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Douglas E. McDougall
John A. Ross

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FROM PRIMITIVE KNOWING TO FORMALISING: THE ROLE OF STUDENT-TO-STUDENT QUESTIONING IN THE DEVELOPMENT OF MATHEMATICAL UNDERSTANDING

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In this paper, we examine the development of inner city middle school students' ideas and the student-to-student interactions and questions that contribute to this development within the context of the Pirie/Kieren model. We analyze data collected from an inquiry oriented, problem based mathematics class in which students were repeatedly challenged to explain their thinking to each other, and defend and justify all solutions. In this instance, we document how one student was able to move from primitive knowing to formalising. Further, we note that this student (and her classmates) were able to use this knowledge several months later when solving a structurally similar problem.

Objectives/Purposes

Prompting students to talk about mathematics is an important goal of education (NCTM 2000; Sfard, 2000; Dorfler, 2000; Cobb, Boufi, McClain, and Whiteneck, 1997). Cobb, (2000) notes that student exchanges with others can constitute a significant mechanism by which they modify their mathematical meanings. Carpenter and Lehrer, (1999) state that "the ability to communicate or articulate one's ideas is an important goal of education, and it also is a benchmark of understanding." (p. 22) Researchers such as those cited above (and others, see for example, Schorr, 2003; Maher, 2002; Shafer and Romberg, 1999) maintain that it is important to provide students with opportunities to discuss their ideas with each other, defend and justify their thinking both orally and in writing and reflect upon the mathematical thinking of others. One important component of this involves students' questioning the mathematical thinking of their peers. This report focuses on the impact of student questioning on the development of mathematical thinking. We do this within the context of the Pirie/Kieren theory for the growth of mathematical understanding (Pirie and Kieren, 1994).

Our central premise is that when students have the opportunity to question each other about their mathematical ideas, both the questioner and the questioned have the opportunity to move beyond their initial or intermediate conceptualizations about the mathematical ideas involved. As students reflect on their own thinking in response to questions that are posed by their peers they have the opportunity to revise, refine, and extend their ways of thinking about the mathematics. As they do this, their earlier conceptualizations and representations become increasingly refined and linked. We stress the role of representations in this dynamic since "the ways in which mathematical ideas are represented is fundamental to how people can understand and use those ideas." (NCTM, p.67) In this paper, we will trace the development of ideas (using the Pirie/Kieren model) and the student-to-student interactions and questions that contribute to this development.

Theoretical Framework

In 1988, Pirie discussed the idea of using categories in characterizing the growth of understanding, observing understanding as a whole dynamic process and not as a single or multi-valued acquisition, nor as a linear combination of knowledge categories. In 1994, Pirie & Kieren described eight potential layers or distinct modes within the growth of understanding for a specific person, on any specific topic. The inner-most layer, called primitive knowing, is what a person can do initially and is the starting place for the growth of any particular mathematical understanding. When a person is doing something to get the idea of what the concept is, he/she is working in the image making layer. A person working in this layer is tied to the action or tied to the doing. Working in the image having layer is when one reaches a “don’t need boundary”, where he/she is no longer tied to the action or the doing. When a person “…can manipulate or combine aspects of ones images to construct context specific, relevant properties” and ask themselves how these images are connected, one is property noticing (Pirie & Kieren, 1994, p.66). When one no longer needs to talk specific and can make a general statement, he/she is formalising. When formalising, “the person abstracts a method or common quality from the previous image dependent know how which characterizes his/her noticed properties” (Pirie & Kieren, 1994, p. 66). This theory is a way to explain understanding and is a useful tool for understanding how understanding grows. The structure of the theory is non-linear, repeating itself with many layers wrapped around.

In 2003, Warner, Alcock, Coppolo & Davis linked this theory to specific behaviors that indicate mathematical flexible thought. Briefly stated, a person exhibiting mathematical flexibility may be characterized as one who displays some or all of the following behaviors: interpretation of their own or someone else’s idea (e.g. through questioning it and thus showing it to be valid or invalid; through using, reorganizing or building on it); use of the same idea in different contexts; sensible raising of hypothetical problem situations based on an existing problem: creating “What if…?” scenarios; use of multiple representations for the same idea; connecting representations (Warner, Coppolo & Davis, 2002). In this study, we will illustrate movement through the first six layers (described above) as we focus on how the transitions from one layer to the next occurred in association with student-to-student interactions and questioning. We will also highlight how these student-to student interactions and questions move students to new representations and the linking of these representations.

Methods

The study took place over the course of 8 months, which involved two visits a week (50 minutes each session), for the first two months, and 3 to 6 visits a month for the remaining 6 months, in a diverse eighth grade inner city classroom with approximately 30 students. The visits were part of a professional development project in which the teacher/researcher (first author), who is a mathematics education researcher at a local university, routinely met with local teachers, planned classroom implementations, and then modeled or co-taught lessons with the teacher. After each lesson, the teacher/researcher would “debrief” with the classroom teacher and a University mathematics education professor (the second author) to discuss key ideas relating to classroom implementation, the development of mathematical ideas, and other relevant issues. During the course of the eight months, several different tasks were explored. The teacher/researcher, along with the classroom teacher encouraged the students to exchange, talk about, and represent ideas; conjecture, question, justify and defend solutions; discuss disagreements and differences; revisit ideas over time; and, generalize and extend their ideas. Generally, the students worked in groups of 3-5, and each group discussed, argued, and ultimately presented its solutions.

During each class session, two cameras captured different views of the group work, class presentations and associated audience interaction. In addition, careful field notes were taken after each session. This study focuses on 8 of the 62 videotapes generated in this manner, as the students explore variations of a task. The problem task was as follows: *John is having a Halloween party. Every person shakes hands with each person at the party once. Twenty-eight handshakes take place. How many people are at the party? Convince us.*

This particular problem entails a context that may suggest a structure that ultimately leads to a solution that is generalizable to a larger class of problems. In this case, such a solution is [n(n-1)]/2.

Episodes were transcribed and coded to identify critical events, which in this case were determined by student-to-student questions and/or interactions.

In the sections that follow, we examine the development of a particular student, Aiesha, by identifying student-to-student questions and/or interactions in the context of the Pirie/Kieren model for mathematical understanding.

Results

Moving from Primitive Knowing to Image Making

Primitive knowing is the starting place for the growth of any particular mathematical understanding, what the student can do initially, with the exception of the knowledge of the topic. In this case, Aiesha begins by shaking hands with a member of her group and then moves to a picture and number representation for her idea (figure 1). She moves to the image making layer (doing something to get the idea of what the concept is), using a picture representation to construct an idea of multiplying the number of people by one less than the number of people to arrive at the number of handshakes. Every time she multiplies, however, she arrives at double the number of actual handshakes in the correct solution. At first, she doesn't notice this mistake and becomes frustrated, explaining that there is no answer. After another student shares his solution, she realizes that the answer is eight and divides her answer to an eight person party by 2. She is working in the image making layer because she is "tied to the action or doing".

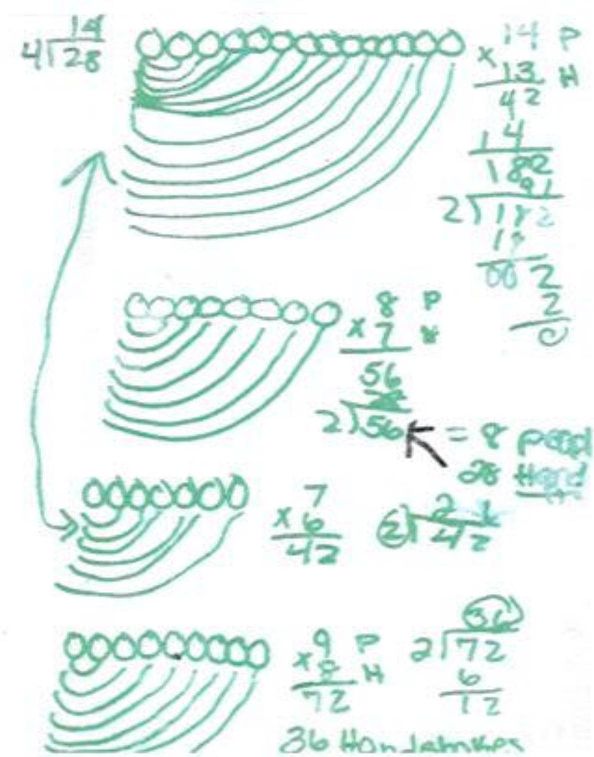

Figure 1- Aiesha's move to image making

Moving from Image Making to Image Having

Two weeks later the students were challenged to begin a new task involving an extension of the original task. In this episode, another group member, Bea, questions Aiesha about her initial representation. Aiesha then restructures her knowledge to generate a representation that is more understandable to her peers. In doing so, she has developed a new and ultimately more useful representation.

Figure 2- Aiesha's initial strategy for finding the number of handshakes when 11 people are at the party

Bea: When you did the demonstration you did up there, I didn't get it. [She is referring to Aiesha presenting figure 1 to the class a few days earlier.]

Aiesha: What do you mean? Bea: All of these lines [pointing to the loops on figure 2]. What about these people [pointing to all of the circles on the right]?

Aiesha: I'm going to show you all. I am multiplying [writing 11 x 10 = 110].

Bea: This person [pointing to the circle all the way on the left] is shaking hands with all of these people, and this is all of his shakes. Right, and how many handshakes is right here [pointing to the first circle to the left]?

Aiesha: Ten.

Bea: And then this one (pointing to the second circle) is going to be nine, right?

Aiesha: And then eight, seven, six, five, four, three, two, one.

Aiesha then begins drawing the chart in figure 3 and explains it to her peers.

Figure 3 - Aiesha's chart for finding the number of handshakes when 11 people are at the party

Aiesha's explanations indicate that she has moved to the image having layer. "At the level of image having a person can use a mental construct about a topic without having to do the particular activities that brought it about." (Pirie & Kieren, 1994, p.66) Aiesha now appears to have an "image" of the handshakes, and is no longer tied to the action of showing each handshake.

Moving from Image Having to Property Noticing

In this episode, two students question Aiesha's idea. This helps her to realize that her drawing shows each person shaking hands twice. Aiesha now begins to consider *why* division by 2 actually works.

