & Tarr, under review; Lee, Powell, Maher, & Weber, in progress; Pratt, 2000), it has been observed that many students use similar approaches to choose the number of trials in empirical probability tasks and often make similar observations regarding the distribution of empirical results with a large number of trials. In each of these studies, students were working in small groups (2-3 students per computer) with probability simulation software in which users can decide how many trials to conduct and examine results in various graphical (e.g., pie graph, bar/pictograph graph) and numerical forms. Students' interaction with the software and each other in the primary studies was videotaped and analyzed for critical events (Powell, Francisco, & Maher, 2003). A constant comparative method (Strauss & Corbin, 1990) was then utilized to look for patterns within the individual studies, followed by an interpretation cycle across data from the different studies (Lesh & Lehrer, 2000).

Making Sense of Patterns Across Studies

Choosing a Small Number of Trials

When students know (or intuitively expect) the underlying probability for each possible event in an experiment (e.g., coin toss should have 50% chance for events, a bag with N marbles has x black and N-x white), they will often use a strategy to imagine a hypothetical experiment of n=N trials and expect an empirical distribution equivalent to the theoretical one. These students are often cued by the known elements of a probability experiment in choosing the number of trials. For example, consider Jasmine and Carmella's (10 years old) work with simulating pulling a marble with replacement from a bag of 2 black and 2 white (2B2W) marbles. The students made a goal of trying to obtain a 50%-50% empirical distribution. After a few sets of 10 trials where the examined the pie graph and data table, they noticed that none had produced a 5B5W result. Jasmine offered, "I wonder if we just pick 4 out, if it would come out as two and two." They ran 2 sets of 4 trials and obtained 2B2W each time (e.g., see Figure 1A). On the third set of 4 trials they obtained 3B1W (Figure 1B).

Jasmine: Three and one. Unlikely. [pie graph displays ¾ black and ¼ white]
Carmella: Well it's actually more likely than what we were doing before [10 trials].
Teacher: And why is it more likely to what you were doing before? [Carmella runs 4 trials.]
Carmella: See look. We got all black….(Figure 1C) But getting all blacks is more likely on four because [pause] there's less numbers, so it would be more likely that we would get that than if we had 10 [trials].



A. Result of 2 Black and 2 White marbles

B. Result of 3 Black and 1 White marbles

C. Result of 4 Black and 0 White marbles

Figure 1. Empirical results from picking a marble with replacement from a bag with 2 black and 2 white marbles.

Jasmine employed a "total weight approach" [TWA where n=N trials] (Drier, 2000a) to reduce the sample size from 10 to 4 because there were 4 total marbles in the bag. Jasmine's intuition to reduce the number of trials in order to increase the chance of getting a 50%-50% distribution of results agrees with the theoretical probabilities calculated (by the author, not children!) using the binomial formula.

$$P(5B5W from 2B2W) = \binom{10}{5}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^5 \approx 0.2461 \qquad P(2B2W from 2B2W) = \binom{4}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 = 0.375$$

In addition, Carmella's reasoning that getting all black marbles with 4 trials is more likely than with 10 trials, is also in agreement with theoretical calculations.

$$P(4B from 2B2W) = \binom{4}{4}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^0 = 0.0625 \qquad P(10B from 2B2W) = \binom{10}{10}\left(\frac{1}{2}\right)^{10}\left(\frac{1}{2}\right)^0 \approx 0.00098$$

Thus, these students' strategy of lowering the sample size was appropriate for meeting *their goal* of obtaining a particular proportion of black marbles in their experimental results. Students tend to have a desire to obtain results that match what they believe to be the expected distribution. In the context of equiprobable chance events, these students seemed to be searching for "evenness" in their empirical results. Although these students appear to be using a representativeness heuristic and falling prey to the "law of small numbers," I believe their thinking is appropriate for the goal they have in mind—obtaining exactly a 50%-50% distribution when they already know the distribution in the population (i.e., what is in the bag of marbles).

The use of lowering the number of trials was used in another research context to meet a different type of student goal. Two seventh grade students (age 12-13), Jerel and Chris, were first learning how to use simulation software to design an experiment for picking a marble out of a bag with replacement from a bag that contained 4 blue, 3 green and 3 red marbles. The students were then asked to run some trials with the software and compare the simulation results with the results when they had pulled marbles from a real bag with the same distribution. The students initially ran several sets of 10 trials using a TWA of n=N. However, while they were running these sets of 10, they had created a goal of seeing which color "won" with each of them cheering for a color (Jerel claimed blue and Chris claimed red). With these sets of 10, some contained more red, some more blue, and one a lot of green. The students quickly decided to run 100 trials and "cheer" for their respective colors. Although the frequency of each color initially changed as well as their relative position in the "race" the students were envisioning, there were many more blue marbles after about 75 trials. At this point Chris noted that blue would win and stopped cheering for his color—of course Jerel still cheered on blue and gloated about his win to Chris. When the teacher asked the class if blue was always the marble picked the most, Chris replied "No, because at the beginning we started getting more red and now we are getting more blue." His reference to beginning may have referred to the earlier trials of 10 when more red were obtained, or the earlier trials in the last set of 100 when there were more red marbles than any other color. Regardless, he was noticing a long-term trend in the results from the simulation. All of this is being said to provide a context for his next goal-directed activity. Chris reduced the number of trials to 10, ran the simulation and obtained 5 red, 2 green, and 3 blue.

What was Chris' strategy? Keeping in mind that Chris wanted to win at the racing game he and Jerel had set up, he appeared to realize that he could win the race (having the highest frequency of red marbles) if he returned to using a smaller number of trials. Chris did not verbalize this strategy; thus, it is being inferred from his actions. However, his earlier statement regarding the frequency of reds in the beginning and the noticing of the long-term tendency for blue to have the highest frequency, provide a lens into what may have prompted his strategy. He appears to have seen the power of small numbers for meeting a specific goal—winning the race with red. Chris' strategy is in accord with the notion that it is with small sized trials where we are more likely to see outcomes with small theoretical probabilities occurring the majority of the time. Whereas, the law of large numbers tells us that the likelihood of this happening limits to zero as the trials size increases.

The use of these well-chosen n=N trials have served these students (in different studies) well in the context of their goal-directed activity when they know the theoretical distributions. Let's contrast this with students who only know that a bag contains N (N typically less than 20) marbles with two or three different colors of an unknown distribution. In this context, students who are asked to find out what they can about the contents of the bag often still employ a TWA and conduct simulations using n=N trials (Stohl & Tarr, 2002; Lee, Powell, Maher, & Weber, in progress). When asked to explain their reasoning, they tend to say that they have a good chance of getting the exact distribution in the bag if they conducted n=N trials. Is this reasoning appropriate? In an inference context, this reasoning can get students in trouble. If the students conduct one set of n=N trials and happen to obtain x black and n-x white marbles, they may stop there and use that to describe the contents of the bag. In this case, ignoring variability can lead them astray. However, some students will conduct several sets of n=N trials, notice variability, and then do some type of "averaging" (not always formal) to estimate the contents of the bag. In many cases, this strategy will work and allows for a somewhat reliable estimate. In this context, the students also have the goal of obtaining something exactly—they want to know *exactly* how many black and white marbles are in the bag—which may or may not have been an instructional goal. Thus in instruction and research, we need to carefully consider the goal the students have in mind when choosing trial sizes. Particularly when using technology, we may encourage students to run a large number of trials and not explore their thinking about using a small number of trials.

Noticing Patterns in the Long Run

In the context of using probability simulation software, and given the freedom to choose the number of trials, students have the opportunity to explore the effects of trial size on the empirical distribution of results. If they know (or expect) a theoretical distribution, they can make comparisons between the two. If the theoretical is unknown, they can begin to use results from a large number of trials to make inferences about that which is unknown. Some students in past studies needed a reason to move beyond using a TWA to run a large number of trials. For example, by slightly changing the marble task such that they do not how many marbles are in the bag (unknown N), students' search for an exact distribution must be replaced with a new goal—finding out which color is most likely or estimating the proportion of each color of marble in the bag. In this way, students have to estimate a relative frequency distribution rather than a frequency distribution that is dependent on knowing N.

Using simulation software affords students an opportunity to observe the dynamic accumulation of data in numerical and graphical forms while data is being generated. This visualization of the data has shown to be a powerful motivator for students noticing variability in short and long term behavior of random events, as in the work of Chris and Jerel. Across several studies, students have expressed their observations about an "evening out" phenomenon [EOP] (first documented in Drier, 2000a, 2000b), also observed by Pratt (2000). This phenomenon often occurs from student-generated goal directed activity, not prescribed by a task or a teacher.

Consider the work of three students [Amanda, Jasmine, Carmella, ages 9-10] who were working together on a fair coin toss simulation. Amanda created a goal of trying to obtain a sample of all heads or all tails. She continuously pressed the Run button to do many trials of 20 to see if she could get an all blue or all gray pie graph. Since she did not use the Clear button to erase the previous set of 20 trials, the number of trials began to accumulate (i.e., 20, 40, 60, …, 200). Even though the data showed some heads and tails, the action of adding more trials to the data set became a pleasurable experience that led to observations about an empirical distribution.

Amanda: Well, it's [size of sectors in the pie graph] staying in the same place pretty much.
Teacher: Why do you think it's staying in the same place?
Amanda: Because…
Carmella: Because she's running it so many times, it's like evening out.
Teacher: Really? Why is it evening out?
Carmella: Because it's so many of them and …
Jasmine: Look how much you've done it [about 1000 trials now]. It's still going.
Teacher: So, Amanda do you think you're ever going to get all blues (heads) or all grays (tails)?
Amanda: No.

Teacher: Why not? Why couldn't we have a pie graph be all blue or all gray?
Carmella: Because it evens out with how many you do.
Teacher: Why did you say that?
Carmella: Because the more you do, the more the chance to even out…with more coins one is still going to be a little bit ahead of the other mostly, but it's unlikely that one [sector in pie] will rise a lot above the other. There's so many it can't do it [pause] it's like it evens out.

Amanda's goal, the visualization with the pie graph, and subsequent questioning from the teacher appeared to prompt Carmella's observation and statement, which appears to be an early verbalization of the law of large numbers from a child's perspective. Consider the similarity in Donovan's (age 12) expression when the EOP was first noticed and stated in a 6th grade classroom discussion about a fair coin simulation.

Donovan: When you program it to do 500 and run it, after awhile you can see the pie graph pretty much staying in the center, so it's [pie graph sectors] always even almost and not going back and forth.

Students who have been given tasks to make inferences about unknown distributions are often able to employ EOP reasoning to help their judgment and defend their confidence in their results. The EOP is a useful tool when students' goal is to generate empirical results *close to* some unknown which they are trying to infer (see Stohl & Tarr, 2002; Lee, Rider, & Tarr, 2005).

Comments

The dynamic nature of simulation tools may contribute to students' creating their own goals and engaging in rich discussion around the results displayed on the computer. Representations that display results from experiments during the simulation process can focus students' attention on variability by "observing the fluctuation of samples… and observing the stabilization of the frequency distribution of the possible outcomes" (Parzysz, 2003, p. 1) within a run of trials. Students' emergent reasoning found across studies [e.g., TWA and EOP] bring to the fore the issue of obtaining "exact vs. close" empirical results. We can not ignore students' tendency to create a goal for themselves to obtain an empirical distribution that exactly mirrors what they expect based on what is known about the theoretical distribution. And when "mirroring" is their goal, using a smaller number of trials can make sense. If students learn that their best chances of obtaining an "even" distribution happens in small, well-chosen trial sizes, they may carry this into the context of making inferences where they do not know the underlying theoretical distribution. When the theoretical distribution is unknown, their goal has to shift to obtain empirical results that are "close enough" to estimate this unknown distribution with confidence. I conjecture that purposefully planning tasks that require this new goal can cause useful perturbations when students are making sense of empirical results from large trials and developing a way to rely on the EOP when the goal is to make inferences.

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THE EFFECT OF REPRESENTATION AND REPRESENTATIONAL SEQUENCE ON STUDENTS' UNDERSTANDING

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This study investigates the effect of representational sequence on students' understanding of mathematical concepts. Pilot studies were conducted with 129 high school students on solving inverse trigonometric identities and with 10 pre-service secondary teachers on representing Simpson's Paradox. Structured activities with a variety of representations and representational sequences were used to examine the impact on students' learning. This study also includes outcomes of surveys of 8 middle school teachers on different aspects of using representations in mathematics classroom. Our ongoing work finds this impact significant and claims that particular representational sequences need to be sensitive to specific content, learning outcomes, student prior knowledge and learning style.

Theoretical Framework

The body of existing research on the role of representation in improving students' mathematical understanding is convincing enough even for educational skeptics. That is why NCTM (2000) included the principle of representation among the five most important process standards of school mathematics. Studying an effect that representations have on students' understanding is critical for effectiveness of teaching mathematics. "We teach mathematics most effectively when we understand the effects on students' learning of external representations and structured mathematical activities" (Goldin & Shteingold 2001, p. 19).

Following Pape & Tchoshanov (2001), we use the term representation to refer to both the external and internal manifestations of mathematical concepts. Within the domain of mathematics, representations may be thought of as internal -- abstractions of mathematical ideas or cognitive schemata that are developed by a learner through experience. On the other hand, representations such as concrete, enactive models (e.g., manipulatives), visual/iconic (pictures, drawings, graphs, etc.), and symbolic/abstract (such as algebraic equations, formulas) are external manifestations of mathematical concepts that "act as stimuli on the senses" and help us understand these concepts (Janvier, Girardon, & Morand 1993, p. 81). Finally, representation also refers to the act of externalizing an internal, mental abstraction. In this paper we are mostly going to talk about the external representations.

A single type of external representation (e.g., visual) by itself doesn't ensure student learning and performance. Numerous studies (Brown & Presmeg 1993, Sylianou & Pitta-Pantazzi 2002) show that visualization is not always associated with mathematical accomplishment. Issues such as content, combination, and sequence of representations play a significant role in developing students' understanding. Lesh, Post, and Behr (1987) argue that not only are distinct types of external representation systems important in their own rights, "but translations among them, and transformations within them, also are important" (ibid., p. 34).

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Teaching with Representation: One Versus Many

In the 1990's, the second author implemented a pilot quasi experimental design with Russian high school pre calculus students (N=70) on solving inverse trigonometric identities such as arctan(1/2) + arctan(1/3) = pi/4 using multiple representations (Tchoshanov, 1997). The experiment consisted of two studies. The first study was focused on the effect of single and combined representational modes on students' understanding and consisted of 3 comparison groups. The first comparison group of students ("pure-analytic", n_1=23) was taught with a traditional analytic (algebraic) approach to trigonometric problem solving and proof. The second comparison group ("pure-visual", n_2=21) was taught with a visual (geometric) approach using enactive (i.e., geoboard as manipulative aid) and iconic (pictorial) representations. The third, experimental group ("representational", n_3=26) was taught with a combination of analytic and visual means using translations among different representational modes.

As reported elsewhere (Pape & Tchoshanov, 2001), the representational group scored 26% higher than the visual and 43% higher than the analytic groups. This experiment also showed that students in the "pure" (analytic and/or visual) groups "stuck" to one particular mode of representation; they were reluctant to use different representations. For instance, students in the pure-visual group tried to avoid any analytic solutions: they were "comfortable" if and only if they could use visual (geometric) techniques. Therefore, we realized that any intensive use of only one particular mode of representation does not improve students' conceptual understanding. Students in the representational group were much more flexible "switching" from one mode of representation to another in search of better understanding of mathematical concept. This observation supports findings from other similar studies; for example, one of the main conclusions from the study conducted by Lesh, Post, and Behr (1987) states that "good problem solvers tend to be sufficiently flexible in their use of a variety of relevant representational systems that they instinctively switch to the most convenient representation to emphasize at any given point in the solution process" (ibid., p. 38).

Sequence of Representations within a Collection

With evidence pointing in the direction that a combination of representations is the best, the next natural question was whether the sequence of representations within that combination may be significant. So the second study (later in the same semester) with the same classes of high school pre-calculus students was aimed at the effect that representational sequence has on students' understanding. Despite the tacitly accepted representational sequence "concrete-visual-abstract" where students first get involved into concrete "hands-on" experiences, then they draw the picture of the problem, and finally they provide formal (abstract, algebraic, or analytic) solution, we considered a variety of representational sequences. According to different types of representational sequence, there were 3 comparison groups (different from those participated in the first study with a total student sample of N=59):

1. The first comparison group (n_1=19) was called "abstract-last" group where activities were structured in a way that students first were engaged in concrete (C) and visual (V) modes of representation and only then to abstract-analytic (A) techniques of solving inverse trigonometric identities (CV-A & VC-A).
2. In the second - "abstract-middle" group (n_2=20) – activities were structured in a way that abstract (A) mode was introduced between concrete (C) and visual (V) modes of representation (CAV & VAC).

Activities in the third comparison group - “abstract-first” group (n_3=20) – were structured in the following representational sequence (ACV & AVC). According to this sequence, first students used trigonometric identities to simplify inverse trigonometric expressions and only then they were involved into activities using concrete and visual representations to illustrate and visually justify what they already proved analytically.

The mean classroom test scores in this study are the following: (1) Abstract-last group 76%; (2) Abstract-middle group - 85%; (3) Abstract–first group - 91%. At first glance, these results contradict the dominant view among educators that mathematical activities should be structured from concrete to abstract in order to develop students’ understanding of mathematical concepts and ultimately - to improve students’ performance. However, Krutetskii (1976) shows that the differences in mathematical performance depend on mostly abstractness-oriented characteristics of the mathematical cast of mind. Students in the Abstract-first group not only outperformed students from other two comparison groups but they were focusing on critical defining conditions (e.g., formalization, symbolization, generalization, curtailment, flexibility, and reversibility) of the problem and valued concrete and visual representation after they were exposed to analytic way of solving inverse trigonometric identities. Using the language of concept image and concept definition (Tall & Vinner, 1981), we may say that students in first two low-performing groups tended to have incomplete concept images without any connection to the defining conditions of the concepts, while students in the Abstract-first group attempted to use critical characteristics to form a concept definition. This result is in some way complementary to findings of Brown and Presmeg (1993), who claim that students with a greater relational understanding of mathematics tend to use more abstract forms of imagery, while students with less relational understanding tend to rely on concrete images. Overall, this study suggests that the students participating developed mathematical understanding that was enhanced not only by the combination (translation among and transformation within) of representations but also by the representational content and sequence.

Selection of Representations within a Collection

For some especially rich mathematical phenomena, the number of distinct representations may be too large to expect a teacher to have time to use all of them. Therefore, it is necessary to learn which representations might be more effective than others, and then form a sequence from those selected. Pilot studies were done by the first author with pre-service secondary teachers (7 at a public research university and 3 at a public comprehensive university) on exploring a sequence of 7 different representations of Simpson’s Paradox, following the examples in Lesser (2001). In this study, the focus was on determining which representations were more effective (and why) in helping students make sense out of a situation which did not seem very intuitive (at least in its first representation – a table of numbers): the results of a comparison were reversed upon aggregation of categories. Understanding this possibility is important for quantitative literacy. Students tended to want to stay with the most concrete and visual representations, but a C-V-A progression may not be expected to apply in the usual manner in the particular case of Simpson’s Paradox. We have followed up on this study and adapted the representations in a way that allows for students to be more active in their construction and interpretation (some of this was piloted in handouts during Lesser (2005) and more of it will be used in a future study with students). Also, we are beginning to explore the implications of the observation that some of the Lesser (2001) representations do not seem to neatly fit into only one of the three categories (C, V, A), suggesting a modified “continuum” model of representations.

Survey of In-service Teachers about Representation

Having observed results with pre-service teachers and pre-college students, the researchers felt that a missing part of the picture was how practicing in-service teachers themselves viewed representations. We conducted a survey in spring 2005 of the mathematics teachers (n = 8) at a public middle school in El Paso County with predominantly Hispanic population (93%) of students, the majority of which are economically disadvantaged (82%). Teachers participating in the interview have diverse teaching experience (3 teachers have 1-3 years of teaching experience, 3 have 4-6, and 2 have more than 6). The main purpose of the interview is to examine how teachers conceptualize various aspects of representation. They were asked to state in their own words what the NCTM (2000) representation standard meant and how it was part of their teaching. They were also asked to state in their own words the meaning of (and give examples of) concepts such as “concrete representation”, “visual representation”, and “algebraic representation”. The examples they gave were analyzed to see if there were clear discrete separations between the categories or if the categories had some “blurring” into more of a continuum of representations. The survey then asked which sequence, if any, teachers thought was “best” (and why) or instead to discuss the manner in which one sequence might be best in some situations and another sequence in others. These answers were compared with the traditional curricular perception that the “best” sequence is “always” C-V-A, in the spirit of the simple-to-complex sequence supported by theories of Bruner (1996) and others. This sequence clearly follows Bruner’s learning model based upon three levels of engagement with representations: enactive (e.g., manipulating concrete materials), iconic (e.g., pictures and graphs), and symbolic (e.g., numerals) (Bruner, 1966). Through early exploration of concrete materials, students are expected to move towards mathematical procedures that are analogous to symbolic procedures.

When the teachers were asked “When it comes to mathematics, what does ‘representation’ mean to you?” there were more than twice as many answers interpreting representation as a verb than as a noun. When asked to characterize their own learning style, many teachers said “traditional”, but the rest of their answer made it clear this word did not have a common meaning. One teacher who described his style as analytical made the interesting follow-up comment: “But I am learning more in geometric representations. This is helping me become more versatile in my teaching.”

Most of the teachers reported receiving very little beyond auditory lecture modes during their high school years, but finally experienced more visual, hands-on and kinesthetic styles when they took certain teacher preparation courses, especially in alternative certification classes. In their own teaching, teachers generally reported using multiple representations, believing that it helped reach more students, with one teacher noting the caveat “Would like to implement more except there is very little time.”

Teachers were then asked to describe in their own words the meaning of (and give examples of) concrete, symbolic/abstract and visual representations. While the researchers had in mind that concrete was something like a table of numbers, teachers generally interpreted the term as physical objects or manipulatives, which is not unreasonable, but two teachers actually classified “formulas and algorithms” as concrete, possibly suggesting that these teachers were not used to classifying or articulating distinctions about multiple representations. The expert view of symbolic representation is formulas, equations, and algebraic notation. One teacher commented that symbolic “incorporates concrete”.

When teachers were asked how they made their selection which representation (or sequence of representations) to use, the answers varied dramatically. Some teachers based the decision on their students (both by knowing their strengths beforehand, and by making adjustments if they still aren't getting it), some on a priori consideration of the content itself (e.g., "If I believe a concept is more visual, I would use a geometric teaching method first"), and some on the time available.

Of course, methods based on the essence of the concept would presume an accurate classification of a phenomenon's representation into a particular representation category. So the survey then asked if there are times when a representation might have strong aspects of more than one of the categories (visual, concrete, symbolic)? Again, student answers betrayed a lack of sophistication in representation classifications, but one student made a comment that revealed a provocative assumption: "There is research that attempts to prove that different representations confuses student[s], but I do not agree, I believe in using a combined approach to teaching Mathematics."

When asked what sequence (of the 6 permutations of the 3 categories) of representations the teachers thought was most effective, 50% chose the "traditional" answer of concrete, visual, abstract/symbolic, while the other half chose various different answers. One student justified her C-V-A choice with "because that's probably how are [sic] brain matures or develops." Once again, teachers were revealing some interesting beliefs about representations that we were not expecting. Another teacher who gave the C-V-A answer gave an explanation that conjured a progression of crystallizing thought: "I think concrete should come first, because it is the most clear picture. It is like connecting their learning to prior knowledge. Then I would put visual next because that is the next most clear concept. Then symbolic will be last because that requires more higher-order thinking skills."

The study suggests that the C-V-A choice is dominant in teachers' perceptions of effective representational sequences. It seems mostly due to their belief that "concrete should come first". This perception might be influenced by works of Bruner, Piaget, and others emphasizing method of ascending from concrete to abstract.

An informal survey of attendees (n = 6) at Lesser (2005) revealed a lack of consensus on the sequence with claims ranging from "sequencing should occur from simplest representation to more complex; start with concrete visual analytic" to "the sequencing is best done by knowing the audience that will be using the data." Also, several attendees at Lesser (2005) noted "blurring" between categories, such as finding visual and abstract features in the trapezoidal, circle graph, vector geometry and probability representations of Simpson's Paradox from Lesser (2001).

An alternative view on representational sequence is presented by Vygotskian fellow Vasilii Davydov. Davydov (1990) first examined the effectiveness of the method of ascending from general to concrete by teaching algebra concepts to elementary school students in the early 1970's in Russia. Studies on Davydov's method have found that "the Russian students (from Davydov's program) have a profound grasp of mathematical structure, confidence, and the ability to extend their knowledge well beyond the levels at which they had been instructed"(Zeigenhagen, 2000). Needless to say, that these students are from regular Russian schools without any selection by criterion of giftedness and a large percentage come from lower socioeconomic environments. Following Davydov's main idea, we argue that in students' conceptual development the representational sequence could be oriented toward ascending from whole to part, from abstract to concrete, from general concept to specific skill. It is not an imperative: the representational sequence depends on different factors such as content, learning objectives, student's prior experience and learning style.

In order to find out teachers' perception of this important issue, we asked them to identify factors that might influence their choice of a specific sequence of representations. Here are the factors that teachers said might influence their choice of a particular sequence (in parentheses is the number of teachers choosing that factor from our list; we also gave them an "other" option, but no one used it): learning style of students (7), teaching/ presentation style of teacher (7), particular math content involved (6), time constraints (6), learning goals (4), and alignment with standardized tests or other assessment (3). In this age of huge focus on high-stakes assessment, it is interesting that that factor was mentioned the least often.

Future Directions

We are planning a case study in which we explore the role of multiple contexts for the same mathematical structure. For the same Simpson's Paradox numbers in the 2x2x2 table of Lesser (2001), the variables of gender, department, and hiring decision can be replaced by other variables. For example, this distance/rate/time problem: "In January, Car A traveled at 37.5 mph for 80 hours of driving while Car B traveled at 25 mph for 20 hours of driving. In February, Car A traveled at 75 mph for 20 hours of driving while Car B traveled at 62.5 mph for 80 hours of driving. Which car had a lower rate of speed for both months combined?" Or a mixture problem: "Ann's box of mixed nuts consists of 80% walnuts priced at 37.5 cents/ounce and 20% cashews priced at 75 cents/ounce. Billy's box of mixed nuts consists of 20% walnuts priced at 25 cents/ounce and 80% cashews priced at 62.5 cents/ounce. Which box of mixed nuts is more expensive?" Or this example from the sports arena: "Sally, a guard on the women's basketball team, made 37.5% of her 80 attempts that were beyond the arc (i.e., 'three-pointers') and 75% of her 20 attempts that were close to the basket (i.e., 'in the paint'). Julie, a forward on the women's basketball team, made 25% of her 20 attempts from beyond the arc and 62.5% of her 80 attempts in the paint. Who had the higher shooting percentage overall?"

This future case study will also include several mathematical scenarios besides Simpson's Paradox for which each scenario can be represented with multiple representations (and contexts). Another example is a table of ordered pairs which would, say, all lie on a line when graphed and fit a $y=mx+b$ algebraic model. Even something as simple as a single number (e.g., one) can be represented with a picture, a word, a fraction, or expressions involving a zero exponent, a trigonometry function, etc. By exploring a variety of mathematical scenarios, we can gain insight into how the traditional C-V-A sequence might behave differently with different mathematical content, or how another sequence might be preferable for particular types or structures of content. In our future studies we also would like to explore the issue of representational "category blurring" or continuity (e.g., visualization as a continuum between concrete and abstract representations) and its impact on students' learning.

Outcomes of this study have direct implications for teacher preparation. Improvement of pre-service teachers' content knowledge depends on learning how to use representations and representational sequences effectively in the mathematics classroom.

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RE-EDUCATING PRESERVICE TEACHERS OF MATHEMATICS: ATTENTION TO THE AFFECTIVE DOMAIN

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Effective mathematics teaching is the result of a complex coordination of specific knowledge and specific beliefs. Too often, however, the emphasis within teacher education programs is placed on the infusion of content knowledge, pedagogy, and pedagogical content knowledge, with only a cursory treatment of the beliefs that, for better or for worse, will govern the eventual application of what has been acquired within these programs. In this paper I present the results of a study that took an opposite approach. That is, it examines the effectiveness of a methods course designed around problem solving in challenging the beliefs of a group of preservice teachers of mathematics. The results indicate that through their experiences with problem solving the participants' beliefs about the nature of mathematics, and what it means to teach and learn mathematics were drastically, and positively, affected. I also present the groundwork for future research that will examine how these beliefs are eventually put into action once these preservice teachers become practicing teachers.

"Prospective elementary teachers do not come to teacher education feeling unprepared for teaching" (Feiman-Nemser et al., 1987). "Long before they enrol in their first education course or math methods course, they have developed a web of interconnected ideas about mathematics, about teaching and learning mathematics, and about schools" (Ball, 1988). These ideas are more than just feelings or fleeting notions about mathematics and mathematics teaching. During their time as students of mathematics they first formulated, and then concretized, deep seated beliefs about mathematics and what it means to learn and teach mathematics. It is these beliefs that often form the foundation on which they will eventually build their own practice as teachers of mathematics (cf. Fosnot, 1989; Skott, 2001). Unfortunately, these deep seated beliefs often run counter to contemporary research on what constitutes good practice. As such, it is one of the roles of the teacher education programs to reshape these beliefs and correct misconceptions that could impede effective teaching in mathematics (Green, 1971).

Beliefs

An individual's beliefs, along with attitudes and emotions, comprise the affective domain (McLeod, 1992). The affective domain is most simply described as *feelings* – how an individual feels about something. In the context of mathematics, the affective domain was introduced to explain why learners who possessed the cognitive resources to succeed at mathematical tasks still failed (Di Martino & Zan, 2001). In this context, the beliefs are what learners believe to be true about mathematics and are often based on an individual's own experiences as learner of mathematics. For example, beliefs that mathematics is "*difficult*", "*useless*", "*all about one answer*", or "*all about memorizing formulas*" stem from experiences that introduced these ideas and then reinforced them. Research has shown such beliefs are slow to form in a learner but, once established, are equally slow to change even in the face of intervention (Eynde, De Corte, & Verschaffel, 2001).

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A qualitatively different form of belief is a person's beliefs about their ability to do mathematics, often referred to as efficacy or self-efficacy. Self-efficacy, like the aforementioned belief structures, is a product of an individual's experiences with mathematics. It is likewise slow to form and difficult to change. Self-efficacy for mathematics has most often been studied in the context of negative belief structures (Ponte, Matos, Guimarães, Cunha Leal, & Canavarro. 1992) such as *"I can't do math", "I don't have a mathematical mind",* or even *"girls aren't good at math."*

In the context of teaching mathematics, beliefs have more recently been involved in explaining the discordance between teachers' knowledge of mathematical subject matter and teacher practice. This research has revealed that the formation of teachers' beliefs about mathematics teaching and learning come from their own experiences as a learner of mathematics (cf. Fosnot, 1989; Skott, 2001). So, a belief that learning mathematics is "*all about learning algorithms*" may manifest itself as a belief that teaching mathematics is "*all about teaching algorithms,*" and an experience that "*all problems have one solution*" may manifest itself in the teaching that "*all problems must have one solution.*"

Research on the learning of mathematics has shown that beliefs are strongly linked to achievement (Ponte, Matos, Guimarães, Cunha, Leal, & Canavarro, 1992). They are gatekeepers to learning. Likewise, research on teaching of mathematics also examines teachers' beliefs and attitudes. The results indicate that these aspects are strongly linked to teaching (cf. Green, 1971). That is, beliefs and attitudes are the harbingers of practice.

Green (1971) classifies beliefs according to three binary partitions. He distinguishes between beliefs that are primary and derived. "Primary beliefs are so basic to a person's way of operating that she cannot give a reason for holding those beliefs: they are essentially self-evident to that person" (Mewborn, 2000). Derived beliefs, on the other hand, are identifiably related to other beliefs. Green (1971) also partitions beliefs according to the psychological conviction with which an individual adheres to them. Core beliefs are passionately held and are central to a person's personality, while less strongly held beliefs are referred to as peripheral. Finally, Green distinguishes between those beliefs held on the basis of evidence and those held non-evidentially. Evidence-based beliefs can change upon presentation of new evidence. Non-evidentiary beliefs are much harder to change being grounded neither in evidence nor logic. Instead they reside at a deeper and tacit level. Green argues that changing learners' belief systems is the main purpose of teaching. The beliefs of preservice teachers of mathematics can be categorized according to any of these partitions.

Cooney, Shealy, and Arvold (1998) characterize preservice secondary mathematics teachers' beliefs systems according to four descriptors: naïve idealists, isolationists, naïve connectionists, and reflective connectionists. These descriptors, they argue, are strong indicators for the likelihood that a given individual is going to change their belief systems. Teachers who fall into the first two categories are much less likely to change than those who fall into the last two categories. Their argument is based on their observation that teachers who reflect on their beliefs and reflect on their practice are already prone to change.

Changing Beliefs

Robust beliefs are difficult to change. However, an abundance of research purports to produce changes in preservice teachers of mathematics. Prominent in this research is an approach by which preservice teachers' beliefs are challenged (cf. Feiman-Nemser et al., 1987). Because beliefs that preservice teachers posses are often *implicitly* constructed from personal experiences as learners of mathematics, challenging these beliefs helps an individual make *explicit* the basis of their beliefs, thereby transforming beliefs from non-evidential to evidential (Green, 1971) and exposing them to the individual for critique and analysis.

Another prominent method for producing change in preservice teachers is by involving them as learners of mathematics (and mathematics pedagogy), usually submersed in a constructivist environment (cf. Ball, 1988; Feiman-Nemser & Featherstone, 1992). Theoretically, this approach serves two purposes. First it models for preservice teachers the ideas of constructivist learning, involving them in experiences which may be completely absent from prior learning encounters. Second, it uses a teaching methodology that repeatedly has been proven effective in promoting construction of new knowledge, new ideas, and new beliefs.

More recently, my own work in this area has shown that preservice teachers' experiences with mathematical discovery has a profound, and immediate, transformative effect on the beliefs regarding the nature of mathematics, as well as their beliefs regarding the teaching and learning of mathematics (Liljedahl, in press, 2002). This approach combines the aforementioned methods by challenging preservice teachers to make explicit their ideas on teaching and learning mathematics in the face of their own experiences with mathematical problem solving in group environments. In what follows I briefly present a research study designed and implemented within this context.

Methodology

Participants in this study are preservice elementary school teachers enrolled in a *Designs for Learning Elementary Mathematics* course for which I was the instructor. This particular offering of the course enrolled 35 students, the vast majority of these students are extremely fearful of having to take mathematics and even more so of having to teach mathematics. This fear resides, most often, within their negative beliefs and attitudes about their ability to learn and do mathematics. At the same time, however, as apprehensive and fearful of mathematics as these students are, they are extremely open to, and appreciative of, any ideas that may help them to become better mathematics teachers.

During the course the participants were submerged into a problem solving environment. That is, problems were used as a way to introduce concepts in mathematics, mathematics teaching, and mathematics learning. There were problems that were assigned to be worked on in class, as homework, and as a project. Each participant worked on these problems within the context of a group, but these groups were not rigid, and as the weeks passed the class became a very fluid and cohesive entity that tended to work on problems as a collective whole. Communication and interaction between participants was frequent and whole class discussions with the instructor (myself) were open and frank.

Throughout the course the participants kept a reflective journal in which they responded to assigned prompts. These prompts varied from invitations to think about assessment to instructions to comment on curriculum. One set of prompts, in particular, were used to assess each participant's beliefs about mathematics, and teaching and learning mathematics (*What is mathematics? What does it mean to learn mathematics? What does it mean to teach mathematics?*). These prompts were assigned after the first week, the fifth week, and the eleventh week of the course. These journal entries were analyzed using an iterative process of coding for emergent themes in order to establish four separate, but related qualities of each participant: their initial belief structures, their final belief structures, the change in their belief structures, and their intentions to act on these belief structures. The results of this analysis was then compared to each participant's journal entry from the twelfth week of the course. In this entry the participants were assigned the task of re-reading their entire journals and then responding to the prompt: *How have your ideas changed through your participation in this course? In particular, how have your ideas about what mathematics is, and what it means to teach and learn mathematics changed?* Because of the need for brevity I present here only some of the results from the analysis of this last entry.

Results and Discussion

When reflecting on what has changed the participants seemed to divide their responses into two subsections – how their beliefs about mathematics have changed, and how their beliefs about learning and teaching mathematics have changed. From the first of these several themes emerged from the data. The latter of these were often expressed in a convoluted testimonial of their own experiences within the course as learners and their newly formed views of themselves as teachers. From these testimonials several themes also emerged. In what follows I present, with exemplars and brief discussion, three of these themes.

Mathematics as a Verb

Almost all of the participants (32) mentioned in their reflective writing that they now see mathematics as something that one '*does*' as opposed to something one '*learns*'. This transition in thinking is nicely demonstrated in Nicole's comments. Early on in the course Nicole makes the statement:

> *I had really bad experiences in mathematics. I was always afraid of not getting it, of not being able to do it. I don't ever want my students to feel like that, I always want them to feel safe, to feel like math is easy.*

Although these comments are student centered and nurturing they are not in synch with contemporary theories of what it means to teach and learn mathematics. Through her own experiences with the learning of mathematics in the context of problem solving, however, Nicole's ideas regarding 'safety' begin to change.

> *I have never before worked for so long or so hard on a problem. It just seemed like we would never get it. In the end the hard work was worth it though. When we finally broke through and saw the answer it felt so great. It felt great knowing that we had solved it … WE had solved it. I know now that it wouldn't have felt nearly as great if Peter [the course instructor and author] told us how to do it. It also wouldn't have felt as great if the problem we had been working on hadn't been as difficult. I definitely want my students to feel this way.*

Not only do we see a change in Nicole's ideas of what constitutes safe mathematics, we also begin to see a desire to transfer her own experiences in doing mathematics into experiences for her future students. Such statements were almost universal amongst the preservice teachers who 'broke through' when solving a problem. The positive feelings associated with moments of discovery are something that they want to share. This transfer of experience from herself into her future students does not end with mathematical discovery, however.

> *I now see how important it is to allow students to work on a problem, to actually do the mathematics, to struggle, and to think.*

In her closing comments Nicole reveals a sentiment that is also shared by the majority of her classmates – mathematics is something to be 'done'. Although not featured prominently in her writing, the subtly shift from thinking about mathematics as something to be learned to something to be done is evident. I have come to refer to this shift as a shift from seeing mathematics as a noun ('*the rules we learn*') to a verb ('*to do, to struggle, to think*').

Humanizing Mathematics

All of the participants mention, in one way or another, how they '*now see the value of mathematics*'. Many are more explicit about the nature of this '*value*', as can be seen in Kevin's comments.

> *I mentioned several times, especially in the beginning, that I thought mathematics needed to be connected to the real world in order for it to be meaningful. Now I'm not so sure that matters. I now see that a problem can become real to me without it being 'real'.*

I have come to refer to this as 'humanizing mathematics through first person experience'. These participants came to see the meaningfulness of mathematics, not through its application to the real world, but through their own experiences with it. How this transfers into their views of the teaching of mathematics can be seen in Kevin's later comments.

> *When I went through school I was pretty sure that most of mathematics is useless. I'm not so sure that this is still true. I now see that mathematics is about the process. I see that it is more important to have students do mathematics than know mathematics. I'm not sure that makes any sense, but I'm just trying to say that now that we have done some problem solving I see that there is real value in just doing it. So, "what does it mean to teach mathematics?" Teaching mathematics means to allow your students to experience doing mathematics first hand.*

This sentiment that mathematics can and should be experienced first hand is more than just a desire that their future students experience the 'joy' of mathematics in general, and mathematical discovery in particular, as Nicole indicated above. Kevin now sees 'doing' mathematical as an end in and of itself. That is, the purpose of 'doing' mathematics is 'doing' mathematics. The mathematical, didactical, and pedagogical implications of such a belief can be disputed. However, in comparison to his initial belief that 'mathematics was useless' it can also be argued that Kevin has improved his belief system as to how it pertains to the teaching of mathematics.

Learning Through Talking

All of the participants wrote about how important the group work had been to their experience of '*doing*' mathematics, and vowed to make extensive use of this in their own teaching. Many of these participants (23) explicitly mentioned that it was the talking that was the thing that was important. This is nicely exemplified in Lena's comments.

> *My whole definition of mathematics broadened beyond problem solving and pattern finding to include communication. At the beginning I flippantly wrote that math is a language, but now I see that it is best developed through use. In other words, I started to see talking as an integral part of learning, and that a math class should be noisy at times.*[1]

This personal experience with the role of communication in the learning of mathematics translates into an explicit intention of how to make use of communication in the teaching of mathematics.

> *I always knew that I would use group work in my teaching, but I never actually thought about it in teaching math. I now see that this is the most important class to have my students*

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[1] It is interesting to note that this view of how best to learn mathematics is shared with research mathematicians (Liljedahl, 2004).

> *working together. It is through their conversations that they will learn things, especially problem solving.*

Lena, like many of classmates, came to see that it wasn't simply the group work that led to learning. More specifically, it was from the conversations that took place during group work that she learned the most. This distinction is further revealed in Lena's reflective writing.

> *But it isn't just group work that is important. I have to give my students many opportunities to communicate mathematically. This can be as simple as answering questions of me to presenting solution to the rest of the class. At every turn I have to try to facilitate communication.*

It is interesting that this view of how best to learn mathematics is shared with research mathematicians (Liljedahl, 2004).

Conclusion

A very powerful conclusion from this study is the impact that the problem solving environment within the class had on the recasting of these preservice teachers' beliefs about what mathematics is, and what it means to teach and learn mathematics. Through their own experiences with mathematics in a non-traditional setting they came to see, and to believe, in the value of teaching mathematics through '*doing*', through '*talking*', and through '*thinking*'. As such, the results contribute to the growing body of research on what exactly it is that elementary school teachers need in the form of teacher education to become effective teachers of mathematics (cf. Ball, 2004).

Future Research

The research presented above relies on the preservice teachers' self-reporting of changes in beliefs while enrolled in a preservice teacher education program. These reports are taken either at different times during a program, and then changes in beliefs are extracted from the comparison of successive reports, or participants are simply asked to report on changes in their beliefs in a reflective manner. This research methodology, while measuring changes in beliefs, ignores changes in practice and, if not treated appropriately, can easily conflates the *intent* of practice with *actual* practice. As such, the research presented here is currently being extended into a longitudinal study in which the participants introduced here are being followed into their first years of practice.

The goal of this research is three fold. First, it is intended to examine how robust the newly formed beliefs are for these preservice teachers. In particular, I will be examining if the newly acquired beliefs can withstand the test of time as well as the pressure of enculturation into an existing school culture. Second, to examine the connection between *intention of practice* (as presented in the participants preservice belief statements) and *actual practice* as observed in the participants subsequent own classrooms. Third, to observe what factors (other than preservice education) affect the belief structures of novice teachers.

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TEACHERS' UNDERSTANDINGS OF HYPOTHESIS TESTING

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The goal of this study was to explore eight high school mathematics teachers' understandings of hypothesis testing as they engaged in a two-week professional development seminar. To this end, we analyzed data collected from the videotaped seminar discussions and the follow-up interviews. We found that teachers' difficulties with hypothesis testing could be explained by a conflation of two sources: their non-stochastic conceptions of probability and their unconventional logics of hypothesis testing. Follow-up interviews suggested that the teachers did not understand the work that hypothesis testing is meant to do. Following these results, we proposed a number of strategies for future professional development.

Research Topic

Past research has found that people have profound difficulties with both understanding and employing the method of hypothesis testing. Albert (1995) and Link (2002) found that students have difficulty recognize the population parameter to be tested in inference scenarios. Albert (1995) also found that the idea of sampling distribution, fundamental to hypothesis testing, was too hard for students to learn. Bady (1979) found that people have a strong tendency to test hypothesis by seeking information that would verify the hypothesis instead of falsifying it.

Examination of popular statistics textbooks suggested that hypothesis testing is typically taught as a multi-step procedure (eg. Yates *et al.*, 1998). Doing hypothesis testing, as shown from the following excerpt, seemed like executing a sequence of actions which does not require any reasoning on the part of students:

> "The first part in this procedure is the statement of the null and alternative hypothesis. Students look for key words and phrases such as "less than", "decreased", "reduces", "greater than", "increased", "improved", and "is different from", as guides in stating the null and alternative hypothesis. In the second part, the critical value of the test statistic necessary to reject the null hypothesis is asked for, which requires that the student recognize the appropriate test statistic, locate the correct tabled value based on the stated level of significance, and supply the correct sign…Finally, the p-value is either read from a table, or is displayed on a graphing calculator screen." (Link, 2002)

We argue that this way of conceptualizing and teaching hypothesis testing might have contributed to students' confusion about hypothesis testing. In this paper, we will explore the difficulties people have as they try to understand hypothesis testing conceptually. To this end, we examined data collected from a professional development seminar, conducted with a group of eight high school teachers, that was designed to investigate their understanding of probability and statistical inference (Liu & Thompson, 2004).

Theoretical Framework and Methodology

Our study was guided by a radical constructivist perspective on human knowledge and human learning. Radical constructivism entails the stance that any cognizing organism builds its own reality out of the items that register against its experiential interface (Glasersfeld, 1995). As

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such, in our study that aimed to understand others' mathematical understanding, it is necessary to attribute mathematical realities to subjects that are independent of the researchers' mathematical realities. This is what Steffe meant when he described the researcher' activity in a constructivist teaching experiment, as that of performing the act of de-centering by trying to understand the *mathematics of the [other]* (Steffe, 1991).

To construct models of others'/teachers' understanding, we adopt an analytical method that Glasersfeld called conceptual analysis (Glasersfeld, 1995), the aim of which is "to describe conceptual operations that, were people to have them, might result in them thinking the way they evidently do." Engaging in conceptual analysis of a person's understanding means trying to think as the person does, to construct a conceptual structure that is intentionally isomorphic to that of the person. In conducting conceptual analysis, a researcher builds models of a person's understanding by observing the person' actions in natural or designed contexts and asking himself, "What can this person be thinking so that his actions make sense from his perspective?" In other words, the researcher/observer puts himself into the position of the observed and attempt to examine the operations that he (the observer) would need or the constraints he would have to operate under in order to (logically) behave as the observed did (Thompson, 1982).

Research Design & Data Analysis

We designed a two-week summer seminar for high school teachers. The seminar was advertised as "an opportunity to learn about issues involved in teaching and learning probability and statistics with understanding and about what constitutes a profound understanding of probability and statistics." Of 12 applicants we selected eight who met our criteria—having taken coursework in statistics and probability and currently teaching, having taught, or preparing to teach high school statistics either as a stand alone course or as a unit within another course. Participating teachers received a stipend.

The research team prepared for the seminar by meeting weekly for eight months to devise a set of issues that would be addressed in it, selecting video segments and student work from prior teaching experiments to use in seminar discussions, and preparing teacher activities.

Table 1 presents demographic information on the eight selected teachers. None of the teachers had extensive coursework in statistics. All had at least a BA in mathematics or mathematics education. Statistics backgrounds varied between self-study (statistics and probability through regression analysis) to an undergraduate sequence in mathematical statistics.

Table 1. Demographic information on seminar participants.

Teacher	**Years Teaching**	**Degree**	**Stat Background**	**Taught**
John	3	MS Applied Math	2 courses math stat	AP Calc, AP Stat
Nicole	24	MAT Math	Regression anal (self study)	AP Calc, Units in stat
Sarah	28	BA Math Ed	Ed research, test & measure	Pre-calc, Units in stat
Betty	9	BA Math Ed	Ed research, FAMS training	Alg 2, Prob & Stat
Lucy	2	BA Math, BA Ed	Intro stat, AP stat training	Alg 2, Units in stat
Linda	9	MS Math	2 courses math stat	Calc, Units in stat
Henry	7	BS Math Ed, M.Ed	1 course math stat, AP stat training	AP Calc, AP Stat
Alice	21	BA Math	1 sem math stat, bus stat	Calc hon, Units in stat

The seminar lasted two weeks in June 2001. Each session began at 9:00a and ended at 3:00p, with 60 minutes for lunch. All seminar sessions were led by a high school AP statistics teacher

(Terry) who had collaborated in the seminar design throughout the planning period. We interviewed each teacher three times: prior to the seminar about his or her understandings of sampling, variability, and the law of large numbers; at the end of the first week on statistical inference; and at the end of the second week on probability and stochastic reasoning. This paper will focus on week 1, in which issues of inference were prominent.

Results

For the purposes of this paper we will focus on two episodes of teachers' discussions during the first week of the seminar, and their responses to an interview question at the end of that week.

The first discussion focused on the idea of *unusualness*. We focused on *unusualness* for several reasons. First, the logic of hypothesis testing is that one rejects a null hypothesis whenever an observed sample is judged to be sufficiently *unusual* in light of it. This logic demands that we assume the sample statistic of interest has some underlying distribution, for without assuming a distribution we have no way to gauge any sample's rarity. This assumption is made *independently* of the sample. It is like a policy decision: "If, according to our assumptions, we judge that samples like the one observed occur less than x% of the time (i.e., are sufficiently unusual), then either our sampling procedure was not random or values of the sample statistic are not distributed as we presumed." Second, we observed in high school teaching experiments that students had a powerful sense of "unusual" as meaning simply that the observed result is surprising, where "surprising" meant differing substantially from what they anticipated. By this meaning, if one has no prior expectation about what a result should be, no result is unusual. Since students infrequently made theoretical commitments regarding distributions of outcomes, their attempts to apply the logic of hypothesis testing often became a meaningless exercise.

To understand the teachers' conceptions of *unusualness*, we adapted the following question from Konold (1994).

> Ephram works at a theater, taking tickets for one movie per night at a theater that holds 250 people. The town has 30 000 people. He estimates that he knows 300 of them by name. Ephram noticed that he often saw at least two people he knew. Is it in fact unusual that at least two people Ephram knows attend the movie he shows, or could people be coming because he is there?

The teachers first gave their intuitive answers. All said it was not unusual for Ephram to see two people he knows. Subsequent discussion focused on the method for investigating the question, and it revealed that only one teacher, Alice, had a conception of unusualness that was grounded in a scheme of distribution of sample statistics. She proposed, as the method of investigating the question, "Each night record how many he knew out of the 250 and keep track of it over a long period of time", which suggested that she had conceived of "Ephram sees x people he knows" as a random event and would evaluate the likelihood of outcomes "Ephram sees at least two people he knows" against the distribution of a large number of possible outcomes.

Other teachers had various conceptions of unusualness. Three teachers, Sarah, Linda, and Betty stated flatly that something is unusual if it is unexpected, and expectations are made on the basis of personal experience. John's conception of unusualness was also subjective and non-quantitative. He justified his intuitive answer: Since Ephram knows 300 people out of 30,000 people in his town, it means for every 100 people, he knows 1 person. On any given night he should know 2.5 people out of 250 people who come to the theatre, given that this 250 people is a random sample of 30,000 in his town. Therefore, it is not unusual that he saw in the theatre at least 2 people he knows. John employed what we call the *proportionality heuristic*: evaluating the likelihood of a sample statistic by comparing it against the population proportion or a statistic of a larger sample. He did not conceptualize a scheme of repeated sampling that would allow him to quantify unusualness. Henry's conception of unusualness was quantitative: He defined unusualness as "something's unusual if I'm doing it less that 50% of the time" This discussion revealed that the teachers, with exception of Alice, had a mostly subjective conception of unusualness, and this conception did not support their thinking in hypothesis testing.

The second discussion focused on the logic of hypothesis testing. The logic of hypothesis testing is similar to the logic of proof by contradiction. In proof by contradiction, we reveal the truth of a statement in question by assuming its logical negation and then bringing this assumption into question by deriving a result that is contrary to the assumption or contrary to an accepted fact. In hypothesis testing, we test the plausibility of *h1* by assuming a rival hypothesis, *h0*, and testing its plausibility in terms of the likelihood of the factual data to have occurred given *h0* is true. A small chance of the factual data with *h0* being true casts doubt on the plausibility of *h0* and in turn suggests the viability of *h1*.

To understand the teachers' understanding of the logic of hypothesis testing, we engaged them in discussion of the following question:

> Assume that sampling procedures are acceptable and that a sample is collected having 60% favoring Pepsi. Argue for or against this conclusion: This sample suggests that there are more people in the sampled population who prefer Pepsi than prefer Coca Cola.

This question was accompanied by a list of 135 simulated samples of size 100 taken from a population split 50-50 in preference. Four of the 135 sample statistics exceeded 60%.

Three teachers, Lucy, John, and Henry, initially took the position that the argument *there were more people in the sampled population who prefer Pepsi than prefer Coca Cola* was false. They based this claim on the evidence that only 2.96% of the simulated samples had 60% or more favoring Pepsi. Their logic seems to have been: If the population was indeed unevenly split, with more Pepsi drinkers than Coke drinkers, then you would expect to get samples like the one obtained (60% Pepsi drinkers) more frequently than 2.96% of the time. The rarity of such samples suggested that the population was *not* unevenly split.

Terry, the seminar leader, pushed the teachers to explain the tension between 1) a sample occurred, and 2) the likelihood of the sample's occurrence is rare under a given assumption. Henry suggested one explanation: The sample was not randomly chosen. John offered another: The assumption (that the population was evenly split) was not valid. Under intense questioning, both Henry and John eventually concurred that the data suggested that samples of 60% or more were sufficiently rare that something must be wrong about the assumption.

One teacher, Linda, insisted that the assumption should not be rejected on the basis of one sample. Her argument was that no matter how rare a sample is, it *can* occur, thus it cannot be used against any assumption. A mixture of beliefs and orientations helped explain why she was opposed to rejecting the null hypothesis, including: 1) A commitment to the null hypothesis. She would reject a null hypothesis only if there were overwhelming evidences against it. Therefore, she opposed to "rejecting the null on the basis of one sample" and proposed to take more samples to see if the null hypothesis was *right* or wrong. 2) A concern for the truth of null hypothesis. Rejecting a null hypothesis, to her, means making a conviction that the null hypothesis was wrong. Because of this belief, she opposed "rejecting the null hypothesis on the basis of one sample" because any rare sample could still occur theoretically. Linda's concern for the truth of null hypothesis is inconsistent with the idea of decision rule. A decision rule does not tell us whether the null hypothesis is right or wrong. Rather, it tells us that if we apply this decision rule consistently, over the long run we can keep the error rate at a reasonably low level.

In sum, the discussion revealed the spectrum of choices the teachers made when facing the question: Do we reject a null hypothesis when a sample is unusual in light of it?

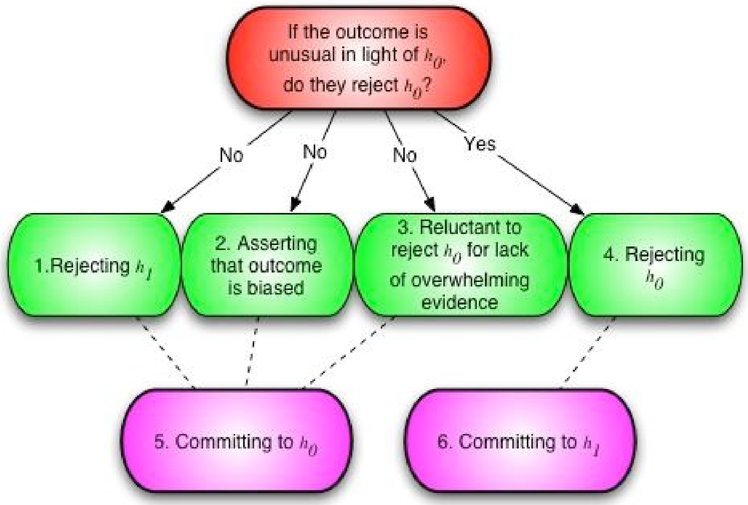

Figure 1: Theoretical framework for the logic of hypothesis testing

This framework captures the varieties of choices the teachers made when a small *p-value* was found. Decisions 1-3 are likely to be made by people who are committed to the null hypothesis, whereas people who are committed to the alternative hypothesis would reject the null on the basis of a small *p-value* (decision 4). The results of the discussion suggested that most of the teachers exhibited a commitment to the null hypothesis (the initial assumption that the population was evenly split), whereas in standard hypothesis testing, one's commitment is to the alternative hypothesis. That is, it is the alternative hypothesis that one suspects is true, and the logic of hypothesis testing provides a conservative method for confirming it.

During the interview at the end of the first week, we gave the teachers this question:

> The Metro Tech Alumni Association surveyed 20 randomly-selected graduates of Metro Tech, asking them if they were satisfied with the education that Metro gave them. Only 60% of the graduates said they were very satisfied. However, the administration claims that over 80% of all graduates are very satisfied. Do you believe the administration? Can you test their claim?

This interview question presents a typical hypothesis testing scenario—There was a stated claim about a population parameter: 80% of all graduates of Metro Tech were very satisfied with the education that Metro gave them. A random sample of 20 graduates found that only 60% of them said they were satisfied. The implied question was, "Will the samples like or more extreme than 60% be sufficiently rare for one to reject the administrations' claim that 80% of all graduates are very satisfied with the education they received?"

Almost all the teachers noticed the large difference between 60% and 80%, and they believed the small sample size was the reason why there was such a big difference. When asked whether they believed the administration's claim, the teachers had different opinions. Two teachers said they did not believe the administration's claim. Four teachers said they did. Henry and Alice based their choice on the fact that 80% was possible, despite its difference to the sample result.

Sarah, however, did not know that 80% was a claim. Rather, she thought it was a sample result. The other two teachers were hesitant in making a decision, with one of them, Lucy, leaning towards not believing the administration.

When asked how they would test the administration's claim, only Henry proposed to use hypothesis testing. The methods other teachers proposed fall into the following categories:

1. Take many more samples of size 20 from the population of graduates (John, Nicole, Sarah, Alice)
2. Take a larger sample from the population of graduates (Alice)
3. Take one or a few more samples of size 20 from the population of graduates (Lucy, Betty)
4. Survey the entire population (Linda)

In sum, teachers' responses on this interview question suggested they did not employ spontaneously the method of hypothesis testing for the situation. Instead, 7 out of 8 teachers proposed methods of investigation that presumed that they would have access to the population, and none of these methods were well-defined policies that would allow one to make consistent judgment. This led to our conjecture that even though the teachers might have understood the logic of hypothesis testing at the end of the seminar, they did not understand the functionality of it. In other words, they did not know the types (or models) of questions that hypothesis testing was created for, and how hypothesis testing became a particularly useful tool for answering these types of questions.

Overall, the results revealed that the majority of the teachers embraced conceptions of probability and logic of hypothesis testing that are incompatible with meanings that will support using it in ways that its inventors intended. Only one teacher conceptualized unusualness within a scheme of repeated sampling, and thus the others did not incorporate the idea of a distribution of sample statistics in their thinking of statistical inference. Most of the teachers did not understand the logic of hypothesis testing. This was revealed in the non-conventional decisions they made when a collected sample fell into the category of "unusual" in light of the initial assumption. These decisions revealed their commitment to the null hypothesis in question. Beyond the complexity of hypothesis testing as a concept, we conjecture that part of teachers' difficulties was due to their lack of understanding of hypothesis testing as a tool, and of the characteristics of the types of questions for which this tool is designed. This conjecture was supported by the evidence revealed in the interview data where only one teacher proposed hypothesis testing as the method of investigation.

Conclusions and Implications

The results revealed that teachers' understandings of probability and statistical inference were highly compartmentalized: Their conceptions of probability (or unusualness) were not grounded in the conception of distribution, and thus did not support thinking about statistical inference. The implication of this result is that instructions of probability and statistical inference must be designed with the principal purpose as that of helping the teachers develop understanding of probability and statistical inference that cut across their existing compartments. This purpose could be achieved by exerting a great amount of coerced effort in helping teachers develop the capacity and orientation in thinking of a *distribution of sample statistics*, which allows them to develop a stochastic/distributional conception of probability, and incorporate the image of distribution of sample statistics in their thinking of statistical inference.

We also learned that part of the teachers' difficulties in understanding hypothesis testing was a result of their tacit beliefs or assumptions about statistical inference, e.g., the belief that rejecting a null hypothesis means to prove it wrong. The implication of this result is that understanding hypothesis testing entails a substantial departure from teachers' prior experience and their established beliefs. To confront these hidden beliefs, we could, for example, design activities that incorporate the theoretical framework (Figure 1) and engage the teachers in discussions of the implications of each potential choice they might make. In having the teachers reflect on the tacit beliefs that might lead them to non-conventional choices, we could help them come to appreciate the logic of hypothesis testing.

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MEMBERSHIP IN COMMUNITIES OF LEARNING: ASSUMING NEW IDENTITIES

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In this work we traced the views of 480 college students on their initial experiences with inquiry based mathematics learning and teaching using two open-ended response surveys. Initially all students expressed difficulty adjusting to the social and mathematical demands of the learning environment and found it difficult to engage in or benefit from group discourse. The students also believed they were unable to construct mathematics independent of an authority. Adjustment to the new class culture was easier for those students with prior knowledge with innovative instruction and those with stronger problem solving skills.

Introduction

Current recommendations for reform in mathematics education emphasize that mathematics classrooms should be transformed into communities (Wenger 1998) in which students engage in mathematical meaning making through discourse and inquiry. In recent years a variety of labels have been used to describe instructional models that nurture such classrooms. Among many include: inquiry based (Borasi 1992), problem based (Duch et al. 1997), discussion based (Boaler 2000), and project based (Blank 1997) instruction. These various instructional models share several fundamental perspectives on learning, teaching, and curriculum. They view learning as a constructive activity and students as active agents in the learning process. They assume learners as mathematicians who debate the accuracy of ideas and resolve differences by testing and comparing methods (Yackel & Cobb 1996). They also assume the teacher as a facilitator of learning. Lastly, the focus of the curriculum is on supporting conceptual understanding of mathematics. These shared features are currently used to characterize a construct, commonly referred to as "reform-minded" practice. There is consensus within the mathematics education community that these reform-minded visions of teaching and learning are drastically different from what has been traditionally practiced in schools and to which many teachers are accustomed. It is widely agreed that making the transition for a traditional to a reformed based practice is difficult for teachers. This however, accounts for only one aspect of classroom life in the presence of change. Our work is grounded in the view that classroom learning environments are not simply constituted by teaching strategies. Rather, classrooms are ecologies that include social and cultural forces exemplified by both students and teachers (Lave & Wenger 1991). Therefore, in order to develop a deep understanding of the ecology of classroom, there is a need to gain an insight into the meanings that participants (students) in this ecology attach to their experiences (Boaler & Greeno 2000). Learning to be a successful student of mathematics involves learning the rules of the school mathematics game and forming a learning identity that fits with the norms of the classroom community (Angier & Povey 1999). Naturally, these identities come to play a role when the students determine legitimacy of educational experiences provided for them, or what they are assigned to do in class (Wenger 1998). Knowledge about learners' frameworks is crucial in assisting the education community in advancing the development of communities of learning. The study on which we report here focused on collecting data relative to two specific questions:

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1) What are the learners' reactions to the instructional expectation that they engage in building mathematical knowledge through discourse and inquiry?
2) What factors influence the learners' reactions to the instructional expectation that they engage in building mathematical knowledge through discourse and inquiry?

Methodology

The data for our analysis come from two surveys administered to 480 students at five different institutions in the USA. The subjects were enrolled in 15 different sections of mathematics courses consisting of: Calculus (3 sections), Linear Algebra and Matrix Theory (1 section), Modern Geometry (6 sections), Discrete Structures (3 sections), and Calculus II (2 sections). All students majored in either a science related field or mathematics. In all these classes, students were encouraged to construct mathematics knowledge through problems and explorations, to form propositions and theorems regarding relationships among mathematical concepts, and to prove their propositions in collaboration with peers. The instructional goal in each of these classes was to provide the students an opportunity to gain experience in acting as a community of learners. These sections were taught by 9 different university professors, each holding a PhD in mathematics or mathematics education, and with a minimum of 10 years of teaching experience. All professors were involved in a two year long professional development (2 summer workshops) sessions during which they were introduced to current goals for instructional reform as well as textbooks and activities that supported reform based goals.

The surveys were administered by a graduate student at each site. The first survey was administered during the sixth week of instruction and the second one during the final examination week at each institution. In additional to obtaining biographical information from each of the subjects (age, gender, status, number of mathematics courses completed at the university, GPA), each survey contained 10 open ended questions which asked students to comment on their experiences in their respective course, challenges they faced throughout the semester, perspectives on the curriculum and teaching they encountered. Lastly, they were asked to comment on those expectations with which they became more comfortable over time and those which remained problematic for them.

The approach to analysis combined categorical aggregation with a constant comparison method (Glaser 1967). First, the information obtained from the surveys were organized along three broad categories including: (1) aspects of curriculum and instruction they found difficult to understand and adopt, (2) aspects of the curriculum and instruction they found beneficial to their learning and to which they easily adjusted, (3) ways in which the course experiences impacted their thinking about mathematics and their own knowledge of mathematics. If students identified issues that did not relate to any of the listed categories they were also noted and then traced through a second round of reading. We marked the number of times each issue emerged throughout the range of data, points on which a majority of the students tended to agree, and issues upon which conflicting views existed among them. The nature of insights and concerns expressed by particular subgroups of students (i.e. females, students most successful in class, students least successful in class) was also marked. The final phase of analysis involved making comparisons across the data according to location and course content. Patterns that persisted were identified and issues that seemingly gained less attention were also noted. In this report we will highlight only those aspects of experiences with which a majority of the students took issues (we defined majority as 80% or more of the entire sample) and will offer an analysis of why these particular issues might have emerged.

Findings

Kenny (1999) argued that individuals assess their educational experiences through frameworks deeply grounded in their socio-cultural beliefs about schooling, subject matter, and appropriate pedagogy. Indeed, students' perceptions on the course content, and their own role in relation to this content, were framed by their beliefs on the nature of the mathematical knowledge and how this knowledge is constituted and shared in educational settings. These views influenced not only their level of engagement in collaborative construction of knowledge but also the extent in which they felt they benefited from such collaboration. These beliefs also shaped students' assessment of mathematics they studied, the curriculum to which they were exposed, the instructional practices they experienced, and the value they attached to their own and peers' contributions. Mathematically, students' particular beliefs about the structure of the discipline determined the extent in which they assumed autonomy and power in constituting mathematical knowledge as a group; and the value they attached to knowledge derived from group discussions. Pedagogically, their views on appropriate methods of teaching framed their expectations of instruction, and how they defined their own role in class.

Is this Legitimate Mathematics?

In the mathematics community legitimate knowledge is characterized by its social and negotiated character (Burton 1999). Yet, this social aspect of mathematics is rarely discussed or shared with students in mathematics classrooms. In schools, mathematical knowledge is commonly portrayed as consisting of universal truths which exist independently of people and which are discovered by mathematicians through a process devoid of intuition, doubt, or debate. Naturally, when exposed to these views of mathematics and mathematics doing over a long period of time, the students begin to view mathematics as a discipline divorced from ordinary human activity and devoid of social, cultural and political considerations (Burton 1999). As a consequence, they begin to form their own identities in relation to the subject—they assume school individuals as receivers, as opposed to producers, of mathematical knowledge. Such experiences had dominated the mathematics education of a majority of the subjects in the study. Nearly 90% of the participants suggested that they felt discussions and arguments did not have a legitimate place in a mathematics classroom. Therefore, they did not find them essential to their learning. Nearly 80% of this group expressed the belief that debates reflected only conflict of "opinions" and "personal feelings" on non-mathematical matters. They believed mathematical truths were already in place, and the universal truths of the mathematics they were to learn could not be judged, questioned, and negotiated. They also believed this mathematical knowledge was to be developed by experts, thus, they felt incapable of generating knowledge. The following comment one of the participants summarizes the group's perspectives.

> In our philosophy or social studies courses we do a lot of debates, we take a position and then try and defend it. I was on the debate team in high school, so I am really good at it myself. But, in a mathematics class, ultimately, the knowledge is there. We can argue over them and all but it is not going to change what is already there. That is the difference between mathematics and other subjects. We debate on things that are gray, like abortion, things like that. In mathematics it is either right or wrong. (Mary)

Although all students stated that they appreciated having an opportunity to share their understandings with peers, they were also reluctant to trust their results as worthwhile knowledge in the absence of the authority of the teacher or a textbook. This sentiment was particularly strong among two groups of students: those who felt least secure in their mathematical ability, and those who had felt successful in traditional classroom culture. The following two statements by two of the participants from 2 different institutions capture the ideas of each group.

> Mathematics for me is about right or wrong. It is something I do and then check to see if I am right. I follow directions to get the answers. I know I am a math minor but when it comes to problem solving and such I am not really good at making sense of it. The best way for me is to study what is there carefully and then reconstruct from examples. If I am not sure what is right or wrong, I ask someone and they explain it to me. (Ken) I think I am really good at math, I have a 3.8 GPA but there is a good chance I may not pass this linear algebra class this semester. I wished he would just do what he is supposed to, tell use what to do, show us a few examples, and then we do the practice. I am not happy that my grade is being jeopardized here (Colleen)

Is this Legitimate Curriculum?

In traditional curriculum, a majority of the tasks are designed so the students must place immediate closure on problems by producing results in a relatively short period of time. To assure success, when dealing with such curriculum, the students must pay close attention to teachers' presentations, take note of examples illustrating how procedures should be used, and learn to follow specific steps to reach results. The students rely on the epistemic authorities, either the teacher or the textbook's answer-key, to measure their own progress and learning. The structure of assignments is such that opportunities for decision making are rare. These experiences lead to forming cultural assumptions that mathematics means following the rules laid down by the teacher; and knowing mathematics means remembering and applying the correct rule when the teacher asks a question (Gregg 1995). In contrast, in a reform based curriculum, students explore mathematics. When confronted with a problem, students must first spend time to understand the parameters of the problem and then devise plans to systemically investigate relationships until a generalizable result is achieved. Indeed, efforts at solving problems may not yield immediate results. According to our participants, developing the skills to enact such relationship with mathematics was difficult. Indeed, adjusting to new curricular expectations was particularly difficult for those students who had experienced considerable success in controlling the demands of more traditional curriculum. They expressed greater need for placing immediate closure on tasks, less patience for ambiguity, and attached little value on the creativity possible in exploration of problems. On the contrary, those students who had become more secure in their problem solving skills appreciated having the opportunity to "explore" mathematics. The following two comments summarize the perspectives of each group.

> This was the first time in my entire education that I had a chance to really explore things and not be afraid of—like being wrong-- or being penalized if I don't have the right answer I am a fool, or that if I don't have the answer right that second then I am really stupid. I was not comfortable with that at the beginning because I thought it was just easier to do the standard questions and memorize things. But after spending hours on the same problem in the computer room I knew better what I was doing. (Nathan)
>
> I think the explorations are confusing. It would be been so much more efficient for her to tell us what we should do rather than asking us to spend hours searching for relationships, not knowing which route is even worthwhile. We could have covered twice as much work, at least twice as much work, if we did the course the regular way. I know that if we discover them we tend to remember them longer but I just think is not efficient (Jeff)

Is this Legitimate Teaching?

Traditional teaching practices are based on the assumption that knowledge must be transmitted from the more knowledgeable person, the teacher, to the least sophisticated individuals, the students (Lave & Wagner 1991). Therefore, traditional mathematics classrooms are teacher centered. In these classrooms, the teachers use the technique of telling students what to do. They set, and reinforce particular codes of social and mathematical conduct in class. They channel students' learning using direct verbal instruction. They assess students' work, praise desirable behaviors and outcomes, and condemn disagreeable performance. In these classrooms the roles of the students are also clearly defined, they come to class daily, watch the teacher demonstrate particular mathematical procedures and then practice those procedures (Boaler & Greeno 2000). Repeated exposure to such practices then shapes the students' identities as school-individuals (Holland et al. 1998), and their relationship with, and expectations of, the teacher. The students learn to look for, respond to, and depend on the teachers' explicit and implicit clues to find pathways to short-term and long-term academic success. Having relied on such clues, and developed a mechanism to use and benefit from them, when these clues are removed, the students might feel less secure in their ability to navigate academic expectations. They become defensive to new signals and feel vulnerable when they cannot quickly make sense of teacher's expectations or respond to them. Over 87% of the participants expressed discomfort with their professor's expectation that they must resolve disagreements or to reach consensus on issues. They had difficulty understanding and adjusting to their teachers' practices. They felt incapable of gaining or generating knowledge in the absence of direct instruction.

> I think mathematics is learned best when the teacher presents the concepts clearly, shows a few examples and then assesses to see who has gotten the ideas. That is how mathematics is different from other subjects. I think this way of teaching is going to put a lot of people at unease, the same way that is putting me at unease. (Erica)
>
> I like for my teacher to tell me what is true and what is right. I don't like my peers, some of them I know less than me, to argue and to discuss. Here we are at the end of the semester and I am not passing this class. I think if she had been more directive I would have done much, much better. (Simon)

Is this Legitimate Social Conduct?

Schools have played a central role in conditioning and shaping social behaviors and social interactions among individuals participating in them. They condition students to view certain conduct as desirable and others as inappropriate (Goodlad 1984). The prominent cultural message reinforced in traditional classrooms is that the students must "listen" to directions provided by the adult (the teacher) in order to learn. Student silence (Freire 1972) dominates the culture of schools and learning is regarded as an individual activity, acquired through silent practice or highly controlled interaction. The teacher controls not only the quality but also the quantity of students' exchange through questions and verbal directions. Peer interactions, unless they fit within the framework provided by the teacher, are not accepted, and in many cased, apprehended. This form of controlled discourse places the teacher and the students at two distinct stations within the space of the classroom culture. Although the distance between the two stations is widened by the grading power of the teacher, the location of each individual learner within the students' station is also separated by a cultural fence that encourages individuality, competition, and intellectual ranking according to those grades. Through years of schooling, the students learn to view the teacher's station as sacred, however, they also develop mechanisms for peaceful co-existence within their own station—they share answers and work and provide one another with emotional support when confronted by the authority figures within the school (Levinson et al. 1996). In contrast with this culture of conformity and silent participation, in problem-discussion based classrooms, collaborative learning takes place. This type of instruction positions students as active agents who are mutually committed and accountable to each other for constructing understanding in their discourse. Students are expected to be co-authors with their teachers, of their understanding of mathematical principles and procedures. Therefore, students must accept the social responsibility of sharing knowledge with peers, debating ideas, and challenging one another. Engagement in mathematical debates seemed to have posed a major social challenge to a majority of the students. 91% of the participants expressed that they felt uncomfortable with public discussions and found it difficult to dispute the views of their peers even when they recognized flaws in their reasoning. Operating from the perspective that their role was to support their peers and not make them look incapable, the students were unwilling to confront one another in the presence of the teacher. They were unsure of how their debates may impact the instructors' assessment of self and peer's work and their own social relationships. The following comment is typical of data provided on both surveys which evidenced learners' vulnerability to this type of mathematical exchange, a sentiment particularly prevalent among female students.

> When somebody goes to the board to show her or his work I think to myself he is probably unhappy that he is up there why should I give him a hard time. Even when I find holes in people's arguments I don't like to be the one to them point out. I have to live with these people in other classes. It does not make sense to me that I should challenge them. It is her job, not mine (Jamal)

Discussion and Implications

Interactions among the students' worldviews on the nature of mathematics, their epistemologies about learning and their past experiences with mathematics, serve as powerful and visible forces on their ability and willingness to take on an active role within the learning environment. These beliefs and past experiences shape the value students placed on group learning and determine the extent in which they participate in, and benefit from, socio-mathematical interactions and group discussions. Our findings suggest that recognizing and accepting mathematics learning as a shared and social activity may not be an immediate consequence of immersing students in such activities.

Boaler (2000), in her studies of students' views on their school experiences in England and later in California documented that those students most successful in discussion oriented classrooms found it difficult to adjust to the demands of traditional instruction. She further argued that traditional instructional methods failed to meet the intrinsic need of those students who had a connected knowledge of mathematics. Here we extent her thesis by proposing that students most successful in traditional classrooms may experience the most difficulty adjusting to the demands of reform based instructional methods. These students, though intellectually capable of handing the mathematical demands in reform based classrooms, can be more resistant to change and more reluctant to accept their new roles. It is plausible to propose that long term exposure to innovative instruction has the potential to refine students' traditional views about mathematics and what it means to know and do mathematics. Naturally, school mathematics provides a cultural system, social structure, and rituals that shape experiences and the identity of the individuals who participate in them. Identities formed in traditional classrooms are fundamentally different from those currently valued. The old identities, although difficult to change, can be enhanced through sustained exposure to new classroom cultures that are discourse oriented. Such experiences allow students to find for the development of new identities, and enhance those characters that traditional classrooms force them to suppress. Certainly, much of students' understanding and beliefs are developed through experience. Not only the content but also how the content is shared contributes to students' beliefs. In order for new educational experiences to have any significant impact on students' epistemological and cultural views, they must be sustained and on going.

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TEACHERS' EVOLVING PRACTICES IN SUPPORTING STUDENTS' MATHEMATICS AND LITERACY DEVELOPMENT

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Sociocultural research on mathematics and literacy frames this interdisciplinary investigation of the evolving practices of secondary mathematics teachers as they seek to understand and support their students' mathematics and literacy development. Teachers' evolving practices included (a) their use of the "Problem of the Day" to engage students in thinking and communicating mathematically, (b) their development of "templates" as scaffolding tools for mediating the literacy demands of the textbook, and (c) their choice to explore student engagement and mathematical communication in connection with their classroom practices.

Purpose of the Study

Significant changes in secondary mathematics curricula involving more opportunities for students and teachers to mathematize situations through talk, texts, stories, pictures, charts and diagrams have arisen from the National Council of Teachers of Mathematics (NCTM) *Standards* (NCTM, 2000) and several curriculum projects funded by the National Science Foundation. These changes pose great challenges to secondary mathematics teachers who are generally underprepared to mediate the intersections between mathematics and literacy (Muth, 1993), and even greater challenges to teachers and students in urban settings, where achievement in literacy and mathematics often lags behind achievement of students in other settings (Schoenbach, Greenleaf, Cziko, & Hurwitz, 1999).

Reform-based curricular materials set mathematics instruction deeply within contextually-based problems. Teachers and learners encounter stories, settings, and tasks within which the potential for the development of mathematical concepts is embedded. The teachers' task is to help students engage in mathematical activity through the use of such contextually complex materials. The materials require the teacher to put learners in situations where the development of significant mathematical concepts is necessary to solve problems. But the rich potential for this development is not necessarily obvious from the materials themselves. In other words, when confronted with reform-based curricular materials, the teacher may not recognize the essential content or may not have strategies for using and supporting students' communication about the central mathematical ideas throughout the tasks. It is not clear from a teaching perspective how to bridge from the written discourse of mathematics that places a premium on condensed symbolic meanings to a spoken discourse that focuses on explanation and justification. This is necessarily more complex when the oral language used in diverse settings draws on varied cultural and everyday life experiences. Furthermore, given a diversity of student backgrounds, like those found among urban students, the mathematical tasks will not resonate with all students in the same way. Thus, the teacher must also be prepared to help all students understand the tasks and the contexts and be able to engage with the tasks. The challenge facing the teacher is therefore substantial.

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In this paper, we address the question:

- How do high school mathematics teachers' practices evolve as they seek to understand and support their students' mathematics and literacy development through reform-based mathematics curricular materials?

More specifically, our questions evolved over time and observation to ask:

- How do practices change when teachers consider literacy in their rationales?
- How do practices change when teachers consider students' responses to literacy tasks?
- How do teachers change their consideration of students' mathematics literacy over time?

Perspectives and Guiding Frameworks

The theoretical perspective of sociocultural research on mathematics and literacy frames this interdisciplinary research. Studies of literacy conducted in the 1970s and 1980s were concerned primarily with understanding learners' cognitive processes and teachers' instructional approaches in a variety of subject-area classes (Alvermann & Moore, 1991). While these studies provided insights about how to facilitate learners' comprehension and knowledge acquisition related to key concepts in specific subject areas, teachers did not implement these recommendations on a dependable basis, largely because of complexities related to the structures of secondary schooling and school culture (Hinchman & Moje, 1998; O'Brien, Steward, & Moje, 1995). More recent research has attended to the complex intersections of adolescent learners, texts, and contexts (Hinchman & Young, 2001). Literacy has come to be seen as multifaceted, involving reading, writing, speaking, listening, and other performative acts—all taking place in certain social settings for certain purposes (Hicks, 1995/1996).

Like other domains of study, mathematics classes at the secondary level require teachers and students to use various kinds of literacies and to participate in various discourse communities specific to the domain (Hinchman & Young, 2001; Hinchman & Zalewski, 2000). Some of these literacy practices have been studied from a cognitive perspective (Friel, Curcio, & Bright, 2001; MacGregor & Price, 1999; Mosenthal & Kirsch, 1993; Muth, 1991). Other recent studies have used a sociocultural frame (Atweh, 1993; Borasi & Siegel, 2000; Lerman, 2001; Sturtevant, Duling, & Hall, 2001) because it accounts for aspects of learning mathematics in complex classroom contexts that a focus on thinking processes alone may not. Understanding how teachers' practices evolve as they strive to teach in ways that engage students in communicative practices is the overarching goal of this research study.

Methods and Data Sources

We are using the methodology of the multi-tiered teaching experiment (Lesh & Kelly, 1999), which allows us to collect and interpret data at the researcher level, the teacher level and at the student level. This multi-level approach is intended to generate and refine principles, programmatic properties (such as interventions) and products that are increasingly useful to both the researchers and the teachers. Central to our analytic approach is the notion that, as researchers, we examine the teachers' descriptions, interpretations, and analyses of artifacts of practices that were developed, examined and refined during the research. It is a multi-method approach, using qualitative approaches to explore sociocultural aspects of mathematics communication, and using quantitative measures of mathematics and literacy performance to discern relative effectiveness of teachers' instructional strategies that attend to the literacy demands of the curricula.

Our research team is comprised of university-based researchers in mathematics education and literacy education, mathematics teachers, and their school administrators, including both principals and other instructional leaders, in a mid-sized urban district in the northeastern United States. This district had recently adopted a Key Curriculum Press textbook series—*Discovering Algebra* (Murdock, Kamischke & Kamischke, 2002), *Discovering Geometry* (Serra, 2003), and *Discovering Advanced Algebra* (Murdock, Kamischke & Kamischke, 2004).

This paper, specifically, draws on data collected as field notes from bi-weekly study group meetings with the mathematics teachers at the high school and two university researchers. All nine mathematics teachers at the school participated in the study group meetings. The majority of these nine teachers had also collaborated with the first author for several years on professional development and implementing new curriculum. During this earlier project, the teachers became very aware of the gap between the literacy demands of the textbook series and the students' struggles to meet these demands. Their colleagues at their feeder middle schools were also struggling with a newly adopted textbook series (*Connected Mathematics Project*) that placed similar literacy demands upon their students. One of the things that rose from this recognized need was a research collaboration among the high school mathematics teachers, the middle school mathematics teachers, and university researchers.

The larger mathematics and literacy research project started officially in August 2003, and the high school teachers entered the project with a commitment to (a) work together as a department to implement investigative activities, (b) communicate with each other about things in their practices that worked and things that did not work, and (c) keep improving student achievement as their top priority. Data from their meetings were analyzed inductively, using open coding, axial coding, and selective coding (Strauss & Corbin, 1998).

Results

Our analysis yielded a story of these teachers' evolving practices in supporting their students' mathematics and literacy development. The story begins with these teachers recognizing that their students were not meeting the literacy demands of the textbook, and moves through iterations of teachers' learning about literacy informing their practices, and their practices informing their literacy learning. We offer several illustrations of the evolving practices of these teachers: (a) "Problem of the Day", (b) "templates", and (c) study group focus.

Principle Finding #1: One activity can change dramatically when teachers consider literacy in their rationales. At the beginning of the first year of the project, the department decided that each teacher would give a "Problem of the Day" (POD) on slips of paper to students as they entered the classroom at the beginning of each class. The department chair explained that they decided to use PODs for several reasons: (a) to serve as a motivator for students to get to class on time since the distribution of the PODs ends when the bell rings, (b) to be a vehicle for reviewing ideas covered in the previous class, and (c) to engage students in explaining their ideas by having the PODs require explanations.

Several months into the first year of the project, the teachers decided there was a need for changing the POD format. Another teacher noted that at the beginning of the year, for some PODs, the teachers were simply asking the students to write definitions of new terms. For one POD, this teacher had asked her students to explain what a box-and-whisker plot is. Since students are allowed to use their books or notes for the PODs, several students wrote "see box plot" since they had simply copied what was written in the glossary in the textbook. As a result, the teachers decided to try to use the PODs to engage students in using the mathematical term instead of just defining the term.

Principle Finding #2: An activity can also evolve dramatically when teachers consider students' responses to literacy. Since the department had used the *Key Curriculum Press* books the previous year and had realized the literacy demands involved in the investigations, the teachers decided to create what they called "templates." These were worksheets that modified the investigations in the textbook, often by separating the questions (instead of having multiple questions to answer for a single section of the investigation), providing space for the students to do their work on the template, and sometimes extending the investigation with questions requiring students to explain their reasoning.

Halfway through the first year, the teachers discussed requiring students to do more reading for the investigation from the textbook instead of putting the text of the problem on the template. The teachers decided to gradually reduce the amount of text on the template and have the students read more of the investigation text from the textbook, modeling strategies for independent reading for them as they took away the supports on the template. They noted that they wanted to keep the organizing features of the template but wanted their students to be able to read independently from the textbook.

Principle Finding #3: Over time, teachers change data collection focus as their understandings of mathematics literacy evolve. For instance, at the start of the project, the teachers were focusing on particular strategies for promoting and supporting students' literacy development in mathematics (e.g., using a word wall to introduce new mathematical terms into the classroom discourse). By the beginning of the second year, the teachers were focusing on more general concerns, such as getting all students actively engaged and improving students oral communication about mathematics.

The teachers generated a research focus for the second year with the overall aim of promoting active student involvement and communication. They also generated a number of measures (e.g., the number of student explanations vs. the number of teacher explanations) of their progress toward this aim and asked the university researchers to collect data on these measures. Their requests for specific data to be considered systematically is beginning to move them much closer to understanding what their students understand about the discourses needed for success in their classrooms.

Discussion

These high school mathematics teachers' practices evolved in several ways that suggest a richer understanding of the discourse processes involved in mathematics literacy. For example, they changed their Problem of the Day to provide students with more of a contextualized opportunity to use mathematics discourse instead of focusing on vocabulary in isolation. They changed their templates to provide students with models and a scaffold for their text reading, and then carefully took parts of the scaffolding away gradually, over time. Finally, they changed their focus from concern for numerical test scores so that they could understand better the language students used as they solved classroom mathematics problems.

We believe that these three changes are very much intertwined. From talking with each other about literacy-related concerns in their classrooms, these urban high school teachers moved from general discussions of good activities to support mathematics development to a desire for specific data to help them understand and support their students' unique usage—and non-usage--of mathematics discourses more directly.

With others who look at subject-specific literacy as a discourse acquisition process, such changes hint of growing understanding that learning mathematics is as much about mathematics literacy as it is about learning mathematics algorithms and processes. And, as others have pointed out, they seem to be realizing that mathematics literacy is about more than isolated vocabulary, reading guides, and writing to explain problem solving as has been suggested by earlier literature on content-area literacy recommendations.

Instead, these teachers' development includes realizing that speaking and listening are as much a part of this process as reading and writing are, and that it is important to consider students' responses to these, too, systematically during instruction. More accurately, addressing mathematics and literacy in reform-based contexts includes finding opportunities for supporting extended and precise language use in oral and written contexts, including helping their urban students learn to read and write such discourses by modeling, and by pointing out differences with tasks they may be asked to complete in other settings.

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A MODELING PERSPECTIVE ON TEACHER LEARNING

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One distinctive characteristic of a models and modeling perspective is the recognition that some of the most important "knowledge objects" that any learner can develop consist of models (organizing structures, concepts, systems of interpretation or conceptual systems expressed using a variety of representations) for making sense of complex problem situations. These systems of interpretation (or models) and their resulting representations and inscriptions provide insight into learners' ways of reasoning and therefore constitute resources for teaching and learning. The importance of models in supporting students' learning has emerged in the course of our ongoing work with teachers and has called attention to the lack of developed models for explaining how teachers learn. We therefore build on the literature on the use of models and modeling providing an understanding of how students' learn to propose a model for explaining how teachers' learn. In doing so, we take the stance of researchers trying to understand teachers' learning models.

One distinctive characteristic of a models and modeling perspective (Lesh & Doerr, 2003) is the recognition that some of the most important "knowledge objects" that any learner can develop consist of models (organizing structures, concepts, systems of interpretation or conceptual systems expressed using a variety of representations) for making sense of complex problem situations. In their seminal book, *How People Learn,* Bransford, Brown, and Cocking (2000) claim that knowledge requires understanding the "facts and ideas in the context of a conceptual framework" and organizing those facts and ideas "in ways that facilitate retrieval and application" (p. 16). Systems of interpretation (or models) and their attendant representations and inscriptions provide insight into learners' ways of reasoning and as such constitute resources for teaching and learning. The perspective we take in this paper builds on the notions of models as a resource for teaching by giving us a *language* for addressing the importance of the development of students' conceptual thinking while concurrently addressing day-to-day pragmatic concerns of teachers. The importance of language has emerged in the course of our ongoing work with teachers and has called attention to the lack of developed models for explaining *how* teachers learn. We therefore build on the literature on the use of models and modeling that provides an understanding of how *students'* learn (cf. Doerr & English, 2003; Lehrer & Schauble, 2000) to propose a model for explaining how teachers' learn. In doing so, we use the term *model* to refer to a teacher's organizing structure or systems of interpretation.

The notion of models also gives us a language for addressing the conceptual thinking of teachers and their students while concurrently providing a mechanism for teasing out the process of learning. Teachers need models of how students' learn to effectively guide their instructional decision-making. They need to understand the big ideas of the discipline, how students' ideas develop, and the important connections between the two (Borko, 2004; Schifter & Fosnot, 1993). We argue here that those engaged in the professional development of teachers need similar models of how teachers learn. For this reason, in this paper we take the stance of researchers trying to understand teachers' learning models.

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Theoretical Perspective

"Models are systems of elements, operations, relationships and rules that can be used to describe, explain, or predict the behavior of some other familiar system" (Doerr & English, 2003, p. 112). In this paper, we are focused on models in which the underlying structure of the model helps clarify the relationships between content, teachers' (and their students') thinking and effective practice. This view of supporting conceptual development of significant mathematical concepts through modeling is informed by earlier research that posits a cyclic approach to model building (Doerr, 1997; McClain, 2003).

It is important to note that the process of engaging in the iterative cycles of model development, critique, refinement is not in service of a solution; it is instead in service of the development of ways of reasoning via models that can lead to generalization (Bransford, 1996; Doerr, 2000, 1996; Lehrer & Schauble, 2000). For this reason, students need multiple experiences that provide them with opportunities to develop mathematically meaningful models. Therefore, a sequence of well-designed tasks is critical in supporting this development. Thus, a modeling perspective leads to the design of a sequence of tasks that has the potential to elicit significant mathematical concepts leading to a system or model that is reusable in a range of contexts (Doerr & English, 2003).

In a similar manner, a well-designed sequence of tasks and/or activities (investigations, cases, videos) is critical in supporting teachers' development of mathematical concepts. The structure or model that organizes these tasks is then built through a cyclic approach where teachers' diverse ways of reasoning as expressed by their models is the basis of development (Doerr & Lesh, 2003). This cornerstone of a model of effective teaching is first secured through the teachers' deep understanding of the content (Ma, 1999; National Research Council, 2001). Once initially secured, the teacher is empowered to engage in cycles of interaction with students during which more sophisticated models of teaching emerge. This is nurtured in the context of a professional development environment that is, itself, organized around meta-level cycles of model building as shown in Figure 1.