Bianca: This person [pointing to the second circle on figure 2] won't shake ten people's hands. But it says every person at the party shakes hands once. [Bianca notices the number 10 written above each circle.]

Edgar: Everyone's not going to shake everyone's hands two times.

In this case, the students' questioning helped Aiesha to notice properties about her representations, thereby prompting her movement to the property noticing layer. This layer may be characterized as one in which the individual "…can manipulate or combine aspects of his/her images to construct context specific, relevant properties." (Pirie, & Kieren, 1994) In this case, Aiesha noticed that her picture representation had double the amount of handshakes, which prompted her to build on her older representation, thereby constructing a new chart (figure 4).

Figure 4 – Aiesha's move to property noticing

Moving from Property Noticing to Formalising

After Aiesha sets up a hypothetical situation by asking Bea what she would do if there was a 500 person party, Aiesha and the other members of her group now spontaneously attempt to generate a more generalized symbolic representation that could work for any number of people or handshakes. For this, they revert back to the original problem. Aiesha draws a chart (similar to figure 4) for an eight person party and constructs a number sentence along with a formula using both words and standard algebraic notation. Ultimately, Aiesha is able to come up with the formula [n(n-1)]/2, which she presents to the class.

We suggest this movement to a symbolic representation moves Aiesha to the formalising layer, creating a general statement, in which a method or common quality from the previous image is abstracted (Pirie, and Kieren, 1994).

Aiesha: N equals the number of people at the party. What I did was n times, well, we're going to do n times n minus one, n minus one in parentheses [*tracing the parentheses with her marker on figure 5*]. First what we have to do, eight, there's eight people, we have to take minus one, so there's seven [*writing 8-1 = 7 on figure 5*]. So, n times n minus one, then you divide that by two. You would multiply eight by seven, then you would divide that whole answer by two.

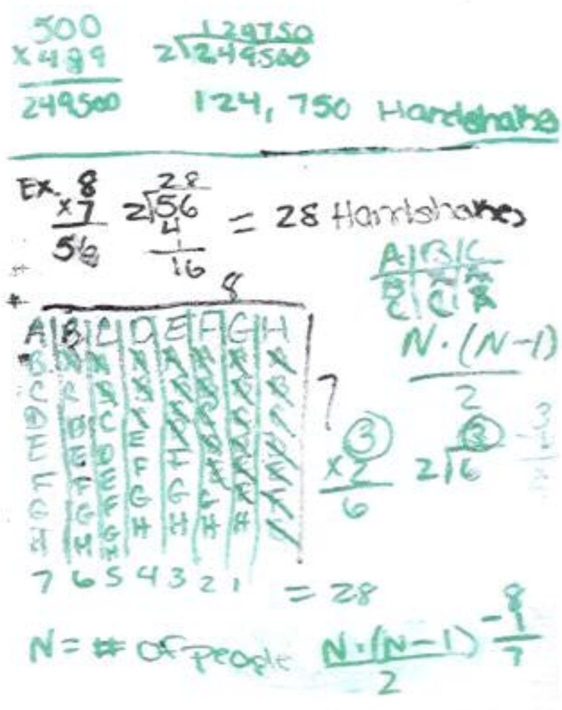

Figure 5: Working in the formalising layer

As Aiesha was presenting her formula, another student, Shaniqua, questioned her and set up a hypothetical situation based on the existing problem. Aiesha showed that her idea was valid, using multiple representations to solve the hypothetical situation (words, numbers, symbols, a chart, picture representation and acting it out), and was questioned into linking these representations to each other.

Shaniqua: [Shaniqua raises her hand during Aiesha's presentation. Aiesha calls on her.] I disagree with something. She said that there was five hundred people at the party and each of those people shake hands with four hundred and ninety nine people's hands [initially directing the comment to the teacher/researcher]. That's not true because if you do that, then you're saying each person shook [now directing the comment to Aiesha]… Ok, let's say there is three people at the party…

Aiesha: Yeah.

Shaniqua: And you are saying that every one of these three people are shaking the same three people's hands? They are shaking the same people's hands?

Aiesha: Do you want to see how that works with three people?

Shaniqua: Yeah. Aiesha describes how to use the formula for three people.

Luis: What's the n? Aiesha: All right, the n equals the number of people, n, three people, right. What you have to do first is 3 minus one, it gives you two. Then you have to do three times two,and it gives you six. You divide two into six and it gives you three. That's how many handshakes.

Aiesha draws the chart for 3 people at the party (see figure 5) and explains it.

Crystal: Why do you use the number two to divide?

Aiesha: All right, I use two because look, when two people (shaking Bianca's hand), it gives you two handshakes (pointing to her and Bianca), but normally…

Aiesha explains that she is initially counting both handshakes, then dividing the second handshake out. She draws the chart (bottom of figure 6) as if she were A and Shaniqua were B. She continues by writing a 2 between two circles (which represent people) on her picture to show the two handshakes that took place between each set of two people (top of figure 6 and the top of figure 2) to answer Crystal's question.

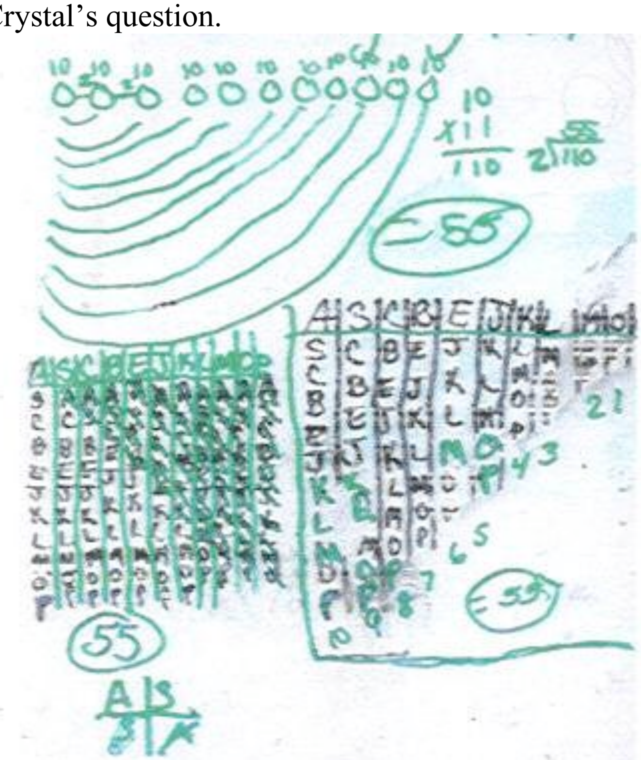

Figure 6: Linking representations to each other

Aiesha was able to show how her formula mapped into her original representation involving circles and loops, the action of actually shaking hands, as well as her chart with letters. Her ability to set up a hypothetical situation about the existing problem, develop multiple representations for the same idea, connect the representations to each other, and ultimately provide a solution that is generalizable indicates that she has reached the formalising layer. Further, some six months later, Aiesha's class was given the opportunity to investigate a task that was structurally similar to this handshake problem. Within a few minutes, Aiesha and her group moved through most of the representations they constructed six months earlier, and reconstructed the formula to generalize, using the correct symbolic notation. Interestingly enough, many students around the room also used the formula Aiesha presented six months earlier for this new task.

Conclusions

We conclude that student-to-student interactions and questions played a central role in Aiesha's movement from primitive knowing to formalising, as well as her movement to linking representations to each other. This ultimately led to her ability to retain and retrieve her ideas when presented with similar types of problems months later, which is a central goal of the teaching and learning process. Of course, we cannot say with complete certainty that these interactions were exclusively responsible for the development of the ideas, however, we believe that our analysis suggests that they played a key role. While it is not possible to draw overwhelming conclusions based on this limited example, we do believe that an analysis of this type has the potential to call attention to the importance of providing meaningful opportunities for such student-to-student interaction.

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IMPROVING PROBLEM-SOLVING JOURNALS: THE MATHEMATICIAN, THE NARRATOR, AND THE PARTICIPANT

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Students' problem solving experiences are fraught with failed attempts, wrong turns, and progress that moves in fits and jerks, oscillating between periods of inactivity, stalled progress, rapid advancement, and epiphanies. Without proper guidance students will tend to 'smooth' out these experiences and, as a result, present stories in their journals that are less reflective of the their process and more representative of their product. In this article I present a framework for guiding students' journaling in such a way that their writing more accurately reflects the erratic to-and-fro of their problem solving process. I also provide a very brief outline of some empirical research that shows how this more structured form of journaling has been used to track preservice teachers' encounters with mathematical discovery.

For mathematicians, problem solving is a process that incorporates not only the logical processes of inductive and deductive reasoning, but also the extra-logical processes of creativity, intuition, imagination, insight, and illumination (Csikszentmihalyi, 1996; Davis & Hersch, 1980; Dewey, 1938; Fischbein, 1987; Hadamard, 1945; Poincaré, 1952). However, as creative a process as problem solving may be, the results of these processes are "encoded in a linear textual format born out of the logical formalist practice that now dominates mathematics" (Borwein & Jörgenson, 2001). This discordance between the process of problem solving and the presentation of its products is nicely summarized in the comments of Dan J. Kleitman, a prominent research mathematician.

> *In working on this problem and in general, mathematicians wander in a fog not knowing what approach or idea will work, or if indeed any idea will, until by good luck, perhaps some novel ideas, perhaps some old approaches, conquer the problem. Mathematicians, in short, typically somewhat lost and bewildered most of the time that they are working on a problem. Once they find solutions, they also have the task of checking that their ideas really work, and that of writing them up, but these are routine, unless (as often happens) they uncover minor errors and imperfections that produce more fog and require more work. What mathematicians write thus bears little resemblance to what they do: they are like people lost in mazes who only describe their escape routes never their travails inside.*
>
> (Liljedahl, 2004a, p. 157)

The discordance between process and product, however, is not a dilemma that is restricted to the domain of professional mathematicians. Students of mathematics also have a difficult time breaking away from the formalist practices of conventions as delivered to them in the form of curriculum, textbooks, and classroom instruction. Adherence to such conventions has resulted in the misrepresentation of mathematical activity and has caused many mathematics students to believe that full rigour is all that mathematics is about (Hanna, 1989); more specifically, 'doing' mathematics is misunderstood to mean 'knowing' mathematics.

In mathematics education treatment of this dilemma has typically revolved around the restructuring of teaching strategies to more accurately reflect the practices of 'doing' mathematics. This has resulted in the popularization of delivery methodologies such as teaching through problem solving (Cobb, Wood, & Yackel, 1991), problem posing (Brown & Walters, 1983), and discovery learning (Bruner, 1961; Dewey, 1916). This refocusing of instructional strategies only works, however, if they are accompanied by a complimentary refocusing of assessment strategies – strategies that value process rather than just product. One popular assessment tool that accomplishes this is the problem-solving journal, which can be used to capture students' problem solving processes both for assessment (c.f. Else, Thompson, & Thompson, 2000) and empirical research (c.f. Zazkis & Liljedahl, 2002). However, as popular as this instrument is it is not without its faults. Students' problem solving experiences are fraught with failed attempts, wrong turns, and progress that moves in fits and jerks, oscillating between periods of inactivity, stalled progress, rapid advancement, and epiphanies. Without proper guidance students will tend to 'smooth' out these experiences and, as a result, present stories in their journals that are less reflective of the their 'travails inside the maze' and more representative of their 'escape route'. In this article I present a framework for guiding students' journaling in such a way that their writing more accurately reflects the erratic to-and-fro of their problem solving process. I also provide a very brief outline of some empirical research that shows how this more structured form of journaling has been used to track preservice teachers' encounters with mathematical discovery, the most elusive and intense of aspects of the problem solving (Barnes, 2000; Burton, 1999; Davis & Hersh, 1980; Hadamard, 1945; Poincaré, 1952; Rota, 1997).

A More Structured Form of Journaling

As mentioned above, literature that detail mathematician's problem solving efforts is unrepresentative of the true process of 'doing' mathematics. One rare exception to this is an account written by Douglas R. Hofstadter called *Discovery and Dissection of a Geometric Gem* (1996) that tells the story of a mathematical discovery with amazing sincerity. It is detailed and complete, from initiation to verification. It tells the story of being lost in a maze, searching for answers, and in a flash of insight, finding the path out. Perhaps the reason that the account is so different is that Hofstadter is not a professional mathematician. He is a college professor of cognitive science and computer science, and an adjunct professor of history and philosophy of science, philosophy, comparative literature, and psychology. As such, he has a unique appreciation for tracking his own problem solving processes.

In analysing Hofstadter's account it becomes clear that one of the reasons that it is so sincere is because of the way in which he incorporates the use of three different voices, a trinity of personas, in telling his tale. I have come to name these personas the *narrator*, the *mathematician*, and the *participant*. These personas are not explicit in Hofstadter's writing in that he does not introduce them, annotate them, or even acknowledge them. Instead they are implicit, emerging from the active analysis of his writing more so than from the passive reading of his writing. Each of these personas contributes to the anecdotal account in a different way. The *narrator* moves the story along. As such, he often uses language that is rich in temporal phrases: 'and then', or 'I started'. He also fills in details of the non-mathematical variety seemingly for the purpose of providing context and engaging content. The *mathematician* is the persona that provides the reasoning and the rational underpinnings for why the mathematics behind the whole process is not only valid, but also worthy of discussion. Finally, the *participant* speaks in the voice of real-time. This persona reveals the emotions and the thoughts that are occurring to Hofstadter as he is experiencing the phenomenon.

To demonstrate these personas, I present a portion of the chapter that contains within it all three voices. Before I do, however, it would be useful to introduce the general context of his mathematical encounter. At the time of writing the chapter, Hofstadter has only recently come to be impassioned with Euclidean geometry and had never been introduced to the Euler line of a triangle. When he did learn about it, however, two things immediately struck him: the connectivity of seemingly different attributes, and the exclusion of the incentre. So, he began a journey of trying to find a connection between the Euler line and the incentre. At the point in the passage presented below Hofstadter has just discovered something about the incentre.

> *One day I made a little discovery of my own, which can be stated in the following picturesque way: If you are standing at the vertex and you swing your gaze from the circumference to the orthocentre, then, when your head has rotated exactly halfway between them, you will be staring at the incentre. More formally, the bisector of the angle formed by two lines joining a given vertex with the circumcentre and with the orthocentre passes through the incentre. (A more technical way of characterizing this property is to say that O and H are "isogonic conjugates".) It wasn't too hard to prove this, luckily. This discovery, which I knew must be as old as the hills, was a relief to me, since it somehow put the incentre back in the same league as the points I felt it deserved to be playing with. Even so, it didn't seem to play nearly as "central" a role as I felt it merited, and I was still a bit disturbed by this imbalance, almost an injustice.*
>
> (Hofstadter, 1996, p. 4)

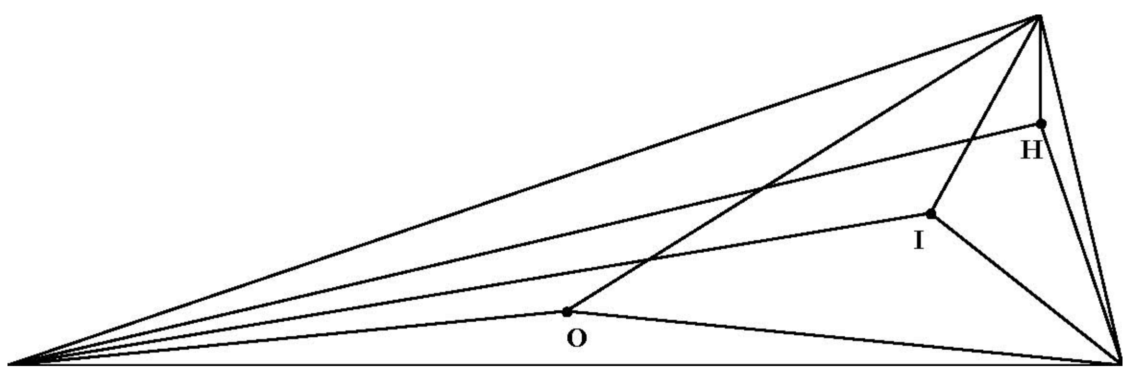

Figure 1: Triangle With Incentre, Orthocentre, and Circumcentre

Even from this brief excerpt it can be seen how the three personas interact with each other, while at the same time presenting different aspects of the mathematical experience. It begins with "One day …", a clear indicator that the *narrator* will be speaking.

> *One day I made a little discovery of my own, which can be stated in the following picturesque way: If you are standing at the vertex and you swing your gaze from the circumference to the orthocentre, then, when your head has rotated exactly halfway between them, you will be staring at the incentre.*

Hofstadter is telling us what he has found in an informal yet descriptive way. This is followed by his *mathematician* persona coming in and formalising this finding in a more precise and mathematical way.

> *More formally, the bisector of the angle formed by two lines joining a given vertex with the circumcentre and with the orthocentre passes through the incentre. (A more technical way of characterizing this property is to say that O and H are "isogonic conjugates".) It wasn't too hard to prove this, luckily.*

Finally, the *participant* reveals how he feels about his finding and what thoughts this find is precipitating.

This discovery, which I knew must be as old as the hills, was a relief to me, since it somehow put the incentre back in the same league as the points I felt it deserved to be playing with. Even so, it didn't seem to play nearly as "central" a role as I felt it merited, and I was still a bit disturbed by this imbalance, almost an injustice.

The interplay present in this passage is typical of the first six pages of the chapter. At that point in the account Hofstadter makes a profound discovery, which is revealed in his last use of the *participant's* voice. After this point there is a brief interplay between the *narrator* and the *mathematician* and then the voice of the *narrator* also disappears forever. The last seven pages of the chapter are comprised of the *mathematician* articulating and proving his discovery.

The most interesting thing about Hofstadter's use of these three voices is what it reveals about the type of journals that my students had produced in the past. At best, these journals had been a combination of the voice of the *mathematician* and the *narrator*. More often than not, however, the journals had been the voice of the mathematician along with a logically reconstructed narrative that presented the logic of the solution rather than the history of the process. Although this had been frustrating I had failed to find a solution to it. Repeated urging to be truthful in their writing sometimes resulted in more detailed narratives with more descriptions of failed attempts and mistaken assumptions, but often only lead to more details of the logical development of the solution. My use of problem-solving journals lacked structure, a framework which the students could adhere to. In particular, the voice of the *participant* was missing, but until I read Hofstadter's account of his discovery I did not even know it could exist. In his description was the missing piece that was required to elevate the students' journaling to the level of detail that was needed to really see their mathematical thinking[1], and to capture their problem solving processes.

Capturing Students' AHA! Experiences

The AHA! experience is a term that captures the essence of the experience of illumination. In the context of 'doing' mathematics it is the *EXPERIENCE* of having an idea come to mind with "characteristics of brevity, suddenness, and immediate certainty" (Poincaré, 1952, p.54). For mathematicians they are an expected and accepted part of mathematical activity, but for students (especially weak and apprehensive students) they are often an unanticipated and pleasant conclusion to a long onerous task or effort (Liljedahl, 2004b, 2002). Although there has been a sizeable amount of research done on this subject (Barnes, 2000; Burton, 1999; Davis & Hersh, 1980; Hadamard, 1945; Poincaré, 1952) most of this research has relied heavily on participants' reflective anecdotal comments as data. There are many reasons for this, the most prominent of which is the elusive nature of the experience – it can happen anywhere at any time, from experiencing illumination in the shower to being awaken in the middle of the night by a good idea. As such, real-time capture is difficult. However, if participants were to record these epiphanies in their problem solving journals as they occur then something very close to real-time capture would be possible. This is the basic premise behind my empirical research in this area.

I introduced this form of journaling to my students (one group of preservice elementary teachers (n=38) and one group of preservice secondary mathematics teachers (n=34)) on the first day of the course. That is, I introduced each of the three personas and what their respective roles were in documenting problem solving efforts. Although there was no mention in what proportions they were to use them, it was made clear that they were expected to incorporate each of these three voices in their problem solving journals. Four weeks into the course their problem solving journals were collected and one specific homework problem was critiqued. This was followed by an in-class formal review of the three personas, examples of their voices, and a review of the expectations regarding the use of the three voices. Other than these moments of instruction (totalling no more than 60 minutes) and the critique of the problem solving journals at the four-week mark, no further class time was devoted to this topic[2].