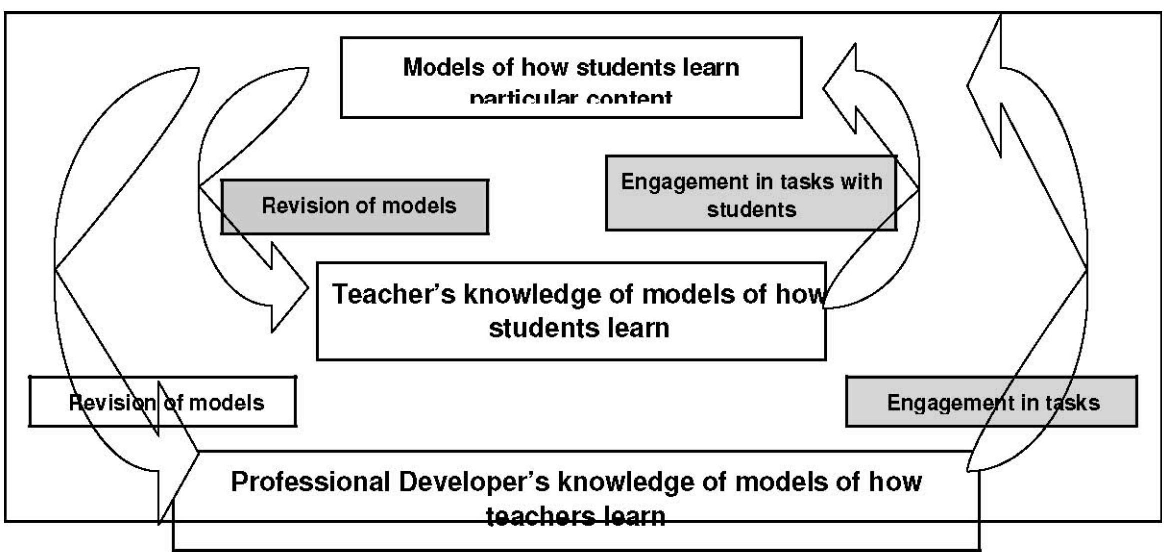

Figure 1. Model of the dynamic interplay of students, teachers and professional developer.

This necessitates first understanding *how* teachers learn so that we can begin to understand the *what* and the *why* of knowledge necessary for effective practice. In this era of high-stakes accountability and demand for highly qualified teachers, we must address these questions in order to be able to provide opportunities for teachers to improve their practice.

Description of the Professional Development Collaboration

Data are taken from a five-year, ongoing professional development effort being conducted with a group of middle-school, mathematics teachers. The collaborative is known as the Vanderbilt Teacher Collaborative at Madison [VTCM]. (Information on the work of the collaborative can be found at www.vtcm.org.) The data consists of videotape of each work session from two (sometimes three) cameras, detailed field notes, and all teacher work. In addition, there are video-recorded teaching sets (cf. Simon, 1999) on each teacher at least once a year, but typically twice. Further, McClain made regular, informal visits to the collaborating teachers' classrooms on a monthly basis. The visits were videotaped.

The envisioned goal of the teacher collaboration was the improvement of teacher content knowledge while supporting teachers' development of a practice that places students' reasoning at the forefront of instructional planning (cf. Peterson, Carpenter, & Fennema, 1989). This implies a role for the teacher that focuses on understanding the varied and diverse models that students are developing to solve tasks. In these instances, the teacher is not trying to guide students to a certain process or solution or ensure that all students understand in a similar manner. The information gained from understanding students' solutions (e.g. arguments and models) is used as the basis of the teacher's decision-making process in the subsequent whole-class discussion and in later planning. This understanding of the students' solutions provides the teacher with the resources necessary to orchestrate a whole-class discussion in a manner that simultaneously builds from students' current ways of reasoning as expressed in their models and that supports the mathematical agenda. This view of whole-class discussion stands in stark contrast to an open-ended session where all students are allowed to share their solutions without concern for potential mathematical contributions. This stance to instruction and the role of the teacher not only represents a fundamental challenge to a core educational practice, but it also represents a fundamental challenge in how we conceive of professional development (Carpenter, Blanton, Cobb, Kaput, McClain, in preparation).

The underlying assumptions of the work were twofold:

1. Teachers need a deep understanding of the mathematics they teach
2. Students' current ways of reasoning should form the basis of instructional decision-making, both in planning and in the day-to-day activity of teaching.

These assumptions guided the choice of tasks and activities for the teacher collaboration and served as a metric for assessing progress. These assumptions were made clear to the teachers who, for their part, viewed the work sessions as about "learning mathematics." The format of the monthly work sessions and summer sessions therefore involved the teachers engaging in statistical data analysis tasks as learners. This was followed by a deliberately facilitated discussion of the teachers' solutions as inscribed in graphs and/or drawings that focused on mathematical issues. In the time between sessions, the teachers posed the task to their students. During the following session the teachers analyzed and critiqued their students' solutions with a focus on students' understandings. The key point to note is that this process involved nested cycles of model building and refinement as shown in Figure 1. The teachers participated with McClain in the first cycle. The teachers then participated with their students in a second cycle.

The third cycle occurred at meta-level and involved reflecting on the activity of the students as expressed in their models. McClain then engaged in a fourth cycle during which she made conjectures about modifications to the task that could better leverage the mathematics.

Results of Analysis

The first work session of the collaboration was organized around a task that involved analyzing data on the number of hours that thirty seventh-graders reported watching television in one week. The task, as posed to the teachers, was to find a way to organize and represent the data so that a recommendation could be made to parents concerning hours of television their students should watch. The teachers expressed no initial difficulty with the task but did ask if a "graph" was a way to "represent the data." Although McClain did not directly answer the question, each of the five groups of teachers subsequently made a bar graph of intervals of data. We make the distinction between a bar graph of intervals of data and a histogram, which is graphed on a continuous axis since this distinction created a lively discussion between McClain and the teachers. For instance, they used bars to represent the number of students who watched between 0 and 5 hours, 6 and 10 hours, and so on. The vertical axis indicated frequency and the horizontal axis displayed discrete categories of numbers of hours. In addition, all groups calculated the mean, mode, median and range and placed it on the chart paper along with the graph. However, these were not the significant aspects of the teachers' work as *they* described it. Deciding on a clever title that clearly described the data and colorfully decorating the graphs were the priority of all groups. As an example, one group drew a large television frame around their graph as if the graph were being shown on the television. As McClain monitored the teachers' activity, she noted the similarity in their work and their attention to artistic detail.

In the subsequent whole-group discussion, each group posted their graph on the wall. When they saw that all groups had created the same graph, they commented with relief, "We must have done it right!" To the teachers, the correct graph *was* the solution. However, none of the groups answered the question. In other words, no one actually analyzed the data; they simply created a graph. When asked to explain their analysis, teachers went through the steps for creating the graph, which they called a histogram, and calculating the mean. When asked where, say, 51/2 hours was placed, they explained that they rounded to six. When asked about a recommendation, all groups pointed to the average, noting that it was what the *typical student watched*.

The number of mathematical misconceptions that emerged from this one task provided a rich setting in which to problematize a focus on the mean and to clarify the difference between a continuous and discrete axis. As the discussion progressed, the teachers were very focused on the *conventions* for the differences between a bar graph and histogram. However, McClain was unable to spark an interest in what the two graphs tell you about the data. Discussions of the mean were equally unproductive, although the teachers took copious notes about what the mean actually represented (e.g. the average number of hours of television watched by 30 7^{th} graders, not the number of hours watched by an average 7^{th} grader). They did agree there was not a *typical* seventh grader although that language still appeared in conversation.

The following session, the teachers shared their students' work on the *How Much TV?* task. In presenting their students' graphs, every teacher focused on labels on the axis and the title. They were impressed by clever, artistic presentations, even when the data were displayed incorrectly. No student had analyzed the data. Although the cycles provided McClain with data on the teachers' current understanding, the process at this point had not offered a means of supporting more sophisticated understandings. The following year, the first task posed to the teachers an analysis of data on the number of hours that forty eighth-grade students reported studying in a week. The purpose of the analysis was to make a recommendation to the Board of Governors about a homework policy.

The task was intentionally similar to the *How Much TV?* task, although none of the teachers commented on this fact. The intent was to use this task to compare the teachers' ways of reasoning with those from the previous year in order to assess the learning of the teachers. In this second task, the teacher models were much more sophisticated and actually represented their thinking as opposed to the first year which were just productions of conventions. As an example, one group took the current district policy concerning homework for eighth graders of 12 hours per week and calculated the proportion of students who studied more than 12 hours and the proportion that studied less. They then formulated an argument for maintaining the current policy based on this data. Other groups created histograms and box and whiskers plots and subsequently used the cut point of 12 hours as a basis for reasoning about the proportion of data above or below the cut point. No group calculated the mean. Further, in their presentations, the representations were *not* the solution. The argument that the representation supported took primacy. Many teachers also argued that there was insufficient data to make a reasonable recommendation; they also needed the grades for these students. They were, in effect, creating design specifications for a study that could adequately answer the question at hand. Here we argue that the model included the inscription and the argument.

It is important to note that the teachers and their students were engaged in monthly statistical data analysis activity in the time between the two tasks reported in this paper. As a result, the ongoing cycles of model building, critique and refinement contributed to the teachers' changing understanding of what it means to analyze data. The full analysis of that data is beyond the scope of this paper. Here, we are focusing on creating a model of how teachers' learn by formulating some conjectures based on the change that occurred in the teachers' models over these two episodes.

A simple overview of the teachers' conceptual models on the first task would entail:

- Analyzing data is synonymous with making a graph.
- The priority in creating a graph is conforming to conventions.
- Graphs are inadequate and must be accompanied by measures of center and spread.
- There is a mapping between all data sets and the correct graph that should be made.

The cycles of (1) first solving the task themselves, (2) then posing the task to their students, and (3) finally engaging in a meta analysis of the students' work were insufficient to cause a perturbation in how the teachers reasoned about statistical data analysis and, therefore, were extremely limited as an opportunity for the teachers to learn. However, there is a significant difference in the teachers' models across the two tasks. On the second task, their presentations to each other were characterized by justification, critique and refinement. The discussion of the teachers' arguments and representations moved from less sophisticated to more sophisticated models, thereby providing numerous instances for previous models to be challenged, critiqued and revised. This was not possible on the first task since all groups created the same representation. The variety in ways of reasoning that emerged created productive mini-cycles that occurred within the first large cycle (e.g. year one), which created numerous opportunities for learning. The teachers expressed a true concern for dealing with the question. The visual models the teachers created were in service of answering the question and were no longer linked to canonical forms of graphs. Although teachers generally agreed about the outcome, extensive time was spent in mini-cycles of model revision based on questions and critiques. The discussions of the students' solutions were equally engaging. They, in fact, contained a variety of mini-cycles as the teachers not only changed their own models of understanding of the mathematics, but also took seriously what the students' models revealed about understandings. A simple overview of the teachers' conceptual models on the second task would entail:

- Analyzing data entails interrogating data to answer a question.
- The model (representation, graph or inscription) is in service of the argument.
- A critical aspect of analysis is returning to the data and asking follow-up questions.

The differences between these two models is the foundation for our claim that learning took place for these teachers.

Some Possible Principles of a Generalized Model

Based on the claim that learning did occur for this group of teachers as evidenced by the changes in their conceptual models, we want to put forth tentative principles leading to a generalized a model of how teachers learn. . The first principle is that *learning occurs when teachers are engaged in mathematical tasks that they view as significant to them and their teaching and that present a real problem for them*. To elaborate, we go back to our earlier statement about the teachers seeing the first task (e.g. How Much TV?) as non-problematic. The task, as the teachers interpreted it, was to map the data to the correct graph. Although they did see the task as something they might try with their students, they viewed it as an "add-on." The task was not viewed as central to their instruction since it did not represent a resource for enabling them to teach conventions for graphs in a better or more efficient way.

Our second principle is that *teachers learn when they assume the role of learner in the context of mathematical investigations*. Unless the teacher is able to take the position of learner, the engagement in the task will likely be from the point of someone being assessed on his or her mathematical knowledge. Teachers in this position will view the result of their activity as a judgment of their worth as a teacher of mathematics. They may be unable to question since questions imply not knowing; this is especially difficult for secondary teachers who (unlike elementary teachers) do have advanced training in mathematics. However, engaging as a learner affords the teachers an opportunity to question, argue and critique in a manner that supports their learning.

Our third principle is that *teachers learn when they engage their students in tasks that highlight their new mathematical knowledge*. This situation provides numerous opportunities for teachers to explore the relevance of their new knowledge to their instruction. It also provides an ideal setting in which to further extend the knowledge through interactions with students who pose new and engaging questions. If we learn best by teaching, then teaching must be a central component in teachers' learning. However, it is not in the day-to-day activity that we can expect this to always occur. It is within the organized activity of first exploring the mathematics themselves, engaging with their students and then reflecting on their activity with a critical eye. In this process, teachers' classrooms become sites for their own and their students' learning.

Concluding Remarks

The complexity inherent in the interrelation of the three principles of the model highlights the difficulties involved in professional development that takes seriously *how* teachers learn. However, the model is in line with Smith's (2000) call for reform in professional development — it provides a model of teacher learning to guide interventions in a manner consistent with research on student learning in mathematics. Taking the stance that a deep understanding of the mathematics one teaches is a *necessary, but insufficient condition* for highly qualified teachers requires that our model attend to teachers' learning of mathematics. Fortunately, the knowledge base on student learning provides an entry into the development of mathematical competence by teachers. The principles we have posited above provide a framework for understanding and explaining how the development of teacher-level models of mathematical competency occurs. Models of how teachers learn the mathematics that they teach is an important first step in theoretically grounding professional development collaborations; these principles themselves should be regarded as a "first cycle" of development of a such a theoretical model and that the principles will be tested, revised and refined as this work continues. Articulating this model of how teachers learn provides an organizing structure for our teacher development efforts and supports the opportunity for theory to emerge from practice in a systematic, disciplined manner.

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THIRD-GRADE STUDENTS, A STANDARDS-BASED MATHEMATICS CURRICULUM, AND ISSUES OF EQUITY

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This paper reports on a study that investigated how patterns of student learning and achievement associated with regular use of a Standards-based curriculum—TERC's revised Investigations curriculum—compare with patterns of student learning and achievement associated with regular use of other more traditional curricula across third-grade students delineated by ethnicity and socioeconomic status (SES). Moreover, in this paper, I explore the following question: How does the revised Investigations curriculum influence ethnic- and SES-related achievement gaps among third-grade students? The Indiana University Content-Focused Third-Grade Assessment was administered to approximately 380 third-grade students from eight schools located within a Midwest urban school district in both the fall and spring. The findings indicate that those students who may benefit from the curriculum the most are not those often considered to need mathematical empowerment the most— the low-SES, minority students— but rather, are the white, higher-SES students.

The ultimate goal of the National Council of Teacher of Mathematics' (NCTM) Standards and likewise of Standards-based curricula is to positively impact *all* students learning and to help *all* students gain mathematical power. Thus, implementation of the Standards has important implications for those students who have the least power in our society, most notably many African Americans, Latinos, and the poor. Because equity is listed first among the core principals of the *Principles and Standards* (NCTM, 2000), implementing a Standards-based curriculum carries with it an implied promise that it will result in the kind of positive and meaningful changes that are necessary to improve achievement among such marginalized groups (Lubienski, 2000a; Lubienski & Shelley, 2003; Martin, Franco, & Mayfield-Ingram, 2004). However, research documenting the benefits and limitations of using Standards-based curricula with these students is still needed to provide insight about its impact on these marginalized groups (Lubienski, 2000a, 2000b; Lubienski & Shelley, 2003; Martin, Franco, & Mayfield-Ingram, 2004). Moreover, as implementation of Standards-based curricula takes place, the implications for marginalized students must become and remain the focus (Martin, Franco, & Mayfield-Ingram, 2004; Secada, 1992).

This paper reports on a study that investigated how patterns of student learning and achievement associated with regular use of a Standards-based curriculum—TERC's revised *Investigation*[1] (*Inv*.) curriculum—compare with patterns of student learning and achievement

[1] *Investigations in Number, Data, and Space* is a National Science Foundation (NSF)-funded curriculum developed by TERC. *Investigations* was one of the first curricula funded by the NSF (in 1990) in that agency's attempt to develop K-12 options that embodied the *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989). In 2001, TERC received another NSF grant to revise *Investigations* over the following five years. The revisions build on the

associated with regular use of other more traditional curricula across third-grade students delineated by ethnicity and socioeconomic status (SES). Moreover, in this paper, I explore the following question: How does the revised *Inv*. curriculum influence ethnic- and SES-related achievement gaps among third-grade students?

Methodology

The study is part of an ongoing, large-scale, longitudinal study being conducted by the Indiana University (IU) Curriculum Evaluation Research Team (see Kehle, Lambdin, Essex, & McCormick, 2004). The approximately 380 participates (third-grade students) are from eight schools (four revised *Inv*. and four comparison schools[2]) located in an ethnically and socioeconomically diverse large city in the Midwest. The revised *Inv*. schools were selected on the basis of their willingness to use the revised curriculum, even if it was in prepublication form, and because they were known by TERC, the author of the curriculum, to have been previously using *Inv*. The local comparison schools were selected to match the *Inv*. schools on SES-, ethnic- and achievement-related criteria. Figure 1 shows the SES of both groups of students. SES in this study was determined by students' eligibility for free or reduced-priced lunch. Thus, low SES represents those students eligible for free and reduced-priced lunches and middle/high or higher SES represents those students not eligible. From Figure 1, it is clear that there are fewer low-SES students in the non-*Inv*. groups than in the *Inv*. group, but both groups do have a high percentage of low-SES students.

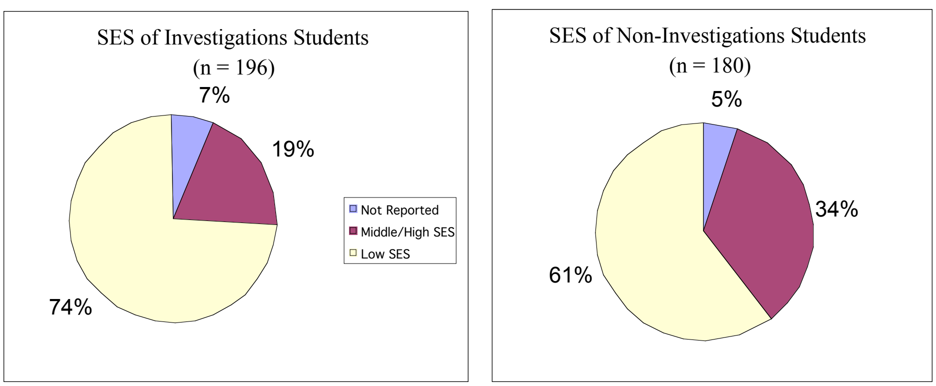

Figure 1. SES of the *Inv*. and non-*Inv*. students.

Figure 2 shows the ethnicity of both groups of students. From Figure 2, it is clear that both the *Inv*. and the non-*Inv*. groups are ethnically diverse. Roughly one fourth of the *Inv*. students are Latino. Over one fourth are African American, and over one fourth are white. Compared to the *Inv*. group, the non-*Inv*. group has a larger percentage of African American students and white students but a smaller percentage of Latino students.

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recommendations of the newer *Principles and Standards for School Mathematics* (NCTM, 2000) as well as the findings from research to date.

[2] The comparison schools used Houghton Mifflin and McGraw-Hill.

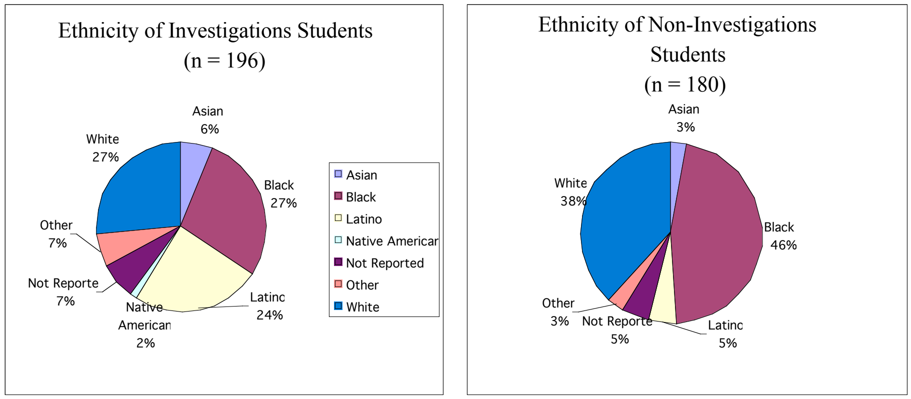

Figure 2. The ethnicity of the *Inv*. and *non-Inv*. students.

Three primary instruments were used to collect data for this study: (a) the Iowa test of Basic Skills (ITBS), (b) a content-focused, third-grade assessment designed by the IU Curriculum Evaluation Research Team, and (c) teacher curriculum logs and a pedagogical survey. The ITBS mathematics and reading subtests were administered to the participants in the fall of the 2002-2003 school year. The results of the ITBS were used to account for any initial differences in scholastic achievement between the *Inv*. and the comparison classes. The IU content-focused, third-grade assessment was also administered early in the fall of 2003 and again in the spring of 2004. This assessment focuses on the content areas of number sense and operations and algebraic reasoning and was designed to capture the growth in students' learning over one school year. It emphasizes problem-solving contexts and authentic tasks over symbolic computation, but it also includes several items involving symbolic computation. The third type of instrument, the teacher curriculum logs and survey, helps describe the curriculum that the teachers implemented in the classroom and control for the fidelity of implementation of the curriculum being used.

Results

I begin this section with an analysis of the overall gains of the *Inv*. and non-*Inv*. students. I follow with the results focusing first on SES-related differences, next on ethnic-related disparities, and then on SES- and ethnic-related differences. As NCTM's Task Force on Mathematics Teaching and Learning in Poor Communities (Cambell & Silver, 2000) claims, "Although race/ethnicity has frequently been used as a basis for examining poor school achievement, . . . poverty is actually a more important demographic factor" (p. 2). Nevertheless, I do present the ethnic-related differences. These results provide the preface for the next portion of the analysis focusing on both the SES- and ethnic-related achievement differences. As Lubienski and Bowman (2002) and Secada (1992) noted, this type of multidimensional analysis is essential to capture the complexity that cannot be caught by unidimensional analysis of achievement patterns.

Overall HLM Results: Inv. Versus Non-Inv.

Using the ITBS to adjust for prior achievement, a two-level hierarchical linear modeling (HLM) analysis found a significant (p < .05) effect associated with the *Inv*. curriculum. On average, students using the *Inv*. curriculum had normalized gains (Hake, 2005) that were .07 higher than students not using the *Inv*. curriculum. For example, a randomly-selected student not using *Inv*. with the average ITBS math score of 170 would expect a normalized gain score of .20, and a similar student using *Inv*. would expect a normalized gain of .27. Variances components estimates for level-1 (the students) of .03 and level-2 (the classrooms) of .004 resulted in an intraclass correlation coefficient of .13. That is, approximately 13% of the overall model variances can be attributed to the level-2, the classroom, attributes. Moreover, in the HLM model, the curriculum variable accounted for approximately 30% of the level-2 variation.

Overall Results: Focus on SES and Normalized Gains

Within a curriculum. When comparing the normalized gains of the *Inv*. students, the results indicate that the higher-SES students' average normalized gain was significantly greater than that of the low-SES students ($t(181) = -3.14$, $p < .01$) (see Table 1). The effect size $g = .57$ indicates a medium effect. A similar analysis of the gains made by the non-*Inv*. students indicates that there was no significant difference between the normalized gains of the lower and higher-SES non-*Inv*. students.

Table 1. *The Mean Normalized Gain Scores on the IU3 Assessment Delineated by SES and Curriculum.*

	Inv.				Non-*Inv*.				Pairwise Comparisons			
	Low SES (n = 145)		M/H SES (n = 38)		Low SES (n = 109)		M/H SES (n = 62)					
Scores	*M*	*SD*	*M*	*SD*	*M*	*SD*	*M*	*SD*	t_{Inv}	t_{Non}	$t_{\text{L SES}}$	$t_{\text{M/H SES}}$
Fall	.23	.16	.32	.18	.19	.18	.23	.19	-3.14**	-1.17	1.62	2.44*

Note. *$p < .05$. **$p < .01$.

Within a socioeconomic group. Delineating by SES and comparing the normalized gains of the low-SES *Inv*. students to those of the low-SES non-*Inv*. students indicates that there were no significant differences in the gains made by these two groups of students over the school year. In contrast, the normalized gains of the higher-SES *Inv*. and non-*Inv*. students were significantly different, again in favor of the higher-SES *Inv*. students ($t(98) = 2.44$, $p < .05$). The effect size g of .50 indicates a medium effect.

Overall Results: Focus on Ethnicity and Normalized Gains

Since 78% of the *Inv*. students and 89% of the non-*Inv*. students are African American, Latino, or white, the analysis presented focuses on these three ethnic groups. Because there are merely nine Latino non-*Inv*. students, I caution the reader to take into account the size of the group when considering the results.

Within a curriculum and between each ethnic group. Comparing the mean normalized gain scores of the three ethnic groups within both curricula indicated that there were significant differences between the mean scores of the different groups (see Table 2). Within *Inv*., the mean normalized gain scores of the white students was significantly greater than that of the African American students ($p < .01$) and the Latino students ($p < .05$). However, there was no significant difference between the mean normalized gain scores of the African American and Latino students. For the non-*Inv*. students, there was only one significant difference ($p < .05$) between the means of the three ethnic groups, and that was between the African American and white students' mean gains.

Table 2. *The Inv. and Non-Inv. Students' Mean Normalized Gain Scores Delineated by Ethnicity.*

	Ethnicities						Pairwise Comparisons		
	African American		Latino[a]		White				
Curriculum	*M*	*SD*	*M*	*SD*	*M*	*SD*	$t_{AA/H}$	$t_{AA/W}$	$t_{H/W}$
Inv.	.20	.14	.22	.14	.29	.20	-.71	-2.78**[b]	2.11*[c]
Non-*Inv.*	.17	.17	.19	.18	.24	.19	-.40	-2.37 *[d]	.50

Note. *$p < .05$. **$p < .01$.

$n_{\text{African American INV}} = 55$. $n_{\text{African American NON}} = 83$. $n_{\text{Latino INV}} = 48$. $n_{\text{Latino NON}} = 9$. $n_{\text{White INV}} = 52$. $n_{\text{White NON}} = 68$.

[a]The small size of the Latino non-*Inv*. group should be considered when interpreting these results.

[b]The effect size g of .52 indicates a medium effect.

[c]The effect size g of .42 indicates a medium-low effect.

[d]The effect size g of .39 indicates a medium-low effect.

Within each ethnic group and between curricula. Table 3 contains the mean normalized gain scores of the *Inv*. and non-*Inv*. students within each ethnic group. As you can see from Table 3, within each ethnic group, the mean normalized gain scores of the *Inv*. students appears to be greater than that of the non-*Inv*. students, but statistically there was no difference between the scores.

Table 3. *The Mean Normalized Gain Scores of the Inv. and Non-Inv. Students Within Each Ethnic Group.*

	Inv.		Non-*Inv.*		
Ethnicity	*M*	*SD*	*M*	*SD*	*t*
African American	.20	.14	.17	.17	.32
Latino[a]	.22	.14	.19	.18	.70
White	.29	.20	.24	.19	.15

Note. $n_{\text{African American INV}} = 55$. $n_{\text{African American NON}} = 83$. $n_{\text{Latino INV}} = 48$. $n_{\text{Latino NON}} = 9$. $n_{\text{White INV}} = 52$. $n_{\text{White NON}} = 68$.

[a]The small size of the Latino non-*Inv*. group should be considered when interpreting these results.

Overall Results: Focus on SES and Ethnicity and Normalized Gains

In this section of the results, I focus on both SES and ethnicity. This proved to be the most problematic analysis statistically because the number of students in many of the groups or subgroups is relatively small (ranging from 5 to 70 students). The small sizes of the groups should be taken into account when considering these results. Even though some of the subgroups are small, capturing the complexity of the different dimensions of the results is essential in creating a complete picture of the overall results in terms of equity.

Within a curriculum and ethnic group and between socioeconomic groups. I began this part of the analysis by comparing the mean normalized gains of the different socioeconomic groups within an ethnic group. For the *Inv*. students, there was no significant difference between the means of the African American and Latino low-SES and higher-SES students (see Table 4). For the white, *Inv*. students however, the higher-SES students' mean was significantly greater than that of the low-SES students ($p < .01$). The effect size g of .82 indicates a large effect.

Table 4. *The Inv. Students' Mean Normalized Gain Scores Delineated by Ethnicity and SES.*

	Low-SES		M/H SES[a]		
Ethnicity	*M*	*SD*	*M*	*SD*	*t*
African American	.19	.13	.25	.21	-.90
Latino	.20	.15	.26	.12	-1.12
White	.24	.19	.40	.21	-2.73**

Note. $^{**}p < .01$.
$n_{\text{African American Low-SES}} = 50$. $n_{\text{African American M/H}} = 5$. $n_{\text{Latino Low-SES}} = 37$. $n_{\text{Latino M/H}} = 11$.
$n_{\text{White Low-SES}} = 36$. $n_{\text{White M/H-SES}} = 16$.
[a]The small size of the higher-SES African American and Latino group should be considered when interpreting these results.

Table 5. *The Non-Inv. Students' Mean Normalized Gain Scores Delineated by Ethnicity and SES.*

	Low-SES		M/H SES		
Ethnicity[a]	*M*	*SD*	*M*	*SD*	*t*
African American[b]	.17	.17	.17	.16	.01
White	.24	.17	.24	.19	-.09

Note. $n_{\text{African American Low-SES}} = 70$. $n_{\text{African American M/H}} = 13$. $n_{\text{White Low-SES}} = 25$. $n_{\text{White M/H-SES}} = 44$.
[a]Because there were so few Latino, non-*Inv*. students (there were nine), they were not considered in this part of the analysis.
[b]The small size of the higher-SES African American group should be considered when interpreting these results.

Table 5 presents the mean normalized gain scores of the different non-*Inv*. African American and white socioeconomic groups. As the table shows, the mean gain scores of the low-SES and higher-SES African American students were both .17. In addition, the mean normalized gain scores of the low- and higher-SES, white students were also the same (M = .24). Thus, there was no significant difference between the socioeconomic groups within the ethnic groups for the non-*Inv*. students.

Within a curriculum and socioeconomic group and between ethnic groups. The actual statistical limitations of the size of the groups proved to encumber the next part of the analysis. That is, there was no statistical difference between any of the groups, but there were some noticeable differences between several of the means. Some evident patterns arose when comparing the mean normalized gains between the low-SES, African-American (M = .19, SD = .13), Latino (M = .20, SD = .15), and white (M = .24, SD = .19) *Inv*. students and comparing the means of the higher-SES, African American (M = .25, SD = .21), Latino (M = .26, SD = .12), and white (M = .40, SD = .21) *Inv*. students. The low-SES groups had much smaller differences in the means between the three ethnicities than the higher-SES groups. The higher-SES, white students' mean appears much greater (not statistically) at .40 than that of the higher-SES African American and white students, which were .25 and .26 respectively. However, because there were so few African American and Latino, higher-SES *Inv*. students, these differences were not significant.

As with the *Inv*. students, there were no significant differences between the mean normalized gain scores of the different ethnic groups within each socioeconomic group of non-*Inv*. students. For the low-SES, African American (M = .17, SD = .17) and white students (M = .24, SD = .17) and for the higher-SES, African American (M = .17, SD = .16) and white students (M = .24, SD = .19), the differences were too small to be statistically significant.

Within a socioeconomic and ethnic group and between curricula. The means of the normalized gain scores of the *Inv.* and non-*Inv.* students from the same ethnic and socioeconomic group were compared. The one significant difference between the means of the different groups was between the higher-SES, white *Inv.* and non-*Inv.* students. Specifically, the mean normalized gain scores of the higher-SES, white *Inv.* students (M = .40, SD = .21) was significantly greater ($t(58) = 2.75$, $p < .01$) than that of their non-*Inv.* counterparts (M = .24, SD = .19).The effect size g of .82 indicates a large effect. Interestingly, the mean gains of the low-SES, white *Inv.* students (M = .24, SD = .19) and of the low-SES, white non-*Inv.* students (M = .24, SD = .17) were the same. Moreover, the mean of the normalized gain score of the higher-SES, white non-*Inv.* students was exactly the same as that of the low-SES, white *Inv.* students with both groups having a mean of .24 and a standard deviation of .19. Though this finding does not fall perfectly under this category, it should be noted that both the low- and higher-SES, white students using the non-*Inv.* curriculum had the same normalized gain scores as the low-SES, white students using *Inv.*

Discussion

The findings of this study add to the discussion about the potential of Standards-based curricula to promote equity. Along with previous studies (Boaler, 2002a; Lubienski, 2000a, 2000b; Lubienski & Shelley, 2003), the findings suggest that implementing Standards-based curricula and pedagogy without paying particular attention to the strengths and needs of low-SES and minority students will not automatically narrow the achievement gap. Moreover, they suggest that using the *Inv.* curriculum has the potential to increase the achievement gap among diverse ethnic and socioeconomic groups. In other words, the findings indicate that those students who may benefit from the curriculum the most are *not* those often considered to need mathematical empowerment the most—the low-SES, minority students—but rather, are the white, higher-SES students. Thus, the findings of this study reaffirm Lubienski's (2000b) assertion that the mathematics education community must "keep equity at the forefront of discussions regarding curriculum and pedagogy"; a curriculum or pedagogical style that is promising for many students may not be promising for those students who need mathematical empowerment the most (p. 480). Even though in this study I explored issues of equity surrounding the use of a elementary Standards-based mathematics curriculum, more studies are needed to investigate the aspects of Standards-based teaching methods that prove to be the most and least beneficial for low-SES and minority students.

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BUILDING AN UNDERSTANDING OF STUDENTS' USE OF GRAPHING CALCULATORS: A CASE STUDY

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This paper investigates the ways in which Advanced Placement Calculus AB students use the graphing calculator when problem solving independently. When students were solving problems on their own they used the graphing calculator in four of the five tool modes suggested by Doerr & Zangor (2002). The students were prompted to use the calculator for five purposes: to skip a step, to get oriented in the problem, to save time, as a new approach, or to check a conjecture. Preliminary findings suggest that students are most often prompted to use the graphing calculator as a visualizing tool when they need a new approach for a problem.

Introduction

How do students use the graphing calculator when they are working on problems independently? Why do they choose to use the graphing calculator as a tool to facilitate the problem solving process? Are the answers to these questions different for boys and girls? These issues have become very important in light of the significant role that the graphing calculator now plays in the instruction and assessment of mathematics. Yet we know very little about the answers to these questions. Thus the purpose of this study is to begin to construct an understanding of how students are using the graphing calculator when problem solving independently.

This study is placed in the Advanced Placement (AP) Calculus AB classroom because all AP students are expected to use a graphing calculator on the College Board AP Exams. Since this is an expectation for all students, the calculator has been incorporated into most AP Calculus classrooms. The tasks chosen for this study are representative of AP Calculus AB free response exam questions. Since interviews took place during the first semester of the school year, the questions were formulated so that they only covered the parts of the curriculum that the students had been exposed to in class. This paper focuses on one of the tasks that the students encountered in the interview process.

Theoretical Framework

It is well documented that students in the United States who use graphing calculators in their mathematics courses outperform those that do not (Ellington, 2003; Meel, 1998; Quesada & Maxwell, 1994; Scheuneman, et. al.; 2002). However, even though technology seems beneficial, there is an open question about whether technology might better suit certain learners. It has been shown that the use of graphing calculators on standardized tests has improved the scores of some groups of students more than others (Scheuneman et.al., 2002; Smith & Shotsberger, 1997). For example, Smith & Shotsberger (1997) conducted a quasi-experimental study in which two instructors each taught one section of college algebra using graphing calculators and one section using a traditional approach. Through quantitative analysis of common exams, the researchers found that the females outperformed the males in all of the graphing calculator sections of the course.

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In a study conducted by the Educational Testing Service (ETS) on the November 1996 and November 1997 SAT I exam, substantial data on graphing calculator use was collected (Scheuneman, et. al., 2002). It was determined that the students who used a graphing calculator on the SAT I exam outperformed those that did not. Additionally, those that used a graphing calculator scored far better than those that used a scientific calculator. It is also significant to note that the girls chose to use a graphing calculator far more than the boys did.

Though the literature provides strong evidence that there is a correlation between graphing calculator use and student achievement, there is no description of how it mediates learning. Doerr and Zangor (2001) conducted a qualitative classroom-based study of one class of pre-calculus students and their teacher. The purpose of the study was to describe how the students and their teacher used the graphing calculator as a tool to construct mathematical meaning out of particular tasks. They determined that in an instructional environment the teacher and students use the graphing calculator in five different modes: as a computational tool, a transformational tool, a data collection and analysis tool, a visualizing tool, and a checking tool. Though this study provided a good description of how the graphing calculator is used in an instructional setting, we still lack a description of how students are using the graphing calculator when they are problem solving outside of the classroom.

The literature provides evidence for the fact that the use of graphing calculator technology does improve student performance on exams. This is especially true for girls, who without the technology, do not traditionally perform as well as males on mathematics exams. Additionally, the literature has begun to describe the different modes of calculator use in instruction, but we do not know how students are using them when they work independently. It is very important to fill this gap in the research so that we can better understand why this technology is such a powerful tool for female mathematics students.

Methods

Participants and Setting

This paper reports on 14 students (6 girls and 8 boys) in the AP Calculus AB course at a midsize suburban high school. The students are all from the same class and have volunteered to participate in this study. The school provided TI-89 calculators for every student in this class. The teacher used the TI-89 calculator as part of the classroom discourse on a regular basis. Furthermore, the teacher verified that this group of 14 students represents the full range of mathematics achievement in the class.

Task

The data reported here come from task-based interviews with each of the fourteen students. The students had the TI-89 available to use during the interview if they chose. The task was the following:

Let f be the function defined by $f(x) = \sin^2 x - \sin x$ *for* $0 \le x \le \dfrac{3\pi}{2}$.

a) *Find the x intercepts of the graph.*
b) *Find the values for which f(x) has a horizontal tangent line*

Analysis

In order to construct an understanding of how the students used the graphing calculator to support their learning, I used both a deductive and inductive coding scheme. I initially coded the data according to Doerr and Zangor's (2000) modes of calculator use as a tool (see Table 1). While coding for calculator use I was also looking for data that did not fit into these codes.

Table 1: Modes of graphing calculator use as a tool

Role of the Graphing Calculator	Description of Student Actions
Computational Tool	Evaluating numerical expressions, estimating and rounding
Transformational Tool	Changing the nature of the task
Data Collection and Analysis Tool	Gathering data, controlling phenomena, finding patterns
Visualizing Tool	Finding symbolic functions, displaying data, interpreting data, solving equations

Initial data analysis indicated that the choice of calculator use was usually triggered by some need the student had. This need made categorizing the use of the graphing calculator as a tool difficult. For example, a student might be using the calculator as a visualizing tool, but the initial purpose was the need to check a conjecture about a function. To investigate this further data was recoded for triggers, reasons that students gave for using the calculator as a tool in a particular way. The triggers that emerged fell into five categories: skipping a step, getting oriented, saving time, starting again, and checking (see Table 2).

Table 2: Triggers for calculator use

Trigger	Description of Student Actions
Skipping a Step	The student does not know how to approach the task by hand, but uses the calculator effectively to meet a goal
Getting Oriented	The student uses the calculator as a starting point with the purpose of getting a feel for the problem
Saving Time	The student knew how to complete the task, but chose to use the calculator to save time
New Approach	The student was having difficulty with the task and chose to use the calculator to try another approach
Checking	The student is using the calculator to confirm results or conjectures, understanding multiple symbolic forms

After coding for triggers, I set out to determine if there was any correspondence between calculator use and trigger. In order to accomplish this I used a matrix like the one suggested by Stake (1995). The matrix was constructed using the four modes of calculator use as a tool to label the columns and the four triggers to label the rows. A tally of each correspondence was completed and recorded in the appropriate cell.

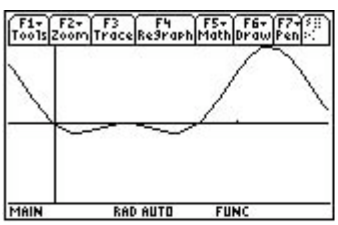
Findings

The students were far enough along in the calculus curriculum that they should have been able to do this problem without the use of the graphing calculator if they chose to do so. All 14 students chose to use the graphing calculator at least once while working on this task. A summary of calculator use as a tool and what triggered its use is shown in Table 3.

Table 3: Calculator use on task #1

	Skipping a Step	Getting Oriented	Saving Time	New Approach	Checking	Total
Computational Tool	2	1	1	0	6	10
Transformational Tool	1	1	4	6	0	12
Data Collection and Analysis Tool	0	0	0	2	3	5
Visualizing Tool	1	1	4	11	6	23
Total	4	3	9	19	15	50

The most common mode of calculator use on this task was as a visualizing tool (46%). This mode was used in a variety of different ways. For example, Shannon used the graph on her calculator to find the zeros of the function (See Fig. 1).



Shannon*: "I saw originally that this was zero [referring to looking at (0, 0)on the graph], so I already found a zero. So I'm just trying to figure out what this one is [she points at the second zero shown on the graph]…if you look at it, it sort of looks kind of close to pi over two…and pi over two is one point five seven.."*

Figure 1: Shannon's graph for part a) of task #1

Sarin also used the calculator as a visualizing tool. However, unlike Shannon who used it to save time, Sarin's purpose was to check his work. He began by finding the x-intercepts by hand. Once he completed that task, he asked if he could use his graphing calculator. He proceeded to graph the function and trace it.

Sarin: *"Well, I wasn't like sure about this function, like the graph of it…and like, and I don't think this will work all the time [referring to the algebraic method he used], so I went back to the calculator and I double checked what happens here. With the graph I traced it, and I found the corresponding x intercepts."*

The calculator was also used as a transformational tool and a computational tool a significant number of times. Calculator use as a computational tool was limited to basic arithmetic. For this problem the students typically used either the "solve", "derive", or "zeros" functions of the calculator as transformational tools.

The students were triggered to use the calculator most often (38%) by the need to find a new way to approach a problem. Many students began to work the problem by hand and when they had difficulty they used the calculator to attempt the problem in a different way. For example, Grace began to find the derivative of *f(x)* by hand. However, she was unsure of her answer and quickly erased her work and then used the calculator to find the derivative (See Fig. 4). When I asked Grace about her reason for attempting the problem in a new way she replied: *"I was trying to do the derivative by hand but then I was getting…I knew the calculator can't make a mistake, so I just went to the calculator and did it."*

The correlation matrix shows that the most common mode of tool use and trigger combination was the use of the calculator as a visualizing tool when a new approach to the problem was needed (22%). Alexander's work above is a good example of this type of situation. It is interesting that the visual approach was often not the first approach for this particular problem. Other interesting correlations were the calculator being used as a computational tool to meet the need of checking (12%), a transformational tool to meet the need of a new approach to the problem (12%), and a visualizing tool to meet the need of checking (12%).

Discussion

It must be noted that this was a pilot study whose purpose was to test out the framework and methodology being used to better understand how individual students use the TI-89 graphing calculator when solving problems independently. With this in mind, I will consider the framework, methodology, and problem choice separately.

I believe the framework used in this study provides an organized lens through which to look closely at student calculator use. The coding scheme used in the analysis was adapted from Doerr and Zangor's (2001) coding scheme for categorizing graphing calculator use as a tool in an interactive classroom setting. The modes of calculator use fit the independent problem solver as well. All calculator use fell discretely into one tool mode.

Any time a student uses a tool when problem solving, there is a reason for choosing that tool. These students often changed their purposes for using the tool, but each prompt fell into one of five categories. These five categories, skipping a step, getting oriented, saving time, developing a new approach and checking seemed to provide a solid description of why students were choosing to use the calculator use as a tool.

With this in mind, I believe that this framework will be useful for any researcher who is hoping to better understand how students are using the graphing calculator as a tool on a task. Given a larger number of participants and a set of well designed problems, this framework has the potential of drawing out differences in the way that subgroups of students use the graphing calculator.

The task chosen for use in this study was not nearly rich enough to draw out differences in the ways that different groups of students might be using the graphing calculator. For future studies it is important to choose problems that are more in-depth and have multiple solution methods so that differences in approach can be studied.

Gallagher and DeLisi (1994) suggest that attributes of questions may influence strategies students choose for solving problems on the SAT-Math test. At the same time, we know that girls have improved more on the SAT-Math test than boys have since the introduction of the graphing calculator use on the exam. With this in mind, it might be helpful to use Gallagher and DeLisi's classification of problems to help choose problems that might result in sex differences in graphing calculator tool use or trigger for use.

Finally, it is often argued that the girls' performance on assessments when using a graphing calculator has improved simply due to confidence (Dunham, 1995; Ruthven, 1990). This may be true, however we need to consider why having a calculator available is a confidence builder.

Future studies should include analysis with this question in mind. Is their confidence higher because they feel they are in a more collaborative situation? Schwartz and Hanson (1992) noted that females prefer a more collaborative and conversational learning environment. Maybe the calculator provides a sounding board while problem-solving independently that isn't present otherwise. It is also possible that having the graphing calculator available influences the students' local affect while problem solving. Goldin (2000) points out the importance of keeping students from slipping into the affective states of anxiety, fear, and despair. Maybe the presence of the calculator is instrumental in keeping students in the preferred affective states of curiosity, puzzlement, and bewilderment that can foster the problem-solving process. Future studies should explore these possibilities to provide a more complete picture of graphing calculator use in the problem-solving process.

Conclusions

Past research has shown that the use of the graphing calculator in secondary mathematics courses increases confidence and competence among all students, but especially girls. Understanding how students use the graphing calculator when problem-solving independently is important for all educators. Understanding how and why students are using the graphing calculators can help educators develop appropriate curriculum materials and make them aware of how calculators are being used effectively and ineffectively. Additionally, we now have a starting point to begin to understand how and why girls use the graphing calculator in different ways than boys. By understanding how females use the graphing calculators, educators will be better informed about what support their female students need in order to push them to complete a higher level of secondary mathematics and possibly go on to pursue math or science related careers.

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DEVELOPMENTAL UNDERSTANDING OF MATHEMATICS WITH ELEMENTARY SCHOOL STUDENTS

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This research describes the process and findings for the development and validation of developmental continua in elementary school mathematics. A major goal was to provide a framework for identifying the phases at which students are operating in mathematics. This framework is based on key concepts and skills matched with phases of development that students pass through to understand mathematics. Three stages of data collection was undertaken to validate the maps and create diagnostic tests that would place a student in one of the phases of development. Findings indicate that deep understanding of some concepts occur later than we think. It also indicates that students seldom consider alternatives when faced with problem-solving activities.

Background

Developmental continua describing the phases that students pass through as they acquire the skills and understanding of concepts associated with given subject are of great use to teachers since they link curriculum, assessment, and instruction. While established developmental continua for elementary language arts have been in use for a number of years, this has not been the case for elementary mathematics.

A major goal of this research was to provide a framework for identifying the phases at which students are operating in mathematics in order to tailor instruction to meet the needs of individual students. This research was conducted across Canada in six provinces. Two questions guided the research study: 1) What are the phases of development that elementary school students pass through as they learn key concepts and skills in mathematics? and 2) How do we identify the phase at which a given student is operating at a particular time.

Although mathematics curriculum guidelines identify learning expectations / outcomes on a grade-by-grade basis, elementary school teachers are faced with the often-daunting task of continuously adjusting the content and pace of instruction to ensure that all students are learning at an appropriate level of difficulty. To assist teachers and students to understand mathematics, three to five concepts and two to three skills per strand were developed. For each concept and skill, behavioral indicators were written describing what a student should be able to understand at each phase.

Theoretical Framework

Expertise in mathematics means more than having acquired a large amount of mathematical information. It also matters how that information is organized. Expert mathematicians have knowledge that is organized around a small number of key concepts that provide the foundation for the discipline. The many connections among these concepts, and the contexts to which they apply, enable mathematicians to generate flexible strategies for solving complex problems.

Similarly, research shows that expertise in teaching mathematics includes developing a deep understanding of mathematical concepts and the relationships among them in order to advance

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student learning (Borko & Putnam, 1995). A significant body of educational research has established the effectiveness of using concept-based organizers to present new knowledge to students (Stone, 1983). If students are able to connect a new concept being taught to previously learned concepts, it is much more likely that the new knowledge will be assimilated.

It would appear that it is beneficial to focus the learning of each mathematics content strand around a few "key ideas," which include both key concepts and key skills. In this way, it becomes easier for students to relate new learning to previous learning. "Key ideas" also simplify the job of the teacher since they provide meaningful organizers that can be used to cluster the lengthy lists of specific expectations/outcomes appearing in curriculum guidelines.

In addition to the need to organize mathematics curriculum guidelines in ways that facilitate teaching and learning, educators are also concerned with understanding how students think about mathematics. One prevailing view is that an improved understanding of how students think mathematically will lead to improved student achievement (Ross, McDougall & Hogaboam-Gray, 2002). Another research project that examined number development was The Rational Number Project, which focused on students' understanding of rational numbers and units of quantity (Carpenter, Fennema & Romberg, 1993). Carpenter et al. point out that there is a clear difference between understanding rational numbers and understanding whole numbers, and that the former is not a simple extension of the latter.

The existing research on phases of development for mathematics provides us with some starting points. It also prepared us to expect that we would have variability during the field-testing stages and contributed to the changes that would be necessary.

Methods and Data Sources

Data collection to validate the developmental maps occurred during two stages of testing in 2003 and 2004. This testing took place in schools in the Toronto District School Board in Ontario, schools in the Calgary Board of Education in Alberta, and schools in New Brunswick. Finally, a smaller third stage of testing, to validate the test items for the diagnostic tests, was conducted in schools in British Columbia, Manitoba, and Nova Scotia in 2004.

Stage 1: In 2002-2003, an author created an initial series of seven developmental maps that sought to describe student learning in each of five strands of elementary mathematics. Students in Kindergarten to Grade 3 answered sets of questions in oral interviews; students in Grades 4 to 6 completed sets of written questions. Test items appeared at more than one grade level since students in any one grade typically span several developmental phases. Approximately 75 students completed each test booklet, so that between 150 and 225 students responded to each item.

Stage 2: In Stage 2 of the research study, responses to the initial sets of questions for each strand was analyzed and revised sets of questions were created. Correlations among responses to questions in the same developmental phase and across developmental phases were examined to establish that the items were, in fact, empirically related, as had been indicated on the relevant map (using a mean >0.5, $r = .40$, and $p<0.05$). Questions yielding low mean scores and questions that were negatively correlated with other items in the same phase or with other items in adjacent phases were further examined and rewritten. The final activity in Stage 2 was to analyze the new data collected using the same procedures as with the data from Stage 1.

Stage 3: Grade-level diagnostic tests were constructed in Stage 3 using questions that remained after the validation process in the earlier stages of the research. The diagnostic tests were administered in three locations that once again represented different regions in Canada: British Columbia, Manitoba, and Nova Scotia.

It is worth noting that the final set of criteria for test item selection, which was based on bivariate correlations was established after careful consideration of a few analytical methods. Two of the options that were seriously explored but rejected were Cronbach's alpha procedure and Item Response Theory.

Findings

Field-test research has validated much of the researchers' original hypotheses regarding the phases of development for the five math strands. All analysis has been completed for all five strands. However, for the purposes of this paper, we will only describe the findings relating to Number and Operations. There were five phases in the Number and Operations developmental map, with five concepts and three skills.

The key concepts for number and operations are:

Numbers tell how many or how much.
Classifying numbers provides information about the characteristics of the numbers.
There are different, but equivalent, representations for a number.
We use a number system based on patterns.
Benchmark numbers are useful for estimating and comparing numbers.
Addition leads to a total and subtraction indicates what's missing.
Addition and subtraction are intrinsically related.
Multiplication and division are extensions of addition and subtraction. Multiplication and division are intrinsically related.
There are many algorithms for performing a given operation. (PRIME ResearchStudy, 2005).

Research for Number indicates that very few students, even at Grade 6, reach Phase 5 (the flexible phase), and many do not even reach Phase 4. This suggests that deeper understanding of some of the topics presented to students may occur later than we think. Another reason might be that the educational system does not regularly provide sufficient conceptual underpinning for students to allow students to reach this phase.

Students were unable to work with fractions and decimals very well. This implies that these topics should be taught using a more exploratory approach in Grades 1 through 5. Specifically, teachers should use manipulatives and technology to help students make the link between conceptual and procedural understanding of fractions and decimals.

Students were able to make comparisons with decimals much easier than with fractions. This is surprising as students are able to name fractions before they are able to name decimals. Students are not able to work with mixed numbers very well. Even at Grade 6, students were unable to distinguish the relationships between improper fractions and mixed numbers. This challenge with decimals continues into late Grade 6. Students are unable to recognize that more decimal places means that there is greater precision.

Communication about student understanding of number is a challenge for many elementary school students in this study. When asked to give explanations about the mathematics concepts, many students recite rules or definitions rather than using reasoning. This finding may help teachers begin with very young children to communicate their understanding using other number representations and concepts.

Research for Operations resulted in a similar finding: very few students, even at Grade 6, reach Phase 5 in the Operations strand because relatively few students are able to work with decimal operations. This is particularly the case in the area of multiplication and division. This also indicates that relatively few students consider alternatives and make explicit choices about how to calculate in particular situations.

We were able to work with very young children in this study. We found that students in Grades 1 and 2 are able to perform some calculations mentally beyond fact recall. Older students were able to understand and perform multiplication more efficiently than their ability to understand and perform division. However, all students were uncomfortable with calculations with decimals higher than thousandths.

When students are asked to create story problems based on a context and using specific numbers or calculations, many students were unable to select a context that made sense for the numbers. For example, when asked to use 12 and 3.01, some students wrote stories that suggested that one person had 12 cars and they wanted to give away 3.01 cars, and wanted to know how many they would have left. These situations were more common with decimals than with whole numbers. This may indicate that students do not see the value in mathematics for its application to real-life situations.

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THE COACHING/MENTORING PHASE IN A PROFESSIONAL DEVELOPMENT PARTNERSHIP PROJECT

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The Professional Development and Curriculum Alignment project (PDCA) is an implementation research study that examines the ways in which the development of teacher mathematical knowledge and instructional practice links to student performance in mathematics. Since January 2000, the researchers have partnered with school districts to create a four-phase professional development model that uses student achievement data and classroom observations to study the phases. The heart of PDCA is the support of teachers' ongoing practice in the demonstration/coaching/mentoring phase. The focus in this presentation is the description and categorization of teacher responses in this phase and the description of coaching moves made by the university partner.

Large-scale assessment and accountability standards have been at the forefront of issues that concern teachers for several years. The No Child Left Behind (NCLB) legislation has raised the stakes even higher. Now federal money allotted to a state is linked to its testing plan at every grade and its efforts to assure "high quality teaching."

The emphasis on the importance and the power of large-scale assessments generates a controversial discussion among educators. Do the tests contribute to the improvement of teaching and learning, or do they dominate and stifle the curriculum? In theory, large-scale assessments have the potential to be positive agents of change in education reform as they evolve from standardized tests and tests of basic skills to more comprehensive assessment plans that also include criterion-referenced tests and performance based tasks. When the large-scale assessment plan provides individual student data, it allows states, districts, schools, and teachers to make instructional decisions that are data driven. This type of assessment can promote systematic change and link student outcomes to professional development, to teacher knowledge, and to implementation of standards-based curriculum.

The literature shows mixed results for the effects of testing in mathematics. (Hancock and Kilkpatrick, 1993; Schorr, Firestone, and Monfils, 2003) Achieve, a nonprofit organization created to help states determine the difference between high-quality and poor-quality standards, assessment, and accountability policies reports, "When they are well devised and implemented, academic standards and tests, and the accountability provisions tied to them, can change the nature of teaching and learning. When they are poorly devised and implemented, [they] can become a distraction and a source of frustration in schools." (Gandel & Vranek, 2001, p. 7) They also conclude that intensive professional development using standards matched by equally rich and rigorous tests that become more challenging at each successive benchmarking grade makes a major difference in the improvement of student performance. This type of professional development requires collaboration between school systems and higher education.

The Professional Development and Curriculum Alignment project (PDCA) is an implementation research study that examines the ways in which the development of teacher mathematical knowledge and instructional practice links to student performance in mathematics

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in Arkansas. Since January 2000, the researchers have partnered with teachers and administrators to create a four-phase professional development model. We use pre/post data from teacher institutes and classroom observation data to study the implementation of the phases. Achievement data indicate the effectiveness of PDCA and determine the direction of future work. The student achievement data are the results of the Benchmark exams administered every spring in grades 3-8. These criterion-referenced tests are challenging in that they assess computation in problem contexts, balance the items across all curriculum strands, and heavily weigh open response items in the raw score.

The beginning phase is the vertical and horizontal alignment of the written curriculum with the tested curriculum. After the teachers identify the development of student learning targets through the grades, in phase two they study major mathematical ideas that unify concepts across curriculum strands (Number, Algebra, Geometry, Measurement, Data & Probability) and select curriculum materials. Phase three is really the heart of PDCA. The demonstration lessons and coaching from university faculty promote transfer of the gains in content knowledge from the institute to the classroom by supporting the teachers' ongoing practice. In phase four teachers, administrators, and university faculty use student assessment data to evaluate and modify curriculum and instruction. Any one of the phases will produce a spike in achievement scores; however, sustained growth requires sustained involvement in all four phases.

A key outcome of the first curriculum alignment phase is the initial change with teachers about computation and basic fact immediacy. As teachers analyze student performance on the released items of the state benchmark tests, they begin to understand that a dependency on pencil/paper drill and worksheets do not correlate well to student performance. Bare computation worksheets need to be replaced by good contextual word problems. The teachers also see that a test item may be related to more than one curriculum strand; the underlying mathematics is really a bigger mathematical idea that connects curriculum strands.

In the content study and curriculum planning phase, which is usually a summer institute, teachers experience major mathematical ideas presented in demonstration lessons for grade bands 3-4, 5-6, and 7-8 so that they can see the growth of a concept across grade levels. The teachers learn to reflect on the lessons from two points of view: the learner and the teacher. The reflection experience will be important in the third phase of PDCA. To meet the challenge of transferring gains in content knowledge to instructional practice, the researchers allot time in the content institute for teachers to examine curriculum resources and materials. Teachers in common grades plan together then share across grades in whole group discussion. An important outcome of phase two is the involvement of teachers in this process that moves them from a traditional curriculum model based on covering a textbook to a dynamic model based on student learning. The degree to which a teacher can describe what students need to learn about a topic in the curriculum correlates to the changes he/she will make in their classroom practice.

This report will focus on the demonstration/coaching/mentoring (D/C/M) phase. Research shows (Smith, 2002; Little, 1993) that high quality professional development should encourage teacher collaboration. PDCA has focused on building a strong collaboration based on trust and parity between the teachers and university educators. The trust is built through three distinct but related processes. First the university educators demonstrate lessons. As observers, the teachers can attend to student learning behavior without having to react to it. Student-centered discourse in which students reason about the mathematics they are learning helps the teacher to begin to learn how to focus on student thinking while they are teaching. The observed classroom interaction and the way that the mathematics is linked to a classroom context change the teacher's knowledge (Fennema & Franke, 1992). As the relationship between the university educator and teacher develops, demonstration evolves into a coaching process. During this phase the lessons are planned together and may be taught as a team. As time passes, the mentoring process begins when the teacher becomes more comfortable with a reformed teaching practice. As a coach, the university educator is more directive; as a mentor, the educator acts in an advisory role. The teacher initiates contact, and the educator provides support, encouragement, and advice. The transition from demonstrating to coaching to mentoring is a multi-year phase that will vary from teacher to teacher with many contributing factors. Weak content knowledge can cause passive behaviors and resistance. Teachers can be reticent to express dislikes or insecurities about mathematics. In fear of exposing their knowledge, they are defensive about their teaching practices (West & Staub, 2003). Some teachers become static in the demonstration phase. They do not believe that they can adapt demonstrated lessons. Teachers who are not threatened by learning and taking risks make significant progress.

We will report four case studies. Two cases are middle schools in districts with about 1500 students each. They are in the outskirts of two large population centers. The middle schools have about 500 students in grades 5-8 with stable student populations from year to year. In Site 1, one of the original PDCA participants, grades 5-7 are housed in one building, and the 8th grade moves to another building. Site 2 aligned curriculum in fall 2001 and added phase 3 in spring 2002. Grades 5-8 are in the middle school building at this site.

Sites 3 and 4 have recently joined the study through the Arkansas Middle School Mathematics Academy grant (ASMSA). These rural communities located close to large towns have student populations of 1200 and 1100, and their middle schools house grades 6-7. ASMSA is one of the middle school grants awarded by the Arkansas Department of Education with NCLB funds in 2004. These grants adapted the PDCA model to participants' needs. Site 2 also joined ASMSA at this time. The alignment phase was streamlined because ASMSA districts had already participated in an alignment activity. In phase two, the Academy participants studied proportional relationships in the five strands and examined four Standards-based curriculum programs (*Connected Mathematics Project, Mathematics in Context, Mathematics, and MathScape*). All demonstration lessons and coaching sessions involved observing investigations from these programs. In phase four, participants and researchers evaluated the four curricula against the newly revised state frameworks and their experiences with different materials in the demonstrations. They chose *Connected Mathematics II* (CMP) as the focus curriculum for the remaining two years of the grant. The 2005 Benchmark data will be presented at the PME conference.

As we gain more experiences from these sites, an interest has emerged in identifying teacher factors related to improved student achievement and in describing the moves that the university faculty partner makes to influence a change in teacher practice. The factors below are related to teacher practice and student achievement:

- Teacher knowledge: Pre/post test data from phase two (summer institutes) shows that teachers with low (below 65 %) scores are more resistant to classroom visits and implementation of demonstration lessons.
- Life-long learner in mathematics: While changes in achievement and practice seem to be independent from teaching experience, they are affected by the teacher's pursuit of professional development (PD). The best case is the teacher who consistently participates in workshops, conferences, or graduate work. This teacher demonstrates curiosity in PD activities and wants to adapt the mathematics for his/her students. The

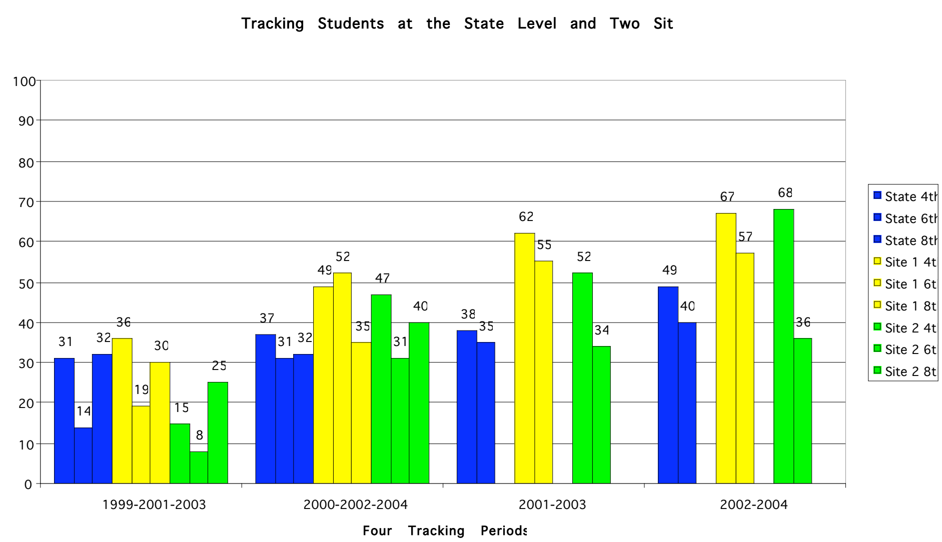
worst case is the teacher who avoids PD and demonstrates reluctance if they do attend required PD.

- Curriculum: There is a continuum in the way teachers understand curriculum. At a low end, they see curriculum as a list of topics and let the textbook drive a teacher-centered curriculum. At the high end, they see curriculum as a framework that organizes mathematical ideas and the student learning targets. The development of connected mathematical ideas drives a student-centered curriculum.
- Assessment: A goal of PDCA is to align the tested curriculum with the taught and written curricula. There also is a continuum in the way teachers understand assessment. At the low end, teachers use traditional end-of-chapter tests that usually test knowledge and skills. They also use many multiple choice practice assessments to prepare for state assessments. At the high end, teachers view assessment as a means to find out what students have learned as opposed to a method of getting a student grade. They value questioning during lessons, performance tasks, projects, warm-up practices, and occasional tests as equal parts of an assessment package.
- Use of demonstrations: At each site we record how many of the demonstration lessons (%) the teachers implement and the way that they adapt a lesson. At a minimum, teachers use provided worksheets or materials. Some teachers focus on getting the lesson technically correct; they mimic the university partner. Other teachers immediately adapt the lesson to their students' needs.
- Administrative support: It is helpful when superintendents and curriculum directors support PDCA, but it is crucial that an in-building administrator communicate expectations consistent with PDCA goals. In the worst case, PDCA has upper level support and only "lip-service" from principals. Ideally, PDCA has the genuine support from both upper level and in-building administrators.

In the table below, we show the pattern of observation ratings of teachers in relation to these six factors. A rating of –1 indicates that the teacher is at the low end of a continuum for a given factor; a 0 indicates that a teacher is midway between the low and high ends—usually we can affect a teacher with this rating; a rating of 1 indicates that the teacher is at the high end of the continuum. If CK + LL + Curric $\leq$ 0, the teacher was stuck in the demonstration phase.

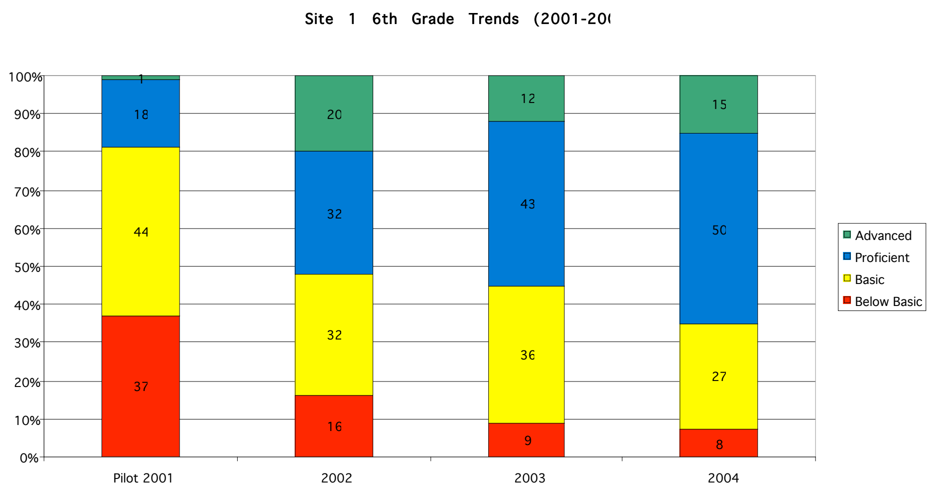
Teacher	Teaching/PDCA (yrs)	CK	LL	Curric.	Phase D/C/M	% use Demos	Type of mimic	Admin. Support
Site 1								
5th	15/2	0	0	1	D	20%	Technical	1
6th	30/6	1	1	1	M	100%	Adapts/Technical	1
7th	4/4	1	1	1	M	60%	Adapts/Technical	1
8th	27/6	1	1	1	C	40%	Adapts	0
8-Alg I	15/6	1	1	1	C	40%	Adapts	0
Site 2								
5th	12/3	-1	-1	0	D	10 %	Good team teaching, not confident solo	0
6th	29/3	-1	-1	-1	D	10%	Technical	0
7th	3/1	1	1	1	C	80%	Adapt	0
8th	5/3	1	1	1	C	20%	Adapts/Technical	0
Site 3								
6th	25/1	0	1	1	C	80%	Technical	1
7th	17/1	1	1	1	C	100%	Adapts	1
8th	10/1	0	0	0	D	10%	Worksheets only	1
Site 4								
7th	5/1	1	-1	0	D	10%	Fair team teaching	1
8th	1 /1	1	1	0	C	90%	Adapts; but not confident in planning	1

The PD data are related to the student achievement data for Sites 1 and 2. The following graph tracks four groups of fourth graders to the sixth grade and to the eighth grade when possible. The tracking is reasonable for the two districts with fairly stable populations. The state data is given as a reference. Each bar represents the percentage of students scoring proficient and advanced (on grade level) on the Benchmark exam.

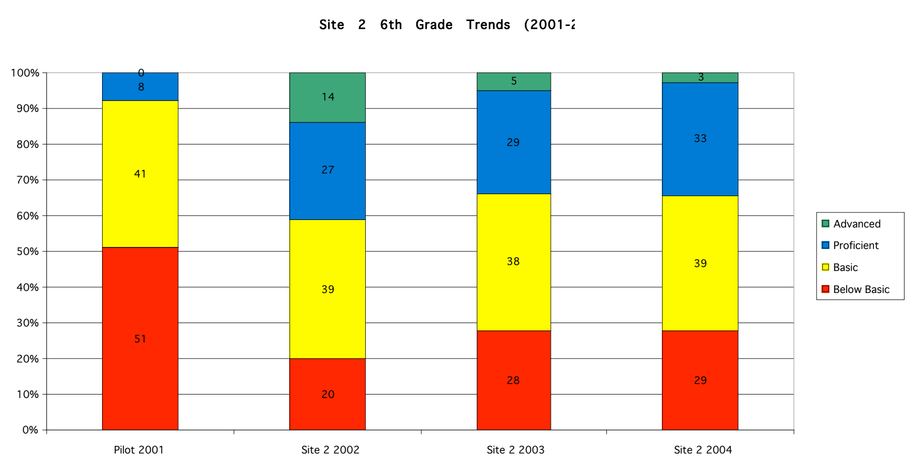


In the 1999-2001-2003 group, the sixth grade pilot data was low across the state. Site 1 data show that the sixth grade surpassed the fourth grade results from 2000 to 2002 and has not had a big drop since then. The eighth grade has been more problematic. At Site 2 there is a big drop in the sixth grade, but there is recovery at the 8th grade level in the 2000-2002-2004 group.

The following graphs show the sixth grade student achievement in more detail.



The Site 1 sixth grade scores are a success story for PDCA. There is sustained growth in the percentage of grade level students (Proficient and Advanced), and the Below Basic category has stayed under 10%. Moving from a traditional grading system and structure that superficially covered many topics to a focus on student success and learning, Ms. H's curriculum plans now include fewer topics in more depth, and she sees the curriculum as five interconnected strands. She takes instructional risks with innovative materials and launches units with rich problems that become reference points for the year. When I suggested *Build It! Festival* from *Great Explorations in Mathematics and Science* to teach geometry and measurement, she thought it would be risky before the Benchmark. As she planned, she realized the strength of the unit and how assessment with performance tasks and projects inform her about what students learn and understand.



The sixth grade teacher at Site 2 has a problem with both the Below Basic and Proficient categories. The percentage of grade level students for 2004 was slightly below the state level.

The school has lost ground in the Advanced category. Mr. C creates a challenge for progress in this middle school. While he has the same teaching experience as the Site 1 teacher, there has been little change in his practice. Mr. C has avoided professional development until PDCA and the AMSMA. This teacher is very skill and textbook oriented, but he does work well with students. He sees the value in a demonstration lesson, but he usually does not incorporate it into his planning. In 2004 the principal and researcher required him to plan a unit from Standards-based material. Mr. C had a student intern in Spring 2005. One researcher took advantage of the intern's enthusiasm for a Standards-based module from *Mathematics in Context, Per Sense*. We observed that Mr. C began to ask more questions about content, and at the end of the school year he opened a team lesson with concept-oriented questions related to Academy inservice meetings.

Both Sites 3 and 4 do well at the 4^{th} grade level. Site 3 performs at the state level in grades 6 and 8 (below 50%), while Site 4 performs slightly below state levels at these grades. Inconsistent views of curriculum have emerged. At Site 3, CMP has been in place for three years. The sixth and seventh grade teachers have embraced the program and have developed a strong sequence of modules. The eighth grade teacher inherited modules that emphasized algebra, and they did not align well with the curriculum framework. In an effort to align material and assessment, she said," I have it! There are 15 weeks of school left and 5 curriculum strands. 15 divided by 5 is 3. I can spend 3 weeks on each strand. Here are my folders." At first, I thought this teacher was hopelessly traditional, but interviews revealed that Ms. J was creative with units that involved art or application problems. In 2005, I will plan a more effective set of modules with this teacher and redirect her creative abilities.

Site 4 is recovering from weak sixth and seventh grade teachers prior to 2004. The principal and I are concentrating on a novice eighth grade teacher/coach who has strong content knowledge and potential. Initially his curriculum plans were textbook driven. His pacing was badly off because he allowed students to hold him back with reteaching. Two demonstration lessons convinced Mr. S that alternative planning was possible. He said, " I can't believe how much more you teach in a classroom than I do. The students really can do more." Instead of using my time in this school for demonstration lessons, the principal and I agreed that I should help the novice teacher plan with CMP material. After using *Comparing & Scaling* and *Stretching & Shrinking*, Mr. S commented, "I can't go back to the textbook. I am slow on the outline, but now I have something to talk about with the kids." The new seventh grade teacher has been fairly effective in a traditional classroom. In 2005, the Site 4 principal hopes that the teachers in grades 6-8 can work together for a more comprehensive implementation of CMP.

When we first started teaching demonstration lessons, we tended to treat the presentation as a "razzle-dazzle" workshop staging. Some teachers treated us as guest lecturers and viewed the demonstration as "time-off." We were frustrated by either the lack of guidance from the teachers or the broad objectives that they e-mailed us—often at the last minute.

Now that we understand these perceptions, problems, and weak grasps of curriculum, we ask better questions related to what teachers and students are doing in classroom. We also have simpler demonstration lessons to include the teachers' classroom materials. While we actually emphasize standards-based investigations, questioning strategies and embedded assessment, we are presenting a lesson that seems replicable to the teachers. It is important to involve teachers in the lesson so that we can move them from demonstration to coaching more quickly.

Do the large-scale assessments contribute to the improvement of teaching and learning, or do they dominate and stifle the curriculum? If teachers believe that testing is a system that they can beat, then the tests do limit the curriculum. However, the data of PDCA suggest that when teachers expect more learning and try more innovative practices to meet the challenges of a state assessment, student achievement and teacher practice are positively impacted. These changes come from teachers who understand the point of studying demonstration lessons and move from the demonstration phase into coaching and mentoring. The researchers will continue to monitor the data in the next two years for ASMSA sites. Two high-achieving PDCA sites have planned PD that delves more deeply into assessment and content issues.

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CONCEPT MAPS: A TOOL FOR ASSESSING UNDERSTANDING?

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This study explores the efficacy of using concept mapping as a framework for examining the understandings of linear algebra concepts held by students. In particular, a three-phase study looked at different types of concept maps and the inherent restrictions they place on a concept map to provide an external representation of student internal linkages. By examining both quantitative and qualitative data connected to the development of concept maps and analyzing this data from the perspective of understanding as generating concept images and concept definitions, as described by Tall, Vinner, and Davis, this study reveals that concept mapping does not always provide, for the students, a teacher, or a researcher, a stable image of student conceptual understanding. In particular, this study points to the conclusion that concept mapping, although a useful instructional tool, can only reveal evoked *connections and even when the same prompt is used within a week, most students reveal some instability where links are either added or subtracted. For some students, wholesale reorganization of concept maps reveal alternative forms of representation that indicate multiple levels of connections that cannot be represented through concept maps.*

Introduction

The assessment of student understanding has been a topic of discussion for many years. As mathematics researchers seek to uncover ways of externalizing the internal connections held by students a variety of tools have been utilized to try to bring this about. One tool, namely concept mapping, developed by science educators has started to be adopted by mathematics educators. Concept mapping activities are considered to be valuable assessments capable of providing explicit and overt representations of the students' knowledge structures (Novak, 1998). In fact, Pearsall, Skipper and Mintzes (1997) have claimed that concept maps "provide a unique window into the way learners structure their knowledge, offering an opportunity to assess both the propositional validity and the structural complexity of that knowledge base." However, not everyone is convinced especially given the wide range of activities that fall under the umbrella of concept mapping.

As McGowen and Tall (1999) put it, "The question as to whether a concept map actually *represents* the inner workings of the individual mind has long vexed the mathematics education community" (p. 2). Some mathematics educators, such as Williams (1998), have argued that concept mapping provides a means of gathering a representative sample of conceptual knowledge. Whereas, as Fruedenthal (1991) posited that internal cognitive structures for mathematical concepts are dynamic and context dependent thereby are insufficiently captured by static representations, one can wonder to what extent will concept maps be influenced by the dynamics and context dependency of students internal cognitive structures?

It is this particular question that drove the three-phased study described in this paper. Specifically, Phase 1 of the study sought to examine validity claims by looking at the impact of requiring participants to produce hierarchical maps and whether participants felt constrained by

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this requirement. The second phase of the study, also looking at validity[1] claims, turned the focus to see if participants who were permitted to construct a concept in any way they saw fit would feel that their maps were indicative of their knowledge. The third phase of the study specifically looked at issues of reliability by examining the short-term stability of participant produced concept maps. In essence, this phase of the study looked at test-retest reliability. Rather than relying on quantitative data drawn the concept maps produced by students and results on pretest multiple-choice exams, as has been the typical research methodology of Ruiz-Primo et al. (2001a, 2001b) and others, this study sought to explore whether students felt their concept maps provided a true representation of the internal structure held in their mind.

The choice of elementary linear algebra content was purposeful since much of the material discussed in the course was new to the students. In many cases such as those described by McGowan and Tall (1999), Park & Travers (1996) or Williams (1998), concept mapping activities served as a tool to assess conceptual change during a learning process. As a consequence, the content being examined by students is dynamic and changing rather than stable. It is the impact of this dynamism and the ability of the concept map to provide a valid and reliable representation of student understanding that is being explored in this study. Linear algebra, with its complexity and introduction of new terms, provided an environment to explore the concept maps ability to provide valid and reliable information when students are involved in a dynamic learning environment.

Theoretical Framework

One model of mathematical understanding, namely understanding as generating concept images and concept definitions, as described by Tall, Vinner, and Davis provides context to this study. According to Vinner (1991), learners acquire concepts when they construct a *concept image* – the collection of mental pictures, representations, and related properties ascribed to a concept. Tall and Vinner (1981) wrote:

> We shall use the term *concept image* to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes…. As the concept image develops it need not be coherent at all times…. We call the portion of the concept image which is activated at a particular time the *evoked concept image*. At different times, seemingly conflicting images may be evoked. Only when conflicting aspects are evoked *simultaneously* need there be any actual sense of conflict or confusion. (p. 152)

Evidently, a concept image differs from a concept's formal definition, if one exists, since a concept image exemplifies the way a particular concept becomes viewed by an individual (Davis & Vinner, 1986). The concept image involves the various linkages of the concept to other associated knowledge structures, exemplars, prototypical examples, and processes. As a result, the concept image is the overall cognitive structure constructed by a learner; however, in different contexts distinct components of this concept image come to the foreground. These excited portions of the concept image comprise the *evoked concept image* that consists of a proper subset of the concept image. This distinction between the image and the evoked image permits one to explain how students can respond inconsistently, providing evidence of

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[1] For the purposes of this discussion, *validity* will be understood here to mean the degree to which a map accurately reflects reality and *reliability* will be understood here to mean the degree to which a map is "repeatable".

understanding in one circumstance and a lack of understanding in another. A learner's description of his or her understandings may supply other discrepancies. In particular, any concept image has a related *concept definition* – the form or words used to specify the concept. This concept definition, however, can differ from the formal mathematical definition of a concept since the concept definition is an individualized characterization of the concept.

In order to elicit these linkages and connections, many researchers such as Williams (1998) have begun to use concept mapping to evoke external representation of student internal linkages. The origin of concept mapping sprung from the work of Joseph D. Novak, a Cornell University researcher who pioneered this tool from David Ausbel's theories concerning the significance of prior knowledge in being able to learn new concepts. Concept maps have been "developed specifically to tap into a learner's cognitive structure and to externalize, for both the learner and the teacher to see, what the learner already knows" (Novak, 1984, p. 40) and has been widely adopted by science educators over the past thirty years but has been underutilized by mathematics educators. In particular, a concept map is a graphical representation of a learners' knowledge structure of a particular concept (see Figure 1). To construct a concept map, ideas first have to be described or generated and the interrelationships between them articulated. Concepts are then placed in a hierarchical order with more general concepts at the top and specific concepts towards the bottom. Linking a concept to another via a linking word or phrase identifies a relationship. Cross links, i.e. across-page connections between concepts, allows for a rich connectivity to emerge. However, hierarchical concept maps are not the only type of concept map that can be generated. In particular, a concept map can be drawn which exhibits a network, webbing, or circular pattern of concepts. Additionally, concept maps may also begin with a specific idea and work out towards a more general idea. Such variability in terms of the structure the concept mapping activity has caused some to question the validity and reliability of various types of concept mapping activities to accurately represent students' internal knowledge structures.

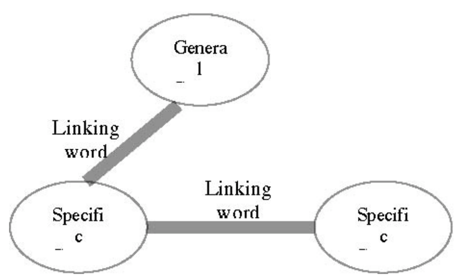


Figure 1: General format of a concept map

Methodology

This study was conducted in three phases with three separate groups of elementary linear algebra students from a Midwest, regional state university during Spring 2000, Spring 2002, and Fall 2003 by the researcher/teacher. In each phase, the same set of 31 linear algebra terms (i.e. *augmented matrix*, *basis*, *consistency*, *determinants*, *cofactor expansion*, *row reduction*, *diagonalization*, *eigenvalues*, *eigenvectors*, *eigenspaces*, *free variable*, *fixed variable, identity matrix*, *inverse of a matrix*, *invertible matrix theorem*, *linear dependence*, *linear independence*, *linear transformation*, *matrix multiplication*, *null space*, *column space*, *row space*, *one-to-one*, *onto*, *parametric form*, *rank*, *row reduction*, *span*, *subspace*, *vectors*, and *vector space*) were given to the students with less than a week to develop their concept map. The first phase asked participants (n = 29) to construct a hierarchical concept map and then they were asked to explain their concept map, address why they organized the map in the way that they did, and what they considered to be the main concept and how does it relate to the other concepts? In addition, they were asked to identify any concepts or links that they would add to their map and other relevant information necessary to understand their particular concept map. In the second phase of the study, participants (n = 17) were asked to construct a concept map but were permitted to do so in any manner they sought fit. Participants in Phase 2 were asked what they learned from constructing the concept map, what changes, if any, would they make to their concept map, what aspects of their thinking about linear algebra concepts does the map not reveal, and finally what surprised them when constructing their maps. The third phase of the study again asked participants (n = 17) to construct a concept map in any manner they saw fit and then within a week construct another concept map containing the same set of stipulated linear algebra terms. Afterwards, the participants were asked to reflect on the concept mapping activity. Quantitative and qualitative data was analyzed using an open coding scheme, for concept maps and definitions of stipulated terms, and adjacency matrices to analyze structure, links, and term usage.

Empirical Data and Analysis

Analysis of Phase 1

The analysis of Phase 1 concept maps focused on the participants perceived reliability of their provided hierarchical concept maps. Although specifically instructed to provide hierarchical concept maps in Phase 1, only 17 participants provided hierarchical maps and the rest of the 29 provided a webbing map or hybrid map (see figure 2). Of the 29 Phase 1 participants, 21 of them identified ways they would change their concept maps by adding more terms (9 participants), forming other connections (9 participants), or reorganizing the structure (5 participants). Phase 1 results indicated that requiring students to produce only hierarchical concept maps appeared too restrictive.

Analysis of Phase 2

Given the results of Phase 1 and the indicated constraint felt by participants in having to construct their concept maps in a hierarchical manner, Phase 2 participants were encouraged to construct their concept maps in whatever manner they thought best exemplified how they connected the terms together. Since over 70% of the participants in Phase 1 indicated they would have changed some elements of their concept maps upon reflection, and perhaps opportunity to diverge from a hierarchical map, the goal of phase 2 was to determine if the removing the restrictions of Phase 1 would alleviate the students desire to change their concept maps.

Interestingly, of the 17 participants in Phase 2 asked to produce a concept map without restricting the nature of the map to just a hierarchical form, nine participants produced hierarchical concept maps, two produced networks, three produced webbings and a circular map, a hybrid map and a disjoint map were produced by individuals. To gain a sense of the variability of structure, figure 3 introduces the various other types of concept maps not previously shown.

Since participants were encouraged to construct their concept maps in whatever manner they thought best exemplified how they connected the terms together, one might wonder if the percentage of participants wanting to change their map might decline in comparison to Phase 1. To gather data on this question, participants were asked if there was anything they would do to change their concept map. Only six of the 17 Phase 2 participants indicated that they would not change their concept map by adding terms, connections, or changing structure although one of these six participants mentioned adding color and making the map neater. One other participant stated that there were "hundreds of ways to change it" but then added, "It is impossible to link everything to where it is connected. Lines would be all over the place." Consequently he decided that he didn't want to attempt to change his map. The other 11 participants indicated that they wished to change something about their map such as adding more terms (1 participant), forming other connections (6 participants), or reorganizing the structure (4 participants). Consequently, Phase 1 and Phase 2 results indicated that even though participants were given close to a week to develop and represent their concept maps they were, in general, feeling that their concept maps did not *fully* represent their thinking about the concepts.

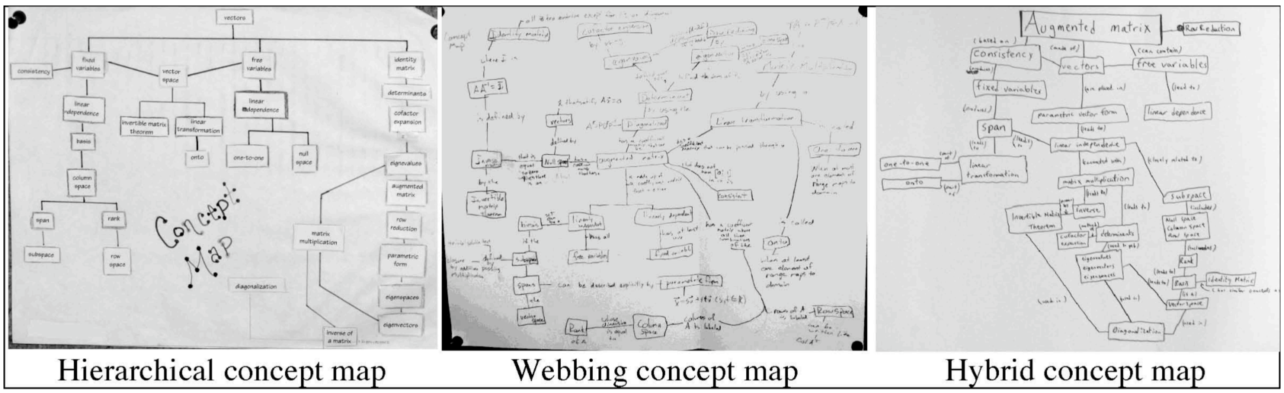


Figure 2: Samples of the types of concept maps from Phase 1

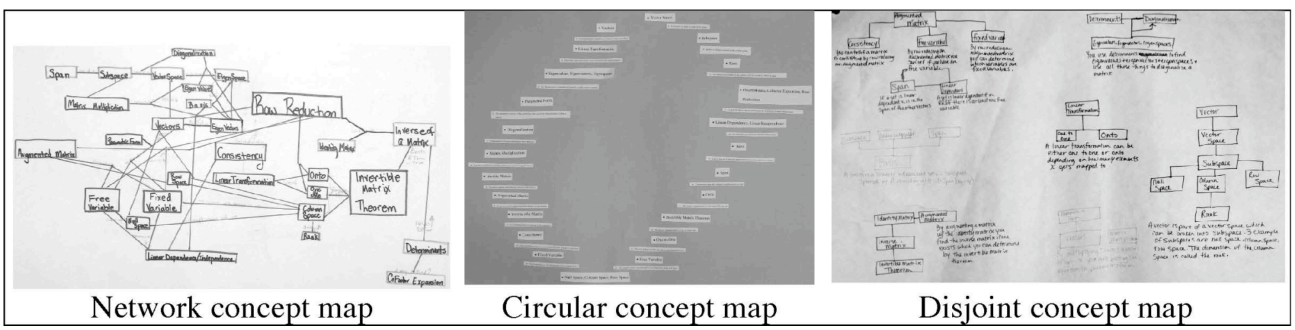


Figure 3: Samples of other types of concept maps from Phase 2

Analysis of Phase 3

Since 70% of the participants in Phase 1 and nearly 65% of the participants in Phase 2 indicated that if given the opportunity to reconstruct their concept map, they would either add additional connections, additional terms, or change the structure of their concept maps, Phase 3 of the study sought to examine the extent to which such changes might occur. Of the 17 participants in Phase 3, only one participant produced the same exact map from the initial formation to the final formation. However, even this participant acknowledged that she could have changed her concept map. In fact she stated,

"I learned from the concept map that there are a lot of different ways that I can connect the concepts of this course together. I could've made my concept map totally different from what

> I did, and it would have made sense. As I went back and reviewed things, I found stuff that connected to each other that I hadn't realized before. It was a good way to get one thinking. "

Even though her concept maps were stable across time, she recognized that she could have constructed different representations and those representations would have been just as valid of a representation as the one she provided.

All of the 16 other participants in Phase 3 altered their concept map in some fashion from their initial construction to their final construction. In particular, the average number of connections rose from 118.2 to 127.4 connections per map. The average number of stable links from initial to final concept mapping activity was 69.76. At the same time, the average number of new links, showing up in the final but not the initial map, was 57.35 and the number of lost links, showing up in the initial but not the final map, was 48.24. These shifts combined with the increase in the number of connections indicates that many of the participants in Phase 3 displayed significant changes from the initial development to the final mapping activity even when that took place only a few days later.

Some of these *new* links arose from participants including additional stipulated terms; however, this change does not account for all of the variability. In fact, six participants increased the number of stipulated terms used, four decreased their use number of stipulated terms, and the rest remained constant. Additionally, ten of the 17 concept maps, during the initial constructions for Phase 3, failed to include all of the stipulated terms whereas seven of the 17 failed to include all of the stipulated terms during the final construction.

Of particular interest are those participants who changed structure from initial to final construction. Six participants fall into this category and one particular participant's maps will be described to exemplify the impact a change in structure can have on interpretations of what knowledge a particular participant holds. Figure 4 provide a sense of the potential variability evident in such a structural change from a hierarchical concept map to a web. Paul radically reorganized his concept map revealing additional connections but also in both case some of the stipulated terms are not evident. In Paul's initial map, terms such as *basis*, *cofactor expansion*, *eigenspace*, and *span* are missing whereas in the final map *cofactor expansion* is missing along with a term, *row reduction*, that was originally included in his initial map. In addition, dimension, a term added to his initial map, was not included in his final map formation. Such instability, along with issues of defining terms in the map, point to the concern about using concept mapping activities to gather information on student understanding.

Examination of Paul's maps revealed that there were 140 stable linkages, 253 new linkages were seen in his final map that were not present in the initial map, and 100 linkages were lost when moving from initial to final constructions. Obviously, when asked to construct his map for the second time, Paul had reevaluated his use of this external representation and decided that a different structure would better exemplify how he organized these terms. In fact, when asked about the change in his maps, Paul stated. "Probably the biggest thing was that it's all interconnected, and impossible to find a true starting point. Many things work off each other sometimes breaking off into individual ideas just to come back together, an example is on the many applications of vectors. It works well in the book, but the [second] map shows how much each one influences the others." As a consequence, Paul was indicating that, even though the first map provided some indications of the linkages he held between the concepts identified on the map, the second map provided a better representation of the connections he held.