They then used this method of journaling to record their efforts, failures, and successes in solving a wide variety of challenging problem solving exercises spread throughout a 13 week *Designs For Learning Mathematics* course. At the end of the course, I introduced them to the idea of an AHA! experience and asked them to write about any such experiences they may have had using a reflective writing style. These reflective accounts were then compared to their 'real-time' problem solving journals to see if there was any evidence of such experiences in their use of the three personas either through some exclamation by the *participant*, some accounting by the *narrator*, or some change in reasoning by the *mathematician*. There was. Of the 36 preservice elementary teachers claiming to have had an AHA! experience in their reflective journals 29 (81%) had clear and discernable evidence of the experiences in their problem solving journals. For the 25 preservice secondary mathematics teachers claiming to have had an AHA! experience the numbers were slightly less with 18 (72%) of them displayed evidence of the experiences in their writing. These numbers were significant in that in previous years there existed little if any corroboration between students' reflective claims of AHA! experiences and the actual evidence of such AHA!'s as presented in their problems solving journals. In what follows I present some passages form two students problem solving journals as well as their reflective journals.

Stephan and Marie both wrote about their AHA! experiences in the context of the *Pentominoe Problem*[3]. Stephan comes to the solution of the problem through an AHA! experience as seen in his problem solving journal.

> *I've got it! I'm sure the half dozen beers have helped but I think I've solved it. Its simple really and I've gotten it because of, believe it or not, golf! The explanation may be muddled but it makes perfect sense (in my head). In golf there are two values when keeping score: the number of shots actually taken and the number of shots relative to par [..] How does this apply to the Pentominoes puzzle? [..] When all the blocks are vertical their sum divided by five will always be a whole number, no matter where they are on the number grid. These vertical blocks are par (E). If you then move a block to the right one then your score changes to +1. If you move it left then it changes to –1 …*

His account begins with the *participant* exclaiming "*I've got it!*". This is followed by a short account by the *narrator* as to where the idea came from, and then the *mathematician* takes over in trying to articulate how and why it works. Stephan's AHA! is corroborated by the following passage in his reflective journal:

> *The AHA! came right after I'd played a round of golf and I was watching golf on TV in the clubhouse. On the screen flashed a player's scorecard and I realized that the very notion of par was the solution to the Pentominoe puzzle.*

For Marie the idea came to her in the hot tub.

> *Yes! I think I have figured it out! I was sitting in my hot tub when I suddenly got the feeling that it wasn't about the numbers but rather about the specific configuration of the shapes. The solution has to do with symmetry! I discovered that the cross is always divisible by 5, and I am pretty sure it is because it is symmetrical. Shoot! This doesn't necessarily work because there are other shapes like "Tee" and "Z" that are always divisible. Why! I really think that symmetry has*
> *something to do with it! But wait … "Tee" is only divisible when it is upright …*

It is clear from this passage that something has occurred to her, even though it does not work out as nicely has she had hoped. From the exclamation of the participant as well as from the change in reasoning be the mathematician it is reasonable to assume that she has had some sudden insight, perhaps even an AHA!. This experience is later confirmed in Marie's reflective journal.

> *The most significant AHA! moment that I had so far is during the Pentominoes puzzle. I was stuck on trying to figure out what the remainder was going to be just by looking at the numbers ... I couldn't possibly imagine that you could memorize all of the possible combinations. I had been working on the problem all day, and struggling with it, and had finally given up trying. I went out for the evening and came home and sat in the hot tub for about half an hour. Even though I wasn't consciously thinking about the problem I think that the ideas were still in my head. I honestly don't know why the idea came to me ... perhaps it was because I was so relaxed and tired and not consciously struggling with the problem, but all of a sudden the solution came to me. [..] obviously this discovery made me feel good because this idea eventually led me to the solution.*

The data was not limited to the quantitative results, however. The use of the three personas in their writing also produced a rich set of qualitative data that provided greater insights into students' experiences with this elusive phenomenon.

Endnotes

1. Andrew Waywood (1992) has done work on creating a developmental model of students' mathematical learning through journaling. In this work he identified three types of journaling within his subjects. They are *recount, summary*, and *dialogue*. *Recounting* is very similar to what I refer to as the voice of the *narrator* and *summarizing* is virtually identical to the voice of the *mathematician*. *Dialogue*, however, is only part of what I refer to as the voice of the *participant*. For Waywood, dialogue is the self-talk that goes on in the journals, through which ideas are revealed. He does not, however, stipulate that *dialogue* contains any expressions of emotions. Both of these characteristics, presentation of ideas and emotions, make up the voice of the *participant*.
2. In a more recent implementation of this form of journaling the student were, over the course of three classes given three problems to solve. The first problem was to be solved and only the solution was to be presented in as precise a mathematical language as possible (the voice of the *mathematician*). The second problem was to be solved, but only the story of how they arrived at the solution was arrived at was to be presented (the voice of the *narrator*). The third problem was to be attempted, but only the feelings they experienced in attempting the problem were to be documented and subsequently presented (the voice of the *participant*). During the fourth class these three journaling styles were discussed and the students proposed that they should be allowed to use all three voices in their journaling. This proposal was then formalized.
3. If a pentominoe is placed on a hundreds chart will the sum of the five numbers that it covers be divisible by five? If not, what will the remainder be? Generalize the solution.

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PROMOTING STUDENTS' COMPREHENSION OF DEFINITE INTEGRAL AND AREA CONCEPTS THROUGH THE USE OF DERIVE SOFTWARE

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The study aimed at investigating what type of mathematical competences are enhanced in students' understanding and solving problems related to definite integral as a result of using Derive Software. Results indicated that some students relied on the use of the software as a means to validate their paper and pencil work, others used the software to graphically represent and calculate approximated areas and a third group of students combined both paper and pencil and software approaches to solve problems but often failed to connect concepts that appeared in the study of the definite integral with basic ideas (and procedures) previously studied (area of simples figures).

Introduction

This study investigates how first year engineering university students performed after they had taken a Calculus course in which they systematically used DERIVE Software to work on a series of tasks that involve numerical, graphic and algebraic approaches. Problem solving activities that involve the study of definite integral were designed in accordance with research results identified in the literature review. In particular, students had the opportunity to use a specially designed Utility File as a means to approximate areas of bounded curves (through the use of rectangles, trapezoids, and parabolic regions). Thus, the study focuses on documenting the extent to which students were able to utilize Derive software in their problem solving approaches. In particular, we were interested in analyzing the type of representations used by the students to understand and solve different types of problems that involve concepts of area and definite integral. Our fundamental research questions were: To what extent do students display relationships between graphic, algebraic, and numerical representations in their problem solving approaches? And what type of difficulties do students experience as a result of using Derive and Utility File? To what extent does the use of Derive Software help students understand concepts involved in the study of definite integral? What type of mathematical representations do students exhibit to understand and solve non-routine problems related to this topic? What types of mathematical competences are enhanced in students' understanding and solving problems related to definite integral as a result of using Derive Software?

Conceptual Framework

Basic ideas that helped frame the study recognize that the use of CAS functions as a cognitive tool for students not only helps to solve problems but also to make sense of and understand mathematical ideas; furthermore, this type of tools provides students the opportunity to generate new mathematical representations that help them investigate relationships associated with a situation or phenomenon under study and to appreciate the balance between formal and informal mathematics (Heid, 2002). Guin and Trouche (2002) introduced the idea of instrumental genesis to explain the process of transforming an artifact (a material object) into an instrument (when students use it as an instrument to solve problems). They say that this process is complex and involves aspects related both to the actual design features of the tool and also to the cognitive process involved in students' appropriation of the instrument to solve problems (the development of instrumentation schema). In this context, it became important to pay attention to the design limitations associated with the use of the software that might have interfered with students work. Drijvers (2002) identifies local and global obstacles that students often display while using a computer algebra environment. In particular, we take the idea that obstacles can be seen as an opportunity for students to reflect on their own learning rather than as a barrier to achieving understanding of mathematical ideas. In this perspective, to analyse how students performed during the study, we followed a conceptual framework in which the use of representations plays a fundamental role in students´ construction of concepts (Goldin, 1998). In addition, a mathematical competence model was adopted from Socas (2001) and used to explain students' level of understanding of basic ideas related to definite integral in which three related phases are identified: the use of a certain language (Semiotic Stage); the use of several registers and their corresponding operations (Structural Stage); and the conversion or transition between different types of representations or registers (Autonomous Stage). In particular, we focus on analysing the type of basic resources and strategies that students utilize when dealing with a particular representation of the problem and also the extent to which students were able to make the transition from one type to another representation (graphic, algebraic and numeric representations). The ideas embedded in this framework played a fundamental role not only in analysing students´ work but also influenced the design and structure of the study. In particular, the design of a Utility File was based on the idea that students could use the Utility File as an aid to calculating a set of definite integrals in which the primitives of the function to be integrated could not be expressed through elementary functions. Thus, the use of the Utility File could help students develop an image of the integration processes and their relationship with the area concept.

Methods and General Procedures

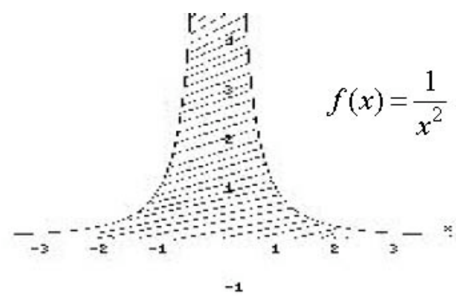
Thirty-one first year engineering university students participated in the study. The study was carried out in a regular Calculus class during one semester, meeting six hours a week with two hours of computer laboratory session. As part of the course, students used DERIVE software to work on a series of problems. In particular, a special Utility File was designed to help students calculate approximations of areas. The goal here was that students could develop a conceptual understanding of integration processes. Ideas like partition, refinements, limits, and approximation items appeared as important during the students` use of the software. Here it was also important for students to recognize linkages between numerical, algebraic and graphic representations associated with the integral concept. To collect data a questionnaire was given to the students at the end of the course. In addition, students took part on task-based interviews in which they had opportunity to reflect on the use of different representations and the use of the software. In particular, the researcher asked students to explain and elaborate their approaches to the problems.