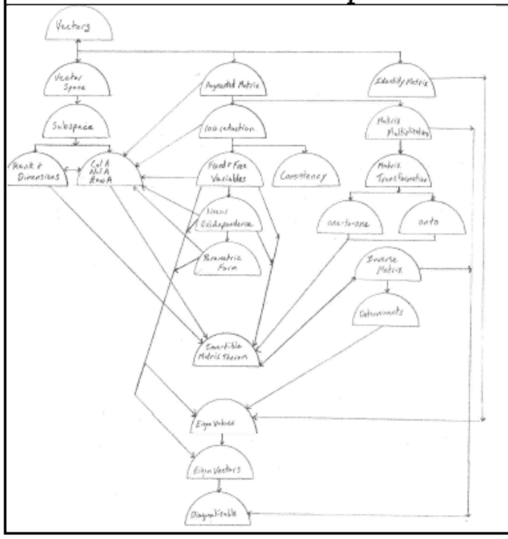
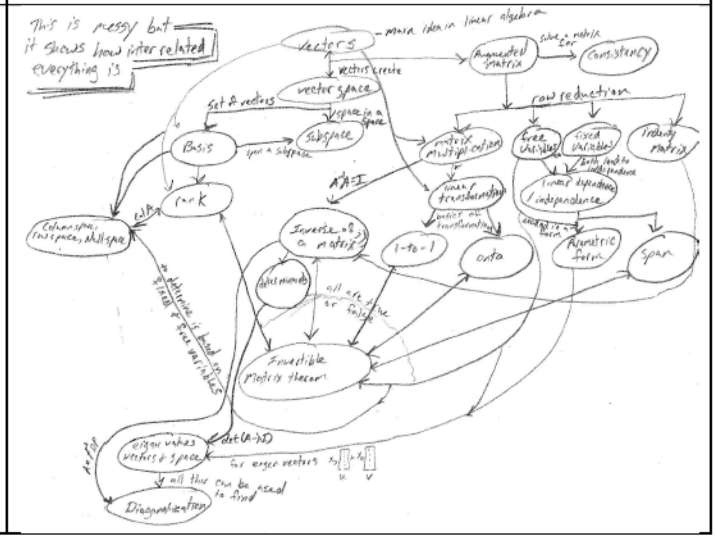
Paul's Initial Map	Paul's Final Map
	

Figure 4: Paul's Initial and Final Maps

Phase 3 results indicated that participants' concept maps were in general susceptible to change across time. Even though the same instructions were provided to these participants and less than a week accrued between map developments, almost every participant changed some aspect of their concept map from the initial to final construction stage. As a consequence, extrapolations of participants' understandings from their initial concept map might not provide a completely accurate picture when compared to similar extrapolations conducted on participants' final concept maps.

Conclusions

Concept mapping is a tool that does provide information about what connections a particular student holds. However, what is evident from this study is that attempting to draw conclusions about what a student understands or does not understand through a concept map may not be reliable. Certainly, it has been shown that a connection that might be evoked at one instant of time might not be evoked the next time. The variability and instability of the developed concept maps points to the conclusion that concept mapping should best be used as an instructional tool rather than relied on as an assessment tool. As some of the participants in the study indicated, the connections that they held in their minds could not be expressed through the concept map medium. Some participants even made arguments similar to the following: "I don't like the uh, … concept of concept maps. I don't really organize stuff in my brain that way I guess, and it always seems that you're not doing justice to the terms by trying to cram them into relations with a couple of other terms." Others complained that concept maps don't capture everything about how they relate the terms. For instance, one participant stated, "It [the concept map] doesn't show a logical step by step flow of how they all work together. It just states how each one is related, not how they combine to solve problems…. I realized that to write all relations I would run out of room and it would look like Hell because the lines would have to cross each other several times." As a result, researchers and teachers need to look at students' concept maps as *evoked* external representations of the students' potential inner connections.

The term evoked is important to remember. In this context, similar to the ideas of an *evoked concept image* described by Tall and Vinner (1981), the *evoked* concept map provides an external representation of connections that a student may hold internally. However, in the process of constructing a concept map, new connections may be elicited that were not present prior to beginning the process of concept mapping. One can hear this echoed in the words of one student, when speaking about his concept map, said, "…it is an accurate portrait of how I saw concepts originally and how light bulbs began to flick on and new connections began to form." Consequently, the assessment tool becomes a learning tool by which new connections are generated. However, light bulbs are not the only byproduct of concept mapping. As one participant put it, "The concept map was a surprisingly effective way of 'putting everything together.' Building the concept map demands that you call on what you know and relate the ideas to each other. This helped me recall things that I had forgotten about earlier in the semester. It also forced me to make decisions about what the 'biggest' concepts were, and what other ideas depended on them. I think all math classes should have a concept map drawn out towards the end of the semester, but it was especially helpful in this class because so many terms and definitions needed to be memorized." Building a concept map asks students to take a whole-brained approach to an entire semester's worth of material. In particular, participants pointed to the construction of the concept maps as helping them to synthesize their knowledge of linear algebra and realize that concepts that seemed unrelated were actually intricately connected.

Consequently, concept mapping needs to be viewed as a tool for researchers and teachers. Just as with any tool, it has its limitations and judgments concerning student understanding needs to be balanced by the fact that this particular tool does not always provide a stable external representation when provoked in a dynamic learning environment. In fact, this study has pointed to the fact that concept maps do not completely expose student understandings. What they do expose is a broad scoped look at what students may or may not be learning and indications about possible mental constructions they have made. From such a broad position, a researcher needs to employ interviews to narrow in on the displayed understandings by gathering additional information. Concept maps may provide a starting point for investigating understanding but only by aggregating information from multiple sources, especially interviews, can one attempt to provide reliable statements concerning the understandings held by a student.

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A CASE OF DISTRIBUTED COGNITION
(OR, MANY HEADS MAKE LIGHT WORK…)

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The following paper examines the cognitive processes that occur in the course of student interactions during collaborative learning activities. We find that conceptual blending and distributed cognition are useful models for interpreting these interactions using the cooperative group itself as the cognitive unit of analysis. Conceptual blending is rarely characterized in the literature as a group phenomenon. This paper advances that notion and describes how these theories can be integrated to provide a framework for understanding the cognitive processes of small groups in a classroom setting.

There is a dynamic process that characterizes the interactions of a small group as it comes to a consensus when exploring a problem. As they think aloud, members of the group continually engage in an iterative process of assembling new concepts. We will describe this process in terms of the notion of conceptual blending developed by Fauconnier and Turner (2002) to account for the cognitive processes of individuals. We will extend their theoretical model to account for the cognitive processes that occur within groups of students engaged in discourse related to problem solving. Conceptual blending takes place in three phases: (1) mapping thoughts from input spaces into a blended space which may include anchoring the concepts using words, pictures, diagrams or other tools, (2) filling in details and coordinating elements from the two input spaces in order to complete the new knowledge structure in a blended space, and finally (3) elaborating or manipulating the newly assembled concept to see what new insights it reveals. Fauconnier and Turner call this last step "running the blend". We illustrate this process using transcript excerpts from an undergraduate modern geometry course.

Theoretical Perspective

Fauconnier and Turner began collaborating on what they eventually called the theory of conceptual blending in 1993 (Fauconnier & Turner, 2002). This theory has its roots in basic ideas about categories and metaphors that were outlined by Lakoff and Johnson (Lakoff, 1987; Lakoff & Johnson, 1980). Fauconnier and Turner's theory of cognition posits the existence of a subconscious process that entails the combining or blending of diverse scenarios or concept spaces (*inputs*), which may or may not be *anchored* to some physical representation (Hutchins, 2002), to form a new stable conceptual model for use in reasoning and problem solving.

A concept space consists of an array of elements and their relationships to one another and can be run, like a script or simulation. It may be organized by a conceptual frame, such as "Christmas party" or a more generic frame such as "evening meal". It is activated as a single unit or chunk. In conceptual blending, two (or more) such concept spaces are activated and crucial elements of each are integrated and *mapped* to a third concept mapping space to form a conceptual blend. For example the "Christmas party" concept space and the "evening meal" concept space mentioned above might both be mapped to a blended space to form the "Christmas dinner" blend. Once the blend is *complete* it can be manipulated to make inferences or answer

Lloyd, G. M., Wilson, M., Wilkins, J. L. M., & Behm, S. L. (Eds.). (2005). *Proceedings of the 27th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.*

questions. This manipulation is referred to as *running the blend*. The blended concept is treated as a simulation that can be run imaginatively according to principles and properties that the input spaces bring to the blend.

Sometimes blending is an entirely mental activity, and other times blends are physically anchored onto an object (as we will see in the instance described in this paper) that represents the common elements of each of the inputs. An example of a conceptual blend that utilizes a material anchor is a clock (Hutchins, 2002), which maps one input space, the flow of time from one day to the next, and another input space, the periodicity of a repeating succession of days, onto a circular dial. Although time goes forward in a linear fashion and each day happens only once, we can conceive of the generic concept of day as a cyclic phenomenon. Just as noon happens each day, the hands of the clock "return" again each day to point at the twelve on your watch face. The advantage of materially anchoring a blend is that the anchor itself, i.e., the clock, can be used to store information, solve problems or gain more information about the system.

Cognitive scientists Hollan, Hutchins and Kirsch (2000) have studied the cognitive properties of groups who must work together cooperatively. They call the activity that takes place "distributed cognition" and cite four core principles of distributed cognition theory:

"people establish and coordinate different types of structure in their environment
it takes effort to maintain coordination
people off-load cognitive effort to the environment whenever practical
there are improved dynamics of cognitive load-balancing available in social organization." (Hollan et al., 2000)

Distributed cognition's focus on the group, along with its tools, artifacts and inscriptions as a unit of analysis (Hutchins, 1995), provides a useful model for interpreting the reasoning processes that take place in small student work groups. In combination with conceptual blending it provides a framework for understanding cognitive group dynamics in the learning environment.

This paper examines the interactions of a small student group in an undergraduate inquiry-based geometry course in light of this theoretical framework—the application of distributed cognition to conceptual blending. We attempt to identify the knowledge structures that emerge as a result of this group-processing phenomenon and determine how they were constructed, and then briefly discuss implications for the design of a learning environment that fosters and supports this process.

The Study

This report is part of a larger study, a teaching experiment (Cobb, 2000; Steffe & Thompson, 2000), conducted at a large university in the southwestern United States. In this geometry course, classroom interactions occurred primarily in three modes: small group work, student presentations to the class followed by discussion, and teacher-mediated whole class discussions. Students sat at circular tables in groups of three or four and the teacher (the second author of this paper) circulated continuously as the groups worked together, listening, asking open-ended questions, and encouraging student progress. Each class session was videotaped using two cameras, one focused on a table of students on the north side of the classroom and the other on a table of students on the south side. This paper profiles the interchange at one of these tables during a single class session about four weeks into the semester, and shows how the concept building process is distributed across group members their tools, artifacts and representations.

Cobb (2002) and Doerr and Tripp (1999), among others, have suggested that a classroom community as a whole is a legitimate unit of analysis when examining student reasoning, and that in such a setting, content knowledge can be seen as an emergent property of group interactions. To see an analysis of the research outlined in this report from the emergent perspective, see Zandieh and Rasmussen (2005). For an description of this mathematical activity from the perspective of horizontal and vertical mathematizing see Rassmussen, Zandieh, King and Teppo (2005).

The three steps of Fauconnier and Turner's model of conceptual blending are identified below in the transcript and instances of the core principles of distributed cognition are noted. The following transcript excerpts illustrate how the conceptual blending process unfolds and occasionally folds back on itself in real time in a typical classroom setting.

Data

By this point in the semester, the students had explored the concepts of straightness, symmetry, and angle on both plane and sphere. During the previous class students had constructed triangles on the sphere for the first time and, from the resulting discussion, had come to a consensus on a basic definition for a triangle that would work on both plane and sphere: three straight line segments that intersect each other at three different points. The teacher's goal for the class session reported here was for students to consider what constituted a triangle on a sphere. Three college juniors, Abby, Nate and Penny (pseudonyms), either mathematics or mathematics education majors, were seated together during this episode.

Orienting and Conjecturing

Teacher: So, what could you get for the sum of the interior angles of a triangle on a sphere? So, that's the question. I'm hoping that in the process of finding the answer to this question that you will create a lot of interesting triangles, not limited to the ones that you've already seen created last class, so that we can share across groups some interesting different looking triangles as well as try to figure out the answer to this question.

Penny left the table for a moment to retrieve a plastic sphere for them to draw on.

Abby: I say anywhere from 180 to maybe 270.

Nate: Based on…?

Abby: Drawing triangles. That's it. I don't think you can have 3 obtuse angles—that's not areal triangle.

Nate: Well, I'll go with the lower bound at 180, but I'm not sure about the upper bound.

Penny returned to the table with a Lenart sphere and some drawing tools.

Nate: Alright, so while you were gone, we decided that the lower bound was 180 degrees.

Abby: Well I decided both. He just didn't agree.

Penny: You decided what?

Abby: That the upper bound is, uh, 270.

In the foregoing brief discussion, students established a setting for the conceptual blending activity that followed. Once the task was agreed upon the blending process could begin.

Mapping and Anchoring

Penny: (preparing to draw on the sphere) Okay. So we have to draw some triangles.

Nate: Yeah. I was looking at how you'd draw the biggest...that was possible…

Penny: You can draw a… I don't know…they (gesturing toward a group at an adjacent table) were saying how they drew one with three obtuse angles, but I don't know how that was possible.

Nate: Well, I'm looking here…for example, if we made the equator and this guy an intersection (holding the edge of the straight line drawing tool to intersect the equator of the sphere). We can make that angle wider and wider and wider (slides the ruler tool so that angle gets wider) till eventually it touches itself or falls on top of itself.

Penny: So are we using the definition that it (the triangle) has to be three intersection points? Nate: Yes. See the trick is…to make this angle as wide as possible, and the remaining two angles as wide as possible.

Nate began by mapping and anchoring his thoughts as he talked about what constituted a triangle. He slid the ruler tool over the surface of the sphere and watched as the angle it formed with the equator changed. This allowed him to offload cognitive effort to the environment – the third principle of distributed cognition. The mapping and anchoring process continued for several minutes more although we have omitted much of this portion of the conversation due to space considerations. Penny drew what Abby and Nate described onto the sphere.

All three students offered conjectures, establishing and coordinating structure as described in the first principle of distributed cognition. Each one adopted a different role as the process unfolded—Nate initially took the lead in guiding the conversation, Penny drew what he and Abby described, and Abby interpreted and critiqued Nate's suggestions. This illustrates the fourth principle of distributed cognition, cognitive load balancing through social organization.

Nate: I think the trick is to approach it two sided. Here, if you look at the intersection we have between the equator and the great circle that you've made, you've got an angle here, you've got an angle here. (Nate points to the two interior angles of a lune, a slice of sphere that is a little narrower than a quarter of a sphere, formed by the ruler tool that has been laid over the top of the sphere so that it intersects with the equator.) We want to maximize that angle (pointing to one of the interior angles of the lune). We've got a bigger angle, we've got a bigger angle, we've got a bigger angle, we've got a bigger angle…until we rotate this around and draw the smallest possible side (he gestures as if to slide the ruler tool whose boundary forms a great circle, around to the opposite side of the sphere).

Penny: Well if, I think the biggest angle's going to be in the center, because then, right?…but perpendicular to this one…(Penny gestures to the angle made by two "longitude lines" that intersect at the north pole of her sphere.)

Nate: No because once you've done it, this angle keeps getting bigger…it's shrinking on this side but it's getting bigger on that side.

Penny: (sliding the ruler tool over the sphere) So just keep going and keep going and keep going…and what angle are we looking at?

Nate: …of course now we've lost track of our triangle…

Nate might have been able to imagine the triangle he wanted, but he could not quite see it on the sphere in front of him. When Penny began to slide the ruler tool over the surface of the sphere he lost track of the vertices of the triangle he was trying to visualize. Abby then tried interpreting Nate's suggestions and coordinating the mapping and anchoring process.

Penny: There's the angle.

Abby: There's that one…

Penny: And then there's this one and there's this one? (Penny does not seem to see how these three angles will connect to form a triangle.)

Nate: Yeah.

Penny: So that's all we want?

Nate: That makes a triangle doesn't it?

At this point Nate's concept was mapped but Abby was still mapping hers.

Abby: You don't have to go to that far of an extreme though. You can do something like this just so that you can see.

Nate: Yeah, but it won't be as big an angle.

Abby: But you will have the three obtuse angles that you can look at.

Nate: Okay. So what can we say about…what those add up to?

Penny: Well…

Nate: Frankly I can't say a lot…(he picks up the angle measurement tool and sets it on the table in front of Penny)…measure it or something…

Completing the Blend

Penny was ready to complete the blend by measuring but Abby was still mapping. In this instance the process folded back on itself from completion back to mapping and then went forward again through completion to running the blend.

Penny: How do you use this thing?

Abby: You see? This is your…other one. This one, this one and this one (touches 3 vertices on the sphere). This is your line.

Penny: No, because this is the set. This is the last one right here. It could be that way too, though.

Abby: It's right there. Because you only have the…this one's only on one edge…there's not another vertex on this—you have to have two on each line…so you need one of these as your vertex, not that.

Penny: Yeah eventually if I carry this along it would run into it wouldn't…oh no…ok, wait, which one?

Abby: Yeah but see look, these two connect right here and then if you turn it they connect at your other point.

Abby's concept of a spherical triangle was finally fully anchored on Penny's sphere. Initially it appeared that there was some confusion between the three students as to which triangle they were talking about. Once this was resolved they were able to move forward with completion. As represented in the second principle of distributed cognition, a great deal of effort was expended during this phase of the conversation to maintain the coordination.

Penny finally completed the blend by identifying each of the elements they had mapped and tracing how they connected to form a triangle, blending the concept of a planar triangle formed by straight line segments with the concept of a closed spherical space, where straight lines are geodesics and every geodesic crosses every other twice.

Penny: Oh, okay. Wow. This is confusing. So this is an angle, right? (She has her thumb and forefinger on the two other vertices, and at last she zeros in on the location of the third vertex of the large, odd-looking triangle she has constructed.)

Abby: Mmhmm.

Nate: Yup.

Penny: Then this one…and then that one? (She indicates the angles she has marked with her thumb and forefinger.)

Nate: Yes. Yeah.

Penny: OK.

Running the Blend

The blend was complete. Everyone was finally looking at the same triangle that Penny had drawn on their sphere. Nate "ran the blend".

Nate: So essentially what that's saying…that's kind of like draping a triangle over the top and spreading it down until it approaches the equator at which point it would no longer be a triangle…So it almost sounds like you could have one that approaches 540 degrees.

After a few false starts Penny managed to measure the three angles of the triangle she had constructed. Each angle measured 160 degrees yielding a sum of 480 degrees. Nate explained that each angle was approaching 180 degrees so the upper bound for the angle sum would be 540 degrees. He talked his team through his manipulation of the conceptual blend of triangle-on-a-sphere to find the answer to the question posed at the beginning of the exercise. In measuring and then summing the angles of the triangle they "ran the blend" for themselves coming to the same conclusion as Nate.

Discussion

The cognitive gymnastics associated with solving a problem of this sort are typically hidden from view, even from the problem solvers themselves. By enlarging the unit of analysis from the individual student to the group, the distributed nature of the cognitive process becomes apparent. Thinking and reasoning were distributed among members of the cooperative group, aided by the use of tools (i.e., the sphere, drawing and measuring tools) and representations (the triangles on the sphere). The act of engaging in conversation and representing with tools and diagrams allowed for the exteriorization of a thinking process that would otherwise remain outside of conscious view. The conceptual blend was assembled as each student contributed his or her thoughts and Penny anchored them with visual representations, until at last, they had a coherent representation of the structure of the problem that they could manipulate to find a solution.

It was evident during this exchange that each of these students understood how to think about sides, angles, and congruence, but not everyone understood which angle they should pay attention to at any given moment. At first Abby and Penny appeared to be operating under the preconception that angle measures had to be 90 degrees or less in order for the construction to be considered "a triangle". Even before Penny begins drawing on the sphere, all three members of the group seem to accept that the sum of the angles of a triangle on a sphere could never be less than 180 degrees.

Progress in achieving goals is accompanied by a repeating sequence of events: inputs or ideas that are mapped onto external representations (i.e., drawn on the sphere), thereby establishing a 'material anchor' for the conceptual blend; completion of the blend by coordinating inputs or making inferences (i.e., these points and these line segments, taken together, form a triangle on a sphere in the same sense that analogous structures would form a triangle on a plane); and finally elaboration—running of the blend to see what new concepts it might generate.

This cycle, recounted above, results in a conceptual blend containing a manipulatable triangle that Nate then mentally 'stretched' around the sphere until it mapped onto the equator. This represents substantial progress in the group's conceptualization of what can constitute a triangle according to the class definition. It seemed clear, at least to Nate, that the sum of angles of a triangle could be greater than the 270-degree upper bound that Abby initially hypothesized. He subsequently suggested that the upper bound must be 540 degrees, and the blend they constructed confirmed that this was plausible.

As a result of their efforts at co-construction, this student group ultimately appeared to have a more coherent concept of what constitutes a triangle on a sphere than their original one. Together they negotiated and constructed a spherical triangle definition. This, in turn, could be used to predict general characteristics of triangles on spheres. They discovered useful ways of representing sides and angles and combining them to form triangles they had never before considered possible. This developing ability to represent extended and shaped their capacity to think and reason about the nature of a triangle on the surface a sphere.

Implications

From the cognitive perspective, a learning environment that is centered on small group work has some advantages. The transcript excerpts above give a glimpse of how small group discourse centered on an open-ended question can help students, first, to articulate, and then, to examine and test their conjectures about the concepts under study. They exteriorized their thought processes in order to negotiate meaning and make sense of the new blended concepts they assembled with their group-mates. Ultimately, this active process of construction produced a knowledge structure that students could manipulate to solve problems, as shown above.

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WHAT CAN WE LEARN FROM LONGITUDINALLY ADMINISTERED PERFORMANCE ASSESSMENTS?

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This paper reports on the student performance assessment data gathered in a long-term professional development initiative designed to improve teachers' mathematical knowledge and instructional practice. Upper-elementary school teachers (grades 4 through 6) in designated low-performing schools participated in professional development focused on improving mathematics instructional practice. Performance assessments were administered to students of these specially prepared teachers to gain an understanding of student progress over time in understanding of particular content areas and in their development of mathematical processes. In this paper, we will share what a longitudinal look through the lens of performance assessment, both quantitative and qualitative, does and does not afford.

Introduction

This paper reports on the student performance assessment data gathered in low-performing, high-poverty schools in a large, urban district. The students were in the classrooms of teachers participating in a long-term professional development initiative designed to improve teachers' mathematical knowledge and instructional practice. The initiative was part of a large-scale reform aimed at supporting students' achievement by improving teacher quality. A partnership was formed among university faculty, business, and school administrators to provide experienced teachers with additional support for relearning the mathematics that they teach and support for developing more effective mathematics instructional practice. The ultimate goal of improving teaching quality was to support student learning.

As part of the school district's reform plan in the lowest performing elementary schools, a group of elementary school teachers had specialized work assignments, teaching only mathematics in grades 4-6. These teachers were designated as *mathematics specialists*. Hired as additional faculty at the sites, these teachers taught three 90-minute classes. Teachers reported having benefited from an exclusive focus on the teaching of mathematics. The teachers participated in intensive professional development, which included university coursework and other support from university and school district personnel.

The teachers completed 6 units of upper-division mathematics and 6 units of graduate teacher education coursework with a focus on mathematics pedagogy, which culminated in the receipt of a certificate. The teachers had on-site coaching by an expert teacher on loan to the university from the school district. The teachers had scheduled, shared, daily professional development time in order to discuss lesson planning, reflect on student work, and help each other with inquiry into mathematics teaching. The program was designed with the features that researchers advocate: 1) long-term engagement with teachers with a focus on deepening teachers' mathematical knowledge and increasing knowledge of children's thinking about mathematics, 2) building connections between professional development and practice, and 3) providing opportunities for collaborative work (cf., California Postsecondary Education Commission,

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2001; Darling-Hammond, 1997; Garet, Porter, Desimone, Birman, & Yoon, 2001; Sowder, Philipp, Armstrong, & Schappelle, 1998).

In the three years of the professional development project, we had the unique opportunity to examine teacher learning, changes in teacher practice, and student learning. The results of teacher learning and practice have been reported in other papers and presentations (cf., Nickerson & Moriarty, in press; Nickerson, 2003; Nickerson, 2004). The focus of the research presented here was to examine student learning, as evidenced in performance assessments.

It was critical that we investigate student learning, but as Hiebert (1999) has suggested, how one assesses student learning depends upon one's goals and values. The aims of the school district reform in which this professional development project was embedded were to meet the close the achievement gap and to prepare all students for algebra. In addition, the school administrators and university faculty shared a goal of supporting students as problems solvers, able to reason and communicate about mathematics. The students in these schools took two assessments in addition to the state-mandated standardized and standards tests. A short computation test was administered each year to assess students' number sense. Three performance assessments tasks were administered each year to gain an understanding of student progress over time in particular content areas and in the development of mathematical processes. These additional tests were used to inform the designers of the mathematics initiative, school district mathematics administrators, and teachers regarding the impact of the instructional work.

Theoretical Framework

We adopt a social perspective on learning. Our assumptions are based on the premise that knowing explicit information is only one part of knowing. With this perspective, knowing mathematics is a matter of competently doing mathematics while participating in valued endeavors (Wenger, 1998). The performance tasks were designed to provide opportunities for students to demonstrate what mathematics they do understand. Performance assessments have the potential to reveal whether students can use the mathematics we believe they are learning (Lesh & Lamon, 1992). Performance assessments share many characteristics of authentic tests. They are designed to: 1) be representative of performance in the field, 2) assess a students' problem-solving repertoire, 3) have multiple levels of access, and 4) require justification of answers (Wiggins, 1989). Performance assessment tasks typically encompass authentic opportunities for multi-step reasoning and conceptual understanding.

Methods and Procedures

At the beginning of the initiative and then at the end of each academic year for the next three years, we administered at each grade level in classrooms with mathematics specialist teachers three grade-level appropriate performance assessment tasks. At the beginning of the first year, Grade 4 students took an end of Grade 3 test. At the end of the academic year, Grade 4 students were administered Grade 4 level tasks. Likewise, Grade 5 and Grade 6 students took grade appropriate tests. The student performance assessments were selected from tests developed by an NSF-funded project, Mathematics Assessment Resource Service (MARS). The MARS assessment is a national test with open-ended items similar to those used on the National Assessment of Educational Progress (NAEP). The assessments were to provide a measure of how we were progressing toward the goal of having all students capable of powerful mathematical reasoning and who were able to justify their reasoning. The tasks were selected in content areas we considered applicable at each grade levels: Measurement (specifically, area &

perimeter), Algebraic Thinking (specifically, identification and extension of patterns and relations), and Number and Operations (solving practical problems with whole numbers, fractions, and decimals). At each successive grade level, similar tasks are extended to a higher level of difficulty. For example, the Algebraic Thinking task at the end of fourth grade requires students to identify and extend a pattern. The Algebraic Thinking task at the end of fifth grade requires students to identify and extend a nonlinear pattern and to graph it. Tasks at both grade levels ask the student to predict the number of pieces needed to construct a future display and to describe how they figured it out.

A point scoring system, similar to a scoring system such as TIMMS, was employed to score the performance assessments. MARS developed a rubric for each item that provides specific direction as to how the points are awarded. The points are typically awarded for product, process, and justification. Lead scorers were trained over three days by CTB-MARS developers. These trainers then trained a group of teachers in scoring. Each time, the trainers work on a set of anchoring papers until the scoring is calibrated. Maintaining scoring reliability is a critical component. At the outset of a scoring session, each task was scored by two people until we reached a high level of confidence in calibration. Thereafter, every fifth paper was double-scored. Scorers discussed any discrepancies and typically a consensus was reached. When it was not, the lead scorer assisted in resolving any differences.

Performance assessments yield complex data and take considerable time and attention to score. The end result is that students' rich responses are scored using the rubric and then reduced to a single score. An abundance of information can be lost in reducing students' responses to a single score (Peressini & Webb, 1999). There were approximately 1100 students who had been with mathematics specialists for three years. We looked at the student performance assessments of students through the lenses of change in quantitative scores to identify trends in student development and to select a subset for careful qualitative analyses.

Quantitative Analysis

The quantitative perspective provided a way of efficiently examining on each of three tasks, the large volume of data. Once the papers were scored with the rubric, the data of individual students was entered into a spreadsheet. For each task, researchers designated stages of performance that correlated with competencies of the task expectations. For example, the 5^{th} grade algebraic thinking item described earlier required students to describe patterns in a set of geometric displays, extend the pattern in a table, graph the results, and justify an extended result. A student in the first stage of performance for this task has some understanding of how to construct a display and to extend a linear pattern. A student in the second stage of performance for this task has an understanding of how to construct displays, extend a linear pattern, and can also plot points on a graph. A student in the third stage of performance for this task can recognize, describe, and justify a conjectured extension to the pattern, in addition to plotting points from data.

With this quantitative perspective we were able to gain information about what percent of students were in the first, second, and third stages of performance in each task. Because we chose similar tasks over grade levels, we were able to gain information about what learning a cohort could or could not demonstrate. Furthermore, we were able to compare cohorts of students. In the second year, we were able to compare current 5^{th} grade students with the prior year's 5^{th} grade students, thus comparing 5^{th} grade students at the end of their first year with a mathematics specialist with student who had been instructed for two years with a mathematics specialist.

Qualitative Analysis

The qualitative perspective allowed a rich analysis on a carefully chosen subset of student papers. Peressini and Webb's (1999) analytic-qualitative scoring framework provided the basis of developing our own framework for analysis. Peressini and Webb developed their framework and a four stage qualitative process for analyzing student responses over a set of mathematics performance assessment tasks. The first phase requires the researcher or teacher to write a qualitative analysis for the students' responses on each performance task. This analysis was elaborated along three broad categories: foundational knowledge, solution processes, and communication. (See Table 1 for a summary.) The second phase involved writing a synthesis comparison across performance assessments. This phase is a synthesis of the knowledge, skills, and strategies that can be identified in the tasks. The third phase involved a distillation of the analysis across tasks to provide a description of a student's reasoning. For Peressini and Webb, the fourth phase was using performance assessments to inform instruction.

Table 1. *Summary of Peressini and Webb's Analytic Scoring Framework*

Foundational Knowledge	**Solution Process**	**Communication**
A. Concepts, facts, definitions B. Procedures and algorithms C. Misconceptions	A. Analysis 1. Understanding 2. Exploration 3. Verify solution B. Reasoning – modes of reasoning including spatial, proportional, abstracting	A. Language B. Symbols C. Dimensions/labels D. Argument

Our research would involve the first three phases described by Peressini and Webb. For the first two phases of the analysis, we utilized Peressini and Webb's framework. Our third and fourth phases involved a longitudinal description of changes and development we could identify in a set of papers. Our framework for the longitudinal qualitative analysis emerged as a result of our attempts to describe a student's work over time individually and then to describe a subset of students' work collectively. Because our analysis was longitudinal, some components of students' solution strategies were less significant and other components proved to be significant. For example, it was generally less informative to include Foundational Knowledge in a longitudinal analysis of these performance assessment tasks because operations with fractions might be elicited in tasks for grade 5 but not in tasks for grade 4 or grade 6. Other aspects proved to be of greater interest, such as the sophistication of solution strategies.

Our framework for analysis appears in the table below:

Table 2. *Summary of Peressini and Webb's Analytic Scoring Framework*

Solution Process-Analysis	**Solution Process-Reasoning**	**Communication**
1. Understanding separate multi-step conditions, draws diagrams, simplify to special cases 2. Exploratory manipulations, systematic solution path 3. Check reasonableness of solution	A. Modes of reasoning: inductive, deductive, spatial/visual, abstraction B. Level of sophistication of reasoning (e.g., relies on models, relies on mathematical representations to solve, numerical, generalized)	Level of argumentation (e.g., nonmathematical, reports procedures, gives rational for procedure, explains thinking)

Results

The data from performance assessments with a quantitative and a qualitative lens allowed us to do a number of analyses, which provided us with a range of information regarding student progress in particular content areas and in reasoning and justification.

We began by looking at the percentage of students at each stage of performance for a given grade and given task. We repeated the analysis of stages for each year and for each task. In the first year of the study for all of the tasks most of the students in were in Stage 1 level of performance. In subsequent years, this analysis was done in terms of stages of performance, we were able to compare the performance of cohorts of students. By the third year, for example, the performance of sixth grade students who had been with a mathematics specialist for one year (the first year) could be compared with the performance of sixth grade students who had been taught by mathematics specialists for three years (the current year). This analysis was done in terms of performance stages and in terms of a numerical comparison of the average correct. With this analysis, we found a significant improvement in roughly three-quarters of the tasks. We could then explore why the increase was not significant for a couple tasks. Perhaps surprisingly, we found improvements in the fourth grade students' performance each year even though in each year, the fourth grade cohorts would have had one year with a mathematics specialist. Each of the fourth graders would have only had a mathematics specialist for one year, yet successive cohorts' performance improved. We attributed this to better-prepared teachers.

Information about the performance levels allowed us to compare Algebraic Thinking items across grade levels for each cohort of students. Increases in each successive stage with successive grade levels indicated a closing of the achievement gap in abilities pertaining relevant to performance assessments.

To gain further insight, we returned and looked qualitatively at several groups of students' performance assessments across time. We separated students into groups representative of their various gains, or lack of gains. For example, one group that was of interest to us were students who individually did not appear to show great gains from grade 4 to grade 6. Another group of interest were students who showed tremendous gains from grade 4 to grade 6. We were interested in seeing what insights we might gain into what changed for them.

As mentioned above, the qualitative analysis enabled us to exam the range of strategies employed by students in solving problems. On the pretests, students, across grade levels, did not have access to solve the problems. They often chose to just add numbers in the given word problem. In the subsequent post-test at the end of one year of instruction with mathematics specialists, students had access to the problems, though with unsophisticated methods. Students approach a problem in which they are asked to determine how many groups of 6 are in 30 (grounded in a particular context) in a number of ways. Methods ranged from modeling the problem by drawing the items and physically grouping them in groups of six to repeated addition to the division algorithm. We characterized these according to levels of sophistication. The least sophisticated successful strategy was having to model the problem exactly as stated. In other words, if the task called for deciding how many school buses were needed to transport students, we considered the need to draw school buses and small representative people the least sophisticated solution strategy.

The students who were increasingly successful and showed great change in quantitative scores were students who had increasingly sophisticated means of tackling problems. What they may have modeled in Grade 4, they could solve with a computation or other abstraction. In the work of another group of students whose work we examined qualitatively, it was interesting to find that the students who scored well through grade 4 and grade 5, but had difficulty in grade 6, were a group of students that had modeled solutions in grades 4 and grades 5, but had failed to generate more sophisticated strategies needed for grade 6.

The qualitative analysis also enabled us to examine how the argumentation evolved for students. Most students lacked experience in argumentation at the beginning of the project. Initially, it was not uncommon for a student to provide an explanation that merely stated, "I thought it in my head." For those who modeled, the explanation was sometimes expressed as, "I saw it." Some explanations consisted of recounting their computational steps without any explanation as to why those steps were employed.

Discussion

Performance assessments are not intended to be a comprehensive assessment of individual student performance but instead over the long term provide a perspective on student growth in understanding of mathematical concepts and how these concepts develop over time. They also can be used to document student growth in mathematical processes. Indeed, performance assessments are sometimes selected as an assessment as a means of influencing instructional emphases on desired processes. Some evidence suggests that using performance assessments promotes desired practices and have led teachers to use innovative instructional practices (Hamilton, 2003).

Performance assessments can be analyzed with both a quantitative and a qualitative lens to provide a longitudinal perspective on students' growth in process and sophistication of argumentation. The performance assessments were costly to administer and to score. The analysis provided a long-term perspective and insight into student growth in understanding of mathematical concepts and student growth in mathematical processes was valuable. It also revealed areas in which we could better support students. This information was then formatively utilized in successive years of teacher support and learning.

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THE POWER OF OPERATIONAL CONJECTURES

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"He is the living God, more so than heretofore, because He is unceasingly constructing ever stronger systems."

Piaget, 1970, p. 141

The purpose of this paper is to illustrate a kind of conjecturing that involves the abduction of mathematical operations: operational conjectures. I share data from a semester-long teaching experiment to demonstrate the power of such conjectures in constructing new fractions schemes.

In considering students' mathematical learning, it is useful to distinguish between three kinds of plausible inference. The first is perceptual judgment (Peirce, 1998), wherein mental operations (interiorized actions) are applied to sensory material in order to transform it into the familiar objects that we perceive. Perceptual judgments occur outside of our awareness and seem so certain that we hardly question them or talk about them except perhaps in philosophy classrooms. We can think about them as observations, as long as we keep in mind that even observations require inference through mental action (p. 228). For example, consider Figure 1. One might perceive the figure as a cube in a corner, a cube with a missing corner, or as two overlapping cubes. Which is the true perception? Spatial operations, such as rotations, translations, and inversions determine our perceptual judgments, even before we are aware of them.

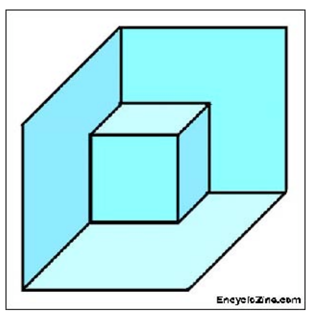

Figure 1. The role of mental operations in perception.

The second kind of plausible inference occurs when students apply concepts to perceived situations. These are exemplified by cases in which students question mathematical ideas in the following manner: "Maybe all the faces have to be the same shape (for a solid to be perfect)" (Lehrer & Curtis, 2000) or, in general, "Does this work?" (Eggleton, 2001). This kind of plausible inference is usually what researchers mean in referring to *conjecture*, and it receives the bulk of researchers' attention (Chazan and Houde, 1989; Arzarello et al, 1998). In fact, the two examples provided above are taken from research reports on conjecturing, but I argue that these are not the most powerful kinds of conjecture, nor are they the most useful to study. The most powerful conjectures involve the *abduction* (Peirce, 1998) of an operation from one way of operating to another.

Peirce introduced abduction as a sort of reverse deduction, whereby one adopts a general rule to explain a surprising observation. For example, a person might be surprised to find several M&M's lying on the floor of a candy store's floor, and further surprised to find that the candies are all green! One might suppose that they were spilled from a bag of M&M's that were all green. In so doing, the surprising event is explained as a matter of course.

When I mention the abduction of operations in this paper, I use the *pattern* of abduction to describe how mental operations can be adopted in order to resolve problematic situations; I refer to such abductions as operational conjectures. In this paper, I will demonstrate how operational conjectures serve as generalizing assimilations and functional accommodations, through which new schemes are created.

Fractions Schemes

Schemes are like strategies, except they are developed and often used outside of one's own awareness. They are ways of operating that Glasersfeld (1998) described as a three-part structure: a recognition template of triggering situations, the operations that are triggered, and an expected result of operating. When acting on a scheme, if expected results do not fit perceived results, one might experience a perturbation, which can result in the modification of the scheme. If this occurs in the context of acting on a scheme, the modification is called a functional accommodation (Steffe, 1991, p. 183). Other times one might use a scheme in a perceived situation that would not ordinarily trigger the scheme. The result might be a modification to the recognition template, referred to as a generalizing assimilation. Steffe has also described a third way in which schemes are modified: metamorphic accommodations (p. 187). Whereas a description of them is beyond the scope of this paper, it is worth noting that metamorphic accommodations can explain the construction of new operations.

Through their work on the Fractions Project and their study of fractions schemes, Leslie Steffe (2002) and John Olive (1999) have demonstrated that partitioning and iterating are fundamental fractions operations, necessary for any meaningful conceptualization of fractions. For example, in utilizing a part-whole conception of fractions, a student understands that fractions are determined by the number of pieces taken up by a fractional part in an equally partitioned whole. In utilizing a partitive unit fraction conception, a student further understands that any unit fractional part can be iterated so many times to re-produce the whole and that this number of iterations determines the relative *size* of the fraction. These students are said to have *a partitive unit fractional scheme*.

According to Steffe (2002), the splitting operation is the simultaneous composition of partitioning and iterating. Students with a splitting operation can solve tasks like the following: "if this rectangular sheet of paper is my bar and it is five times as big as your bar, make your bar." Finding an appropriate solution requires the student to understand that she can use partitioning to resolve a situation that is iterative in nature. Namely, by partitioning my bar into five parts, she can obtain a part that can be iterated five times to reproduce my bar. Note that this situation is not necessarily a fractional situation. In fact, the student subject of this study demonstrated that he could solve the above task, but he did not seem to have any fractions schemes available beyond that of comparing the number of visible parts in a fraction to the number of visible parts in a whole: a *part-whole fractional scheme*.

Methods

In considering a student's potential for learning through conjecturing, it seems that mathematical power is determined by his available operations. To illustrate the point, I introduce a grade 6 student named Josh. He possessed remarkable mathematical potential relative to his actualized mathematical schemes, because he had developed a splitting operation before he had constructed any fractions schemes.

Josh was one of six grade 6 students to participate in pairs in a semester-long teaching experiment. I selected these six participants from a group of eleven students identified by their classroom teachers as students who might benefit by participating in the project. I met with each of the eleven students alone, once, for an initial interview to determine their levels of development in terms of fractions schemes and operations. During the interviews, I posed problems using string and construction paper, asking questions like the one described above.

Once I had selected the six participants and determined the level of development for each of them, I formed student pairs. Josh and his student partner, Matthew, formed one of two lower-level pairs: those who had developed part-whole fraction schemes but who had not yet developed a partitive unit fractional schemes (as one pair of students had). I pulled Josh and his student partner from their math classes once or twice per week, for a total of seventeen 45-minute teaching episodes. During these episodes, I acted as both teacher and researcher, posing tasks intended to elicit student conjectures about fractions while building models of the students' learning. A computer fractions environment called TIMA:Sticks mediated my tasks and the students' actions.

The sticks in TIMA:Sticks could be drawn to any length, partitioned into parts, cut, pulled out as parts from another stick (leaving the original stick in tact), copied, joined together, covered (hidden), compared to the size of a ruler that the students could specify, and measured relative to that ruler. More information about TIMA:Sticks software can be found in Steffe and Olive, 1996. In particular, Olive's paper describes the available functions of the program, the rationale for creating them, and their affordances and limitations. With respect to the present paper and occasioning conjectures, the environment mediated actions that were similar to the students' actions in the initial interviews, using string and construction paper. In fact, the software was designed as a virtual string environment. However, the measure function that was available to the students often served as an authority in determining the fractional sizes of pieces and in testing conjectures.

During the teaching experiments, there was a witness who observed each teaching episode and provided feedback to me on the effectiveness of my interactions with the students. We videotaped each of the teaching experiments with two cameras: one focused on the students and me, and the other focused on the computer monitor. I reviewed the tapes after each teaching episode in order to refine my models of the students' learning and to design new tasks for the next teaching episode. My models consisted of schemes that appeared to be available to the students, as well as hypothetical reorganizations of those schemes. I tried to design tasks eliciting conjectures that would provoke those reorganizations. I identified instances of conjecturing activity during a second round of retrospective analysis, using the models I had constructed of the students' schemes. In particular, I looked for instances in which the students had formed a mathematical goal but had no available way of operating with which they were confident the goal could be satisfied. The following examples elucidate the process and provide evidence for the power of Josh's splitting operation in forming new schemes through abduction.

Josh's Construction of a Partitive Unit Fractional Scheme

Josh and Matthew appeared to be operating with fractions at the same level, although Matthew did not seem to have a splitting operation available. The following protocol occurred during the second teaching episode with the pair, 28 February, 2003. In the middle of the episode, Matthew had accidentally partitioned the left half of a two-halves stick into two parts, and I asked the students to consider what one of the resulting parts (the leftmost fourth) would measure. Josh responded that it would be "one third… if it was even." When Matthew said that he thought it would be one-third anyway, Josh agreed. But after they pulled out the part and measured it to be "1/4," Josh reiterated his initial qualification as an explanation for the surprising result: "because it's not even." This response fit the pattern of abduction, but Josh's expressed confidence in the reason indicates that his response was a based on a perceptual judgment rather than an operational conjecture. Still, his agreement with Matthew that the part could be one-third indicates that *one-third* was not determined through iteration in the ruler and served as contra-indication that he had constructed a partitive unit fractional scheme.

The following protocol picks up as we revisited the measure of one of the smaller parts, which had been pulled out of the stick. In each protocol, T-R precedes my statements and actions, as the teacher-researcher; likewise J and M precede the statements and actions of Josh and Matthew, respectively.

T-R: If I measure this [pointing to a copy of the left-most piece, which had been pulled out from the partitioned whole]?

M: One-fourth.

T-R: How do you know that's one-fourth?

M: Because we already measured it.

T-R: Okay…

J: Let's see. Because… them two look the same [pointing to the two fourths in the partitioned whole]; you could put one more [partition] in there [pointing to the middle of the right half].

Josh's final statement indicates an operational conjecture because he was acting on the stick in a novel way. He knew that one-fourth meant that the piece in question should be one out of four equal pieces making up the ruler (a part-whole conception), but there were only three visible parts. His operational conjecture, then, was to use partitioning (segmenting) to create the fourth equal part in the whole from the three unequal parts, which explained the surprising measure (one-fourth) of the piece. He conjectured (**Conjecture J1**) that he could produce the desired fraction by partitioning the larger part into two parts.

Because Josh was a splitter, partitioning and iterating were inverse operations for him so that one-fourth might become an iterable unit. In other words, Josh's abduction of his partitioning operation within the context of conceptualizing a unit fraction could yield partitive unit fractions. Corroborating evidence of this occurred later in the episode when, for the first time in my observations of him, Josh was able to estimate the fractional size of a given piece in the absence of a partitioned whole.

While the students closed their eyes, I pulled one fourth from a four-fourths stick and covered everything except for the one-fourth piece and the ruler. When the students opened their eyes, Josh looked at the piece and the ruler for a moment and said, "that's one-fourth of it." It seems that Josh had mentally iterated the piece within the ruler to segment the ruler into four parts, thus determining that the piece was one out of four equal parts in the ruler. It seems that Josh's operational conjecture *did* engender a novel use of iterating as well. This would be a matter of course if Josh's conjecture had actually been an abduction of his splitting operation, which was a composition of partitioning and iterating. In drawing the reasonable conclusion that this was the case, I have formed my own abduction.

A similar event occurred at the end of the episode. The witness had intervened with a task in which he produced a copy of the ruler and made an arbitrary mark on it that was about one-third across it. When he asked the students whether he had marked one-half, Josh looked at the marked stick for a moment and replied, "that's a third of the stick because you can put another one in there [pointing to the middle of the larger part of the stick]." These new ways of acting indicate that Conjecture J1 had resulted in a functional accommodation of Josh's part-whole fractional scheme and a new way of operating that was like that of a partitive unit fractional scheme. I attributed the latter scheme to Josh once I was able to determine that Josh was indeed using iteration (and not simply partitioning) to determine fractional sizes and that this new way of operating was relatively permanent.

Josh's Generalizing Assimilation of His Partitive Unit Fractional Scheme

After two more teaching episodes of working with Matthew, scheduling conflicts required me to reassign the partners in the lower-level pairs, so Josh began working with Sierra. During the ensuing teaching episodes, I observed more instances of Josh estimating the fractional sizes of pieces (one-fifth, one-ninth, etc.) without a visibly partitioned copy of the ruler. Josh seemed to mentally iterate the pieces within the ruler to determine the fractional sizes of the pieces, treating them as iterable units and partitive unit fractions. I hypothesized that Josh's novel use of partitioning would engender a novel use of iterating: a partitive unit fractional scheme. Josh's actions in his seventh teaching episode (24 March, 2003) indicate that he had begun constructing partitive unit fractions, corroborating my hypothesis. Furthermore, in the following protocol, taken from that same teaching episode, Josh formed yet another operational conjecture, leading to the development of a more general scheme.

After producing a four-eighteenths stick, Josh was surprised (as was Sierra) to find that the computer measured the stick to be "2/9." The following protocol documents the students' attempts to explain the surprise, including Josh's actions illustrated in Figure 2.

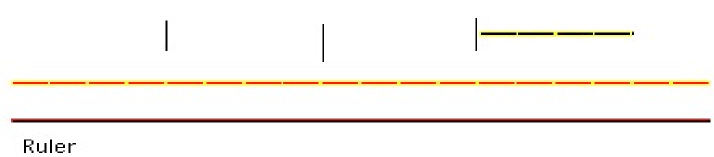

Figure 2. Josh's marks for iterating a composite fraction.

S: 2 times 9 is 18 and there's eighteen [parts] over there.
T: Ah. Okay. So maybe that will help you.
J: [Meanwhile, Josh was dragging the fraction stick across the top of the eighteen-eighteenths stick making marks as illustrated in Figure 2]
T: [to Sierra] Do you know what he's doing?
S: [shakes her head, negatively]
T: Josh, what are you doing? Sierra's not sure. I'm not either. [after a few seconds of silence] It looks like you had an idea…

J: Remember last time? We had like, say, one of these bars equaled up to three things [dragging the four-eighteenths fraction between each pair of marks that he had just made]. You remember?

T: Oh! Okay. So, you were hoping that it might equal up to something.

J: Yeah. [continues dragging the fraction in the other direction until it reaches the right side of the rightmost mark, as illustrated in Figure 2] It wouldn't work.

T: It doesn't work? Why doesn't it work?

J: Because I would have two left over [pointing to the two rightmost parts in the eighteen-eighteenths stick].

Sierra's initial explanation was an abductive relation of the numerator and denominator to the eighteen visible parts in the whole. Such a relation might help her establish an invented rule or procedure about fractions, but it was not an operational conjecture about fractional measures because it did not appear to involve a novel use of fractional operations, such as partitioning, iterating, splitting, disembedding, or unitizing (see Steffe and Olive, 1996 for a complete description of these operations). On the other hand, Josh seemed to unite four parts and iterate them in order to affirm their measure of two-ninths, just as he would iterate unit fractions with a partitive unit fractional scheme.

Josh's iteration of the four-part unit in justifying its measure was an operational conjecture. He conjectured (**Conjecture J2**) that the four-eighteenths stick would fit evenly into the ruler to establish it as a simpler fraction, namely two-ninths. Although his operational conjecture did not enable him to successfully complete the task described above, in a subsequent task Josh was able to operate similarly to justify why five-twentieths measured as one-fourth. He was able to explain his reasoning to Sierra in the following manner.

J: [to Sierra] There's a fifth. [dragging the five-twentieths stick across the ruler] And then you put that one right there, and that's another fifth. So, that'll make that ten. Put that one right there, it'd be fifteen. Put that one right there, it'd be twenty. That's one fourth… There's four of these little things right there [pointing to the places he had dragged the five-twentieths stick] going into that.

The novelty of this way of operating is indicated by Josh's difficulty with language in initially referring to the one-fourth stick as one-fifth (presumably because it contained five parts). Still, he was able to keep track of the units of units (fourths, each made up of five twentieths) as he iterated the composite unit. By then Josh had fully developed a partitive unit fractional scheme with which he could operate consistently, reliably, and confidently. The operational conjecture of Conjecture J2 was a generalizing assimilation of that scheme, in which he was coordinating units of units. He had abducted more than a single operation, his conjecture involved the abduction of an entire way of operating (his partitive unit fractional scheme) to a new class of situations (those involving composite units).

Josh was able to use similar ways of operating to anticipate the measures of unsimplified fractions and to produce fractions equivalent to a given fraction, even when the given fraction was a non-unit fraction (e.g. two-thirds). And by the end of the teaching experiment, he appeared to be operating with advanced schemes, such as a *commensurate fractional scheme*, *a partitive unit fractional scheme for composite units*, and a general *partitive fractional scheme*. These schemes are described in detail in Steffe, 2002.

Results of the Study

It is the explicit goal of PME-NA 2005 to support the development of frameworks that, in turn, support research and learning. In this short paper, I have sketched such a framework and demonstrated its promise in terms of describing and promoting student learning. In particular, among the schemes and operations that Steffe and Olive identified in their Fractions Project, the splitting operation stands out as especially powerful because it contains both partitioning and iteration operations that can be used in abduction to construct many other fractions schemes. Analysis of the extensive teaching experiments that I conducted for my study support this conclusion beyond the examples provided here. Among the four students identified in the lower level of fractions development, Josh was one of two with a splitting operation. These two students (especially Josh) made immense progress relative to the other two lower level students. By the end of the teaching experiment, Josh was operating on par with the strongest student in the high-level pair (the other student who was also a splitter) and was more advanced than the other high-level student, who did not have a splitting operation.

The results of my study indicate that there are general operations (such as the splitting operation) that can be abducted to various situations, thus advancing students' conceptual learning. In forming Conjecture J1, Josh's operational conjecture engendered a functional accommodation of his part-whole fractional scheme that eventually resulted in a partitive unit fractional scheme. In forming Conjecture J2, his operational conjecture amounted to a generalizing assimilation of his partitive unit fractional scheme, using the scheme in situations that involved composite units and simplifying fractions.

Beyond inductively applying concepts and operations to perceived situations, the abduction of operations helps to explain how operational systems themselves become more powerful. Josh was not simply drawing and testing conclusions about the TIMA:Sticks environment, he was constructing a stronger system of fractions throughout the teaching experiment. The universe of his fractions knowledge had increased, not by accumulating new facts, but by becoming a more powerful system of operating. Likewise, no universe in which we act is a static body from which we acquire new facts and perceptions. Rather, the potential for new knowledge is based on the systems of operating that we construct. And this operational knowledge (God's too, according to Piaget) is ever increasing.

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PERFORMANCE DIFFERENCES OF PROFICIENT AND NON-PROFICIENT STUDENTS ON A STANDARDS-BASED TEST

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The differences between proficient and non-proficient students on standardized tests are often explained by three factors: reading difficulties, limited mathematics content knowledge, and test bias. This report is part of a larger study that found no statistically significant differences between non-proficient students who were Hispanic (low SES), African American (low SES), and Anglo (high SES). This study examined questions on a standards-base test in which the performance of non-proficient students and proficient students were statistically different. Findings indicate that non-proficient students often have an incomplete understanding of words, misinterpret copulative words like "or" or lack tested mathematics content knowledge.

Emphasis on standardized testing has reinforced perceptions of a "gap" in the school achievement of diverse groups of students (Holt & Campbell, 2004). Researchers have begun to investigate these perceptions (Olson & Ellerton, 2004; Powell, 2004; Thurber, Shinn, & Smolkowski, 2002). Powell suggested that standardized achievement tests scores do not reflect the true attainment of students from different social, economic, or ethnic groups, implying that the tests may be biased. Not surprisingly, students from different socioeconomic classes interpreted test questions imbedded in a context of money differently (Solano-Flores & Trumbull, 2003). Solano-Flores and Trumbull suggested that the conceptual understanding of students from low-socioeconomic groups may be higher than the state's standards-based state achievement test indicated. They encouraged researchers to further investigate the extent to which large-scale tests reflect students' conceptual understanding.

Olson and Ellerton (2004) investigated whether a standards-based test reflected students' conceptual understanding for students classified as proficient and non-proficient. They found a moderate relationship between the test score for students classified as proficient and their conceptual understanding ($r = .63$, $p = .007$), but no relationship between non-proficient students' test score and their conceptual understanding ($r = .29$, $p = .053$). Contrary to Powell's assertion, there were no significant test performance differences between non-proficient students who were Hispanic, African American, or Anglo.

Mayer and Hegarty (1996) suggested that the difficulties that many students have solving mathematical problems rest in the comprehension process. They describe two basic types of problem solvers, novice and skilled. In a research study that mapped brain activity during problem solving, the novice problem-solvers were attentive to the numbers while reading the problem and performed arithmetic operations to arrive at an answer. Skilled problem-solvers initially read the problem slower and focused on qualitative relationships within the problem to understand the situation, create a representation, and devise a solution. Unlike the novice problem-solver, skilled problem-solvers successfully transformed problem situations into mathematical representations.

Students with scores on standards-based tests that are below the state benchmark for proficiency miss more questions than students classified as proficient, suggesting that they may be

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novice problem-solvers with difficulty comprehending problems. Proficient students successfully answer more questions correctly, indicating that they learned the mathematics content and gained the problem solving skills of a skilled problem-solver. To investigate factors that contribute to errors in the problem solving process made by low-performing students, this study examined test questions and students' responses in which the performance of proficient and non-proficient students differed significantly. Specifically, this study sought to answer two questions. First, do proficient and non-proficient students have the same impediments to correctly solving a mathematical problem? Second, what attributes of questions or students' understanding of mathematical ideas are implicated to non-proficient students' lower test scores?

Theoretical Framework

Research (e.g., Cobb, Wood, & Yackel, 1970; Kumpulainen & Mutanen, 2000) suggests that social activities influence the learning of mathematics as individuals negotiate meaning through interactions. From the perspective of symbolic interactionism (Blumer, 1969), an individual interprets another person's words and gestures to create meaning. Shared meanings are constructed through a dynamic process of creating and re-creating meanings as individuals interact with each other. People attach particular meanings to words, symbols, and gestures through these social interactions (Voigt, 1996). As the individual interacts with others (e.g., teachers, peers, parents) words may acquire a single meaning or multiple meanings. Attention to the meanings that words and symbols acquire is particularly important in mathematics because many mathematical words also have a common usage that is quite different from their mathematical meaning.

The general population and many teachers conceptualize mathematics as a domain in which words and symbols have unambiguous meanings and problems have a single correct solution. However, research indicates that pictures, problems, and text problems can be interpreted in different ways (Neth & Voigt, 1991; Solano-Flores & Trumbull, 2003; Voigt, 1996). Students use background knowledge and experiences to interpret these symbolic representations. They may interpret a situation quite different from the teacher's intended problem simply because the students did not recognize that some words have multiple meanings or nuances. It is critical for students to select an appropriate meaning for words on large-scale tests if the test is to reflect students' mathematical proficiency. This study explored the problem solving strategies of students with different levels of proficiency to identify where in the problem solving process difficulties arose and whether their interpretation of words and symbols led to incorrect responses.

Methods

Sixty-one eighth-grade students from three urban-middle schools in Western U. S. who used a reform mathematics curriculum participated in this study. The students were classified as either non-proficient or proficient based on the state's standardized test from the previous year. A treatment standards-based test was constructed using Colorado's standards-based framework. The treatment test resembled Colorado's large-scale achievement test and included both multiple-choice and constructed-response questions. Each student was interviewed within one week of completing the test to determine his or her conceptual understanding of the problem. These understandings were coded as full, partial, or no understanding.

Student responses were analyzed for impediments in the problem solving process using the Newman error analysis methodology (Newman, 1983). The impediments were described using six categories (reading, comprehension, transformation, process skill, encoding, or carelessness) to indicate an error factor. A reading error was recorded when students did not correctly read a problem aloud. A comprehension error was coded when a student could not restate the problem in his or her own words. A translation error was recorded when the student could not represent the problem with appropriate mathematical symbols or produce a representation that led to a correct solution. Process errors occurred when the student did not correctly perform the selected operation or symbolic manipulations. An encoding error occurred when the student correctly solved the problem but selected the wrong answer. Newman found that these five error factors are hierarchical and the remaining error, carelessness, may occur at any stage in the problem solving process.

Chi-square statistical tests were conducted to identify the questions in which the error factor was statistically different between the non-proficient and proficient students. These questions and students' responses were further analyzed for mathematical misconceptions and linguistic factors that influenced students' interpretation of the questions.

Results and Discussion

To determine whether proficient and non-proficient students had similar impediments when solving problems, descriptive statistics were used to analyze the types of errors made by these two student populations. Proficient students made fewer errors (M = 5.5) than non-proficient students (M = 14.2) and analysis of the error factors indicated a similar pattern of errors. Both groups had few reading or comprehension errors (less than 10%), with the majority of errors (80 to 84%) due to transforming the problem situation into a representation (pictorial or mathematical symbols). These finding support Mayer & Hegarty (1996) assertion that skilled problem-solvers are better able to transform problems into representations; they made fewer errors. However, the patterns of error factors for the non- and proficient students indicate that the leading error factor was identical, suggesting that both groups may benefit from mathematics instruction that focuses on translating problem situations into representations.

To tease out the attributes of questions that led to a lower test performance by non-proficient students, Chi-square tests were conducted on the 20 questions. This analysis identified eight questions in which the error factor was statistically different between the two student populations. Six of these questions utilized a multiple-choice format and the remaining two questions used a short constructed-response format. Qualitative analysis of students' responses during the interviews revealed two linguistic factors that impacted student performance. These factors included a narrow understanding of the meanings of words and misinterpretation of copulative words.

Narrow Understanding of Words

The proficient students recognized that words have a variety of meanings with different nuances. For example, a problem which asked students to find an approximation of a 15 percent tip (see Figure 1) was identified by a one-sample chi-square test to be problematic for only the non-proficient students (χ^2 (2, N = 44) = 16.40, p < .001). Eighty-seven percent of the non-proficient students made an error translating the problem into a mathematical strategy that led to a correct solution compared to ten percent of the proficient students who made a similar error. Qualitative analysis of students' explanations revealed that both groups of students substituted the word estimation for approximation while explaining their solution strategy. However, the word estimate held different meanings for the two student groups.

Of the following, which is the closest approximation of a 15 percent tip on a restaurant check of \$24.99? A) \$2.50 B) B) \$3.00 C) C) \$3.75 D) D) \$4.50 E) E) \$5.00

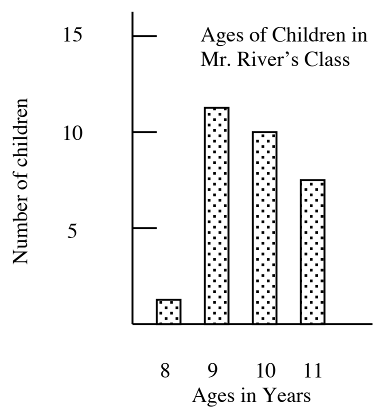
Figure 1. Restaurant tip question which was designed to test students' understanding of approximation and percentage.

Half of the non-proficient students guessed because they "never got percents" and were unable to solve the problem, indicating an error from limited knowledge of mathematics. The other non-proficient students who made a translation error used a similar strategy. Typically they explained, "I know that 15 percent of \$15 is about \$2. So \$25 is about twice as much so I'd leave a \$4.00 tip I picked \$4.50 because when you estimate you always go higher so you have enough." (interview, March 2004). These non-proficient students interpreted estimation as a mechanism to ensure that an individual had enough money to complete a mathematical procedure; rounding off or a mental math strategy. In contrast, the proficient students interpreted estimation as using one of two mathematical procedures; rounding off or a mental math strategy. Proficient students explained, "I rounded off \$24.99 to \$25 and calculating the 15 percent tip by multiplying 25 by .15." or "I knew that a ten percent tip is about \$2.50 and then I added \$1.25 [half of \$2.50] to equal \$3.75."

Both the proficient students and many of the non-proficient used their knowledge of estimation and percentage to determine an appropriate tip. But, these non-proficient students did not arrive at the expected answer because they *always* rounded up to make sure that there was enough money. In addition, the non-proficient students did not rely on a mathematical procedure. Instead they used their understanding of number sense to estimate; \$2 is the tip for \$15 and \$25 is about two times 15, therefore, the tip is about \$4. The proficient students relied on a calculation by either rounding off a number or decomposing one of the numbers before completing the calculation and arrived at the expected response of \$3.75. These results indicate that proficient students have multiple ways to interpret a word (e.g., estimation) that may vary with a problem context and that they connect these meanings with mathematical computations. While many of the non-proficient students demonstrated an understanding of both approximation and percentage, they had a limited interpretation of the word estimate which was linked only to having enough money to make a purchase. This limited interpretation led them to round up to \$4.50 instead of selected the response that was in fact closer to their estimate of \$4.00.

Interpreting Words with a Mathematical Meaning

Proficient students were more likely than non-proficient students to interpret words with a mathematical meaning. For example, 95% of the proficient students correctly responded to a question which required students to interpret a graph (see Figure 2) compared to 33 percent of the non-proficient students who correctly answered this question. Both student groups correctly restated the question and extracted data from the graph. A one-sample chi-square test indicated that the number of translation error for non-proficient students was statistically significant (χ^2 (4, N = 21) = 10.17, p = .04). Qualitative analysis indicated that non-proficient students made two mistakes interpreting words with a mathematical meaning and suggested an incomplete understanding of the word *most* and *or*. Following is a brief discussion of these two errors.



The graph above shows how many of the 32 children in Mr. Rivera's class are 8, 9, 10, and 11 years old. Which of the following is true?

A) Most are younger than 9
B) Most are younger than 10
C) Most are 9 or older
D) All the above are true
E) None of the above are true

Figure 2. Half the non-proficient students incorrectly answered this multiple-choice question.

Monica (pseudonym) was an articulate student with an incomplete understanding of the word *most*. She explained, "For A, most are 9, so most can't be younger than 9. For B, two out of three, the 9 and 10 year olds are younger than 11, no it's two out of four that are younger than 10. C is wrong because some children are 11 and older. So the answer is E." Then Olson asked, "What is meant by the word most?" Monica gave an example, "More. Like if you add up the numbers you get a bigger number. For C you'd add up the 9 and 10 year olds." Olson asked, "Why did you add up just the number of 9 and 10 year olds?" Monica responded, "Because it didn't say to add the number of 11 year olds too." We interpreted Monica's responses to indicate a meaning of most that limited the summation to two quantities.

Most of the non-proficient students had difficulty interpreting the copulative word *or*. After Jivonne (pseudonym) explained his response of E, Olson probed "Which students are ten or younger?" Jivonne responded, "The nine and eight year olds." Olson asked, "What about the ten year olds?" Jivonne answered, "They're ten." Jivonne, as other non-proficient students, did not interpret the copulative *or* with an inclusive meaning. During language acquisition, individuals gradually learn the proper use of prepositions and copulatives (Halliday & Hasan, 1989) The word *or* usually indicates an exclusive choice as in the question, *Are there more 9 year olds or 10 year olds*? Jivonne interpreted *or* as a copulative with an exclusive meaning and did not recognize the particular nuance where *or* also indicated an inclusive meaning. This error was not a reading or comprehension error, but is a linguistic issue that indicates a high order mistake made before individuals develop mature language usage. These findings suggest that focusing on reading will not increase non-proficient students' achievement. But, class discussions about different mathematical interpretations of words and multiple representations may support both proficient and non-proficient students.

Summary

Mayer and Hegarty (1996) found that the linguistic structure of a problem sometimes required students to apply an inconsistent interpretation of a key word. For example, in the following problem students must comprehend the problem context to correctly interpret the key word, less. *A pen costs 50 cents at Buy-It which is five cents less than at Supers. What does a pen cost at Supers?* One would expect that the novice problem-solvers (non-proficient student) to focus on the numbers and key word *less* to solve the problem by subtracting five cents from 50 cents. Mayer and Hegarty labeled problems which used key words with a meaning different from the learned meaning as "inconsistent" (p. 41). They found that novice problem-solvers focused on numbers and key words within a problem and incorrectly translated inconsistent problems into a solution strategy. However, this study found that the problem of translating a problem into an appropriate solution strategy went beyond interpreting key words in a problem context suggested by Mayer and Hegarty. Non-proficient students were unable to attach an appropriate mathematical meaning to words used in common language and to interpret copulative words like *or* with both a mathematical and common use meaning.

Solano-Flores and Trumbull (2003) found that students of different socioeconomic status tended to interpret words differently. For example in a money problem, students who lived in poverty interpreted the resource of money as limited and when a parent had "only $1.00 bills" that the resource was exactly one dollar with not access to additional money (p. 4). They suggest that multiple-choice test questions may not reflect students' conceptual understanding because of the sociocultural context assumed by problems. While students' socioeconomic background may influence their interpretation of problems, it is clearly not the only factor that impacts students' responses on large-scale tests. In this study, both students who lived in poverty and those with adequate resources had similar problems interpreting words.

This study along with research conducted by Mayer and Hegarty (1996) and Solano-Flores and Trumbull (2003) indicate that educators need to pay more attention to how students interpret words. Too often it appears that teachers assume that students are able to select an appropriate meaning for a word because they can use it appropriately in common language. But in math, words assume particular meanings that are derived from mathematical relationships within the problem and words may assume a meaning that is inconsistent with a learned key word association. These findings suggests that focusing instruction on key words may be misguided and that teachers need to spend considerable time discussing how words attain different meanings depending on the context of a problem. In addition, teachers need to be sensitive to the sometimes subtle nuances of meanings that words acquire in mathematics.

This study found that low-performing students had a limited understanding of language and attached a single meaning to words while proficient students understood different nuances of words and selected an appropriate meaning from a set of possible meanings. Additional research is needed to identify words and contexts that interfere with low performing students' interpretation of problems in such a way that a response does not reflect their mathematical proficiency. Solano-Flores and Trumbull (2003) suggested that increased attention is needed to address linguistic diversity and the testing of English Language Learners (ELL). We suggest that additional research is needed to investigate how ELL students learn different meanings of words in English (e.g., common vs. mathematical meanings). In addition, research is needed to determine whether ELL students in bilingual programs develop multiple meanings of words.

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COMMUNICATION IN CLASS – THE CORE OF TEACHING?

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Teachers' questions and pupils' responses form an essential part of a lesson in school. Mathematics is not about getting answers, but about developing pupils' insight into relationships and structures. Learning mathematics means, among other things, learning to use a specialized conceptual language e.g. in talking and writing as well as in reasoning and problem solving. While the role of communication in classroom cannot be overemphasized it has to be noticed that the level of teachers' listening matters. Here we will develop a hierarchic structure to classify teachers' listening. The classification structure is used to evaluate teachers' levels of listening.

Theoretical Background

Teachers' questions and pupils' responses are essential parts of a lesson in school. Traditionally, it is thought that a pupil's answer shows explicitly what he/she knows. This has led to the situation that pupils expect the teacher to look for a correct answer that they expect to be in their teacher's mind. The constructivist idea, however, emphasizes that it is the teacher's task to help pupils in constructing their knowledge and understanding of concepts and mathematical thinking (cf. Davis & al. 1990).

Social-cultural research has emphasized studies on classroom discourse (e.g. Hufferd-Ackles & al. 2004). The main goal of talking mathematics in classroom is to understand and extend one's own thinking as well as the thinking of others. While the role of communication in classroom cannot be overemphasized, it has to be noticed that communication needs a listener as well as a speaker. When the teacher wants to pay attention to his/her pupils' understanding and thinking process he/she has to listen carefully and interpretatively to the pupils.

Our aim is to concentrate on communication in mathematics teaching, and especially on teachers' listening skills. One basic demand for genuine exchange of ideas is that the participants are listening with understanding to each other. Therefore, we decided to study how mathematics teachers actually listen to their pupils.

Teachers' Listening

When using discussion as a teaching method, teachers are not always listening carefully what their pupils are saying, but having their own presentation in the first place in their mind (cf. Pehkonen & Ejersbo 2004). This phenomenon is also known from earlier research in the form of teachers neglecting to use such answers that do not fit into their instructional plans.

Listening has been in the center of communication research more than fifty years (cf. Stewart 1983, Burley-Allen 1995), but in mathematics education it has a shorter history. Today one may find some studies on communication with the focus on listening, among them: Davis (1997) reported a collaborative research project with a middle school mathematics teacher, and gave some examples how the teacher listens. She described the teacher's evaluative listening, interpretative listening and hermeneutic listening in mathematic lessons in eight-grade class. Furthermore Coles (2001) has analyzed one teacher's mathematics lessons, using Davis' (1997) levels of listening. His observation was that both the teacher's and his pupils' listening

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developed with such teaching strategies that slowed down situations in the class, and offered room for discussions.

Peressini & Knuth (1998) made a thorough analysis of a teacher's discourse in his highschool mathematics classroom and of an educator's discourse in a university mathematics education classroom. The researchers were impressed how the teacher strived to listen to his pupils and to make sense of what they were saying and the thinking that grounded their mathematical discourse. Nicol (1999) explored prospective teachers' learning to teach mathematics. She analyzed during mathematics lessons how elementary student teachers asked questions, how and what they listened and how they responded to pupils' answers. As a consequence, she points out teachers' difficulties, challenges and tensions in listening while teaching.

Empirical Study

We are starting a research project the aim of which is to find out on which level teachers in the Finnish comprehensive school (grades 1–9) listen to their pupils' answers during mathematics lessons. In order to find interesting research questions and to develop analysis methods for our further results, we have carried out this pilot study. Especially we try to develop a proper taxonomy for the levels of listening.

In this pilot study we have looked through some videotaped mathematics lessons from different Finnish teachers at grade five and eight, in order to find out how they listen to their pupils in normal mathematics classes. On the basis of the literature, i.a. Stewart (1983), Burley-Allen (1995), Davis (1997), and applying our own experiences as teachers and teacher educators we formed the following classification structure that contained five levels of listening from a pupil's point of view: 1) Not listening, 2) Listening selectively, 3) Evaluative listening, 4) Interpretative listening, 5) Empathic listening (Pehkonen & Ahtee 2004).

For an observer, it is not easy to interpret the levels of teachers' listening, since thinking happens in teachers' mind. Selective listening means that the teacher listens a part of a pupil's answer, but not all. Evaluative listening contains the teacher's evaluation on the correctness of a pupil's answer, i.e. its compatibility with the teacher's "correct" answer. In interpretative listening the teacher strives to understand a pupil's answer in his/her own framework, i.e. as a mathematics teacher, and to interpret it in a positive spirit.

Empathic listening differs from interpretative listening in that now the teacher tries to understand and value a pupil's ideas, although they might be strange and new to the teacher. Then the pupil and the teacher try to understand the topic from a new view point. Empathic listening has been critized impossible to implement (e.g. Stewart 1983), since then the listener should be able to switch-off his/her own feelings and thinking. Therefore, we have decided to call the highest level in the classification structure as open listening.

Earlier research results (e.g. Pehkonen & Ejersbo 2004) indicate that the level of teachers' listening to their pupils is usually very low.

Results

We looked the videotapes first alone, and then the lessons were transcribed. After that we picked up together some typical episodes on two-way communication. First we classified the episodes separately, and after that discussed together long enough so that we ended with the classification shown here. In the following we present with the aid of some episodes on which level teachers are listening to their pupils. However, during the classification we noticed that the classification structure is too rough, and therefore, we use sub-classes in some cases.

Description of Listening Levels

In the following, the levels of listening with their sub-classes are described more carefully, and they will be used later to analyze the communication episodes given. The hierarchy of listening levels is based on teachers' level of awareness and thinking. We have tried to describe the levels of listening so exactly that also other persons could apply the classification structure and reach similar results.

1. Not listening

A teacher's non-listening is surely typical in almost all lessons. The teacher often ignores what he/she hears, because he/she wants to proceed with the topic, and because a pupil's question or comment may lead to a side-track for a long time. Or he/she may have in his/her mind an idea he/she wants immediately to present to pupils, and therefore, he/she will not ponder what the meaning of the pupil's question or comment would be. Especially during lessons in middle school, there are often situations when pupils are making improper comments, in order to get others' attention. As a consequence on this level of listening, we concluded to extract two sub-levels: Firstly, the teacher does not even hear pupils' comments or questions; he/she hears without listening (1a). Secondly, the teacher hears pupils' comment or question, but he/she ignores (1b) the utterance.

2. Selective listening

The teacher is trimmed to listen only the questions concerning the topic to be dealt with. For example, an inexperienced teacher tries to listen only to such pupils from whom he/she may expect "correct" answers. He/she experience often all kind of disturbance as a threat for his/her teacherhood. There are connections with strong control, discipline requirements and defence mechanisms.

3. Evaluative listening

Often a teacher has in his/her mind an answer (the model answer) which he/she expects from pupils and with which he/she compares pupils' answers. Therefore, the teacher's evaluation can be a simple verbal accepting utterance, as Right or Good, or it could be also only a short nod, a head's shake or a break before moving to the next question. In a more elaborated evaluative listening, the teacher comments the pupil's answer, e.g. by transforming the terms and expressions used into a correct form. Thus, we will separate a simple evaluative listening (3a) and a more elaborated evaluative listening (3b).

4. Interpretative listening

In the interpretative listening, a teacher interprets and understands a pupil's answer within his/her own thinking. He/she does not have the model answer in his/her mind. For example, he/she may repeat the pupil's answer with other words; thus he/she processes the pupil's answer, and therefore, interprets it. Also here we may separate a simple interpretation (4a) and a more elaborated interpretation (4b).

5. Open listening

Here a teacher strives to understand a pupil's thoughts from his/her world, and not only to place them into his/her own "model thinking". Open listening requires a conscious effort from the teacher to hear, follow and understand the pupil's ideas. This level represents the most open situation from the pupil's viewpoint – the pupil is not expected to think in a certain way, but he/she has freedom to develop his/her own new ideas.

Examples of Listening Levels

In the following, we give some episodes that are selected from the videoed lessons and represent the two-way communication in the class: usually the teacher asks, and the pupils answer. The episodes are selected so that different levels are represented as many-sided as possible. The episodes are analyzed, and the teacher's level of listening coded and reasoned.

Episode 1

Here the pupils are independently solving problems from the textbook. The teacher checks one problem together with the whole class, in order to ensure that everybody knows what to do. After that he tells the class to continue solving problems.

1 Teacher: Now, do the problem C.
2 Carl: Where is it?
3 Teacher: It is on the page 23.
4 Peter: How does it go?
5 The teacher starts to go around the class looking at the pupils' working.

Here the teacher first clearly listened to Carl's question (2), but not any more to Peter's question (4). The first one might be selective listening (level 2), whereas in the case of the second one he/she ignored the question (4); thus listening happens on the level 1b.

Episode 2

The topic of the lesson was fractions and their transformations. Instruction goes forward with short questions and answers.

1 The teacher writes the fraction 20/8 on the blackboard.
2 Teacher: Can we transform 20/8 to a mixed number?
3 Heli: Yes.
4 Teacher: How many wholes will it give?
5 Lina: 2
6 Teacher: And how many parts are left?
7 Simon: 2
8 The teacher writes 2 2/8 on the blackboard.
9 Teacher: And can we do something to this number?
10 Sam: 4/8
11 The teacher corrects 2 4/8 on the blackboard.
12 Teacher: Sorry. A good remark.

Heli's answer (3) to the first question was straight according to the teacher's expectation. Therefore, we could suppose that the teacher is acting on the level of evaluative listening, but using a simple evaluation (level 3a). Listening of the next answer (5) can be placed also on the same level (level 3a). In the case of Simon's answer (7), the teacher automatically accepts it without thinking; therefore, we conclude that she was hearing Simon's answer (7) without really listening (level 1a). Sam answers to the teacher's question (9) by correcting the mistake made by the teacher on the blackboard. Therefore, the teacher has to connect Sam's answer (10) with her earlier question (6), and thus she has listened to Sam on a higher level (level 4b).

Episode 3

Now decimal numbers are dealt with in the lesson.

1 Teacher: Is it allowed to add zeros after the decimal point wherever?

2 Maria: Yes, to the end.

3 Teacher: Yes. Not between wherever. To the end one can add zeros appropriately.

The teacher accepts on Maria's answer (2) and sharpens it. This example shows a more elaborated evaluation (level 3b).

Episode 4

This is a part of a mathematics lesson on grade 8, where the topic is the use of percentages. In the episode we will find many-level listening.

1 Teacher: If 12 hens from the henhouse are on the yard eating, and 60 % are inside, how many hens altogether are there in the henhouse?

2 Tina: You cannot say so that 60 % are inside.

3 Teacher: How many percents of the hens are on the yard, if 60 % are inside?

4 Tina: Ask again.

5 Teacher: If 60 % of the hens are inside, how many percents are then outside?

6 Jane: They are 12.

7 Teacher: In percents?

8 Jane: 40

9 Tina: How can you change like that?

10 Teacher: Well Jane. How did you get that 40?

11 Jane: Well, if there are 60 % inside, then there are 40 % left from the whole 100 %.

12 Teacher: Yes. Now we know, how many hens there are. Now we will continue.

The teacher changed her first question (1) into a simpler form, when Tina announced in her first comment (2) that she did not understand the question at all. Then Tina asked the teacher to repeat her question (4). Jane gives firstly the number of the hens as an answer (6). Jane's second answer (8) was quitted only with a nod. Finally the teacher asked Jane to reason, how she had concluded her answer (10). Thus the teacher got little by little the answer in the form she wanted (11).

In this episode, the teacher's listening was mainly on the levels of evaluation and interpretation: Based on the answer (2) she interprets that Tina did not understand the situation at all (level 4a). The teacher has no problem to interpret Tina's comment; the experienced teacher divided her question into two parts. In the case of the answers (6 and 8), listening is evaluative (level 3a). The teacher's reaction to Tina's comment (9) shows that the teacher interpreted her answer as non-understanding (level 4a), and therefore, asked Jane to give reasons to her answer. The teacher's last listening to Jane's answer (11) was simple evaluative (level 3a).

Discussion

The classification of teachers' listening into five main levels is based on videotaped mathematics lessons; we ended up with the classification structure given in Figure 1. In practice we noticed that the five-step classification structure for teachers' listening was too coarse, and we divided some levels into two sub-levels.

In our pilot study that contained only ten teachers and from each 1–3 lessons, teachers' listening was mainly on the levels 1, 2 and 3a. Only in some cases, the teachers reached the level 3b, and very rarely the level of listening seemed to come up to the level 4 (interpretation). In our data, there was no case that could have been classified on the level 5 (open listening). It is worth noticing that in our episodes we show examples from each level of teachers' listening that we were able to find in the lessons, and therefore, it is no way representative.

As earlier stated, the level of a teacher's listening to his/her pupils seems to be in the best cases evaluative (e.g. Pehkonen & Ejersbo 2004). But the learning conception compatible to constructivism would demand that teachers listen to their pupils also on the levels of interpretative and open listening. Until then pupils begin to pay attention to their own conceptions and their deviation from the way shown in mathematics. For the teacher, open listening is absolutely important, in order he/she could perceive how pupils interpret matters and what kind of difficulties pupils might have in understanding the topic. Thus he/she gets hints via which he/she can help pupils to check their conceptions and thinking.

Not listening hearing without listening ignoring
Selective listening
Evaluative listening simple evaluation more elaborated evaluation
Interpretative listening simple interpretation more elaborated interpretation
Open listening

Figure 1. The hierarchic classification structure of teachers' listening to their pupils.

Listening belongs to teachers' pedagogical skills, but it has had fairly little attention in teacher education programs. For example, if a teacher is not careful enough, he/she can develop a habit to use selective listening and concentrate only on listening correct answers. Thus pupils are led to schematic thinking. The communication between the teacher and pupils will improve, when the teacher shows that he/she tries to understand what pupils mean. Then pupils are more ready to cooperate, i.e. to express also their spontaneous thoughts. The teacher's task is no more to be sure that pupils understand subject matters in a certain way, but the aim is to make a change possible in pupils' thinking. Emphasizing authority or embarrassing pupils will inhibit the formation of free communication situations.

The question of teachers' levels of listening needs more research, since this seems to be in a key position when implementing open teaching methods. Especially interesting it woul be to investigate the levels of teachers' listening in lessons with different working methods as well as in different school forms, in order to reach a reliable description of the state-of-art in teachers' listening. Our next step will be to collect from different grade levels a big sample of videotapes, and to apply the classification structure of listening levels.

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TEACHERS' BELIEFS IN THEIR INSTRUCTIONAL CAPACITY: THE EFFECTS OF IN-SERVICE

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A randomized field trial of 106 grade 6 mathematics teachers examined the effects of professional development program on teachers' beliefs about their instructional capacity. Teacher efficacy was selected as the outcome measure because teachers with high efficacy beliefs are more likely to implement challenging instructional ideas and have higher student achievement. Previous research suggests that mathematics education reform depresses teacher efficacy. We found that a mathematics in-service oriented around reform elements contributed to teacher expectations about their ability to meet classroom management challenges in the mathematics classroom. The effect of the in-service on teacher efficacy for classroom management was consistent across various measures of prior PD experience. The results from mandated assessments showed that student achievement in mathematics, but not in Reading and Writing, increased following the in-service.

Objective

To measure the effects of in-service on grade 6 mathematics teachers' beliefs about their instructional capacity.

Research Perspectives

Previous research, reviewed in Ross (1998), provides convincing evidence that teacher beliefs about their instructional capacity have a substantial impact on teacher willingness to implement instructional reform and on student achievement. Teachers who believe that they will be able to bring about student learning (i.e., who have high teacher efficacy) choose more challenging goals, are more likely to take responsibility for student outcomes, and persist in the face of failure (Bandura, 1997).Teachers with high teacher efficacy are more likely to try out new teaching ideas, particularly techniques that are difficult, involve risks, and require control to be shared with students (Ross, 1992; Czerniak & Schriver-Waldon, 1991; Dutton, 1990; Moore, 1990; Shachar & Shmuelevitz, 1997), characteristics of instruction that have been frequently attributed to standards-based mathematics teaching (e.g., Ross, McDougall, Hogaboam-Gray, 2002).

Teacher efficacy has a substantial impact on student achievement. Teachers with high expectations about their ability to help students learn produce higher student achievement in mathematics, regardless of whether teacher beliefs about their capacity are measured at the individual teacher level (Ashton & Webb, 1986; Herman, Meece, & McCombs, 2000; Moore & Esselman, 1994; Muijs & Reynolds, 2001; Ross & Cousins, 1993) or at the professional community level (Ross & Gray, 2004;Goddard, 2001; Goddard, Hoy, & Hoy, 2000; Goddard & LofGerfo, 2004).

Teacher efficacy tends to be a stable teacher characteristic, formed in preservice and in the early years of teaching (Woolfolk Hoy & Spero, 2005), that continues at the same level unless disturbed by fundamental changes in the conditions of teacher work. Implementation of math

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education reform exemplifies such a fundamental change. Smith (1996) argued that mathematics education reform threatens teacher beliefs about their instructional ability in multiple ways. For example, traditional mathematics programs provide clear direction about what teachers need to do to create learning-- the teaching task is narrowed to presenting procedures and assigning practice. Simply restating the rules constitutes an explanation to students. Applications to real world problems that could create confusion and uncertainty can be avoided. In contrast, in the constructivist approaches that distinguish mathematics reform, teachers cannot avoid real world problems and they may find themselves dealing with mathematical issues they have not mastered themselves. In addition, traditional math teaching gives teachers a strong sense of their effectiveness because students do not know formal math until teachers tell them. Student success can be easily attributed to teacher's competence. In standards-based mathematics classrooms the teacher is no longer the sole knowledge source but a facilitator of student constructions, making the teacher's contribution to learning difficult to disentangle from the students'. Ross, McKeiver, and Hogaboam-Gray (1997) found that mathematics teachers who had to change their instructional practices in response to mandated curriculum changes experienced a sharp deterioration in teacher efficacy because they could no longer predict the learning outcomes of their strategies. In this study the decline was temporary; teacher beliefs about their capacity resurged when teachers compiled student performance evidence indicating that their new methods were effective. Gabriele et al. (1999) found that as teachers implemented mathematics teaching reform sources of information about their effectiveness changed from student mastery of algorithms to focus on changes in student thinking. Gabriele et al. speculated that in-service that focused on helping teachers detect changes in student thinking and provided opportunities for collegial talk about these changes might contribute to improved teacher efficacy.

We tested Gabriele et al.'s speculation about the positive effects of PD on teacher efficacy. Studies of teacher efficacy effects of PD with control groups are rare. Fritz, Miller-Heyl, Kreutzer and MacPhee (1995) examined the effects of 20-24 hours of PD focused on developing teachers' personal self-esteem, internal locus of control and communication skills. Teachers were given K-8 curriculum materials to promote student outcomes such as drug and alcohol awareness. Treatment teachers had higher teacher efficacy scores on the post- and delayed posttests. Effects were strongest for teachers identified as high users of the curriculum materials. Fritz et al. attributed the PD effects to the content of the program. The researchers argued that although they compared volunteers to a convenience sample of control teachers, their claims of a program effect were valid because the two groups had equivalent teacher efficacy scores on the pretest. Edwards, Green, Lyons, Rogers, and Swords (1998) found that a peer coaching program had a small positive effect on teacher efficacy Although treatment and control group teachers had equivalent teacher efficacy scores on the pretest, the treatment group had completed significantly more in-service credits and sample mortality was significantly higher in the treatment than the control group.

Given that high teacher efficacy facilitates implementation of standards-based mathematics teaching, we designed a PD program that had as one of its goals the strengthening of teacher efficacy. The guiding research question for the study was "Will teacher PD that attended to sources of teacher efficacy information increase teacher efficacy beliefs?"

Methods and Data Sources

We conducted a randomized field trial in which the population of grade 6 teachers in one Ontario, Canada school district (N=106), was randomly assigned to early or late participation in a mathematics in-service. The dependent variable was a teacher survey consisting of 12 teacher efficacy items adapted for mathematics teaching from Tschannen-Moran and Wolfolk Hoy (2001): 4 items for *efficacy for engagement,* 4 items for *efficacy for teaching strategies,* and 4 items for *efficacy for student management*. We used this measure because of its high reliability, evidence of concurrent validity with the Rand items and Gibson and Dembo (1984) scales (Tschannen-Moran & Wolfolk Hoy, 2001; 2002), fidelity to the prevailing conception of teacher efficacy (Tschannen-Moran et al., 1998), and concerns (expressed by Deemer & Minke, 1999; Guskey & Passaro, 1994) about the factor structure of the most frequently used alternative measure, Gibson and Dembo. We also administered a battery of student and teacher measures (not reported) to establish the equivalence of the two groups. The teacher efficacy survey was administered at the outset of the project and at the mid-point (after the early group had received the in-service but before the late treatment participated). The independent variable was in-service group: we contrasted the early to the late group using General Linear Modeling. We examined multivariate and univariate effects. At the end of the in-service, students completed mandated assessments in mathematics that mainly consisted of open-ended performance tasks.

The PD consisted of one full day, followed by three 2-hour after-school sessions. Communication of mathematical ideas was the organizing theme, chosen because it impacts multiple aspects of mathematics teaching. At each session organizers modeled constructivist teaching using rich mathematical tasks appropriate to grade 6. Teachers completed the tasks in small groups, discussed alternate strategies and solutions likely to be generated by their students, identified key mathematics concepts in the tasks, and devised strategies for supporting student thinking. Between sessions teachers used the same or similar tasks in their own classrooms. In sessions 2-4 teachers brought samples of student work collected in their classes to support teacher strategy discussions.

The PD was designed to enhance teacher beliefs about their capacity. Teacher efficacy involves an appraisal of the difficulties of the teaching task, weighed against an assessment of personal competence (Tschannen-Moran et al., 1998). The strongest predictor of teacher efficacy is mastery experience, i.e., success in the classroom (Bandura, 1997). Our first strategy for increasing teacher opportunities for mastery experiences was to strengthen competence by incorporating features of effective mathematics PD identified in Hill's (2004) review: active learning by teachers, using examples from classroom practice, collaborative activities modeling effective pedagogy, providing opportunities for reflection, practice and feedback, and focus on content. By increasing competence we anticipated that teachers would be more successful in the classroom, according to teachers' usual criteria, which would enhance teacher efficacy.

Our second strategy for increasing mastery experiences was to redefine success. For example, instead of defining a lesson as successful if most students obtained the right answer using conventional algorithms, we urged teachers to focus on the depth of conceptual understanding that students reached and on the extent to which students contributed to the construction of their knowledge. To influence teacher criteria, we provided teachers with a rubric containing ten dimensions of mathematics teaching. For each dimension there were four levels of teacher practice, ranging from transmission teaching to standards-based teaching. We selected three dimensions for special attention and experienced teachers modeled standards-based teaching using grade 6 tasks. While modeling, presenters encouraged teachers to judge their success in terms of familiar standards (e.g., student use of mathematical language) and those less familiar (e.g., students' invention of problem solving procedures and sharing, explaining and justifying their solutions). When debriefing between-session practice, we focused on these new standards for judging success. In doing so we tried to reduce teacher perceptions of the difficulty of the instructional task and increase beliefs in their ability to teach in new ways.

In addition to the strategies for strengthening teacher opportunities for mastery experience, we also incorporated into the PD opportunities for teacher efficacy enhancement through vicarious experience, social persuasion, and management of affective states.

Results

We conducted an exploratory factor analysis (principal axis with promax rotation) on the teacher efficacy items because Woolfolk Hoy (n.d.) reported that the item loadings tend to vary from one administration to the next. We found that one item fit the instructional strategies dimension better than the student engagement dimension. The remaining items all loaded at least .25 on the scales identified by Tschannen-Moran and Woolfolk Hoy (2001).

There were no pretest differences between the groups on any of the teacher or student variables (for details see the first author). We conducted a multivariate analysis of covariance using GLM. The dependent variables were the posttest scores on the three teacher efficacy variables. The covariates were the comparable pretest scores. The independent variable was experimental condition. The multivariate results showed that pretest teacher efficacy accounted for 23-46% of the variance in posttest teacher efficacy. In-service group explained 5.8% of the variance.

Examination of the pretest adjusted means in Table 1 showed that teachers who had received the in-service scored higher than teachers who had not done so on all teacher efficacy variables. However, the univariate results indicated the in-service group effect was significant on only one of the teacher efficacy variables, accounting for 5.7% of the total variance. Specifically, PD participation maintained teacher expectations in their ability to meet classroom management challenges in the mathematics classroom, while the expectations of those who had not yet received the PD declined. We also found that the effect of the in-service on teacher efficacy for classroom management was consistent across various measures of prior PD experience.

Table 1
Post Teacher Efficacy Scores Adjusted by Pretests

	Adjusted Posttest Mean	
	Treatment Group	Control Group
Engagement Efficacy	4.23	4.15
Inst Strategies Efficacy	3.98	3.90
Class Management Efficacy	4.56	3.78

We examined whether the effect of the in-service on teacher efficacy for classroom management was moderated by prior PD experience. We ran separate univariate analyses of covariance in which the dependent variable was the posttest teacher efficacy for classroom management score, the covariate was the pretest score on the same indicator, and the independent variables were attending at least one mathematics conference or taking one university mathematics course, experimental condition, and the interaction of condition and covariate. In both analyses (not shown), treatment group membership continued to be a significant predictor of posttest efficacy for classroom management. Mathematics conference attendance, participation in university mathematics, and the interaction of group membership with teacher background variable were not statistically significant.

We examined year over year changes in the annual assessments conducted by an independent testing agency (EQAO) for the provincial government. In this analysis we collapsed the treatment groups into a single category since all grade 6 teachers had participated in the PD by the time of the provincial assessment. We found that grade 6 mathematics achievement increased significantly from 2003 to 2004 [$t(6105)=3.73$, $p<.001$], although the effect was small (ES=.095). In contrast there were no significant differences in grade 6 Reading [$t(6105)=0.839$, $p=.401$]or Writing scores [$t(6105)=1.749$, $p=.080$], assessments completed by the same students at the same time. This result supports research that found that changes in teacher efficacy foreshadow changes in student achievement (Ross, Hogaboam-Gray, & Hannay, 2001).

Conclusions

This paper contributes to our understanding of reform in mathematics education at two levels. First, it demonstrates that teacher efficacy can be enhanced through PD. The increases were small (about 6% of the variance) but were robust across teacher background variables. We found, as other researchers have done, that teacher efficacy was highly stable in the group that had not yet received the PD (the pre-post correlation was r=.70). Given (i) the stability of the construct, (ii) the influence of teacher efficacy on teacher willingness to implement mathematics education reform, and (iii) the well-demonstrated effect of teacher beliefs about their ability on student achievement, even small changes in teacher efficacy are noteworthy.

Second, this study demonstrates that enhancements in teacher efficacy foreshadow improvements in student achievement. Again, the effects were small (representing 10% of the variance in achievement) but the external examinations covered the whole year's work, not just the mathematics content covered by the PD.

The study was limited by its short duration and brevity of PD experiences. Many of the studies reporting substantial impacts on teachers involve small samples interacting with PD deliverers frequently (e.g., weekly) for extended time periods (e.g., for more than one year). However, the amount of PD time provided by this project is comparable to the PD experienced by most teachers.

The most important direction for further research is to explore why we found significant increases in teacher efficacy for classroom management but not for the other two components of the Ohio State Scale. We speculate that this finding will replicate in other studies in which teachers are implementing mathematics education reform. Standards-based mathematics teaching requires that teachers share control of the classroom agenda with students, a requirement that is a core challenge to traditional practice. We suspect that in this context teacher confidence begins to emerge with their ability to manage classroom behavior and that confidence in instructional strategies and in engaging students follows only when confidence in management is consolidated. But this pattern may not generalize beyond the specific context of mathematics reform. It may be that the PD in other subjects may impact on different dimensions of teacher efficacy beliefs.

Acknowledgements

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TEACHERS' DIFFICULTIES IN ADAPTING TO THE USE OF NEW TECHNOLOGIES IN MATHEMATICS CLASSROOMS AND THE INFLUENCE ON STUDENTS' LEARNING AND ATTITUDES

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We present results from a larger on-going research studying the implementation –that is part of national project sponsored, since 1997, by the Mexican Ministry of Education– of technological tools in middle-school mathematics classrooms (children 12-15 yrs. old). We focus here on teachers' difficulties in adapting their teaching practice to the use of technological tools and the effect of their approach and attitudes for students' learning. A small sample of the participating schools was used to carry out a detailed analysis of teachers' practices, and students' attitudes and learning, combining field observations and interviews with quantitative research. We analyzed 9 teachers' performances with 13 study groups (615 students) and 8 control groups (324 students), in two schools. How well teachers adapted to the use of the proposed pedagogical model, as well as their attitudes, affected students' learning and opinions of the tools.

Introduction

The results presented in this paper are part of the research associated to an on-going national project, known as EMAT –Teaching of Mathematics with Technology, that has been sponsored, since 1997, by the Mexican Ministry of Education, and whose objective is to implement technological tools into the mathematics classrooms of Mexican public middle-schools (children 12-15 yrs. old). We begin by giving a description of this project, present an overview of its evaluation and associated research, give the methodology we used with a sample of teachers and students to evaluate teacher's practice and students learning, and then give some results.

The EMAT Project

The aims of the project, as stated in its literature (Ursini & Rojano, 2000), are to promote the use of new technologies, using a constructionist approach, to enrich and improve the current teaching and learning of mathematics in Mexico. A study carried out in Mexico and England (Rojano et al., 1996) involving mathematical practices in science classes, revealed that in Mexico few students are able to close the gap between the formal treatment of the curricular topics and their possible applications. This suggested that it was necessary to replace the formal approach of the then official curriculum, with a "down-up" approach capable of fostering the students' explorative, manipulative, and communication skills. Thus, a main part of the EMAT project was to design activities and a pedagogical model for incorporating the use of technological tools to mathematics teaching, that emphasized exploratory and collaborative learning.

This design of the pedagogical model, the choice of tools, and the activities (which were collected in a book, for each tool, of activity worksheets) was carefully carried out, by expert national and international researchers in Mathematics Education, taking into account results from international research in computer-based mathematics education to the practice in the "real world". The role of the technology is emphasized as that of support tools that mediate action.

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Assuming that mediation modifies the process of knowledge construction, the computational instruments (Balacheff & Kaput, 1996) were conceived as mediational tools for students' construction of concepts. In its first phase (1997-2000), the project researched the use of Spreadsheets, Cabri-Géomètre, SimCalc, Stella and the TI-92 calculator, with nearly 90 teachers and 10000 students in 8 states during that first period (see Moreno et al., 1999). In the second phase (2001-2005) the use of some of the tools used in the first phase (Spreadsheets, Cabri-Géomètre, and the TI-92 calculator) has been expanding gradually in the national public school system, and new tools have been incorporated. The main tool that was added in the second phase was the Logo programming language; this decision was taken at the suggestion of both national and international advisors who evaluated the first phase and pointed out that there was still the need for more expressive activities (such as programming), on the part of the students.

In addition to the choice of tools and design of the activities, an important part of the EMAT proposal is the pedagogical model. In particular, much of the philosophy and pedagogy underlying the design of mathematical microworlds (Hoyles & Noss, 1992) was present in the design and recommendations for what are called the "EMAT laboratories". Thus, emphasis was put on the changes in the classroom structure (Ursini & Rojano, 2000), such as the requirement of a different teaching approach and the way the classroom needs to be set up: from the physical set-up of the equipment, to the collaboration between students, to the role of the teacher, to the pedagogical tools (e.g. worksheets). In particular, the pedagogical model emphasizes a collaborative model of learning, with students working in pairs or teams for each computer (and the classroom computers set-up in a horseshoe fashion) for promoting discussions and the exchange of ideas. The teacher's role is that of, on one side, promoting the exchange of ideas and collective discussion; at the same time, acting as mediator between the students and the technological tools (the computational environment), aiding the students in their work with the class activities and sharing with them the same expressive medium (the tool).

The Evaluation of the Project and its Associated Research

Over the past 7 years, the project has been evaluated using both global and local levels of assessments. As Moreno et al. (1999) describe, the global level focuses on understanding the educational system as a complex model that includes teachers, headmasters, authorities and parents as essential elements, whose observations also form part of the assessment of the project's collective ways of thinking about itself. On the other hand, the local level concentrates on the specific learning of students, and the use of the tools with respect to student profiles combining both quantitative and qualitative (e.g. longitudinal case studies) research methodologies. (Many other associated investigations are also being carried out, such as those researching gender differences.)

Results from the evaluation of the first phase showed that although, in general, the project was groundbreaking in changing the role of the teacher and the traditional passive attitude of children, it also had its challenges and difficulties (Ursini & Sacristán, 2002). Among the issues noted in the evaluation of the first phase were (despite the attempts to train teachers as profoundly as possible in the use of the tool and of the pedagogical model):

- lack of adequate mathematical preparation on the part of the teachers;
- lack of experience working with technology by both teachers and students;
- difficulties in adapting to the proposed pedagogical model;
- teachers' lack of free time to prepare anything outside the established curriculum.

A more detailed analysis of these issues was carried out in the second phase of the project, looking in particular at the impact of these issues on students' learning, as described below.

Teachers' Practices and Attitudes and the Relationship with Students' Learning

Description and Methodology

In the academic year 2002-2003, we took the opportunity, during the implementation of the Logo programming tool in the second phase of the project, to carry out a multi-level analysis of teachers' practices and students' attitudes and learning, combining field observations and interviews with quantitative research. In particular we investigated: (i) The ways in which the student and teacher materials are used; (ii) teacher's performance during the Logo sessions; (iii) children's attitudes, and (iv) children's mathematical performance both in standardized tests and through their academic scores.

We had 21 academic groups (7 groups for each of the 3 years of middle-school): 13 study groups involving 615 students and 8 control groups (324 students), with 9 teachers (all volunteers), in two schools. The study groups each had a 50 min. session per week devoted to the Logo programming activities. The control groups didn't, but otherwise followed the same curricular structure, often with the same teacher as the study groups. (When possible, the study groups were chosen because they had a *lower* mathematical performance in the first 3 months of the year than the control groups; it was hoped that they would benefit from the use of the tool.)

A team of four researchers carried out, for all the study groups, "participatory observations" (Brousseau, 1999) of each session where the Logo EMAT activities were used (approximately 25 to 30 sessions per study group); "passive observation" of several "regular" mathematics lessons for both the study and control groups was also done. We were concerned in not only observing teachers' practices and students' attitudes and performances during these sessions, but also the teacher-student interactions. Several researchers have designed classroom observation protocols to assess teaching practice (e.g. Piburn et al., 2000; Lawrenz et al., 2002). Inspired by some of these works, taking into account the pedagogical model established for the EMAT project, and being aware of the limitations that teachers have shown to have in the previous evaluations of the project (as described above), we designed the following set of main observational categories for teachers (some of these also included sub-categories not given here):

1. Mathematical proficiency shown of the day's topic;
2. Clear understanding of the day's activity;
3. Technical proficiency shown of the Logo language and the use of the tool;
4. Class Handling – Group Handling / Balance between control and allowing freedom to explore;
5. Explanations, Guidance and Technical Help
6. Making explicit the mathematical connections (between the Logo context and school mathematics);
7. Familiarity with the activity handbook and the teacher's handbook;
8. Use of the pedagogical recommendations contained in the teacher's handbook;
9. Class and activity preparation;
10. Self-confidence;
11. Improvement in the overall performance of the teacher in the implementation of the activities, with respect to the previous sessions.

Each technology-based session, teachers were rated according to these categories using a scale of 1 to 5; this gave us a set of 25 to 30 scores per teacher, per category. This provided a map of the progress of the teachers in the different categories; in addition, we were also able to average them both by category and overall, at the end of the academic year. These scores were complemented by qualitative data: field notes and interviews. This allowed us to rate teachers' overall performance and to carry out case studies of each of them.

In addition to these qualitative observations of the teachers, we designed standardized mathematics tests that were applied to the students in both the study and control groups at the beginning and at the end of the school year (739 out of the original 987 students, took the final test – some students dropped out of school during the academic year while others were absent on the testing days). We also looked at children's and the groups' academic scores (Grade Point Averages) throughout the year.

We complemented the quantitative study of student's learning with students' observations: In each study group we selected a random sample of 3 students from the low, medium and high achieving ranges based on their academic scores in the first 3 months of the school year. These three students (in each study group) were followed as case studies throughout the technology-based sessions. For this we used some of the methodologies described in Watt & Watt (1993).

At the end of the academic year, we carried out structured interviews of all the participating teachers and of the case-study students.

Overview of Results

In terms of technological abilities and familiarity, five years after the Emat project was first put into practice, teachers and students were now much more familiar with computers, even if it was their first year working in the Emat project. This is a cultural change not related to the project in particular, as computers become more and more prevalent in society. Nevertheless, four of the 9 teachers that we observed still had difficulties and lacked technical ability and self-confidence in the use of both the computer and the tool. Interestingly, all of these teachers were over 40 years old (with one of them in fifties); we point this out because in the larger Emat study we have noticed that older teachers have much more difficulty adapting to the use of technology. In our sample, two of the five teachers who were more proficient in the use of the technology, were also over fifty, but they were very technological-oriented and used computers in almost all their work. But the other three teachers were the three youngest teachers of our sample.

In terms of the use of the tool itself (in this case Logo) and the activities, the teachers had been given intensive preparation courses, but this was not enough to make them highly proficient in the use of the tool and they were expected to work more on their own (they were provided with teacher's handbooks). But one of the main problems we observed was that teachers hardly ever prepared their technology-based sessions and activities; (as was observed in the larger Emat project, teacher's often work two or even three shifts and lack free time to prepare anything outside the stablished curriculum). This meant that they were very unfamiliar and often a lacked understanding (in terms of didactical recommendations and aims, as well as mathematical content) of the activities they were giving to the students. Very often they just picked an activity at the beginning of the session with no previous analysis or reflection; seldom did we see teachers prepare the technology sessions beforehand (a clear exception to this was Jesus, of whom we give some more details further below), although this slowly changed over the course of the academic year as the teachers became more enthusiastic with the use of the tool. In other words, as their appreciation of the benefits of the use of the tool grew (as well as ability and self-confidence in the use of the tool), they became more interested in understanding and preparing the activities and technology sessions. It was a case of learning over the course of their practice.

Another main area we looked at was how well teachers adapted to the proposed pedagogical model. As stated above, this model breaks with the traditional teaching approach; it promotes exploration, collaboration and discussion on the part of the students; and the teacher's role is that of guide and mediator, and of one who should also make explicit the mathematical content found in the activities.

In the beginning, almost all the teachers had difficulties putting into practice the model. We found two extreme cases: On one extreme, some teachers did not feel comfortable with the changes in class structure and wanted to control what the students did: in one case, the teacher (whom we will call here Gilberto, and of whom we discuss further below) did not allow the students to freely explore and carry out the activities; he even dictated to the students what they had to type on the computer; in that group, many students said they had no clear understanding of what they were doing. Another teacher was very strict with discipline, which sometimes prevented students to freely engage in discussion and collaboration.

On the other extreme, we found some teachers who left students almost totally on their own, without any guidance nor sometimes even interest, on how to carry-out the activities, This also meant there was no technical assistance if the students needed it, so the students helped themselves. As with other problems, this behavior was influenced by the fact that the teachers were themselves unfamiliar with the activities, and lacked self-confidence. However, in these cases, the students were able to carry out the activities on their own quite well, but there were no whole classroom discussions mediated by the teachers to allow the group to share their results and to understand the mathematical elements involved in the Logo programming context.

In fact, at the beginning, most teachers, in both extremes, seldom engaged in class discussions, and therefore did not make explicit the mathematical content found in the activities. Again, this was a consequence of their lack of familiarity and understanding of the activities.

Nevertheless, although it took a while, the attitude of the teachers changed and most of them became gradually more comfortable with the pedagogical model and began to understand it. Towards the end year, in most classes (except, notably, Gilberto's) students were allowed to freely explore the activities but the teacher still maintained some control of what was to be covered and engaged students in whole class discussions of the knowledge explored.

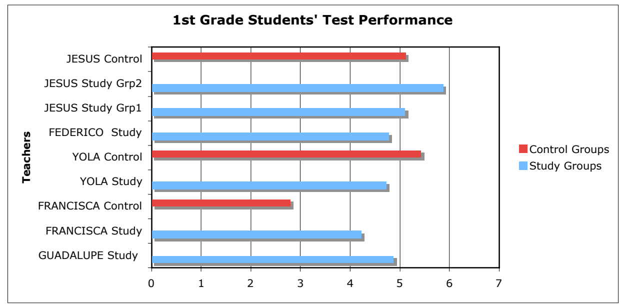
In the next section we explore how the teacher's attitude and ability to adapt to the pedagogical model appears to have influenced students' learning and opinions of the tool. It is worth pointing out that the benefits of the use of the tool for students are very hard to assess. On the one hand the knowledge that they work with is highly situated within the technological context. On the other hand, the results derived from the student data we collected were highly inconsistent. Nevertheless, there does seem to be a correlation between students' learning and the teacher's ability and attitude towards the technology and the tool.

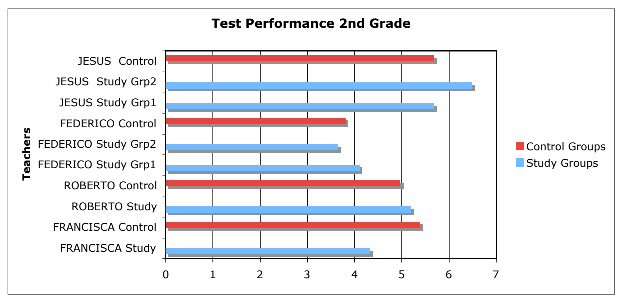
Relationship Between Teacher's Practice and Attitude and Students' Learning – Results from the Teacher Case Studies

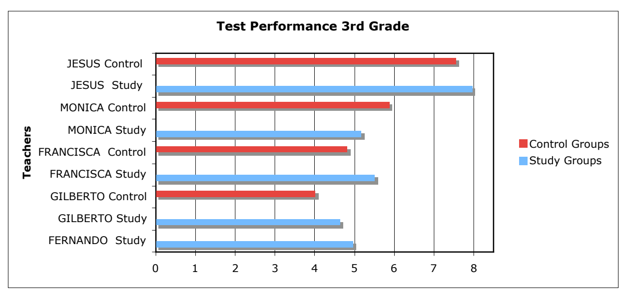
First, is the case of Jesus who was the overall highest-rated teacher (see table below). Initially, he did not want to participate in the study, but he discovered the potential of the Logo activities during the teacher-training workshop and he became highly motivated and tried to follow the proposed pedagogical model. He had four study groups and three control groups (two study groups for 1st grade and one each of 2nd and 3rd grades; and three control groups of each grade). At the end of the year, in both the standarized tests and academic scores, *all* the study groups scored higher than the control groups. Furthermore, all the interviewed students in his study groups felt very positive and enthusiastic about the tool, and were aware of the mathematics they had learned. One of his students said: “The aim of programming I think is to interact with the algebra, because variables [in Logo] are related to the unknowns in algebra and that allows us to develop the process in Logo.” This teacher’s students, after working with Logo, also demanded a different use of some of the other Emat tools, such as programming macros in Cabri.

Teacher	*Overall Teacher Score (out of 5)*
Jesus	4.8
Monica	4
Guadalupe	3.8
Yola	3.8
Fernando	3.2
Roberto	2.6
Francisca	2.2
Federico	2
Gilberto	2

Teachers’ scores.







At the other extreme was Gilberto, who was the lowest-rated teacher (except in the mathematics proficiency category): He had a good mathematics proficiency, and his students had good scores in the mathematics standarized pre-tests as compared to other teachers’ groups. However, this teacher didn’t show any motivation for the use of technology and he ignored the proposed pedagogical model (as already stated above, he didn’t allow students to explore on their own and “dictated” what students should type on the computer); he also had very poor technical abilities but didn’t show any interest in improving (in particular, he didn’t attend a refresher workshop that was offered to these teachers). Although we only wanted volunteer teachers to participate at that stage of the study, we believe that this teacher only participated in the study because his peers did, and this was reflected in his attitude. In the standarized tests at the end of the year, as well as in the academic scores, his study group scored worse than all the other groups in that grade. More importantly, in the interviews, many of his students had negative comments on the use and benefits of the tool explaining that they found the Logo activities boring, useless, or confusing; these were the *only* students of all the ones we interviewed that didn’t feel positively about the tool.

Another extreme case is that of Federico. Federico had very poor mathematical proficiency and a lack of interest in the use of the tool, and he never prepared any of the lessons. Not surprisingly his students had the lowest test scores of all the groups.

A middle case is the case of Roberto. Roberto had great initial difficulties in putting into practice the tool, as he didn't prepare his sessions at all and had very little technical proficiency in the use of the tool, so the students were often without guidance during the activities. On the other hand, he had a positive attitude towards the use of technology and had a willingness to learn, so he would ask students to show him how they had solved the activities and learned from them. His students had fair scores in the tests, with the study group having better scores than the control one, and his students had positive opinions of the tool.

Summary and Concluding Remarks

In general, many of the teachers lacked experience working with the tool, and had initial difficulties in adapting to the proposed pedagogical model; but their attitude towards the use of technology was crucial in their motivation to learn and gradually change their practice. (We would like to research this further, using attitude measuring instruments). We found a correlation between teachers' performance during the technology-based sessions (teacher's class preparation, their understanding and use of the proposed pedagogical model, their experience working with technology, etc.) and children's performance in the math tests (compared with the control groups). These results are not surprising –Noss & Hoyles (1996) emphasize the influence of the setting on children's performance and mathematical behavior– but they provide evidence of the importance of teachers attitudes and pedagogical implementation of the materials, for students' learning, particularly when new technologies are first incorporated into the practice.

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A SLIPPERY SLOPE BETWEEN EQUIVALENCE AND EQUALITY: EXPLORING STUDENTS' REASONING IN THE CONTEXT OF ALGEBRA INSTRUCTION INVOLVING A COMPUTER ALGEBRA SYSTEM

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This study explores the reasoning of high school students who participated in a classroom teaching experiment in which a computer algebra system was used to promote reflection on equivalence of algebraic expressions. Results of a post-test indicate that many students experienced difficulties teasing equivalence and equality apart. Moreover, results suggest that reasoning about expressions in terms of their having a common form played a significant positive role in enabling students to ascertain whether given expressions are equivalent.

Background and Perspectives

Although the idea of equivalence of algebraic expressions lies at the heart of transformational activity in algebra, it is not typically placed at the foreground of algebra instruction. The research literature on algebra learning, while extensive (Kieran, 1992), includes relatively few inquiries into students' understandings of algebraic equivalence (Ball, Pierce, & Stacey, 2003; Kieran, 1984; Pomerantsev & Korosteleva, 2003; Steinberger, Sleeman, & Ktorza, 1990). Consequently, relatively little is known about students' understandings of algebraic equivalence and, in particular, how those understandings might play out in the sense they make of transformational work in algebra.

In a comparison of novice and more advanced high school students' performance on tasks dealing with equivalence of linear equations, Kieran (1984) found that the latter group showed little awareness that the equation-solving processes they were able to execute proficiently preserve solutions. In a study using a much larger sample of eight-and ninth-grade students, and employing tasks similar to those used by Kieran (1984), Steinberg, Sleeman and Ktorza (1990) found that while most students knew how to use transformations to solve simple linear equations, many did not spontaneously relate this knowledge to the production of equivalent expressions. Ball, Pierce and Stacey (2003) developed an instrument designed to assess students' abilities to quickly recognize equivalent algebraic forms—the "Algebraic Expectation Quiz" (ibid., p. 4). On the basis of the performance of a sample of fifty students on this test, administered before and after students' progression from 11th -to 12th -grade and during which time they participated in algebra instruction involving the use of a computer algebra system (CAS), the researchers reported that "recognizing equivalence, even in simple cases, is a significant obstacle for students" (ibid., p. 4). Pomerantsev and Korosteleva (2003) presented compelling evidence that difficulties related to understanding algebraic equivalence can extend well beyond the post-secondary level. Their study (ibid.) involved the use of a diagnostic test administered to a large (N=416) sample of students enrolled in different stages of their K-8 teacher preparation program at a major American university. Test items were designed to assess students' abilities to discern and use structural aspects of algebraic expressions; results of this research revealed serious difficulties in doing so, cutting across students in all groups contained in the sample.

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Within the last decade or so a fairly coherent research program has emerged out of the French *didactique* tradition of research in mathematics education. A number of these studies have explored the use of computer algebra systems (CAS) in mathematics classes at the high school or college level (e.g., Artigue, 2002; Guin & Trouche, 1999; Lagrange, 2000). These researchers argue that CAS can be used as a tool to promote the co-development of both conceptual understanding and technical proficiency among students, provided that technical aspects of mathematical activity are not ignored. For instance, Lagrange (2000) frames the idea of mathematical technique as a bridge between task and theory, in the sense that as students develop techniques in response to certain task situations, they concomitantly engage in theory-building and reflection on the technical aspects of their activity in relation to the mathematical ideas addressed in the task. In a study touching on the idea of equivalence, Artigue (2002) drew on students' work involving the passage from one given form of expression to another to illustrate how the research team attended specifically to the fact that "equivalence problems arise which go far beyond what is usual for the classroom" (p. 265). Artigue's study employed a CAS as a "lever to promote work on the syntax of algebraic expressions, which is something very difficult to motivate in standard environments" (ibid.), asserting that it forces students to confront issues of equivalence and simplification. Similarly, Nicaud et al. (2004) foreground the importance of equivalence in algebra, framing it as "a major reasoning mode in algebra; it consists of searching for the solution of a problem by replacing the algebraic expressions of the problem by equivalent expressions" (pp. 171-2).

It is against this backdrop, which highlights the importance of the notion of algebraic equivalence as well as students' reported difficulties therein, that we situate the current study.

The Study

Goals and Participants

Our study is part of an ongoing three-year project involving 5 classes of 10^{th} -graders who have been following an integrated curriculum involving algebra as part of the course of study since Grade 7. Inspired by, and wishing to build upon, the recent research emerging out of France, a central aim of this project is to inform our understanding of high school students' emerging ideas about algebra in relation to their engagement with instruction that integrates the use of CAS with traditional paper-and-pencil work. More specifically, the aim of this study is to gain insight into students' reasoning about the idea of equivalence in the context of instruction designed to foster the emergence of such reasoning.

The particular group of 15 students featured in this brief report comprised a class at a private school in a major metropolitan center in eastern Canada. The students had learned basic techniques of factoring and solving linear and quadratic equations during the previous year and had used graphing calculators on a regular basis. However, they had no prior experience using symbol-manipulating calculators. Results of a pre-test indicated that these students were quite skilled in algebraic manipulation.

Method of Inquiry

Our research team conducted teaching experiments in these classes. This entailed engaging students in a sequence of eight instructional activities designed to fit into their program of study, which was taught by their regular classroom teacher, and to integrate TI-92 Plus calculators with traditional paper-and-pencil algebra work.

The idea of algebraic equivalence was an underlying conceptual thread running through the design of the activities, with a particular subset of three of the activities designed specifically to support student reflection on this idea explicitly in relation to the transformation of expressions and the solving of equations. The activities were each designed to take up to two class periods. Each activity was punctuated by parts, each part included presentation of student work and discussion of the main issues raised by the tasks in the given part. Tasks were of three types that involved either work with CAS, or with paper/pencil, or were of a reflective nature. Each activity was accompanied by a teacher version that included suggestions for classroom discussion. In designing these tasks, the research team took into serious consideration both the students' background knowledge and the fact that the tasks were to fit into an existing curriculum; but we also moved to ensure that they would unfold within a particular classroom culture that gave priority to discussion of serious mathematical issues.

The data corpus generated for this study includes: audio-video recordings of the classroom lessons centered around the activities and of individual interviews conducted with selected students after each lesson; students' written work on activity sheets and a post-test; field notes generated by members of the research team who were present during the unfolding of the classroom lessons.

Addressing Algebraic Equivalence: Foregrounding the Idea of Common Form

The sequence of 3 activities in which the idea of equivalence was explicitly addressed unfolded over 4 hour-long classroom lessons held on consecutive days, starting approximately five weeks into the academic year. The research team designed these activities in consultation with the classroom teachers who implemented them. These consultations together with the team's examination of the Grade 10 mathematics textbook used in the class indicated that the activities would constitute students' first encounter with equivalence and its relation to algebraic expressions, transformations, and equations as an *explicit* idea of reflection.

In broad terms, the intended conceptual progression of these activities was to have students develop connections between equivalence of expressions, addressed in the first two activities, and equivalent equations, addressed in a third activity. Details of this sequence entailed having students use numerical evaluation of expressions, and comparison of their resultant values, as the entry point for discussions of equivalence. The impossibility of testing all possible numerical replacements in order to determine equivalence motivated the use of algebraic manipulation and the explicit search for common forms of expressions—an idea highlighted in classroom discussions. Discussions also included attention to restrictions on equivalence. The relation between equivalent/non-equivalent expressions and equation solutions was then explored in both CAS and paper-and-pencil tasks. An outline of the content sequence of these activities is shown in Figure 1 (Kieran & Saldanha, 2005). In these activities, the idea of *equivalent expressions* was eventually defined formally as follows: "We specify a set of admissible numbers for x (e.g., excluding the numbers where one of the expressions is not defined). If, for any admissible number that replaces x, each of the expressions gives the same value, we say that these expressions are equivalent on the set of admissible values".

Activity 1: Equivalence of Expressions
- Part I (with CAS): Comparing expressions by numerical evaluation
- Part II (with paper/pencil): Comparing expressions by algebraic manipulation
- Part III (with CAS): Testing for equivalence by re-expressing the form of an expression – using the EXPAND command
- Part IV (with CAS): Testing for equivalence without re-expressing the form of an expression – using a test of equality
- Part V (with CAS): Testing for equivalence – using either CAS method

Activity 2: Continuation of Equivalence of Expressions
- Part I: Exploring and interpreting the effects of the ENTER button, and the EXPAND and FACTOR commands
- Part II: Showing equivalence of expressions by using various CAS approaches

Activity 3: Transition from Expressions to Equations
- Part I (with CAS): Introduction to the use of the SOLVE command
- Part II (with CAS): Expressions revisited, and their subsequent integration into equations
- Part III (paper/pencil): Constructing equations and identities
- Part IV (with CAS): Synthesis of various equation types

Figure 1: Outline of content of the three activities addressing equivalence.

Shortly after completing this sequence of activities, students took a post-test designed to query their understandings of the content addressed within these activities. A post-test question pertinent to the coming discussion is shown in Figure 2.

Q.5 The following equation has $x = 2$ and $x = 2/3$ as solutions:

$$x(2x-4)+(-x+2)^2 = -3x^2+8x-4$$

(i) Precisely what does it mean to say that, "the values 2 and 2/3 are solutions of this equation"?

(ii) Use the CAS to show that: (a) the two values above are indeed solutions, and
(b) there are no other solutions.

What I entered into the CAS	What the CAS displays and my interpretation of it

(iii) Are the expressions on the left- and right-hand sides of this equation equivalent? Please explain.

Figure 2. A post-test question (see part iii) that revealed interpretation of equivalence.

Data Analysis and Results

Our report focuses on results of a preliminary analysis of students' post-test responses.[1]

[1] See Kieran & Saldanha (2005) for a discussion of students' interaction with the CAS in relation to their emerging understandings of equivalence. In that report we describe the unfolding of the activity sequence and selected instructional episodes in greater detail.

Analytical Method

Analysis of these data unfolded in a manner that initially involved mutually influential and overlapping phases, but that eventually coalesced into more independent and identifiable broad stages. Following Saldanha (2004) and Thompson, Saldanha & Liu (2004) we began, before examining any data, by identifying relevant dimensions of each question to which students who had engaged in the activity sequence might be expected to attend. Two overarching criteria were used to determine whether a dimension of a response to a particular question was deemed relevant, for our purposes: the dimension ostensibly reflected 1) a certain sensitivity to a key idea addressed in instruction, and 2) aspects of the kind of understanding targeted in instruction—that is, understandings of the idea(s), addressed in particular question, that we considered to be desirable and in line with the aims of instruction. Our initial determination of relevant dimensions was then revised and refined through a process of negotiation as the research team began to examine and take into account some of the post-test data. The set of relevant dimensions for each post-test question converged to a final form as our examination of the student data became increasingly directed and systematic.

In a final stage of analysis each of the authors and a third member of the research team independently coded all students' responses to the post-test questions, for correctness and for whether they indicated correct or incorrect attention to the relevant dimensions. For instance, for part (iii) of Question 5 of the post-test (see Figure 2) we considered three relevant dimensions of a response: a) its correctness; b) appeals (implicitly or otherwise) to the idea of numerical equality of both expressions for each numerical replacement value of x; c) appeals to the algebraic idea of common form.

Initial disagreements in independent code assignments were few and usually rooted in differing interpretations of student explanations that lacked *explicit* reference to a key idea. These disagreements were resolved by a process of comparison and negotiation and a 100% consensus on the coding was achieved.

We should add, further, that while this scheme constituted the bulk of our method for a first analysis of the data, it was also accompanied by a less systematic documentation of student responses. Here we flagged selected responses as indicative of conceptions that appeared to depart significantly from those we envisioned as instructional endpoints.

Result 1: Disentangling Equivalence and Equality

Our analyses of post-test responses revealed that many students' emerging thinking about equivalence was tightly bound up—indeed, arguably confounded—with notions of numerical equality. This is illustrated by the following, incorrect, response to Question 5(iii) of the post-test, shown in Figure 3:

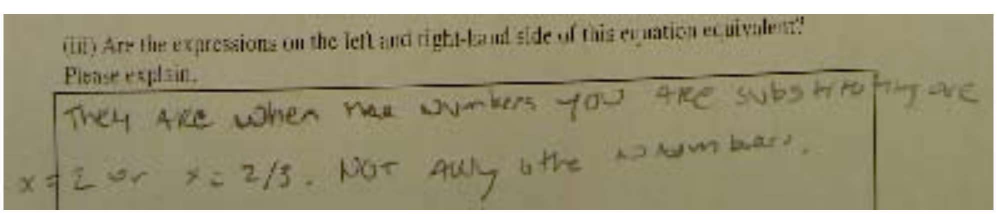
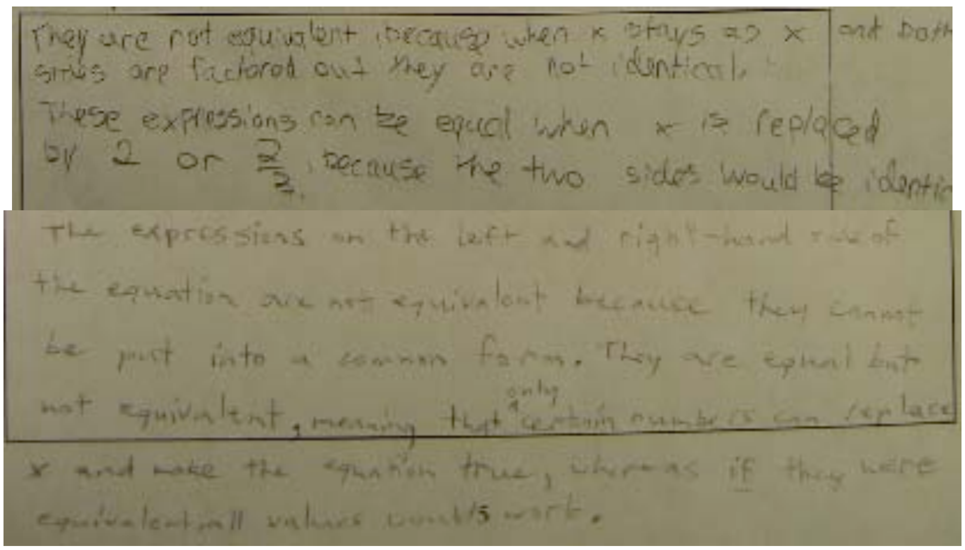


Figure 3. A response suggesting a murkiness between equality and equivalence.

We detect in this illustrative response a certain precariousness between the ideas of equality and equivalence. The response is consistent with someone having assimilated the idea of equivalence into a conception rooted exclusively in numerical equality: this becomes clear if the reader simply replaces the word "equivalent" with "equal". On the negative side, the response contains little evidence of an understanding of equivalence as a structural property of pairs of algebraic expressions. There is also no evidence of an understanding of algebraic transformations as an indispensable tool for converting either/both expressions into identical form and thereby eliminating the need to demonstrate equality for all values of x by exhaustive evaluation and comparison.

The response above is in apparent contrast to those like the two shown in Figure 4, both of which suggest attention to parts of these ideas and a clearer distinction between equivalence and equality:


They are not equivalent because when x stays as x and both sides are factored out they are not identical.
These expressions can be equal when x is replaced by 2 or $\frac{2}{3}$, because the two sides would be identic

The expressions on the left and right-hand side of the equation are not equivalent because they cannot be put into a common form. They are equal but not equivalent, meaning that only certain numbers can replace x and make the equation true, whereas if they were equivalent all values would work.

Figure 4. Responses suggesting clearer understandings of the relationship between equality and equivalence.

Result 2: The Role of Common Form

In addition to these results, we examined the distribution of students' responses to Question 5(iii) in relation to the idea of common form. Our findings, tabulated in Figure 5, reveal a compelling correlation: a very large proportion (89%) of students who gave a correct response made some correct reference to the idea of common form in the post-test, while none (0%) of those who answered incorrectly did so.

	Correct response	Incorrect response	Total
Correct reference to common form	8	0	8
Incorrect reference to common form	0	2	2
No reference to common form	1	4	5
Total	9	6	15

Figure 5. Distribution of students' responses to post-test Question 5(iii) (see Figure 2) in relation to references to common form.

The results presented in Figure 5 indicate that the idea of expressions having a common form, when understood well and used correctly, played a significant positive role in students' abilities to assess equivalence of expressions. Indeed, it would appear that sound reasoning about common form was an almost sufficient condition for enabling students to correctly assess equivalence of expressions.

Result 2 is encouraging in its suggestion that our instructional sequence oriented a significant proportion of students to common form as a salient idea and to its role in understanding algebraic equivalence. At the same time, however, Result 1 suggests that this was not the case for a somewhat smaller, though arguably still significant, proportion of students. Two questions naturally arise for us as a result of these mixed findings; the first is how the thinking of students in these two groups might differ significantly in ways that could account for the differences we documented. A second, perhaps related, question is whether aspects of the instructional sequence and engagement with it might have hindered some students in making the conceptual advance from equality to equivalence that we intended.

These two clusters of results now point us in a clearer direction for a next phase of our study: they suggest that we need to consider in greater detail, and by triangulation with our other data sources, how the thinking of students in these two groups might differ significantly. In addition they impel us to question our instructional design assumptions and decisions with regard to both the specific tasks we set to support students' transition from equality to equivalence and the engagements we envisioned around those tasks.

Conclusion

Findings from our preliminary analysis of students' written post-test responses suggest, on one hand, that the idea of common form—a notion that was given primacy in instruction—played a significant positive role in helping students assess the equivalence of given algebraic expressions. On the other hand, we also found ample evidence that distinctions between equality and equivalence were, at best, murky for many students. This murkiness was present despite their having participated in an instructional sequence designed specifically to support their clarifying the distinctions and negotiating the conceptual advance from equality to equivalence. This last alerts us not to underestimate what might be conceptually entailed in relating equality and equivalence in a way that also clarifies distinctions between them. Indeed, it points to the possibility that those entailments are not insignificant for all students and it reminds us that instruction needs to be sensitive to that possibility.

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DELVING INTO CONCEPTUAL FRAMEWORKS: PROBLEM SOLVING REPRESENTATIONS, AND MODELS AND MODELING PERSPECTIVES

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What are the fundamental features that characterize a conceptual framework in mathematics education? What type of questions do we need to formulate and discuss in order to evaluate and contrast the potential associated with different frameworks? What vision of mathematics is endorsed or appears as important in particular perspectives? What types of tasks are used to promote learning within those perspectives? What instructional environments favor students learning? These questions were used as a guide to examine three important conceptual frameworks widely used in research and practice in mathematics education: Problem-solving, representations, and models-modeling perspectives.

An important aspect to evaluate research results in mathematics education is to analyze the extent to which a set of principles associated with the conceptual framework embedded in the study are consistently utilized to reach and explain those results or findings. The existence of a diversity of frameworks to guide and direct research studies in the discipline makes necessary to review and contrast main tenets and principles that are sometimes implicitly used to support and present research results. Indeed, the editors of Educational Studies in Mathematics (ESM) (2002) suggest to "carry out comparative surveys of several theories, in particular of theories that purport to provide frameworks for dealing with the same related areas, topics and questions" (p. 253). How can one examine or contrast relevant theories in mathematics education? Delving into the fundamental of a theoretical perspective demands the development of an inquiry framework to guide the process of analysis. Thus, initial questions that can help organize and orient the analysis of a conceptual frame include: What does it mean to evaluate a conceptual framework? What are the fundamental themes that any theoretical perspective needs to address or include? What type questions do we need to pose and discuss in order to evaluate the potential of a particular conceptual framework? How can we identify strengths and limitations in research studies that are endorsed by a specific theoretical frame? In particular, we are interested in discussing possible connections of some theoretical perspectives with curriculum, learning scenarios, and forms of evaluating students' mathematical competences. That is, the extent to which principles and concepts associated with theoretical perspectives inform and support instructional practices. In this context, Greeno, Collins & Resnick (1996) stated that:

> …The role of theory in practice is not to prescribe a set of practices that should be followed, but rather to assist in clarifying alternative practices, including understanding of ways that aspects of practice related to alternative functions and purposes of activity (p.40).

We focus on discussing three research perspectives that are often utilized to support and orient the development of research studies in mathematics education:

(i) Studies based on problem solving perspective (Schoenfeld, 1992; 1994). Problem solving has been the focus of substantial research in mathematics education during the last three decades. This perspective has influenced notably distinct curriculum proposals (NCTM,

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2000) and it is important to reflect on the extent to which its principles appear consistently in formulating both research and practice projects.

(ii) Studies that rely on the recognition of representation and visualization as important aspects in students' comprehension of mathematics. In particular, the use of several "registers of representation of mathematical objects and the coordination of those registers to develop students' understanding of the discipline" (Duval, 1999). Representing mathematical ideas plays a fundamental role in developing understanding in mathematics. Thus, students' ways to represent and connect mathematical knowledge function as a vehicle to understand that knowledge deeply and use it in problem solving situations.

(iii) Research done under the umbrella of "models and modeling perspective" (Lesh & Doerr, 2003). In this perspective it is recognized that modeling activities are important for students to reveal their various ways of thinking and favor the development of their conceptual systems as a result of solving the activities.

Choosing themes to structure and organize information associated with each perspective necessarily reflects a position regarding what it might (or might not) be relevant to examine or look around the frameworks. Unfortunately, in mathematics education there has been a great interest in developing new conceptual frameworks and little work has been done around evaluating or testing the existing perspectives. Thus, the task itself of evaluating the perspectives by focusing on particular topics or questions seems to be an issue that needs to be part of the academic agenda of the discipline.

> [It] has become the norm rather than the exception for researchers to propose their own conceptual framework rather than adopting or refining and existing one in an explicit and disciplined way. This prolific theorizing …may also mean that theories are not sufficiently examined, tested, refined and expanded (ESM, 2002, p. 253).

We commence by examining the extent to which each perspective deals with a set of questions related to the nature of mathematical knowledge and learning, ways to describe and characterize mathematical competence, types of problems or learning activities that are important to promote students' learning, and ways to evaluate and communicate students' mathematical knowledge. We are interested in discussing what each perspective informs about students' mathematical competences and what aspects may be common or shared by those perspectives. In this context, we recognize that any conceptual framework or perspective constantly needs to be examined, refined or adjusted in terms of the development of tools (particularly technological tools) that influence directly the ways students learn the discipline. At the end, we identify elements of an emerging frame that takes into account the use of dynamic representations in the process of developing and understanding students' mathematical ideas. In particular, we emphasize the use of this type of representations as a key element to reconstruct and develop mathematical ideas or results. Thus, students' learning is conceived as an ongoing and continuous process that is enhanced with the use of technology.

Processes of Inquiry

How can we recognize the existence of a particular theoretical framework? How is it constructed? Should any research report be embedded in a particular framework? What tools are important to evaluate strengths and limitations of that framework? What type of sources informs about the principles and basis associated with a particular framework? These questions reflect the kind of difficulties that might arise when one tries to examine closely the use of those a perspective as means to structure, organize, and guide research and practice in mathematics education. We recognize that the task itself of evaluating the robustness of a particular framework might focus on analyzing different issues and consequently take different directions. The focus, scope and direction of the analysis is determined by the themes to analyze, the questions to discuss, and the sources or material chosen to examine. Greeno, Collins, & Resnick (1996) for example, review and contrast three general perspectives about cognition and learning that have been developed in psychology research (empiricist, rationalist, and pragmatist-sociohistoric). They chose to focus on analyzing theoretical issues associated with each perspective related to the nature of knowledge, the nature of learning and transfer, and the nature of motivation and engagement. Regarding educational practices based on those perspectives, they discuss aspects related to design of learning environments, analysis and formulation of curricula, and assessment. In the same vein, Schoenfeld (2000) contrasts differences in using terms like theory or models in mathematics, science, and education and he identifies criteria to evaluate empirical or theoretical work in mathematics education. The criteria include: Descriptive power (what is it important in the domain), explanatory power (how and why things work), scope (what can it be covered?), predicting power (whether the theory can specify some results in advance), rigor and specificity (how well defined are the elements and relationships within the theory?),…, and multiple sources of evidence (use of different evidences to reach and explain the same result). To evaluate the extent to which a particular framework fulfills the Schoenfeld's criteria demands the identification of features of the domain embedded in that framework. In general, a framework is known in terms of its uses or application and it seldom addresses directly aspects regarding its construction, nature, or development.

In this context, we organize our inquiry process by selecting a set of questions to examine and discuss issues related to the discipline, its practice and development; the process of learning it (what does it mean to learn mathematics); the students' participation in learning (how does learning take place); and evaluation of students' learning. The sources and materials that we chose to analyze represent seminal work associated with each perspective. However, we do not intend to do an extensive literature review of each perspective, instead we focus on analyzing what we judged are some representative sources.

At the beginning, we organize the discussion around fundamental themes that includes: (a) features of mathematics knowledge (how can the discipline be characterized? What are the tools to develop and understand mathematics?); (b) learning environments: What does it mean to learn mathematics? What conditions favor students' learning? And (c) level of explanation of the process involved in students learning of new concepts: How do students enhance or construct mathematical knowledge beyond their current knowledge? These questions are discussed in terms of analyzing statements endorsed by each perspective. Later we contrast differences and similarities among them, and we argue about the need to readjust constantly tenets and principles around the theoretical perspectives.

Results and Discussion

We organize our inquiry around themes that were taken as a reference to formulate questions that orient and guide the analysis. These themes are analyzed in detail to identify main differences or contrasts among the perspectives. At the outset, it is convenient to identify the scope of each perspective. That is, it is important to recognize the focus and type of explanation favored or taken in each perspective to explicate students' mathematical behaviors. In particular, there is interest to discuss the students' processes of developing or constructing mathematical knowledge. Two main trends were identified: Perspectives that explain the development of students' mathematical competences in terms of discussing global tendencies based on the identification of resources, strategies, and metacognitive behaviors; and those that pay attention to the students' microscopic behaviors shown during the processes of understanding particular mathematical ideas. Thus, the scope provides the context to discuss other themes.

Scope of Each Perspective

There is evidence that problem-solving perspectives seem to be useful framework to analyze global aspects of students' mathematical competence. For example, problem solving research results often recognize the importance for students to conceptualize a vision of mathematics consistent with the practice of the discipline, to monitor their problem-solving processes and to develop basic resources to comprehend and solve nonroutine problems. However, this perspective fails to explain in detail ways for students to actually develop this type of students' mathematical proficiency. Problem solving dimensions (basic resources, cognitive and metacognitive strategies, beliefs and affective components) become important to characterize students' mathematical competences in general terms (the extent to which students exhibit them) but fall short in explaining ways in which students develop those dimensions.

Representation perspectives focus on explicating micro-behaviors around the students' learning processes that involve description of ways that students transit from the use of one representation to another. Here it is recognized that representation and visualization play a fundamental role in thinking and learning mathematics. In this framework, the use of semiotic systems and the understanding of how they function during the students' learning become important aspects to explain the process of learning.

Models-modeling emphasizes the subjects' construction of conceptual systems (models) but offers little information regarding how students themselves develop new knowledge to construct more robust models.

Regarding the type of mathematical vision endorsed by each perspective, it is possible to recognize features associated directly with the practice or development of mathematics with the problem solving perspective. That is, mathematics is seen as a science of patterns that is developed or learned within an environment that favors and encourages processes of inquiry or reflection that lead to the understanding of phenomena through the use of mathematical resources. Models-modeling perspectives see the discipline as system of relationships that can be expressed through models. Thus, conceptual systems, cognitive systems and models are fundamental ingredients to explain students' processes of understanding mathematical ideas. Representation perspectives identify mathematics as semiotic representation system that deals with mathematical ideas and their transformations based on the use of different registers of representations.

The Role of Problems

What types of tasks or problems are used, within the frame, to explore, promote, and document students' learning? This question becomes important to analyze purposes and ways to use the problems during the research. "…[I]t is so important to justify the choice of the mathematical tasks used in a research, not just in terms of the general goals and theoretical framework of the research, but in terms of the specific characteristics of the task" (Sierpinska, 2004, p. 10). In this context, problem-solving perspectives recognize nonroutine tasks as a vehicle for students to exhibit their ways of thinking and problem solving behaviors. These problems are embedded in diverse contexts and demand the use of resources and strategies that may lead students to solve them and eventually to pose new questions or problems. In particular, nonroutine problems provide opportunities for students to explore distinct ways of solution, to use diverse representations, to formulate conjectures, to present arguments and to communicate results (Schoenfeld, 2002).

The problems or activities used in models-modeling involve open ended tasks, presented in realistic and meaningful contexts for students. "The activities are inbuilt with ways for students to realistically assess the quality of their own ways of thinking without predetermining what their final solution should look like" (Lesh & Yoon, 2004). Indeed, a particular feature of this type of tasks is that there is no one particular solution that all students need to achieve, rather plausible solutions models depend on set of conditions that students judge to be important to consider while approaching the task.

Representations perspectives rely on the use of problems in which students can use multiple representations to discuss relationships and mathematical properties. Thus, numeric, geometric representations often can be expressed in distinct ways to recognize particular properties. In addition, problems or phenomena that can be analyzed in terms of algebraic, numeric, or graphic representations become important in this perspective to grasp key mathematical concepts associated with the situation in study. The next table represents a summary of general features associated with each perspective.

Perspectives	**Mathematics Features**	**Type of Tasks**	**Processes of Learning**	**Learning Environmen ts**	**Evaluation**
Problem-Solving	Mathematics as a science of patterns Direct relationships between mathematical practice and students learning Mathematical thinking involves the formulation of questions, conjectures, relationships, and the use of distinct types of arguments.	Nonroutine tasks that include problems to be solved during the class time, homework problems and projects. Trans-forming routine tasks into non-routine activities through processes that involve formulation of questions.	Problem solving dimensions: Basic resources, cognitive and metacognitive (monitoring and self-control) strategies, beliefs' systems (affect).	Classroom as a mathematical microcosm. Classroom as mathematical communities Students work in small groups, whole group participation The instructor as a scaffolding	Solution processes of nonroutine problems. Students competences in mathematical processes that involve: Representations Communication Conjecturing Formulation of questions Distinct types of arguments Monitoring
Representations	Distinction between mathematical objects and their representations Mathematical thinking expressed through systems of semiotic representations	Tasks that involved the use of multiple representations.	Coordination of represent-ations Transit from one representation to the other (meaning). Operations within the same register; conversion of registers; and discrimination of registers.	Problem solving environments to promote students' construction of representations of mathematical ideas and their connections.	Evidence that students display connections between registers. Recognition of the same object through different representations.

Models-Modeling	Mathematics as a system of relationships useful to understand and make sense of distinct phenomena. Solving a situation or task leads to the construction of tools for thinking. Mathematics is seen as a system with elements, operations, rules, and relationships.	Tasks embedded into distinct contexts. Solutions involve explanations, descriptions, interpretations, representations, operations, algorithms, arguments, extensions, revisions, adjustments, etc.	Learning involves the construction of models or conceptual systems. Learning is expressed through a sequence of modeling cycles that might evolve from being non-stables models to robust and stables models.	Learning environments are designed around the discussion of model-eliciting tasks. Students often work in pair of groups of three and the teacher functions as a monitor during the sessions.	Students' development of conceptual tools to be used in solving family of problems. Student-self evaluation: The student becomes "the client" who reviews and assesses his/her results and those of others.

To what extent does each perspective recognize that the students' use of technology to understand and solve mathematical problems demands for a re-examination of basic principles to explain the development of students' mathematical competences? We observe that main research results associated with each perspective come from examining students' work that involves the use of paper and pencil (Schoenfeld, 1992, Duval, 1999) and a few cases from students using Excel to work on thought-revealing activities (Lesh & Doerr, 2001). We recognize that different tools may offer distinct possibilities for students to interact with mathematical tasks. For example the use of dynamic geometry software may become an important tool to identify invariants or relationships associated with particular problem of phenomenon by constructing a dynamic representation of the problem, while the use of Excel may result a useful tool to represent relevant information of the problem and relationships in tables to detect patterns or visual behaviors. In this context, it becomes important to examine the extent to which the questions that students ask, the representation they utilize and the arguments they use to support their results with the use of technology are consistent with those that appear in paper and pencil approaches. Our position is that more data need to be generated and analyzed to actually characterize the type of mathematical thinking that emerge when students systematically used technology in their processes of understanding mathematical ideas. As a consequence, frameworks that explain students' mathematical competences need be adjusted constantly in accordance to what students develop in their problem solving approaches that incorporate the use of distinct technological tools.

Remarks

What features are relevant in mathematics knowledge? How do students learn new mathematical knowledge? How can students construct or develop mathematical concepts beyond those they have learned? What processes entail the students' abilities to articulate their competence in learning mathematics? What type of tasks becomes relevant for students to develop mathematical thinking? What instructional conditions are important for students to learn? These types of questions seem to be relevant to examine the scope and explanatory power associated which each perspective. The extent to which each perspective addresses explicitly themes involved in these types of questions became important during the analysis of the sources related to each perspective. Indeed, issues that emerge during the analysis and were important to structure and ponder the information to be analyzed involve:

(i) *Scope of the framework*, here it was evident that problem-solving and models-modeling perspectives focus on explicating general or macro cognitive processes around students' learning; while the representation perspective seems to focus on explaining particular learning behaviors.

(ii) *Sources and ways to inform*, that is, the nature itself of each perspective differs since the problem-solving perspective, for example, addresses directly issues related to mathematics knowledge (what is mathematics?) and ways to create a mathematical microcosm in classroom, while the representation perspective only deals with these components implicitly.

(iii) *The need for an evaluation tool*, an initial difficult arose when deciding what to look for in each perspective, that is, the existence of distinct perspectives in the field requires to develop ways to examine and contrast their fundamental principles and tenets, in terms of evaluating the type of contributions to understanding relevant problems of the discipline.

(iv) *Formulation of questions*, a common feature associated to the three perspectives is that students develop, construct and transform their own understanding as a result of posing relevant questions and pursuing them through different means and constantly revising them within a learning community.

(v) The use of *technology* has influenced notably the ways how students represent and examine mathematics knowledge, and frameworks needs to re-adjust their principles in accordance to the types of transformations that are produced with the use of technological artifacts in students' learning.

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STUDENT UNDERSTANDING OF ACCUMULATION AND RIEMANN SUMS

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Student understanding of the conceptual structure of the Riemann sum definition of a definite integral is a topic on which little research has been done. This research uses the ideas of approximation (finding over and under estimates, determining a bound on the error, and finding an approximation accurate to within any predetermined bound) to develop a strong conceptual understanding of accumulation in college calculus students.

The conceptual structure of the Riemann sum definition of a definite integral includes a complex blend of multiplicative and proportional reasoning, accumulation and covariation, and limits or nets. We hypothesize that a deeper understanding of the structure of a Riemann sum is invaluable to students because 1) many real world application problems involve functions that are nonintegrable, 2) several numerical methods for approximating integrals, such as the trapezoid rule, midpoint rule, and Simpson's rule are based on the Riemann sum structure, and 3) even setting up an appropriate definite integral requires an understanding of the underlying structure. Thus, understanding Riemann sums will help students to understand more complex methods of numerical integration.

Oehrtman's initial studies on calculus students' spontaneous reasoning about limits showed that their ideas and language about approximations naturally reflected many formal aspects of limit structures that are often assumed to be too difficult for introductory calculus (Oehrtman, 2003). Additionally, these ideas are amenable to instruction and, in initial trials, were more useful for students' reasoning about novel problems than were formal language and symbols for the same structures (Oehrtman, 2004). Based on the results of this research, we have developed a series of activities for calculus students to explore concepts of sequences, series, limits of functions, the definition of the derivative, the definition of a definite integral, the fundamental theorem of calculus, and Taylor series. In each case, questions about approximations engage students in the appropriate limit structures for each of these concepts by asking students to provide algebraic, graphical, numerical, and contextual representations for answers to the following general questions: What are you approximating? What are your approximations? What are the errors? What are bounds for the errors? How can you find an approximation to any degree of accuracy required? This paper will analyze calculus students' problem solving process when given this type of approximation task within the context of accumulation.

Methods/Subjects

The subjects for this study were seven students currently enrolled in a calculus workshop in a large public university in the southwestern United States. The calculus workshop consisted of 16 students who were also enrolled in either Calculus I or Calculus II. Class sessions were videotaped, and four days of data are analyzed in this study. During these four days, the students worked on an activity that introduced the concept of accumulation. The class was divided into three groups and each group was given a different contextual problem, although all three

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problems dealt with accumulation and discrete approximations. The seven students in this study were given the problem below.

> *A uniform pressure P applied across a surface area A creates a total force of F=PA. The density of water is 62 lb per cubic foot, so that under water the pressure varies according to depth, d, as P=62d.*
>
> a) *Draw and label a large picture of a dam 100 feet wide and extending 50 feet under water.*
> b) *Approximate the total force of the water exerted on this dam.*
> c) *Find an approximation accurate to within 1000 pounds.*
> d) *Write a formula indicating how to find an approximation with any pre-determined accuracy, ε.*

Six of the seven students in this study worked in a group together on a regular basis throughout the course. Three were female and four were male. Two of the students were present for all four days of class, three students were present for three days, and two were present for two of the days. Both authors were involved in the research development and data collection. Oehrtman was the professor of the course and the principal researcher, and Sealey was one of several graduate students working with the professor.

This research is based on a teaching experiment methodology (Simon, 1995). Thus, a hypothetical learning trajectory was created. Before the implementation of the activity and the problem stated above, the students had worked with several activities developing their understanding of approximation structures and their facility using these structures to explore calculus problems. For example, a previous activity led students to approximate the value of pi using an infinite alternating series. The students found overestimates and underestimates, discussed "how close" their approximation was to the actual value of pi, and were able to determine how many terms were needed to obtain various degrees of accuracy.

Students were expected to be familiar with the structure and use of approximation language (error, error bound, overestimate, underestimate, etc.) throughout the problem solving sessions. In regards to the approximation structure, students were expected to be able to:

- recognize that the pressure was not constant so that a simple product formula could not be used without further justification
- realize that the pressure is roughly constant over areas of the dam at similar depths (i.e., long horizontal rectangular strips) so that the force on these individual regions could be approximated as a product of their area and the pressure at some point in that region
- determine that the total force on the dam is equal to the sum of the forces on the individual regions considered so that the total force could be approximated by adding the approximations for the individual regions
- compute several approximations using different partitions of the dam
- recognize the need for determining an overestimate and an underestimate for the amount of force on the dam
- realize that using the smallest pressure in each region (i.e., at the top) would result in an underestimate and using the largest pressure in each region (i.e., at the bottom) would result in an overestimate
- understand that the exact force is bounded by the overestimate and underestimate

- recognize that the terms in the under and over estimates are the same except for the first term of the underestimate (zero) and the last term of the overestimate
- realize that the size of the error is bound by the last term in the overestimate
- find bounds for the error for specific approximations that they have computed
- understand that better approximations can be found by increasing the number of terms in the sum (or equivalently, decreasing the size of the interval)
- find approximations with a pre-determined accuracy
- explain in general how an approximation with any pre-determined accuracy can be found
- understand that the exact answer is the limit of the sum as the number of terms approaches infinity (or the size of the interval approaches zero)
- understand that the limit of the sum of products is a definite integral

Most of these goals were expected to be developed by the students with minor input from the professor and researchers.

Theoretical Perspective

The theoretical perspective used in this analysis is taken from the problem solving frameworks of Carlson & Bloom (2005) and Schoenfeld (1992). Students' knowledge base, problem solving strategies, monitoring and control, beliefs and affects, and practices will be described using the language of Carlson & Bloom's framework who describe problem solving as a cyclic pattern consisting of four stages: orienting, planning, executing, and checking. During the orienting phase, students are reading the problem and making sense of it. They may draw graphs or diagrams, write down formulas, and categorize the problem. The second phase, planning, involves developing an approach to finding the solution. During this phase, there is a sub-cycle called the "conjecture cycle," which consists of mentally proposing various solution strategies, imagining what will happen when using them, and evaluating whether or not they will be productive. During this phase, successful problem solvers monitor their progress and abandon solution strategies that do not appear helpful. The third phase, executing, consists of carrying out the procedures planned in phase two. Finally, in the fourth phase, checking, the students interpret the solution against the original problem, verify that the solution seems reasonable, and reflect on the "efficiency, correctness, and aesthetic quality of the solution" (Carlson & Bloom, 2005).

Data

As expected, the students began the activity in the orienting phase. They spent approximately four minutes reading the question several times, drawing pictures of the situation, and writing down the formulas they believed to be relevant on their whiteboard. They quickly made the transition into the planning phase, but then moved back and forth from planning to orienting as they simultaneously developed solution strategies and became accustomed to the problem.

Laura: So we have to figure out the…well, approximate the force on the dam…right? So to figure out the pressure, we'll just to…the depth is 50 feet, right? (pause) It's just 62 times 50? Is the pressure?	Orienting
Joel: I guess. Yeah. Wait. Density is 62 pounds per cubic (foot).	
Laura: So that underwater, the pressure varies according to depth, d, as P equals 62 d. Oh wait—	
Joel: So it would be the integral from 0 to 50, wouldn't it? (To Prof.) Are we allowed to do integrals for this, or what?	Planning
Prof: Um…no. I want you to approximate.	
Laura: How sad.	
Laura: so it doesn't affect it like above the 50 foot mark, does it?	Orienting

Students in this study often fell back to an orienting phase although this was not described in Carlson & Bloom's study on research mathematicians. In addition to the difference in expertise of the subjects between these two studies, this difference in observed problem solving behavior may partially be attributable to other factors. First, the students in this study were monitored periodically by the researchers who sometimes rejected their plan as illustrated in the excerpt above. Also, in this study, the students worked on the problem during four class periods spanning eight days, and it was necessary to reorient themselves to their problem and the work that they had done during previous class periods. Finally, as mentioned previously, the students in the group varied slightly from day to day because of attendance. Thus, it was always necessary to explain to the different students what was done during the previous session.

Also interesting in the transcript above are the students' first attempts at solving the problem. Laura initially wanted to simply multiply 62 times 50 (depth) for the answer, but soon realized that the pressure varies according to depth. Joel's first attempt at solving the problem was to use an integral. Here the students recognized the problem as something similar to what they had seen somewhere else, most likely in a calculus class, and wanted to apply the rules they "knew." This knowledge appeared to be largely procedurally based, however, because the students later incorrectly set up the integral to check their approximation and had great difficulty determining the appropriate function to integrate. Eventually, after computing an approximation using the pressure as the average pressure (the pressure in the middle of the dam), the students were able to approach the problem from a more conceptual basis, describing the underlying product structure of the definite integral to determine the appropriate integrand. On the whiteboard they wrote $\int_0^{50} 62d \cdot 100\Delta d$ (and later changed this to *dd*) and labeled 62*d* as pressure and 100 delta *d* as area.

For an approximation, the students determined that they would use $\sum_{i=0}^{49} 100 \cdot 62d$ which was obtained by dividing the 50 foot dam into 50 one foot subintervals, and assuming constant pressure over each subinterval. During day 2, the students spent time checking their work from day 1 and reorienting themselves to the meaning of $\sum_{i=0}^{49} 100 \cdot 62d$.

Joel: We're missing a variable here, aren't we? Because A would be 100 times d, wouldn't it?	Monitoring
Doug: Well (pause)	
Joel: It would be the contact area, right? (pause) Force equals pressure times area—pressure at a certain point (pause) because the pressure's not constant over the whole area of the dam.	Reorienting
…	
Joel: Wouldn't our A be delta d? It says (reads) uniform pressure, P, applied across a surface area A. So that surface area would be 100 times delta d, and since our delta d is 1, that's probably why we didn't put it in here…right?	Reorienting/Monitoring
Doug: yeah	
Joel: maybe that's why	

Note that both Doug and Joel were present during the prior session, but they still required lots of time to understand what they did previously. Later, the professor asked the students to explain to him how they were thinking about $\sum_{i=0}^{49} 100 \cdot 62d$.

Prof: [pointing to $\sum_{i=0}^{49} 100 \cdot 62d$ *]. And you would be using the pressure at what locations on each of those slices?*
Jill: at the bottom
Jeff: It would be the top…cause you're starting at zero.
Jill: Oh cause it's like…at the top…ok, you're right.
Prof: So when you plug in i=0, what force, what pressure are you using?
Joel: So that's our underestimate? (pause) Is that an underestimate then?
Jill: Well…ok, because if you're—if you're counting it at zero
Jeff: It should be.
Jill: Yeah, it should be an ***under****estimate.*

The important thing to note here is that the students used the term "underestimate" without being prompted by the professor. They saw this quality of their approximation on their own, and even had a strategy of how to use it. Immediately following the above transcript, the students continued with the planning phase given Jeff's suggestion:

Jeff: Couldn't we just do an overestimate and then…
Jill: Could we go 0 to 49 and then
Jill/Jeff: 1 to 50
Jeff: and that would give us our…
Prof: If you do 2 different things, you could get an under and an over--
Jill: yeah, ok.
Jeff: Then we should be able to move from there

Following the above dialogue, the students entered the execution phase and performed the necessary calculations to obtain an overestimate and underestimate. Right away, the students dealt with the issue of error and error bound and quickly determined that a more accurate approximation would be found if they used more sections or "made your delta *d* smaller". They proceeded to plan and execute a solution strategy using intervals of half a foot (instead of 1 foot). During the remainder of this class period and most of the next period, the students calculated several approximations (overestimates and underestimates) using successively smaller intervals. It is evident that they understood how to make an approximation accurate to within a predetermined bound.

On day three, the students realized that the terms in the overestimate and the underestimate were the same with the exception of the first and last terms. Thus, they were able to easily determine that they would need 7750 partitions to find an approximation accurate to within 1000 pounds. Because their calculators could only compute 800 partitions at a time, the students needed to find a way to calculate 7750 partitions. Laura suggested that they split up the 7750 into sections of 800 (0 to 799, 800 to 1599, etc.) and add up the results for the final approximation. Each student in the group calculated at least one of the sections, involving every group member in the solution.

While analyzing the videos, it was not apparent that the students ever engaged in a checking phase as described in the Carlson and Bloom framework. Although they seemed to evaluate each other's solution strategies and accept or reject them openly, they rarely checked the work they did as a group. However, on day four, the students were asked to present to the rest of the class what they had done on their whiteboards. On this day, the students claimed that they had used lots of checking mechanisms along the way.

Jeff: that was just a check thing down there? [pointing to whiteboard]
Joel: yeah
Jeff: and that was another check thing
Joel: we did lots of checks along the way
Jeff: yeah, there was another check over there for the same thing. And these two drawings are the exact same

At this point in the dialogue, Jeff noticed that several of their calculations and solution strategies produced the "same thing". For example, he noticed that the answer they got when they integrated was the same answer they got when they used the average pressure to calculate the total force. However, there is not evidence in the data that shows that the students intended each strategy to produce the same result. It is clear that they were pleased when their answers were comparable, but they seemed more surprised than affirmed. For example, at one point on day three, Jeff was surprised to learn that the average of the group's overestimate and underestimate was also equal to the exact answer that they computed with the integral. Later, on day four, he calls this a "check". These students either recall their original intentions differently than they played out in class or their use of the word "check" differs from its use in the final phase of Carlson's problem solving framework. Here they appear to describe a means of confirming that two methods produced the same answer.

Another interesting phenomenon that was found in the data is the students' ideas about how this problem related to integrals. As mentioned earlier, the students initially wanted to use an integral to solve the problem. It seems likely that they chose this method because the problem was similar to those they solved in their calculus classes using integrals, and not because they understood (at that point) the structure of accumulation and definite integrals. On day 3, the students began a discussion about the integral and their problem.

Laura: well like when you do something like an integral—
Several: it's different
Laura: it's different?
Irene: yeah
Jeff: it's different with the integral, I think it was
Doug: uh huh, because you're—
Prof: so why is it different with the integral?
Doug: (inaudible. Speaking to Laura)
Jill: because the integral's like 0 and 800 and those are like—you go in between the two [points to spot in the air when she says 0, points to higher spot when she says 800 and moves finger up and down repeatedly, without pausing, when she says in between the two], where this one [points to whiteboard] is like [moves finger as before but pauses every few inches]…you're going like—
Jeff: yeah, it uses the exact

Jill seemed to have indicated that the approximations they did using summations involved only a finite number of pressures at a finite number of depths [indicates the depths of the dam with her finger and pauses to represent specific depths], while the integral would use every pressure and every depth in the interval [moves her hand without pauses]. Whether or not her group members saw things as she did is unclear, but they certainly accepted what she said without further discussion. Also, Jeff knew that the integral "uses the exact" pressure at each depth as opposed to the approximations obtained by the summation that assumed a constant pressure over a small interval.

Conclusion

Although the title "Riemann sums" was not stressed by the professor or the researchers, the students in this study seem to have a good understanding of the concepts involved. There is no evidence at this point, however, that the students realize that the limit of these sums results in the *actual* quantity that they were approximating, and not merely an approximation that was bounded by very small error. However, the data shows that the students have a good understanding of the concept of limit, even though they were not using the word "limit". The students were able to find overestimates and underestimates and understood that the actual value lies within this interval. Also, when asked to find an approximation accurate to within epsilon, the students were able to determine a formula that calculated the number of intervals they needed to use. Thus, for any epsilon, they could find an approximation such that the error was less than epsilon.

It has been shown that the students in this study recognized both the multiplicative and additive structures of an accumulating rate and used a Riemann sum to approximate the force of water against a dam. More importantly, the students understood that by using more terms in the sum (and thus smaller intervals), a smaller bound on the error could be obtained. It has also been shown that the students understood that the limiting effect produced approximations that were accurate to within any predetermined bound. These ideas are of approximation have the same structure as the formal definition of the limit. We have revised the hypothetical learning trajectory in two ways. First, we will use context problems with non-linear integrands so that the exact value is not intuitively computable from a simple product. This would ensure that the students focus on approximations and the error bound. We anticipate it would also engage the students more completely in the checking phase of the problem solving framework. Second, toward the end of the activity, we would emphasize the terminology of Riemann sums, discuss the limit, and explore the relationship to area under a curve.

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CHARACTERISTIC FEATURES OF DISCOURSE IN A SKETCHPAD CLASSROOM

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Numerous studies emphasized the opportunities offered by dynamic technology software for reforming the teaching and learning of mathematics. Research has well documented the impact of dynamic technology upon the cognitive development component of learning. Our study turns to a regular classroom to investigate the role of Sketchpad in shaping communication and understanding of geometrical concepts. We present a case of engaging Sketchpad as a tool to support the regular class curriculum. Our findings record interesting developments in the practices used to identify quadrilaterals as well as in the practices that are endorsed for making claims about quadrilaterals. We argue that the use of Sketchpad does not simply sustain existing classroom practices more effectively, but instead gives rise to a characteristic discourse in the classroom that emphasizes a pragmatic view of geometric proof.

Context and Objectives

Over the course of the past fifteen years, the presence of new kinds of dynamic mathematics software has provided both teachers and researchers with many opportunities as well as challenges. The first set of challenges taken on by researchers was to determine the extent to which such software could improve geometry teaching and learning, as traditionally conceived: thus, the teacher might ask, can *Sketchpad* help me teach what I've been teaching before more effectively? And, the researcher might ask whether *Sketchpad* helps students perform as well, or perhaps even better, on tasks designed for non-technology environments. In fact, the majority of early studies related to the use of dynamic mathematics software for geometry focused on improvements (or not) in student learning, and were conducted in classroom contexts in which students had individual access to the software. More recently, both teachers and researchers have shifted to forms of research that are increasingly sensitive to changes in the way of doing mathematics that dynamic technologies support or impose. This has also led to some less ideologically-motivated research questions, which have prompted researchers to consider some of the unique phenomena associated with dynamic mathematics environments, including the characteristic forms of reasoning and problem solving that such environments give rise to in student work—and that have no counterpart in traditional learning environments. For example, the possibility of dragging in *Sketchpad* seems to lead students to the characteristic practice of looking for counter-examples (Mariotti, 2000; Yu & Barrett, 2002). This and similar practices are seen to arise from interactions between the students and the tool, each one shaping the other, thus acknowledging the complex process through which people come to perceive and use tools.[1]

[1] Such an approach devotes renewed attention to the process through which people learn to use tools. Formerly, many proponents of mathematics education technologies have touted technologies that minimize or even eliminate this process, perhaps in order to support the ideologically-motivated desire to not let the technology 'get in the way' of the mathematics learning.

According to Vérillon and Rabardel (1995), this process includes the development, or appropriation, of schemes of action for the tool by the subject as well as the elaboration of the tool itself. The former process has been called "instrumentation" and the latter "instrumentalisation." In articulating and investigating these processes in educational technology settings, researchers such as Artigue (2002) and Trouche (2000) focus explicitly on how the technologies under consideration (in this case, computer algebra systems (CAS)) change teachers and students ways of doing mathematics.

How might dynamic geometry environments change the ways of doing mathematics? Consider a commonly-seen action in *Sketchpad* that involves collapsing a triangle by dragging one vertex so that it is collinear to the other two. This action on a triangle is natural in that it follows continuously from manipulating the triangle into different configurations (isosceles, equilateral, obtuse, etc.). Such an action will lead to new questions. For example, a teacher might ask whether the collapsed triangle is still a triangle—a question that rarely, if ever, appears in textbooks. New questions can also give rise to new forms of reasoning, or new kinds of warranted arguments. A student might argue, for instance, that the collinear points do not form a triangle "because there is no hole." This argument is perhaps more appropriate for arguing why a geometric configuration is not a triangle than the one that might otherwise be used, in a non-dynamic environment, namely, "because there are not three sides."

Despite focusing on some of the new opportunities afforded by the use of *Sketchpad*, very little of the research has provided explicit accounts of the effects of the tool on the school geometry itself, that is, on the classroom discourse that emerges as a result of the tool's presence. Thus, the goal of the present study was to investigate the characteristic features of classroom discourse that the introduction of *Sketchpad* would bring about, and to analyze the implications such changes would have with respect to the values and goals of traditional geometry instruction, which relies heavily on purely deductive conceptions of proof.

Conceptual Framework

We adopt a "participationist" view of mathematics learning, which conceptualizes learning in terms of participation in certain well-defined practices rather than in terms of acquiring knowledge (cf. Lave & Wenger, 1991). Within this broad framework, we focus in particular on the discursive practices in which specific mathematical communities participate. According to the discursive approach, thinking can be conceptualized as a special case of the activity of communication, or as a type of discursive activity (see Sfard, 2000, 2001a,b). And learning is defined as the process through which one changes one's discursive ways in a certain well-defined manner. In this view of learning, symbolic tools and other artifacts that are used in the process of learning do not merely operate as external representations that can be discarded once their jobs as construction aids are completed.

The word discourse here is used to denote "any act of communication, whether verbal or not, whether with others or with oneself, whether synchronic, like in a face-to-face communication, or asynchronous, like in exchange of letters or in reading a book" (Sfard, 2001b, p. 370). Sfard has proposed that discourses differ one from another in at least four basic features of communication: keywords and their use; visual mediators, routines, and endorsed narratives. By focusing on these features of communication, on the changes they undergo or the particular characteristics they possess, researchers can describe mathematical thinking and learning in particular environments. In this paper, we will be particularly interested in describing the characteristic forms of mathematical thinking occasioned by the presence of a new tool. We will be able to describe these by paying careful attention to the four features of communication. In fact, we will be focus primarily on two of these features—routines and endorsed narratives—since we would like to examine changes in mathematical thinking with respect to the geometric arguments made in the classroom.

The changes in routines and endorsed narratives that occur with the introduction of *Sketchpad* cannot be separated from the other components of discourse present in the classroom. Indeed, changes in routines will rely heavily on the keywords and visual mediators that *Sketchpad* uniquely offers. For instance, *Sketchpad* provides different visual mediators than do static diagrams, since these mediators can be directly manipulated by the continuous dragging of the mouse. Therefore, as we will see in the classroom transcripts, statements such as "this triangle turns into that one" are both natural and appropriate in environments where triangles can be dynamically manipulated, and effectively morphed from one shape into another.

In considering the impact of the introduction of *Sketchpad* in a classroom situation, we will be focusing on the characteristic aspects of discourse the use of *Sketchpad* occasioned. In particular, we will focus on the ways students identified, compared and characterized quadrilaterals. The discursive activities in which they engaged will allow us to identify characteristic forms of thinking and reasoning that the use of *Sketchpad* brings about.

Research Context

Our research population consisted of two teachers— one of them completing his one year of internship as a student teacher in the school— and one class of twenty-three grade ten students enrolled in a geometry course.

The research setting was a high school in the Midwest attracting mostly middle-class students. The mathematics courses at this school are often described as being reform-oriented, both by the teachers in the school's mathematics department and by members of the teacher education faculty at the University. Both the teachers expressed a constructivist orientation toward the teaching of geometry. In interviews prior to the research, they emphasized their wish to work with the ideas that the students would generate, and avoid designing activities that would prevent students' instinctive ideas from emerging.

Our data consists of classroom videos, interviews and emails. We video-taped the one-hour periods of geometry over the course of four weeks. Most lessons consisted of a combination of whole classroom discussion, with the teachers leading the discussions, and small group interactions, during which students worked on specific problems or tasks either individually or in small groups.

We also video-taped and transcribed all the meetings between the researchers and the teachers (four in total). In addition, we were given access to all the email communication between the Teacher and Student-Teacher, as this was their primary mode of lesson planning. We also administered three written questionnaires to both teachers over the course of the study, one prior to its inception, one at the end, and one approximately halfway through. The questionnaires included a series of questions about the perceived uses of *Sketchpad*, as well as the perceived advantages and disadvantages of using *Sketchpad*. Finally, the Teacher emailed the researchers frequently while observing the Student-Teacher's classes, with questions about *Sketchpad* and requests for sketches. These requests often included descriptions of the purposes the Teacher had for using *Sketchpad*.

For the purposes of this research report, we have chosen to focus on the first three classes of the unit, which involved three hours of videotape. During these classes, the students were introduced to quadrilaterals, to their properties and definitions, and worked on problems that required them to identify, compare and classify various types of quadrilaterals.

Given our research objective, to describe the characteristic geometric discourse occasioned by the use of *Sketchpad*, our data analysis consisted in examining the utterances of the teachers as well as the students, particularly in cases where *Sketchpad* was in use. We focused on the four basic features of communication identified by Sfard and Lavie (forthcoming), documenting instances of keywords, visual mediators, routines and endorsed narratives (with a particular focus on the two latter features) made in the presence of the computer and the screen graphics, and that seemed to be occasioned by the use of *Sketchpad*.

Results

In our analysis, we have focused particularly on the procedure of substantiating narratives about class-membership of geometrical figures. We witnessed a shift in the routines of identification figures from the purely deductive procedure of identifying critical properties (look at the figures and check whether it fulfills the definitional requirements) to the transformation-based procedure supported by dynamic tools (take a figure that is unquestionably a certain shape, and check whether you can transform it continuously in such a way that in all intermediate stages the transformed figure preserves the properties that justify calling it that figure). The latter procedure supposes an epistemological shift in identifying the objects involved and the ways in which they are considered identical.

We focus on the evolution of the identification process to pinpoint some of the ontological messages that the different formulations implied, that is, messages about the nature of the objects being identified. Most prominently, named-shapes, such as the rectangle, were talked about as being a multitude of objects, changing over time, rather than a single object described discursively. Related to this was the status of these shapes as existing in time, living and breathing as other everyday objects do. Such shapes were permitted to undergo the kinds of transformations permitted to everyday objects, and in particular, transformations that did not change a certain set of their defining properties. As a result, not only were students encouraged to think of specific named-shapes, such as the rectangle, as being continuously transformable into other rectangles, and sometimes squares, but they were encouraged to think of specific named-shapes as being a multiplicity of shapes, which included examples quite different from the prototypical ones on their worksheet.

By examining the types of the routines both implicitly and explicitly endorsed, we were able to gain some insight into the routines uniquely afforded by *Sketchpad*. These routines challenge the ones traditionally used and accepted in school geometry. Similarly, addressing the question “Can I make <this shape> into <that shape>” provides another alternate routine of identification with a clear pragmatic basis. By showing that a wide range of instances of <this shape> can be made into <that shape>, one can answer the question “Is <this shape> a <that shape>?” Once again, this involves developing a logic of dragging in which a sufficient range of <this shape> is generated and tested. In fact, those who use *Sketchpad* to work on geometry problem engage in precisely these kinds of routines. They test canonical examples of a given configuration as well as pathological ones. Thus, for example, when testing a conjecture about a triangle, the experienced *Sketchpad* user will test acute-angle triangles, obtuse-angle triangles, right-angle triangles and degenerate triangles. The ability to decide how to search the range of continuous examples so easily generated in *Sketchpad* is crucial to successful problem solving, and relatively unique to dynamic environments. This ability involves some discursive mediation, and does not rely entirely on visual-based routines. It also requires some reasoning by continuity, which has never been germane to the set of routines required in the routines of identification of traditional school geometry.

"Reasoning by continuity" is a form of geometrical reasoning that enacts Poncelet's Principle of Continuity: the properties and relations of a geometrical system or figure, be they metric or descriptive, remain valid in all of the successive stages of transformation during a motion that preserves the definition properties of that figure or system. The Poncelet principle finds, perhaps for the first time, a physical embodiment through the use of *Sketchpad*, where dragging qualifies as a continuous motion that can preserve the system's initial properties. Although we can change orientation and size, and even modify certain magnitudes to the point where they vanish or change sign, the metric and incidence properties of the initial construction remain unchanged. The principle of continuity allows us to expand the study of geometrical objects (like particular quadrilaterals) to characterizing their behavior under motion, while guaranteeing that motion does not take us out of the geometry in which we placed our objects. Of interest for our study is ,for example, the fact that the principle of continuity guarantees that any member of the set of parallelograms may be obtained from another via dragging.

We show how the dynamic features of *Sketchpad* promoted a shift in the nature of questions and actions along the way, placing the visual in a central role in the process of exploration, a shift applauded by many mathematicians (see, for example, Davis and Anderson (1979) and Cunningham and Zimmerman (1991)), and effectively expanding the discourse of geometry. Our research suggests that reasoning by continuity may be a resource to help students expand their mathematical arguments.

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LOCATING IRRATIONAL NUMBERS ON THE NUMBER LINE

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Can the exact location of $\sqrt{5}$ *be found on the number line? In this report we consider the answers of a group of preservice secondary school teachers to this question, in light of their general conceptions of irrational numbers and their representations. The results indicate strong reliance on decimal approximation of irrational numbers. Pedagogical implications are considered.*

Background: Snapshot from Research Literature

Prior research on irrational numbers is rather slim. A small number of researchers who investigated students' and teachers' understanding of irrational numbers reported the difficulties that participants have in identifying the set membership, that is, in recognizing numbers as either rational or irrational (Fischbein, Jehiam, & Cohen, 1995), in providing appropriate definitions for rational and irrational numbers (Tirosh, Fischbein, Graeber, & Wilson, 1998) and in using different representations flexibly (Peled & Hershkovitz, 1999).

Of particular interest here is the study of Arcavi, Bruckheimer and Ben-Zvi (1987) related to their work on using history of mathematics to design pre-service and in-service teacher courses. These researchers report several findings on teachers' knowledge, conceptions, and/or misconceptions regarding irrational numbers. One of the most striking discoveries from their study is that there is a widespread belief among teachers that irrationality relies upon decimals. Arcavi et al. report that 70% of the participating teachers knew that the first time the concept of irrationality arose was before the Common Era (Greeks). However, although the majority knew "when" it arose, very few also knew "how" it arose. This became particularly apparent when they were asked to order chronologically the appearance of three concepts: negative numbers, decimal fractions, and irrationals. Majority believed that decimal fractions preceded irrationals in the historical development. The authors concluded that this not only indicated the lack of knowledge about the relatively recent development of decimals, but more importantly, it indicated that the origin of the concept of irrationality is conceived as relying upon decimals, and not connected to geometry as occurred historically (commensurable and incommensurable lengths). Arcavi et al. (1987) point out that "the historical origins of irrationals in general, and the connections to geometry in particular, can provide an insightful understanding of the concept as well as teaching ideas for the introduction of the topic in the classroom" (p. 18). This particular connection to geometry is of our interest here.

This report is part of the ongoing investigation on understanding of irrational numbers. Previously we focused on formal and intuitive knowledge of irrationality as well as on representations of irrational numbers (Sirotic & Zazkis, 2004; Zazkis & Sirotic, 2004, Zazkis, 2005). Here we limit our focus to the geometric representation of irrational length as specified by a point on a number line.

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Research Setting

The Task: Show How You Would Find the Exact Location of $\sqrt{5}$ on the Number Line

The task was designed in order to investigate understanding of the geometric representation of an irrational number. In particular, we were interested in what means the participants would use in order to locate $\sqrt{5}$ on the number line precisely. It is said that to every real number there corresponds exactly one point on the real number line. One may find this difficult to believe if one has never seen an irrational point located on the number line, especially considering the fact that the number line is everywhere dense with rational numbers. Further, we included $\sqrt{5}$ in this item rather than the "generic" $\sqrt{2}$ assuming that the latter would lead some participants to automatic recall from memory, rather than construct the required length.

The number line drawing given to students with this task was intentionally set in the Cartesian plane with a visible grid to simplify the straightedge and compass construction (i.e., there is no need to draw a perpendicular line at 2). It is intended to aid in the invoking of the Pythagorean Theorem in the efforts to construct the required length. The expected response is shown in the Figure 1.

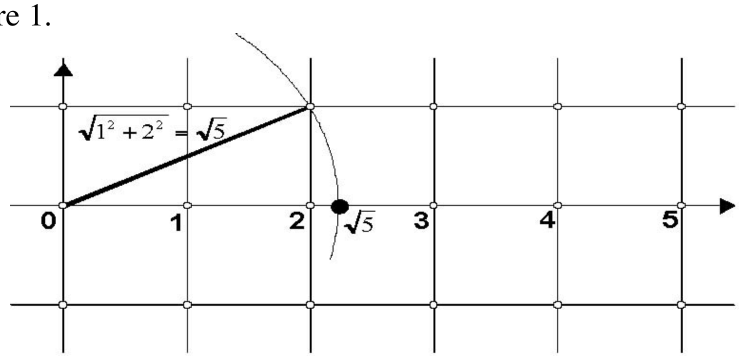


Figure 1. Geometric construction of $\sqrt{5}$

We were interested to see whether participants will use this conventional or a similar approach or whether they will resort to thinking in terms of decimal expansions.

Participants

Participants of this study were 46 preservice secondary school teachers enrolled in the professional development course "Designs for Learning: Secondary Mathematics". They responded to the item above as a part of a written questionnaire that included several additional items related to irrationality. Following the completion of the questionnaire 16 volunteers from the group participated in a clinical interview in which their responses and general dispositions were probed further.

Analysis of Responses

The geometric representation of irrational numbers was strangely absent from the concept images of many participants. The common conception of real number line appeared to be limited to rational number line, or even more strictly, to decimal rational number line where only finite decimals receive their representations as "points on the number line". This is in agreement with the practical experience that finite decimal approximations are both convenient and sufficient, which could be the source of these conflicts.

Table 1 summarizes the results of the written responses to the task. The responses fall into five distinct categories: an exact location of the point using the knowledge of Pythagorean Theorem, more or less fine decimal approximation, very rough approximation (between 2 and 3), responses related to graphing of a related function, and an outright claim that this is impossible to do. Next we exemplify and examine some representatives of each category.

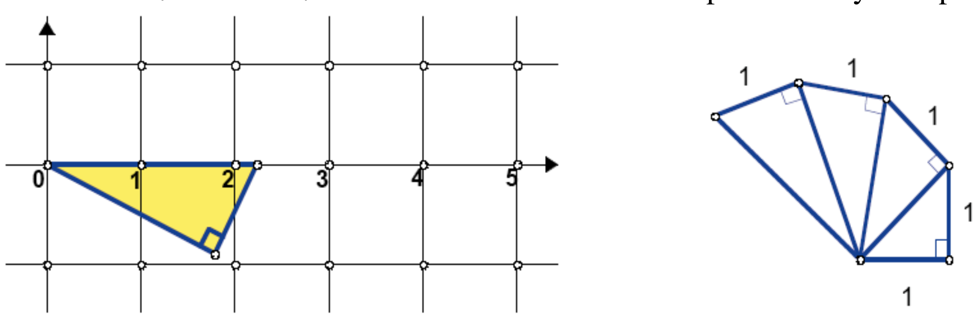
Response category	**Number of participants [%]**	
Exact, using Pythagorean Theorem	9	[19.6%]
Decimal approximation using one or more digits after the decimal point	18	[39.2%]
Very rough approximation, i.e. "between 2 and 3"	6	[13%]
Other response (for example, using graphs of $f(x)=\sqrt{x}$ or $f(x)=x^2-5$)	6	[13%]
Responses arguing "you can't"	4	[8.7%]
No response	3	[6.5%]

Table 1: Quantification of results for the construction of $\sqrt{5}$ (n=46)

Geometric Approaches

Ten participants (out of 46) used geometric approaches, nine of which we classified as precise. We presented above what could be considered a conventional geometric approach. Indeed, it appeared in the work of four participants.

Two other valid geometric approaches were found. One of them is a slight variation of the previous response. Instead of determining the placement of $\sqrt{5}$ by construction it uses a "ready made" right triangle with the side lengths of 1 and 2. This approach was described as "making the hypotenuse $h=\sqrt{1^2+2^2}=\sqrt{5}$ lie on the number line" and presented by four participants.



(a) (b)

Figure 2: Locating $\sqrt{5}$ by (a) a "ready-made" right triangle (b) using successive triangles

The other valid geometric approach is the familiar spiral of right triangles constructed by the successive application of Pythagorean Theorem with one of the legs always equal to 1 and the other leg equal to the hypotenuse of the previously constructed triangle (Figure 2b). This construction is a more generalized version of the conventional geometric approach in the sense that a square root of any whole number can be constructed in this way. It might not be the most efficient construction, but it spares one from having to think about what two perfect squares add up to the required square of the length of the hypotenuse. Only one participant used this approach.

The next response, suggested by one participant, uses geometric approximation (Figure 3). It seems to involve "eye-balling" of when the partial pieces in square A will make a whole squared unit: Area A = Area B, where A is a square. $\sqrt{5} \times \sqrt{5} = 5 \times 1$.

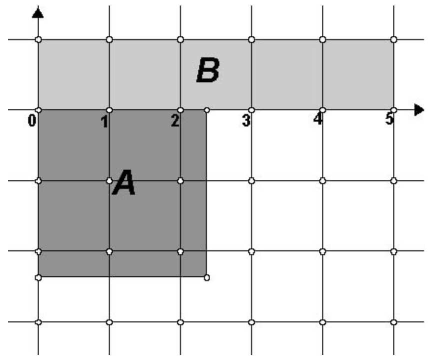


Figure 3: Locating $\sqrt{5}$ by "eye-balling" the areas

Numerical Approaches

Next we present a range of responses from the written part, arranged by the degree of accuracy. Twenty-four participants (over 52%) offered an approach based on the decimal expansion of $\sqrt{5}$. We start with those who offered a very rough approximation, and end with those who demonstrated a genuine striving for accuracy.