The course was problem solving oriented and students were constantly asked to respond to questions concerning to the concept of area and definite integral. In general, three related phases distinguish the instructional approach used in this course: (i) The instructor's presentation and discussion with the whole class of ideas around fundamental concepts involved in the definite integral themes; (ii) students worked in pairs on a series of problems in which they used the DERIVE software to approach them. Students worked in the computer laboratory and each pair handed in a written report; and (iii) the instructor reviewed students´ approaches and discussed what students did during the session with the whole class. These three instructional phases appeared consistently throughout the course for one semester. Some conceptual ideas that emerged during students´ discussions include themes like limit of sums (Riemman´s approach), area of bounded regions and their relation to the fundamental theorem of calculus. In particular, the relationship between a given function f(x) (and its integral function F(x) $(F'(x) = f(x))$ .In addition, students used the software to graphically function F) (x (F) (’ = xf represent functions and evaluate definite integrals.

To analyse students´ understanding of fundamental ideas related to definite integral concepts, at the end of the course, students were asked to work on a questionnaire that included 10 non-routine problems. In general, the problems were organized into three groups in accordance with the following characteristics:

a. Problems in which there was a geometric representation and students were asked to determine, whenever possible, the area of some regions (students had to analyse features of the graph in order to identify corresponding integration limits). Otherwise, students needed to provide a mathematical argument to explain why it was not possible to calculate the area. An example illustrating the type of problem in this group is,

Calculate the area of the shaded region. If you think that it cannot be done, then explain why not.



$f(x) = \frac{1}{x^2}$

b. Problems in which students received an algebraic expression to find the integral and it was important to graphically represent it in order to solve them. Here, it was important for students to identify the location of the region (above or below x-axis) and in some cases recognize discontinuities of the function. An example of a problem from this group is,

Determine the limited area between the *x* axis and the function

$f(x) = 2x^4 - 2x^3 - 14x^2 + 2x + 12$

c. Problems in which there was a statement and students needed to discuss whether it was false or true. An example from this group is,

Is the next proposition false or true? (justify your response)

$$If \int_a^b f(x)\,dx \quad \int_a^b g(x)\,dx \;\; then \;\; f(x) \quad g(x) \;\; \forall x \in [a,b]$$

In addition to solving the questionnaire both with and without the use of the software, six students were later interviewed.

Results and Discussion

To characterize students´ competences it was important to analyze what students did in each of the items of the questionnaire; in general, three student profiles emerged from the students´ responses to the questionnaire. Later, the analysis of the information gathered through the students´ task-based interviews provided a basis to confirm the presence of those students´ problem-solving profiles.

(1) Students who were grouped in the first profile (E1, E3 and E5) showed a tendency to use the software as a tool to carry out algebraic operations or to find points of intersections of the curve with X-axis. That is, their use of the software focused mainly on calculating algebraic or numeric operations involved in the problem or situation without including a graphical approach. In those problems that involved graphic representation, generally they had difficulties making sense of the situation and tried to use persistently algebraic methods. However it was important to observe that this group of students tried to examine the validity of general relationships (problem of the third group) by analyzing particular examples graphically which led us to consider that for this group of students the way how the problems were stated influenced their use of the tool. Moreover, these students seem to perceive the process of solving definite integral as the application of some rules or procedures by taking into account the context of the problem. For example, when the student E1 was asked to explain the meaning of definite integral he answered:

S (Student): If I have the integral

$$\int_b^a X^2 \cdot dx$$

then to calculate it (pointing to the integrand), I would get the primitive, that is, I would obtain; and this means that if I have function (drew

$$f(x) \rightleftarrows F(x)$$

if I solve this (left side) I get this (right side), likewise if I get the derivative of this (right side) I get this function (left side). Thus, I get the derivative of, I would get

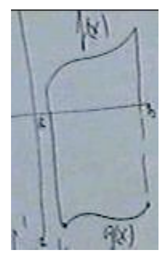
$$\frac{\cancel{3}X^2}{\cancel{3}} = X^2$$

The student' comments involved the use of the Fundamental Theorem of Calculus. However, he does not rely on those methods widely used in laboratory practices. It might be that class instruction did not influence the students' way of understanding the concept of definite integral.

(2) A second group of students generally recognized the importance of finding areas of limited curves through the idea of approximation. They were aware of the need to get better and better approximation by the process of refining a partition within an interval. However, they have not developed a clear understanding of how the process of selecting a particular partition on the interval was done. Particularly when they tried to do this task without using of software. They also associated the definite integral concept with a set of procedures to calculate its value and failed to identify particular necessary conditions to apply those procedures. It seems that students considered the use of the software a tool that allows them to facilitate those approaches that involve only the use of paper and pencil. Indeed, the software was used often as a means to support what they had done with paper and pencil. Another important result is that when students worked on a problem embedded in a graphical representation, they were often able to identify limits of integrations and ways to calculate areas of limited regions; however, when the problem was expressed in an algebraic form they seldom relied on graphic representations to solve the problem. Here the use of the software seemed to be sufficient to solve the problem. In one

problem, that involved calculating the area of a region bounded by simple triangles (calculate $\int_{-3}^{4} |x+1| dx$), students immediately began to apply integral definite methods rather than calculating the area through simple formula (bh/2); in particular, during the interview one student (E6) mentioned that since the topic was definite integral, then they had to solve it in that way.

(3) A third student profile is associated with those students who successfully apply the idea of approximation to determine areas of bounded regions. They were fluent not only in deciding what type of partition to take on the interval but also in using algebraic tools to carry out the operations involved in calculation of the corresponding areas. This group of students showed a clear disposition to use the Utility File designed to approximate areas. In general, they identified and properly used important information connected with both algebraic and graphic representations in order to calculate definite integrals. There is evidence that these students grasped the relationship between area and definite integral concepts. This was evident in the way they used the Utility File to approximate bounded areas. Here, students recognized that calculating integrals goes beyond applying a set of formulae or using a particular software command, it involves a process that they could visualize through the use of the Utility File. However, when they were asked to examine general statements about properties of functions and their relationships with the definite integral they failed to provide a coherent argument to support their claims. In particular, they seemed to lack problem-solving strategies (analyzing particular cases, providing counterexamples, or using graphic representations) to make sense or interpret this type of problems. For example, during the interview when the student E2 was asked to reflect on a possible relationship between the graphic of two functions and their integrals, he provided contradictory arguments:



R (Researcher): If you have these two functions (see picture) with f(x) > g(x), Is the integral of f(x) □ the integral of g(x) in the interval (a, b)?

S: this is greater (Pointing to the g region) this area is greater than this one (pointing to the g region in comparison to f region) but it is negative. Here he pauses and seems to be thinking,

R: ¿Then what?

S: ¿I am checking whether f(x) □ g(x)?

R: We have assumed that f(x) > g(x).

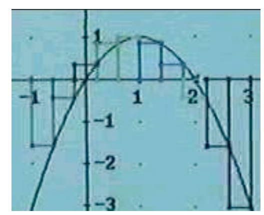
S: The values here (pointing at the segment between a and b) when evaluated under the function we get positive values (he drew a line under the graph of f) These (values) will give you negative Ys (he drew a line above the graph of g). Here I can verify that this is true (pointing to the inequality of the functions). These values are positive and these are negative values (pointing to the graph of f and g)…

However, it seems that the student is not convinced with his initial response. He confuses the use of the inequality sign between the functions and between the integrals seen as area. This group showed a strong inclination to use the software to approach all the problems without considering a graphic representation which is and often necessary to solve them. This group experienced serious difficulties in solving problems that involved proving or rejecting some propositions. In addition, they seem to think that the software provides them not only with an efficient way to approach the problem but it also becomes the only way to actually solve the problems correctly. Indeed, during the interview when these students were asked to explain graphically the meaning of what they had obtained through the software, they were not able to explain on their own what happened when the value of the integral was negative. Again, these students, in general, seemed to use the software mechanically to make calculations and failed to identify and relate embedded representations in order to make the transition between and within them. To illustrate this type of behavior, we show part of an interview with a student E4 who decided to use the Utility File to approach a problem that can be easily solved directly. During the interaction between the researcher and the student it becomes clear that he shows fluency in the use of the software but experiences difficulties in interpreting his work.

To calculate the area limited by the function and the X-axis, the student chose the command RECT_EXTREMO_DERECHO(a,b,n) from the Utility File (see Camacho & Depool, 2003a, for more details of Utility File) and substitute values of "a" by -1, "b" by 3 and "n" by 10, then calculate the matrix and represent the rectangles.

S: This is the area taking the right side of this interval (that is the interval [-1,3]), here we can draw the rectangle, but we need to provide a number for x.

R: Well, you may notice that you are considering from -1 to 3, but we want to find the area in the interval from 0 to 3. That is, I am asking you to find the area of this shaded region (pointing at the figure)



S: ¿from 0 to 3?

R: Yes.

S: I can do that by a numerical method as well.

R: What are you doing?

S: Calculating numerically (He selects the command MEDIDA_EXTREMO_DERECHO (a, b, n)). We are going to do it by drawing rectangles. I substitute "a" by 0, "b" by 3 and "n" by 10 and the result is – 0.495 Why is the value negative?

R: I was going to ask you why it is negative

He tries to use the Utility File again but it does not work, that is, he doesn't get a response.