- Some participants circled a "big blob" around the area of expected location and said "somewhere around here".
- Between 2 and 3. I have no idea of the exact location, but it's closer to 2 than to 3.
- I used my calculator and found that $\sqrt{5} \approx 2.23$. Also $\sqrt{5} = 5^{1/2}$. To plot the point I found the $\sqrt{5}$ midpoint between 2 and 3, then between 2 and 2.5, then plotted $\sqrt{5}$ roughly at 2.25.
- There are 5 whole numbers between 4 and 9 (perfect squares), and since 5 comes after 4 it will be 1/5 the way between 2 and 3. (Note: In this response we note an example of 'overgeneralization of linearity' (Matz, 1982), a response that stems from what can be seen to $\sqrt{5}$ hold true in linear relationships. In particular, the location of $\sqrt{5}$ is said to be obtainable using a linear interpolation between the two neighbouring perfect squares.)
- Divide the section between 2 and 3 into 10 equal parts, find the two neighbouring tick-marks that correspond to just below and just above 5 when squared. Then divide this segment into 10 parts and repeat the process until you get better and better approximation.
- Closest perfect square is 4, $\sqrt{4}$ =2, so it is a little over 2. For greater accuracy, we would try more digits.

$2.3 \times 2.3 = 5.29$	(too high)	$2.2 \times 2.2 = 4.84$	(too low)
$2.23 \times 2.23 = 4.9729$	(too low)	$2.24 \times 2.24 = 5.0176$	(too high)
$2.238 \times 2.238 = 5.008644$	(too high)	$2.237 \times 2.237 = 5.004169$	(still too high)
$2.236 \times 2.236 = 4.99696$	(too low), etc.		

Function-Graph Approach

This type of response was found among three participants. These approaches assume what is to be found; that is to say, they assume the availability of an accurate graph, from which the required length would be simply read off, instead of finding a way to construct such length. It should be noted that one of the three participants who offered this kind of response admitted his doubts about the validity of such approach.

- Using functions, such as a sketch of $f(x) = x^2 - 5$ and then looking at the zero of this function $x^2 - 5 = 0$. A statement "if my graph is <u>absolutely accurate</u>, I will find the exact location" accompanied this approach.
- Similar approach as above, only using $f(x) = \sqrt{x}$ and then looking at the value of this function at $x = 5$ on the graph (the ordinate distance).

Impossible?

Some participants questioned the validity of the assignment. Most likely the word "exact" triggered the following responses.

- $\sqrt{1} = 1, \sqrt{2} \approx 1.4, \sqrt{3} \approx 1.7, \sqrt{4} \approx 2, \sqrt{5} \approx 2.3$ I don't think you can find the <u>exact</u> location of $\sqrt{5}$ looking at the number line because it is a huge decimal form number. I do believe there is a way by using calculus, but I'm not sure how to do it.
- This is a trick question as $\sqrt{5}$ is irrational, it <u>cannot be placed exactly</u> on the number line, because its digits are infinite.
- Can I find the exact location without knowing the rest of ∞ digits? You can't.
- Divide on calculator. There is no exact point like that.

Real Number Line Versus Rational Number Line

Since only 9 out of 46 prospective teachers (19.6%) were able to locate $\sqrt{5}$ on the number line accurately, we investigated what may be the reason for these difficulties. A rather striking observation is that the vast majority of participants perceive the number line as a *rational* number line. It turns out that those arguing "you can't" and those that used a more or less fine decimal approximation hold this perception. This can be concluded from the interviews where we probed for a precise, not approximate, location of $\sqrt{5}$. Under such demand, all participants that previously offered a decimal approximation later concluded it cannot be done. In other words, the common opinion was that it must be rounded before it can be located.

Next, we look at the range of responses from the clinical interviews that may shed some light on why locating $\sqrt{5}$ is perceived to be so problematic.

(responding to the question about whether $\sqrt{5}$ can be found on the number line precisely)

Anna: No, because we don't know the exact value, because .0 bigillion numbers ending with 5 is smaller than .0 bigillion numbers ending with 6. They're two different numbers, right, so <u>because it never ends we can never know the exact value</u>.

Kyra: Yeah, yeah, like you would never be able to finally say okay, this is where it is, because <u>there are still more numbers that you're reading off your irrational number</u>. But if you're using this scale of, you know, 1, between 1 and 2 is 2 cm or something, there's only so much precision that you can make with that point that you draw on there, <u>like I can't make it as precise as anirrational number</u> or, you know. . .

Precise Location: What Can Be Gained?

Among those participants who were able to find the precise location of $\sqrt{5}$ we found there was a sense of security that such number indeed existed. Their understanding seemed much more robust. Perhaps we could even say that the availability of a geometric representation aided them in the life cycle of concept development towards its final stage of encapsulation. This is in contrast with many others who offered the decimal approximation approach, where the number was seen as a process, stuck in its making forever. The following excerpt with Stephanie exemplifies this view.

Stephanie: Yeah. Okay, what I am thinking of, because somehow you can build this triangle and this triangle exists, this is another interpretation of the irrational number, so this segment represents the length of that hypotenuse, represents square root of 5, because this triangle exists. So it should be something what is, like we can touch, I don't know.

At some point students need to become aware that there is a profound distinction between the exact value of an irrational number and its rational approximation. We suggest this is better done sooner than later. Our findings indicate deep misconception and apparent confusion of some students who do not understand the distinction between π and $\frac{22}{7}$, as an example of irrational number and its rational approximation. Similarly, in the study of Arcavi et al., classifying $\frac{22}{7}$ as an irrational number was reported to be a common and persistent error. Furthermore, students need to be aware of the effects of premature substitution of irrational values by their rational approximations in partial results during calculations, both in the sense that this complicates calculations and creates problems of cumulative error. However, students' awareness of this important distinction between the irrational number and its rational approximation is unlikely to be attained if it is not within the active knowledge repertoire of their teachers.

Pedagogical Considerations

A significant part of school curriculum is focused on the notion of number. The concept of a number line appears early in elementary school and aids in ordering numbers and introducing integers and operations with integers. As rational numbers are dense, the idea that they do not "cover" the continuous number line presented a challenge to mathematicians. The formalization of this idea and a formal definition of real numbers is presented through the introduction of Dedekind cuts and is beyond what is normally presented in school. However, in school curriculum today we expect students to accept, intuitively, the idea of one-to-one correspondence between real numbers and points on a number line. We rely on explaining real numbers as "all the points on the number line". It is important to be aware of the fact that 2500 years have passed from the "discovery" of irrational magnitudes to the formal construction of the set of real numbers. It would be unreasonable to expect that what took centuries of mathematicians' work to develop could be acquired by students in a few session of classroom exposure.

The concept of an irrational number is inherently difficult, yet understanding of irrational numbers is essential for the extension and reconstruction of the concept of number from the system of rational numbers to the system of real numbers. Therefore a careful didactical attention is essential for proper development of this concept. We believe that emphasis on decimal representation of irrational numbers, be it explicit or implicit, does not contribute to the conceptual understanding of irrationality. And with irrational numbers one is faced with infinite decimal numbers of a special kind – numbers that cannot be written down or known fully. On this note, Stewart (1995) challenges the wisdom of calling irrational numbers *real*; that is, how can something be real if it cannot be even written down fully? In this sense, geometric representation should come almost as a relief in the process of learning about irrationals.

To be able to capture infinite decimals with something finite and concrete, and as simple as a point on the number line, even if this is only possible for a certain category of irrationals (constructible lengths), should help in taming the difficult notion of irrationality. It is our contention that exposing students to the geometric origins of irrationality and placing more emphasis on the geometric representation of irrational numbers can aid students in several ways. Firstly, they are likely to become more sensitive to the distinction between the irrational number and its rational approximation. Further, the geometric representation of irrational number may indeed contribute in encapsulating the diffficult notion of irrationality form its process stage to its object stage. It is both accessible to the learner (required is the knowledge of the Pythagorean Theorem) and yet revealing of the idea that to every number there corresponds a (single) point on the number line. As such, students' attention is drawn to yet another, more concrete, representation of the irrational number, as an object – a point on the number line, an irrational distance from 0 – and away from the never-ending process of construction in time, as often perceived through the infinite decimal representation.

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CHARACTERIZING LINKS AMONG PRE-SERVICE TEACHERS' COLLECTIONS OF PROPORTIONAL REASONING REPRESENTATIONS

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We report on the approaches used to link instructional representations of proportional reasoning topics made by three pre-service teachers who are in the beginning of their mathematics education program. By characterizing the links used in their collections of representations, we are offered a glimpse into the structures of their emerging teacher knowledge: the knowledge of content and pedagogy they bring to their education programs. Five categories of links are explored: symbolic, applied, sequential, descriptive, and conjunctive. These categories are further classified as having mathematical or pedagogical properties, according to the definitions provided by Shulman (1986) and Grossman (1990). By examining the ways pre-service teachers use links in their collections of representations, we are provided with a view into their thinking about teaching and learning as they transition from experienced students to beginning teachers. Furthermore, preferences for using particular types of links provide a multi-layered view of their emerging teacher knowledge.

Focus

The purpose of this preliminary case study is to characterize the explicit links made by three pre-service teachers among their collections of proportional reasoning instructional representations. These collections were created during their first methods course. Links are defined as mappings made by a pre-service teacher between two or more different instructional representations of a particular mathematical concept. Examination of these links, which were prepared within the context of learning mathematics pedagogy, informs teacher educators about the diverse ways pre-service teachers think about teaching mathematics through the use of instructional representations. We are further provided a glimpse into the structures of pre-service teachers' knowledge they bring to teacher education programs: their 'emerging teacher knowledge'. We assume mathematics pre-service teachers have a competent background in learning mathematics as students, but we acknowledge there is a difference between knowing mathematics and knowing mathematics for teaching. Pre-service teachers' use of explicit links reveals characteristics of their instructional approaches as they transition from experienced student to beginning teacher.

Definitions of representation abound in mathematics education research, but can generally be classified as either external or internal. Goldin and Shteingold (2001) describe external representations as signs, characters, or objects that represent something else, and internal representations as the mental images that students create in their own minds about a particular mathematical idea or concept. In this study, we define instructional representations as the external representations used by teachers to communicate mathematical ideas to students.

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Background

The use of instructional representations is considered to be a viable tool for connecting content and pedagogy and for assessing subject matter knowledge of pre-service teachers (Ball, Lubienski, & Mewborn, 2002; Berenson & Nason, 2003; Wilson, Shulman, & Richert, 1987). The importance of translating and moving between and among different instructional representations is also considered to be a necessary component of teacher knowledge. Successful teachers who are able to contribute to students' understanding of a concept are ones who are able to translate and link different instructional representations (Lesh, Post, & Behr, 1987; Orton, 1988).

Historically, teachers' behaviors have been used to evaluate their knowledge. When evaluating teachers' knowledge, Shulman (1986) claims that using behaviors as a way of evaluating effective teaching is not a viable practice; teachers should be evaluated and taught ways of making informed judgments. Research-based teacher education programs should "draw upon the growing research on the pedagogical structure of student conceptions and misconceptions, on those features that make particular topics easy or difficult to learn" (p. 14). This study aims to explore the pedagogical structures and their relation to the content knowledge of beginning pre-service mathematics teachers.

Theoretical Frameworks

This study is based on the frameworks of Shulman (1986) and Grossman (1990) and their constructs that teachers' knowledge base consists of several different, but interrelated, components. Shulman (1986) characterizes three categories of teacher content knowledge: subject matter knowledge, pedagogical content knowledge, and curricular knowledge. Subject matter knowledge consists of knowledge about the structures and content of the subject matter. Pedagogical content knowledge is the knowledge of how to teach the subject matter. Regarding the use of representation, Shulman claims that part of pedagogical content knowledge consists of the use of multiple representations and ways teachers use them to foster student understanding. Curricular knowledge consists of knowledge about the programs designed for teaching a subject and available teacher resources. Grossman (1990) includes curricular knowledge under the category of general pedagogical knowledge and claims that pedagogical content knowledge is the link between subject matter knowledge and general pedagogical knowledge. This particular link guides our analysis of pre-service teachers' representations.

Methodology

An instrumental case study design was employed to examine the multiple ways beginning pre-service teachers link different instructional representations of a given concept. The three participants were chosen for this study based on purposeful sampling where diverse cases are analyzed for emergent patterns (Creswell, 1998). The three participants in this study, Laura, Joseph, and Kate (pseudonyms) are undergraduate students enrolled in the first of four methods courses they will take in their teacher education program. These pre-service teachers were chosen for this study because they represent the diversity of students typically enrolled in this methods course. Many of the undergraduate students who enroll in this course are seeking their first degree, while several others are enrolled in licensure-only programs. Laura and Joseph are both seeking degrees while Kate is a licensure-only student. Laura and Kate are secondary pre-service teachers and Joseph is a middle grades pre-service teacher. These three pre-service teachers illustrate the possible range of students enrolled in this teacher education program.

During this study, the researchers acted as co-teachers in an undergraduate methods course for pre-service teachers preparing to become middle or secondary school mathematics teachers. The class met for 110-minute sessions once a week for 14 weeks. Throughout the course, the pre-service teachers engaged in multiple homework assignments and classroom activities based on the goal of deepening their mathematical content knowledge while addressing their education in pedagogical methods. This study focuses on weekly homework assignments where the pre-service teachers created their own collections of instructional representations. Each week, they were required to develop a collection of four different instructional representations about a particular mathematics concept. Their collections were based on the idea of introducing the following proportional reasoning topics: pi, ratio and proportion, slope, scaling, similarity, right triangle trigonometry, and the unit circle. The pre-service teachers were not instructed to make any connections between their four representations. Their collections of representations were to be chosen from the following categories: numerical and symbolic, definitions, contexts, visual, problems, manipulatives, and activities/tasks. These collections were intended to provide the pre-service teachers with sources of instructional materials to use once they enter classrooms as teachers. The participants' instructional representation assignments serve as the primary sources of data.

Results

Using pedagogical content knowledge as an organizational construct for categorization, the links made by the pre-service teachers were categorized according to their focus and structure. Five categories of links emerged: *symbolic, applied*, *sequential*, *descriptive*, and *conjunctive*. These categories are further classified as having mathematical properties or pedagogical properties, according to Shulman (1986) and Grossman (1990) and their definitions of general pedagogical knowledge and subject matter knowledge. These two classifications provide further insight into beginning pre-service teachers' pedagogical content knowledge and how they think about teaching the mathematics they have already learned as students.

Symbolic Links

Symbolic links occur when symbolic notations are linked to a numerical representation. This type of link is commonly found in mathematics textbooks where a formula is provided and then illustrated with a numerical example. Symbolic links are also used in traditional mathematics instruction, where students are provided with a formula and definition and then shown several examples which they are expected to follow during routine practice exercises. An example of a symbolic link is found in Kate's instructional representations of slope. She provided several different ways of representing slope symbolically, using both words and symbols. After presenting these different symbolic representations for finding the slope of a line, she continued by providing graphical examples where these symbolic notations were utilized with numerical values to determine and predict slope. She used the following labels on her graphs:

Graphs given to find actual value for slope.

Graphs given to predict slope.

She provided visual representations of graphs of lines and provided examples of finding the numerical value of the slope by replacing the symbolic expressions with numerical values. She used visual graphs in such a way as to help students learn how to find the value of a given slope and how to use graphs to predict the value of a given slope.

Applied Links

Applied links refer to the transference of one instructional representation to its use in a problem or activity. That is, one representation, such as a definition or symbolic notation, is applied as a tool for solving a mathematics problem or in an activity. For example, in Laura's representations of similarity, she provided a numerical representation to explain similar triangles and proportions. She labeled the lengths of the sides of two triangles and provided both the numerical and the symbolic notation of the resulting proportions of those sides to show similarity. She then applied these representations to an activity where students would construct their own similar triangles by stating:

Continue the above numerical representation with some hands on activity.

She planned for students to then be given several isosceles triangles and construct their own triangles similar to the given ones based on the lengths of its sides. Students would use the numerical and symbolic representations provided to guide them in their constructions of similar triangles.

Symbolic links and applied links are both further classified as having mathematical properties characterized by the structure of the links. Symbolic links are purely mathematical because they replace symbolic notation that is used to represent distinct mathematical concepts with numerical values. Applied links rely on the transference of mathematical properties from one instructional representation to the utilization of that representation in the solving of a problem or task.

Sequential Links

Sequential links impose an order to the arrangement of instructional representations for use in instruction. These links are stated explicitly with the goal of introducing one instructional representation before another one. For example, in Laura's instructional representations of slope, she presented a definition and a symbolic representation, and then introduced an activity representation by stating:

To be done after introducing the definition and symbolic representation of slope.

Laura continued to put an order to her instructional representations. In her last slope representation, after illustrating the above activity, she describes a context representation for slope by beginning:

After introducing the idea and definition of slope, mention that hills and mountains have slope as well.

Descriptive Links

Descriptive links focus on a particular qualitative property among different instructional representations of a particular concept. Descriptive links illustrate relationships between representations without the use of numerical values. One example can be found in Joseph's instructional representations of scaling (Figure 1). Joseph introduced scaling by creating a "shrink ray gun" and "shrinking" the Statue of Liberty. He continued using the shrink ray gun metaphor by using the descriptor "shrunken" in each subsequent scaling representation. He first used this descriptor in his context representation:

How many new shrunken versions could fit in the bigger original…?

He also used the descriptor in both his numerical representation and in a problem. In his numerical representation, he stated:

Let's say the length of the shrunken torch she holds is 1.5 feet. If the original version was about 1200 times bigger than the shrunken one, then her torch must be 1200 times bigger as well.

In a problem representation involving similar triangles, he explained:

These two triangles may not look the same, one is bigger than the other and facing a different direction, but as you'll find out the small one is just a shrunken version of the big triangle.

Descriptive links and sequential links are classified as having pedagogical properties. These types of links are used to help with students' understanding or to guide instruction. Sequential links are used as directions for instruction and are directed to the teacher, not the student. Descriptive links are image-inducing words used to help students focus on the particular properties of an image of a concept.

Last week, I took a trip to New York City and decided to visit Liberty Island to see the Statue of Liberty. When I was there I took out my awesome Shrink Ray and zapped the Statue of Liberty. Bwa ha ha! Now the Statue of Liberty is the size of a normal woman. If you had to take a guess, about how many times bigger was the orginal Statue of Liberty compared to the new shrunken version of it. (i.e. – how many new shrunken versions could fit in the bigger orginal Statue of Liberty?)

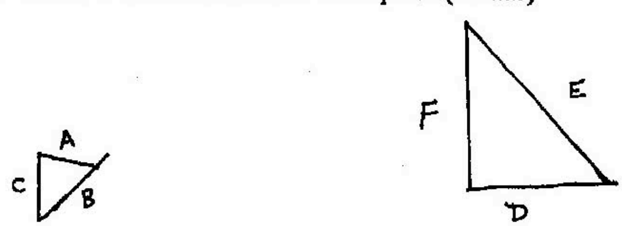
This example represents a concept in math known as SCALING!

Scaling is used when we use proportions to measure the sides of two similar objects. (*Definitions*)

Lets go back to our Statue of Liberty example…let's say the length of the shrunken torch she holds is 1.5 feet. If the orginal version was about 1200 times bigger than the shrunken one, the her torch must be 1200 times bigger as well.

1.5ft x 1200 = 1800foot tall torch! (*Numerical and Symbolic*)

How about we look at another example…(*Visual*)



These two triangles may not look the same, one is bigger than the other and facing a different direction, but as you'll find out the small one is just a shrunken version of the big triangle. (*Problem*)

Figure 1. Joseph's descriptive links.

Conjunctive Links

Conjunctive links refer to the use of one instructional representation in tandem with another. These types of links show direct relationships between two or more representations. In Kate's representations of slope, she provided a symbolic representation and used it in conjunction with a visual representation by explicitly stating (Figure 2):

Use with visual representations.

These visual representations were the same graphs addressed in the description of symbolic links above. Furthermore, Kate's last slope representation was an activity/task where students learn to estimate slope by looking at examples of various graphs of lines. These were not the same graphs used with her symbolic representations. She constructed a separate page of graphs for students to view and explicitly stated:

See graph paper.

Numerical & Symbolic (use with visual representations)

$$\text{slope} = \frac{\text{rise}}{\text{run}}$$

$$\text{slope} = \frac{\text{vertical change}}{\text{horizontal change}}$$

$$\text{slope} = \frac{\text{change in y-coordinates}}{\text{change in x-coordinates}} \quad \frac{(y_2 - y_1)}{(x_2 - x_1)}$$

$$\text{slope} \ \frac{\Delta y}{\Delta x}$$

Special cases

horizontal lines- slope = 0 (zero)

vertical lines- no slope

Slope is indicated by the letter *m* in equations for lines.

Figure 2. Kate's conjunctive link.

Conjunctive links can be classified as having mathematical properties, pedagogical properties, or both, depending upon the context in which they are used. As an example, in Kate's use of a conjunctive link, she mapped related properties between the two representations: symbolic and visual. Yet, this method of illustration can foster students' comprehension of the connections and relationships among different representations of the same concept. Therefore, Kate's use of conjunctive links can be classified as both mathematical and pedagogical.

Conclusion

Each participant in this study exhibited preferences towards the use of one type of link over another. Laura consistently used sequential links in her collections of instructional representations. Joseph utilized descriptive links on several occasions and Kate provided several uses of conjunctive links. Further research is warranted to determine if these pre-service teachers continue to use the same links as they progress through their teacher education programs.

By examining the ways pre-service teachers use links in their sets of instructional representations, we are provided with a glimpse into their thinking about teaching and learning as novice teachers. Preferences for particular types of links also provide a view of their emerging teacher knowledge base. It should be noted that the characteristics used to categorize these pre-service teachers' representational links have overlapping features. For example, some applied links could also be categorized as conjunctive links. Furthermore, more than one type of link may be revealed in a given collection of four instructional representations. For example, Kate used both symbolic links and conjunctive links in her collection of slope representations. Further study is warranted to find what the relationship is between preferences for types of links and pre-service teachers' beliefs about teaching and learning, and if there are other categories of links used by pre-service teachers. Also, other categories may emerge from the study of expert in-service teachers.

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A PROCESS MODEL FOR CHANGE IN ELEMENTARY MATHEMATICS TEACHERS' BELIEFS AND PRACTICES

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This research report focuses on the process of change in beliefs and practices experienced by practicing elementary school teachers during a sixteen-session course using two modules from the Developing Mathematical Ideas (DMI) materials. Using three case studies to represent the variations in changes in beliefs and practices observed among the course participants, we develop a process model for teacher change. Using this model, we analyze the ways in which teachers expressed interest in change, problematized their beliefs, experimented with possible solutions, and reflected on experimental results leading to changes in beliefs and practices. The results of our analysis indicate that variation in changes across course participants can be explained by variations in levels of their engagement in particular aspects of the opportunities offered during the course.

Introduction

Paradigmatic and systemic changes recommended in the standards of the National Council of Teachers of Mathematics (NCTM, 2000) and supported by teacher enhancement grants from the National Science Foundation involve complex issues associated with teacher beliefs, teacher knowledge, and changes in teaching practices. Friel and Bright (1997) integrated and summarized a number of issues of particular importance for these changes, including: (1) working to change teacher beliefs is the starting point for most professional development, and (2) changing beliefs and experimenting with teaching practices are intertwined in an iterative process.

This study (1) provides a metaphor-based framework for organizing, describing, and analyzing teachers' beliefs about knowing, learning, and teaching mathematics; (2) explores a model for the processes of teacher change; and (3) reports effects of a specific teacher development program on participating teachers' beliefs. The context of this study is a one-semester graduate course consisting of 16 three-hour sessions using two modules from the innovative elementary mathematics professional development program *Developing Mathematical Ideas* (DMI) (Schifter, Bastable, & Russell, 1999a and 1999b).

Conceptual Framework

Bullough (1992) and his associates (Bullough & Stokes, 1994) suggested that *metaphors* provide powerful mental images or models and are able to briefly summarize the elaborate and complex theories, assumptions, and understandings upon which people act. Discussions by Stigler and Hiebert (1999) and Ma (1999) suggested that traditional metaphors still dominate the culture of mathematics education. Research from social constructivist perspectives provides the basis for creating alternative metaphors for knowing, learning, and teaching mathematics. From this literature we created a set of six metaphors to describe teachers' beliefs and practices (see Table 1).

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Table 1. Metaphors for Knowing, Learning, and Teaching Mathematics

Topic	Traditional School Mathematics	Standards-Based School Mathematics
Knowing Mathematics	**Toolbox:** Knowing mathematics is having a toolbox filled with a collection of facts, definitions, rules, and efficient computational procedures (e.g., standard algorithms) to be applied in computing correct answers by matching the intended, well-practiced tool to a familiar type of problem.	**Flexible Problem-Solving with Understanding:** Knowing mathematics is having and being able to flexibly use a complex, interconnected web of understanding of concepts, procedures, and problem-solving experiences to convert new, nonroutine, culturally valued, real-world problems into mathematical abstractions that can be solved using concept-driven, sensible strategies.
Learning Mathematics	**Behaviorist:** Learning mathematics involves memorization and practice, which strengthens mental associations between the generalized knowledge and specific procedures that have been demonstrated by the teacher and the typical problem types to which those preferred procedures are routinely applied.	**Social Constructivist:** Learning mathematics involves constructing a complex web of knowledge through social negotiation of meaning for mathematical language and symbols; construction of shared understandings of mathematical concepts that can generate possible problem-solving strategies for nonroutine problems; and developing flexibility in thinking and communicating mathematically through participation in a cultural community.
Teaching Mathematics	**Master:** Teaching mathematics consists of direct instruction (ala master-apprentice or master-disciple) in which the teacher shows students preferred procedures; tells facts, definitions, and rules; assigns practice of these generalized responses; assigns applications to particular contexts (e.g., word problems); and tests for computational speed and proficiency by counting correct answers to familiar problems.	**Facilitator:** Teaching mathematics consists of posing worthwhile mathematical tasks, facilitating students' problem-solving efforts, questioning students' understanding and thinking, and orchestrating discourse to facilitate and guide students' construction of more complex understandings; and facilitating students' reflection on their experiences so that they build connections among their context-specific conceptions and produce generalizations that are sufficiently well-connected to particular contexts and experiences that they can be assessed as being generative of new strategies for solving new problems in unfamiliar contexts.

Guskey (1986) provided a simplified model of the process of teacher change that explained the inter-relationship between changing beliefs and changing practices. Clarke and Hollingsworth (2002) proposed an interconnected model of teacher professional growth that emphasized two processes of change: reflection and enactment. In applying Dewey's (1933) model of reflective thought, Mewborn (1999) emphasized the importance of problematizing teaching situations. Both Dewey (1933) and Guskey (1986) discussed the importance of interest in change.

Synthesizing this literature resulted in a model of professional growth with four sequenced elements: (1) initial interest in change; (2) problematizing current practices and proposing solutions; (3) exploring and testing alternative practices; and (4) reflective analyses of benefits. The analyses of the case studies in this study led to further elaboration of the details within this model.

Methods and Evidence

Data for this study were taken from (1) audio taped whole-class and small-group discussions during the course; (2) field notes; (3) written materials collected as part of the course from the 13 participating teachers; (4) a group interview with participants conducted by an independent evaluator following the course; (5) a post-course observation of several participants' teaching practices in their regular classrooms; and (6) a post-observation interview on changes in practices resulting from the DMI course.

Analysis of the data proceeded in a manner consistent with a naturalistic inquiry approach (Lincoln & Guba, 1985). This analysis attended specifically to teachers' beliefs (especially images and metaphors for teaching) and participation in change opportunities.

First, two researchers read the transcripts independently and identified emergent themes. They paid particular attention to teachers' beliefs (especially images and metaphors for teaching) and participation in processes of change. As these researchers discussed their reading of the transcripts, common themes began to emerge and they developed codes for these themes. Second, these same researchers re-read the transcripts and coded the conversations according to the emergent themes. The next step in the analysis of the data involved the isolation and validation of the major themes wherever they appeared in the data by triangulation across the various data sources and across time. Lastly, these themes were used to analyze all of the other data from the study.

At the end of the semester following the conclusion of the DMI course, the third author observed several of the DMI course participants' as they taught in their classrooms and then interviewed them about changes in their teaching practices they attributed to the DMI course. These interviews were audio taped and transcribed. This data provided an additional source for teachers' reflections and for triangulation and confirmation of the conclusions from the primary data collected during the DMI course.

Results and Discussion

As we analyzed the data we noticed that the impact of the course varied across the participants. To illuminate this variation, we wrote three cases of individual teachers (indicated by the pseudonyms Christine, Linda, and Paula). At the time of the DMI course, Christine was teaching third grade for the second consecutive year. She had previously taught for ten years in the resource room with students she characterized as two or three years behind grade-level in mathematics. Linda had taught kindergarten for 16 years and was teaching first grade for the first time. Paula had been teaching fifth grade for six years and had previously taught grades two and four.

These three cases represent a continuum of engagement in the process of change. In many ways, the cases of Christine and Paula represent the two extremes of the course's impact, while the effects of the course on Linda are similar to those of the majority of course participants. On the one extreme, Christine started with very traditional beliefs about knowing, learning, and teaching mathematics, and made the least progress toward alternative beliefs and practices. More typical of the results of course participants, Linda started with very traditional beliefs and practices and made significant shifts in both, although only beginning to understand the coherent alternative provided by the standards-based perspective. At the other extreme, Paula started the course already wondering about children's understanding of mathematics and made the most real progress by connecting additional complexity to her previous understanding and developing a robust and coherent philosophy of standards-based mathematics education.

We used our set of six metaphors (see Table 1) to summarize these teachers' beliefs and practices at the beginning and end of the DMI course and as points of reference for change (see Table 2). These three cases also focused on varying degrees of participation in the above described four components of change which emerged from our synthesis of the literature and preliminary analysis of the data.

Table 2. Changes in Metaphors/Beliefs

Name (pseudonym)	**Beliefs About *Knowing* Math**		**Beliefs About *Learning* Math**		**Beliefs About *Teaching* Math**	
	Beginning	**Ending**	**Beginning**	**Ending**	**Beginning**	**Ending**
Christine	**Toolbox:** Successful in math; liked algorithms, correct answers, and automaticity	**Toolbox:** Enriched by allowing some student-invented strategies if they led to traditional algorithms	**Behaviorist:** Emphasized repetitive practice; felt successful bringing resource students up to grade level	**Behaviorist:** Focused on additional procedures rather than concepts or student thinking	**Master:** With room to improve teaching skills and become more exciting	**Master:** With some showing of invented strategies (allowed students to become part of the show-and-tell)
Linda	**Toolbox:** Not very successful in math; felt safe by following traditional emphases on speed and efficiency	**Flexible PS/U and Toolbox:** Liked number sense and invented strategies for mental math and seeing relationships; held on to some traditional tools and emphases	**Behaviorist:** Focused on correct answers without under-standing	**Social Constructivist:** Began to value each voice; concerned about student explorations, thinking, and understanding	**Master:** Looking for additional strategies and activities	**Facilitator:** Increased ability to ask good questions, rephrase what students said, support explanations, and assess errors in thinking
Paula	**Toolbox and Flexible PS/U:** Very successful with algorithms but noticed others were not; had some interest in problem solving and understanding	**Flexible PS/U:** Focused on understanding big ideas, problem solving, and critical thinking	**Behaviorist:** Some interest in sense making	**Social Constructivist:** Recognizing more individual responsibility for learning that occurs through dialogue and written representations	**Master:** Focus on textbook sequence; looking for more successful strategies and a more individual focus	**Facilitator:** Pose interesting problems; use questioning to find out what students think; and lead them to build deep, well-connected understandings

Our analysis indicated that participants' beliefs about knowing, learning, and teaching mathematics are interrelated. See Table 3 for a summary of our appraisals (as low, moderate, or high) of these three teachers' levels of initial interest in change, problematizing and posing possible solutions, experimenting with alternatives, and reflective analysis of benefits and making changes. Our analysis of these variations across the course participants also suggests that teaching practices were unlikely to be problematized without first problematizing beliefs about knowing and learning mathematics. Without problematizing beliefs about knowing and learning mathematics, interest followed the typical focus on incremental change within current teaching metaphors rather than significant experimentation with paradigmatic changes. Important elements of interest seen to be essential to the problematizing process include curiosity, interest in improving student learning, dissatisfaction with current beliefs and practices, and/or recognition of differences among teachers' beliefs and practices.

Table 3. Interest in Change, Problematizing, Exploring, and Analyzing

Name (pseudonym)	**Interest in Change**	**Problematizing Current Beliefs and Practices and Posing Possible Solutions**	**Exploring/Testing Alternative Practices**	**Reflective Analysis of Benefits and Changing Beliefs and Practices**
Christine	Low: looking for spice	Low: confident in current success with low-performing students	Moderate: tried asking children to explain their strategies	Low: questioned benefit of children explaining thinking; saw limited usefulness for alternative strategies, only as temporary transitions to traditional algorithms
Linda	Moderate: looking to develop greater confidence in her own understanding of mathematics; curious about children's thinking different from her own	High: questioned her own understanding, the process by which she had failed to learn to understand math, and her traditional approach to teaching; developed a strong interest in change to develop number sense and relational thinking	High: questioned students' thinking; kept track of children's responses; supported children with problem solving and communicating thinking; assessed to understand students' processes and conceptual understanding	Moderate: gained confidence in her ability to problem solve; became concerned about individual students' understanding; changed her goals for learning; broadened her view of the curriculum
Paula	High: dissatisfied with traditional practice; interested in exploring whys of new practices	High: concerned with inequity of traditional practices; interested in teaching for understanding	High: modified curriculum and teaching to focus on big ideas, discourse, and written representations; attended to individual children's thinking and understanding	High: concluded questioning develops deep understanding in mathematics and other content areas; students who understand are able to analyze their own errors; formed a new philosophy of mathematics education

An additional element of the change process evident in the study included the contexts in which these change processes occurred. In the case of the DMI course, these contexts included two conceptual laboratories: (1) the DMI classroom and (2) the participants' school classrooms. Change processes in the DMI classroom involved what Clarke and Hollingsworth (2002) referred to as the teacher's personal domain, while change processes occurring in the school classrooms involved the teacher's domain of practice. Being able to problematize, experiment, and reflect in both the DMI class and one's own classroom provided an iterative element with a short turn-around time which supported the change process. However, participants with low interest or low levels of engagement in these change processes, particularly in problematizing, made only limited changes in their beliefs and practices. Conversely, those participants who had higher levels of initial interest and more fully engaged in each of the change processes, experienced greater changes in their beliefs and practices.

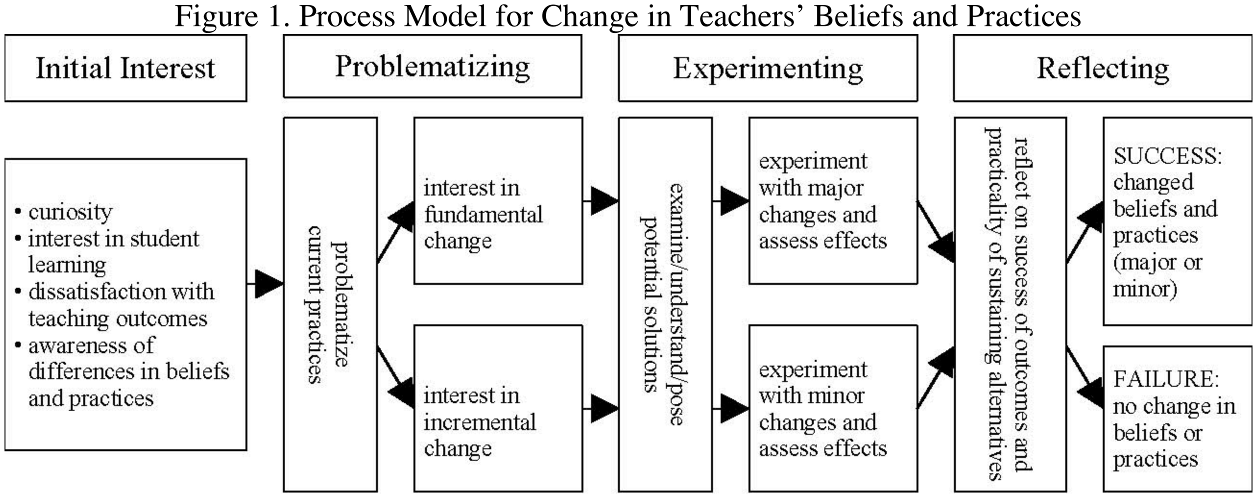
Conclusions

This study concludes that variations in change during the DMI course result from varying levels of engagement in the processes of change identified in the following model (see Figure 1). This study shows that this process model provides a useful synthesis of previous research on teacher change, fairly represents the experiences designed into the DMI course as taught during this study, and explains the variations in changes we noticed across participants in the course.

It is clear to us that participants' beliefs about knowing, learning, and teaching mathematics are interrelated. Metaphors for each of these components of one's beliefs provide succinct ways of categorizing otherwise complex and varied sets of beliefs. The results of this study suggest that teaching practices are unlikely to be problematized unless beliefs about knowing and learning mathematics are problematized first. Without problematizing beliefs about knowing and learning mathematics, interest tended to follow the typical pattern of remaining focused on incremental changes within current teaching metaphors rather than participation in the paradigmatic changes offered by the DMI course. This model also indicates some elements of interest that precede the problematizing process, including curiosity, recognition of differences, and dissatisfaction with current beliefs and practices.

It is encouraging to see the progress some teachers made in changing their beliefs during the DMI course even though they were initially looking for additional strategies to use while teaching within their existing metaphors. This study has taught us that examining the processes through which such teachers changed their beliefs can be useful for understanding both the nature of such changes and the variations across individuals participating in the same learning opportunity.

Figure 1. Process Model for Change in Teachers' Beliefs and Practices



Relationship to Goals of PME-NA

This research relates closely to PME-NA Goal 3: "To further a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implications thereof." Specifically, this study offers an important contribution to the continuing discussion on researching teachers' beliefs and changes in practices for teaching mathematics in the elementary grades. Numerous research and short oral reports during the 2004 Annual Conference addressed the topics of mathematics teacher beliefs, teacher education, teacher professional development, and changing teaching practices.

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TEACHING ASSISTANTS' KNOWLEDGE AND BELIEFS RELATED TO STUDENT LEARNING OF CALCULUS

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Teachers' knowledge and beliefs shape decisions they make while planning and carrying out instruction for their students. Knowledge and beliefs associated with student thinking (e.g., typical solution paths, typical difficulties) appear to be especially powerful influences on teachers' practices and a productive site for professional development. These issues, however, have not been examined in many content domains, in fine-grained detail, or in connection to literature on student understanding in those domains. This study examined mathematics graduate student teaching assistants' knowledge of student thinking for important concepts from calculus. Findings indicate that participating teachers did not possess rich knowledge of student thinking and that, in general, they were unable to generate solution paths other than the one they had used to solve the problem. In addition, teachers asserted a variety of (sometimes contradictory) relationships between students' correct solutions and their understanding of the problems' central concepts.

Introduction and Objectives

Among factors shaping teaching practices, knowledge and beliefs appear to be especially influential (Ball & Bass, 2000; Ball, Lubienski, & Mewborn, 2001; Borko & Putnam, 1996; Calderhead, 1996; Hill, Rowan, & Ball, 2004; Thompson, 1992). Studies have documented K-12 teachers' content knowledge, pedagogical content knowledge, and the roles of such knowledge in how teachers learn from professional development (Ball, 1990; Ma, 1999; Sfard, 1991). In addition, it appears that attention to knowledge and beliefs related specifically to student thinking and understanding is an especially effective approach to professional development (Carpenter et al, 1989; Fennema et al., 1996).

A base of research about student understanding of particular concepts has made such professional development (PD) possible. Until recently, this research was primarily limited to topics at K-12 levels. There is now sufficient research at the undergraduate level that PD for college teachers focused on student learning is feasible. In addition, since particular areas of knowledge of student learning have been identified as especially powerful influences on teachers' practices, there is a need to examine teachers' decision making in detail. To enrich the research base on teachers' knowledge and beliefs and to inform the design of PD, this study examined two questions: What *knowledge* do college instructors have of student thinking for key concepts in calculus and concepts important to the learning of calculus? What do teachers take as evidence that students understand – in particular, what *beliefs* do college teachers have about the relationship between correctness of answers and understanding of concepts?

Research Design and Methods

Data for the study came from task-based interviews conducted with mathematics doctoral students who were employed as teaching assistants (TAs) in the mathematics department of a large, public university in the Midwestern U.S. TAs were interviewed using tasks related to

Lloyd, G. M., Wilson, M., Wilkins, J. L. M., & Behm, S. L. (Eds.). (2005).
Proceedings of the 27th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.

concepts that are central to calculus (limit, derivative) or that influence the learning of calculus (function, interpretation of representations) and for which research on student thinking exists (Bezuidenhout, 2001; Monk, 1994; White & Mitchelmore, 1996). TAs had varied amounts of college-level teaching experience (1-8 semesters).

TAs were interviewed individually. For each concept, the semi-structured interview consisted of four parts where TAs were:

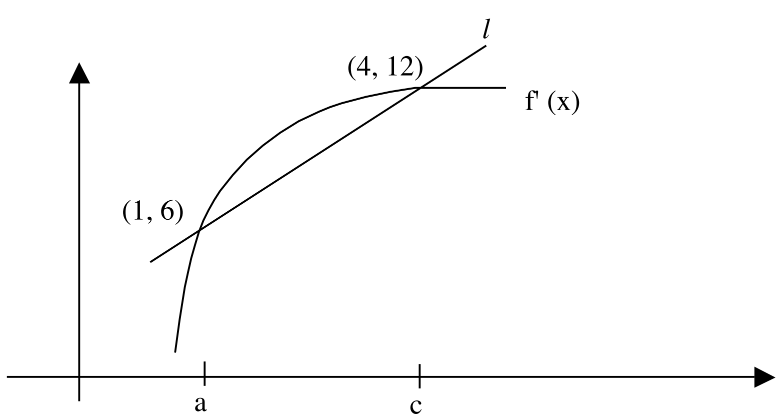
(1) asked to solve 1-2 related tasks and to describe the strategies they used to solve the tasks. Tasks were selected from research articles or modeled after those used in research on student understanding.
(2) asked to describe concepts central to the tasks.
(3) questioned in detail about how they thought students would approach the task, what typical student solution strategies might be, and what difficulties students might have while working on the tasks.
(4) asked specifically what they would feel confident assuming a student understood if the student solved the task correctly and what they would infer about understanding of the concepts if a student did not generate a correct solution.

Interviews for each task lasted 30-60 minutes. 12 TAs were interviewed about each task. Interviews were audio recorded and transcribed and TAs' written work was collected.

Although tasks related to all of the concepts listed above were used in the study, data reported here come only from interviews involving function, derivative, and interpretation of representation tasks. More specifically, the findings presented in this paper are based on data from interviews about the following two multi-part tasks:

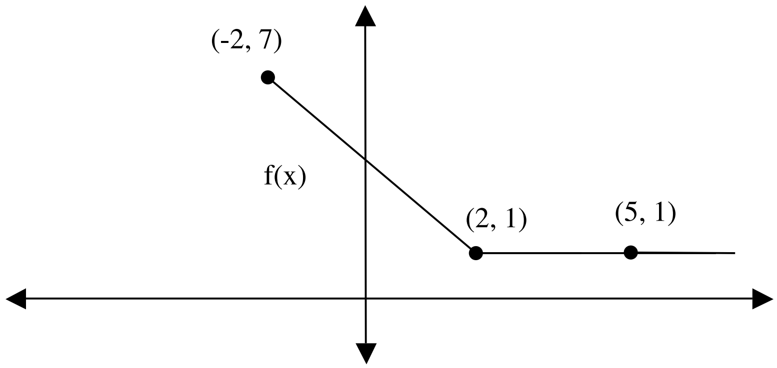
(1) Sliding secant, from Monk, 1994.

Below is the graph of the function f (x) and a secant line l.



(a) Find the slope of the secant line l.
(b) If point a moves along the x-axis toward point c, does the slope of l increase, decrease, or stay the same?

(2) Piecewise linear functions and derivatives
Below is the graph of a function f (x):



(a) Sketch f' (x)
(b) Find an expression for f (x)
(c) Find an expression for f' (x)

Data analysis utilized findings from research on student understanding of these tasks and concepts and techniques for identifying themes (Strauss & Corbin, 1990; Yin, 1989). This approach was used to catalog TAs' knowledge of strategies and difficulties for each task. Cross-task analysis was used to examine the data for patterns in beliefs TAs stated about evidence for student understanding.

Findings

Knowledge of Task

When solving these tasks, many TAs were able to provide not only appropriate procedures, but could also describe and explain their thinking as it related to their well-developed understandings of the concepts. TAs displayed not just the knowledge necessary to obtain the correct answers but also a rich understanding why their answers were correct and what concepts were involved. In some cases, however, TAs had difficulty with tasks. In particular, nearly half of the participants encountered minor difficulties with parts (a) and (b) of the Piecewise Linear task.

Knowledge of Student Strategies

TAs displayed some knowledge of student thinking, but it was not extensive and did not reflect what is known from research on student understanding. Although approaches TAs generated for the problems were correct (most of the time), they fell far short of the breadth of solution strategies that have been documented in research on student understanding for such tasks. In particular, when asked to describe student strategies, TAs claimed students would approach the task in the same way that the TA had solved it (during the earlier portion of the interview). While some TAs qualified their responses, stating that students' solutions would be a "less sophisticated" version of their approach, most did not offer any other potential solution paths. This was true even when TAs were asked specifically if students might pursue other approaches. Their responses included: "Well, I always imagine a student would do exactly what I would do;" "I can't think of any other way they might try to solve this problem;" "I can only think of what I did. What else would you do?"

Also of note was the fact that TAs' knowledge of strategies appeared to come from observations they had made when their students had worked similar tasks in courses the TAs had taught. Their knowledge of strategies did not appear to come from their own decomposition of the task or knowledge of student strategies for related tasks. For example, when asked if there were typical difficulties or mistakes students would have with the task, TAs who could not think of any said things such as, "Not that I remember." When TAs did know of such issues, they often stated them in sentences such as, "As far as I can remember, the mistakes they made were…" or others that indicated they were recalling specific experiences from their teaching.

Knowledge of Student Difficulties

As with student strategies, TAs were able to generate some difficulties they felt would be typical for students, but their repertoire was mostly limited to "forgetting." For example, when describing difficulties students would have with part (a) of the Sliding Secant task (finding the slope of the given line), TAs said things such as, "They just forget what is the definition of slope, it should be the rise over run." Other responses that did not directly implicate forgetting as a difficulty indicated that memorizing particular formulas was the key to getting the correct answer and therefore difficulties could stem from a failure to memorize adequately. For example, one such response was, "That's basically something you have to memorize. So I don't know if there's any simple way to remember it except memorize."

Research on student difficulty with the Sliding Secant task (Monk, 1994) indicates that students are not equally successful on the two parts of the task. Students are considerably more successful finding the slope than they are on the second part (determining how slope of a line changes as one of its endpoints changes). Some TAs did perceive the two parts as being of different difficulty levels but many thought students would perform similarly on both parts.

Once TAs described the difficulties they thought students would have, they were asked to discuss why they thought those difficulties occurred. In most cases, TAs were unable to provide any explanation for why students struggled in the ways they had described. In some cases, TAs reiterated their descriptions of the difficulties and in others they seemed perplexed by the question and were unable to form any answer.

Beliefs About Relationships Between Solution Correctness and Understanding

In addition to gathering information about TAs' knowledge of student strategies and difficulties, questions in the interview were designed to specifically probe TAs' beliefs about the relationships between students' answers and students' understanding of concepts in the tasks. These questions included the following:

- If a student produced a correct answer, what would you feel comfortable assuming he or she understood?
- If a student understood the ideas, how likely is he or she to produce the correct answer?
- If a student produced an incorrect answer, what would you assume about what he or she understood?

In response to the first question, TAs indicated a belief that generating a correct answer was a strong indication that students understood the concepts in the task. For example, for one TA, a correct answer to the first part of the Piecewise Linear task indicates that, "they had a good understanding of the link between the slope of a straight line and its derivative and they know how to come up with a graph based on that." For part (b) of the same task, another TA asserted that if a student produced a correct answer, that would imply, "they have an excellent understanding of taking derivatives of piecewise functions." Another TA said of part (b) of the Sliding Secant task: "Getting [it] correct means they intuitively understand what it means to graph a function." These responses, typical of the interview data, indicate that TAs are inclined to attribute understanding to students as long as they are able to generate correct answers.

While TAs felt that correct answers were strong indicators of student understanding, they also believed that if students understood the concepts, they were highly likely to generate correct answers. For example, the following was a typical responses when asked how likely students were to get correct answers if they understood the ideas: "I just cannot see how you can understand what slope means and what linearity means and not be able to calculate slope if I give you two points."

Beliefs that TAs expressed in response to the first two prompts stand in stark contrast to research findings on student understanding of calculus as well as findings from research on student understanding in other areas. Numerous studies have shown that students who generate correct answers do not necessarily possess strong understanding of underlying concepts and students who understand mathematical ideas are not always able to utilize their knowledge to solve problems (Selden, A., Selden, J., and Mason, 1994; Selden et al, 2000).

Data from the third prompt ("If a student produced an incorrect answer, what would you assume about what they understood?") were somewhat more complex. Two findings were apparent from the analyses. First, TAs believe that incorrect answers are not a particularly strong indicator of a lack of understanding. TAs said that students could easily make mistakes leading to incorrect answers while still holding strong understanding of the ideas. One TA, for example, said, "I think there's a lot of different ways you could go about getting a wrong answer and hold a reasonable understanding of the concept." The second finding pertains to a relationship between answers and understanding that was noticeably absent from the data. Very rarely did TAs assert that a student's incorrect answer was an indication of a lack of understanding.

One way of representing the relationships between correctness of answers and understanding, as articulated by the TAs, is the following:

If the answer is correct, the student understands the ideas [$C \Rightarrow U$]

If a student understands the ideas, the answer will be correct [$U \Rightarrow C$]

If the answer is not correct, the student is still likely to understand [$\neg C \Rightarrow U$]

All three of these relationships were common in TAs' descriptions of answers and understanding. Absent from the set of relationships was the following:

If the answer is not correct, the student does not understand the ideas [$\neg C \Rightarrow \neg U$]

TAs often stated that more than one of these beliefs was valid for interpreting students' work. For example, some TAs asserted that both the second and third statements were valid. TAs did not express the inherent contradiction in holding these beliefs simultaneously. In addition, the underlying structure of these beliefs does not follow rules of logic that graduate students in mathematics are typically well versed in. For example, if understanding implies correct solutions will occur ($U \Rightarrow C$), then according to rules of logic, not getting a correct answer should imply not understanding ($\neg C \Rightarrow \neg U$). This, however, was not how participants believed correct answers and understanding were related.

Conclusions and Implications

Participants' limited understanding of student thinking indicates that strong understanding of mathematical content does not necessarily enable teachers to generate student solution strategies or anticipate difficulties. TAs' inability to generate alternative approaches, even when pressed to do so, has important implications. For example, such TAs are unlikely to recognize such strategies in their students and may be unable or not inclined to validate students' correct reasoning when it is not the same as their own.

There are also potential implications of TAs' limited ideas of why students encounter difficulties while solving problems. For example, TAs who cannot understand what a student has done may be unable to find a different way to explain the ideas or help the student in a manner that takes into account their particular incorrect understanding. In other words, having a correct "diagnosis" of the source of the difficult can assist teachers as they help students correct their understanding. Furthermore, if teachers cannot anticipate difficulties then they cannot teach with those in mind, clarifying potential miscommunications before they occur, or be "on the lookout" for problems when they first arise (so, for example, they do not go unchecked until an exam).

Participants' beliefs about relationships between answers and understanding also have implications for research on teacher cognition and practice. The complex relationships between answers and understanding point to some powerful underlying beliefs that may provide insight into teachers' instructional decisions in college mathematics courses. There appear to be some patterns in how TAs make inferences. Correct student work seems to be strong, convincing evidence of solid understanding of ideas. Incorrect work, however, is taken to be weak, unconvincing evidence of a lack of understanding. TAs seem to have a "high bar" for *not* understanding – their null hypothesis is that students understand. This contrasts quite sharply with beliefs typical of mathematics educators where there a high bar is set for what counts as evidence of understanding and the null hypothesis is that students do not understand – a condition that requires considerable evidence to disconfirm.

From these findings, it appears that TAs might be using assessments (i.e. exams, problem sets) for means other than determining what students understand about course content for the purposes of assisting students in avoiding errors (other than those of a calculational variety). Of course, students can and do make calculation errors that can lead to incorrect answers, but if TAs are inclined to view these incorrect answers as instances of memory lapse or a night of inadequate studying alone (and not tied to deeper issues of understanding), then there are implications for the purpose of assessment and teaching. Smaller exams and problem sets (or other formative assessments) are wonderful opportunities for instructors to gauge student progress and can be used for designing future lessons. Noticing a pattern in student performance can help instructors alter teaching practice in the future to aid students in correcting conceptual errors or anticipate future ones. Furthermore, if TAs believe that students inherently understand despite incorrect answers then TAs might not feel the need to examine his or her practices in the classroom, but instead consider the responsibility of learning as falling primarily to the students.

Participants' predilection to presume students understand even when they give incorrect answers is a potentially powerful site for PD. TAs exposed to research about student thinking as part of PD activities might examine and refine their beliefs. The underlying logical structure of claims made about answers and understanding may also provide productive avenues into the examination of what counts as evidence of student learning that is especially interesting to mathematics graduate students and faculty.

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ORGANIZING COMMON GROUND TO SUPPORT COLLABORATIVE INQUIRY

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Despite extensive efforts to enhance students' mathematics learning experiences in classrooms, teachers continue to face great challenges when trying to engage students in more collaborative, inquiry-oriented practices. This limited success can be attributed, in part, to an underdeveloped understanding teacher's role in organizing students' successful participation in these practices. This paper reports findings from an in-depth, longitudinal case study of a highly accomplished teacher who consistently supported students' participation in collaborative inquiry practices. Analyses suggest that a central aspect of the teacher's role is organizing common ground (Clark, 1996). This construct may provide conceptual guidance for teachers as they strive to support learning environments aligned with current calls for reform.

Purposes and Perspectives

Recent advances in sociocultural learning theories have led to the conceptualization of classrooms as communities of practice. From this perspective, mathematics is understood as "a set of practices of inquiry and sense-making that include communication, questioning, understanding, explaining, and reasoning" and learning mathematics is understood as "increasing participation in an expanding range of such practices" (Greeno & MMAP, 1997, p. 104). Correspondingly, a new vision of "good practice" has emerged (NCTM, 2000), which encourages and implores teachers to organize more participatory, discussion-based lessons to develop students' mathematical proficiency. Organizing students' participation in collaborative practices however is very challenging (Ball, 1993; Heaton, 2000; Schifter, 2001). The math education community continues to work to further understand this kind of teaching as well as how to support teachers in implementing such practices (Boaler, 2002).

The main research question of this study was: What is the role of the teacher in organizing student participation in collaborative learning practices? *Learning practices* are the primary means by which students are provided opportunities to further their understanding of mathematical ideas and concepts (Cohen & Ball, 2000), for example, by explaining one's idea, completing practice problems alone, and working with others on a novel problem. Collaborative learning practices comprise a subset of learning practices characterized by student-student interactions. *Collaboration* is evidenced by participants' continued efforts to "construct and maintain a shared conceptualization of a problem" (Roschelle & Teasley, 1995, p. 70, cited in Dillenbourg, 1999). In a math class, this means that students are jointly developing an understanding of the topic. They offer their thoughts, attend and respond to each other's ideas, and develop some shared meaning or understanding[1]

[1] Collaboration is distinct from cooperation. For example, students can cooperate to accomplish a task, such as creating a poster or other work product, but not engage in collaboration with respect.

This paper reports findings of an in-depth case study of a teacher and math class that regularly engaged in collaborative inquiry activities. The students worked collectively, solving challenging problems and forging new understandings of mathematical concepts. They attended to and extended each other's ideas, thereby engaging in a highly generative kind of intellectual activity not often supported in mathematics classrooms. One key finding was the critical importance of the teacher's role in organizing *common ground* (Clark, 1996) to support collaborative inquiry practices. In this paper, I explicate the construct *common ground*, arguing that common ground shared *among* students and comprising *students'* mathematical thinking is critical to collaborative work. I then discuss some of the teacher's deliberate pedagogical actions that helped organize and maintain common ground shared among students over time.

Data and Modes of Inquiry

Data for this study was collected during the 2000-2001 school year. The focal case was Ms. Nelson's ninth-grade pre-algebra class. Ms. Nelson was a highly experienced and accomplished teacher (e.g., National Board certified) and consistently organized collaborative inquiry activities that attended to the development of students' conceptual understanding. Her exceptional skill at organizing collaborative work, regardless of specific course or class composition, was confirmed by members of the Stanford Mathematics Teaching and Learning Study[2] (Boaler, 2003a, 2003b). The sampling strategy then was *purposive* (Yin, 1994) and the case was chosen to offer insight into teacher practices that support mathematically intensive collaborative learning environments.

At the time of the study, Ms. Nelson had 22 years of experience and had won multiple awards for her teaching. The class comprised 20 lower-attaining students who had experienced primarily traditional modes of instruction in prior math classes. The research was conducted from an interpretivist paradigm, drawing upon ethnographic methods described by Eisenhart (1988). Data collection included videotapes and observations of lessons (115 hrs; 66% of class time); interviews with students (19); student questionnaires (3); teacher interviews (4); and video viewing sessions with the teacher (7). In addition Ms. Nelson and I had frequent conversations after lessons.

Data analysis followed principles of grounded theory (Strauss & Corbin, 1990), and required multiple passes through the data corpus. Patterns and themes were identified, and particular teaching strategies that fostered student engagement in collaborative activities were codified. A refined set of codes was subsequently applied to transcribed segments of whole-class discussions. As a final step, analyses were refined and validated by reviewing videotapes and analyses with Ms. Nelson. She confirmed that the results presented an accurate portrayal of her practice, both the description of the practice and how it functioned to support students' participation. For a more detailed description of the methods and findings, see Staples (2004).

Results

Common Ground

Analyses of Ms. Nelson's lessons revealed that a central aspect of her role in supporting collaborative inquiry activities was to *establish and monitor a common ground* among students comprising their mathematical ideas. *Common ground* comprises the suppositions, ideas, and

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objects that individuals take as *mutually* held or recognized among the participants in an activity setting (Clark, 1996). In a mathematics class, common ground might comprise (among other things) the algebra problem on the board, a student's solution, and the question the teacher just posed. It also includes mathematical terminology, normative practices, the class's shared history, and a host of other mutually recognized understandings. Although these too play a critical role in supporting the collaborative inquiry, I focus less on these aspects in an effort to foreground processes *during* collaborative inquiry that support the accumulation of common ground in moment-to-moment interactions over the course of a lesson. It is common ground that allows participations to make sense of others' contributions and to coordinate their efforts to continue working on mathematics together. Common ground constantly accumulates as the activity advances, with participants recognizing new elements that can be taken as mutually recognized.

While common ground provides the basis for all joint activities (Clark, 1996), including collaborative inquiry, the kinds of ideas, concepts, and routines that are part of a common ground vary from activity to activity, and classroom to classroom, depending on the requirements of the setting's activities. In Ms. Nelson's class, common ground had two important characteristics that were critical in supporting collaborative mathematics. First, common ground significantly comprised *students' ideas and ways of thinking*. Second, there was deliberate attention to ensuring these ideas were shared *among students*.

These two characteristics are not required of more traditionally organized instructional activities, as student-student interactions and communication are not taken as central components of instruction. In more traditional pedagogies, the focus is on establishing a common ground between the teacher and each individual student (for example, see Edwards & Mercer, 1987), and the common ground primarily comprises the teacher's (or textbook's) mathematical ideas and ways of understanding. Even in more student-centered classrooms, many teachers elicit students' ideas, but do not attend to ensuring that the ideas become part of a common ground for collaborative work. By contrast, collaborative inquiry work *requires* that students attend to one another's thinking. As Schwartz (1999) notes, "for collaboration to occur, it is necessary for the collaborators to have a model of one another's thoughts and ideally for the collaborators to have a shared set of models" (p. 200).

Another important component is that common ground is defined as that which an individual *assumes others* hold in common with her. This points to the problematic nature of accomplishing joint activities, such as collaboration, as each participant's actions are based on assumptions about what comprises common ground. These assumptions are essentially inferences made based on the available evidence, and they may or may not be valid. It is this common ground—an assumed common ground—that is acted upon as collaborative inquiry work proceeds. This places certain demands on a teaching environment, to which the teacher must attend.

In her teaching, Ms. Nelson constantly directed her efforts towards establishing and maintaining a common ground. Common ground provided the foundation for the class's communications and kept students positioned to contribute to the inquiry activities and further their mathematical understandings. In the remainder of this paper, I focus on particular instructional strategies that helped establish and monitor a common ground among students comprising their mathematical ideas.

Teaching to Support a Common Ground

Three categories of pedagogical strategies that Ms. Nelson used to organize a common ground were identified. I describe each of these, simultaneously offering an analytic description of her practice and revealing the function of each of these categories in supporting common ground and hence collaborative inquiry.

Establishing a shared intellectual context comprising student ideas

As noted above, common ground comprising students' ideas, shared among the students, provides the basis for collaborative work. Ms. Nelson also organized classroom exchanges and activities so that students were regularly afforded multiple opportunities to access and make sense of other students' ideas. Two primary characteristics of classroom exchanges that supported a shared intellectual context were: *repetition* (multiple statements) and *the use of multiple modes of representing* the ideas (e.g., visual symbolic, visual graphical, auditory), which are both forms of redundancy. Like many teachers, Ms. Nelson constantly invited students to share their ideas and explain their answers. She also attended to how these ideas were presented publicly so that they became part of a common ground. Students were regularly asked to "go up" to the board to explain their idea while making a record of it, or to point to an extant diagram or representation to facilitate their explanations. In addition, Ms. Nelson frequently had students repeat their ideas, or repeated them herself, verbatim. This public display and repetition of ideas made students' thinking more available to others for collaborative work. Furthermore, a public record of a contribution provides strong evidence that it is part of common ground and is reasonable to assume that others are aware of the idea. Visual representations were particularly important in supporting common ground. They were enduring records of students' ideas and the relationships among the ideas. Contrast this with the ephemeral nature of words.

A subtle but important aspect of Ms. Nelson's work was that, as she directed the sharing of students' ideas, she actively strove to make the students' *thinking* and *ways of reasoning* available to others in the class. For example, during one lesson, students were asked to find sets of three values that summed to 90. Frank had put his solution up on the board. Zoe wanted to explain how she found a new set of three values by manipulating a set of values Frank had through a process of adding and subtracting so as to maintain equivalence. In doing so, she wrote over and partially erased Frank's original values. Ms. Nelson asked her to "show how" she changed the values and prompted her to rewrite Frank's values so the class could follow her thinking. Thus both results and thinking processes were communicated and recorded, to the extent possible. This attention to making students' thinking visible opened more space for collaboration. It provided students with insight into the presenter's thinking processes, creating a broader basis for the collaborative work and more opportunities to consider others' reasoning and to discuss, reflect on, and critically examine various methods and approaches.

Maintaining continuity

Another important aspect of Ms. Nelson's work to support collaborative learning practices was *maintaining continuity* of common ground over time. As a common ground accumulates, it must remain shared among the participants in the class. Such a requirement places particular demands on the teacher and how instruction is organized. Students differ in the background knowledge they bring to a task, the rate at which they process information, and how they come to understand an idea, to name but a few characteristics. The teacher then must employ strategies to accommodate the array of students in her class. She needs to maintain an eye on the collective and the individuals within the collective, keeping students in a position to contribute to the discourse as it evolves over time. To be sure, it is impossible to achieve complete alignment among students, and indeed, joint activity generally proceeds without complete mutual understanding (Clark, 1996). However, there needs to be a sufficient basis for continued work.

To maintain continuity, Ms. Nelson proactively monitored the common ground. She elicited evidence that students had some common basis for problem solving and pursuing indicators of misalignment. She asked students to restate their contributions, slowed discussions to linger on an idea, and asked for clarification and elaboration as necessary. It is easy for students to become misaligned, which inhibits communication and jeopardizes collaboration. Often teachers gloss over indications that students are not following a line of reasoning. Edwards and Mercer (1987) attribute this to the practicalities of joint action in a classroom that get prioritized in an effort to sustain the teacher's agenda or make progress towards the completion of an activity. Even if a result or outcome is subsequently explained, students have lost an opportunity to engage with the mathematical thinking and reasoning that went into the development of the idea.

Ms. Nelson used a variety of *coordinating mechanisms* (Clark, 1996) to support continued accumulation of common ground. She regularly emphasized the purpose of the current activity, identifying the role of a particular student's contribution or positioning an event with respect to the overarching goal of their collaboration. An awareness of purpose supports collaborative work as it provides additional information to help students make sense of the new contribution and to understand how it is a contribution. It also aligns students' efforts in considering the same issues, attending to the same problem, and contributing their energies towards a common goal. For example, as students went to the board, Ms. Nelson often explained, reiterated, or clarified the purpose of the student's upcoming participation. As one instance, Ms. Nelson remarked, "Come up, please," to Ron and then continued, addressing the class: "Ron says that there are more diagonal lines. That Oscar didn't put enough in". These kinds of pedagogical moves helped students make sense of new contributions and created links between what was about to take place and the work they had already done. Similarly, she highlighted what they had figured out and the current state of their inquiry. Some of the above strategies (e.g., board use) also serve as coordinating mechanisms, aligning students' attention and participation.

In other reform-oriented classrooms, I have often observed teachers attending to making an idea shared only *after* it has been developed by one or two students. Thus there is a period of misalignment (discontinuity), followed by work to establish the result as part of a common ground. This approach certainly has educative value. However, it has two important limitations. Mathematically, it affords students access to the final product only and not the opportunity to consider some more subtle points that went into the development and refinement of the idea. This affects collaborative activity. During this period of misalignment, only a small number of students are positioned to contribute, thus undermining the class's collaborative inquiry as a collective activity.

Developing students' awareness of common ground

Collaborative exchanges are greatly facilitated when students recognize that part of their role is to attend and respond to others' thinking, and share ideas in a manner that others can understand—in other words, when they recognize they are contributing to a common ground. Collaboration is further supported when students are aware that others might hold alternate ideas and not share the same background knowledge and understandings and thus target their explanations accordingly. Hall and Rubin (1998) capture this orientation towards others, with its attentiveness to audience, with the concept *explanatory empathy* (p. 223). Explanatory empathy they argue is needed for, as well as developed through, collaborative work, and can prompt students to consider their ideas from a different perspective (namely that of their audience).

The students in Ms. Nelson's class participated in ways that demonstrated explanatory empathy and an attention to a common ground. They referenced one another's ideas and willingly shared partial ideas with the expectation that someone else might build on or extend their thinking. As students explained and justified their ideas, they took as part of their responsibility to explain clearly enough so that they could be understood by their peers. Not surprisingly, this is something Ms. Nelson deliberately worked to develop over time. For example, she often set up the possibility(s) for the next interaction to explicitly encourage continued attention to a particular student's idea. For example, in response to Ken's raised hand as Jay struggled to articulate his idea, Ms. Nelson responded: "OK, do you wanna explain some more, Ken? Or do you have a question for Jay?" At the beginning of the year, she also frequently highlighted for students when their attention to others' ideas and ways of thinking helped to promote the class's understanding, thus reinforcing the value of this kind of work. Through her actions, she fostered norms that facilitated students' attentiveness to one another's ideas. These norms, however, did not preclude the need for Ms. Nelson to continue to actively support student-student exchanges.

Conclusions and Implications for Practice

In this paper, I proposed the establishment and maintenance of a common ground among students, comprising their ideas, as an important aspect of the teacher's role in organizing collaborative inquiry activities among students. I focused on the instructional practices of one highly accomplished teacher as she worked to establish ideas as part of a common ground, maintain continuity during the development of ideas (thus keeping students' positioned to collaborate), and develop students' awareness of common ground.

Common ground may be useful in providing conceptual guidance for teachers as they work to engage students in collaborative inquiry learning activities. It offers a focus for the teacher's pedagogical moves. Although reform documents often position the teacher as a *facilitator* in collaborative classrooms, little direction is given for what that looks like or how it is accomplished (Chazan & Ball, 1999). Frequently this role is interpreted to mean taking a somewhat laissez-faire approach in the classroom. Focusing on establishing and maintaining a common ground may be a productive way for teachers to think about and approach teaching. It offers a very active conceptualization for the role for the teacher. It further streamlines the potential demands placed on the teacher in supporting a collaborative inquiry learning environment. Rather than require attention to 25 individual students' ways of thinking, and how to coordinate among them, the teacher is required to attend to a single common ground and organizes her efforts around establishing and maintaining common ground, shared among students, comprising their ideas and ways of thinking. Finally, the characteristics of common ground as a dynamic, fluid construct that accumulates over time and one that is based on mutual assumptions help emphasize the need to find ways to coordinate students and organize instruction to support the ongoing development of a common ground and thus collaborative inquiry.

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STUDENTS' USE OF STANDARD ALGORITHMS FOR SOLVING LINEAR EQUATIONS

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This paper describes a study in which students' use of standard algorithms for solving linear equations were investigated. Of interest was whether students would discover a standard algorithm in this domain on their own, and whether the rate of discovery varied for two instructional interventions. 130 6th grade students participated in a week-long 'math camp' where they learned symbolic methods for solving equations. In a 2x2 design, some students were asked to re-solve previously completed problems using a different strategy, and other students were shown a brief demonstration of worked-out examples of solved equations. Results suggest the 'solve it another way' intervention was successful in encouraging students to discover and use a variety of equation-solving strategies. However, viewing a demonstration of a standard algorithm did not increase the chances that students would use the demonstrated algorithm on the post-test.

Doing mathematics frequently involves the use of algorithms. Knowledge of algorithms is an important component of our educational goals for US mathematics students at all levels (Morrow & Kenney, 1998; National Research Council, 2001). Our goals with respect to algorithms are for students to not only know how to use certain standard algorithmic methods but also to know why algorithms are effective and/or efficient (Dowker, 1997; Dowker, Flood, Griffiths, Harriss, & Hook, 1996), to understand why algorithms work (Morrow & Kenney, 1998; Skemp, 1978), and to be able to select from a variety of known algorithms, depending on which is most appropriate for a given problem (Star, 2001, 2002, 2004; Star, in press). This last capacity, which I refer to as *procedural flexibility*, is the focus of the present work. A common finding on national and international assessments is that too many US students lack procedural flexibility; too many students only know algorithms by rote and thus have difficulty when faced with novel or unfamiliar problems (e.g., Beaton, Mullis, Martin, Gonzales, Kelly, & Smith, 1996; Schmidt, McKnight, Cogan, Jakwerth, & Houang, 1999).

Curricula and teachers at all levels have been challenged in the attempt to focus on the development of procedural flexibility. Current reform documents suggest that this outcome may be achieved by the use of invented algorithms and/or the delaying of (or elimination of) explicit direct instruction on standard algorithms (Lampert, 1986, 1992a, 1992b; National Council of Teachers of Mathematics, 2000). Research in the elementary grades has shown that this approach has great promise (Carpenter & Moser, 1984; Carroll, 2000).

However, at the middle and high school level, less emphasis has been given to the learning and teaching of algorithms and/or the development of procedural flexibility (Morrow & Kenney, 1998; Star, in press). The relative lack of focus on procedural knowledge at the high school is particularly surprising, given the increased prevalence of algorithms in the high school mathematics content areas, the presence of longer and more complex algorithms in high school, and students' historical difficulties in developing understanding of the algorithms typically used in high school algebra.

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One possible explanation for the (relative) lack of attention to students' learning of algorithms is that the commitment to the use of non-standard, alternative algorithms and the sharing and comparing of standard and non-standard problem-solving strategies may not be as widespread or robust at the middle and high school level as at the elementary school level. Furthermore, in elementary school it is widely believed that there is value in letting students discover standard and non-standard algorithm, as opposed to receiving direct instruction on the use of such algorithms (e.g., Carpenter, Franke, Jacobs, Fennema, & Empson, 1998). It is unclear whether the benefits of discovery of algorithms extend to high school topics.

This paper describes a small exploratory study that intended to investigate some of the issues. Students with little or no prior knowledge of symbolic methods for solving linear equations were allowed to explore problem-solving strategies. The research sought to document the kinds of symbolic approaches that these students discovered. Specifically, we investigated the impact of two instructional interventions (in a 2x2 design). In the first intervention (the "alternative ordering" paradigm), some students were asked to re-solve problems that they had previously completed, but using a different ordering of steps (Star, 2001). In the second treatment (direct strategy instruction), some students were shown a demonstration of several worked-out examples.

Elsewhere we have reported on this data (Star, 2004; Star & Madnani, 2004; Star & Rittle-Johnson, 2005); in this paper, we address the following questions. First, do students discover a standard algorithm for solving linear equations when allowed to work largely on their own? Second, is there a difference in the discovery of and use of a standard algorithm for either of the interventions? Each of these questions speaks to the development of procedural flexibility – how and when students develop multiple approaches to solving equations, and how and why students choose to use particular strategies on certain problems.

Method

During the summer after 6th grade, 130 students (82 girls, 48 boys) volunteered to attend a 'math camp' where they worked on a series of linear equations. All students attended one-hour sessions for five consecutive days (Monday through Friday) during the summer. Students sat individually at tables for the duration of each session, which was held in a large seminar room on the campus of a local university.

A pretest was administered to all students during the first session on Monday. Students were given 10 minutes for the pretest; all students finished in the allotted time. The pretest assessed students' knowledge of formal linear equation solving strategies. Immediately following the pretest, students received a scripted 30-minute benchmark lesson on linear equation-solving transformations. In this lesson, students were introduced to the four basic transformations used to solve linear equations: combining like variable or constant terms (COMBINE), using the distributive property (EXPAND), adding or subtracting a constant or variable term to both sides of an equation (MOVE), and multiplying or dividing both sides of an equation by a constant (DIVIDE). Students were not given any strategic instruction on how transformations could be chained together to solve an equation. Rather, the focus in instruction was strictly on pattern recognition: identifying which transformation could be used for particular patterns of symbols, and how that transformation was correctly applied. For example, students were shown the symbols, $2x + 3x$, and instructed that the COMBINE step allowed this expression to be rewritten as $5x$. The lesson provided students both guided and independent practice at using each of the four transformations.

For the next three one-hour problem-solving sessions (Tuesday, Wednesday, and Thursday), students worked individually at their own pace through a booklet containing 31 linear equations to be solved. The booklet was divided into three sections, and students completed only the problems in one section during each class. If a student finished the day's problems before the end of the class, he/she was instructed to close his/her booklet and sit quietly.

If a student became stuck while attempting a problem, a helper answered the student's questions in a semi-standardized format. Specifically, the helper corrected the student's arithmetic mistakes (e.g., if the student multiplied 2 by 3 and got 5, the helper pointed out this error) or reminded him/her of the four possible transformations and how each was used. Helpers never gave strategic advice to students, such as suggesting which transformation to apply next or whether one method of solution was any better than another method. Students independently came up with their own choices for which transformations to apply.

Students were randomly assigned to condition. Students in the alternative-ordering condition were given a problem that they had just solved and were asked to re-solve it using a different ordering of steps. Instead of solving the same problem twice, students in the single-solution condition completed a different but isomorphic problem. For example, all students solved the problem, $4x + 10 = 2x + 16$. Students in the alternative-ordering condition were given this same problem again and were asked to solve it using a different ordering of steps. The students in the single-solution condition were given a structurally equivalent problem, $6x +9 =3x + 12$, instead. After solving problems on their own for one session, students in the direct strategy instruction conditions received a brief, eight-minute period of strategy instruction. The researcher solved three equations on a blackboard, using the strategy that led to an efficient solution to each problem. Students were not given a justification for why a particular strategy was chosen.

During the final session on Friday, students completed a posttest. Students were given 50 minutes to take the posttest.

Analysis

A review of widely-used and traditional US algebra textbooks (e.g., Dolciani, Swanson, & Graham, 1992; Foerster, 1990) indicated that a standard (and often explicitly-taught) algorithm for solving linear equations consists of the following steps. First, EXPAND or distribute any terms of the form a(bx + c). Second, COMBINE or combine like variable and constant terms on the left and right sides of the equation, transforming the equation to the form ax + b = cx + d. Third, MOVE or collect variable terms to one side and constant terms to the other side, transforming the equation to the form ax = b. Finally, DIVIDE or divide both sides by the coefficient of the variable term. Students' written solutions were evaluated on whether or not this standard algorithm (SA) was discovered and how frequently it was used on the study post-test. Students' other strategies for solving equations were also categorized.

Results

Use of the SA

With respect to students' discovery of the standard algorithm, participants fell into three groups. First, a small minority of students (referred to as "early users") started using the SA almost immediately. These students did not use the SA on the pre-test, but they did use it by the first time it was possible to assess its use (on a sufficiently complex problem on the first problem-solving day). 9% of students (12 students) fell into this category. There was no evidence that these students knew the SA (or any other symbolic method for solving equations) prior to the study nor did they receive any instruction during the study on any method for solving equations. Yet these students were able to use the SA very quickly. The majority of students – 86 participants (66% of all students; 73% of non-early users) – did not use the SA on the post-test. The remaining 32 students (27% of all participants; 25% of non-early users) discovered and used the SA on the post-test.

As expected, discovery of the SA was a valuable asset in students' problem solving. The use of SA was correlated with higher performance on the post-test. Early users were the top performers on the post-test, averaging 94% correct on post-test equations. SA discoverers averaged 76% correct, while non-SA users averaged only 59% correct. Differences between all groups were significant, $p < .01$.

Effect of Condition on Discovery of SA

Viewing a demonstration of worked-out examples of solved linear equations (including one solved using the SA) did *not* increase the chances that a student would discover and use the SA. 16 of the 32 non-early-user students who used the SA on the post-test saw such a demonstration, while the other 16 did not see a demonstration; of the 86 students who did not discover the SA, 49 viewed a demonstration while 37 did not. In other words, 30% of students who did not see a demonstration of a worked-out example discovered the SA, while only 25% of those who saw a demonstration made this discovery.

With respect to the alternative-ordering treatment (being asked to re-solve previously completed problems but using a different ordering of steps), participating in the treatment *decreased* the likelihood that students would use the SA on post-test problems. Of the 32 students who discovered the SA, only 12 were asked to resolve previously completed problems during the problem-solving sessions. In other words, 19% of treatment students discovered the SA, while 36% of non-treatment students did; this difference is significant, $p < .05$. A closer analysis of students' strategies indicates that students who were asked to generate alternative ways to solve equations during the problem-solving sessions were more likely to use these other, perhaps more innovative ways on the post-test rather than the SA.

Discussion

The goal of the present study was to investigate students' discovery of a standard algorithm for solving linear equations. Standard algorithms offer students an efficient and widely-applicable strategy for solving certain kinds of problems, yet the exclusively reliance on a single strategy such as a standard algorithm is not consistent with student learning goals as articulated by recent policy documents (National Research Council, 2001). In addition, one typical way in which standard algorithms are taught, by an explicit demonstration of the algorithm, is considered by some to be inconsistent with recent pedagogical reforms (National Council of Teachers of Mathematics, 2000). An unexplored issue that is quite relevant to the debate around standard algorithms is whether students discover and use a standard algorithm on their own.

The results of this study indicate that a significant minority (about a quarter) of students discovered the SA for solving linear equations largely on their own. Given that there is little or no data that explicitly explores this issue – whether or not students discover a standard algorithm when left on their own to do so – it is difficult to know whether the proportion of SA users found here should be considered high or low. In favor of a high interpretation, recall that students began the study with little or no prior knowledge of symbolic strategies for solving problems in this domain. Yet many came to use the SA after only three one-hour individual problem-solving sessions, with only a short period of instruction (including no instruction on strategy use) and essentially no feedback. These students initially generated other (often inefficient and idiosyncratic) strategies for solving equations, and they subsequently made efficiency modifications that resulted in the use of the SA. On the other hand, if one believes that the SA is a powerful equation solving tool that all students should know and be able to use, the finding here that only a quarter of students achieved this goal is clearly inadequate. Pure discovery, under this interpretation, is not an appropriate technique for maximizing student learning of the SA.

Results also indicate that using the SA was beneficial to students' problem solving. Those who used the SA on the post-test solved more problems correctly than those who did not. This finding confirms the perhaps intuitive observation that the SA is an important tool for students – it is a reliable and reasonably efficient method that works regardless of problem conditions.

In addition to investigating whether or not students discovered the SA, this study also explored the effectiveness of two instructional interventions on SA usage. The first intervention involved a short demonstration of several problem-solving strategies, including the SA. While one might predict that students who view a demonstration of the SA would be more likely to subsequently use it as compared to those who do not see a demonstration, this was not found to be the case. Seeing a worked out example of the SA did *not* improve the likelihood that students would use the SA on the post-test. One explanation for this finding is that students who received the demonstration had not developed sufficient fluency in the use of the equation solving operators to make sense of the demonstrated strategies. This explanation is consistent with the work of Schwartz and Bransford (1998), who call such fluency differentiated domain knowledge. Schwartz and Bransford (1998) found that when students have developed differentiated domain knowledge, direct instruction is particularly beneficial. In the absence of such basic fluency, students are not able to integrate a demonstration of a worked-out example into their existing knowledge base.

The second instructional intervention investigated here was the alternative ordering paradigm, where students are given a previously-solved problem and are asked to generate another, different solution strategy. This treatment has been found to have a beneficial effect on students' flexibility and conceptual knowledge in prior studies (Star, 2001, 2002, 2004; Star & Rittle-Johnson, 2005; Star & Seifert, 2005a, 2005b). But in the present study this intervention was not found to improve students' ability to discover the SA. We take this non-finding as a suggestion of a possible trade-off in the initial stages of learning between the goal of flexible use of multiple strategies and the goal of mastery of a standard algorithm. It seems that flexibility is fostered by activities such as the alternative ordering task, where students are asked to generate multiple strategies and reflect upon their similarities and differences (Star, 2001; Star & Rittle-Johnson, 2005). However, if one's instructional goal is quick learning of a standard and efficient algorithm, then our results suggest that repeated practice on collections of similar problems might be more effective.

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IMPROVING ON EXPECTATIONS: PRELIMINARY RESULTS FROM USING NETWORK-SUPPORTED FUNCTION-BASED ALGEBRA

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This paper reports preliminary analyses comparing results on the state-administered 8th Grade and 9th Grade algebra Texas Assessment of Knowledge and Skills (TAKS) for a treatment and a control group. The treatment group consisted of 127 students from algebra classes at a highly diverse school in central Texas taught by two relatively new teachers using a network-supported function-based algebra (NFBA) approach as integrated with the ongoing use of an existing school-wide algebra curriculum. The control group was comprised of 99 students taught by two more-senior teachers in the same school using only the school-wide algebra curriculum. The intervention consisted of implementing 20-25 class days worth of NFBA materials over an eleven-week period in the spring of 2005. Because the students were not randomly assigned to the classes, the study is a quasi-experimental design. Using a two sample paired t-Test for means, statistically significant results for the treatment group (p-value one tail = 0.000335 > α= 0.05) were obtained. We can conclude the NFBA intervention was effective in improving outcomes related to learning the algebra objectives assessed on the 9th Grade TAKS.

Introduction

To date, the multiple-strands based approach to curricula promoted by the National Council of Teachers of Mathematics (1989, 2000) has not displaced the single-strand Algebra I course as gatekeeper in the educational system of the United States. If anything, the standard, "stand-alone", Algebra I course is now even more central at many levels, including in state curricula (e.g., minimum course requirements and exit exams) and in nationally administered tests (e.g., the new SAT tests). As a result, improving student outcomes related to the content of the traditional Algebra I curriculum is, perhaps, the single most strongly felt need relative to secondary mathematics education. Given the raised expectations regarding introductory algebra, we look to ask if there are ways of systematically improving on expected student outcomes in ways that move beyond the current overemphasis on addressing performance shortcomings with remediation? Our study looks to move in this direction. As illustrated by the results for our control group, past student performance on state-administered tests tends to be predictive of future testing outcomes. In our effort to identify approaches that are likely to improve expected student outcomes, not maintain them, we compared paired 8th and 9th Grade TAKS results for the students in our study and asked the question: Do the students in our treatment group outperformed their peers in the control group on the algebra objectives tested on the state administered, ninth-grade, Texas Assessment of Knowledge and Skills (TAKS)? Did the network-supported function-based algebra intervention have the effect of improving on expected student outcomes?

Our intervention centered on the use of function-based algebra as supported by generative activity design in a next-generation classroom network technology (i.e., the TI-Navigator 2.0 network combined with classroom sets of TI-84 plus calculators). We call this approach

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network-supported function based algebra (NFBA). After providing some background for our study we report our results. Because the students were not randomly assigned, the study is based on a quasi-experimental design. Using a two sample paired t-Test for means, statistically significant results in outcomes for the treatment group (p-value one tail = $0.000335 > \alpha = 0.05$) were obtained.

Background

There are three strands of analysis that are brought together in framing our study: (1) using function-based algebra (FBA) in a way that speaks more directly to the *structural* aspects of a standard introductory algebra curriculum, (2) situating this version of a function-based approach relative to *generative activity design* as supported by the capabilities of next-generation classroom networks (Stroup, Ares & Hurford, 2005) and (3) explaining our use of performance on previous high stakes mathematics testing to evaluate the effectiveness of the algebra-specific interventions implemented for this study.

Function-Based Algebra Revisited – Emphasizing Mathematical Structure

In ways that highlight the idea of function, affordable technologies like the graphing calculator have long been recognized to have the potential to substantively alter the organization of teaching and learning algebra concepts. Indeed, a number of approaches to pursuing function-based algebra (FBA) are discussed in the research literature (for an overview see Kaput, 1995). Many of these approaches were developed as part of an ambitious, and still ongoing, effort to fundamentally reorganize school-based mathematics to focus on *modeling*. For curricula, this would mean that the formal set-theoretical approaches to defining function that had come to be associated with the "new math" movement would be downplayed and largely replaced by an approach highlighting how functions can be used to model co-variation – i.e., how one variable is related to, or co-varies with, another variable. Computing technologies like the graphing calculator were to support significant engagement with, and movement between, *representations of functions* in symbolic, tabular and graphical forms. Indeed a technology-supported engagement with these "multiple representations" of functional dependencies, especially as situated in motivating "real world" contexts, has come to typify both *what function-based algebra is* and *why function-based algebra it is expected to be effective* with learners.

In the United States this modeling-based approach to FBA informed the development of the "standards-based" mathematics curricula funded by the National Science Foundation and then incorporated into various levels of "systemic reform" initiative also supported by NSF. These systemic reform initiatives, anticipating the language associated with the more recent No Child Left Behind legislation, were to "raise the bar" and "close the gaps" in student performance. The significance of this modeling-focused alignment notwithstanding (e.g., the State Systemic Initiative in Texas played a considerable role in the State-wide adoption of graphing technologies for algebra instruction and assessment), in day-to-day practice a modeling-focused approach to FBA has fallen well short of displacing much of what still constitutes the core of traditional algebra instruction. In part, the feedback from educators seems to be that as powerful as "real world applications" might be in motivating some students, the "bottom line" is that abstractions and formalisms are what continue to be emphasized on standardized exams and thus are what teachers feel considerable pressure to engage. Among school-based educators who are feeling enormous pressure to improve testing outcomes, modeling-based FBA is simply not seen as sufficiently helpful in addressing the core "structural" topics of a standard algebra curriculum. In framing our study, however, it is important to underscore that this perceived shortcoming is *not* a limitation in the potential power of using a function-based approach, but is only a limitation in a particular implementation of FBA that is, itself, principally motivated by the goal of making modeling the overarching focus of school-based mathematics (and far less by improving outcomes related to learning introductory algebra). For our study we take the strong position that while emphasizing modeling should continue to be important, a function-based approach also has enormous potential to improve student understanding of the structural aspects of introductory algebra. To make this point both with teachers and in our materials development, we have found it helpful to advance the following deliberately provocative, but still sincere, claim: *When viewed through the lens of a larger sense of what FBA can be, nearly 70% of a standard introductory algebra curriculum centers on only three big topics*. These three big topics are: *equivalence* (of functions), *equals* (as one kind of comparison of functions), and a systematic engagement with aspects of *the linear function*. This approach, as it is to be investigated in this study, builds on ideas associated with FBA introduced by Schwartz and Yerushalmy (1992) (see also Kline, 1945).

A major strength of this more structurally-focused, function-based algebra is that it allows for a consistent interpretations of both *equivalence* and *equals* in ways that students can use to understand the seeming ambush of “rules for simplifying” and “rules for solving” typically presented early-on in a standard algebra curriculum. If the expression x + x + 3 is *equivalent* to the expression 2x + 3, then the function f(x) = x + x + 3 and the (simplified) function g(x) = 2x + 3, when assigned to Y1 and Y2 on the calculator, will have graphs that are everywhere coincident. They will also have paired values in the tables that are, for any values in the domain, the same. Students will say “the graphs” are “on top of each other.” This “everywhere the same-ness” associated with equivalence then will be readily distinguished from *equals*, as just one kind of comparison of functions. Equals comes to be associated with the value(s) of the independent variable where the given functions intersect (and > is associated with where one function is “above” another; < where one is “below”). The students will understand from looking at the graphs that the function f(x) = 2x and the function g(x) = x + 3 are clearly *not* equivalent (they are not everywhere the same). But there is one value of x where these functions will pair this x with the same y-value (the students will say there is one place where the functions are “equal” or “at the same value”). Graphically, equals is represented as the intersection in a way that is quite general and that readily extends beyond comparisons of linear functions (e.g., $-x^2 + 2x + 8 = x^2 - 4x + 4$).

This distinction between equivalence and equals is helpful because in a standard, non-function based, algebra curriculum rules for simplifying expressions and rules for solving systems of equations are introduced very near each other and, not surprisingly, often become confounded. In addition students will feel like they have no ready way of checking their results, other than asking the teacher. In marked contrast, using a function-based approach, as supported by the use of a combined graphing, tabular and symbolic technology like a graphing calculator, students can readily “see” the difference between these ideas and can use these insights to make sense of results from “grouping like terms” as distinct from “doing the same thing to both sides”. This then allows the students to test their own results, using the technology, for either simplifying or solving. For simplifying they can ask themselves if the resulting simplified function is everywhere “the same” as the given function? For solving systems of linear equations they can ask did their attempts to “do the same thing” to the linear functions on both sides of the equation preserve the solution set (i.e., the x-value at the intersection)? Having students be able to distinguish and make sense of these two core topics in a standard algebra curriculum is significant and illustrates the power of FBA to help with structural aspects of a standard Algebra I curriculum. These ideas were emphasized in the materials we developed.

Of course, a modeling-oriented approach to FBA can be helpful in supporting student understanding of the third of the big three topics: a systematic engagement with aspects of the linear function. But herein we want to continue to illustrate some elements of a less modeling-centric engagement with FBA. As a result, we will illustrate implementing aspects of studying linear functions using generative activity design as supported by new network technologies. The effectiveness of this structural approach to FBA, without network capabilities, has begun to be established (cf., Brawner, 2001). We now move on to consider the role new network technologies can have in further enhancing function-based algebra.

Supporting Generative Design with TI-Navigator 2.0 ™

Briefly, generative design (cf. Stroup, Ares & Hurford, 2005) centers on taking tasks that typically converge to one outcome, e.g., "simplify 2x + 3x", and turning them into tasks where students can create a space of responses, e.g., "create functions that are the same as f(x) = 5x." The same "content" is engaged for these two examples, but with generative design a "space" of diverse ways for students to participate is opened up, and the teacher, based on the responses, can get a "snapshot" of current student understanding (so, for example, if none of the functions the students create to be same as f(x) = 5x involve the use of negative terms, the teacher can see in real time that students may not be confident with negative terms and can use this information to adjust the direction of the class). To illustrate how generative design and NFBA can help with the third of the three core topics in a standard algebra curriculum, we'll briefly sketch some of the activities we used in our intervention.

The Navigator 2.0™ system allows students to move an individual point around on his/her calculator screen and also have the movement of this point, along with the points from all the other students, be projected in front of the class. In one introductory activity students are asked to "move to a place on the calculator screen where the y-value is two times the x-value". There are many places the students can move to in satisfying this rule, and this is what makes the task generative. Often the majority of the points are located in the first quadrant and this gives the teacher some sense of where the students are in terms of confidence with negative x-and y-values. This exploration of a rule for pairing points does describe a function and this approach to creating functions is not dependent on co-variation (indeed, should the teacher want to discuss it, this activity can be used to highlight a set-theoretic approach to defining a function). After observing that "a line" forms in the upfront space, all the points then can be sent back to the students' calculators and can act as "targets" for creating different functions on their calculators (in Y1= , Y2 =, etc.) that include ("go through") these points. Then the students can send up what they consider their "most interesting" functions. A space of often quite interesting equivalent expressions is thereby created and shared in the upfront-space. To further explore ideas related to linear functions, students also can be given a rule like "move to a place where your x-value plus your y-value add up to 2." Again a "line" forms but now when the points are sent back to calculators, the students are pushed to explore ideas related to moving from a linear function in standard form (i.e., x and y summing to 2) to the same function being expressed in slope-intercept form (the form the students must use on the calculators in order to send a function through the points). Again, these and many other structural ideas found in a standard algebra curriculum can be explored using network-supported function-based algebra.

Improving on Expectations

As is mentioned earlier, the intent of the No Child Left Behind Legislation in the United States is to “raise the bar” of what is expected of all students and to “close the gaps” in performance of currently underserved populations. The effort is to be forward looking as higher expectations and measurable progress are to present a tight system of positive feedback in driving demonstrable improvement in educational outcome. Even in a time of heightened political partisanship in the United States, this vision is still seen as compelling and potentially unifying. But as systems theorists (cf., Senge, 1994) are quick to remind us, a challenge in implementing major structural reforms is ensuring that the intended dynamics meant to both characterize and drive the change – in this case positive forms of feedback between raised expectations and measurable outcomes – are not themselves overwhelmed by unanticipated and unintended consequences of what may be well-intending implementation. Relative to learning algebra, one widely used strategy is to preserve the current approaches to teaching algebra and then address shortcomings in student outcomes with remediation. The problem is that remediation, almost by definition, is an inherently *backward* looking and *corrective* strategy. Its role is to fix what is seen as broken, not to drive forward progress. Relative to mathematics education, with more and more effort at each grade level (especially in underperforming schools but also in lower “tracks” in higher performing schools) spent on correcting for past or anticipated shortcomings (e.g., “reviewing” material not mastered from previous years, funding remediation classes during the school year and/or in the summer, or spending considerable class time practicing test-taking skills) attention to proactive strategies (strategies that improve on expected outcomes) is being compromised. From a structural point of view a *positive* feedback loop – like that between raised expectations and measurable progress found at the heart of the NCLB legislation – needs practical *forward looking* and *forward acting* strategies to be effective.

To make the case for NFBA being an example of one such strategy, we look to compare our treatment group outcomes on the 9th Grade TAKS algebra objectives relative to what might be expected based on previous performances on the 8th Grade TAKS.

The Study

Research Question

Does the network-supported function-based approach outlined above improve the performance of the treatment group in statistically significant ways relative to the performance of control group peers?

The Sample

The study participants were 226 students from a diverse high school in central Texas. All the students were enrolled in “non-repeater” (non-remedial) sections of Algebra I and nearly all the students were in ninth grade. Two relatively junior teachers were assigned by the department chair to the experimental group and two more-experienced teachers were assigned to the control group: 127 students were in the treatment group and 99 students were in the control group.

Activities

In their Algebra I class, the treatment groups used a NFBA over nine weeks of instruction in the spring of 2005. The treatment and control groups kept their curricula on the same topics but the experimental group used the NFBA materials, on average, approximately two days a week.

Methods

The raw 8th and 9th grade scores for the State-administered TAKS tests were obtained for the students participating in the study. The 8th grade TAKS was taken before the intervention and the 9th grade TAKS scores for the algebra objectives were collected after the intervention.

Analyses

The raw scores on the 8th grade TAKS and the algebra items on the 9th grade TAKS were converted to percent-correct results. Table 1 and Figure 1 show the comparison of the means for the 8th and 9th grade TAKS for the treatment and control groups.

	Treatment	*Control*
8th GRADE TAKS SCORES	53.8	56.4
9th GRADE TAKS SCORES (Algebra Items)	57.9	56.1

Table 1. Mean TAKS Score Results for Treatment and Control Groups

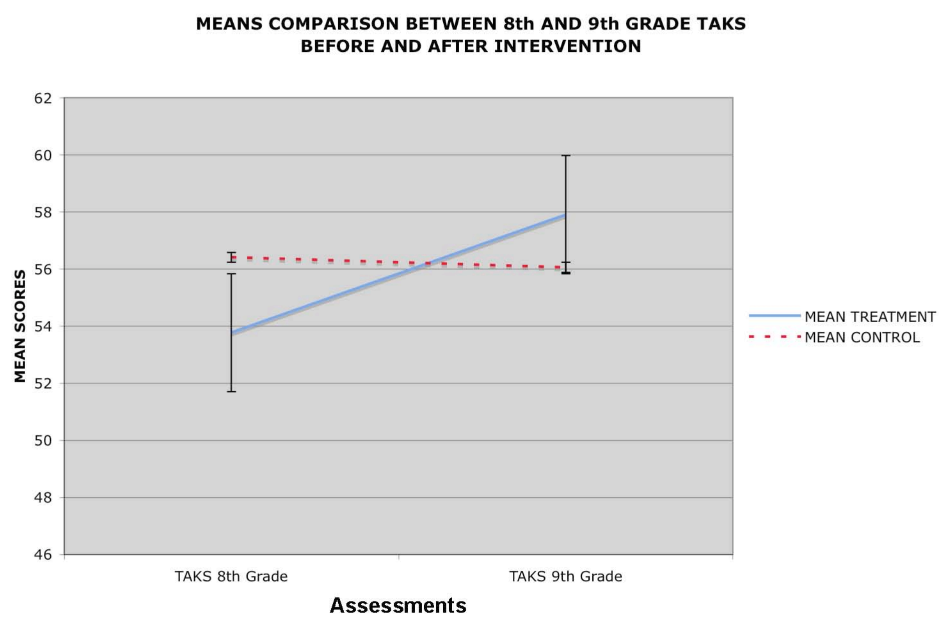

Figure 1. Means Comparison of TAKS Results

We implemented *two* approaches to study changes attributable to the intervention: (1) comparing the student performances between the treatment and control groups first before the intervention (8th Grade TAKS) and then after the intervention (9th Grade TAKS for Algebra Items); (2) comparing the *paired* student performances before and after the intervention for the control group and then the treatment group.

First approach: Comparison Between Treatment and Control Group Results First Before and then After the Intervention

Using this first approach no statistical difference was found between treatment and control groups' results either before or after the intervention. The graph in Figure 1, however, suggests a need for additional analyses. On the graph it is clear that, although no statistically significant differences were found using the given methods, the treatment group started off about 2% lower than the control group on the average 8th grade TAKS scores. Then after the intervention the plot of the 9th grade results shows that the students in the control group maintained almost the same average on the 9th grade TAKS score (the dotted line is almost completely horizontal, showing no change) whereas the treatment group's graph shows appreciable improvement, approximately 4%. This suggests the possibility of comparing *paired* scores before and after intervention, for the control and the treatment groups separately, using a two-sample paired t-Test for the means.

Second approach: Comparison of Paired TAKS Scores Beforeand After Intervention for the Control Group and then for the Treatment Group

We performed a two sample paired t-test for means for the *control group* to look for changes in TAKS scores before and after intervention. As might be suspected from examining graph for the control group in Figure 1, the results of the t-test show no evidence that the means for the control group before and after the intervention are different (p-value one tail = 0.402 > α= 0.05). As a result we can conclude that the students in the control group maintained consistent averages for the 8th grade and 9th grade algebra TAKS scores. There was no statistically significant improvement. This result is consistent with the sense that absent changes in practice, performance in one year is likely to predictive of performance in the next.

When we implemented a two sample paired t-test for the means for the treatment group, the results (p-value one tail = 0.000335 > α= 0.05) provided strong evidence of differences in means before and after intervention. This suggests the students in the treatment group improved significantly in paired results on the 8th and 9th Grade TAKS. Considering that the treatment and control groups were comparable, that no improvement was shown for the paired 8th and 9th grade TAKS scores in the control group, and that improvement was shown for the paired 8th and 9th grade TAKS scores in the treatment group, we have strong evidence to say that this improvement in TAKS scores was an effect of the intervention. Network-supported function-based algebra does appear to have been proactively effective in improving student outcomes.

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CHARACTERIZING THE DEPTH OF PROSPECTIVE SECONDARY MATHEMATICS TEACHERS' KNOWLEDGE

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Existing research suggests the need for improvement in the mathematical understandings of those who teach mathematics (Ball, 1990a; Ball & McDiarmid, 1990b; Cooney, 1999). Although some have characterized the types of knowledge mathematics teachers need (Shulman, 1986, Kennedy, 1998) there does not seem to be much research characterizing the depth of that knowledge. Using weighted mean as our context we identified characteristics of prospective secondary mathematics teachers' mathematical thinking that would help us make distinctions concerning the depth of their knowledge. After analyzing four students' interview tasks, as well as their classroom work, we noticed three characteristics that helped us distinguish the depth of their mathematical conceptions. These characteristics include: 1) Specificity of methods used to solve a problem, 2) Conceptual meaning attached to procedures, 3) Flexibility in meaning attached to mathematical ideas.

Existing research suggests the need for improvement in the mathematical understandings of those who teach mathematics (Ball, 1990a; Ball & McDiarmid, 1990b; Cooney, 1999). Underlying this focus seems to be the hypothesis that higher levels of mathematical knowledge by teachers will have a positive effect on students' mathematical achievement. This hypothesis begs two very important questions which need to be addressed before one can begin to think about drawing a causal link to student achievement: 1) How can the mathematical knowledge of those who teach mathematics be characterized? 2) What types of mathematical knowledge in teachers have the greatest promise for improving mathematics achievement of students?

Although some have attempted to characterize different types of mathematical knowledge needed by a secondary mathematics teacher (Shulman, 1986; Kennedy,1998) there seems to be little research that addresses the two questions we posed. Our research goal was to characterize prospective secondary mathematics teachers' conceptual understanding of weighted mean.

Method

Participants in this study were prospective secondary mathematics teachers who had completed between six and seven college courses in mathematics beyond calculus. These students were enrolled in a mathematics education course the focus of which was enhancing the prospective secondary mathematics teachers' mathematical understandings of data analysis topics. During the class students were asked to reason through tasks involving weighted mean using a range of representations, including the fulcrum of a balance beam (a model in which a meter stick with weights at specific locations was balanced on a fulcrum) and the center of balance for a weighted polygon (a model in which cutouts of polygonal regions were weighted at vertices with plastic beads) (Kranendonk & Witmer, 1998). Also, students were asked to reason about weighted mean tasks for which no physical model was provided. Data sources included photocopies of student work, classroom and small group video recordings, and verbatim,

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annotated transcripts of interviews that were conducted after students had engaged in related activities in class.

Tasks

At the beginning of the interview, the prospective secondary mathematics teachers were given a piece of paper on which was drawn the quadrilateral as shown in Figure 1. The dots at each vertex were used to represent the number of weights placed at each vertex. Each student was then asked to find the balance point of the quadrilateral. A typical student method would be to “collapse” the weights at A and B to the midpoint, in essence keeping the balance point of the quadrilateral the same, doing the same for the weights at C and D, and then repeating the process for the weights at the new positions. A similar task was posed asking students to locate the balance point for a weighted triangle.

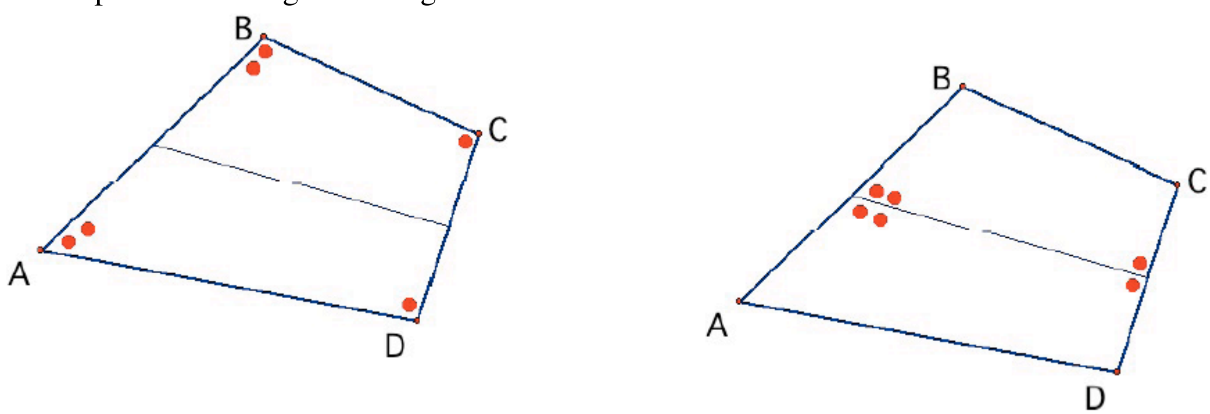

Figure 1. Diagram used in the quadrilateral task. The left side shows the diagram as presented to students and the right side shows the altered diagram after students have collapsed weights along sides AB and CD.

Another task prospective teachers were given was to find the mean for data presented in different graphical representations. Examples of diagrams used in a histogram task and a dot plot task that were used are shown in Figure 2.

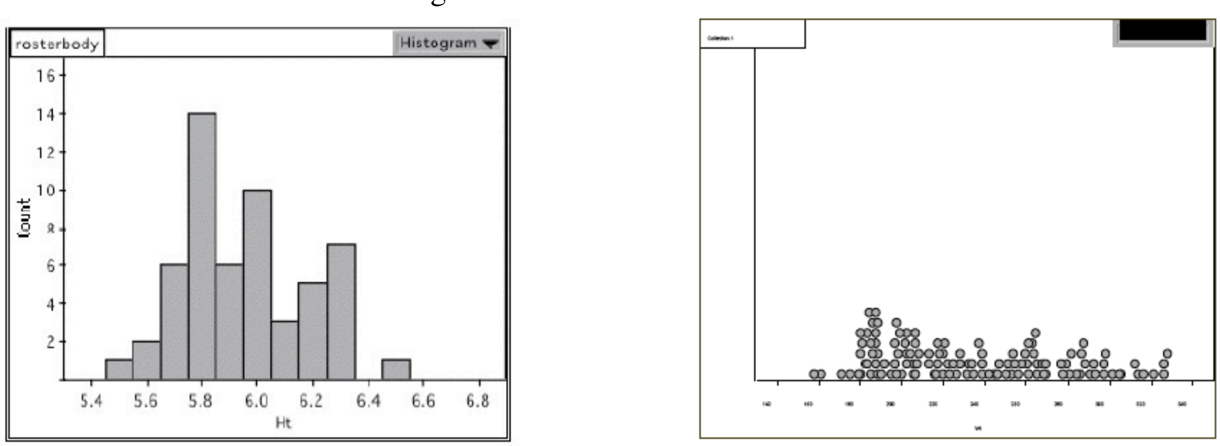

Figure 2. Diagrams used in a histogram task and in a dot plot task. Students were asked to find the mean.

Analysis

We collected interview data from on hour-long interview with each of 16 students in the class. Each interview was videotaped and audiotaped. Verbatim transcripts were produced and annotated with screen shots of computer screens, student writing, and student gestures. We analyzed the data line-by-line searching for evidence of conceptual understandings that underpinned students' approaches to the problems we posed.

Results

The analysis reported in this paper centered on four of the 16 students we interviewed. Their pseudonyms are Jeff, Jenny, Jim, and Niles. We chose these four because they seemed the most willing to share their thinking, and we believed, as a consequence, that they would provide us with the greatest opportunity to characterize understanding of weighted mean. In our analysis of the ways prospective teachers understood weighted mean, several conceptions of weighted mean characterized the understandings of students who completed the interview tasks with ease:

Calculation of weighted mean involves balancing forces in opposing directions;

1. The sum of the products of the weights and the distances from the weighted mean is the same in opposing directions; and
2. The weighted mean for a set of data stays the same when some of the data values change as long as the sum of the directed changes in those data values is zero.

It was our observation that robust understandings of weighted mean were characterized by students' reasoning from these strongly held foundational conceptions of weighted mean. Our question was how students who held and used such robust understandings would differ from those who did not. The following features seemed to characterize the general approaches of students with robust understanding of weighted mean.

These characteristics include:

- Generality of methods used to solve a problem
- Conceptual meaning attached to procedures
- Flexibility in meaning attached to mathematical ideas

Our intention is to define the meaning we have attached to each of these characteristics and to provide contrasting examples of these three that exemplify the differences in prospective secondary teachers' mathematical understandings.

Discussion

Generality of Methods Used to Solve a Problem

One distinction we recognized within the data involved prospective teachers' choice of method for solving a problem. For some of the students, the method chosen worked only for special cases. For example, in the balance beam model one student developed an algorithm that was not conceptually based and was overly specific. The algorithm seemed to have been a serendipitous combination of two methods that had been developed in class. It so happened that this odd combination of methods yielded a correct answer in spite of the lack of a conceptual base. As a consequence, the method did not work when it is extended from the one-dimensional balance beam model to the two-dimensional weighted triangle model. In particular, Jenny used such a method in her work on finding the balance point of the quadrilateral. She moved the weights at A and B to the midpoint of AB so that there were four weights at the midpoint and repeated the same step for segment CD so that there were two weights at the midpoint of CD (see

Figure 1). In the passage shown below she is describing her method for placing the balance point along the segment connecting the midpoints of AB and CD.

268 I need in order to find the balance point I need to find is
269 that the weight times the distance over the side is even
270 with weights times distance on that side [Student uses her
271 left and right hands to show the figure, this side and that
272 side]. So since I have if you will. I can actually write the
273 weight as a fraction also. And I have 4/6 here and I have 2/6
274 here so pretty much what I'm doing I'm getting it to be 1
275 equals 1. So I know that in order to do that I would have to.

276 This like x_1 and x_2 [Student writes down $\frac{4}{6}\frac{x_1}{6} = \frac{x_2}{6}\frac{2}{6}$]

She indicated that x_1 is 2 and x_2 is 4 which satisfied her conception of "1 equals 1" because the fractions on the left and the right side of the equal sign sum to one. Jenny had combined a discussion of the balance beam model with that of the principle, products of the weights and the distances from the weighted mean is the same on both sides of the balance point. Although Jenny verbalized the rule, "the weight times the distance over the side is even with weights times distance on that side" (Jenny, lines 269-270), her symbolic statement in line 276 above seemed to imply that she used neither the weight nor the distance in her formula, borrowing instead from the discussion of proportional distance that occurred during the class discussion of the balance beam model.

As a result of her limited conception, her method did not extend to a different case. When asked to find the balance point of a triangle, Jenny worked with two points at a time, because, she explained, "Well because it's a lot easier because if you try to balance three points at a time it's going to be a lot more complicated" (Jenny, lines 503-505). She attempted to apply her method to a triangle with one weight at two vertices and two weights at one vertex, writing

$\frac{x}{4}\frac{1}{4} = \frac{y}{4}\frac{2}{4} = \frac{1}{4}\frac{x}{4}$ (see Figure 3).

The method fell short because it did not account for the need to 44 44 44balance forces in exactly opposing directions. Jenny's algorithm was not grounded in a conceptual understanding of weighted mean but instead in a malformed rule that did not extend beyond the one-dimensional model from which it was derived.

Figure 3. Jenny's drawing of triangle along with symbolic representation.

In contrast to Jenny, Jim, seems to have a much deeper conception. His method was conceptually based and worked when it was extended to other tasks. The following description is Jim's rationale for the location of the balance point along either segment BC or AD shown in figure 1.

106 Because [25 second pause] I guess the main idea with that
107 is that in order to find any sort of balance point anywhere

you would need to have the distance on one side equal the distance on the other between the block. Like in the example we have here we have two that are three away. Then distance on this side here would be six because we have two at that distance (see Figure 4).

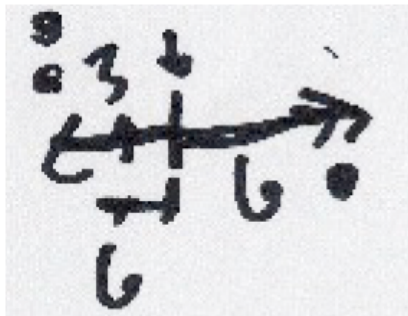

Figure 4. Jeff's depiction of equal distance on both sides.

Jim's method seemed to suggest that he understood that maintaining balance required that forces in opposing directions be the same. On a task later in the interview, Jim generalized this method to place the balance point when there were more than two locations for the weights and in the context of a different representation. In his work to find the mean of a histogram he had constructed (see Figure 5), he indicated that 2.5 would be the mean since the sum of the distance from 2.5 to each of the points to the left was the same as the sum of the distances from 2.5 to each of the points to the right. (Jim, Lines 668-690). An important indicator of Jim's understanding was that this approach, never having been presented in class, seemed to be of his own creation.

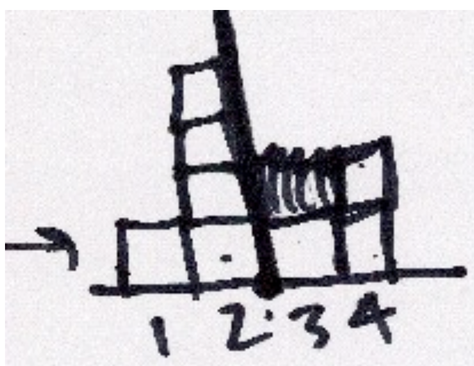

Figure 5. Jeff's histogram

Conceptual meaning attached to procedure

Another distinction we recognized within the data is the extent to which students were able to articulate a conceptual explanation concerning why a particular procedure worked. For example, the concept underlying the procedure of finding the balance point of a balance beam with weights and the procedure of finding the balance point of a weighted polygon is the same. In order to achieve balance the sum of the horizontal components (as well as the sum of the vertical components) of the forces involved must equal zero. The difference is that finding the balance point of a balance beam involves only the horizontal vector component whereas the balance point of a weighted polygon involves both the vertical and horizontal components. For example, Niles, even when pressed for a justification by the interviewer, described the "collapsing" procedure only as a series of steps to be followed. Niles' initial procedure involved moving the weights to the midpoint regardless of the number of weights at each endpoint. His procedure seemed to have evolved from his having encountered only a weighted polygon with equal weights on adjacent vertices. Although he had successfully solved many balance beam problems, his lack of a conceptual base meant that he did not generalize the balance principle to the weighted polygon. This suggested to us that this was simply a procedure and that he did not have a grasp of the underlying concept.

In contrast to Niles who provided no conceptual explanation concerning why the "collapsing" procedure works, both Jim and Jeff provided evidence that suggested they had a conceptual understanding of the procedure. Jim, in describing why one could collapse the points along the edge AB, stated, "…this is like a summary of like the balance point. The smaller balance point and working towards the larger balance point is a summary of A and B. It's a way of representing A and B at one point instead of two" (Jim, lines 457-463). Similarly, Jeff stated, "If you find the points that would balance it would be like the segments with the weights on the end. If you bring the weights onto that point it will stay balanced" (Jeff, lines 106-109) and later stated, "so if you would move your weights back out you should still be able to balance that quadrilateral at that point" (Lines 195-198). All of these statements suggest that Jim and Jeff have a conception that this procedure worked because the balance of the object is maintained through the "collapsing" procedure. Jeff provided further evidence of his conceptual understanding of this procedure by describing its relationship to vectors.

273 It's kind of like where you go back to physics where like
274 with vectors. They act in two directions on the X and Y
275 axis
276 but the result is a vector that is not on the X or Y-axis. Like
277 you have like if you had like a Y-axis going here
278
279 like forces that want to tilt along that axis and forces that
280 want to tilt along the other axis. And then because the
281 weights are different it ends up going like depending on
282 where the weights are it's going to tilt. So it's kind of like
283 you're balancing out the forces that want to make it tilt one
284 way or the other way so that it stays at equilibrium.

Flexibility in Meaning Attached to Mathematical Ideas

Another characteristic that seemed to characterize the depth of a prospective secondary mathematics teacher's thinking involves their ability to see the same mathematical idea in a variety of settings. In other words, their conception was not tied to a particular representation or a specific set of information. Both Niles and Jenny had difficulty connecting procedures for finding weighted mean that involved distance (e.g., finding the fulcrum for a balance beam) to ones that involved location (e.g., finding the weighted mean for a set of values on a number line). For example, Niles, when asked by the interviewer to describe the relationship between the concept of balancing and the procedure of finding the mean of data shown in the dot plot (see appropriate figure) stated, "I want to have an even number of weights to the right of the line as I do on the left…9 times 183 is all these weights times distance would have to equal what I have on the right side and left side" (Niles, lines 886-892). His statement "9 times 183" referred to the location times the number of values at that location, whereas, the "weight times distance" principle to which he referred calls for a distance rather than a location.

Jenny faced a similar difficulty when asked to think about finding the mean weight of two people who weighed 160 and 320 pounds.

> 1288 Like okay since 160 is half the weight of 320 I would say
> 1289 you would have to have two people on 160 to average out
> 1290 the distance. You'd have to have two people at 160 to
> 1291 balance out that one person on 320 if you wanted the
> 1292 distance to be directly in the middle the weight to be
> 1293 directly in the middle.

She proceeded to calculate the average of the three points and seemed dismayed when the result did not match the midpoint of 160 and 320 (Jenny, lines 1343-1345). As with Niles, she reached a point of confusion due to the fact that she placed locations, 160 and 320, into the "weight times distance" principle; a rule in which distance refers to distance from the mean, not location.

On the other hand, Jim showed flexibility in his understanding of a mathematical idea in that he was able to extend his concept to a variety of different settings. When prompted to describe how he could use the "collapsing" strategy he used on the quadrilateral to find the mean of the dot plot (see figure 3) Jim stated, "…you could maybe split these into the first four points. Or we could even make it simpler the first two points" (Jim, lines 912-915). He proceeded to further verbalize his thinking,

> "And then put two weights on wherever the balance point ends up and then balance these two points on that balance point…so like there's one weight and average them and make wherever the balance point two weights and go through that for everything until you find the total" (Jim, lines 927-950).

Jim's verbalization indicated that his conceptualization of the "collapsing" method was not tied to a particular representation. Jim's work throughout the interview provided strong evidence that he was able to articulate how his concept could be applied to a completely different representation and setting.

Conclusion

Our purpose was to identify characteristics and provide examples of prospective secondary mathematics teachers' explanations that were based on deeper mathematical understandings, as well as ones that are not. We described and exemplified three salient characteristics: 1) generality of methods used to solve a problem, 2) conceptual meaning attached to procedures, and 3) flexibility in meaning attached to mathematical ideas. For each of these we have provided examples that represented contrasts in the depth of mathematical understanding associated with each of these characteristics. Our hope is that the characteristics we have identified will serve as a lens through which one could describe the depth of the mathematical knowledge that a prospective or practicing secondary mathematics teacher exhibits in dealing with a particular concept.

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THE RELATIONSHIP OF MATHEMATICS ANXIETY OF ELEMENTARY PRESERVICE TEACHERS WITH MATHEMATICS TEACHER EFFICACY

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The study investigated the relationship between mathematics anxiety and mathematics teacher efficacy among elementary preservice teachers. Participants included 28 elementary preservice teachers at a mid-sized university in the southeastern United States who had just completed a mathematics methods course. Data sources included the Mathematics Anxiety Rating Scale, Mathematics Teaching Efficacy Beliefs Instrument, and interviews. Findings revealed a significant, moderate negative relationship between mathematics anxiety and mathematics teacher efficacy (r = -.440, p < .05). In general, the preservice teachers with the lowest degrees of mathematics anxiety had the highest levels of mathematics teacher efficacy. The interviews indicated that efficaciousness toward mathematics teaching practices, descriptions of mathematics, and basis for mathematics teaching efficacy beliefs were associated with mathematics anxiety.

Introduction

The National Council of Teachers of Mathematics (NCTM) has presented a vision of reform mathematics based upon constructivist approaches that has far-reaching implications for teacher practices in the mathematics classroom (2000). Teachers are the crucial component to the success of the current reform movement in mathematics education (Battista, 1994). Teacher implementation of effective instructional practices in mathematics, such as those prescribed by NCTM, has been linked to level of mathematics anxiety of the teacher (Bush, 1981; Karp, 1991). This leads to particular concern among elementary preservice teachers, as among this population mathematics anxiety is prevalent (Hembree, 1990). Such prevalence causes concerns regarding their teaching effectiveness in mathematics, as well as the potential for passing this anxiety to their students (Buhlman & Young, 1982; Sovchik, 1996; Trice & Ogden, 1986). In response to this problem, studies have been conducted which examine the relationship of mathematics anxiety to other constructs. However, minimal research has been conducted examining mathematics anxiety and its relationship with mathematics teacher efficacy. Research focusing on teacher efficacy has linked efficaciousness of the teacher to classroom instructional strategies, willingness to embrace educational reform, commitment to teaching, and student achievement. Given the implications of mathematics anxiety and teaching efficacy upon significant educational variables, further study should occur which examines the relationship between these two constructs. The following research questions were investigated:

1. What is the relationship between mathematics anxiety and mathematics teacher efficacy among elementary preservice teachers?
2. What are the perceptions of elementary preservice teachers with high and low levels of mathematics anxiety toward their skills and abilities to teach mathematics effectively?

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Literature Review

Mathematics Anxiety

Mathematics anxiety is considered to be more than a dislike towards mathematics (Vinson, 2001). It has been defined as a state of discomfort that occurs when an individual is required to perform mathematically (Wood, 1988), or the feeling of tension, helplessness, or mental disorganization an individual has when required to manipulate numbers and shapes (Richardson & Suinn, 1972; Tobias, 1978). Mathematics anxiety can lead to a very debilitating state of mind and develop into the more serious mathematics avoidance and mathematics phobia (Tobias, 1978).

Studies have indicated that mathematics anxiety has implications for teacher practices in mathematics (Bush, 1981). Teachers with high mathematics anxiety use more traditional teaching methods, such as lecture, and concentrate on teaching basic skills rather than concepts in mathematics. These teachers devote more time to seatwork and whole-class instruction and less time to playing games, problem-solving, small-group instruction, and individualized instruction. These teachers also dominate the mathematics classroom and nurture a dependent atmosphere among students (Karp, 1991). In addition, teachers with high mathematics anxiety avoid teaching mathematics (Trice & Ogden, 1986), as well as perpetuate this negative attitude toward mathematics among their students (Swetman, 1994). Such negative attitudes toward mathematics affect student performances in mathematics (Hembree, 1990; Ma, 1999).

Teacher Efficacy

Teacher efficacy was derived from Bandura’s (1997) conceptualization of self-efficacy which is defined as individuals’ judgments of their capabilities to accomplish certain levels of performance. Using Bandura’s theoretical framework, teacher efficacy is considered by many researchers to be a two-dimensional construct (Enochs, Smith, & Huinker, 2000; Gibson & Dembo, 1984). The first factor, personal teaching efficacy, represents a teacher’s belief in his or her skills and abilities to be an effective teacher. The second factor, teaching outcome expectancy, is a teacher’s belief that effective teaching can bring about student learning regardless of external factors such as home environment, family background, and parental influences.

Teacher efficacy has been correlated to such significant variables as classroom instructional strategies and willingness to embrace innovations. Inservice teachers as well as preservice teachers who have high teacher efficacy use a greater variety of instructional strategies (Riggs & Enochs, 1990; Wenta, 2000). Highly efficacious teachers are more likely to use inquiry and student-centered teaching strategies, while teachers with a low sense of efficacy are more likely to use teacher-directed strategies such as lecture and reading from the text (Czerniak, 1990). Teacher efficacy is also a significant predictor of mathematics instructional strategies (Enon, 1995). In addition, teachers with high teaching efficacy are more likely to try new teaching strategies, particularly techniques that may be difficult to implement and involve risks such as sharing control with students (Hami, Czerniak, & Lumpe, 1996; Riggs & Enochs, 1990).

Although there are many studies concerning teacher efficacy, there is limited research on mathematics teacher efficacy, specifically with preservice teachers. Most of the studies on preservice teachers and mathematics teaching efficacy have examined the effects of a mathematics methods course. Elementary preservice teachers’ participation in a mathematics methods course corresponded to significant increases in mathematics teaching efficacy (Cakiroglu, 2000; Huinker & Madison, 1997; Wenta, 2000).

Methodology

The study involved 28 elementary preservice teachers (26 females and 2 males) at a mid-sized university in the southeastern United States. Of the 28 participants, 86% were between the ages of 18 and 23 years and 14% were 24 to 29 years of age. The participants were enrolled in two sections of a 3-semester hour undergraduate mathematics methods course offered by the Elementary Education department and taught by full-time faculty in that department. Prior to taking the mathematics methods course, the participants had completed nine hours of college mathematics courses, most frequently including Precalculus Algebra and Mathematics for Elementary Teachers. Of the 28 participants, 4 were chosen to participate in individual interviews.

During the last week of classes, the preservice teachers completed two surveys. One was designed to measure the degree of mathematics anxiety – the Mathematics Anxiety Rating Scale (MARS). The MARS is a 98 likert scale item instrument consisting of brief everyday life and academic situations pertaining to mathematics (Suinn, 1972). Richardson and Suinn (1972) reported a test-retest reliability coefficient of .97. Evidence of validity was provided by a study that revealed negative correlations between the MARS and the Differential Aptitude Test (Suinn, Edie, Nicoletti, & Spinellli, 1972). The other survey was designed to measure the level of mathematics teacher efficacy – the Mathematics Teaching Efficacy Beliefs Instrument (MTEBI). The MTEBI consists of 21 likert scale items, 13 on the Personal Mathematics Teaching Efficacy subscale and 8 on the Mathematics Teaching Outcome Expectancy subscale (Enochs, Smith, & Huinker, 2000). The two subscales are consistent with the two-dimensional aspect of teacher efficacy. These items were used for this study with a slight modification of wording. The Personal Mathematics Teaching Efficacy subscale addresses the preservice teachers' beliefs in their individual capabilities to be effective mathematics teachers. The Mathematics Teaching Outcome Expectancy subscale addresses the preservice teachers' beliefs that effective teaching of mathematics can bring about student learning regardless of external factors. Reliability analysis produced an alpha coefficient of .88 for the Personal Mathematics Teaching Efficacy subscale and an alpha coefficient of .75 for the Mathematics Teaching Outcome Expectancy subscale (n = 324). Confirmatory factor analysis indicated that the two subscales are independent, adding to the construct validity of the MTEBI (Enochs, Smith, & Huinker, 2000).

An interview protocol was developed by the researcher based upon the MARS and the Personal Mathematics Teaching Efficacy subscale of the MTEBI. The interview protocol was used to gather in-depth information on the participants' perceptions of their skills and abilities to teach mathematics effectively as well as how their mathematics anxiety may have affected these perceptions. Three experts in the field of mathematics education and research examined the protocol and offered suggestions for improvements, thus establishing content validity.

The participants chosen to participate in the interview portion of the study were the two who scored the highest on the MARS with the highest degree of mathematics anxiety and the two who scored the lowest on the MARS with the lowest degree of mathematics anxiety. The four participants participated in semi-structured interviews within one week of completion of the mathematics methods course. The interviews were conducted at the researcher's office at the university with each interview lasting approximately 45 minutes. The researcher assured the preservice teachers of their confidentiality and obtained consent to tape-record the interviews. The data from the interviews were transcribed from the audiotapes.

In order to determine the relationship between mathematics anxiety and mathematics teacher efficacy, scores obtained on the MARS and MTEBI were analyzed using the Pearson product-moment correlation. In analyzing the data from the interviews, grounded theory was used as prescribed by Strauss and Corbin (1998) through the use of open, axial, and selective coding processes.

Results/Conclusions

The results of the analysis, as indicated on Table 1, reveal a significant, moderate negative relationship between mathematics anxiety and mathematics teacher efficacy among the preservice teachers (r = -.440) when combining both subscales on the MTEBI ($p < .05$). This relationship indicates that the preservice teachers with lower mathematics anxiety generally had higher mathematics teacher efficacy, and the preservice teachers with higher mathematics anxiety generally had lower mathematics teacher efficacy. Furthermore, when analyzing the Personal Mathematics Teaching Efficacy subscale of the MTEBI in relationship to the MARS, the data indicate a significant, moderate negative relationship (r = -.460, $p < .05$). This relationship indicates the preservice teachers with lower levels of mathematics anxiety generally had stronger beliefs in their abilities and skills to be effective mathematics teachers. In addition, the preservice teachers with higher mathematics anxiety had lower judgments of their abilities and skills to be effective mathematics teachers. When analyzing the relationship between the Mathematics Teaching Outcome Expectancy subscale of the MTEBI and the MARS, the data indicate no relationship (r = -.019, $p < .05$). Therefore, there is no significant relationship between the preservice teachers' levels of mathematics anxiety and their beliefs that effective teaching can bring about student learning of mathematics regardless of external factors.

Table 1
*Pearson Product Moment Correlations between Mathematics Teacher Efficacy Scores and Mathematics Anxiety Scores**

MTEBI	MARS total score
Personal mathematics teaching efficacy MTEBI subscale	-.460**
Mathematics teaching outcome expectancy MTEBI subscale	-.019
Both subscales of MTEBI combined	-.440**

*Note: *n = 28.*
***Correlation is significant at the 0.05 level (2-tailed).*

The results from the surveys indicated that the preservice teachers with stronger beliefs in their skills and abilities to teach mathematics effectively generally had lower levels of mathematics anxiety. Existing research conducted by Wenta (2000) involving interviews with elementary preservice teachers suggested a link between mathematics teacher efficacy and mathematics anxiety, and this research is a validation of this negative relationship. However, this study indicated no relationship between mathematics anxiety and preservice teachers' beliefs that effective teaching can bring about student learning of mathematics regardless of external factors. Previous studies have indicated that preservice teachers prior to student teaching often have an unrealistic optimism toward teachers' abilities to overcome negative external influences (Hoy, 2000). Perhaps the preservice teachers in this study, with such limited teaching experience, held this unrealistic optimism which may have influenced the relationship between their mathematics anxiety and mathematics teaching outcome expectancy beliefs. It is also noteworthy that researchers have voiced concerns about the meaning of teaching outcome expectancy beliefs and the interpretation of this factor (Morrell & Caroll, 2003; Tschannen-Moran, Hoy, & Hoy, 1998).

In analyzing the data from the interviews, three themes emerged related to perceptions toward effectiveness in teaching mathematics, which included descriptions of mathematics, basis of efficacy beliefs, and mathematics instructional practices. The preservice teachers, based upon their level of mathematics anxiety, had very different descriptions of mathematics which were linked with their past experiences with mathematics. The preservice teachers with high mathematics anxiety focused upon mathematics in school, such as timed tests and pop quizzes, which imply a memorization of mathematics procedural knowledge. In contrast, the preservice teachers with low mathematics anxiety mentioned a parent who was a positive role model in mathematics and focused upon the processes of mathematics such as problem-solving and brainteasers. Previous studies have indicated that past experiences with mathematics such as negative experiences in the classroom and mathematics presented as a rigid set of rules are contributors to mathematics anxiety (Dossel, 1993; Tobias, 1990). Similarly, the preservice teachers in this study with high degrees of mathematics anxiety emphasized negative experiences within school mathematics as well as mathematics focused on procedures.

The interviews also indicated that all of the preservice teachers expressed positive perceptions regarding their skills and abilities to be effective mathematics teachers, even the preservice teachers with the highest degrees of mathematics anxiety. But they had a different basis for these perceptions, which were either mathematics content-oriented or student-oriented, and were linked with their past experiences with mathematics. Bandura (1986) has asserted that efficacy beliefs are most influenced by an individual's previous experiences. The basis of the preservice teachers' efficacy beliefs with low mathematics anxiety was mathematics content-oriented. Both preservice teachers expressed a confidence in understanding mathematics implicitly associated with their past experiences, which translated into perceiving themselves as effective mathematics teachers. The basis of the preservice teachers' efficacy beliefs with high mathematics anxiety was student-oriented. Both felt a sense of understanding with students who struggle to learn mathematics based upon their own past experiences which led them perceive they would be effective teachers of mathematics. A similar finding was determined by Trujillo and Hadfield (1999) who conducted interviews with five mathematically anxious preservice teachers. The researchers found that despite the preservice teachers' mathematics anxiety, the preservice teachers were confident and optimistic as to the possibility of setting aside their fears in order to develop into effective mathematics teachers.

Regarding mathematics instructional practices, all of the preservice teachers expressed a strong sense of efficaciousness towards the use of real-life situations and manipulatives in mathematics. The NCTM (2000) supports this assertion on the use of authentic situations in mathematics and encourages teachers to "focus on important mathematics - mathematics that will prepare students for continued study and for solving problems in a variety of school, home, and work settings" (pp. 14-15). In addition, the use of manipulatives is consistent with the reform vision of NCTM (2000), and research has indicated the learning occurs best when students have a meaningful context for mathematical knowledge. Previous research indicates that mathematically anxious teachers tend to use more traditional approaches to mathematics instruction (Bush, 1981; Karp, 1991), but the efficacious views of all of the preservice teachers towards authentic situations and manipulatives in mathematics concur with the reform vision of NCTM.

Previous studies have indicated that mathematics methods courses have been effective in reducing mathematics anxiety and building mathematics teacher efficacy among elementary preservice teachers (Cakiroglu, 2000; Huinker & Madison, 1997; Tooke & Lindstrom, 1998; Wenta, 2000). Certainly these two constructs should be addressed in mathematics methods courses, and the results of the interviews in this study seem to suggest that preservice teachers need experiences within mathematics methods courses which address their past experiences with mathematics. Providing a self-awareness of negative experiences with mathematics may be a building block towards reducing mathematics anxiety and increasing mathematics teaching efficacy. It has been suggested that mathematics anxiety may be reduced by addressing these past experiences with mathematics by allowing preservice teachers opportunities to discuss and write about feelings towards mathematics (Furner & Duff, 2002). In addition, Bandura (1986) asserted that in order to build efficacy beliefs, emotional states, such as anxiety, should be addressed. He purported that individuals gauge their degree of efficaciousness by the emotional state they experience as they consider an action. Therefore, addressing the mathematics anxiety of elementary preservice teachers is imperative for building mathematics teaching efficacy. Previous studies have indicated that mathematics methods courses which include the use of manipulatives to make mathematics concepts more concrete, as well as a focus upon developing conceptual knowledge before procedure knowledge, have been effective in reducing mathematics anxiety (Vinson, 2001). Bandura (1986) also asserted that two strong means of building efficacy beliefs are through mastery experiences and vicarious experiences. Most certainly, mathematics methods courses should allow preservice teachers to have mastery experiences through actual mathematics teaching experiences as well as vicarious experiences, such as observing role models teach mathematics.

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AN ANALYSIS OF THE FLOW OF MATHEMATICAL IDEAS AMONG STUDENTS AND GROUPS IN PROBLEM SOLVING

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The flow of mathematical ideas among 17 students working in 3 groups in an open-ended, problem-solving activity over a 14-hour period was analyzed revealing 8 emergent patterns characterized as initial event, interaction, and consequence. We utilize a trace of the development of ideas related to data collection of students engaged in solving a modeling problem to illustrate 4 of these patterns. Factors contributing to the development, retention, or abandonment of currently held ideas, as well as the rejection or acceptance of proposed ideas are considered.

Objective

We report the presentation and development of mathematical ideas in a group problem-solving activity. While examining the flow of these student ideas within and among groups, we identify factors affecting their growth and durability.

Theoretical Framework

Meaningful mathematical activities must necessarily involve the consideration, evaluation, and utilization of mathematical ideas by students. In order for teachers to attend to the representations and thinking processes outwardly manifested during the development of these ideas, it is essential that the teachers listen carefully to the students and be aware of the complexity of the negotiation of such ideas (Davis, 1996). Awareness of students' mathematical ideas enables teachers to present alternative interpretations, challenge students to resolve conflicts, and empower students to build on their own ideas (Maher, Davis, & Alston, 1992). The dynamic exchange of ideas is an integral part of the vision of the National Council of Teachers of Mathematics (NCTM) for school mathematics. It is through the expression and communication of mathematical ideas that these "ideas become objects of reflection, refinement, discussion, and amendment" (NCTM, 2000, p.60).

By analyzing the flow of the mathematical ideas among students and groups, the teacher may gain insight into the problem-solving process and the criteria used by the students when evaluating a new idea. An example of a trace of the flow of ideas is found in Steencken and Maher (2002), in which the authors followed a large group of students working with fractions over many classroom sessions. Videotapes provide a useful tool in that we are able to use several cameras to capture the conversations and activities of the students in real time in a large group, and are not limited by the perspective of a single teacher or observer. Based upon the analysis of the videotapes of each session, the researchers planned follow-up activities and constructed narratives illustrating the growth of the children's understanding of fraction ideas from session to session and day to day. We have advanced the concept of the trace of the flow of ideas and provided a framework in order to follow and present the developing ideas of students videotaped while engaged in an open-ended, problem-solving activity.

Our research is guided by three questions: What representations do the students develop in order to utilize, implement, and communicate mathematical ideas? What factors, if any,

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contributed to the development, retention, or abandonment of an idea currently being utilized? When ideas are presented to others, what factors, if any, contributed to their rejection or acceptance?

A common theme that emerges across the research on collaborative learning is that communication for the sharing of ideas among students is essential (Sfard, 2001). Students' ideas must be presented to others for building upon the ideas to occur. There may be an initial hesitancy on the part of students to 'copy' other students' ideas. However, as a community of learners emerges, the participants recognize that using another's idea is important. The expectation is that the learners will appropriate each other's ideas (Brown & Renshaw, 1999; Roth, 1996). The first step in the appropriation and building upon of the mathematical ideas of others is communication. The presenter of the idea must communicate the essential information of the idea clearly and explicitly, and the receiver must be oriented to the presenter's situation, be prepared to hear at that moment, and respond in a meaningful manner (Baldwin & Garvey, 1973; Barron, 2000, 2003; Davis, 1992; McNair, 1994). Meaningful responses by the active listener may include acceptance of the idea, a request for its clarification, elaboration, or justification. In addition, the listener may repeat the idea to confirm details; she may engage in negotiating the merits of an idea; or she may situate or experiment with it (Barron, 2000; Hung, 2001). A student's response to the presentation of an idea, as well as a group's desire to utilize an idea may be affected by factors including the influence of an outsider or authority figure, the tenacity of its presenter, or by a friendship with the originator (Chick & Watson, 2002). Initial lack of response to or rejection of an idea is characteristic of less successful groups. Nevertheless, students may still listen carefully to ideas unobserved by teachers and researchers. Further, they may be influenced by these ideas and utilize or build upon them at a later time (Barron, 2000, 2003; Davis, Maher, & Martino, 1992; Maher & Martino, 1992).

Method of Inquiry and Data Sources

The data used for this study were collected during a summer mathematics institute for high-school students involved in a longitudinal study of the development of mathematical thinking and proof making (Maher, 2005). Seventeen students participated in a two-week NSF funded Summer Institute[1] to work on several open-ended mathematics problems dealing with modeling growth and movement. They met in a large open area of their local high school's library for four hours each day. Data for this study were collected over a 14-hour period on four of these days (July 7, 8, 9, and 12, 1999) during which time the students were asked to model the growth of a spiral shell by expressing the length of the radii from the center of the spiral to a point on the spiral as a function of the angle of rotation, theta, from the center using polar coordinates (Speiser & Walter, 2004). Figure 1 provides a statement of the problem and the picture of the ammonite shell.

The students were seated in two groups of six and one group of five around three circular tables. They were provided with paper, rulers, and TI-89 graphing calculators. Three cameras, one at each group, and a fourth roving camera recorded the activity of each group. Videotapes, observer notes, student work, and calculator screenshots provide the database for the analysis.

The method of analysis of the data involved the use of grounded theory (Strauss & Corbin, 1998) and the partitioning of the process into smaller steps utilizing the seven non-linear phases described in Powell, Francisco, and Maher (2003). Analysis included the recording of notes for summary purposes leading to the synchronization of events using a central time code (CTC), the identification of mathematical ideas, the construction of narratives tracing these ideas, the coding of the analytic notes and narratives for critical events and transitional phases in the use of ideas, and the recognition of distinct patterns concerning the consideration, development, appropriation, pursuit, and abandonment of specific mathematical ideas.

This task is about a spiral shell. The shell here is a fossil Placenticeras, an ammonite which fell to the bottom of a shallow sea 170 million years ago near what is now Glendive, Montana, and was buried in a mudslide.

Several people can join forces to build a solution together.

1. In your photo, locate the center of the spiral, and then draw a ray from this center, pointing in any direction you like. Having chosen a center and a ray, we can now use polar coordinates to describe the spiral of the shell.

2. Make a table of r as a function of θ, for the spiral of the shell, using the photo and a metric ruler. (Let's measure distances in centimeters and angles in radians.) Based on the information you have gathered, what can you say about r, as a function of θ?

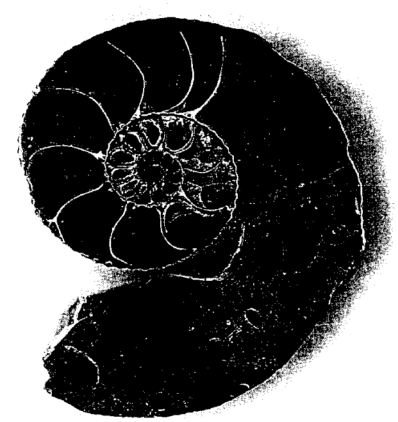

Figure 1. The Placenticeras problem.

Results

The emergence of eight distinct patterns and their analysis became the central focus of the research. We utilize a trace of the development of ideas related to data collection to illustrate four of these patterns.

Wednesday, July 7, 1999

The students in the study were given the statement of the problem and the picture of the Placenticeras and were asked to model the growth of the spiral shell. The students at Table 1, Ankur, Aquisha, Brian, Jeff, Romina, and Shelly began by locating the center of the spiral. Shelly called a researcher over to the table and asked him about "number two" of the statement of the Placenticeras problem. The researcher asked the students, "Have you worked in polar coordinates?" After the students indicated they had not, the researcher drew a ray, the polar axis, from a point, the center. Then the researcher plotted another point, and located the second point by drawing a radius, r, from the center to the point. He labeled an angle, theta, with its vertex at the center, and the radius and polar axis as sides. Following this, the students asked for some guidance as to how to do the problem. The researcher told them to concentrate on the spiral of the shell, and asked Aquisha, "Can you tell me what you're doing?" Aquisha explained how she had made a grid like the x- and y-axes on the picture of the spiral and had drawn a radius to the very end of the spiral (see Figure 2).

The students at the table discussed with the researcher how to answer the question using what Aquisha already had done. Shelly restated the question to the researcher.

Shelly:	So you just want us to see how theta and r relate. Like–
Researcher:	–Yeah. That's the question–
Shelly:	–if theta's one thing, how much would r be?
Researcher:	Yeah, yeah.
Shelly:	Alright, alright.

Jeff: So, then we're doing angles in radians.

Researcher: Sounds like you're ready to run. Yeah. Forward.

The researcher then left the table. Immediately Romina said to the other students, "I still don't get what we're doing." Shelly agreed, "I'm still a little shaky." Jeff explained that they should make a dot on the ammonite and then draw a line out to it, much like Aquisha did, and the researcher had shown when explaining polar coordinates. Shelly said, "We're allowed to move, so I'm going to see what everyone else is doing." Shelly walked to Table 2, and then to Table 3. She returned to her table and told the members of her table that she had found nothing helpful.

Romina asked Milin of Table 2 how he was measuring the angle. Milin said, "I'm just gonna use, like, 360, 720, and then 90, and like, whatever." After talking to Milin, Romina reported to her table, "So they're just going around the spiral." When Brian asked for clarification of what she meant, she did not explain in any more detail.

Aquisha shared her work with the students at Table 1 again. Shelly asked if they could use trigonometric functions to solve the angle. The students pursued the idea of using trigonometric functions with Aquisha's grid for the remainder of Wednesday's session.

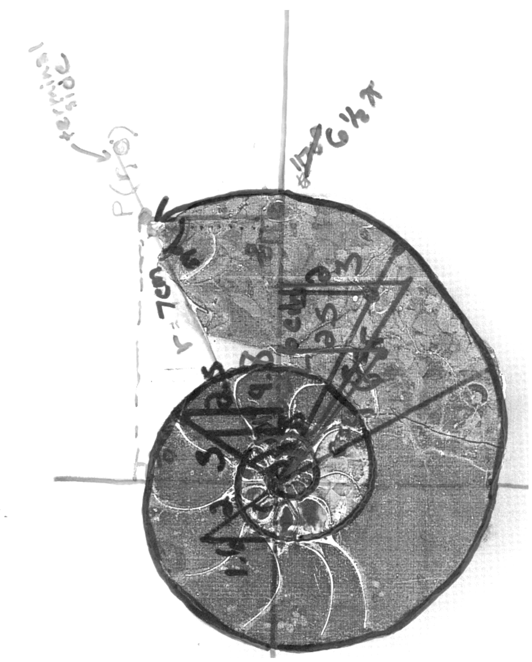

Figure 2. The work of Table 1.

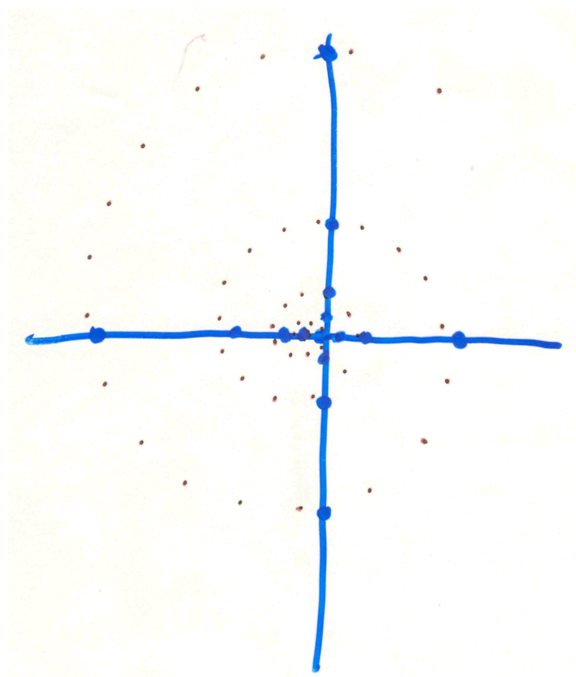

Figure 3. Table 2's 90-degree dot method.

By the end of Wednesday's session, the students at Table 2 had successfully collected all of their data using what we call the 90-degree dot method. Mike had placed a blank overhead transparency over the picture of the spiral and had drawn many dots along the length of the spiral. Milin mentioned that he was not going to use that many dots, but rather would use only the measurements of the radii that corresponded to the points of intersection of the spiral with perpendicular axes that he had superimposed on the spiral as shown in Figure 3. He paired the lengths of these radii with their respective angles of rotation and recorded them in a chart.

Thursday, July 8, 1999

The students at Table 1 continued their work with data collection. They expressed dissatisfaction with their results on three separate occasions. The first time was at 12:25 into the session. After an angle was calculated to use in their data, Romina said, "that doesn't help at all." The second was at 28:26 of the session, when Romina and Jeff decided the radius they were working with would not help them. The third was at 52:40, when Jeff declared, I don't think what we're doing is making any sense."

Meanwhile, Mike at Table 2 had created a quadratic model for the growth of the spiral, which Matt of Table 2 had displayed on the overhead projector screen for all to see. Following Jeff's declaration of lack of success, Shelly visited Table 2, and returned reporting that "[the students at Table 2] used the numbers we have to find an equation." Jeff and Shelly went to Table 2 to learn the 90-degree dot method. Romina initially protested, "we can't, we're not going to do exactly what they did." However, after Brian and Jeff discussed it with her, she agreed to pursue the 90-degree dot method because, in her words, "we're not going anywhere."

Discussion

Four specific examples are found in the above account of the students' development of ideas related to data collection that illustrate patterns that emerge surrounding the flow of mathematical ideas:

1) *Initial event*: Students at Table 1 lacked understanding of the problem and polar coordinates. *Interaction*: Researcher explained the idea of polar coordinates. *Consequence*: The students looked for a way to use the researcher's explanation of polar coordinates in their solution to the problem.
2) *Initial event*: Aquisha placed a grid and radius on the spiral. *Interaction:* Researcher asked Aquisha to explain what she was doing and encouraged her to share with others at the table. *Consequence*: Aquisha continued to pursue her idea, and explained it to the students at her table a second time. After this the students decided to use her idea with trigonometric functions.
3) *Initial event*: Students at Table 1 were developing the idea of using the trigonometric functions with Aquisha's grid. *Interaction*: Students at Table 2 had successfully collected all of their data. *Consequence*: The students at Table 1 continued to develop their idea.
4) *Initial event*: On Thursday, the students at Table 1 were dissatisfied with their progress and expressed a lack of success. *Interaction*: Shelly sought out a successful idea at Table 2 and told the students at Table 1 about the students at Table 2 successfully building a model of growth for the Placenticeras. *Consequence*: The students at Table 1 pursued the 90-degree dot method.

The students developed several representations in the process of collecting data, each further removed from the initial visual representation with which they were presented as shown in Figure 1. The students at Table 1 worked directly with the picture of the Placenticeras as shown in Figure 2. All of their rays, angles and measurements were still drawn upon the photocopy they had received from the researchers. The students at Table 2 first created a representation of the spiral consisting of many dots, and then abstracted the dot spiral reducing the radii to be measured to those that were convenient to the grid superimposed upon the dot spiral. The representation in Figure 3 retained the identity of the spiral while moving away from the physical photocopy. Such a representation was powerful in its simplicity and elegance.

The four examples outlined above illustrate some of the factors that contribute to the development, retention, abandonment, or acceptance of ideas. When the students at Table 1 initially lacked understanding or success, they reached out to a researcher (pattern 1). The students attempted to utilize the new information that the researcher had shared, perhaps due to his role as an outsider with authority (Chick & Watson, 2002). The researcher encouraged Aquisha to share her work with the other students at the table. Such encouragement may have led to her tenacity to share the idea with the students again later, despite an initial lack of response from the other students (Chick & Watson, 2002). Her tenacity resulted in the group integrating Aquisha's idea with Shelly's idea to use trigonometric functions to collect data (pattern 2). The use of her idea may also illustrate an example of students hearing an idea, but not building upon it until a later time (Davis, Maher, & Martino, 1992). After the positive feedback of the group towards Aquisha's use of a grid and Shelly's use of trigonometric functions, the students further developed the ideas, illustrating the strengthening of the interest in an idea based upon positive feedback (Barron, 2000, 2003). While the students at Table 1 were involved in the development of this method they did not seek out others' ideas (pattern 3). However, after they had expressed dissatisfaction with their progress and expressed a lack of success with the idea several times, Shelly visited Table 2 seeking a successful idea. The friendships of the members of Table 1 with the members of Table 2 facilitated and added validity to the ideas of the students at Table 2 (Chick & Watson, 2002). As seen in the research of Brown and Renshaw (1999) and Roth (1996), Romina initially declined to 'copy' the idea of the others 'exactly.' However, Shelly's testimonial to the continued success of Table 2 and the lack of success at Table 1 was enough to entice Romina to "do exactly what [Table 2] did," since "we're not going anywhere" (pattern 4).

Conclusions

We report on some of the factors that contributed to the development, retention, abandonment, or acceptance of ideas in students' problem solving. The conditions of the study encouraged students to move freely between and among groups. This movement extended communication and resulted in students' examining the ideas of other groups and reporting back to the initial group. Sometimes, the knowledge gained affirmed earlier choices; other times, it did not and the idea was ignored. The willingness of students to talk about and share ideas gave opportunities for receiving encouragement and positive feedback. The tenacity of students about their ideas gave opportunities for others to consider it at a later time. Clearly, some ideas survive and not others. One explanation is that all ideas do not have the potential for success. For ideas that do, they must be heard, pursued, developed, appropriated, and then implemented. This process explains, at least in part, the durability of potentially good ideas and their growth in individuals and groups.

We also found that dissatisfaction or a lack of success with one's own or another's idea, and/or the success of another's idea contributed to the abandonment of an initial idea and appropriation of another's idea.

There are clearly multiple and complex reasons why some ideas are durable over time. Further research is recommended to determine the extent of these or other influences in the flow of mathematical ideas.

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DISCORDANT IMPLEMENTATION OF MATHEMATICS CURRICULA BY MIDDLE SCHOOL MATHEMATICS TEACHERS

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We report the extent of textbook use by 39 middle school mathematics teachers in six states, 16 utilizing standards-based (NSF-funded) curricula and 23 using publisher-generated curricula. Results indicate that both sets of teachers placed significantly higher emphasis on Number & Operation, often at the expense of other contend strands. Location of topics within a textbook represented an over-simplistic explanation of what mathematics gets taught or omitted. Most teachers using an NSF-funded curriculum taught content intended for students in a different (lower) grade, and both sets of teachers supplemented with skill-building worksheets. Implications for documenting teachers' "fidelity of implementation" are offered.

Background

One of the truisms of teaching in the United States is the autonomy provided to teachers in making decisions for the benefit of their students. US teachers decide the extent to which they utilize curricular materials in planning and implementing mathematics instruction. Numerous studies document teachers' heavy reliance on the mathematics textbooks in what content to teach and when to teach it (Grouws, Smith & Sztajn, 2004; Robitaille & Travers, 1992). Moreover, for many teachers the textbook provides the scope of mathematical topics that students have an opportunity to learn as the content in the book typically determines what mathematical topics will be selected when planning and delivering lessons (Floden, 2002). Given the strong relationship between curricular materials and students' opportunities to learn, there is a need to examine the extent of textbook use by mathematics teachers. In this paper we report findings on extent of textbook use by middle school teachers in implementing mathematics instruction.

Methodology

Sample

During 2003-04, 39 Grade 7 and 8 mathematics teachers from 11 schools (in six states) participated in data collection. School selection was based on the type of middle school mathematics curriculum employed. Specifically, 16 teachers in five schools utilized one of three standards-based curricula developed with funding from NSF (*Connected Mathematics Project*, *Mathematics in Context*, and *MATHThematics*) while 23 teachers in six schools utilized publisher-generated curricula (*Saxon, Glencoe, Houghton Mifflin, Harcourt, Southwestern*). Textbooks developed with funding from NSF differ from publisher-generated textbooks in that they were designed to focus on a smaller set of mathematical topics, treat these topics in depth, and utilize instructional strategies such as hands-on learning and student discussion; publisher-

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generated textbooks were generally organized around 2-page lessons that include worked-out examples and practice sets on a variety of topics.

Procedures and Instruments

Several instruments were used to document teachers' use of curricular materials. An *Initial Teacher Survey* documented teachers' background (e.g., education, teaching experience), beliefs, and practices using mathematics textbooks. *Textbook-Use Diaries* described teachers' use of curricular materials in planning and enacting mathematics instruction during three 10-day intervals (October, January and March). Table-of-Contents Implementation Records (TOC) reported the amount of the textbook "covered" (only the lessons taught directly from the textbook) during the school year and were coded in relation to learning expectations in Principles and Standards for School M*athematics* (NCTM, 2000). *Teacher Interviews* focused on use of curricular materials in deciding *what* to teach and *how* to teach it. *Classroom Observations* of teachers were made three times during the year, were scheduled in advance, and focused on the extent to which the content and presentation of mathematics lessons were influenced by curricular materials.

Results

Analysis of the Written Curriculum

Prior to an analysis of the *enacted curriculum* (i.e., content that is taught), we examined the *written curriculum* (i.e., set of lessons available to teachers in their textbooks). Because the NSF-funded curricula were developed to offer alternatives to publisher-generated textbooks (Trafton, Reys & Wasman, 2001), it was not surprising to find differences between textbooks in terms of the distribution and placement of mathematics content. In particular, curriculum analyses revealed notable differences between the two types of written curricula and the relative emphasis placed on each of four content strands: (a) Number & Operations, (b) Geometry & Measurement, (c) Algebra, and (d) Data Analysis & Probability.

Type of textbook	Number & Operations	Geometry & Measurement	Data Analysis & Probability	Algebra
NSF-funded	20	30	21	30
Publisher-generated	34	23	15	28

Table 1: Mean percent of each type of textbook devoted to four content areas of mathematics.

As noted in Table 1, approximately 20 percent of the lessons in the three NSF-funded middle grades mathematics textbook series focused on Number & Operations in contrast to almost 34 percent of lessons in Publisher-generated textbooks. Analysis of Variance (ANOVA) revealed such differences are significant (p = .0014). Likewise, 30 percent of lessons in the NSF-funded textbooks focused on Geometry & Measurement and this exceeded (p = .00021) similar content emphasis in the Publisher-generated textbooks (difference significant, p = .00021). Likewise, Data Analysis and Probability received a larger portion of pages in the NSF-funded materials. The portion of lessons devoted to Algebra was about the same for both types of textbooks.

Analysis of the Enacted Curriculum

Although teachers reported frequent use of both types of textbooks, they did not necessarily follow their textbooks page-by-page. Nonetheless, the amount the textbook "covered" in one school year was similar. The mean percentage of textbook lessons taught was 60 % for NSF textbook users, 69 % for Publisher-generated users.

Given that both groups of teachers taught approximately 6 or 7 in 10 textbook lessons, they ostensibly made decisions regarding which lessons to teach and which lessons to omit. Figure 1 shows the percent of textbook lessons taught by teachers in the sample. Teachers from both groups reported implementing most of the lessons (about 80%) from the Number & Operations strand. Algebra lessons were second most likely to be presented to students (about 70%) and Geometry lessons third (about 60%). Lessons related to Data Analysis & Probability were least likely to be presented to students, particularly in classrooms using publisher-generated textbooks. Given that, for both curriculum types, fewer lessons in Data Analysis & Probability were available than any other content strand, it is particularly notable that such lessons are also the least likely to be taught.

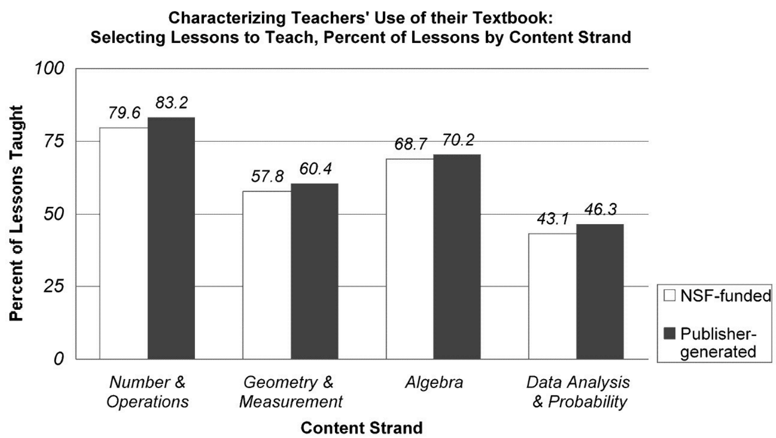

Figure 1: Percent of textbook lessons taught, by content strand.

Teachers' decisions about what to teach (and not to teach) differed from the written curriculum with respect to the distribution of mathematics content. Figure 2 reports approximately 23 % of lessons taught from the NSF-funded textbooks and over 42 % of lessons taught from publisher-generated textbooks focused on Number & Operations, these means are significantly different ($p = .00024$).

For Geometry & Measurement, the 27 % of lessons taught from NSF–funded textbooks exceeded significantly ($p = .0042$) the 20 % for publisher-generated textbooks. Moreover, lessons in Data Analysis & Probability comprised only15 % of the enacted NSF-funded curriculum and 10 % of the publisher-generated curriculum, these are likewise significantly different ($p = .021$). The percent of Algebra lessons taught was not found to be significantly different.

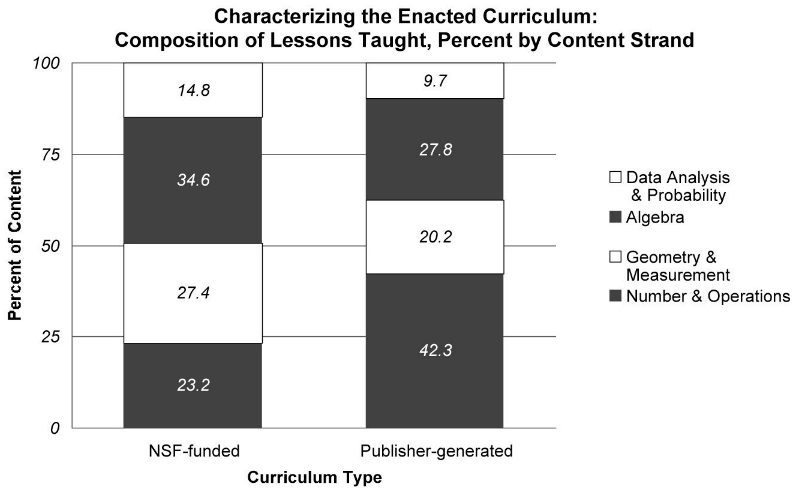

Figure 2: Enacted curriculum, by content strand.

Analysis of Emphasis Indices

The Emphasis Index was calculated to measure the relative emphasis teachers placed on each content strand. To determine the emphasis Teacher A placed on Number & Operations, the percentage of lessons in Number & Operations taught by her was divided by the percentage of Number & Operations lessons in the textbook she used. For example, if Teacher A taught 77 out of 98 (79 %) lessons in her textbook, including 41 of 42 (98 %) lessons in Number & Operations, then her Emphasis Index for Number & Operations is 0.98 ÷ 0.79= 1.24, greater than 1, meaning she placed more emphasis on this content strand than would be expected given the composition of the textbook. An Emphasis Index less than 1 represent less emphasis than would be expected.

In Figure 3, both sets of teachers placed greater emphasis on Number & Operations with mean indices of 1.33 and 1.25 for NSF and publisher-generated textbooks, respectively. Both means are significantly greater than 1 (p = .00081 for NSF-funded textbooks and p < .0001 for publisher-generated textbooks).

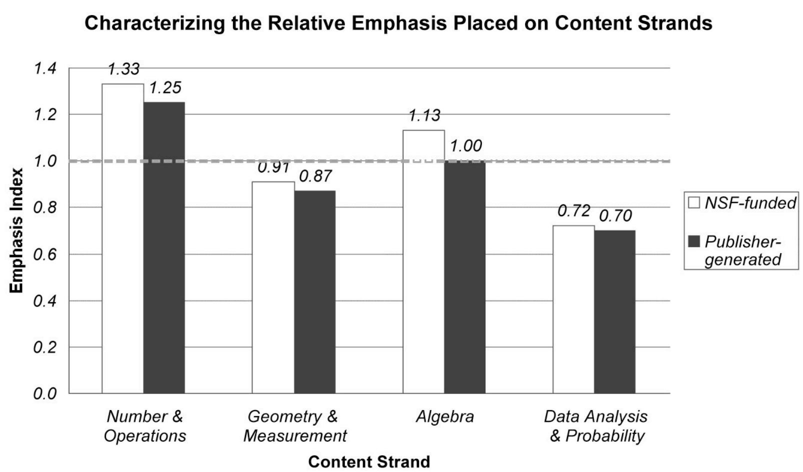

Figure 3: Emphasis Indices for each content strand.

These data are particularly noteworthy for the publisher-generated textbook users whose written curriculum places more emphasis on Number & Operations than on any other content strand. For the users of NSF-funded textbooks, a significantly larger emphasis ($p = .0042$) was placed on Algebra than would be expected given their textbook composition. With reported indices of 0.91 and 0.87, both sets of teachers placed less emphasis on Geometry & Measurement although this index is significantly less than 1 only for publisher-generated textbook users ($p = .0041$). With mean Emphasis Indices of 0.72 and 0.70, the NSF-funded and publisher-generated textbook users placed significantly less emphasis on Data Analysis & Probability ($p = .0025$ and $p = .00073$, respectively).

Analysis of the Omitted Curriculum

Among teachers using publisher-generated curricula, significant differences were observed in the composition of written and omitted curricula. Lessons in Number & Operations comprise almost 34% of publisher-generated textbook lessons but only 16.6% of the omitted curriculum ($p < .0001$). In contrast, lessons in Data Analysis & Probability comprise only 14.74% of the Publisher-generated textbook lessons but represent 29% of the omitted curriculum ($p = .002$); similarly, Geometry & Measurement constituted a significantly higher percentage of the omitted curriculum (29.41%) than of the written curriculum ($p = .027$). Finally, Algebra lessons had approximately 25% of the omitted curriculum for both sets of teachers, but these data were not significantly different than the percentage of Algebra lessons comprising the NSF-funded and Publisher-generated curricula. Indeed, no significant differences were determined for the set of teachers using an NSF-funded curriculum.

Teachers' Selection of Content to Teach, Omit

Teachers' relatively heavy emphasis on Number & Operations, and inversely less emphasis on Geometry & Measurement and Data Analysis & Probability, raises the question, "Is the selection of content to teach or omit a function of its location within the textbook?" Our analysis indicates that the sequence of topics within a given textbook series represents an over-simplistic explanation of what mathematics content gets taught or omitted.

According to recent market share data, *Glencoe Mathematics: Applications and Connections* (Collins et al., 2001) is the most widely used textbook series in middle grades mathematics (Weiss et al., 2003). In our study, 9 teachers in two different states used these series. The selection of content to teach and omit was not uniform.

Figure 4 shows that none of the 9 teachers taught the final several lessons in their textbook. However, 6 of 9 teachers omitted content from the *first* chapter of the textbook and *all* but one teacher omitted content from the first half of their textbook. Teacher 73, for example, did not teach more than 50 % of lessons from the first half of her textbook, this includes *all* eight lessons on Data Analysis & Probability. Teacher 20, only taught 2 of 7 lessons in Geometry & Measurement (Chapter 10) but taught *all* 7 lessons in Number & Operations (Chapter 11). Thus, the relatively emphasis placed on these two content strands cannot be attributed exclusively to their placement within the textbook, or to a school policy, since both teachers taught at the same school.

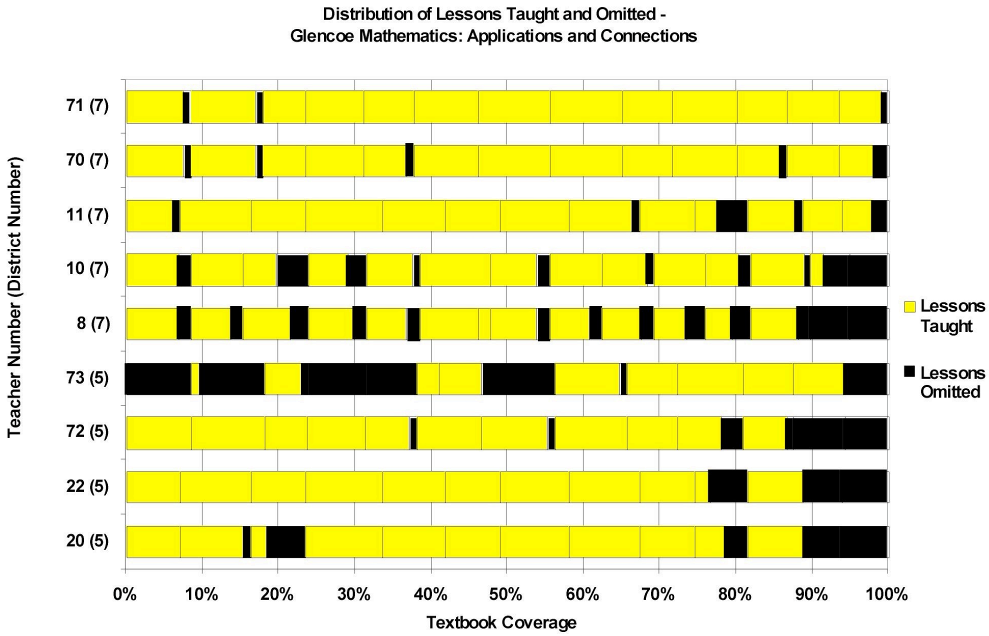

Figure 4: Distribution of Lessons Taught or Omitted by teachers using *Glencoe Mathematics: Applications and Connections* (2001).

Additional evidence of discordant implementation is evident among teachers using other textbook series. In the case of Saxon, two teachers (Grades 7 & 8) taught almost the same percentage of textbook lessons (60%), however their selection of content was remarkably distinct. A first-year teacher taught *all* of the first 76 lessons in the textbook and *none* of its final 56 lessons. In contrast, a veteran teacher taught 71 of 123 lessons but chose them throughout the entire sequence of the textbook. Notwithstanding these differences, both teachers taught most lessons in Number & Operations (80% and 69%) but far fewer in Geometry & Measurement (45.2% and 57.6%). Surprisingly, *all* teachers of *CMP* and *MiC* utilized modules that are intended for students in a different grade. That is, all five Grade 7 teachers (from three districts, in different states) selected modules written for Grade 6. Moreover, all five Grade 8 teachers taught at least one module from Grade 7. The modular organization of these two curricula readily enabled these teachers to "pick and choose" which modules to employ or skip. In contrast, *MATHThematics* is structured in a more traditional, non-modular fashion, perhaps explaining why *none* of its seven teachers (from two districts in two states) taught from a book for a different grade.

Teachers Perception of the Role of the Textbook

Teachers in the study participated in one face-to-face individual interview to explain the role the district-adopted textbook played in their classroom. About half of the teachers indicated that the textbook is a strong determinate of what mathematics is taught and in what order. The other half indicated the strongest determinant of what is taught is the state or district curriculum framework and mandated tests (district or state level) which document student learning associated with the curriculum framework. One teacher said, "Since this is in line with [state standards], it is a 'no-brainer' that this determines what [textbook] we use." Another said, "I base all of my teaching out of the textbook, but the choice of textbook was very deliberate in that we compared it to what our district goals were." Five teachers indicated that they carefully monitored the state/district curriculum as they used the textbook. If there were topics not covered in the textbook then these teachers sought out other resource material to supplement it. One teacher said, "I look at … objectives and then my text and see if there are things that are extra (not in book but on test), then I might supplement and go to other sources."

A few respondents said they relied totally on the textbook. One teacher said, "I use it as a crutch…I'm letting [it] dictate what I teach." One teacher indicated that the district curriculum standards are "somewhat helpful, but kind of vague, and the textbooks are the most concrete, so it is the major influence — the textbooks are really the main thing." Several teachers indicated that, while the textbook guides *what* topics to teach, it has less influence on *how* they teach mathematics. For example, one teacher said, "the text drives what gets taught, but not how it gets taught." On the other hand, over 80% of the teachers interviewed indicated that the textbook is their primary or main resource for planning and teaching mathematics.

Patterns in the responses were similar regardless of the type of textbook used in the 11 districts. Eleven teachers (seven using publisher-generated textbooks and four using NSF-funded textbooks) indicated less reliance on the district-adopted textbook. Most said they use the textbook as an occasional resource for practice problems, homework assignments, or examples. These teachers indicated they draw from their own experience and knowledge in teaching mathematics. Two teachers using NSF-funded materials said the textbook is "helpful" for particular resources such as "real-life approaches," emphasis on conceptual understanding, and promotion of "active learning." Only two teachers in the sample (one using each type of textbook) indicated that the textbook was "not helpful" or played a "small role" in planning or teaching mathematics in her classroom.

Discussion

Mathematical Sciences Education Board (2004) recently stipulated that comparative studies of curricular effectiveness must include reports of "implementation fidelity" including the extent of use of curricular materials. In this study, we found varying degrees of implementation fidelity but despite such variability, both teachers of NSF and Publisher-generated curricula placed disproportionate emphasis on Number & Operations at the expense of other content strands.

The Emphasis indices show that lessons related to Number and Algebra receive greater attention than would be expected based on the distribution of lessons in the textbooks' tables of contents, while the Geometry & Measurement as well as Data and Probability lessons receive the least attention. Such trends may help explain some of the results of the Third International Mathematics and Science Study – Repeat (TIMSS-R). In particular, TIMSS-R reveals that US eighth-grade students' scores in both Geometry *and* Measurement were significantly below the US overall score while their scores in Fractions and Number Sense were significantly higher (Mullis et al., 2000). Such differences in relative performance among content areas may be related to one or more factors, including "emphases in intended curricula or widely used textbooks, strengths or weaknesses in curriculum implementation, and the grade level at which topics are introduced [and] differences in the match between the implemented curriculum and content measured by the test may also be a factor." (Mullis et al., 2000, p. 99).

Interview data revealed that instructional decisions about what to teach are also influenced by state or district curriculum frameworks that emphasize the learning of basic facts and skills. The resulting enacted curriculum differs markedly in its relative emphasis on mathematics content strands. The data suggest that the district-adopted textbook strongly influences both *what* is taught and *how* it is taught to middle school mathematics students. Coupled with the high frequency of textbook use by teachers, these data suggest that textbooks likely impact students' mathematics experience in important ways. Moreover, the type of mathematics textbook being used seemed to matter little in the extent to which it serves as a resource for teachers and students.

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MIDDLE GRADES STUDENTS' PRECONCEPTIONS OF INFINITY

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The main purpose of this research is to investigate middle grades students' preconceptions of infinity through the series of structured activities. The research consists of 2 studies. Student sample for the study-1 included 41 middle grades students selected from different local El Paso, TX school districts. They participated in the intensive summer class on Visual Calculus. Students were tested in their understanding of concepts of infinity and limit at the beginning and at the end of the class. They were also engaged into concept-definition task ("Write your definition of Infinity"). Study-2 focused on middle grades students' concept-images of infinity. 152 6th graders were involved into "Draw Infinity" task. The study found that middle school students have variety of initial ideas (that deserve close consideration) about infinity even though they usually don't start a formal study of infinity until high school or college level pre-calculus course.

Theoretical Framework

The challenge of studying children's preconceptions (initial ideas) of infinity is mostly based on the fact that our intuition of infinity is intrinsically contradictory. In some sense it is counterintuitive because our logic is naturally adapted to finite objects and events (Fishbein, Tirosh, & Hess, 1979, Clegg, 2003, Maor, 1991).

Majority of studies on students' conceptions of infinity and limit were conducted at the high school and college level. The focus of our study is middle school students. "The child's conception of space" by Piaget and Inhelder (1956) is one of the first studies on young children's understanding of infinity even though Piaget's interest was not in infinity per se but in child's conception of continuity and limit. Piaget used tasks such as subdividing a geometrical shape (e.g., a segment) to study children's conception of continuity. With respect to the theory of stages of child's development they found that at the pre-operational stage children could not continue subdivision very far. In the concrete operational stage children could continue a large but finite number of divisions. Only in the formal operational stage, which corresponds to the middle grades level, children are capable to continue the subdivision indefinitely and they can recognize the limit of this process as a point.

In studying children's preconception of infinity an important starting question is: What does it mean to have an initial idea of infinity? To answer this question, first, we need to define what infinity is. A very broad definition could be found in Webster's dictionary: "Infinity is unlimited extent of time, space, or quantity" (Webster's Dictionary, p. 586). So, one can say that the initial idea might be a cognizance of *unlimited process* such as the continuous subdivision of a segment or a cognizance of an *endlessness of a sequences* such as the set of natural numbers. Infinity could be perceived as a *'process'* (continuous subdivision) and as an *'object'* (set of natural numbers is infinite). Infinity could be considered in different contexts: numerical vs. geometric, static vs. dynamic, counting vs. measuring (Monaghan, 2001).

In this study we examine students' preconceptions of infinity from perspectives of students' performance in infinity-related problem solving activities as well as students' concept definitions and concept images of infinity. "The terms concept definition and concept image distinguish

Lloyd, G. M., Wilson, M., Wilkins, J. L. M., & Behm, S. L. (Eds.). (2005).
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between a formal mathematical definition and a person's ideas about a particular mathematical concept, such as function" (Tall & Vinner, 1981). A concept image is a mental structure associated with a concept. It might include mental pictures and cognitive representations associated with properties of the concept. A concept definition is a form of oral or written description used to specify the concept. We used two main instruments to engage students in externalization of their concept images and concept definitions of infinity. The first instrument is 'define infinity' task in a form of survey with the following statement "Write down your definition of infinity." This instrument was used to analyze students' concept definitions of infinity. The second instrument used for examining students' concept images of infinity is 'draw infinity' task.

Research Design

This research consists of two studies. Study-1 dealt with middle grades students' understanding of Calculus concepts and students' concept-definitions of infinity. There were 41 middle grades students from local school districts who participated in Visual Calculus study in summer-2004 (20 students - in Team C, and 21 - in Team E). Both groups received the same content, instruction, and assessment. Students were tested in their understanding of concepts of infinity and limit at the beginning and at the end of the class. Tests were consisted of two conceptual problems each.

Along with the pre-and-post assessment we used student's journal writing as a source for qualitative data collection. We asked students to write down their definitions of infinity/ infinite (concept-definition task). We were interested in what are the middle grades students' concept-definitions of infinity? We were also interested in how students perceive infinity? Do they perceive it as a process or an object? Do they see it as a number or as a concept? What context do they use when they define infinity and limit?

Study-2 examined middle grades students' concept-images of infinity. The sample of the second study included 152 6^{th} grade students from one of the urban El Paso middle schools. The instrument used – "Draw Infinity" (concept-image task).

Data collection and analysis for both studies were performed by 3 independent experts using self-designed coding sheet.

Results and Conclusions

Study-1 had two components: (1) students' learning of calculus concepts; (2) analysis of students' responses on "Write down your definition of infinity" task. There was an improvement in students' performance from the pre-test (Mean = 13.37, SD= 14.94) to posttest (Mean= 58.07, SD=12.38) $F(1, 39) = 313.24$, $p<.01$. The difference between tests was statistically significant $F(1, 39) = 3.79$, $p<.05$. We contribute this improvement to students' active engagement in a sequence of conceptually structured projects. Projects were based on the existing middle grades CMP curriculum (e.g., Cookie Monster problem and its modifications) and were implemented through conceptual discourse and use of multiple representations and connections.

Results of students' performance on "Write down your definition of infinity" task showed that students' primary preconception of infinity is a process (97.6%), something which goes on and on. Examples of this preconception could be the following excerpts from students' journal logs: "Infinity is something that keeps on going forever and ever and it never stops getting bigger or smaller", "Infinite is never ending ever." As you notice students may use different words "infinite" and "infinity" to express the same meaning. "The two words 'infinite' and 'infinity' often appear interchangeable in children's talk. It should not be assumed, though children's usage is often consistent, that 'infinity' always refers to an object and that 'infinite' always refers to a process." (Moreno & Waldegg, 1991, p. 212-213).

Some of the students were able to capture both the process and object views in their definitions. 24.3% of students offered an attempt to present an 'object' view of infinity. One student, for example, wrote "Infinity is a number that no one can reach." Or "My understanding of infinity is that it is a number that adds up to a number that is so high you don't know what it adds up too". Another student attempted to represent infinity in more philosophical way: "It is eternity."

Almost a third of students (31.7%) knew the symbol for infinity: "The symbol for infinity is a horizontal eight -∞ ." However, 4% of students used variety of other symbols including '8' or '~' to represent infinity. Here is another interesting student's perspective on infinity - "Infinity is unknown". There were only few cases (4.8%) where students perceived infinity as a concept "Infinity is a matter of never ending concept." 34.1% used numeric context to define infinity, 65.9% - descriptive/ situational and only 2.4% - geometric context.

Study-2 was entirely focused on 6^{th} graders (n=152) concept-images of infinity using "Draw infinity" task as a main instrument. Table 1 summarizes middle grades students' concept-definitions and concept-images of infinity.

Preconception of infinity	Define Infinity Task %	Draw Infinity Task %
'Process' view of infinity	97.6	72.4
'Object' view of infinity	24.3	28.9
'Context' of infinity • numerical • geometrical • descriptive/ situational • symbolic	 34.1 2.4 65.9 31.7	 31.6 49.3 78.9 26.3
No response	0	2.6

Table 1. Middle grades students' preconceptions of infinity

Results of this study didn't show significant differences in students' performance on concept definition and concept image tasks concerning 'object' view of infinity, using numerical and symbolic contexts to represent infinity. However, we found significant differences regarding the 'process' view of infinity: the number of students using 'process' view dropped (25.4% decrease) on the 'draw infinity' task compare to the 'define infinity' task. There is also a significant increase (46.9%) on number of students using geometric context to represent infinity on the 'draw infinity' task versus the 'define infinity' task. Another difference occurs in using descriptive/ situational context to represent infinity. More students used this context in the 'draw infinity' task than in the 'define infinity' task (the difference is 13%).

Compare to the 'define infinity' task, in the 'draw infinity' task students used more variations under numeric context (figure 1). Majority of students (48%) preferred to use a set of natural numbers to explain infinity. Other numeric representations included: numeric table (18%), finite large number (20%), infinitely large/ small number (2%), non-terminating repeating decimal (2%), and other numeric representations (10%) such as a bunch of zeros. The study showed that students' concept-images of infinity is content-dependent. For instance, there were some students who used infinitely large number and non-terminating repeating decimal (topics covered in 6th grade mathematics curriculum) to represent infinity. However, none of students used non-terminating non repeating decimal (topic that is not addressed until Algebra-1) to represent infinity.

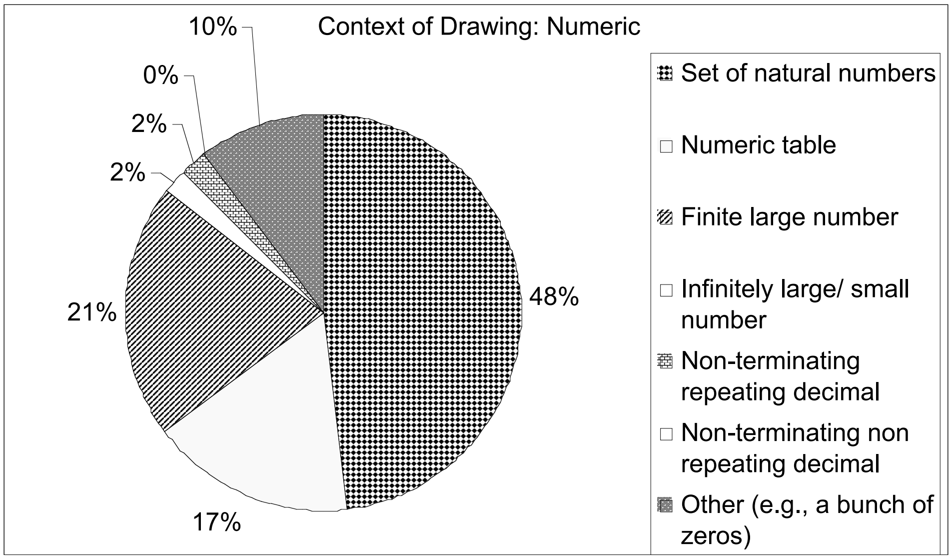

Figure 1. Variations within numeric context of representing infinity

Variations in geometric contexts included: geometric shape, circle, concentric circles/ squares, number line, ray/ arrow, spiral, endless curve/ line, set of objects (dots, circles) (figure 2). It was interesting to observe that students prefer to use circle or circular shapes (including concentric circles and spirals) to other geometric shapes to represent infinity. Number of students used quadratic or rectangular shapes (including concentric squares) to draw infinity. There were very few cases when students used shapes other that circular or rectangular ones.