R: Why do you think you get a positive value on one side and negative value on the other side

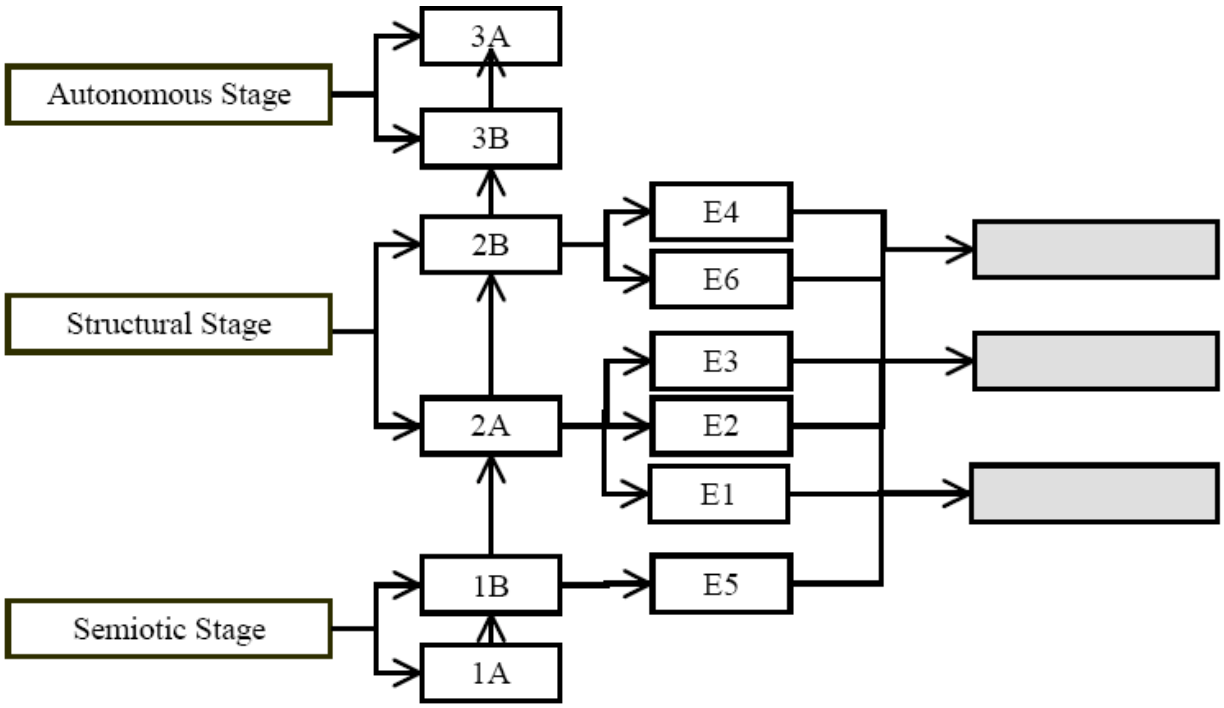
S: Why is that?

R: What do you think?

S: ...(Silence)

Students' tendency to approach the problem through the use of the software is evident. At this stage the interviewer directs the dialogue to understand the meaning of this process and eventually the student seems to comprehend what he is doing.

The following table represents a relationship identified in students' mathematical competence and categories (see Camacho & Depool, 2003b) and their profiles.



It is important to recognize that the use of the software helped students understand basic concepts involved in the study of definite integral. For example, when they used the Utility File, students had the opportunity to work on various examples to approximate area values associated with bounded regions. Here, they focused on examining aspects that include the relationship between the domain of the involved function and feasible partitions to approximate corresponding areas. In particular, the use of the Utility File provides students with basic elements to visualize the concept of limit involved in calculating definite integrals. In addition, students, in general, used the DERIVE software to graphically represent functions and to calculate integral. In this context, the software became an important tool for students to identify intersections of the graph and x-axis and the position of the region (above or below x-axis). In some cases, they utilized the software as a means to validate results that they had obtained through paper and pencil procedures.

An important issue that emerged from analysis of students´ work is that in order to comprehend and explore connections and relationships between concepts connected with the study of definite integral, they need to make the transition, in terms of meaning, between the distinct representations of the concept. For example, when dealing with the graphic representation of a function it was important to explain the position of the graph and its relation to the sign associated with the value of its integral and also to the process of approximation to the value of its area through a numerical approach (area of small rectangles). In addition, students need to develop problem-solving strategies that help them think of cases beyond those embedded in particular problems. That is, it was evident that when students were asked to work on problems that involved general statements, they experienced serious difficulties in constructing examples or counter-examples that could help them to understand and explore the situation in general terms. In this respect, it becomes important to design and implement instructional activities that include the use of the software in problem-solving contexts in which students have opportunities to develop basic problem solving strategies (including metacognitive ones). Finally, implementation of students learning activities with the use of technology should value the ways in which students present and communicate their results. In this study, the use of task-based interviews functioned as tools for reflection by which students had opportunities to reveal their ideas and at the same time to explain and examine in detail connection between the various representations of the problem. At this stage, it was observed that students had not often thought of those connections and the very questions asked by the interviewer (researcher) became an opportunity for students to enhance their understanding. In this regard, it was important that the class itself should be seen as a community that demands constant reflection from each of its members. Remarks: Although the use of the software provided an interesting instrument for students to free themselves from memorizing formulae or calculation procedures it is also important to recognize that students need time to mature and develop a firm conceptual understanding of the definite integral. In particular, students need to pay attention to the process of transforming and connecting relationships among graphic, algebraic and numerical representations. This seems to be a crucial step in order for students to develop deep understanding of the definite integral concept. In addition, students need to develop a set of problem-solving strategies that would help them decide when to use and monitor the work done through the use of the software. Students´ task interviews not only provided important information regarding students´ competences but also became a tool for reflection by which students had the opportunity to extend their knowledge.

Endnote

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RELATED RATES PROBLEMS: IDENTIFYING CONCEPTUAL BARRIERS

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Related rates problems require students to have a strong understanding of differentiation, function, and variable, and how these concepts apply beyond strictly computational problems. It has been suggested that students do not fully understand the concepts of differentiation or variable (Orton, 1983; White & Mitchelmore, 1996; Clark et al., 1997). The intent of this study was to gain a better understanding of what obstacles calculus students must overcome to gain a conceptual understanding of related rates problems and to suggest activities that facilitate students' ability to confront and successfully complete these types of problems. It was found that students do not actively engage in transformational reasoning while solving related rates problems.

Calculus is a challenging course for most undergraduate students, and calculus word problems appear to be a particular source of trouble (White & Mitchelmore, 1996, Selden, Selden, & Mason, 1994; Martin, 1996, 2000; Ferrini-Mundy & Graham 1991). There are particular topics that are usually difficult for students to understand, and even the best students have trouble with non-routine problems (Selden, Selden, & Mason, 1994; White & Mitchelmore, 1996). Martin (1996) indicated that related rates problems are not only difficult, but that even robust problem solvers likely will not achieve a conceptual understanding of these problems without sufficient guidance.

Background

Problems that apply the concepts of calculus such as those of related rates emerge as a source of frustration for students and pedagogical complexity for instructors (White & Mitchelmore, 1996; Martin, 1996, 2000; Clark et al., 1997). Martin (2000) conducted a study investigating students' difficulties with geometric related rates problems. In attempting to understand students' difficulties with these related rates problems, Martin broke down the procedure for solving them into seven steps. She then classified these steps as either conceptual or procedural as follows:

1. Sketch the situation and label (Conceptual)
2. Summarize the problem and identify given and requested information (Conceptual)
3. Identify the relevant geometric equation (Procedural)
4. Implicitly differentiate the geometric equation (Procedural)
5. Substitute specific values and solve (Procedural)
6. Interpret and report results (Conceptual)
7. Solve an auxiliary problem, e.g. solve a similar triangles problem before being able to use the volume of a cone formula to relate the variables (Varies) In her study, Martin found that the problems that appeared to be the easiest for students were the ones that required only the selection of the appropriate geometric formula, differentiation, substitution, and algebraic manipulation. The most difficult questions were those that required Step 7, solving an auxiliary problem. She also indicated that the conceptual steps are more difficult for students than the procedural ones. However, Martin concluded that students' poor performance on these types of problems has links to difficulties with both procedural and conceptual understandings.

It should be noted that while Step 3 is listed as procedural, there is a conceptual component embedded in it, understanding function composition. Carlson, Oehrtman, and Engelke (submitted) and Carlson (1998) provide evidence that first semester calculus students did not have an object view of function composition and experienced frustration when confronted with a composition problem. Step 4 also has an embedded conceptual component, the chain rule.

Much of Martin's research supported the findings of White and Mitchelmore (1996). White and Mitchelmore studied students' understanding of related rates and extrema calculus problems. Their study used differently worded versions of four problems: two problems focused on related rates, one was a maximization problem, and one was a minimization problem. The four versions of the problems ranged from a word problem that required the student to model the situation and come up with the appropriate relation to an almost strictly symbolic version that merely needed to be manipulated. It was found that students performed better when there was less need for translation from words to symbols.

White and Mitchelmore's (1996) study also showed that students have a tendency for a "*manipulation focus*, in which they base decisions about which procedure to apply on the given symbols and ignore the meaning behind the symbols. Interview comments showed that manipulation focus errors were not just bad luck, but that students were actively looking for symbols to which they could apply known manipulations."(p. 88) The researchers also describe two other forms of the manipulation focus: 1) the *x,y syndrome*, in which students remember a procedure in terms of the symbols first used to introduce the concept without understanding the meaning of the symbols; and 2) the students fail to distinguish a general relationship from a specific value.

Simon (1996) indicated that transformational reasoning is a critical component of mathematics learning and understanding. The idea behind transformational reasoning is that students can create a mental model that can be manipulated to see and understand relationships. Simon states, "Central to transformational reasoning is the ability to consider, not a static state, but a dynamic process by which a new state or a continuum of states are generated." (p. 201) I conjecture that transformational reasoning is an essential reasoning ability that is needed to effectively solve related problems, since students must understand how particular quantities are changing in the given problem. Covariational reasoning as described by Carlson, Jacobs, Coe, Larsen, and Hsu (2002) is a particular type of transformational reasoning that focuses on a student's ability to coordinate change in one variable with change in another variable.

There is substantial research with evidence of students obtaining process-oriented strategies (Sfard, 1992; Martin, 1996, 2000; Vinner, 1997) and just attempting to manipulate symbols (Orton, 1983; White & Mitchelmore, 1996) to achieve an answer. Much of this would appear to stem from deficiencies in the conceptual understandings of the underpinnings of calculus such as the concepts of function (specifically composition), variable, and derivative.

Theoretical Perspective

There is evidence that students do not understand the concept of variable, and therefore learn processes for solving problems that involve pushing around symbols that have no meaning (Martin, 1996, 2000; White & Mitchelmore, 1996). It has also been suggested that that students do not understand the concept of rate of change, average or instantaneous, and thus do not understand the concept of derivative, much less the chain rule (Orton, 1983; Clark et al., 1997). To successfully solve a related rates problem, students should be proficient with their understanding of variable, functions (particularly composition), geometric properties, and implicit differentiation (specifically how it is related to the chain rule).

From the literature review, it appeared that some of the major obstacles students encounter in solving related rates problems are: inability to draw a picture that correctly represents the situation in the problem; not knowing what geometric relation is appropriate; not understanding implicit differentiation, the chain rule (Orton, 1983; Clark, et al, 1997); manipulating symbols that have no meaning (White & Mitchelmore, 1996; Sfard, 1992); having a process-focused view of mathematics. These obstacles formed a framework around which in-class and take-home activities were developed, intending to enhance the introduction and instruction of related rates.

Methods

Dubinsky's (1991) suggestion for an approach to fostering conceptual thinking in mathematics has the following four steps: observe students, analyze the data, design instructional materials, and repeat the process until stabilization occurs. Following this suggestion, students were given three activities: one that focused on solving functions for a designated variable (finding geometric formulas utilizing composition) and two specific to related rates. The related rates tasks were designed to develop a deeper sense of the problem solving process and to draw attention to changing versus constant quantities. After the unit on related rates was completed, student interviews were conducted. Students were asked to solve two or three related problems they had not seen before while being video taped.

The first two questions directly parallel ones from their homework and serve as a baseline to determine understanding the procedural aspects of the problem solving process. The questions the students were given are:

1. A plane flying horizontally at an altitude of 3 miles and a speed of 600 mi/hr passes directly over a radar station. When the plane is 5 miles away from the station, at what rate is the distance from the plane to the station increasing?
2. A spherical balloon is to be deflated so that its radius decreases at a constant rate of 15 cm/min. At what rate must the air be removed when the radius is 9 cm?
3. Coffee is poured at a uniform rate of 20 cm^3/sec into a cup whose inside is shaped like a truncated cone. If the upper and lower radii of the cup are 4 cm and 2 cm, respectively, and the height of the cup is 6 cm, how fast will the coffee level be rising when the coffee is halfway up the cup?

The video recordings were transcribed and analyzed using open and axial coding techniques.

Results

Three major student difficulties emerged from the data: algebraic and/or geometric deficiencies, student fixation on the procedural steps, and failure to recognize and consider general relationships. I theorize that the thread that binds these together is a lack of transformational/covariational reasoning applied to the problem. Students did not engage in mental activities to build a conceptual model of the important relations and did not actively engage in covariational reasoning at the beginning of the problem, as is evidenced by their construction of static diagrams which appeared to result in their to relying on procedural steps. Their reliance on implementing procedures without a rational foundation were also evident in what often seemed to me their random use of algebraic techniques and misguided geometric associations.

Students had particular problems recognizing when to use the similar triangle relationship; they did not understand the power of substitution and function composition; and they were not effective in determining what algebraic procedures to implement to arrive at the most appropriate defining relationship. Computational errors led to incorrect solutions; geometric misconceptions led to incorrect models. As is seen in the transcript excerpt below, Jen wrestled with whether air is the same thing as volume in the balloon problem.

Jen: Ok. So, I'm going to start by implicitly differentiating that cause I don't know what else to do (laughs). And then, we want to know, radius decreases, I know dr/dt, and I want to do when r = 9. So, then I'm solving, but then I'm solving for dv/dt, right?

INT: Why do you think you're solving for dv/dt?

Jen: At what rate must the air be removed? So, that would mean, that the volume, that the rate at which the volume is getting smaller? INT: Ok. Jen: Or, when the radius is 9 cm, yeah, because they gave us the rate at which the radius decreases. Hmm. (makes an unhappy face)

INT: What do you not like right now? Jen: That I don't know dv/dt. Like I don't know what I'm solving for. Actually, like if I solve for dv/dt, then I'm going to need to do something else, I think. Because I want to know the rate at which air needs to be removed. And I'll just know, like if I put in, the things I know, then I'm just going to get the rate at which the volume decreases, but how would I find the rate at which air needs to be taken out? Or are they the same thing and I'm just making this complicated?

INT: Do you think the air is the same thing as the volume?

Jen: Well, that's what the balloon is filled with. That's why I guess maybe it's the same thing. Must the air be removed…

INT: Tell me more about this conflict that you have.

Jen: Well, I want to know the rate at which air needs to be removed, but I don't know if that's the same as the volume, the rate at which the volume is decreasing. Or does, no. It has to be right, because I don't have anything else to…unless I'm totally off. I don't have anything else to work with. Like no other numbers they give. Like I thought maybe it would be dr/dt, but they give me dr/dt so I can't be solving for that. When the radius is 9 cm. I don't know, cause they want to know, either the rate at which it's decreasing or the rate at which air is being removed to keep the radius decreasing at a constant rate. I have to keep 15 and I have to keep 9, so I must be solving for dv/dt.

It appears Jen is thinking about how the problem can be solved using her internalized procedure, but there are cognitive conflicts in how she understands volume and her experience with balloons. Without a robust conceptual structure of the problem, followed by the active engagement of her mind in covariational reasoning, she was only able to apply a procedural approach to the problem.

Students appeared to focus on the three procedural steps that Martin outlined. They generally drew a diagram and labeled the constants, chose a formula and differentiated it, then plugged in values. This abbreviated procedure works well on standard problems, particularly ones that do not require an auxiliary problem to be solved as in the balloon problem. The students did not struggle greatly with the plane and balloon problems; this may be the result of having seen similar problems before.

One of the critical components of solving related rates problems is being able to diagram and visualize the situation. Students appeared to be proficient at drawing an appropriate diagram. However, difficulties arose when students began their labeling phase. Students labeled their diagrams with the constants given in the problem; however, no attention was given to the quantities that are changing. After drawing a diagram, students immediately looked for a geometric formula that would fit the situation, differentiate it, and plug in values. The focus was on the procedural steps. What became apparent, particularly in the cup problem, was that almost nowhere did students account for the general relationship between the radius and height of the cone. They did not appear to have a model that indicated these two quantities were continually changing in relation to each. This illustrates a lack of covariational reasoning.

When the students did not know an explicit formula off the top of their heads, they would ask the interviewer or want to consult their text; the students lacked confidence about their ability to solve these problems during the interviews. As is evidenced in the data from the cup interview problem, students frequently were frustrated when they could not find a formula for that particular shape. When the book did not provide an explicit formula, they frequently went to an alternate idea such as viewing the cup as a trapezoid that has been spun around. After an alternate idea didn't seem to pan out, students were discouraged and some were ready to give up. In almost every case, the interviewer needed to prompt the student to consider the whole cone before any real progress could be made.

When a problem required the student to think beyond the classic examples from class discussion and homework, the abbreviated procedure did not work so well. Troubles arose for students when a nonstandard question was posed as is supported by the interview excerpts below.

Betty: Ok, um, so then, how fast will the coffee level be rising when the coffee is halfway up the cup? Um, so that would make the height 3 cm. So, first thing we have to do is take the derivative. Um, we want to find the change in the height. So, um, cause we know the volume, the derivative of the volume. So, first we'll solve for the height cause that's what we're looking for.

Rather than attending to the dynamic nature of the problem (not engaging in transformational reasoning), she jumped right to differentiating her chosen formula. Betty is demonstrating a manipulation focus; she is looking for a formula that she can manipulate to find an answer. Later in the interview, she appeared to experience cognitive dissonance when her attempt to use the chain rule resulted in the realization that one could not have two variables in the formula.

Betty: Then I … don't need to solve for this separate derivative of r. 'Cause I think you still need to apply the chain rule, I think, don't you?

INT: How are you applying the chain rule?

Betty: To that one, right?

INT: Um hmm.

Betty: Well, we don't want, this is hard. We don't want two variables in our formula…

INT: No?

Betty: So we need to get rid of one of them. And I know I want the derivative of h, because how fast the height is changing. Then wait, I'm not, no I am cause how fast the level of coffee will be rising when the coffee is halfway…so I know the h is 3 here, but I want to be able to put that in.

Betty's lack of transformational reasoning causes her to struggle with the algebraic and geometric aspects of the problem.

Another student did not draw diagrams without prompting; he immediately searched for a formula to differentiate and plug values into. His procedure worked for the plane and balloon problems, but created a major obstacle when attempting to solve the cup problem.

One student, Harry, did solve the cup problem successfully. He tried numerous approaches before reaching his solution. First he tried to think of it as a whole cone with the bottom removed and then decided that was going to be hard, there had to be a "trick."

Harry: And then, um, ah…I know there's something to do with like, similar triangles in this we could say, but then we're going to lop off this little, this thing doesn't exist for the liquid's going in there all automatically, so I suppose…yeah. It would make sense to find the dimensions of this cone here, maybe with the full volume of it, and then I could subtract this part (the little part at the bottom), and then that would give me the volume of just the cup itself.

He next tries a trapezoid approach. Eventually, he decides that viewing the cup as a whole cone with an adjusted height for the coffee is the way to go.

Harry: Well, originally I thought I couldn't pour coffee into a regular cone, because that's going to change the answer, but it won't because it's not going to matter because the coffee is always coming in at a constant rate, and even if I poured it into this imaginary cone, and it kept going in there, that once it got to this level here (bottom of cup), and that's kinda where the problem starts, you know what I mean?

INT: Uh, huh.

Harry: And then it's going to increase, so I just need to change the number, so instead of finding the height is 3, halfway up there, I could just find it as, um, 9, on the whole cone.

While Harry does solve this problem successfully, he does not begin the process with covariational reasoning and must try numerous approaches before coming up with an appropriate solution.

Discussion

As Martin found, when students were required to solve an auxiliary problem, they struggled more. It is the solving of an auxiliary problem that requires the student to investigate and understand the general relationship between changing quantities. This suggests that the students are not using transformational/covariational reasoning in solving the problems; they do not have mental models that can be manipulated to play out the situation at hand, and thus merely manipulate known symbols without regard for their meaning. This lack of transformational reasoning appears to foster students' dependence on the procedural steps, which in turn highlights students' deficiencies in basic algebraic and geometric knowledge.