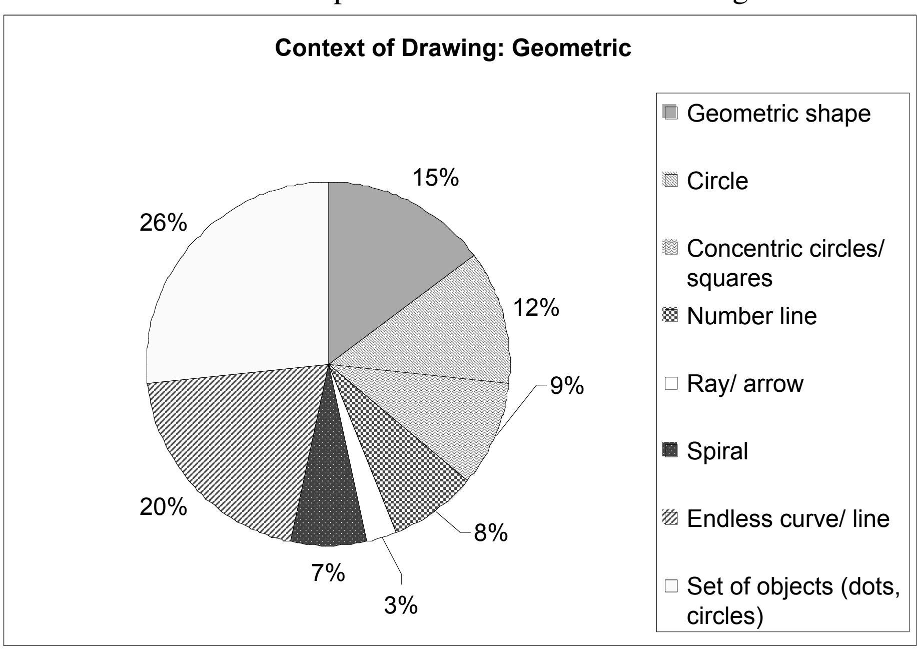

Figure 2. Variations within geometric context of representing infinity

Analysis of students' drawings shows that the increase in number of students using descriptive/ situational context in the 'draw infinity' task is mostly due to the use of science/ space context to visually represent the concept. Students used the context of universe/ galaxy (51%), solar system (16%), and even black hole (3%) as their concept images of infinity (figure 3).

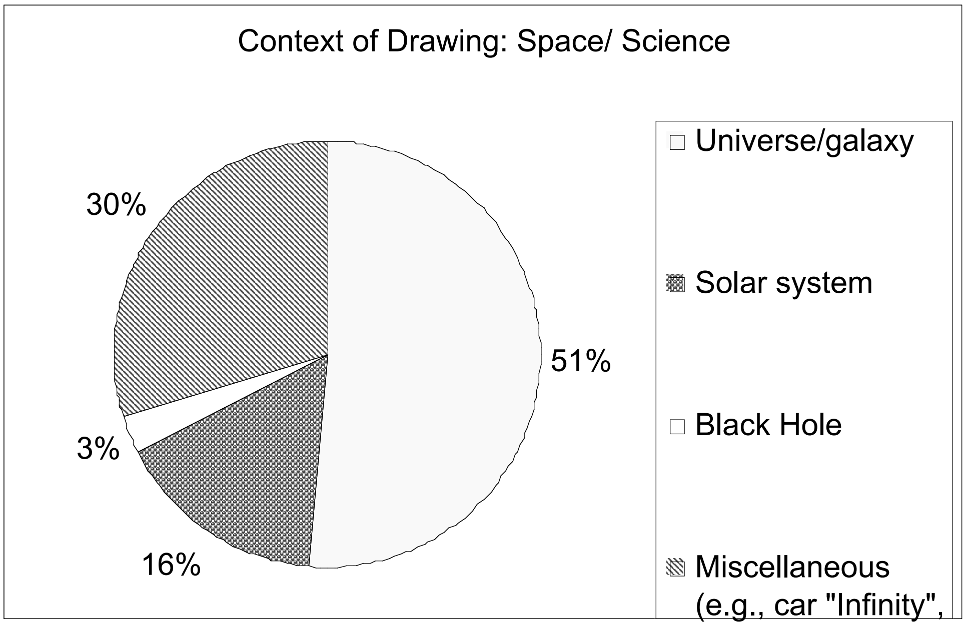

Figure 3. Variations within 'space/ science' context of representing infinity

One may pose a question "Why do we need to know middle grade students' preconceptions of infinity?" In some sense, it is too early to talk about 6th graders' perceptions of infinity since they are going to be formally introduced to infinity at a high school level (grades 10-12). However, it is known that concept images are "built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures" (Tall & Vinner, 1981, p. 151). Therefore, it is important to provide middle school students with relevant experiences to build their concept images of infinity which will help them to understand infinity at a more rigorous level later on.

Infinity Procept: Process-Object Duality of Infinity Concept

We conducted a pilot survey asking different age group students as well as college instructors to answer the question on the "Cookie Monster" problem: "The cookie monster sneaks into the kitchen and eats half of a cookie; on the second day he comes in and eats half of what remains of the cookie from the first day; on the third day he comes in and eats half of what remains from the second day. If the cookie monster continues this process, will he ever eat the entire cookie?"

Surprisingly enough, the responses we have from different groups were similar: about 2/3 of respondents say "No", 1/3 – say "Yes", and there are a very few who responded "Yes and No". Is it a coincidence that this distribution almost matches the process – object distribution in the 'draw infinity' task (72.4% and 28.9%, see table 1)? We can't answer this question until we conduct additional studies on concept images and concept definitions of infinity using different age groups including college instructors. So far, we can draw some hypothetic conclusions.

These conclusions could be based on the theory of 'process-object duality' of mathematics concepts (Dubinsky & McDonald, 2001). Many concepts in mathematics could be viewed both as a process and as an object. For instance, concept of function could be perceived as a process (as computational or procedural aspect of finding an output value for a given input value) and/or an object (as a result of a computational process or as an entity that can be acted upon by other procedures, e.g. differentiation, integration). If we assume that majority of individuals responding "No" to the Cookie Monster problem have predominant 'process' view of infinity then it explains their responses: from the 'ordinal' perception of infinity, the partial sums of the series

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots\frac{1}{2^n}+\ldots$$

will always be less than 1: $\frac{2^n-1}{2^n}<1$ If we assume that "yes – respondents" have the 'object' 2^n view of infinity then their thinking could be based on a 'cardinal' perception of the above series 1 as an object (a sum of an infinite geometric series) which is: $S=\frac{\frac{1}{2}}{1-\frac{1}{2}}=1$

Based on this example, here is a second reason for why do we need to study middle school students' preconceptions of infinity - we need to recognize the preconceptions early and start addressing them so students will have more comprehensive view of fundamental mathematical principles and concepts. If pre-or-miss-conceptions are not recognized and addressed, then students' traditional experiences could easily build strong 'narrow-minded' mental scripts in their cognition that could be harder to remove later. Existing research shows that students enter calculus courses with a broad spectrum of misconceptions about infinity. These misconceptions are still not addressed through the course and "the first year of a calculus course has a negligible effect on students' conceptions of infinity" (Monaghan, 2001, p. 244).

Next reason is concerned with building students' concept images and concept definitions of infinity using research-based approaches on dual nature of mathematical concepts (Sfard, 1991). In a way, understanding a dual nature of mathematical concepts will help students to become flexible when they will face more abstract and complex mathematical ideas. "In order to be able to deal with mathematics flexibly, students need both the process and object views of many concepts, as well as the ability to move between the two views when appropriate" (Selden, 2002). Concept that could be viewed both as a process and an object is called procept (Tall, 1991). We consider a proceptual perspective as a tool to help students at their earlier stages of learning to understand and overcome contradictory and counterintuitive nature of infinity concept.

Finalizing our concluding remarks, the main recommendation from this research is to consider possibilities of recognizing and addressing middle school students' preconceptions of infinity and limit through the infusion of conceptually rich infinity-related discussions and activities into middle school curriculum so the students will be more prepared to understand these concepts in-depth at the high school and college level.

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MODELS OF TEACHING TO PROMOTE MATHEMATICAL MEANING: UNPACKING DISCOURSE IN MIDDLE GRADES MATHEMATICS CLASSES

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This paper reports on the development of models of teaching that were outgrowths of a study on discourse in middle grades mathematics classes. Grounded theory methodology was used to move from the construction of a model of the flow of discourse, to multi-level analysis of relationships of forms of talk and verbal assessment, and, ultimately, to models of teaching that promote discourse on a continuum from univocal (conveying meaning) to dialogic (constructing meaning through dialogue). Three specific cases are highlighted that represent deductive (associated with univocal), inductive (associated with dialogic), and in/deductive (i.e., mixed) models of teaching.

Introduction

Recent reform efforts have identified communication as essential to the teaching and learning of mathematics (NCTM, 2000). However, communication alone cannot guarantee learning—research has demonstrated that the *quality* and *type* of discourse greatly affect its potential for promoting conceptual understanding (Kazemi & Stipek, 2001). Mathematical meaning-making is more frequently associated with *dialogic discourse* (i.e., constructing meaning through give-and-take communication) than with univocal discourse (i.e., one-way transmission of knowledge) (Wertsch & Toma, 1995). A model of the flow of discourse for mediating mathematical meaning within middle grades mathematics classrooms was developed and previously reported (Truxaw & DeFranco, 2004). Since current research supports the conviction that effective *teaching* is a significant, if not *the most* significant indicator of student achievement and success (Darling-Hammond, 2000; Sanders, 1998), this study sought to further uncover the teacher's role in the orchestration of meaningful discourse. Using the model of discourse as a foundation, multi-level analysis of purposively selected cases was undertaken. This paper describes three resulting *models of teaching* that promote discourse on a continuum from univocal to dialogic.

Background

Sociocultural theory, with its contention that higher mental functions derive from social interaction, provides a meaningful framework for analysis and discussion of discourse as a mediating tool in the learning-teaching process. Verbal exchanges between more mature and less mature participants may develop back and forth processes from thought to word and from word to thought that allow learners to move beyond what would be easy for them to grasp on their own (Vygotsky, 2002). For example, in mathematics classrooms, conversations between the teacher and the student may provide mediation that may, in turn, promote mathematical meaning-making. When considering language as a mediator of meaning, it is critical to consider the two main intentions of communication—that is, "to produce a maximally accurate transmission of a message" and "to create a new message in the course of the transmission" (Lotman, 2000, p. 68), characterized as univocal and dialogic discourse, respectively (Wertsch, 1998).

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The model of the flow of discourse for mediating mathematical meaning—that provided the foundation for the current study—was framed by sociocultural theory, but also was built from research related to classroom discourse (Truxaw & DeFranco, 2004; Truxaw, 2004). For example, analysis included attention to basic structures such as moves, exchanges, sequences, and episodes (Lemke, 1989; Mehan, 1985; Sinclair & Coulthard, 1975). The move, exemplified by a question or an answer from one speaker, is recognized as the "smallest building block" (Wells, 1999, p. 236). An exchange is made up of two or three moves and occurs between speakers (typically including initiation, response, and follow-up moves). A sequence contains a single nuclear exchange and any exchanges that are bound to it. The episode is the level above sequence and represents "all the talk that occurs in the performance of activity" (p. 237).

Additionally, socio-linguistic coding strategies were researched, adapted, and applied. To enhance the analysis, forms of talk and verbal assessment within whole group discussion were identified (Truxaw & DeFranco, 2004; Truxaw, 2004) that included the following: *monologic talk* (i.e., involves one speaker—usually the teacher—with no expectation of verbal response), *leading talk* (i.e., occurs when the teacher controls the verbal exchanges, leading students toward the teacher's point of view), *exploratory talk* (i.e., speaking without answers fully intact, analogous to preliminary drafts in writing) (Cazden, 2001), *accountable talk* (i.e., talk that requires accountability to accurate and appropriate knowledge, to rigorous standards of reasoning, and to the learning community) (Michaels, O'Connor, Hall, & Resnick, 2002), *inert assessment* (assessment that does not incorporate students' understanding into subsequent moves, but rather, guides instruction by keeping the flow and function relatively constant), and *generative assessment* (assessment that mediates discourse to promote students' active monitoring and regulation of thinking about the mathematics being taught).

Methods and Procedures

The participants were a purposive sample of seven middle grades mathematics teachers (grades 4 through 8) who were identified as having characteristics indicative of expertise (Darling-Hammond, 2000), including representatives from three specific groups: teachers who had achieved *National Board for Professional Teaching Standards* [NBPTS] certification in Early Adolescent Mathematics, recipients of the *Presidential Award for Excellence in Mathematics and Science Teaching* [PAEMST], and teachers recommended by university faculty. Data were collected via semi-structured interviews, classroom observations, field notes, audiotapes, and videotapes. Mathematics lessons were observed, field notes written, and classroom discourse audiotaped and videotaped. Audio recordings from interviews and observations were transcribed and coded.

Data collected from each participant were analyzed both individually and using constant comparison methods (Glaser & Strauss, 1967) so that each set of data would provide additional evidence to inspect, test, and refine the theory and models being developed. The transcripts from the classroom observations were coded on several levels—for example, line-by-line coding of moves was accomplished using schemes adapted from Wells (1999) and sequences were coded using strategies developed during a pilot study (Truxaw, 2004). Next, individual sequence maps (i.e., diagrams representing the flow of forms of talk and verbal assessment within a sequence) were created by applying the coded data to a preliminary graphic model of classroom discourse. Maps and coded transcripts were inspected, compared, adapted, and synthesized to develop an overall model of the flow of classroom discourse.

Building a model of the flow of discourse provided a theoretical foundation as well as analytical tools to address the research question reported in this paper—that is, *what models of teaching can be developed to illustrate discourse in middle grades mathematics classes?* Fine-grained analysis of the seven individual cases was undertaken. To begin to unpack what the teaching looked like, certain sequences and instructional episodes were identified that stood out as potentially informative. Particular focus was paid to sequences that included the following: evidence of discourse that tended toward univocal function; evidence of discourse that tended toward dialogic function; and evidence of discourse that tended toward dialogic function, but then shifted back toward univocal function. Models of teaching were built through multi-level analysis of sequence maps, lesson transcripts, and interview transcripts.

Results and Discussion

Three models of teaching were uncovered that promoted discourse on a continuum from univocal to dialogic—that is, *deductive* (associated with univocal), *inductive* (associated with dialogic), and *in/deductive* (associated with both univocal and dialogic). The deductive model begins with a frame of reference (e.g., a problem) and works from top down—rules, definitions, and procedures are applied to the frame of reference until the individual problem is solved. In the deductive model the goal is the solution of a specific problem or problems. The inductive teaching model represents cycles, rather than linear series of steps. It begins with a frame of reference (e.g., a rich problem) and builds meaning cyclically/recursively through inductive processes. The frame of reference is referred back to multiple times from a persepective that progressively builds understanding. The goal is not the solution of the original problem, but rather, the development of generalized meaning through the cyclical/recursive process. The in/deductive model begins with a frame of reference and includes opportunities for exploration and hypothesis-building. When solutions are suggested, explanations are requested and presented, but then the cycle stops. Although the in/deductive model begins similarly to the inductive model, in the end, the goal is to solve the individual problems, rather than to generate new meaning—that is, it finishes more like the deductive model. Next, the three models will be described within the context of the learning episodes representative of each.

The Deductive Model

The deductive model (see Figure 1) derived from a learning episode that consisted of one sequence in a lesson taught by Lydia (all names are pseudonyms)—a seventh grade mathematics teacher and a PAEMST recipient. The lesson focused on reviewing for a “celebration of knowledge” (i.e., a formal assessment). Lydia stated that the purpose of the learning episode was to review concepts and procedures related to simplifying fractions (post observation interview, October 2003). Specifically, the students were asked to simplify the fraction 12/21 (see Figure 1-A). Lydia led students through steps that included finding factors, common factors, the greatest common factor, and equivalent fractions (see Figure 1 B & C)—until the problem was solved (see Figure 1 D). Although the sequence included embedded exchanges that referenced real world analogies of fractions concepts, Lydia directed and led the students through these, maintaining control and conveying the information via the verbal exchanges. The discourse in this sequence tended toward univocal. The method of instruction was predominantly deductive, or “top down”, with the general rules being explained and then applied to the specific case. The sequence used exclusively leading talk and inert assessment to lead students to meaning, rather than having them generate meaning for themselves.

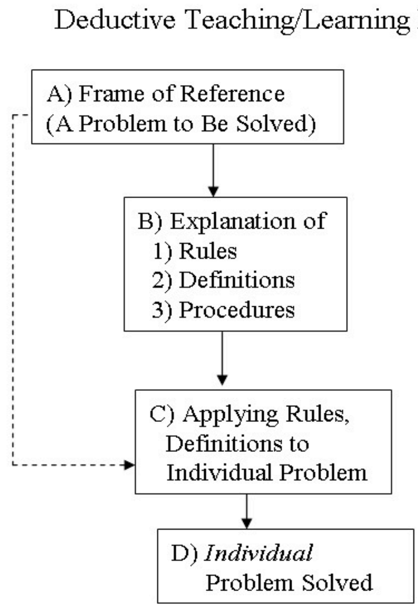

Figure 1. Deductive model of teaching.

The Inductive Model

The inductive model (see Figure 2) was built from a learning episode consisting of four sequences in a lesson taught by Jacob—an eighth-grade mathematics teacher with NBPTS certification. A problem was introduced in sequence one (i.e., “What is the sum of the reciprocals of the prime or composite factors of 28?”), establishing a frame of reference (see Figure 2-A). In sequence two, common understanding of key terms was developed (e.g., prime and composite) (see Figure 2-B-1), while in sequence three the problem was investigated in small groups and a solution was presented by a student (see Figure 1-B-2&3). By consensus, the class agreed that the sum of the reciprocals of the prime and composite factors of 28 equaled 1, which provided a new basis for meaning-making (see Figure 2-C). Within the first three sequences, all four forms of talk and both inert and generative assessments were used, but the overall function of the discourse was univocal in nature—that is, its main purpose was to establish common understanding. Although one might imagine that the learning episode would be complete with the presentation and acceptance of a solution, instead, the first three sequences served as a springboard for sequence four. The fourth sequence built from the first three, using the problem as a frame of reference for developing, testing and revising hypotheses; exploring connections between the problem’s solution and other concepts (e.g., abundant numbers, deficient numbers, and perfect numbers); constructing revised frames of reference; and demonstrating students’ understanding related to the original problem and the revised hypotheses.

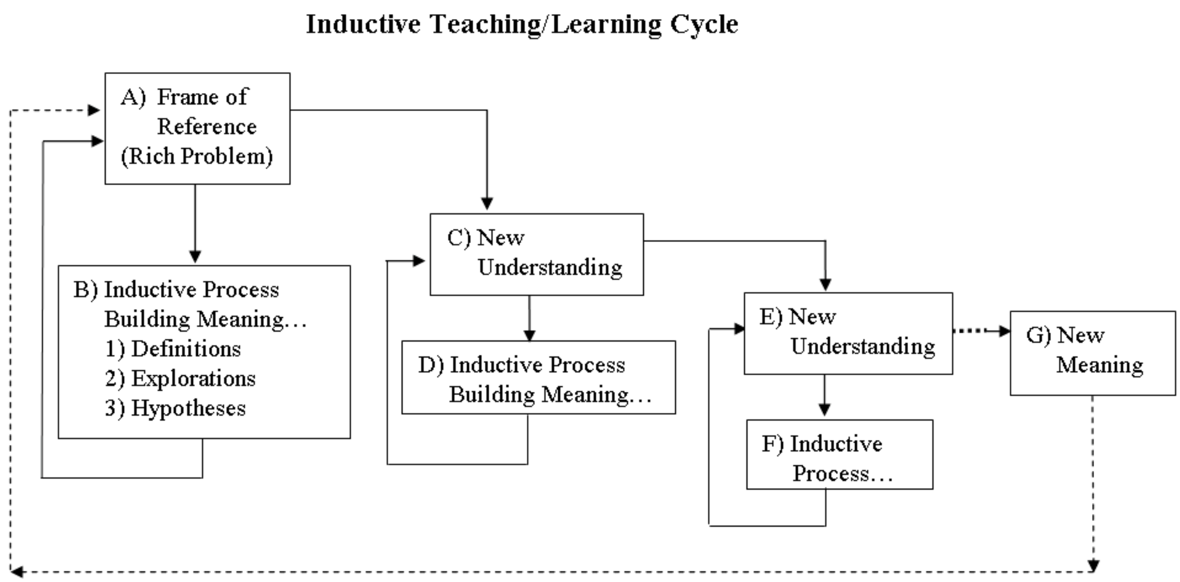

Figure 2. Inductive model of teaching.

The cyclic nature of the discourse (i.e., recursively establishing common understanding, exploring, conjecturing, testing, and revising hypotheses) was used to progressively build new meaning (see Figure 2D-2G). The discourse in the fourth sequence was particularly complex—including multiple instances of leading, exploratory, and accountable talk and both inert and generative assessments. Also of note is that accountable talk and generative assessment occurred more frequently in sequence four than they had in the previous three sequences. The method of instructive was predominantly inductive—that is, moving from specific cases toward more general hypotheses and rules. The discourse in the learning episode (i.e., sequences 1– 4) moved from relatively univocal (while building common understanding) to relatively dialogic (as the common themes were used to build new meaning).

The In/Deductive Model

The in/deductive model (see Figure 3) was based on a learning episode consisting of two sequences within a lesson taught by Sam—a seventh grade mathematics teacher with NBPTS certification. This model appeared to begin with a dialogic stance, but then shifted back toward univocal function. In this learning episode Sam developed a frame of reference by introducing problems that were intended to help students make connections between fractions and decimals, moving toward generating an algorithm for multiplying decimals (post-observation interview, October 2003). First, Sam asked the students to consider whether 1.25 times 0.5 would be greater than or less than one (see Figure 3-A). He asked for estimates from the whole group, but then gave students time to explore and discuss in small groups. He then asked representatives from each table to share their hypotheses (whether the result would be greater than or less than one) (see Figure 2-B). Members of one group demonstrated new meaning when they opted to change their answer after hearing others' explanations (see Figure 3-C). The class developed consensus that 1.25 times 0.5 would be less than one. Next, the second part of the problem was addressed—that is, whether 1.25 times 0.5 would be greater than or less than one half (see Figure 3-D). Again, exploration and discussion occurred in small groups. Representatives from the groups then presented their answers and explanations. When a consensus was reached, Sam said to the class, “Everybody agrees it’s greater than a half. Okay. *Convince me*. Why do you think it’s going to be greater than a half?” The request, “*Convince me*,” seemed generative—that is, providing potential for meaning-making. In the ensuing exchanges a student explained the *procedures* he used to compute the problem, saying, “…what you’re really doing is one and a quarter divided by two.” Sam followed up by reinforcing that “dividing by two and multiplying by a half is the same thing” (see Figure 3-D). Sam then asked, “Okay. So, why does that work out to be greater than a half?” The student responded with the *answer* to the problem, “I think that gets you 75 hundredths.” Sam followed up with, “…because if this reminded you of a dollar, dividing by 2 gets you 50 cents?” The student answered, “Yes.” Sam followed up with, “…And then a little bit more?” The student said, “Yeah.” Sam said, “All right,” ending the learning episode (see Figure 3-E).

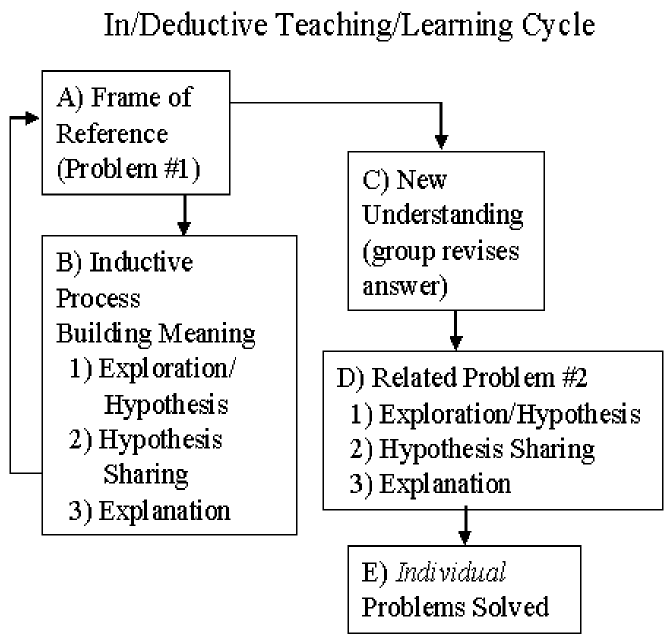

Figure 3. In/deductive model of teaching.

The in/deductive model provided clues to both the inductive and deductive models. For example, the functions of discourse and, in turn, the models of teaching, seemed to be affected not only by frame of reference (i.e., the problems), but also by the teachers’ choices of verbal interventions. Both the in/deductive and the inductive models began with seemingly rich frames of reference and included opportunity for exploration and hypothesis building. When Sam said, “Convince me,” the student’s responses focused predominantly on procedural, rather than conceptual, explanations. Instead of pressing with further generative assessments, Sam chose to accept the explanations as is, and end the learning episode. In other words, when the individual problems were solved and procedurally explained, the cycle ended—similar to the deductive model. This in/deductive model suggests relationships between teacher’s choices in follow-up moves and outcomes of the discourse—in this case, potentially dialogic discourse reverted back to univocal function.

Final Remarks

The models described in this paper suggest questions for further investigation. For example, current research suggests that the learning *task* may prove to be important in understanding the nature of the classroom discourse. There is evidence that a rich problem is more likely to provoke dynamic discourse than procedural tasks (NCTM, 2000). The inductive model (built from Jacob's teaching) involved a rich problem. In contrast, the deductive model focused on reviewing specific mathematical procedures. The in/deductive model began with what appeared to be rich problems, but ended up being procedurally solved and explained. This may imply that although rich problems may enhance dialogic discourse, they alone are not sufficient.

Along with the task, the teacher's *intent* (as indicated by verbal interventions and post-observations interviews) may influence the flow of the classroom discourse. These teachers made choices in their follow-up moves that were influenced by their intentions. Jacob (inductive) more consistently used generative assessments to press responses toward conceptual understanding (Kazemi & Stipek, 2001); Lydia (deductive) consistently used inert assessments that maintained univocal flow and function of the discourse; and Sam (in/deductive) used some assessments that appeared generative (e.g., "Convince me"), but when students offered procedural explanations, he didn't continue to press toward a more conceptual understanding. In post-observation interviews, the participants expressed different intentions related to their teaching. Jacob said that his intention was to assist the students in what he called "guided discovery" (post-observation interview, October 2003). Lydia talked about reviewing for a formal assessment—"I knew what I had written up on that celebration [i.e., test]—the information I wanted to know that they understood and find out what they didn't understand, which is what our whole class was about" (post-observation interview, October 2003). Sam said that he sought explanations from his students. However, when asked why he did this, he said that usually it was to find out if students really knew answers or were just guessing (post-observation interview, October 2003).

Another potential issue is the influence of the teacher's mathematical *content* knowledge on the discourse and the model of teaching—for example, how important is the teacher's mathematical content in his/her facility to ask focusing questions? The backgrounds of the teachers highlighted in this study include the following: Jacob had an undergraduate degree in economics, secondary teaching certification in mathematics, and a master's degree in mathematics education; Lydia had an undergraduate degree in mathematics and secondary certification in mathematics; Sam had nearly completed a master's degree in mathematics education. During the classroom observations of these teachers, it appeared that their understanding of the mathematics being taught helped them to ask meaningful questions; however, there was a sense that content knowledge alone did not necessarily foster high press exchanges or dialogic discourse. The teachers' intentions influenced what they chose to do with their content knowledge and with the given task. Clearly, more research is warranted to further investigate potential models of teaching and to explore possible reasons why the discourse and associated teaching and learning progress as they do.

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STUDENTS BUILDING MATHEMATICAL CONNECTIONS THROUGH COMMUNICATION

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This paper reports results from a study of high-school students engaged in small-group mathematical investigations in after-school sessions. Knowledge that individual participants brought to the tasks as well as the joint construction of new knowledge by group members contributed to successful problem solving; all students contributed to illustrating the isomorphic relationships among several problem tasks. Communication between students helped them, as a group, to build and extend their mathematical knowledge.

Introduction

We describe here a session where, through collaboration and discussion, a group of students extended their understanding of some features of Pascal's Triangle by identifying connections between two isomorphic combinatorics problems with different surface features. The question that guided us was: How did communication among students in a problem-solving group contribute to the group's success in making sense of problems in combinatorics?

Theoretical Framework

Maher and Sfard, in stressing the crucial importance of communication in mathematical thought, provide the theoretical framework for this study. Maher (1998) notes that, in communicating about their ideas, students develop and consolidate mathematical understanding. According to Maher, communication enables students to share personal mental images and provides for the comparison of representations. She points out that in discussion, there are opportunities for the reorganization of ideas; opportunities for deeper understanding of similarities and differences emerge as individual ideas are made public for further discussion and possible revision. She emphasizes that the process of justifying one's thinking is an essential component of mathematical reasoning. According to Sfard (2001), students learn to think mathematically by participating in discourse about ideas – arguing, asking questions, and anticipating feedback. Our view is that the process of communicating ideas and providing support for those ideas are necessary prerequisites for making suitable connections between problems of equivalent structure.

Method of Inquiry and Data Sources

This research uses data from a longitudinal study (Maher, 2005) that has followed the mathematical thinking of a group of public school students from first grade through college. Data for this analysis are taken from an after-school problem-investigation session involving four students during the sophomore year of high school (March 1998). Two cameras captured the conversation, actions and written work of the participants. Videotapes, student work, and researcher notes provided the data for the analysis. Session transcripts were prepared and verified and events related to student discussion of Pascal's Triangle and combinatorics problems were selected for analysis. Here we examine in detail events from this session in which students

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formed connections among combinatorics problems and Pascal's Triangle. We describe the mathematical ideas that emerged from the discussion.

In this problem-solving session, four students explored the addition rule for Pascal's Triangle and how it relates to two isomorphic problems in combinatorics, the pizza problem (How many pizzas is it possible to make when there are n toppings to choose from?) and the towers problem (How many towers n cubes tall can be built from Unifix cubes when choosing from two colors, say white and blue cubes?). In previous sessions, students had explored those two problems in depth, had determined the general answers to both problems (2^n), and had observed that the answers to those problems can be found in Pascal's Triangle. (The numbers in row n of Pascal's Triangle – $C(n,r)$, where r goes from 0 through n – give both the number of pizzas with exactly r toppings when there are n toppings to choose from and also the number of towers n cubes tall containing exactly r blue cubes.) The students were also aware from classroom work of a relationship between Pascal's Triangle and the binomial coefficients.

Results

During the course of this problem-solving session, the group explored the relationships among the binomial expansion, the towers problem, the pizza problem, and Pascal's Triangle. The following episodes show how the group worked together and formed connections between mathematical ideas through discourse. In particular, we note how students first communicated their ideas about two relationships (how the towers problem is related to Pascal's Triangle and the binomial expansion and how the pizza problem is related to Pascal's Triangle) and provided support for those ideas, and then they described how those two isomorphic problems (the towers problem and the pizza problem) are related to each other.

Episode 1: Connecting Towers to the Binomial Expansion and Pascal's Triangle

Space limitations preclude including a full transcript of this episode, and so the beginning of the episode is summarized here. In this episode, the students used towers built from blue and white cubes to represent $(a+b)^2$. They showed how row 2 of Pascal's Triangle, representing $(a+b)^2$, can also represent these towers. Their original model is shown on the left side of Figure 1; a modified model is shown on the right side of Figure 1. (They built this model after the researcher requested a model that made it clear that all possible 2-tall towers were accounted for.) The all-blue tower represented a^2, the two blue-white towers represented $2ab$, and the all-white tower represented b^2. The students also modeled row 3 of Pascal's Triangle as shown in Figure 2, and they discussed how the towers represented by row 3 can be generated from the towers represented by row 2. This episode lasted about four minutes and included over 100 conversational turn units; these are described by Powell (2003) as "tied sequences of utterances that constitute speakers' turn at talk and at holding the floor" (p. 55). Jeff and Ankur were each responsible for about 30% of the conversational turns in this episode, while Michael and Romina each accounted for about 20%.

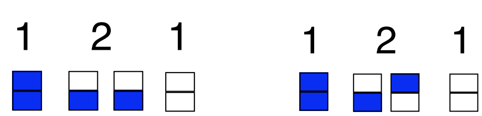

Figure 1. Towers representing row 2 of Pascal's Triangle (two configurations)

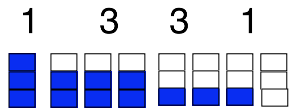

Figure 2. Towers representing row 3 of Pascal's Triangle

In the remainder of the discussion with the researcher, the students explored the connection between adding cubes to towers and multiplying by (a+b): Adding a blue cube to the top of a tower can be considered equivalent to multiplying a polynomial by a, and adding a white cube can be considered equivalent to multiplying by b. This segment lasted about a minute and consisted of about 30 conversational turns. Highlights of this discussion are shown below. The transcript is in the right column and our interpretation of the discourse is in the left column.

Michael begins to explain the connection between multiplying by (a+b) and adding cubes to towers, but he has a procedural question.	Michael: OK. Then we, you like multiply a plus b. It would be like first- I don't know which one is first.
Romina answers his question.	Romina: a.
Michael demonstrates how multiplying by a is like adding blue cubes to towers.	Michael: a, a is first. It would be like putting one of these [blue cubes] on top of each tower. [Michael places a blue cube on top of each of the towers shown in Figure 2, creating the four towers shown in Figure 3.]
Ankur starts to describe what to do next. Note: since b represents white, Ankur misspeaks when he equates b with blue.	Ankur: Now you're distributing the b, distributing the blue.
Michael reiterates the link he just made. He goes on to describe what comes next: adding white cubes (which represents multiplying by b).	Michael: But you still have those [the 3-tall towers with the blue cubes on top], you know. And you just stick a bunch of whites on there. 'Cause you're, you're like-
Jeff makes explicit the connection between the 2-tall towers generated by Michael's process and the 3-tall towers representing row 3 of Pascal's Triangle. Refer to Figure 4.	Jeff: And that would represent that one right there. These two that you just made would represent these two here. That would represent that. And that's how you break it down.

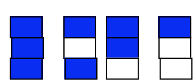

Figure 3. Building some of the three-tall towers by adding cubes to the two-tall towers

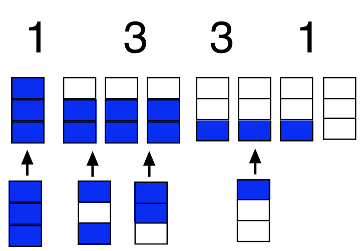

Figure 4. Association between three-tall towers and two-tall towers with an added blue cube

Episode 2: Connecting Pizzas and Pascal's Triangle

This episode began with a researcher question about the relationship between towers and pizzas. The students did not answer that specific question during this episode, but they did describe how the numbers in Pascal's Triangle can represent different pizzas. For example, since row 2 represents all the possible pizzas when there are two available toppings, the three numbers in row 2 (1 2 1) represent respectively the 1 plain pizza, the 2 one-topping pizzas, and the 1 two-topping pizza. This episode lasted about one minute, with 54 conversational turns to which Ankur, Jeff, and Romina contributed about equally. Highlights follow.

	Researcher: What pizza problem would we be talking about here and how would that be the same as that? [The researcher indicates all the 2-tall towers.] If you had to make up a pizza problem to model this row [row 2, which is 1 2 1], what's the pizza problem?
Ankur suggests a link.	Ankur: Three toppings.
Jeff challenges that link.	Jeff: No. There's only, how could there be three toppings?
Ankur suggests a different link.	Ankur: Two toppings.
Jeff accepts the correction but asks what the exact connection is.	Jeff: All right. What would *a* squared be, though? Like what and what?
Romina suggests a connection: row 2 of Pascal's Triangle is linked to two pizza toppings.	Romina: Well, so, that [row 2] would be two different toppings, right?
Ankur makes the connection between a plain pizza and row 0 of Pascal's Triangle.	Ankur: Wait a minute. *a* squared. This [1 in row 0 of Pascal's Triangle] is a plain pizza. That's it. This is no toppings.
Jeff asks what the exact connection is.	Jeff: Wait. Where, where is that? What's that on?
Ankur offer a concrete representation as a clarification.	Ankur: The one. [Ankur has placed a single white cube on the table.]
Romina makes the link explicit.	Romina: The 1 all the way at the top.
Jeff agrees with this link.	Jeff: All right. All right. We got plain.
Ankur builds on his link between pizzas and row 0 and Romina's link between pizzas and row 2.	Ankur: This [row 1, which is 1 1] is one topping.
Jeff suggests one way to make the link concrete.	Jeff: Uh. Peppers.
Romina agrees with Ankur's connection between row 1 and pizzas.	Romina: One topping.
Jeff agrees too.	Jeff: One topping. Yeah that's one topping.
Ankur expands on the link between the two numbers in row 1 and the two possible pizzas that can be built when there is one possible topping.	Ankur: Peppers. It's either peppers or without peppers.
Romina agrees with Ankur's link.	Romina: No peppers.

Jeff also agrees.	Jeff: Yeah.
Ankur follows up on Romina's earlier connection between row 2 and pizzas.	Ankur: And this [row 2, which is 1 2 1] is two toppings. Peppers, um, pepperoni.
Romina builds on her earlier link.	Romina: Yeah, that [the first 1 in row 2] could be like plain.
Jeff indicates that he understands.	Jeff: Plain.
Romina continues to expand on the link between row 2 and pizzas.	Romina: And all. [Romina indicates the last 1 in row 2 of Pascal's Triangle.]
Ankur offers a concrete representation.	Ankur: It's like peppers, pepperoni.
Jeff agrees.	Jeff: Yeah. Both.
Ankur finishes describing the connection between row 2 and pizzas.	Ankur: And plain. [Ankur points in turn to the 1, 2, and 1.]
Jeff challenges the connection.	Jeff: Wait. That's five, though.
Ankur justifies his assertion.	Ankur: No, it isn't. It's still four, because peppers and pepperoni is the same thing as pepperoni and peppers.

Episode 3: Connecting Towers and Pizzas

In this episode, Michael described how to form a connection between towers and pizzas – by associating one color with the presence of a topping and the other with its absence. The other three participants follow up by listing specific connections. This discussion lasted less than one minute and included 48 conversational turn units. Michael and Ankur were each responsible for about one-third of the conversational turns, with Jeff and Romina together contributing the remaining one-third. Highlights are given below.

Three of the students (Ankur, Jeff, and Romina) have just asserted incorrectly that the 1 at the top of Pascal's Triangle represents 1*a*.	Researcher: Is that, is that what that 1 [at the top of Pascal's Triangle] means? 1 *a*?
Jeff confirms it.	Jeff: Yeah.
Romina confirms it too.	Romina: Yeah.
The researcher recalls the correlation between the binomial expansion and the towers problem.	Researcher: I thought you told me that this [a blue cube] was *a*, 1 *a*, and this [a white cube] was 1 *b*. That's what you told me before.
Ankur expresses doubt about what the 1 means.	Ankur: I don't know what the 1 is.
Romina proposes a different link.	Romina: The 1 is just 1.
Ankur accepts this proposal.	Ankur: The 1 is just 1.
Jeff accepts it too.	Jeff: Yeah.
Romina expands.	Romina: It's not an *a* or a *b*. It's nothing; it's just 1.
Ankur expands.	Ankur: It doesn't represent anything in the $a + b$.
	Researcher: OK. Well. Let's put that aside for now. Why don't we start with-
Jeff seems to be proposing that they look at row 2.	Jeff: We'll start with 2.

But Ankur actually refers to row 1.	Ankur: All right. It's 2. It's one topping. Either pepperoni or without pepperoni. It's two possibilities.
The researcher again brings up the towers problem.	Researcher: Now wait. Now I'm lost again. What, what, what was this? Did we move from here to here [from the towers representing row 1 to those representing row 2]?
Ankur tries to suggest that there is no link between the colors of the cubes and the pizza problem.	Ankur: The colors don't, don't look at the colors.
Michael disagrees.	Michael: No, no, no.
Ankur reiterates: The colors of the cube are not related to pizza toppings.	But the colors [of the towers] don't specifically represent anything.
Romina agrees that there is no link.	Romina: Yeah.
Michael says that there is; he states the connection: one color represents the presence of a topping and the other represents its absence.	Michael: Yes. It does. [Michael points to the blue cube.] Topping. [Michael points to the white cube.] Or no topping.
Ankur immediately retreats from the earlier assertion of no link; instead he builds on this proposed connection.	Ankur: OK. So all of the whites are no topping?
Michael agrees.	Michael: Yeah.
Romina asks about a link between a specific tower and a specific pizza.	Romina: This [the 3-tall tower with all white cubes] is a whole no topping? [The students are discussing the towers shown in Figure 2.]
Michael agrees that Romina has described a tower that is equivalent to a plain pizza.	Michael: Then this [the same 3-tall tower] is a plain pizza.
Romina offers another link between a specific tower and a specific pizza.	Romina: This [a 3-tall tower with two blue cubes and a white cube] is a two-topping.
Michael builds on Romina's link.	Michael: With a choice. If you had a choice of three toppings. This [the blue-white-blue tower] would be a pizza with two different toppings, without the other, third topping.
Romina indicates understanding.	Romina: Oh. With the one. Ooh. That's what I was asking.
Jeff builds on the connection made by Michael.	Jeff: Two, two toppings. Well, yeah. Well, if you're just saying that this [the 3-tall tower with all white cubes] is the pizza with three no toppings, it's plain.
Romina agrees with Jeff's link.	Romina: It's just a plain pizza. … [The students now move on to discuss the towers shown in Figure 1.]
Jeff builds on Michael's earlier link.	Jeff: Yeah, so this [blue-blue tower] is choice of two using two. This [blue-white tower] is choice of two using one.
Ankur joins Jeff's explanation of the connection.	Ankur: Two using one.

Jeff builds the connection further.

Jeff: This [white-blue tower] is choice of two using the other one.

Ankur finishes building the link between two-tall towers and two-topping pizzas.

Ankur: And that's [white-white tower] using nothing. [Figure 5 summarizes this discussion.]

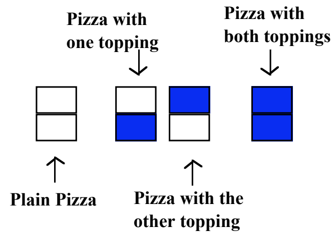

Figure 5. Connecting two-tall towers to two-topping pizzas

Conclusions/Implications

In the first two episodes, the students communicated their ideas about Pascal's Triangle and its relationship to the towers problem and the pizza problem; they supported their ideas through demonstrations and by offering specific examples. Specifically, in episode 1, all four students participated in using what they knew about the binomial coefficients and Pascal's Triangle to describe how the towers problem and the binomial coefficients are related. In episode 2, they described how the answers to the pizza problem can be found in Pascal's Triangle.

In the last episode, they described a new connection – the isomorphism between the towers problem and the pizza problem. After some early uncertainty, they proceeded with a comprehensive description of the relationship between the pizza and towers problems. They did this by building both on their earlier discussions and on Michael's crucial insight: the idea that a blue cube can represent the presence of a pizza topping and a white cube can represent its absence. We believe that the framework for the development of this new connection was provided by their earlier discussions of and justifications for their ideas about Pascal's Triangle and its relationship with these two problems. Communicating their ideas and providing support for those ideas helped them to make this connection between problems of equivalent structure and to build their understanding of that isomorphic relationship.

Acknowledgement

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SOLUTION STRATEGIES USED BY HIGH SCHOOL STUDENTS TO SOLVE WORD ALGEBRAIC *RATE* PROBLEMS

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In this article we report on the strategies utilized by high school students for solving word algebraic rate problems. The proposal of this study is to contribute with theoretical and empirical elements to the classification of the problems proposed by Guzmán, Bednarz & Hitt (2003). The results of our study show that the order of difficulty encountered in the process of solution by the students for solving problems coincides with the expectations of those researchers. However, with regard to the strategies we identified, the majority of the students who participated in this phase of the study demonstrated arithmetical strategies.

Introduction

A series of problems within the process of teaching and learning of algebra has been the center of attention of various research studies in recent years. The international community of mathematics education has identified different approaches aimed at making this learning meaningful for the students; this investigation was developed with a focus on the problem solving perspective, in particular those problems called word algebraic *rate* problems, which we will describe below.

The results demonstrated in this article are taken from a larger project whose proposal is to contribute with empirical and theoretical elements to the Bednarz & Janvier (1994, 1996) theory, in particular the classification of word algebraic *rate* problems suggested by Bednarz, Guzmán & Hitt (2003). In the larger study, we are also interested in corroborating the hierarchy of word algebraic *rate* problems established by these authors (*ibid*.) in relation to their complexity. We wished to document and analyze the type of strategies the students used for solving these different classes of problems.

The proposal of this article is to contribute with only empirical elements to the theoretical classification of word algebraic *rate* problems proposed by Guzmán, Bednarz & Hitt (2003). The emphasis in this article is placed on the analysis of distinct strategies used by high school students for this particular class of problems.

Theoretical Framework

Bednarz & Janvier (1994, 996) found that considering problem solving as a perspective to understand the algebraic thought processes of the students implicated the questioning of the basis of the word problems that they were using. These researchers proposed a theoretical tool (*grille d'analyse*) that permits the classification of word problems utilized in the teaching and learning of arithmetic and algebra. One of the three classes of word problems that they identified was of *rate*.

The word *rate* problems are those that involve a relationship of comparison between non-homogenous magnitudes (Bednarz & Janvier, 1994), of which PROBLEM 1 is an example. In this problem two non-homogenous quantities can be observed (employees and salaries) related by *rate* (salary per employee). The symbolism created by these authors to sustain their theory is described in Vargas & Guzmán (2000).

Guzmán *et al*. (1999), Guzmán *et al*. (2003) and Bednarz *et al*. (2003) studied in-depth word algebraic *rate* problems and analyzed those that are frequently encountered in mathematics textbooks. These authors (*ibid*.) classified these problems[1] and predicted, in a theoretical manner, a certain order of complexity from the mathematical-relational structures underlying each problem, or in other words, the nature of the relationships between the quantities of the problem, known and unknown, and the linking of these relationships. In their study, these researchers identified eight classes or categories of problems. Two problems typical of Categories II and III, according to the classification of these authors, are described in the methodological section (PROBLEM 1 and PROBLEM 2).

Methodology

Founded in the theoretical studies of these authors (*ibid*.), we undertook an analysis of the problems contained in textbooks used in secondary mathematics education in Mexico (older versions from 1944-1979 and recent versions from 1994 to date). From this analysis and the problems identified by Bednarz *et al*. (2003) in high school textbooks, 17 problems were selected for this study. We designed worksheets, initiated a piloting phase and are now analyzing the strategies used by high school students to solve the problems. The sources of information are films of the regular mathematics course, sound recordings, teacher observations and reports written by the students.

The school where the study was undertaken is a technical high school within the public education sector in Mexico (SEP).There were 45 students in the group studied. These students were elected because they were finishing their algebra course during the first semester of high school. Their basic knowledge of algebra had been acquired in said high school algebra course and one algebra course in secondary school.

Each session had a duration of an hour and a half. During this time, the students were given general instructions, along with worksheets, each with a problem. They solved the problems in teams, and once the problems were solved, a member of the team explained the results to the entire group. All of the participants then discussed the solution strategies that were used and came to an agreement, when possible, about the presented solution.

Because of space limitations, we will here present only the results of two problems from Categories II and III (problems 1 and 2 respectively). These categories are typical of the textbooks used both in Mexican and Canadian schools (Guzmán *et al.,* 2003). The problems are the following:

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[1] The word *problem* herein specifically denotes a word algebraic *rate* problem.

PROBLEM 1. *175 employees, both men and women, work in a factory. The men earn 42 pesos per day and the women earn 35 pesos per day. If the daily payroll is $6,825.00, how many men and how many women work in the factory?* (Beristáin & Campos, 1978, p. 239). The problem pertains to Category II.

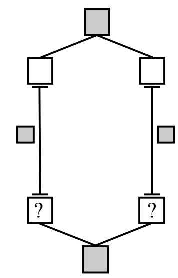


Figure 1. Scheme of the PROBLEM 1. This type of scheme was explained by Guzmán *et al.* (2003).

The problems of this category contain, explicitly, information with respect to the relationships between homogenous quantities; in this case, 175 is the total of men and women who work in the factory and $6,825.00 is the total that all of the employees (both men and women) earn. These relationships are binary operations, specifically, of addition. In this case, the *rate* is known and is “salary per employee (man or woman)”, 42 pesos/man and 35 pesos/woman.

PROBLEM 2. *A small bottle and a large one contain 18 liters of oil, 4 large bottles and 12 small ones contain 120 liters. How many liters of oil are contained in each bottle?* (Martínez, Struck, Palmas, & Álvarez, 2001, p. 146). The problem pertains to Category III, Subcategory IIIa.

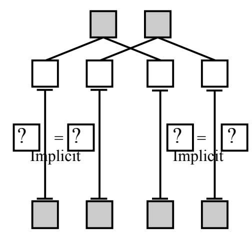


Figure 2. Scheme of the PROBLEM 2. This type of scheme was explained by Guzmán *et al.* (2003).

In this problem, the *rate* is “liters of oil per bottle (small or large)”. It is not known how many liters of oil the small bottles contain or how many liters of oil the large bottles contain. This *rate* relates two non-homogenous quantities: liters of oil and bottles (small and large). The structure of the problem involves operations: the total quantity of liters of oil that are contained in the four large bottles and 12 small bottles is known (120 liters). The total quantity of liters that a small bottle and a large bottle is also known (18 liters). In the problems of this category, the idea of variation of homogenous quantities is encountered (Bednarz *et al.,* 2003). The quantity of bottles varies and, in consequence, the total quantity of liters of oil.

The analysis of the strategies of the students for solving the problem was undertaken following some of the categories proposed by Bednarz & Guzmán (2003): *Identification of the structure of the problem, numerical trial, numerical play and transformation of the structure of the problem.*

Each one of the categories mentioned can be described as the following:

Identification of the structure of the problem (S). It is said that a student identifies the structure of the problem with the detection of the known and unknown quantities of the problem, such as the number and type of relationships involved.

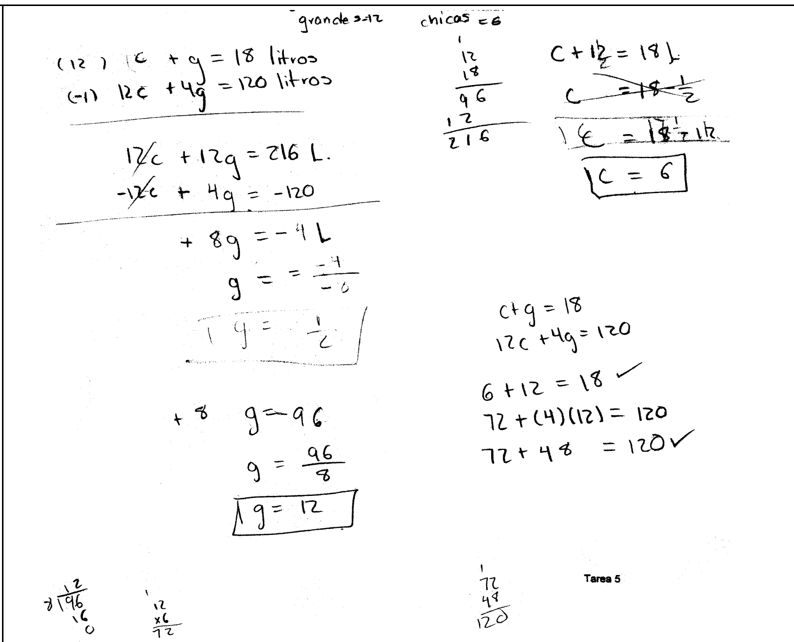
Numerical trial (NT). It is said that a student proceeds with numerical trial when quantities are considered to be possible results. These are then worked with as if they were the solution to the problem and with the manipulation of the relationships involved, confirmation is made as to whether these are, in fact, the results. If confirmation is not made, then a new process of selecting quantities as possible results is initiated.

Numerical Play (NP). It is said that a student uses numerical play when unable to identify the structure of the problem, that is to say, when it is difficult to understand the number and type of relationships involved, such as the number of known and unknown quantities. In general, this student is working with numbers without understanding (without taking into account the relationships), searching to obtain a possible result.

Transformation of the Structure (TS). It is said that a students uses Transformation of the structure when he or she is unable to identify the structure of the problem. In general, this student erroneously interprets the problem, which is common, and changes the type of relationships involved or considers only certain quantities of the problem.

Results

The principal strategies identified in the process of solution of Problems 1 and 2 were the following:

Algebraic language with *Identification of the structure*. Various teams solved through algebraic language, that is to say, they wrote and solved equations (Figure 3).	 Figure 3. Strategy (*S* and algebraic language) used by the students for solving PROBLEM 2.

Numerical trial with *Identification of the structure*.

Various teams estimated certain initial numeric values that they thought would comply with the implied relationships in the problem (Figure 4).

They took from these and made adjustments through the numerical essay.

One of the teams that used this strategy in PROBLEM 1, affirmed (as recorded on the audio tape): “and with the first operation”, which is to say, they estimated an initial value as a possible result and immediately confirmed it in the first operation undertaken (Figure 5).

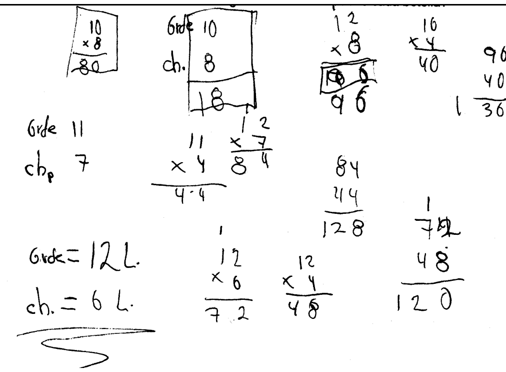


Figure 4. Strategy (*NT* and *S*) used by the students for solving PROBLEM 2

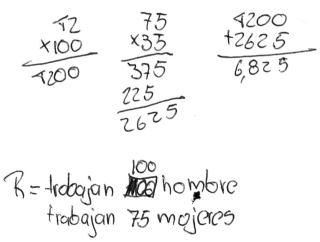


Figure 5. Strategy (NT and S) used by the students for solving PROBLEM 1.

First procedure: *Transformation of the structure*. Second procedure: *Numerical trial* with *Identification of the structure*.

Some of the teams, at first, did not identify all of the relationships involved in the problem and did operations. Afterwards, they identified all of the relationships and used Numerical trial, solved the problem (Figure 6).

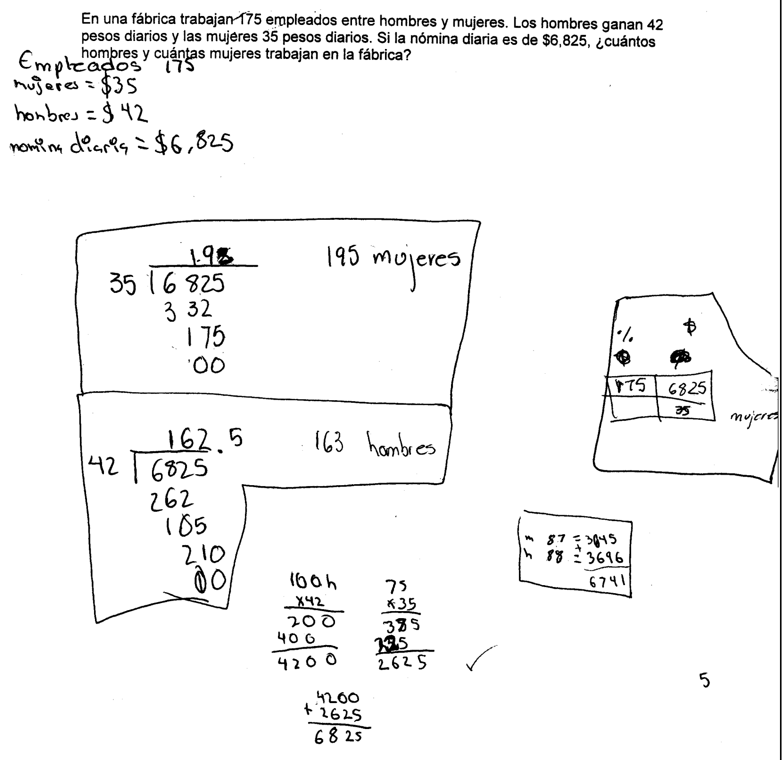
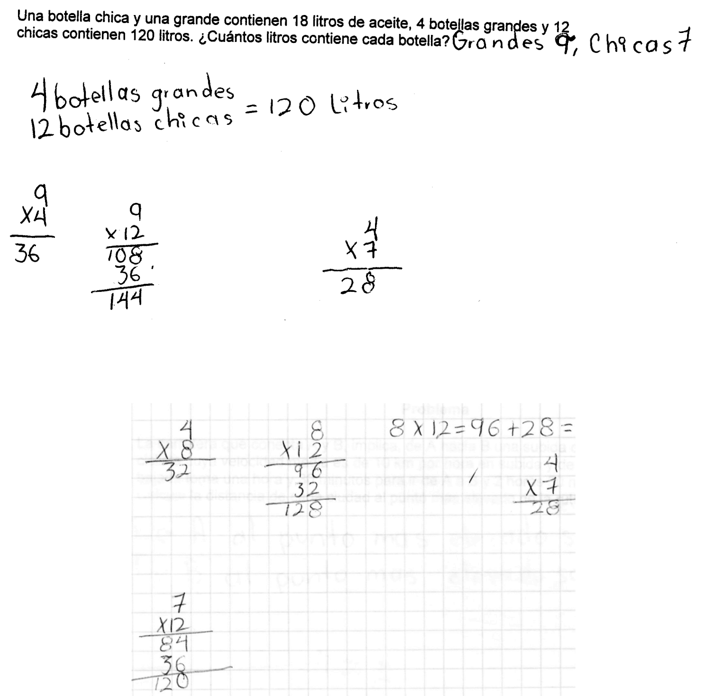
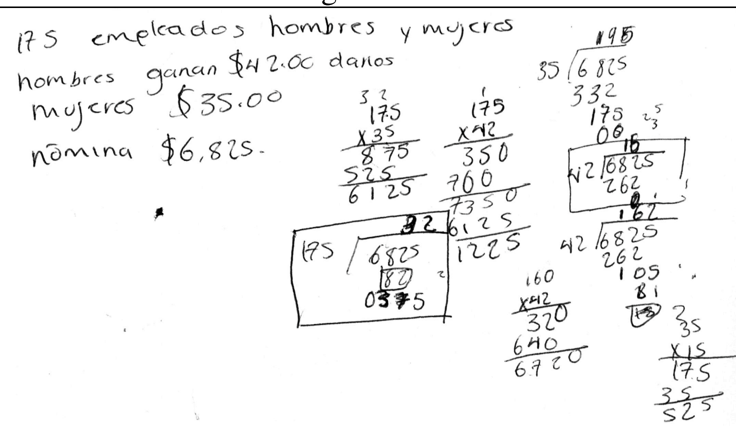


Figure 6. Strategy (*TS*, *NT* and *S*) used by the students for solving PROBLEM 1.

Numerical trial with *Transformation of the structure* of the problem. This team transformed the structure of the problem. Adding to the existent relationships the relationship of the “contents of the large bottle the same as the contents of the small bottle”. They estimated initial values as possible results and used *Numerical trial*. They did not obtain the correct results (Figure 7).	 Figure 7. Strategy (*NT* and *TS*) used by the students for solving PROBLEM 2.
Numerical Play with *Transformation of the structure*. This team finished the operation with the quantities found in the problem, without taking into account the relationships involved.	 Figure 8. Strategy (*NP*) used by the students for solving PROBLEM 1.

In PROBLEM 1, the teams of students did not use algebraic language. All of the teams that approached the problem did so through arithmetic. The characteristic strategy utilized in this problem was *Numerical trial* (Figure 5).

The characteristic strategy of the teams for approaching PROBLEM 2 was algebraic language with *Identification of the structure* (Figure 3), which approximately 50% of the teams used. Another frequently used strategy used was *Numerical trial* (Figure 4).

Preliminary Conclusions

It is notable that the majority of the students who participated in this first phase of the study used informal strategies – not those taught – to solve the proposed problems (Problems 1 and 2). It is known that such strategies are not part of formal mathematics curriculum, however, they are frequently used by students before being introduced to algebra (Bednarz & Guzmán, 2003) and even when they have received courses of this type, as was demonstrated by the population of this study when approaching the problems.

In some way these informal strategies help the students *connect* (in the sense of the classification of the problems proposed by Bednarz & Janvier, 1996) the known and unknown quantities of the problems. In this way, the majority of the student participants in this study identified, in an arithmetical context, the structure of the problem. As to the difficulties that these same students had in solving the problems, we could say that they counted on the translation of the statements from the algebraic language, although the procedures they used led to successful solution of the proposed problems.

The empirical results obtained to date do not greatly aid in the prediction of the order of the complexity of the problems. In this respect little can be said for the moment, other than that Problems 1 and 2 pertain to non-complex categories (in order of difficulty) and the students found them easy to solve.

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SAME VIDEO, DIFFERENT UNDERSTANDINGS: CO-CONSTRUCTION OFREPRESENTATIONS OF CLASSROOM INSTRUCTIONAL PRACTICES WITHIN A PROFESSIONAL TEACHING COMMUNITY

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In this paper, I illustrate that classroom instructional practices that come to be constituted by the teachers in context of professional development sessions cannot be always easily linked to the instructional practices that were recorded to create video materials used by the teachers. I draw on empirical data from collaboration with a group of middle school mathematics teachers to illustrate how different classroom instructional practices were co-constructed when teachers analyzed same classroom videos in a PD session in two different points in time. I then suggest that attention to both the established question and resources that the teachers draw on may aid our understanding and anticipation of teachers' co-constructions of classroom instructional practices in professional development settings.

Using classroom video in teacher education is becoming widespread and has been the focus of research by a broad body of scholars. Records of practice captured on video media help teachers and university collaborators "ground the conversation in ways that are virtually impossible when the referents are remote or merely rhetorical" (Ball & Cohen, 1999, p. 17). Multimedia technology, capable of organizing large bodies of data and linking classroom videos with additional materials, has been argued useful in creating representations of teaching practice that are rich enough while remaining "workable" to allow for meaningful analyses of teaching practice (Lampert, 2000). While aware of remarkable possibilities brought to teacher education and research on teacher learning by classroom video recordings, I would like to draw attention to relation between such video recordings and classroom instructional practices they come to represent in teacher professional development sessions.

I will draw on empirical data from our[1] longitudinal collaboration with a group of middle school mathematics teachers to illustrate how different classroom instructional practices were co-constructed when teachers analyzed same classroom videos in a PD session in two different points in time. Against this background, I will introduce a framework for understanding teachers' co-constructions by attending to both questions that are seen by the teachers as important to address and resources they draw on in the process of addressing these questions.

Theoretical Background

Part of the study I draw on was devoted to exploring how can researchers with limited access to and influence over teachers' work environments help the teachers establish and sustain PTCs

[1] Author is part of a research team involved in a larger project focused on analyzing the commonalities and contrast between two sites. The past and current collaborators include Paul Cobb, Kay McClain, Maggie McGatha, Teruni Lamberg, Chrystal Dean, Qing Zhao, Lori Tyler, Jose Cortina, and Melissa Gresalfi.

where productive ongoing teacher learning could happen. To this end, we engaged in an endeavor of highly interventionist nature, where we aimed consistently at perturbing teachers' current views of their practice by engaging them in activities in which it would be possible for these teachers to exercise alternative perspectives on teaching and learning.

The general methodology the research team adopted when collaborating with the teachers falls under the heading of a design experiment (Brown, 1992; Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003). This methodological approach is considered particularly useful in studying the development of practices that, while potentially powerful, do not typically occur in situ. Studying such practices then requires that researchers engage in instructional design in order to support development of forms of participation that are the focus of their investigation (Cobb & McClain, 2004). Choice of theoretical perspective and interpretive tools to guide both the research and design was a pragmatic one. It had to provide the researchers insight into continuous developments in studied practices and guide the design of further interventions. Theoretical considerations that treat learning and knowing as situated, inherently social, and distributed across people, materials and representational systems (Greeno, Collins, & Resnick, 1996) have proven useful in this regard. A specific example is a framework described by Cobb and colleagues (Cobb, McClain, Lamberg, & Dean, 2003) that coordinates individual teachers' learning with the development of collective practices of the PTC as they are situated in the institutional setting of a school and school district. This framework was developed out of practical needs to account for teachers' learning in the social context of the PTC as it is enabled and constrained by the broader context of the institution.

Within a family of situative perspectives, meanings that symbols or representations come to take are neither necessitated by their design, nor can a meaning be chosen at will by a meaning making individual. Rather, meanings are viewed as contextualized constructions of interacting systems of participating individuals, along with tools and artifacts at their hand. To demonstrate that this is a crucial distinction from a perspective of designing for teachers' learning, I will illustrate how the same segments of classroom video came to represent strikingly different classroom instructional practices in the group's interactions in different points in time. I suggest that when accounting for the differences in teachers' co-constructions of classroom instructional practice, it is useful to take into consideration how the two activities got constituted in the session from the teachers' perspective. I will draw on Dewey's (1910) conceptualization of reflective thought and discuss two important aspects of teachers' activity that shape instructional practices that classroom videos come to represent – what becomes a question worth addressing in the group's activity and what kinds of resources the teachers choose to draw on in answering that question.

Design Experiment Background

The data are taken from a longitudinal collaboration with a group of 9-12 middle-school mathematics teachers that work in a large urban district in the southeast United States. The school district serves a 60% minority student population and is located in a state with a high-stakes accountability program. During each of the five years of our collaboration, we have conducted 6 one-day work sessions and extended summer session. In the first 18 months of the collaboration, the group evolved to a PTC with a joint enterprise, mutual engagement, and a shared repertoire (Wenger, 1998).

To support the teachers' development of instructional practices that place students' reasoning at the center of their instructional decision-making, we have engaged the teachers in a series of activities that focused on teaching and learning of statistics. We built on a statistical data analysis instructional sequence that was designed, tested, and revised during a prior classroom design experiment conducted with seventh-grade students (McClain & Cobb, 2001). Initially, we engaged the teachers in a sequence of statistical data analysis activities in which they were placed in the role of students. Our goal was both to enhance their own statistical understanding and to provide them with genuine experiences of participating in classroom instruction organized around their reasoning. Against this background we later introduced videos from the seventh-grade design experiment classroom. We conjectured that the teachers' prior engagement with the statistics tasks as learners would aid them in interpreting the interactions in the design experiment classroom. It is important to note that we did not attempt to teach teachers how to interpret or analyze video. Our goal was instead to support teachers' engagement in conversations about the students' mathematical development and aspects of the classroom instructional practice that were supportive of such development.

For purpose of this paper, I use preliminary findings from analyses of collected data that are currently underway (for some more detailed analyses, see Zhao, Visnovska, & McClain, 2004). At this point, I do not use the data to make claims about learning of the PTC. Rather, I attempt to illustrate ways in which outlined perspective may aid our interpretations of teachers' participation in PTC sessions.

Same Video, Different Understandings

The two about 10 minute segments of edited classroom video were first used on the second day of a three-day summer work session that occurred eleven months into the collaboration. Two years later, we used both pieces of video during a summer session again. Following illustrations outline classroom instructional practices that were represented by the segments of video from the teachers' perspective on these two occasions. Rather than trying to bring a comprehensive image of the teachers' co-constructions of classroom practices, I contrast how (a) students' attention to classroom instruction and (b) the teacher's competence in managing the class were constituted during the two activities.

In the summer of 2001, after 7 prior full-day work sessions, the researchers asked the teachers to analyze the classroom interactions as captured in video recordings to determine the students' perception of their obligations. In other words, we tried to orient the teachers to focus on the ways of participating that were constituted as normative in the statistics class. While attempting to address the question, the teachers focused on a range of other issues. From their perspective, the most critical was the issue of classroom management. The teachers attended to what they perceived as the students' misbehavior, their lack of attention to instruction, and the teacher's lack of control. Rachel's contribution best illustrates the way in which the video teacher's competence was constituted in that summer session

> If I was an administrator here in [the district] – and I just look at some of my own evaluations – if I had the same lesson as [on this video] I think I will have a score of 3 or 2 [on scale 1 to 5, where 5 is best] just for the . . . number of times that [the teacher] called females to participate. . . . There was a lot of math being taught, but for the number of students that were actually participating, it didn't look like a lot to me. And if our principals were watching it [they would say to me] "Hmm, I don't know, because everyone was not actively involved. Because you had some kids looking [around], had some kids leaning back, they weren't . . . talking or anything but you can't see them to be paying attention." (Rachel, 6-5-2001)

The group considered Rachel's concern to be an important one and regarded apparent student non-participation as highly problematic aspect of discussed classroom instructional practice.

In the summer of 2003, the researchers first asked the teachers to analyze interviews in which the video students commented on their obligations in statistics design experiment class and in their algebra class. The teachers concluded that while not being very motivated and engaged in their algebra class, the students seemed highly motivated in statistics. Against that background, we asked the teachers to analyze classroom videos to establish how was development of the students' interest in statistics supported. At this time, the teachers developed a very different view of the non-participating or "distracting" students who previously captured most of their attention.

Lisa: We talked about the boy who had to get the hood off his head [on the teacher's request]. He might seem like he was not engaged but then he took off.

Ellen: It bothered me how many kids' heads were on the table, so maybe it was a morning class. And they took off.

Lisa: They can be listening, not having a thing to say. The discussion sparked when they had something to say. (6-28-2003).

The teachers' observations of students who did not seem to engage in the class at some points in the lesson were immediately followed by mention of these same students' legitimate participation in later classroom events. The teachers pointed out that it would be difficult for these students to sustain their later participation if they had not paid attention earlier in the class. In the rest of the session, the teachers tried to avoid basing their judgments about students' engagement solely on a temporary *appearance* of students' lack of paying attention.

In contrast to constructing the video teacher as lacking control in the classroom, the teachers now viewed her instruction as a thoughtful, positive example of classroom practice.

> The teacher would set goals, and accountability, and expectations. …She clearly stated what was going to happen and what she expected to happen and throughout the [lesson] she held students accountable for participating in discussion. And it was done in very – what we consider a positive way. (Amy, 6-28-2003)

In their discussions in the summer of 2003, the teachers identified aspects of the video teacher's practice that they saw as exemplary and worthy of emulation.

My purpose for including the abbreviated excerpts was to illustrate that the teachers did not merely focus on different characteristics of the video teacher's classroom instructional practice in the two discussed sessions. Rather, their two co-constructions of practice were incongruous in several important points.

Accounting for the Difference

Previous illustration leads us to question a direct relationship between classroom video and classroom instructional practices that it came to represent when used in the professional development sessions. Consequently, researchers' own interpretation of classroom instructional practices captured in video materials does not form a sufficient basis for anticipating teachers' participation in planned activities. That highlights a need for a means of conceptualizing how video activities became constituted by the group of teachers in the two sessions. While there are clearly alternative ways of accounting for the illustrated contrast, they often differ in terms of their usefulness from perspective of designing further collaboration. For example, it might be argued that the teachers did not know how to analyze classroom video when they first encountered it in the professional development session. This interpretation, however, does not further our understanding of the nature of this activity from the teachers' perspective. It is therefore of little help to the research team in both gaining insights to past developments and subsequent planning. I will now discuss some considerations that proved more useful in furthering the research team's insights.

When we asked the teachers to examine classroom videos in the two summer sessions we conjectured that the teachers would analyze provided videos by selecting pieces of information that they found relevant to answering respective questions and by searching for warrants of their claims. In other words, we conjectured that the group would use video as a means of *reflecting* on classroom instructional practice. In his treatment of reflective thought as a major characteristics that distinguishes intellectual human activity, Dewey (1910) delineated two subprocesses involved in every reflective operation: "(a) a state of perplexity, hesitation, doubt; and (b) an act of search or investigation directed toward bringing to light further facts which serve to corroborate or to nullify the suggested belief" (p.9). I will examine both these points in turn, bringing to the fore first the negotiation of a worthwhile question for investigation. Later I will discuss resources that the teachers' draw on when addressing the question at hand.

Worthwhile Question

The two discussed activities of analyzing classroom video, as constituted by the teachers in the two sessions, differed significantly in terms of importance or relevance that the teachers ascribed to the questions they were to investigate. In the summer of 2003, the teachers – perplexed by experienced difficulties in motivating their students – came to consider the question of cultivating students' interests in the process of instruction as relevant and worth investigating (Zhao et al, 2004). They co-constructed classroom instructional practice in terms of characteristics that could have been supportive to development of students' statistical interests.

In contrast, in the summer of 2001, the teachers' focus quickly shifted from the proposed question aimed at classroom norms established in the video classroom to *evaluating* the students' participation and the video teacher's classroom instruction (Visnovska, MAP)[2]. This was the case in spite of significant researchers' efforts of re-directing the teachers' activity. It was this alternative focus, rather than the proposed one, that guided the teachers' analyses, and in that way contributed to classroom instructional practice that the video came to represent in the subsequent session discussions.

Resources

When a perplexity has been experienced and demand for its solution guides the teachers' actions, the importance of *resources* for addressing the question comes to the foreground. The teachers' prior experiences in similar situations, as well as applicable "lenses" or tools that could provide them with insights into investigated situation certainly belong amongst the most important resources. I use the term "resource" here in the sense of *what gets used as a resource* by the teachers when they engage in an investigative activity. Consequently, what the teachers

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[2] While the researchers were aware of importance the proposed question plays in establishing the nature of activity within the group, anticipation of whether a particular question will be seen by the teachers as a perplexing issue worth of investigation was not a trivial matter. The research team underwent significant learning in this respect that we documented elsewhere (Zhao et al., 2004).

know but do not draw upon when solving a problem does not constitute a resource for them in that situation. In similar manner, artifacts present in the teachers' physical space might not get used by the teachers as resources for furthering their inquiry.

Examining the resources for which the teachers reached in the two activities of analyzing classroom videos brings the broader contexts, within which the teachers' different co-constructions of classroom instructional practice can be better understood, in the picture. In the summer of 2003, the teachers used the segments of classroom video as a resource that allowed them to examine classroom interactions in useful detail. They paid attention to the teacher's moves and students' contributions, and to ways in which the students interacted with each other. In this process, the teachers reviewed parts of video that they found intriguing, and used the video to search for situations that would illustrate their point. The teachers' co-constructions of classroom instructional practice that the videos came to represent in that activity reflected the teachers' familiarity with the episodes as well as relative importance that the teachers' came to see in different aspects of classroom interactions.

On the other hand, the teachers used the classroom video differently in the summer of 2001. They viewed the classroom episodes, laughing on students who turned in their desks or put their head down. This time, they did not use videos as a resource for their investigation into classroom instructional practices. However, they capitalized on measures of evaluation of classroom instruction that were used by the school administrators in their district. These measures were external to the video activity as planned by the researchers and to the teachers' and researchers' common history of professional development sessions. Nevertheless, they constituted an important resource for the teachers' sense making of someone else's classroom instruction at that point in the collaboration. The teachers' co-constructions of classroom instructional practice that the videos came to represent this time reflected ways in which a good classroom instruction was defined within the institutional setting of the teachers' school district.

While neither the classroom videos nor the measures for evaluation were the sole resources that the teachers drew on in the two activities, I used these to illustrate how the resources the teachers draw on in their activity might be seen as contributing to classroom instructional practice that videos come to represent in professional development sessions.

Realization that the teachers might draw on resources that shape their work environment when engaging in PTC activities is clearly a positive one – it suggests that the teachers treat these activities as related to their own practice. While this is an important resource for the researchers who design for learning of a PTC, it also places additional demands on their work. It is crucial that the researchers develop tools that aid their anticipation of ways in which planned activities will become constituted in the session.

Discussion

I illustrated that classroom instructional practices that come to be constituted by the teachers in context of professional development sessions cannot be always easily linked to the instructional practices that were recorded to create video materials used by the teachers. I then suggested that attention to both the established question and resources that the teachers draw on may aid our understanding and anticipation of teachers' co-constructions of classroom instructional practices in professional development settings.

How classroom instructional practice comes to be portrayed in a PTC sessions in different points in time can be informative with respect to documenting learning of the PTC. Little (2003) made a similar point when she outlined constructs of publicly available features of practice, horizon of observation, and categories and classifications as ways to capture learning of groups of teachers working together in their schools. Relating the two proposed perspective would likely bring additional insights.

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PURPOSEFUL CHOICE: BUILDING MATHEMATICS THROUGH INQUIRY

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Mathematical work by high-school students on a task designed to challenge university honors calculus students' abilities to model exponential growth is examined in detail. This study adds to our understanding of the purposeful choices high-school students make in mathematical modeling and how those choices resemble expert problem solving strategies.

Introduction

Previous research provides insight into the evolution of a mathematical task designed to challenge university students' abilities to model exponential growth (Speiser & Walter, 2004). In this paper, we examine details of high-school students' mathematical work on that task. The problem situation suggests the use of polar coordinates, a topic with which these students were not familiar. Little research has been published about the details of high-school students' mathematization work with exponential functions, particularly when students are not given direct instruction in the use of polar coordinates to solve problems. In the few studies examined, students were extensively instructed on the mathematics needed to solve problems involving exponential functions (Doerr, 2000). Without explicit instruction, students purposefully choose mathematics they already know to invent strategies that accurately portray mathematical relations and anticipate more advanced solution strategies. This study adds to our understanding of high school students as "transitional" problem solvers and of the specific mathematics they purposefully choose to reason from when investigating exponential growth in a spiral shell.

Perspective

People make choices as innately endowed agents of their own actions. A fundamental emphasis of school mathematics programs should be to provide learning conditions that not only recognize individual agency, but also foster and promote learner-initiated inquiry. This emphasis focuses on the learner and the perception of collaborative negotiation in problem solving as each learner makes choices within a learning community.

Purposeful choice in problem solving is especially evident in the work of experts. Expert problem solvers generally work on small pieces of a problem, rather than dealing with all of the complexities of the problem at the same time. In well-structured, narrowly defined domains, content experts exhibit excellent memory, classify problems according to mathematical structure, and use correct algorithms and procedures (DeFranco & Hilton, 1999). In complex domains, problem solving experts clarify goals; organize, represent, and interpret with varying levels of flexibility; monitor self and group solution progress; and analyze and evaluate solution strategies (DeFranco & Hilton, 1999; Graesser et al., 1992). Novice problem solvers rush to use a memorized algorithm, focus on superficial features of a problem and interpretation of explicit material, and lack evaluative monitoring of progress (DeFranco & Hilton, 1999).Transitional problem solvers, in contrast to experts and novices, exhibit expert-like problem solving behaviors in complex domains but may not have extensive content knowledge (Walter, 2004).

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We know very little about the specific mathematical choices high-school students make when they purposefully develop problem solving strategies rather than being instructed on a particular solution strategy by a teacher. In classrooms, teachers' pedagogical choices are pragmatic complexities that frequently channel problem solving activities along narrowly imagined learning trajectories and often substantially set aside learner-initiated inquiry. Learner-initiated inquiry validates individual agency and repositions pragmatic complexities as appropriate issues for all classroom participants.

Mathematizing a spiral shell using polar coordinates, as a mathematics task, originated in September, 1991 as a realistic problem situation to challenge honors calculus students (Speiser & Walter, 2004). The fossilized spiral shell, a Placenticeras, provided the inspiration and name for a mathematics task that has been given to high-school precalculus students (Walter, 2001, 2004; Walter & Maher, 2002), undergraduate students (Speiser & Walter, 2004), graduate students, preservice secondary mathematics teachers, and practicing elementary teachers participating in a three-year professional development project. Various solution strategies emerged as problem solvers with different mathematical backgrounds interpreted the problem in different ways. For example, Speiser and Walter found that choices by honors calculus students' to place the polar origin of the spiral at different locations resulted in different sinusoidal wobbles and students crafted equations that appropriately accommodated the different wobbles.

How might high-school students, who were unfamiliar with polar coordinates, construct mathematical models for a spiral shell? What choices of prior mathematics would high-school students make to approximate polar coordinates? Would high-school students' work resemble the work of "expert" honors calculus students (Speiser & Walter, 2004)?

Method

This paper focuses on the mathematics work of one group of six high-school students mathematizing the fossilized remains of a spiral shell. The evolution of one solution was selected for presentation here. After examining the actual fossil, students were given a photocopy of the shell and asked to locate the center of the spiral, draw a ray in any direction they chose, and use polar coordinates to find the radius of the shell as a function of theta (Speiser & Walter, 2004).

Videotape of student problem solving sessions, student written work, students' calculator screen and memory downloads, and researcher field notes provided detailed data sources. Videotapes were transcribed and transcriptions were verified for accuracy. Transcripts were linked to video with time codes (hr:min:sec:frame). Time code and the first initial of the speaker's name reference transcript data presented here. All data sources, including mathematical language, notation, presentations, gesture, and calculator data, were together analyzed to determine the mathematics content of students' solutions and whether problem solving choices in learner-initiated inquiries resembled expert, transitional, or novice performances. Comparisons to expert solutions, as presented in Speiser and Walter (2004), provided a basis for anticipation of and approximation to more advanced solution strategies.

Data and Analysis

Students chose to superimpose a spiral onto the shell to provide a visual structure of the shell from which they could reason. Students demonstrated novice performance when: (1) focusing on superficial features (00:08:11:10), (2) a non-productive suggestion for notation was offered and dropped (00:08:11:10), and (3) trying to interpret explicit material provided in the task (00:18:27:21).

00:08:11:10	M	Can I draw the line straight up?
00:11:05:00	M	Why don’t we do, r is x and theta for y?
00:18:27:21	S	Okay, what does table r as a function of theta mean?

One of the students, M, exhibited transitional problem solving performance by: (1) monitoring self and group progress by suggesting successfully that the group agree to select one location for the center of the spiral (00:08:47:15), (2) clarifying goals with respect to finding a radius (00:14:12:27), (3) building from understanding the radius of a circle to the concept of multiple radii of a spiral (00:15:48:17), and (4) organizing, representing and interpreting flexibly by “folding back” (Pirie & Kieren, 1994) to a circle in order to build understanding of the spiral (00:18:38:22; 00:18:47:02).

00:08:47:15	M	You want to all agree on one center?
00:14:12:27	M	I, what I get from it is, like, you’re finding the radius, but, like, are you going from the center to here? [*points to the outer trace of the spiral]*
00:15:48:17	M	So you have four [radii] going from the center to the top [outer trace of the spiral].
00:18:38:22	M	Okay, say this [the radius of a circle] is point five, right? And that [*points to the spiral*] was the circle or whatever, would we measure like radians?
00:18:47:02	M	It would be point five centimeters, right? Around the circle.

M demonstrated evidence of an expert strategy, working with a separate piece of the problem, when she chose to build representations for radians based on a diagram she drew of a circle with radius 0.5 centimeters. Referring to the diagram created by M, another student, R, provided evidence of transitional performance with efficient recall of an algorithm for circumference in the familiar domain of the circle (00:18:51:26). R’s question revealed transitional content knowledge about multiple revolutions in the spiral and flexible use of the algorithm of the circumference of a circle in the unfamiliar content domain of a spiral.

00:18:51:26	R	Isn’t there like six pi radians [in a circle] and should it just be six pi times point five?

Only one student in the group resisted efforts to use circles as a structure for the model of the spiral. But without content expertise with polar coordinates, all six students moved forward to reason about a model for the spiral from the structure of concentric circles (00:19:27:18) and right triangle trigonometry.

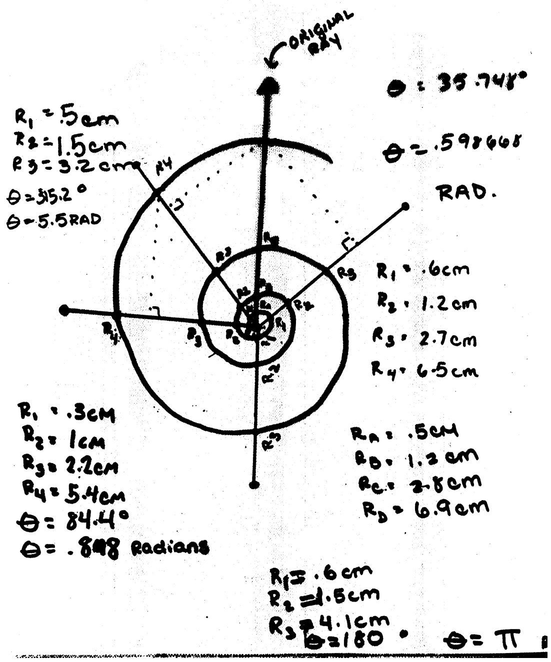

Figure 1.

00:19:27:18 M I don’t think...I know it’s not perfect, but I think that everyone should just, you know, assume, say it is a circle. So why don’t we have one circle that’s point five centimeters and then one circle with one point two centimeter radius?

Students proceeded to measure radii of concentric circles, calculate angle measures they associated with each radius, and organize the collected numerical data into a chart (see Figure 2).

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θ (RAD)	$R_{1(A)}$	$R_{2(B)}$	$R_{3(C)}$	$R_{4(D)}$
0	.5cm	1.2cm	2.8cm	6.9cm
.59 RAD	.6cm	1.2cm	2.7cm	6.5cm
.848 RAD	.3cm	1 cm	2.2cm	5.4cm
π RAD	.6cm	1.5cm	4.1cm	X
5.5 RAD	.5cm	1.5cm	3.2cm	X

θ_T = 6.28 RAD

Figure 2.

Results

Students chose geometric, numerical, algebraic, linguistic, and gestic representations to accurately structure the mathematical relations evident to them in the spiral shell. They collaboratively determined that geometric representations were central to their solution strategy, Their geometric representations included the traced spiral of the shell, rays, circles, circumferences of circles, angles of triangles and central angles of circles, concentric circles with multiple radii, and right triangles.

After questioning one another as to whether the angle or the radius should be designated as the independent variable, students chose theta as the independent variable, and radius as the dependent variable based on their knowledge of the language "as a function of" to relate two variables. Students wrote equations showing the relationships between the independent and dependent variables they had chosen, but did not approximate the data with an exponential equation of the form $r = r_0 a^\theta$. Students used trigonometric relationships to calculate length of radii, inverse trigonometric functions to determine angle measures in degrees, and dimensional analysis to convert angle measures from degrees to radians (see Figure 1). The strategies and representations developed by the students resulted in a solution that demonstrated mathematical relations between a discrete set of ordered pairs, rather than an equation for any radius length as a function of theta. However, the students' work did contain several mathematical strategies and representations evident in expert solutions (Speiser & Walter, 2004). Expert solutions include tracing the spiral, locating the center, drawing rays from the center with angles in increments of $\frac{\pi}{4}$ radians and θ ranging from zero to 6π, and fitting an equation of the form shown above (Speiser & Walter, 2004).

The high-school students chose to adapt their language over time and in conjunction with the evolution of their solution to reflect the language provided in the written task, such as when "this little hole" was eventually referred to as "the center", "this thing" was articulated as "the spiral", and "radius of this thing" changed to a more accurate individual identification of multiple radii with subscript notation. Student gestic representations included circular and spiraling motions with their hands.

Students' purposeful choice to use concentric circles as approximations to the spiral was based on their understanding of the properties of circles and provided the basis for their frequent use of "the circle" when referring to "the spiral." Students chose concentric circles to approximate the spiral because they were not familiar with polar coordinates, but were familiar with mathematical properties of circles. Those properties of concentric circles most closely matched the characteristics they recognized in a spiral.

Implications

A major goal in mathematics education is for learners to recognize and use familiar mathematics to problem solve in unfamiliar problem situations. When given opportunities to initiate and sustain inquiry without instruction on solution strategies, high-school students can purposefully choose from prior experiences to build toward understandings of more advanced mathematics by recognizing similar mathematical relationships in seemingly dissimilar situations. As transitional problem solvers, high-school students lack the content knowledge to structure solutions that expert problem solvers create, but high-school students can flexibly interpret, organize, and accurately represent problem situations in unfamiliar, complex domains.

Increased awareness of the choices students make to build inductively from familiar to unfamiliar mathematics offers mathematics educators insight into how students learn specific concepts. Examining the mathematical choices that students make in structuring solutions may assist in the identification of concepts that students need to more fully understand. Choice and learner-initiated inquiry are actions of personal agency, both of which are essential for individual engagement in productive learning communities. The pragmatic complexities of teachers' pedagogical choices should comprise learner-initiated inquiry and learner choice as central issues in classroom practices to build expert-like problem solving in students.

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UNDERGRADUATES' USE OF EXAMPLES IN A TRANSITION TO PROOF COURSE

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In this paper, we report an exploratory study in which we examined the ways in which, and the extent to which, eleven undergraduates used examples in constructing proofs. Our main findings are that: a) six of these students used examples regularly in their proof constructions while four others did not consider examples for any of their proof constructions, b) the six students who used examples did so for multiple purposes, including understanding mathematical statements, determining the truth value of an assertion, and generating an understanding of why an assertion was true, and c) these six students often failed to construct proofs despite their use of examples, in part because their choice of examples was inappropriate or because they could not express intuitive, example-based arguments within the language of formal proof.

Introduction

A primary goal of advanced mathematics courses is to improve undergraduates' abilities to construct formal proofs. Numerous research studies have documented undergraduates' difficulties in this regard (e.g., Moore, 1994; Weber, 2001), and there is a growing body of research on their specific difficulties with proof (Harel & Sowder, submitted). While there has been considerable research on undergraduates' difficulties with proof, some researchers have noted that there has been comparatively little work about the processes that undergraduates use when they construct proofs, and that more research of this type is needed (e.g., Harel & Sowder, submitted; Weber, 2001). In this paper, we examine just one aspect of undergraduates' proof construction; in particular, we investigate the ways in which undergraduates use examples of mathematical concepts to aid them in constructing proofs about those concepts.

Although proofs in the advanced mathematics classroom are expected to be formal and rigorous, proof construction is not solely a deductive process (Weber & Alcock, 2004). Several researchers have argued that the consideration of examples can be an important component of proof construction. In his discussion of the ways in which mathematicians solve problems and construct proofs, Polya (1957) cites several heuristics that involve the consideration of examples, such as "looking at simple cases" to see why a general assertion is true. Building on Polya's work, Schoenfeld (1985) illustrates the variety of ways that this strategy can be implemented to construct proofs. Garuti, Boero, and Lemut (1998) argue that considering examples can not only help students to determine whether an assertion is true or false, but that this can and should also play a role in constructing a proof of this determination. Alcock (2004) interviewed five mathematicians about their proving practices and associated pedagogical issues. These mathematicians indicated that examples were useful in helping them understand the meaning of mathematical statements, generating ideas for how a statement might be proven, and verifying that inferences drawn within a proof are valid.

The research literature suggests that using examples could help undergraduates construct proofs, but that undergraduates often do not use examples for this purpose. Dahlberg and Housman (1999) report that undergraduates who generate examples when learning about a concept tend to develop richer and more accurate images of that concept. Gibson (1998)

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illustrates how inspection of examples in real analysis (in the form of diagrams) helps some students overcome impasses in their proof-writing when they cannot decide how to proceed. However, Moore (1994) found that undergraduates in a transition-to-proof course often could not prove statements about new concepts because they were both disinclined and unable to generate examples of those concepts. The mathematicians interviewed by Alcock (2004) lamented that the undergraduates in their transition-to-proof courses rarely seemed to use examples when working on proofs.

At present, we are not aware of systematic studies on how undergraduate mathematics majors use examples when they write proofs. The purpose of this paper is to present an exploratory study that addresses this issue. The specific research questions that will be addressed are: To what extent do undergraduates in a transition-to-proof course generate and make use of examples in their proof construction? For what purposes do these undergraduates use examples in their proof construction? What factors inhibit these undergraduates from using examples productively in their proof construction?

Methods

Participants

Eleven students were interviewed individually at the end of a course entitled "Introduction to Mathematical Reasoning", a transition-to-proof course (cf., Moore, 1994) designed to facilitate students' transition from calculation-oriented mathematics to abstract, proof-based mathematics.

Procedure

The participants were first given the following two problems.

Relation task. Let D be a set. Define a relation ~ on functions with domain D as follows. $f \sim g$ if and only if there exists x in D such that $f(x) = g(x)$. Prove or disprove that ~ is an equivalence relation.

Function task. A function $f{:}\mathbf{R} \rightarrow \mathbf{R}$ is said to be increasing if and only if for all $x, y \in \mathbf{R}$, ($x > y$ implies $f(x) > f(y)$). A function $f{:}\mathbf{R} \rightarrow \mathbf{R}$ is said to have a global maximum at a real number c if and only if, for all $x \in \mathbf{R}$($x \neq c$ implies $f(x) < f(c)$). Suppose f is an increasing functions. Prove that there is no real number c that is a global maximum for f.

The participants were presented with these tasks one at a time on separate sheets of paper, and were asked to describe what they were thinking about as they attempted to complete these tasks. They worked without assistance from the interviewer until they either completed the task to their own satisfaction or reached an impasse and could not proceed. The interviewer then asked them about why they had taken specific actions and, if appropriate, about why they now found it difficult to proceed.

Students were then given a copy of their second exam in the course (taken one week before the interview) and asked to recall their thinking while completing two of the questions from the exam. Finally students were asked reflective questions about their proving practices. These questions first started out very generally (e.g., do you have any general strategies that you use when you write a proof?) and then became increasingly focused (e.g., when you are writing a proof, are you thinking of examples, the rules given to you in the course, or some combination of both?).

Analysis

Each interview was audio-taped and transcribed. For each task, the three authors of this paper independently located each time an individual mentioned or made use of an example. There was a high level of agreement on these codings and each disagreement was discussed. Codes were then generated to describe the purposes for which the participants used the examples. These codes consisted of: understanding a statement (US), evaluating whether an assertion was correct (ET), and constructing a counter-example (CX). Each instance of example usage was coded independently by the three authors and any disagreements were discussed. Finally, for each proof construction task, each proof that the student produced was coded as being a valid proof (V), a valid proof with the exception of minor errors (E), or an invalid proof (I). A more thorough description of the methodology of this study is provided in Alcock and Weber (in press).

Results

Table 3 presents a summary of participants' performance and example usage on the proof construction tasks, and the mid-term examination questions that were discussed.

		RELATION TASK		**FUNCTION TASK**		**MID-TERM**
Student Name	**Course Section**	**Example Use**	**Score on proof**	**Example Use**	**Score on proof**	**Example Use**
Andy	A	Yes (US, IL)	I	No	I	Yes (CX)
Brad	A	Yes (US, GP)	E	Yes (US, GP)	I	Yes (US)
Carla	A	No	I	No	V	No
Dan	A	No	I	Yes (US, GP, IL)	I	Yes (CX, US)
Ellen	B	No	V	No	V	Yes (GP)
Fay	B	No	I	No	E	No
Gail	B	No	I	No	I	No
Hannah	C	Yes (US, ET, GP)	I	Yes (US, TV, GP)	I	Yes (ET, CX)
John	C	No	I	No	I	No
Karen	C	No	I	Yes (US, GP)	I	Yes (ET, US)
Lisa	C	Yes (ET, US)	I	Yes (US)	I	Yes (ET, US)

(Purposes of examples use given in parentheses).

Table 1. Summary of students' use of examples.

To What Extent Did Participants Use Examples?

Our data suggest that Andy, Brad, Dan, Hannah, Karen, and Lisa regularly use examples when they are writing proofs. Each of these students used examples on at least one of the two proof construction tasks and each used an example in answering at least one of the questions on the mid-term examination. Further, in the reflective sections of the interview, each student described the use of examples as part of their proving process, indicated that they used examples frequently, and were able to articulate the ways in which they used examples.

In contrast, we found no evidence that Carla, Fay, Gail, and John used examples regularly. These students did not consider examples in either proof construction task, nor in the questions that we discussed from their mid-term examination. When writing proofs, these participants either reasoned deductively from relevant definitions or attempted to apply proving procedures that they had learned in class. Further, when asked to describe their proving processes, these students did not describe using examples. In fact, Carla and Fay stressed that they found example usage in their proof-writing to be problematic so they generally avoided this mode of reasoning. The remaining student, Ellen, was less consistent in her example usage.

For What Purposes Did Participants Use Examples?

Participants used examples for multiple purposes. The most common reason that participants used an example was to understand a mathematical statement. When participants read a statement pertaining to a class of mathematical objects, they would often see how that statement pertained to a single object to obtain a better sense of what that statement was asserting. For instance, when participants read the definition of a mathematical concept, they would sometimes attempt to understand the definition by constructing a single object that satisfied it. We illustrate this with the following excerpt from Brad working on the relation task:

B: Uh…trying to think what the question's asking, sorry… Equivalence relation…*mumbling*…okay. So D subset […] Alright, I'm just going to like write out some examples to try and…like, set a D. And then…yes, write out a function or two. I don't know if that's going to help me.

[Brad writes $D = \{1, 3, 5\}$ $f(x) = x^2$ $g(x) = x$]

B: *Pause*. Would this be an example? Like where f of x is equal to 1, and g of x is equal to 1…and since x is 1, like 1 is in the domain, f is related to g?

In this episode, Brad appeared initially unsure as to the nature of the relation that was being defined. After a brief pause, he introduced two functions that satisfy this relation. Later in the interview, Brad refers back to his example when he is trying to determine what it would mean for the relation to be reflexive and symmetric.

A second purpose that students used examples for was in attempting to determine if a given assertion was true. For instance, on her mid-term examination, Lisa was asked to prove or disprove that if n was an integer greater than one, n was divisible by a prime. Before offering a proof on her exam, Lisa wrote:

n = 2 2/2 = 1 √ n = 3 3/3 = 1 √ n = 4 4/2 = 2 √

When asked to describe her thinking on this question, Lisa replied:

L: I was just trying to use um… induction like if you know n and n+1 then you know, um, you can keep going. So I, um, did the first few and they worked. So I just tried to set it up for n+1.

When asked to describe their proving processes, a number of participants indicated that checking examples to see if the statement they were proving was true was one of the first things that they did. For instance, when Dan was asked how he begun his proof attempts, he stated that, "The first thing that I do is see whether or not I think it's true or false … just by working out examples in my head".

A third purpose that undergraduates used examples for was to generate proofs. When constructing a proof of a general assertion, participants would sometimes inspect a specific example with the purpose of understanding *why* that assertion should be true. They would then attempt to use the gained understanding as a basis for constructing a formal proof. This is illustrated in the excerpt below, as Ellen describes the thought processes that she used to answer the second question on her mid-term exam. On this question, a relation was defined on **R** x **R** such that (a, b) is related to (c, d) if both $a \le c$ and $b \le d$. The question asked the student both to prove that any two-element subset of **R** x **R** had an upper bound with respect to this relation and to find the least upper bound of {(-1, 2), (3, -4)}. The diagram she used to generate this solution is presented in Figure 1. Her description of her solution to this problem is given below:

E: For the second part where it says prove any two element subset of R-squared has an upper bound, um I guess I realized that it would be um... like I sort of skipped it and went to the [next] part where he gave actual numbers. And I just realized that I could draw a graph of it. So I did over here [referring to the graph in Figure 1] and I realized the upper bound would be the corner of the sort of rectangle that it would make … so I just thought that any upper bound would sort of just be to the right and up of your two points. So for this one, it was obvious that it was the corner of the square. The least upper bound would be sort of on the lines and then um, so I went back to the second part then and [inaudible] since it's real numbers … I guess pairs of real numbers so either way you can go an infinite amount. So for any um two pairs that he takes, you can always go um larger I guess. So that's why there's an upper bound.

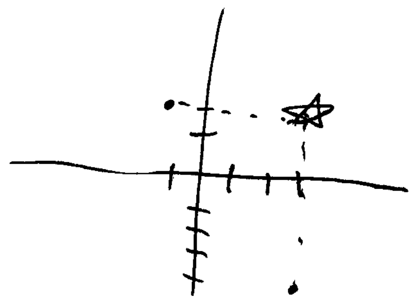

Figure 1. Ellen's diagram of finding a least upper bound for {(-1,2), (3,-4)}.

In the interview segment, Ellen describes how she was able to generalize the technique she used to find an upper bound for {(-1,2), (3,-4)} to prove that any pair of ordered pairs would have an upper bound. Later in the interview, Ellen described the ways in which visual examples help her generate proofs.

E: [Visual examples] just help me understand why it [the statement to be proven] is true. And what makes it true. And so, by that I sort of see what I have to prove within the proof. But the actual way to prove it comes from the definitions and things that we learned in class.

Finally, when undergraduates believed a statement was false, they would search for counterexamples to disprove it. As the use of counterexamples is well known, for the sake of brevity, an episode of counterexample use will not be provided here.

What Difficulties Did Participants Have With Using Examples?

In the preceding sub-section, we described some ways that undergraduates successfully used examples to aid in their proof construction. However, Table 1 indicates that there were many cases in which participants were unable to construct proofs. Collectively, on proof construction tasks, the undergraduates produced a correct or mostly correct proof only 5 out of 22 times, a finding that once again demonstrates the serious difficulties undergraduates have with proof. In cases in which undergraduates used examples, the undergraduates were able to produce a valid or mostly valid proof only one time in nine attempts.

Our data suggest two factors that inhibited participants from using examples effectively. First, participants frequently examined inappropriate examples. For instance, when Andy was attempting the function task, he chose the functions $f(x) = x^2$, $g(x) = x^4$, and $h(x) = x^6$. He concluded that these functions were "related' since they were all parabolas. However, these functions were not the same, and therefore the relation was not an equivalence relation. This led Andy to conclude that the relation defined in the task was not an equivalence relation. Likewise, in considering increasing functions, several participants considered the function $f(x) = x^2$. In some cases, such as Andy's, participants' inappropriate choice of functions appeared in part due to a poor concept understanding or an inaccurate image of the concept that they were looking to instantiate. However, in all cases, the participants did not *check* to see if the examples they constructed matched the definitions or appropriate conditions specified in the problem statement.

Second, participants sometimes were able to construct a convincing intuitive argument for why an assertion should be true, but were unable to formalize this argument in the form of a rigorous proof. On the functions task, Brad, Hannah, Karen, and Lisa all drew prototypical graphs on increasing functions and were able to provide coherent arguments based on their graphs for why such functions cannot have global maxima. However, they were generally at a loss for how they would even begin to express their argument as a proof. This is illustrated with excerpts from Hanna's interview.

> H: Okay, well, drawing the graph of what some…*f* of *x* may look – some increasing *f* of *x* may look like [refers to graphs of a prototypical increasing function and a function with a global maximum]. So increasing so it's just going on to infinity. Forever and ever and ever increasing. For the second one, if it's going to have a global max, at some *c*, That means everything, of the rest of it, of the rest of the graph is under that *c*, because that's the maximum. *Laughs*. Wherever the rest of the graph is, it has to be under there. So…what you just need to be able to *prove* [with comedic frustration] is that wherever this *c* happens to be, this is just going to be – this, this could just even equal this *c* and you're…home free. Because…as *f* of *x* equals *f* of *c*…then, you've already proved that this…that you can't have a global maximum.

Students who reached such an impasse often expressed extreme frustration that they could not prove a statement that was so "obvious" to them. Hannah, for instance, complained, "It's just so obvious a second grader could tell you this… but that doesn't mean you have to … pin it down and prove it".

Discussion

One finding from this paper was that some participants consistently used examples in their proof construction while others almost never did. This finding suggests that participants in this study had qualitatively different *reasoning styles*. Further, our study did not indicate that students who used examples were more successful than those who did not. There were successful students in our study who used each reasoning style (cf., Alcock & Weber, in press) and there were unsuccessful students with each reasoning style. The finding that there are different reasoning styles in advanced mathematics courses has been reported elsewhere in the literature. Pinto and Tall (1999) distinguish between "formal learners" who learn concepts via memorization of definitions and logical deduction and "natural learners" who use their images of concepts to give meaning to their definitions. Alcock and Simpson (2004, 2005) note that some students in real analysis use visual reasoning consistently while others rarely do. Pinto and Tall, as well as

Alcock and Simpson, found little correlation between students' reasoning style and academic success.

Although examples helped some participants understand statements and construct proofs, there were many instances in which they were not useful. One cause of participants' difficulties was that they did not check if examples satisfied appropriate conditions. Although this is a serious problem, it seems more tractable than other difficulties reported in the mathematics education literature. We suggest that relatively explicit instruction on what conditions examples should satisfy to be useful might be able to at least partially address undergraduates' difficulties in this regard. Participants also had difficulty formalizing example-based arguments into formal proofs. This difficulty is more serious, and we conjecture that part of the participants' difficulty was due to their inability to form connections between the definitions of mathematical concepts and the examples of the concept that they constructed. This illustrates the complexity of example use in proof construction and suggests to us that more research should be done on the processes of example-based proof construction and more attention should be paid to these processes in advanced mathematics classrooms. We conjecture that providing more explicit attention to the processes of choosing appropriate examples and using them to generate proofs might benefit students who are having these difficulties.

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DECIPHERING STUDENTS' DEVELOPING CONCEPTIONS OF FUNCTIONS IN A COLLABORATIVE COMPUTING ENVIRONMENT

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This paper uses data from a pilot study of a classroom wireless handheld computer system to investigate students' developing understanding of functions. Set in the applied curricular context of cryptography, the handheld network allowed students to encode and decode text messages using simple polynomial functions, and required them to coordinate different representations on their linked devices in order to collaboratively solve decoding problems. The paper reports on aspects of students' emergent function conceptions, and examines those conceptions in the context of previous work on student understanding of functions. The findings presented suggest that the multi-representational computing environment and the applied context of the problem-solving tasks supported students' engagement with functions as both processes and objects, and that student groups gradually developed problem-solving techniques that drew on increasingly sophisticated conceptions of function.

Many observers of contemporary mathematics education have argued that functions should be seen as the cornerstone of the secondary mathematics curriculum (Leinhardt, Zaslavsky & Stein, 1990; Harel & Dubinsky, 1992; O'Callaghan, 1998). Others have further suggested that such a central role for functional relationships follows from the advent of classroom computing tools capable of rapidly and dynamically integrating algebraic, graphical, and tabular representations of those relationships (Schwartz & Yerushalmy, 1992; Kieran, 1993). The handheld computer learning environment described in this paper represents an attempt to integrate these multiple representations with the collaborative affordances of networked devices in order to investigate the ways such a system might support students' learning about functions.

The same breadth and complexity that account for the widespread relevance of functions to the school curriculum also pose considerable difficulties for specifying student learning with regard to the concept. Aspects of the understanding of functions might include formal definitions involving set correspondence or ordered pairs, translations among representational modes, interpretations or procedural manipulations within one of those modes, and so on. The distinction between functions as *processes* and as *objects* provides one way to sort through much of this conceptual complexity. Taking this view, Sfard (1992) proposes that students might be expected to encounter and grasp operational aspects of functions—the process conception—first, and only gradually see those operations reify into abstract structures, or objects, which might themselves undergo operations. Briedenbach, et al. (1992) expand this framework to include *prefunction* and *action* conceptions of function, where viewing functions as actions amount to a "pre-process" perspective focused on the step-by-step evaluation of an algebraic expression.

The central claim of this paper is that the collaborative multi-representational environment and the applied context of the problem-solving tasks may paint a somewhat different picture of student's learning of functions from those emerging in more conventional instructional settings. In particular, the curricular unit developed for this study was set in the context of cryptography, and emphasizes a conception of functions-as-codes that may promote an object perspective.

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Moreover, the deployment of those code functions across several representational modes may further encourage the object view; other researchers have suggested that while symbolic representations of functions tend to make the process aspects salient, other representations such as graphs might more readily invite an object perspective (Schwartz & Yerushalmy, 1992).

Schwarz and Dreyfus (1995) argue that computer-based multi-representational environments can lead to an ontological shift in learners' perspectives on functions. In their account, actions within a given representational setting, such as rescaling a graph or reordering a table, create different *representatives* of the same function within that setting. By allowing learners to flexibly engage an array of such representatives rather than limiting them to the partial perspective provided by a fixed object in a specific setting, such as the graph of a function restricted to a particular viewing window, these actions invite learners to consider those properties of a given function that remain invariant across different representatives. Similarly, Slavit (1997) suggests that as students perform comparable procedures and encounter common features across varying notational systems and types of functions, they develop an understanding of functions as characterized by properties according to which they can be analyzed and classified. This property-oriented view of functions represents one path by which students' conception of functions might reify into abstract objects characterized by such properties.

Informed by work with another computer-based multi-representational learning environment, Moschkovich, et al. (1993) make the case that many problem contexts, particularly those that they would identify as most rich and worthy of curricular emphasis, encourage or require students to alternately engage both object and process perspectives on functions. Moreover, such tasks should encourage use of connections among several function representations, including symbols, tables and graphs as well as real word contexts and verbal descriptions. The authors argue that understanding of functions might be envisioned in terms of a learner's ability to flexibly move in a grid spanned along one dimension by the process and object perspectives, and along another by an array of representational modes.

Thompson (1994), however, cautions against drawing hasty conclusions about the nature of connections drawn between different forms of representational activity, asserting that "the core concept of 'function' is not *represented* by any of what are commonly called the multiple representations of function, but instead our making connections among representational activities produces a subjective sense of invariance" (p. 39). He argues that while a knowledgeable adult may perceive a mathematical object such as a function being represented by, for example, a table or graph, a student considering the same representation will not necessarily recognize the same object, let alone the continuity of that object across other representations (Thompson & Sfard, 1994). Instead of cultivating an object conception of functions by highlighting the invariance of those objects across different domains, attempts to engage learners with multiple representations of functions may simply leave them with disconnected conceptions of those various representations. Rather than focusing on abstract objects, instructional activities involving multiple representations should emphasize aspects of specific situations which students might themselves be able to represent, and to perceive across different forms of representational activity (Thompson, 1994). The remainder of this paper will investigate the extent to which the cryptography context and the computing environment provided such a situation for students to engage and represent functional relationships.

The *Code Breaker* Pilot Study

This paper reports on a project involving the development of both software and curricular materials for a middle school mathematics unit, and the piloting of that unit for five weeks with a diverse group of 100 students in four summer school classrooms. The unit introduced the topic of algebraic functions through the applied context of cryptography. A handheld software program titled *Code Breaker* provided students with a suite of tools—graphs, function tables, letter and word frequency charts—for analyzing various representations of the relationship between clear and encrypted text. By assigning the numbers one through twenty-six to letters in the alphabet, *Code Breaker* allowed students to input simple polynomial functions which map the letters to a set of output values comprising the numerical alphabet for the cipher text. The teacher used a desktop computer as a central server to organize students into groups so that each handheld in the group was synchronized to show the same function. Thus, when one student altered the algebraic function on her computer, the corresponding graph, text and tables simultaneously changed not only on her device, but also on her group mates' devices.

This paper will summarize from a detailed analysis of the problem-solving work of two groups, A and B, comprised of four students each. I draw from video data of student interactions in a total of thirty-two decoding "events" undertaken by the two groups over a period of nearly three weeks. These events, ranging from two to thirty minutes in duration, began when a group downloaded a new code to break, and concluded when they either solved or stopped working on the code. All decoding events were transcribed and coded across several analytic categories. These video records, along with partial server logs of student decoding activity, researchers' notes, and written records maintained by the groups, allow for reconstruction of the problem-solving strategies employed by the groups.

In order to characterize the ways in which the groups found increasingly sophisticated ways to break codes, I distinguish three levels of code breaking strategy. According to this framework, groups using level one approaches to decoding made use only of the "general features" of an encoded message, such as the size and sign of values in the coded text, without considering how the encoding function would map specific input values to specific outputs. By contrast, second level strategies involved the specification of a single ordered pair—one association between a letter in the standard alphabet and a number in the encrypted message. Similarly, groups employing level three code-breaking techniques specified and made use of multiple ordered pairs associated with a given code.

The hierarchy among these approaches reflects their relative code-breaking efficacy. While level one inferences usually (though not always) led students to precisely identify the exponent of an encoding function, they rarely determined fewer than several possible values for the coefficient, and provided virtually no guidance in identifying the constant. By contrast, given a known exponent and a suspected coefficient produced by first-level inferences, groups who achieved level two insights could then determine a corresponding constant, allowing them to either specify the correct function or eliminate a possible coefficient. Those groups who successfully combined multiple ordered pairs were able to specify the function given only a correct exponent. In other words, the higher the strategic level demonstrated by a group, the more parameters of an encoding function they were likely able to deduce.

Table 1: Levels of Code Breaking Strategy

Level	**Approach**	**Examples**
One	Approximating candidate function parameters based on "general features" of an encoded message: sign and magnitude of numerical values, fit of graph to fixed viewing window	Deduced an encoding function with even coefficient and constant values from a message made up of exclusively even numbers; deduced a linear encoding function with approximate coefficient four (y=4x+b) from graphing window domain [0,26] and range [-10,100]
Two	Specifying candidate function parameters based on a single ordered pairing of an ordinal letter-value input and a coded text output value	Given the candidate y=4x+b above, group identified the least-valued number in the message (e.g. -3) as an A, and uses the ordered pair (1,-3) to deduce b=-7.
Three	(Uniquely) specifying candidate function parameters by simultaneously considering multiple ordered pairs in the code	Given an encrypted message with two single-letter words, represented by the numbers 8 and 88, group assumed 8=A and 88=I, then used ordered pairs (1,8) and (9,88) to uniquely specify linear function y=10x-2 and quadratic y=x2+7

This hierarchy appears to correspond to the development of the groups' approaches to breaking codes over the course of the unit. Tables 2 and 3 summarize key details from the respective lists of codes that Groups A and B attempted to solve during the unit. Because the groups each attempted to decode some messages that were assigned by the teacher to all students, and some that were encrypted by other groups, the lists are not identical, but they do each feature a set of codes that are generally similar and in several cases the same. The functions used to encode these messages grew more complex as each group progressed through the tasks, featuring gradual increases from first to third degree polynomial functions, and the eventual introduction of codes with offsets (which shifted the mapping between the letters and their ordinal values). Many of these increases in code complexity were accompanied by changes in the ways the group went about trying to solve them, as each group began breaking codes using only level one inferences, and eventually experimented with level two and then level three approaches.

Table 2: Codes and Strategies (Group B)

Code	Encoding Function	Level	Solved	Approaches
B1	Linear	**1**		General only
B2	Linear	**1**	Yes	General, incorrect ordered pair
B3	Linear	**1**	Yes	General only
B4	Linear	**2**	Yes	General, ordered pair
B5	Linear	**2**	Yes	General, ordered pair
B6	Linear	**2**		General, ordered pair
B7	Linear	**2**	Yes	General, ordered pair
B8	Quadratic	**3**	Yes	General, two ordered pairs
B9	Quadratic	**3**	Yes	Two ordered pairs
B10	Linear, Offset	**2**	Yes	General, then ordered pair to determine offset
B11	Linear, Offset	**1**	Yes	General only
B12	Quadratic, Offset	**2**	Yes	General, ordered pair
B13	Cubic, Offset	**1**		General only
B14	Cubic, Offset		Yes	(Could not be determined)
B15	Cubic, Offset	**2**	Yes	General, one ordered pair

The first seven codes Group B attempted to solve were all based on linear functions, and gradually introduced parameters with increasing magnitudes and negative values (B2, for example, was encoded with the function y=2x, while B7 was based on y=6x-14). While two of the first three codes yielded to level one strategies, B4 (y=-4x+7) proved much more resistant. After fifteen minutes of sustained but unsuccessful efforts with level one approaches, they drew on a level two insight that allowed them to decode the message. They used similar approaches fairly effectively on the remaining linear codes. Events B8 and B9 marked the first appearances of both quadratic encoding functions, and of level three decoding strategies. The group's particular use of these level three approaches, however, hinged on certain specific features of those two codes, including the absence of an offset. Over their final six decoding events, all of which featured codes with offsets, the group was unable to adapt their level three approach to these new challenges, though they did find other ways to solve all but one.

Two of the most notable features of Group A's decoding events come by way of comparison to Group B: they implemented second-and third-order strategies much less frequently than their counterparts, and they broke considerably fewer codes. Whereas the girls utilized second-or third-order approaches in ten of their fifteen decoding events, the boys did so in only four of seventeen. While Group A attempted to specify an ordered pair in events A2 and A4, they did not correctly do so until event A9, nor did they repeat the feat until event A15. Nonetheless, the level three decoding strategy they successfully implemented in the last was arguably more effective than the one Group B utilized, proving equally effective for codes with and without offsets.

Table 3: Codes and Strategies (Group A)

Event	Encoding function	Level	Solved	Approaches
A1	Linear	**1**		General only
A2	Quadratic	**1**		General, incorrect ordered pair
A3	Linear	**1**	Yes	General only
A4	Linear	**1**		General, considered one incorrect pair
A5	Linear	**1**	Yes	General only
A6	Linear	**1**	Yes	General only
A7	Linear	**1**	Yes	General only
A8	Linear	**1**	Yes	General only
A9	Linear, Offset	**2**		General, tried ordered pairs
A10	Linear, Offset	**1**		General only
A11	Linear	**1**		General only
A12	(Unknown)	**1**		General only
A13	Cubic, Offset		Yes	(Could not be determined)
A14	(Unknown)	**1**		General only
A15	Quadratic, Offset	**2**		General, max
A16	Cubic, Offset	**2**	Yes	General, ordered pair
A17	Cubic, Offset	**3**	Yes	General, used tables to match multiple pairs

Likewise, while Group B successfully solved codes in thirteen of fifteen events, Group A managed only eight of seventeen. This difference, however, may largely reflect greater persistence on the part of Group B. While the girls generally stayed at a code until they had solved it or until the end of class, the boys often abandoned one message fairly quickly in favor of another. Group A's tendency to switch codes midstream also helps to account for the less smooth progression in the complexity of the codes they attempted. Both the quadratic encoding function they encountered quite early, in event A2, and the linear function without offset they engaged in event A11 after having solved many similar codes were associated with encrypted messages they had begun decoding before finishing others.

Discussion

Despite these differences, Groups A and B generally displayed quite similar patterns with regard to their developing code-breaking proficiency, each beginning to solve problems with level one techniques only and then gradually experimenting with and incorporating higher level strategies into their approach. Inasmuch as these levels capture the trajectory along which each group learned how to decrypt messages in the *Code Breaker* environment, they prompt questions about the relationship between that trajectory and the nature of students' learning about the functions on which the codes were based. In particular, to what extent might these groups' steady advancement from first to third level decoding approaches reflect their gradual reification of the function concept?

Use of the techniques I have identified as level two approaches to breaking codes would likely require students to understand functions as actions or processes. Level two strategies make an explicit connection between the parameters of a candidate function and the mapping between one plaintext input and an associated cipher text output. In other words, they require seeing the candidate as a process or algorithm through which the input value is mapped to a specified output. In doing so, students would certainly be demonstrating "the ability to plug numbers into an algebraic expression and calculate" associated with an action conception of functions (Briedenbach, 1992, p. 251). Inasmuch as students enacting these level two strategies for determining an encoding function recognized the uniqueness of this mapping from ordinal input to encrypted output, they could also be said to have developed a process conception of function.

By recognizing that a candidate function must provide a mapping that holds simultaneously across multiple ordered pairs, groups employing level three approaches may well have been demonstrating more object-like conceptions of function. Level three strategies hinge not only on identifying multiple pairings of a letter in the alphabet and a number in the coded message, but also on using those multiple pairings *in combination* to determine the encoding function. In other words, those strategies make use of the fact that encoding functions are not simply mappings between singular input-output pairs, but also fixed relationships between a set of input values (the ordinal values of letters of the alphabet) and a set of output values (the numbers in the code). Such a set-mapping perspective of encoding functions is strongly suggestive of an object conception.

While these characterizations of groups' successive development through level two and three strategies appear compatible with the theory of reification, level one decoding approaches complicate the picture. When students working with algebraic and tabular representations of the coding situation who inferred, say, a negative linear coefficient from an array of negative numbers in the coded text likely demonstrated an action or early-process conception of function. However, students' use of the graphing feature in the software, even relatively early in their use of the handheld software, routinely featured inferences that did not require specifying ordered pairs, yet suggested elements of an object perspective. Such inferences relied heavily on the fixed dimensions of the graphing window; because the x-axis spanned the set of alphabetic input values and the y-axis automatically adjusted to fit the range of output values in a coded message, students effectively worked to match up these sets by adjusting the candidate function parameters to achieve a good "fit" to the window. In other words, they worked to identify graphs that appropriately represented the complete set of ordered pairs comprising the code they sought to break. Just as students observed by Schwarz and Dreyfus (1995) were able to recognize invariant function properties by adjusting window dimensions in a graph, Groups A and B appeared to recognize and compare among members of parameterized classes of candidate functions as they worked within a window fixed by the set-mapping of the encoding function.

To be sure, as Thompson's (1994) caution reminds us, these students were not necessarily *seeing* functions as objects when they examined the graph in this way. They were, however, *making use* of object-like properties of candidate function graphs in order to specify the encoding parameters. I do not want to argue that such use indicates those students had achieved an object perspective. Rather, and unlike the multi-representational connections against which Thompson warns, these engagements with the graphs of functions may demonstrate the aptness of the coding situation for representing function in ways that might contribute to cultivating object conceptions. Grounding functions in both the context of codes and an array of representations *of those codes-as-functions* may have provided students with object-like foundations to their level one decoding activity, on which they could later build more sophisticated process-like level two and object-like level three insights. In a subsequent paper, I will provide a detailed examination of several episodes of students' work with these representations at each of the strategic levels in order to more fully characterize the interplay between their developing code-breaking expertise and their understanding of the function concept.

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A THIRD-GRADER'S DEVELOPING NUMBER SENSE: A CASE FOR OLIVIA

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This paper is a preliminary report of a teaching experiment that was conducted with one 3rd grade child. Our aim during these sessions was to support Olivia's understanding of number. From the outset of the sessions, we necessarily needed to move beyond purely cognitive accounts to address the range of challenges that Olivia faced as she developed and refined her understandings. This report outlines several themes that began to emerge during the sessions. These themes include Olivia's developing mathematics disposition, the socially-situated nature of the sessions, and Olivia's mathematical progress.

This paper is a report of a teaching experiment that was conducted with one 3rd grade student, Olivia. We worked with Olivia over an 8-month period for a total of 25 one-hour sessions. Our goal during each of the sessions was to support Olivia as she developed and refined ideas about early number concepts. At the same time, we hoped to develop a better understanding of those activities that supported in part her continued progress. Additionally, we hope to develop an account of which activities within and across sessions that contributed to Olivia's successes. As we do so, we also trace a *potential sequence of activities* that might be implemented in other teaching situations. On another level, we address concerns about how the teacher-researcher (or teacher) might support students who experience tremendous difficulties stemming from 'unforeseen' factors such as the child's school math experiences, the conceptual challenges the child faces, and so on. Because Olivia did not necessarily perceive mathematics as a sense-making activity, we had the added challenge of capitalizing on instances during the sessions that made it possible for her to make shifts in how she perceived her own mathematical activity. As such we had the unique opportunity to trace, seemingly incrementally, Olivia's conceptual as well as *emotional* progress. As she developed more efficient and sophisticated ways of reasoning, she appeared to develop a mathematical disposition in which she began to view knowing and doing mathematics as a sense-making activity in which she manipulated experientially-real mathematical objects in flexible ways.

Before reporting our analysis, we briefly address several methodological issues that framed our efforts. To do so, we first consider issues related to our initial assumptions from the outset of the teaching experiment. We then provide information about the research methods that we employed. Following this discussion, we discuss three overarching themes that have emerged during our analysis of the first several teaching sessions.

Theoretical Considerations

One of our goals during the teaching sessions was to support Olivia as she continued to develop and/or refine her understandings of number concepts. Often teaching experiments are designed to develop models of children's conceptual schemes (e.g., counting schemes, Steffe, von Glasersfeld, Richards & Cobb, 1983). Our intent during the interview sessions was to identify shifts in Olivia's number schemes using Steffe et al.'s (1983) preexisting model for

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children's number development. As such, our focus, although informed by our understanding of the quality and nature of children's numerosity, was primarily on how to support Olivia's continued development. To this end, we were particularly interested in those activities that might allow Olivia to draw on and build on her current understandings as she engaged in various games and activities.

The teacher-researcher's role during these sessions was one of continually moving from choosing and adapting activities that might support Olivia's continued development. During the sessions, we introduced various mathematics games and activities designed for young children (Kamii & Housman, 2000), however it was often necessary to adapt these games for our particularly purposes for a given session. Which games to introduce and what changes, if any, needed to be made, were sometimes decided a priori, whereas at other times, changes were made necessarily "on the fly" in order to best support Olivia's ongoing mathematical activity (cf., Steffe & Thompson, 2000).

Data Collection and Analysis

Data included videotaped recordings of each of the sessions as well as expanded notes made by the teacher-researcher after each of these sessions. The teacher-researcher's expanded notes documented the activities that were introduced as well as any changes or adaptations that she made to those activities during the sessions.

As the teacher-researcher documented the activities, she developed initial hypotheses about those activities or teaching situations that may have been particularly significant, that is, instances that may have become possible learning opportunities for Olivia.

These preliminary analyses were used to refine, to develop and in some cases, to refute initial hypotheses as the researchers carefully analyzed each of the videotape recorded sessions. As a consequence of conducting additional analyses using methods that fit with Glaser and Strauss' (1967) constant comparative method, we further clarified those situations that supported Olivia's continued progress.

Findings

Olivia's Developing Mathematical Disposition

One of the important issues that surfaced during the initial teaching sessions related to Olivia's mathematical disposition. Initially, Olivia appeared to view her mathematical activity as that of giving correct answers in which the solutions, themselves, seemed to hold explainable power that were beyond justification. It was as if she saw no need to justify or explain her ideas or solution methods. In a few instances, when she did offer explanations after being prompted to do so, she explained the procedures that she used, albeit she used these procedures incorrectly. Additionally, at times, we were puzzled by the methods she used to solve tasks for which the answer was not readily known. When she did not have a way to solve a task, she sometimes invoked (or perhaps developed) coping strategies in which she offered answers that, from an observer's point of view, might be nonsensical. Her solutions, as they were, helped us to clarify what we inferred to be her current understandings. To illustrate our point, we recount a scenario that occurred with some regularity during the first several sessions as we played Double War. You may recall that this game is played using a deck of playing or numeral cards. Each player receives the same number of cards to begin play. The object of the game is to win all of one's opponents' cards. During a round of play each player places two cards face-up simultaneously and determines who has the largest sum. The player with the largest sum wins the round (i.e., collects all the cards played during the round). The game is over when one player has collected all of the cards.

On several occasions, when Olivia showed a pair of tens (the largest pair of addends in this game), she gave an answer of 11. She did so with confidence, almost instantaneously. When prompted to explain how she arrived at this answer, she had difficulty doing so. At one point, however, when explaining how quickly she arrived at 14 when 10 and 4 were played, she explained that she dropped the zero from the 10 (the numeral in the upper corner of the playing card) and put the 1 and 4 together to make 14. Her explanation was informative for several reasons. Certainly, as a consequence of giving this explanation, we had insight into how she possibly derived the answer of 11 (by juxtaposing the two ones from each numeral, 10). Secondly, we see an instance in which Olivia may have invented a strategy for reasoning about numbers, per se. However, when she employed this strategy she did not add the two quantities, but instead juxtaposed the two numerals, 1 and 1 for instance, to arrive at her answer of 11. Given the fact that Olivia did not know combinations for sums up to 10 such as 3 + 3 = 6, her *self-invented procedure* was particularly problematic. Further, unless she had opportunities to develop a strategies in which she operated with numbers as experientially-real mathematical objects, instances such as the one we describe here might have contributed to her view that doing and knowing mathematics did not involve sense-making.

Viewing the Teaching Sessions as Social Accomplishments

We were aware of some of the difficulties that Olivia faced in her regular classroom. However, without knowing with some confidence how Olivia came to have this view of her mathematical activity, and the school setting in which she continued to building her understandings, we cannot speak to these or possibly other factors that contributed to her current perceptions. What we did become increasingly aware of during the sessions was how quickly she shifted from solving tasks in two different ways. Sometimes she solved tasks by drawing on her current understandings. In other instances, she invoked, from our view, solution methods in which she used meaningless procedures. Typically, if we presented paper and pencil tasks or posed bare number sentences, she offered solution methods that we could characterize similarly to our early Double War example.

Her responses to these types of tasks caused us to immediately refine our goals during the sessions. We were pressed to ensure her that the types of activities that we presented during these sessions were different from those that she engaged in at school. Of course, we did so indirectly by choosing activities that did not mirror the types of activities that she may have engaged in during the school day. Also, as we worked with Olivia, we encouraged her to give explanations in which she explained what she was thinking about and how she solved various tasks. In doing so, we also communicated to her that we valued her ideas and that we expected her to reason about numbers as mathematical objects.

Thus, we began to appreciate anew what Steffe and Thompson (2000) refer to as the teacher-researcher's role being responsive and intuitive. At times, it was necessary to move beyond are initial goal of understanding Olivia's mathematical activity and how we might support her continued progress. In these instances the teacher-researcher needed to constitute and reconstitute with Olivia our roles and our expectations during the sessions. In some cases we did so explicitly; in other cases we did so implicitly. As such, not only was it necessary to consider Olivia's understandings across the sessions, but also we needed to consider how the teacher-researcher and Olivia mutually established what constituted engaging in meaningful mathematical activity. For these reasons, in retrospect, we have begun to view the teaching sessions as social events in which Olivia had opportunities to construct and/or refine her understandings of number.

To illustrate our point we consider an episode that occurred during the second teaching session. During this episode, Olivia and the teacher-researcher played *Counters in the Bowl*. To play this game, players take turns covering some or all of the counters and then they prompt the other players to determine how many counters are hidden under the bowl. Prior to the game, the players agree on the number of counters that they will use. During the episode that we will consider, Olivia and the teacher-researcher decided to play the game with 6 counters. We enter the dialogue after the teacher-researcher has posed the next problem (three counters are showing and 3 counters are hidden under the bowl). As the dialogue proceeds, Olivia decides how many counters are under the bowl.

Olivia: I know this is 3 (covers the 3 visible counters with one hand).
Teacher: Uh-huh.
Olivia: There're 2 more (holds up 2 fingers) under there.
Teacher: Let's see (points to the bowl).
Olivia: (Tries to lift the bowl.)
Teacher: (Holds the bowl down so that 3 counters remain covered) Can you figure it out? Can you figure it out with out looking? Let's see.
Olivia: I said, "Hmm. Is that right /Teacher: 3?/ or wrong (jesters by pointing to her head as if she is thinking)?"
Teacher: 3…can you count to figure it out?

To explain how many counters were hidden under the bowl, initially, Olivia tried to lift the bowl to see how many were under it. We suspect that she did so to respond to the teacher-researcher's comment, "Let's see." She then realized she needed to provide a different explanation. So when the teacher-researcher asked her to figure out if 2 counters were under the bowl, she was obliged to offer an different explanation. As such, note that she immediately explained that she wondered if the answer was right or wrong. Her explanation is puzzling at first glance because we are not certain if she realizes that her answer was wrong. Because of the teacher-researcher's response, she may have inferred that indeed her answer of 2 was not correct. As such, this might explain why she chose to give the response that she did. On the other hand, perhaps she was not sure how to explain her answer. Certainly there may be other factors that contributed to her response at this juncture. As the dialogue continued, we develop some sense of how the teacher-researcher interpreted her response. Note that the teacher-researcher immediately prompted (and in fact, actually spoke over) Olivia and asked her to count number of counters that might be under the bowl. For this reason, we suggest that the teacher-researcher interpreted Olivia's response to mean that Olivia was unsure of how to explain her answer.

As we consider this brief exchange, we note that this situation is a possible juncture where the teacher-researcher and Olivia might begin to establish what constituted an acceptable mathematical explanation (Yackel & Cobb, 1996). The extent to which the teacher-researcher saw this as an important instance that she might capitalize on is apparent by her response to Olivia's explanation. Note that the teacher-researcher actually spoke over Olivia to prompt her to consider the three counters. As she did so, she implicitly communicated to Olivia that her line of reasoning (about the correctness of her answer) was indeed not an acceptable explanation.

The extent to which Olivia might adjust her mathematical activity at this juncture remains unclear. As the dialogue continues, it will be important to determine if Olivia can participate in a different type of activity, an activity in which she reasoned with numbers as mathematical objects in sensible ways. We reenter the session as Olivia (and the teacher-researcher) determined that there were, in fact, 3 counters under the bowl.

Olivia: 3 (holds up hand with no fingers extended), 4 (holds up one finger), 5 (holds up a second finger).

Teacher: And we know we have to have 6. 3, 4 / Olivia: 4/ (holds up one finger), 5…. /Olivia: 5/ (holds up second finger and then holds up third finger).

Olivia: 6.

Teacher: (Continues to hold up three fingers) How many [counters] are underneath there?

Olivia: 3.

Teacher: Wow, let's see if you are right. Ah, you are. Fantastic!

In the above exchange, note that in response to the teacher-researcher's prompt, Olivia was obliged to count-on to determine the number of counters under the bowl. Interestingly, as she did so, she stopped after holding up two fingers. We suspect that she stopped counting after she had 2 fingers extended because the two fingers she held now represented her answer of two. As the exchange proceeded, in response to Olivia's count, the teacher-researcher reminded Olivia that there were 6 counters altogether, and began to count-on as Olivia restated the number words she had counted. By doing so, the teacher-researcher, too, was obliged to support Olivia as Olivia made the count to find the correct number of counters under the bowl. So by the end of this exchange, the teacher-researcher and Olivia had together constituted the answer of 3. By doing so, they had also mutually constituted what counted as an acceptable explanation. Continuing with this line of reasoning, the teacher-researcher's comment at the end of this exchange gives further support for this interpretation. Note that she exclaimed, "Fantastic!" By making this comment, she communicated explicitly not only that the answer was correct, but also the way in which they had derived this answer (and explanation) were particularly valued. In this case, then what constituted an acceptable explanation included counting-on and keeping track using one's fingers as one did so.

This exchange is one of many instances in which the teacher-researcher and Olivia constituted and reconstituted what counted as an acceptable explanation. As Olivia routinely offered these types of explanations, the teacher-researcher and Olivia began to take such explanations as common ways to solve tasks and to explain one's ideas (i.e., taken-as-shared). As such, the teacher-researcher continued to reconstitute norms (what Yackel & Cobb, 1996 refer to as sociomathematical norms) by stating that she particularly valued Olivia's solution method. In retrospect, when we considered Olivia's explanations across subsequent sessions, we realize that instances such as the one we have offered here were important and contributed in part to Olivia's progress during the sessions.

As an aside, we note that instances such as the *Counters in the Bowl* episode were occasions for Olivia to make possible shifts in her view about what it means to know and do mathematics. As she participated in these sessions, she had the opportunity to develop a different view about her own mathematical activity—one in which drawing on her own understandings to determine solutions for problems for which the answer was not readily known. These and other exchanges were important and contributed in part to Olivia's perceptions of herself as a mathematical learner.

Olivia's World of Number

During the first sessions, when finding sums using a pair of dice (with pips), Olivia immediately recognized the various patterns without counting the individual pips. However, to sum the pips from a pair of dice, she typically counted-on. (She usually did not use this method when adding 1 more.) As she counted, she did not always count efficiently. For instance, she did not typically count-on from the larger number to make the count. During the second and third sessions, however, Olivia began to count from the larger number more and with more regularity. In fact, she sometimes self-corrected herself if she began her count with the smaller number. As she began to use this strategy more frequently, she began to employ it to sum larger numbers. For instance, she used this strategy to solve, 10 + 5, one of the more difficult tasks, during the second session while she played Double War. To determine how many she had, she first said the number word, five. Then she proceeded to count-on correctly from 10, extending one finger at a time as she said the number words, 11, 12, 13, 14, and 15.

More generally, Olivia had not yet committed to memory many of her sums up to 20. Also, she did not consistently add one or two more, mentally. Interestingly, as she engaged in several of the activities during the first several sessions, she began to consistently apply a plus-one strategy to find sums up to 10. By the second session, for sums up to 8, she sometimes added two more by verbally counting-on. By the fourth session, she consistently count-on two more, often without using her fingers to keep track of how many she counted.

Although Olivia continued to use counting strategies throughout the first several sessions, she began to develop more consistent approaches as she did so. Additionally, she began to use methods that were slightly more sophisticated. In some instances she might recall a number relation for sums such as 2 + 2 or 5 + 5. She also began to count-on verbally and sometimes mentally to determine sums. These shifts were still tenuous by the end of the third session. We suspect that the activities that she engaged in contributed in part to the progress, albeit incremental, that she made.

Some Instructional Design Issues

One of the goals during the teaching sessions was to develop a sequence of tasks that might support Olivia's continued progress. We knew that we needed to pose tasks that were markedly different from standard paper and pencil tasks. We also needed to develop tasks to support Olivia as she developed more efficient counting methods (e.g., counting all, counting-on from the largest). In addition, we also targeted ideas related to adding one and adding two more to a given number. By posing these types of activities, we hoped that Olivia might begin to build relationships among the different number combinations in which one of the addends was 1 or 2. We hypothesized that she could use her counting methods to make these relationships. We also hoped that over time, she would no longer need to count to determine these sums.

We used several games during the sessions, including Double War, Cover-up and All but 7, Plus-one Bingo, Plus-two Bingo, Plus-one Cover-up, and Nickelodeon. Here we speak to the utility of using Double War to illustrate the sequence of activities that proved to be useful.

Double War proved to be a very useful game to use during the early sessions. Because Olivia had difficulty counting efficiently, we chose to play a modified version of the game. As such, we used the numeral cards for 1-5 initially, and quickly included the 10-card. As we have illustrated in our earlier discussion, when we introduced the slightly revised version of the game (inserted the 10-card), Olivia, had opportunities count-on from 10 accurately as early as the second session. Our hope was that as she continued to play this game, we would introduce all the numerals, 1-10 to play the game.

Final Comments

In our discussion we have highlighted several themes that emerged during the first several sessions of a teaching experiment that we conducted with one 3rd grade student. Here we briefly summarize our discussion and offer some additional comments.

Our comments relate to our emerging view that the teaching session is a social event in which the teacher-researcher and the child negotiate tasks, their (differential) roles, and what constitutes genuine mathematical activity. As was the case with Olivia, we were pressed to make sure that she understood what our expectations were for solving tasks, for making sense of tasks, her mathematical activity, and so on. The extent to which we needed to focus our efforts in these ways was a surprising result. We suspect that there are reasons, some beyond the scope of our discussion, that account for this phenomenon. Here, our remarks address one of our concerns. It is clear that Olivia struggled a great deal during the sessions to make sense of and to build new ideas. As we worked with her, however, we were not convinced that her struggles were bound to what she could or could not understand. At times during the sessions, it seemed as if she had never considered that one might engage in mathematical activity to make sense of or to explain one's ideas. Such perceptions, for students such as Olivia, might have dire consequences. When we picture what might have happened to Olivia as she advanced to 4th grade had she not worked with us, we are grateful that she began to make steady progress during the sessions. Accounts such as the story of Olivia should remind us to continue to advocate for children who experience tremendous difficulties engaging in school mathematics.

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WHAT ARE STATE-LEVEL MATHEMATICS TESTS ACTUALLY MEASURING?

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The purpose of this study is to investigate the relationship between students' success on state-mandated mathematics proficiency tests and students' conceptual and problem solving knowledge related to mathematics. In particular, this study considers student scores from Virginia's Standards of Learning (SOL) tests to investigate whether students who pass these tests possess competence in mathematical problem solving and conceptual understanding. Further, this study investigates the validity of the SOL test scores as measures of student competence in these mathematical processes. Findings from the study suggest that a passing score on the SOL test may not be indicative of proficiency in conceptual understanding and problem solving and thus brings into question the validity of the SOL tests as adequate measures of these processes.

Introduction

In the United States, many states have developed sophisticated assessment programs used to evaluate student achievement and hold schools, teachers, and students accountable for learning important content. *The No Child Left Behind Act of 2001* has put further pressure on schools and school districts to make sure that all students are learning. Virginia, like many states in the US, has developed their *Standards of Learning* (SOL) (Board of Education [BOE], 1995) and associated tests to evaluate the success of schools and teachers in enabling their students to meet these standards. In Virginia, students are administered mathematics achievement tests in grades 3, 5, 8 and end-of-course tests for high school courses (e.g., algebra I, algebra II, and geometry). These tests are based on the Virginia SOL (BOE, 1995) and are meant to assess students' proficiency related to the Standards. In order for schools in Virginia to reach and maintain accreditation they are expected to have a school-wide passing rate of at least 70% on each of the tests.

The mathematics SOL (BOE, 1995) identify six content strands of mathematics important for students to learn: Number and Number Sense, Computation and Estimation, Measurement, Geometry, Probability and Statistics, and, Patterns. In addition, the Standards point out that, "students must *gain an understanding* of fundamental ideas in [mathematics]…and develop proficiency in mathematical skills" (p. 1, italics added). Further, the standards declare that: "Problem solving has been integrated throughout the six content strands. The development of problem-solving skills should be a major goal of the mathematics program at every grade level" (p. 4). The SOL assessments were developed to assess what students are learning in relation to the Standards. The validity of these tests was then evaluated by a committee of testing and measurement experts brought together by the Virginia Board of Education (Hambleton, et al., n.d.).

The validity of test scores is related to the degree to which the scores can be used for their intended purpose. Based on the statements from the SOL document (BOE, 1995), the intent of the SOL tests is to measure the degree to which students develop their mathematical skills, conceptual understanding, and problem-solving in the six content areas. The construct validity of a test is a measure of the degree to which the test measures what it was intended to measure. For example, the construct validity of a mathematics test is often assessed by correlating test scores with a known and accepted standardized mathematics achievement test. Using students' Stanford 9 Achievement test scores it was found that the correlations between the SOL test scores and the Stanford 9 test scores provided some evidence that the tests had construct validity (Hambleton et al., n.d.).

Content validity is related to the degree to which test items are providing evidence about the actual SOL. Domain validity, a special kind of content validity, measures the broadness with which the SOL are being assessed. In other words, at what process levels are the SOL being evaluated. Because of the format of the SOL mathematics tests, that is, all items follow a multiple-choice format, Hambleton and others (n.d.) pointed out that this format may not be suitable for assessing all of the goals of the SOL. For example, if the SOL called for a student to show their conceptual understanding of a mathematical idea or show their ability to solve a nonroutine problem, the design of the SOL tests may not be adequate to assess these process standards. Hambleton and others (n.d.) recommended that further evidence be provided "on the extent to which the test specifications match the SOL, and where they do not, [indicate]…the steps that are in place for insuring that the areas of the SOL not covered on the assessments are taught and assessed in other ways?" (p. 3-4)

The goal of this study is to investigate whether the Virginia SOL tests indeed provide valid measures of the process standards (i.e., conceptual understanding, problem solving), and whether students who pass the SOL tests actually possess conceptual understanding and problem solving knowledge. In other words, this study investigates the following questions: (1) Do students who pass these SOL tests possess competence in mathematical problem solving and conceptual understanding? (2) Are the scores from these tests valid measures of students' competence in these process skills?

Methods

Sample

This study involved three cohorts of fourth grade students (N = 2377) from one school district in Virginia.

Measures

Students' scaled scores as well as proficiency ratings (i.e., pass/fail) from the Grade 3 SOL test were used as one measure of mathematic achievement. The items on these tests use a multiple-choice format. The Grade 3 SOL tests are administered in the spring of the school year.

An additional measure of student achievement was gathered using the *New Standards Reference Examination* (NSRE), published by Harcourt, Inc. The NSRE was administered to students in fourth grade in the spring of the school year. The NSRE is based on the *New Standards Performance Standards* (National Center on Education and the Economy [NCEE], 1997) and assesses students' mathematical knowledge in the different content strands of mathematics advocated by the NCTM (NCTM, 2000) and other professional organizations. In addition, the NSRE assesses student performance for three process levels: skills, conceptual understanding, and problem-solving. The skills component assesses students' performance on basic mathematical procedures and techniques. The conceptual component assesses students' understanding of mathematical processes and ideas. The problem solving component assesses students' ability to reason mathematically and apply their mathematical knowledge to problem situations. The items on the NSRE use multiple formats including multiple choice, short answer, and open-ended. Students' scaled scores as well as performance standard level were used in this study. Performance standard levels are based on the *New Standards Performance Standards* and are reported in five levels: 'achieved the standard with honors', 'achieved the standard', 'nearly achieved the standard', 'below the standard', and 'little evidence of achievement'. For the purpose of this study students' performance levels were collapsed into two categories: 'achieved the standard' (i.e., achieved the standard with honors, achieved the standard) and 'did not achieve the standard' (i.e., nearly achieved the standard, below the standard, little evidence of achievement).

Results

Using data from all three cohorts of students, Pearson's correlation coefficients were calculated for the relationship between the Grade 3 SOL math test scaled scores and the scaled scores from the three components of the NSRE. Correlations were all found to be statistically significant and within the range of .64 to .81 (see Table 1).

Table 1.
Correlations Between Scores on the Grade 3 Virginia Standards of Learning Mathematics Test and Scores on the Three Components of the New Standards Reference Exam.

	N	Skills	Conceptual	Problem-Solving
Cohort 1 SOL	779	.64	.81	.74
Cohort 2 SOL	798	.68	.77	.68
Cohort 3 SOL	800	.69	.78	.66

Note: All correlation coefficients were statistically significant at α = .001.

In addition, proficiency ratings for the Grade 3 SOL math test (i.e., pass/fail) were compared to the performance standards (i.e., achieved standard/did not achieve the standard) for the three components of the NSRE: skills, conceptual, and problem solving (see Table 2). The proficiency ratings across the district for the SOL test were 80%, 82%, and 87% for cohorts 1, 2, and 3, respectively. These passing rates represent proficiency as a district. The performance for the *skills* component of the NSRE was comparable although lower. For cohorts 1, 2, and 3, 68%, 69%, and 71% of the students 'achieved the standard,' respectively. However, for the *conceptual* component of the NSRE, only 36%, 42%, and 45% of the students achieved the standard, respectively. For the *problem solving* component of the NSRE, only 22%, 24%, 31% of the students achieved the standard.

When considering only those students that passed the Grade 3 SOL test, only 80%, 79% and 80% of the students in cohorts 1, 2, and 3, respectively, achieved the standard for the skills component of the NSRE (see Table 3). While only 45%, 50%, and 51% of the students who passed the Grade 3 SOL test achieved the standard for the conceptual component, respectively. Finally, only 27%, 29%, and 35% of the students who passed the Grade 3 SOL test achieved the standard for the problem solving component, respectively.

Discussion and Implications

The goal of this study was to consider whether success on state proficiency tests is indicative of students' development of important conceptual understanding and problem solving knowledge related to mathematics. Based on the relatively high correlations between test scores on the SOL tests and the NSRE one might conclude that the SOL test scores have construct validity and that success on the SOL tests indicate conceptual understanding and problem solving ability. However, correlations between two scores merely indicate the strength of the relationship between the scores in terms of rank order, that is, how well one can predict one score from another. Correlations do not take into account any criterion or standard; in other words, while high scores on one test may be predictive of higher scores on a second test, it may be the case that higher scores on the second test are not high enough to meet the criterion set for proficiency. This seems to be the case for the Grade 3 SOL tests. Based on the magnitude and positive direction of the correlations between the scores from the SOL test and the NSRE, students who have higher scores on the SOL test tend to have higher conceptual understanding and problem solving scores based on the NSRE; however, overall, a passing score on the SOL test does not indicate that students have developed the conceptual understanding and problem solving skills necessary to achieve the standard as outlined by the *New Standards Performance Standards.*

Table 2.
Percentage of Students Rated 'Proficient' for the Grade 3 Mathematics SOL Test and Percentage of Students 'Achieving the Standard' for the Three Components of the NSRE.

	N	SOL	Skills	Conceptual	Problem-Solving
Cohort 1	779	80	68	36	22
Cohort 2	798	82	69	42	25
Cohort 3	800	87	71	45	31

Table 3.
Percentage of Students Rated 'Proficient' on the Grade 3 SOL Test that 'Achieved the Standard' on the NSRE.

	N	SOL	Skills	Conceptual	Problem-Solving
Cohort 1	779	80	80	45	27
Cohort 2	798	82	79	50	29
Cohort 3	800	87	80	51	35

At best, it seems that the Grade 3 SOL test is a measure of students' proficiency at performing basic mathematics skills and procedures. Schools and school districts that have successfully met the 70% passing rate may be lulled into a false sense of success. As indicated by Hambleton and others (n.d.), there may need to be additional evidence, beyond the multiple choice tests, to indicate whether students are meeting the SOL in the broadest sense, including conceptual understanding and problem solving ability. Aligning curriculum to standards without taking into consideration the process standards may deprive students of important and essential mathematics learning. Instruction focused on skills and procedures, which may be emphasized on the SOL tests, may limit students' mathematical development. When the state assessment program drives the curriculum, as is often the case with high-stakes accountability programs, it is important that the assessments used provide valid measures of what is essential for students to learn. Based on the findings of this study the SOL tests do not seem to provide valid measures of proficiency for all of the essential components of mathematics learning.

In order to make sure that students are not missing out on important mathematical competencies, states should use tests that better assess students' conceptual understanding and problem solving. Some states actually use the NSRE for their state proficiency test (e.g., Vermont). Standards provide students, teachers, and schools with important content and process goals for learning. Holding students, teachers, and schools accountable for meeting these goals is also important. However, the assessment programs used by states should serve as a way of helping reach these goals and not ultimately limit what students learn. Further, the tests should help diagnose students' misunderstandings instead of providing a false sense of success by indicating that students are mathematically proficient when in fact they have not developed important conceptual knowledge and problem solving skills.

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DISCOURSE AND PROTOTYPE DEVELOPMENT AMONG MIDDLE SCHOOL STUDENTS IN A DYNAMIC GEOMETRIC ENVIRONMENT

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The purpose of this study was to document and describe how middle school students conceptualize the dynamic on-screen representations of quadrilaterals using Doerfler's concept of prototype and carrier (Doerfler, 2000). To explore these questions, a researcher-as-teacher teaching experiment (Ball, 2000) was conducted. Two seventh grade classes were taught one unit in quadrilaterals using the Shape Makers (Battista, 1998) curriculum written with Geometer's Sketchpad (Jackiw, 1998). A total of eight students were videotaped as they worked on their computers. The data for this study consisted of video taped records, student written work and student interviews. A modified form of iterative video taped analysis (Lesh & Lehrer, 2000) was used to analyze the data. The results show that the students' construction of prototypes was not only facilitated by the dynamic geometric environment, but was also based on the context provided by the nature of the activity and the corresponding discourse between students and teachers.

Theoretical Framework

Within the last decade, rapid developments in computer technology and mathematical software have given students of mathematics new means by which mathematical objects and relationships can be visualized. For example, dynamic geometric software allows students access to mathematical shapes that may be manipulated and morphed while preserving the defining properties that make up that shape. The purpose of this study was to document and describe how middle school students conceptualize the dynamic on-screen representations of quadrilaterals in an interactive hypermediated environment using Doerfler's concept of prototype (Doerfler, 2000).

Doerfler (2000) describes three types of prototypes that are applicable to this investigation into student discourse and thinking in a dynamic geometry environment. They are figurative prototypes, operative prototypes, and relational prototypes. In constructing figurative prototypes, the learner's interaction with the concrete carrier leads to a perceptive understanding focusing on the properties, distinctive features and internal relations of the concept. The second type of prototype, operative, focuses on the operations on the concrete carrier. In geometry, possible operations would be isometric and non-isometric transformations. Relational prototypes refer to the construction of specific relations using a concrete carrier. In a paper and pencil curriculum, unlike figurative and operational prototypes, these relationships may not be immediately perceivable, but are mediated and constructed cognitively. For example, the fact that a square is a trapezoid would represent a relational prototype making links between a square and trapezoid. Such a conclusion transcends both figurative and operative understanding of the concept of a square in that a relationship with another figure has been established. While in a traditional geometry curriculum, this relationship can be meaningfully constructed only through mental deductive reasoning, the use of dynamic geometric software allows for this relationship to be meaningfully constructed through visual inductive reasoning.

The construction of prototypes involves the perceptive, discursive and cognitive interaction with, and the manipulation of, some kind of mathematical model. The model, known as a concrete carrier (Doerfler, 2000), may be an object, a drawing, or a mathematical expression be it physical or imagined. For example, in the case of geometric *Shape Makers* (Battista, 2003), the carrier is a computerized onscreen representation of geometric objects. The relationship between concrete carriers and prototypes is particularly important in a dynamic geometry environment. Furthermore, although each type of prototype consists of certain characteristics with varying degrees of complexity, there is not a hierarchical or sequential relationship between them. In other words the acquisition of one type of prototype is not a prerequisite, nor does it necessarily lead to the construction of another. Rather they are networked in a non-linear manner to give meaning to more complex concepts.

According to Doerfler (1991), the development of meaning has a holistic aspect, corresponding to the prototype for the respective concept, in which development is not divided up into more elementary particles or ideas to be progressed though sequentially. Rather, to attain meaning, suitable concrete carriers must be developed and presented to the students in a manner and context to promote the construction of the appropriate prototype. The use of dynamic geometry provides us with a new set of carriers for geometric concepts not seen in either of their non-dynamic technology based predecessors or traditional pencil and paper constructions. While the same formalizations may not be constructed it is possible that equivalent and meaningful conceptualizations could be developed (Doerfler, 1991, 2000). Using this theoretical framework, the following questions were investigated: What was the nature of the discursive activity of students in a dynamic geometric environment? How did this discursive activity contribute to their construction of meaning in a geometric environment? Did the use of a dynamic geometric environment allow for the concurrent construction of figurative, operative, and relational prototypes in this meaning making process?

Empirical Framework

In order to investigate these questions, a naturalistic (Moschkovich & Brenner, 2000) researcher-as-teacher teaching experiment (Ball, 2000) was conducted. The researcher assumed the full teaching responsibility of two middle school mathematics classes for six weeks. The classes were grade seven and situated in a large Midwestern middle school. The two classes consisted of one advanced topics course, and one regular math course. Using the school mobile computer lab, the classroom was transformed into a computer based hypermedia learning environment in which two students shared one computer. The curriculum used for the teaching experiment included the use of *Geometer's Sketchpad* (Jackiw, 1991) and *Shape Makers* (Battista, 2003). A total of eight students, four from each class, were videotaped as they worked on their computers. The primary data source for this study consisted of close-up videotapes of the students working at their computers. Another video camera was positioned at the back of the room to capture the whole class environment. Other data sources consisted of students' written work, researcher-teacher reflection, and videotapes of the students' exit interviews.

The analysis of the video tapes used an Iterative Refinement Cycle (Lesh & Lehrer, 2000) model in which four interpretive cycles were used on the data. The first cycle, global viewing, was used to reorganize the video data to recreate the approximate lesson sequencing for each pair of students. The second cycle was an editing stage in which any conversation unrelated to the mathematics or whole class discussion was edited out. The third cycle consisted of an analysis of each student pair in chronological sequence. The purpose was to observe the progression and the students' development of geometrical concepts and any related discourse. What resulted from this third iterative cycle was a detailed account of very specific episodes in the investigative activities for each student pair. This cycle contained two important interrelated interpretive elements, an identification of issues pertaining to the study's research questions, and the creation of detailed, interpretive transcripts. The fourth cycle consisted of an analysis across student pair on a particular activity. The purpose was to observe the commonality and differences across student pairs for the activities. The common activities chosen for this further layer of analysis was based on the third interpretive cycle. Finally, throughout the entire iterative viewing and interpretive process, the analysis of the other data source, such as video-taped whole class discussions, video-taped exit interviews, and student written work, was interspersed to help inform the context and nature of the discourse among student pairs.

The creation of transcripts was an important part of the methodological framework in that the process supported the analytical viewing by providing a written record of the previewed data and any associated interpretations. What made the creation of transcripts an interpretive process, as opposed to a clerical process, was the important relationship between what the students said, and what was done on the computer in relation to the discursive context. Therefore, the creation of the transcripts was not just a record of what was said, but also consisted of how it was said, any associated actions to what was said, and an interpretation based on the conceptual framework.

The transcripts were written in a prose style experimental writing form (Richardson, 1998). According to Richardson, experimental writing is rooted in poststructuralist thought in which language; subjectivity, social organization and power are linked to make sense of the subjective interpretation of ones experiences. Furthermore, this poststructuralist context directs the researcher to write from a particular point of view at a specific time thus freeing the researcher from attempting to write a single text in which everything is said to everyone. Rather than serving as a means of "telling" about one's experiences, writing becomes a way of knowing, and a method of inquiry and analysis (Richardson, 1998). Writing therefore becomes expressive, rather than productive (Denzin, 1998).

All quotations were taken directly from the video data and reflect exactly what the students said. As the transcripts were taken, they were put into a prose form to help facilitate the analytic process. In this process, prose writing cues were added, with embedded analysis based on the study's conceptual framework in order to give the reader a sense of the nature of the discourse as it related to the on-screen actions, and the meaning-making process. In this way, the transcription process transcended being an objective recording process, becoming an integral part of the interpretive cycle.

Results and Discussion

The results of this study support the notion that the development of different prototypes (figurative, operative, and relational) for a concept may be concurrent, and that this development is dependent on the nature of the activity, and the nature of the discourse associated with the activity. For example, many of the activities required students to make various shapes out of a specific Shape Maker. This action of making, and the corresponding discourse about 'making' provided a context that enabled students to construct relationships between the different types of quadrilaterals. Moreover, these activities helped mediate figurative and relational prototypes.

The following discussion will focus on the progressive schematization of the rhombus for two of the eight student participants, Chloe and Lucas. Chloe and Lucas had a fairly healthy working relationship. Their participation in the study as a case study pair was based on a recommendation by the their classroom teacher. Before the study, Lucas and Chloe were already paired up as learning partners and had an established working relationship. Lucas was a quiet, mild mannered, reserved boy. He was small for his age and had a high-pitched voice, which caused one student to refer to him teasingly as "squeaky voice." While he objected to the comment, his overall reaction seemed temperate, reflecting an easy-going disposition. On the other hand, Chloe had a stronger, outgoing personality. She smiled and talked much, processing her ideas in an extroverted manner.

Like most of the students in their classroom, Chloe and Lucas had very little prototype development for the rhombus as indicated by their pre-study assessments and video data taken on the first few days of the study. In general, of all the shapes, the rhombus seemed least familiar to most students. One possible reason could be, that unlike the square, rectangle, and parallelogram, the rhombus tends to have less exposure in the K-6 geometry curriculum. Another possible reason is that the name rhombus is not associated with any particular characteristic or figure. For example, the name parallelogram reflects the two sets of parallel sides that define it, and the kite may be visually associated with shape of the flying toy kite. However, in spite of their unfamiliarity, Chloe and Lucas were able to establish property-based figurative prototypes for the Rhombus Maker using relational prototypes to the more familiar square. The following case story was taken from the second day of the research study during an activity called 'Can You Make It.'

Day 2 – Can You Make It

"I'm good," said Lucas, commending himself. Lucas grabbed the Trapezoid Maker and began to experiment with it. After unsuccessfully trying to get it to fit on the mouth (G), he then took the Trapezoid Maker and turned it into square F (Figure 1). "Oh, I'm good."

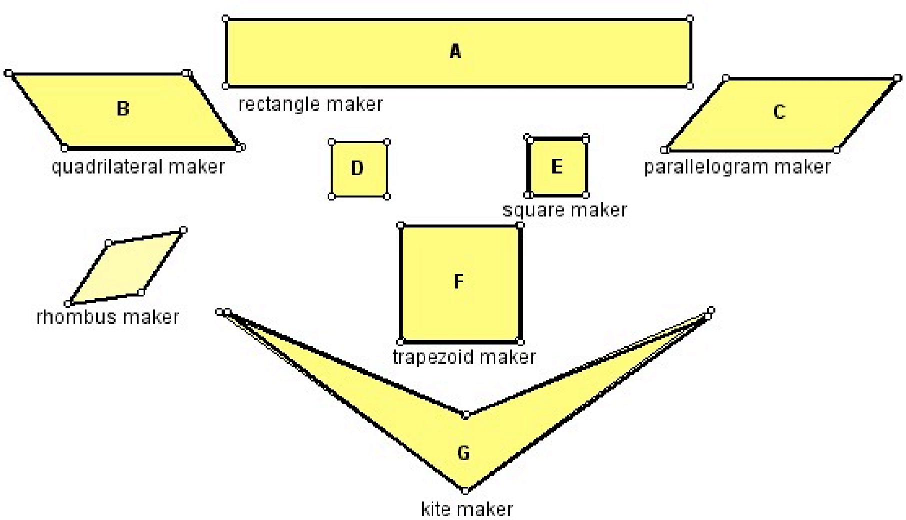

Figure 1. Chloe and Lucas, can you make it?

With only one shape left, square D, and the Rhombus Maker remaining, Lucas and Chloe attempted to turn the Rhombus Maker into a square. When they succeeded, thus finishing the worksheet, they exclaimed, "We got it! We got it all done!"

Lucas began to highlight the Shape Makers and turn them into different colors. Chloe tried to get him back on task, "We have to fill this [student sheet 2] in now. Fill this in!" As they filled in the sheets Lucas asked, "Why is it [square-D] rhombus?"

"Because it [Rhombus Maker] always has four equal sides, but not always 90 degrees…ah… 60 degrees," replied Chloe.

"No. 90 degrees," corrected Lucas.

"No, 60…"

"Because there are four…" began Lucas

Correcting herself Chloe said, "I'm thinking about triangles."

"Not always 90," repeated Lucas.

In the previous case story from Day 2, Chloe and Lucas used the Rhombus Maker to make a square while completing the activity "Can You Make It." The activity required that the students tell why they used that particular Shape Maker. In their reasoning, Chloe stated, "Because it [Rhombus Maker] always has four equal sides, but not 90 degrees," reflective of a figurative prototype for the rhombus. The construction of this figurative prototype for the rhombus appeared to have been supported through a relational prototype with the square shape mediated through the carrier's (Rhombus Maker) actions. Since the Rhombus Maker was in the form of a square, Chloe seemed to notice that like a square, a rhombus had to have four equal sides. However, unlike the square, it could have angles other than 90 degrees, as determined by shifting the Rhombus Maker so that it not longer had right angles. It was this association with the Rhombus Maker making the square that helped mediate the construction of this figurative prototype for the rhombus shape. In other words, the students did not internalize the fact that a rhombus had to have four congruent sides as a defining property of the rhombus. Rather, the meaningfulness of a rhombus having four congruent sides was as a result of the context in which the carrier (Rhombus Maker) was situated. In this case, the context was that the Rhombus Maker was in the form of a square.

When the context created by the association between the Rhombus Maker and square was removed, both Chloe and Lucas were unable to recognize or correctly identify the Shape Maker. This was illustrated by an activity that was done on Day 3, "Identify the Hiding Shape Makers." In this activity, all seven on-screen Shape Makers started out as congruent squares. While all of the Shape Makers were in the form of a square, the square shape was a temporary state. The goal was to determine which Shape Maker made up each square by manipulating each Shape Maker and identifying them by their associated properties, actions, or combination of both. In the following excerpt, Chloe and Lucas were not able to correctly identify the Rhombus Maker in spite of their discussion on the previous day when they determined that, "it [Rhombus Maker] always has four equal sides, but not always 90 degrees." The excerpt will show that in the absence of the relationship between the Rhombus Maker and square, the figurative prototype, that a rhombus has four equal sides, was lost upon the students.

Day 3 – Hiding Shape Makers

Lucas grabbed shape E, the Rhombus Maker, and began to morph it (Figure 2).

"That's Kite Maker," guessed Chloe incorrectly.

"No I think this is the parallelogram," said Lucas, also incorrect.

"The Kite Maker, its always in a diamond," reasoned Chloe.

"Yeah, so is a parallelogram," added Lucas.

"No its not," objected Chloe.

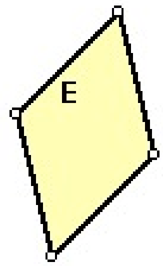

Figure 2. Unknown shape

Lucas continued to carefully check each vertex to determine the actions and nature of the unknown shape E. Disagreeing with Chloe's conjecture that shape E was a Kite Maker, Lucas pointed out, "With Kite Maker…one side can be short and one side can be long…that's parallelogram. E is parallelogram." His statement about the sides of the Kite Maker was reflective of a figurative prototype for a kite.

Since the context or activity did not draw the students' attention to the lengths of the sides of shape E, neither of the students seemed to recognize that the all the sides on shape E were congruent. This was in contrast to the previous day when the Rhombus Maker was in the form of a square, a context that helped mediate figurative aspects of the rhombus through a relational prototype with the square.

While both students were incorrect in their identification, of the Rhombus Maker, both were correct in pointing out figurative aspects of the Rhombus Maker. Chloe's focus was on the "diamond shape" that was often used to describe both the Rhombus Maker and the Kite Maker that led her to misidentify shape E as the Kite Maker. Lucas's focus was on the two sets of parallel sides that remained invariant with the Rhombus Maker that led him to misidentify the shape as the Parallelogram Maker. Without realizing it, the students separated out two related aspects of the figurative prototype for a rhombus shape, that a diamond shape with two sets of parallel sides is a rhombus. Furthermore, calling the shapes made by the Rhombus Maker a kite or a parallelogram are in fact correct categorizations when using inclusive definitions in that any shape made by a Rhombus Maker is a kite and a parallelogram. Rather than continue to investigate the unknown shape E, Rhombus Maker, Chloe moved ahead in the activity by trying to identify some of the other Shape Makers.

Later, towards the end of activity, both Chloe and Lucas then misidentified the unknown shape A to be the Rhombus Maker. The correct Shape Maker identification for shape A was the Parallelogram Maker. Looking over their near completed sheets, Chloe concluded by the process of elimination, "[Shape] A has to be rhombus."

"Let's check," suggested Lucas. Turning it into a parallelogram, with two long sides and two short sides, Lucas incorrectly concluded, "Yep, it's the rhombus."

When Lucas began to drag one vertex of shape A, it initially moved like a Rhombus Maker, preserving the congruence of all four sides. When he dragged a second vertex, he turned the unknown shape into a parallelogram with two short sides and two noticeably longer sides contradicting Chloe's statement from Day 2, "…it [Rhombus Maker] always has four equal sides, but not always 90 degrees." In spite of this fact, neither student changed their identification of shape E, keeping the incorrect identification of the Rhombus Maker. By not having the square as a referent shape to compare with, as on the previous day, the figurative prototype, that a rhombus had to have four equal sides, was lost in this day's activity resulting in the incorrect identification of Shape Maker E and Shape Maker A.

The results also suggest that the prototypes for a concept are not discrete entities, but may be interconnected and networked to make sense of a more complicated relationship or activity. Finally, the classification of prototypes is not an exacting science in which certain verbal statements are assigned to a specific prototype. In other words, certain verbal statements taken in context may reflect various kinds of prototypes for a concept. Furthermore, the results of this study suggest that the use of prototypes is contextually based, dependent upon both the carriers and the nature of the activity in which the carriers are used. The identification of prototypes requires an understanding of the context consisting of the nature of the activity, the concrete carrier utilized and the associated discourse between the cognizing beings.

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SUPPORTING TEACHERS' LEARNING THROUGH THE USE OF STUDENTS' WORK: CONCEPTUALIZING TEACHER LEARNING ACROSS THE SETTING OF PROFESSIONAL DEVELOPMENT AND THE CLASSROOM

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Looking at students' work has increased in popularity in mathematics teacher professional development as a promising means of supporting teachers' learning (Ball & Cohen, 1999; Kazemi & Franke, 2003; Little, 2003). Because students' work constitutes an indispensable aspect of teachers' instructional practices, it is typically considered to help relate teacher professional development directly to teachers' daily practices (Ball & Cohen, 1999). However, up to this point, little conceptual work has been done to articulate the theoretical underpinning of this approach. In this paper, I draw on my experience working with a group of middle-school mathematics teachers to begin to conceptualize this issue and more broadly, the issue of supporting teachers' learning across the setting of professional development and the classroom.

Introduction

It is widely understood among mathematics education researchers[1] that the ultimate goal for teacher professional development is to bring about changes in classroom instructional practices and thus improvements in students' learning of mathematics. Many researchers therefore propose that teacher learning in the setting of professional development needs to be closely tied to teachers' experiences, needs and practices that transpire in the setting of the classroom (Ball & Cohen, 1999; Nelson, 1997; Borko, 2004; Schifter, 1998). Such learning, as Ball and Cohen (1999) point out, needs to be reflected upon, refined and regenerated in teachers' classroom practices. This poses a challenge to researchers; that is, how to support teacher learning across the two distinct settings in a coherent manner so that teachers perceive what they learn in professional development as meaningful and relevant to their classroom practices. To accomplish this, many researchers, when designing for professional development, explicitly generate ideas or conjectures to "connect" the professional development to teachers' classrooms. Among various approaches, examining students' work has increased in popularity in the field of mathematics teacher education professional development (Ball & Cohen, 1999; Little, 2003). To illustrate, Franke and Kazemi (2003) report,

> "…students work, as a tool for professional development, has the potential to influence professional discourse about teaching and learning, to engage teachers in a cycle of experimentation and reflection, and to shift teachers' focus from one of general pedagogy to one that is particularly connected to their own students." (p.3)

However, although the use of students' work is conjectured to be an effective means of supporting teachers' learning, its theoretical underpinning has yet been explicated. Up to this

[1] In the remainder of the paper, I use the word *researchers* to refer to teacher educators and instructional designers who engage in supporting teachers' learning via the means of professional development.

point, little conceptual work has been done to spell out the complexity involved in using students' work in professional development and more broadly, supporting teacher learning across the two distinct settings of professional development and classroom practice. My goal in this paper is modest. As an illustrative case, I utilize my experience working with a group of middle-school mathematics teachers to begin to conceptualize how using students' work supports teacher learning by situating it within an issue of broader significance; that is, supporting teacher learning across the setting of professional development and the classroom.

Background

Data are taken from our ongoing collaboration with a group of middle-school mathematics teachers in a large, urban district in the southeast United States. This school district serves a 60% minority student population and is located in a state with a high-stakes testing accountability program. The long-term goal of the research team in working with the teachers is to support their development of instructional practices which place students' reasoning at the center of their instructional planning and decision making. To this end, we have engaged the teachers in activities developed from a statistical data analysis instructional sequence that was designed, tested, and revised during prior NSF funded classroom design experiments conducted with middle-grade students (Cobb, McClain, & Gravemeijer, 2003; McClain & Cobb, 2001). During our collaboration, the research team[2] has conducted monthly work sessions (one day) and extended summer sessions (typically three days). For the purpose of this paper, I focus on the third year (2002-2003 academic year) in our collaboration when the research team proactively attempted to support teachers to focus on students' reasoning through the use of students' work.

Theoretical Framework

The analysis is guided by a framework described by Cobb and colleagues (Cobb, McClain, Lamberg, & Dean, 2003) that coordinates individual teachers' learning with the development of collective practices of the Professional Teaching Community, as they are situated in the institutional setting of a school district (cf. Dean, 2004). This framework was developed out of practical needs to account for teachers' learning in the social context of the Professional Teaching Community as it is enabled and constrained by the broader context of the institution.

Data Sources and Method of Analysis

Data consist of videotapes of all work sessions accompanied by a set of field notes and copies of all teachers' work. Modified teaching sets (cf. Simon & Tzur, 1999) were also collected on each teacher at least twice a year. This entailed videotaping each teacher's class and conducting follow-up audio-taped interviews that focused on issues that emerged in the course of instruction. The observed classroom session served as a context within which the teacher could be oriented to address issues that are of relevance to the researchers' goals.

The approach followed for analyzing data collected from the work sessions and the teaching sets involves a method described by Cobb and Whitenack (1996), an adaptation of Glaser and Strauss' (1967) constant comparative method. The initial orientation for a retrospective analysis is provided by the tentative and eminently revisable conjectures that are developed both prior to and while actually conducting the classroom design experiment. The method involves

[2] The research team was composed of the author, Paul Cobb, Kay McClain, Jose Luis Cortina, Chrystal Dean, Teruni Lamberg, Lori Tyler and Jana Visnovska.

continually testing and revising conjectures while working through the data chronologically. This constant comparison of conjectures with data results in the formulation of claims or assertions that span the whole data set but yet remain empirically grounded in the details of specific episodes (cf. Cobb et al., 2003).

An Illustrative Case

The research team's overarching goal was to help teachers learn to focus on students' reasoning and place it at the center of their instructional decision-making. In order to support this learning, we engaged the teachers in statistical and data analysis activities that typically followed the pattern of (1) solving a selected task from the statistics sequence during the work session, (2) teaching the same task with their students after the session, and (3) bringing students' written work to the following work session for group discussion.

Each task from the statistics instructional sequence typically involves comparisons of two data sets. For example, one task involves comparing data from the life span of two brands of batteries and making a recommendation to other consumers based on the analysis. When enacted in the classroom, the instructional activity was typically comprised of (1) a whole-class discussion in which the teacher and students talked through the data creation process (Cobb & Tzou, submitted), (2) an individual or small-group activity in which students worked to analyze data, and (3) a whole-class data analysis discussion which focused on students' diverse ways of analyzing data (Cobb et al., 2003). Ideally, the data analysis discussion should be conducted in a separate lesson so that the teacher will have enough time to carefully examine the students' work generated in the individual or small-group activities and capitalize on it in orchestrating the data analysis discussion.

Throughout the third year of our collaboration, the research team focused on the data creation process and the subsequent data analysis discussion. We conjectured that by examining these two aspects of instruction the teachers would come to see their current practice as problematic. In particular, they would focus on what is entailed in orchestrating a deliberately facilitated whole-class discussion that builds on students' diverse ways of reasoning and that builds towards the envisioned long-term learning goal. Consequently, this would make it possible to create situations where students' reasoning could be a focus of discussion. The focus on the data analysis discussion led the research team to design a series of activities to further support teachers' learning. For example, these activities involved watching videos of participating teachers co-instructing a statistics lesson, examining students' work and discussing possible ways to orchestrate the ensuing whole-class discussion by capitalizing on the diverse ways of students' reasoning. From the researchers' perspective, students' work constitutes records of students' mathematical reasoning, the very instructional aspect that the research team strived to bring to the center of teachers' instructional planning and decision-making. We conjectured that focusing explicitly on students' work would open up opportunities to help teachers gain insights into the diversity in students' reasoning and later use it as a resource to inform the subsequent whole-class data analysis discussion and build towards the envisioned learning goal. In order to orient the teachers when examining their students' work, we posed the following questions:

- What are the different solutions that you can identify from your students' work?
- How would you categorize students' solutions according to their levels of sophistication?
- How would you, as a teacher, build on these different solutions? Which solutions would you choose to talk about in class and why?

The activity of examining students' work resulted differently than the research team had expected. Analysis of the data generated from the work sessions indicates that even though many teachers were engaged in this activity and some were able to categorize students' work in terms of levels of sophistication, they did not view this activity as a preparation for the ensuing whole-class discussion of data analysis. When teachers looked at students' work, their primary focus was evaluative, trying to identify whether students got it or not. Students' work, for these teachers, became an assessment tool for each lesson rather than a resource for planning for the ensuing instruction. This orientation that teachers took towards students' work became more evident when the question of "how are you going to build on students' different solutions" received almost no response but only puzzled looks from the teachers. The conversation within the work group came to a halt. As a result, the researchers had to explain what was entailed in building on students' solutions as we understood it. In doing so, we made explicit connections between students' work, the orchestration of the subsequent whole-class discussion and the relatively long-term goal for the instructional sequence. However, the ensuing conversation within the work group revealed to us that the teachers' image of a whole-class discussion involved no more than a series of student presentations in which no attempt was made to capitalize on students' current understanding. For example, the teachers would ask every group of students to present their solutions even when the groups shared the same reasoning. The order that the teacher arranged for the presentations was often random, not necessarily relating to the level of sophistication in students' solutions.

The teachers' understanding of the utility of students' work and the ensuing whole-class discussion was unexpected to the researchers and led us to the realization that there was something about these teachers' instructional world that we had yet understood. In order to generate data to seek for an explanation and to inform our collaboration with the teachers, we conducted a series of modified teaching sets (Simon & Tzur, 1999) with all participating teachers. A central principle that guided our analysis of the teaching sets was to assume that teachers' perspectives of teaching and learning and specific instructional practices they developed in their classrooms are always reasonable and coherent in the context of their landscape of teaching.

The analysis of the modified teaching sets revealed that most teachers were unable to explicate how their students learned and what supported their learning. For these teachers, there was a black box between teaching and learning. The phenomenon that the same classroom instruction always resulted in different learning outcomes for different students only added to the teachers' uncertainty of how individual students learned. As teachers expressed a sense of limited control in the process of supporting students' learning, they indicated two aspects in their instruction that constituted the their realm of influence. The first aspect concerned making sure that students were provided with sufficient opportunities to engage in instructional activities as intended by the teacher. The common strategies that the teaches employed included using different forms of presentations (e.g. different visuals or manipulatives), providing students with enough problems to practice or enough time to process the information, or breaking down the mathematics problems into small steps. The second aspect was centered on making sure that the students would attend to such learning opportunities. Students' paying attention was highly valued in all of the observed classrooms. For many teachers staying on task was synonymous with learning. Students' failure in understanding the mathematics was typically accounted for in terms of their lack of focused attention or, sometimes, their unwillingness to concentrate on the mathematics. As a result, the teachers arranged their instruction in a way to ensure that students would be on task.

The analysis of the collected teaching sets in addition to the work session data enabled the research team to formulate conjectures as we strived to understand the world of teaching from the teachers' perspective. Only in doing so, we began to understand the rationale behind teachers' orientation towards the use of students' work and the subsequent whole-class discussion of the statistical data analysis. For many teachers, an instructional unit was often composed of one single activity rather than a series of activities that encompass starting points, envisioned overarching goals, and conjectured means of support towards these goals**.** Students' work, for many teachers, constituted no more than a product of learning, an indicator of whether the previous instructional activity was successful or not. In other words, it was the end point of each instructional unit. As a result, students' work was not perceived by the teachers as an instructional resource for *prospective planning*. Rather, it was used as a tool for *retrospective assessment*. The diverse ways of students' reasoning as reflected in students' work only served to reveal the elusiveness of individual students' learning and carried no pedagogical value for the teachers. Understanding students' reasoning via the means of analyzing students' work was not only alien but also irrelevant to the teachers' classroom experiences.

From the teachers' perspective, it was unmanageable to address the diversity in students' reasoning on a whole-class scale. Typically, the teachers would provide individualized instruction when students were working in small groups, which, from the teachers' view, was the primary situation where learning was supposed to happen. The whole-class discussion following the group work was not considered a major learning opportunity for students. It was, however, an occasion for students' to present the *product* of their learning and opportunistically learn from each other's presentations. The teachers did express their valuation for the end-of-class presentation in that they viewed it as an opportunity both to build students' social skills and self-esteem and to let students *see* what other groups did.

The research team's attempt to understand the world of teaching from a teacher's perspective generated insights for us to understand teachers' orientation towards students' work. At the same time, it provided leverage for the team to conjecture possible ways of proceeding in our future collaboration with the teachers[3]. Whereas our overarching goals in supporting teachers' learning remained unchanged, the conjectured means of supporting such learning were adjusted as a result of our deepened understanding of *the world of teaching from a teacher's perspective*, or what I call *the instructional reality* of teachers.

Towards a Reconceptualization

The findings from our collaboration with teachers indicated to the research team there is more to understand when using students' work to support teachers' learning in professional development. At the same time, such reconceptualization necessitates rethinking supporting teachers' learning across the setting of professional development and the classroom. By drawing on existing theoretical constructs in the literature, I will develop a foundation for the proposed reconceptualization from the notions of *consequential transition* (Beach, 1999) and *instructional reality* (Zhao, Visnovska, & McClain, 2004).

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[3] The analysis of how the understanding of teachers' instructional reality informed the research team's design to support teachers' learning is documented in Zhao et al. (2004).

Consequential Transition

Beach (1999) uses *consequential transition* to understand how learning evolves and generalizes as individuals participate in activities across different settings. Beach defines consequential transitions as the "developmental changes in the relation between an individual and one or more social activities" (p.114). In the setting of professional development, teachers as well as researchers participate in and contribute to the activities collectively; in the classroom, teachers and students act together to shape the classroom activities. When teachers make a shift from classroom teaching to participating in professional development, a transition takes place. Such transition can be consequential in terms of teachers' learning when a *bi-directional interplay* between the two settings becomes established; that is, teachers' participation in professional development is oriented towards reworking their classroom practices, and their classroom teaching constitutes the context for them to make sense of, reflect on and apply what they learn in professional development.

One of the methodological tools Beach proposes for analyzing consequential transition is called *developmental coupling*, which "encompasses aspects of both changing individuals and changing social activity" (Beach, 1999, p.120). Beach also clarifies that developmental coupling necessarily involves artifacts that reify practices and transcend different social activities in which people participate. He argues that it is in the transitions between different activities that new forms of reasoning come into being. Therefore, methodologically, the two activities involved in the transition need to be examined in juxtaposition.

This methodological concept of developmental coupling carries important implications for conceptualizing the process of supporting teacher learning across different settings. For example, it implies that in order to understand how teachers looked at students' work in professional development activities, one needs to examine how students' work is used in teachers' classroom practices. Teachers' reasoning with students' work in each setting needs to be explicated and understood in juxtaposition.

The current conceptualization of supporting teacher learning via the use of students' work typically rests on one rationale; that is, because students' work constitutes an indispensable aspect of teachers' classroom practice, professional development activities involving the use of students' work will naturally contribute to connect teachers' learning in professional development to their classroom practices. In addition, from many researchers' perspective, students' work constitutes records of students' reasoning, therefore, engaging teachers in conversations about students' work would eventually help them focus on how students think or reason mathematically. However, several issues remain unaddressed in this conceptualization: Questions such as, how do teachers normally use students' work in their classroom practices, what pedagogical value do teachers attribute to students' work in their daily instruction, or is it possible that teachers look at students' work differently than researchers. Answers to these questions are critical in that they reveal the nature of teachers' classroom practices, which, I argue, should constitute the basis for conceptualizing the supporting teachers' learning in the setting of professional development.

Instructional Reality

Developmental coupling provides an important theoretical underpinning for conceptualizing the bi-directional interplay in teacher learning across different settings. In my view, researchers also need to probe into the world of teaching and characterize it in a way that can lead to tentative design conjectures. In this regard, I propose to use the notion of *instructional reality*

(Zhao et al., 2004). It encompasses the perspectives that teachers hold towards teaching and learning, the instructional challenges and frustrations that they encounter and their explanation of them, the obligations of being a teacher as they understand and the valuations they hold towards specific aspects of their instructional world. In other words, this notion allows researchers to speculate (and perhaps gain an understanding of) how teaching looks from a teacher's perspective.

This notion of instructional reality requires that researchers make an explicit commitment to take the perspectives and instructional practices that teachers develop as reasonable and coherent from their own viewpoints (Simon & Tzur, 1999; Heinz, Kinzel, Simon, & Tzur, 2000). Operating on the basis of this assumption enables researchers to avoid characterizing teachers and their practices solely in terms of deficits. For example, the observation of a seemingly insensible or ineffective instructional decision made by a teacher does not merely conclude with a negative assessment of the teacher's competence. Instead, it becomes the focal point that the researchers need to account for so that it can be seen as a reasonable and coherent component within the landscape of the teacher's instructional reality. It is this explanation of what teachers do and why they do it that, I argue, can provide valuable guidance for researchers in designing to support teachers' learning. For example, in the illustrative case, the different orientation that the teachers took towards the use of students' work became a point of reference when the teaching sets were generated and analyzed. The research team then formulated conjectures to explain this orientation by situating it within the broader landscape of teaching, resulting in a coherent account from the teachers' perspective.

The notion of instructional reality expands the focus to include not only issues about teachers' conceptualization of mathematics teaching and learning—what is mathematics, how students learn, and what supports their learning (Heinz et al., 2002)— but also other aspects that significantly affect teaching from a teacher's point of view – for example, how to motivate students (cf. Zhao et al., 2004). Additionally, in the notion of instructional reality, what teachers do and how they justify their practices are considered as significantly influenced by the particular instructional resources they use in the classroom (e.g., textbook, state-mandated curriculum, copies of students' work). More broadly, the notion of instructional reality situates teaching within the institutional context in which teachers develop and refine their practices. The institutional contexts in which teachers work significantly affect how they approach teaching and learning, and therefore, constitute a resource for researchers when explaining what teachers do and why they do it (Cobb et al., 2003; Elmore, 2000; Spillane, 2001).

To summarize, on the one hand, the tool of developmental coupling makes visible the bi-directional interplay between teachers' classroom practices and their learning in the setting of professional development. The notion of instructional reality, on the other hand, characterizes teachers' practice in a way that allows this bi-directionality to be spelled out and, as a result, informs the design of professional development. Taken together, they constitute a foundation for the proposed reconceptualization, the bi-directional interplay between the two settings.

Summary

We, the research team, have learned that teachers' use of students' work in professional development cannot be fully understood without considering how the same activity is constituted in teachers' classroom practices. The illustrative case in this paper indicates that using students' work to support teacher learning is much more complicated than it is currently conceptualized.

The notion of bi-directional interplay between the professional development activities and teachers' classroom practices seems potentially useful in unpacking such complexity.

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