In the tasks that were developed, an attempt was made to help students become proficient in creating an appropriate relationship by focusing on the generalities of the diagram. The results thus far indicate that minimal progress was made in that direction. Students are still inclined to label their diagrams with constants, neglecting the quantities that are changing. A new activity utilizing Geometer's Sketchpad has been developed that highlights properties of similar figures and allows for a physical manipulation of objects. Supporting tasks will be developed to promote their building a mental structure of the situation; including prompts that promote their identification of the changing quantities and ability to coordinate the changes in these quantities in their mind. They will be piloted in the coming academic year. There are also plans to revise and study the related rates specific activities, as well as further develop tasks focused on function composition.

Conclusion

Related rates problems are difficult for calculus students. A series of tasks was designed to help students overcome the difficulties that arise. The difficulties appear to stem from the students' inability to build a conceptual model of the situation, to identify the relevant relationships and to appropriately coordinate changes in these objects in their mind; not applying transformational/covariational reasoning in solving the problems. This resulted in their heavy focus on applying the procedural steps and ignoring the relationships between quantities that are changing in the problem. This suggests that students' transformational reasoning skills and how they apply them, particularly in the diagramming phase, are critical to the successful completion of a related rates problem.

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A SERIES PROBLEM IN GEOMETRIA AND ARITHMETICA: THE REPRESENTATION SCHEMES OF PRESERVICE SECONDARY MATHEMATICS TEACHERS

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The teacher stands before her students and removes four cubes, with side lengths from 1 cm to 4 cm from her bag. She then returns the cubes to the bag, shakes it, and slowly withdraws from the bag … the four cubes? No, she withdraws not cubes but a single square block with side length 10 cm. It is difficult for the students not to be amazed by this extraordinary feat of conversion of 4 cubes into a square. Her students ask, "How is this possible?" Before answering their question, we describe the methods for our inquiry and how the literature on mathematical representation and history has influenced our work. Next, we present an analysis of one of the series problems represented arithmetically and geometrically, and our students pre-service math teachers) report the rich perspective they developed, of their personal representational schemes, from their explorations of these types of problems.

Purpose of the Study

In this paper we report on the development of geometric understanding by students for series problems and the implications for mathematics learning and teaching. Our work is framed by the view that posing and analyzing rich tasks for students provide windows into their thinking with ramifications for curriculum and instruction. As a result of observations of what students say and write, and how they represent mathematical situations, researchers make decisions about appropriate ongoing investigations to clarify or validate early assertions.

Studies (e.g., Vinner, 1989; Tall, 1991) have consistently shown that students' understanding is typically analytic and not visual. Two possible reasons for this are when the analytic mode, instead of the graphic mode, is pervasively used in instruction, or when students or teachers hold the belief that mathematics is the skillful manipulation of symbols and numbers. It is clear from the literature (e.g., Lesh, Post, & Behr, 1987; Janvier, 1987; NCTM, 2000) that having multiple ways – for example, graphic and analytic – to represent mathematical concepts is beneficial.

Our contention is not that one student's representational scheme is superior to another, only that students often construct vastly different personal and idiosyncratic representations which lead to different understandings of a concept. Because student-generated representations provide useful windows into students' thinking, it is productive for teachers to value these personal representations.

Methodology and Data Sources

Twenty-eight students (pre-service high school mathematics teachers) from one senior-level mathematical problem solving class participated in the study. Analyzing their responses to Presmeg's (1986) theoretical framework, we determined that some of the students were non-visual learners and that others tended to process information visually. From the class, we chose two students – one visual and one non-visual – for audio-taped interview sessions to develop case studies. Students in the class responded to written and oral tasks and questions, and the case studies consisted of students' responses to questions about the classroom activities. In general, the aims of our study were to arrive at a comprehensive understanding of the role of students' personal and idiosyncratic representations in their learning and to develop general theoretical statements about their learning processes.

We explored students' thinking on tasks designed to probe their different ways of understanding and representing series problems. Using multiple sources of qualitative data – audiotapes of interviews with students, transcripts of those tapes, researchers' field notes, worksheets of case study students, and two researchers' journals – case study analyses were undertaken to identify patterns and changes in students' thinking with respect to their understanding. In particular, we report here how their work on these series problems presented geometrically influenced the ways they thought about teaching. Analyses of taped sessions included coding of events, triangulation of qualitative data, and identification of distinct strands.

Mathematical Representation

Students in sixth-century B.C. Greece concentrated on four very separate *mathemata*, or four subjects of study: arithmetica (arithmetic), harmonia (music), geometria (geometry), and astrologia (astronomy). "This fourfold division of knowledge became known in the Middle Ages as the 'quadrivium'" (Burton, 1997, p. 88). To these early Greeks, arithmetic and geometry were as separate as music and astronomy; mathematicians were to realize, of course, that arithmetic and geometry are not separate, and that some intriguing mathematics lies at their intersection. This report attempts to explore the beauty and richness of viewing one problem from arithmetic and geometric perspectives.

There is a belief among mathematics educators (e.g., Janvier 1987; Lesh, Post, & Behr, 1987) that students benefit from being able to understand a variety of representations for mathematical concepts and to select and apply a representation that is suited to a particular mathematical task. The National Council of Teachers of Mathematics (NCTM) reinforces this belief: "Different representations support different ways of thinking about and manipulating mathematical objects. An object can be better understood when viewed through multiple lenses" (2000, p. 360). Students develop mathematical power as they learn to operate on mathematical objects and to translate from one mathematical representation to another.

Recently, Aspinwall and Shaw (2002) reported their work with two students with contrasting modes of mathematical thinking – Al, whose mode was primarily visual, and Betty, whose mode was almost entirely symbolic. Their contention was that students often construct vastly different personal and idiosyncratic representations, which lead to different understandings of concepts. Given problems presented graphically, Betty generally found it nearly impossible to think about the problem in graphical terms; thus, she translated from the graphic representations to symbolic representations, or equations, in order to make sense of the problems. Once she completed analytic operations on the symbols, she translated the problem back to the graphic representations required for the tasks. Al, however, operated directly on the graphic representations without having first to translate to symbolic representations. Betty and Al showcased two very different ways of solving problems, but the study suggested that if students could move freely between the visual (geometria) and the symbolic (arithmetica), their mathematical understanding would be much richer and their problem-solving abilities more robust.

From Cubes to a Square Block: An Arithmetic Perspective

Let us return now to the question posed by the students: how is it possible that four cubes, with side lengths ranging in size from 1 cm to 4 cm, can be magically transformed into a single square block with side length 10 cm. From an arithmetic perspective, this problem can be represented by the following equation, $1^3 + 2^3 + 3^3 + 4^3 = 10^2$. One can further examine this relationship by determining what would happen if only one cube, with side length 1 cm (1 cm^3), is placed in the magic bag? What if two cubes, with side lengths 1 cm and 2 cm (1 cm^3 and 8 cm^3, respectively), are placed in the bag? What happens with three cubes, with side lengths 1 cm, 2 cm, and 3 cm (1 cm^3, 8 cm^3, and 27 cm^3, respectively)? What if four cubes, with side lengths 1 cm, 2 cm, 3 cm, and 4 cm (1 cm^3, 8 cm^3, 27 cm^3, and 64 cm^3, respectively), are placed in the magic bag? When we investigate these questions, we notice something intriguing.

- 1 cube $1^3 = 1 = 1^2$, which is true.
- 2 cubes $1^3 + 2^3 = 9 = (3)^2$, which is true.
- 3 cubes $1^3 + 2^3 + 3^3 = 36 = (6)^2$, which is true.
- 4 cubes $1^3 + 2^3 + 3^3 + 4^3 = 100 = (10)^2$, which is true.

Then the students ask, "Does placing consecutively larger cubes into the magic bag always produce a square block with this intriguing property; that is, does the following equality always hold: $1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$?"

An inductive approach is sufficient to show that this relationship is true for any natural number, n. To use this approach, one must first show that it works for the 1st natural number. The second part is the induction step; that is, one must show that if the expression is true for an arbitrary natural number, say k, then the equation must be true for the next consecutive natural number, k+1. If this induction step can be proved, we clearly have a domino effect, that is, if the equation is satisfied for n=1, then the equation is satisfied for n=2, and if the equation is satisfied for n=2, then the equation is satisfied for n=3, and so on throughout all the natural numbers.

Geometria with Arithmetica

Let us now explore this generalized problem from a combined arithmetic and geometric perspective, which we have termed "geo-arithmetic." The problem is shown geometrically in Figure 1. First we consider the square, in Figure 2, that is size $(1 + 2 + 3 + \cdots + n) \times (1 + 2 + 3 + \cdots + n)$. We can divide this large square into smaller squares and rectangles, and calculate the areas of these squares and rectangles based on their dimensions – lengths and widths. But we will add the areas separately based on their placement in groups that we will designate as the Diagonal, Bricked, Vertical-Line, Dotted-Line, and Horizontal-Line regions. (See Figure 2) Finally, we will demonstrate that the sum of each of these regions is a cube so that the area of the square is the sum of the cubes.

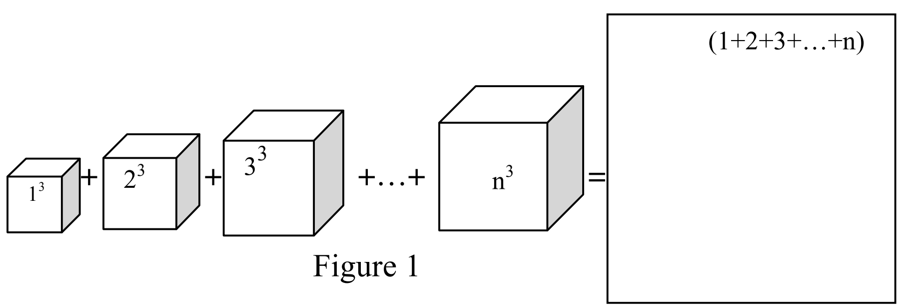

Figure 1

n

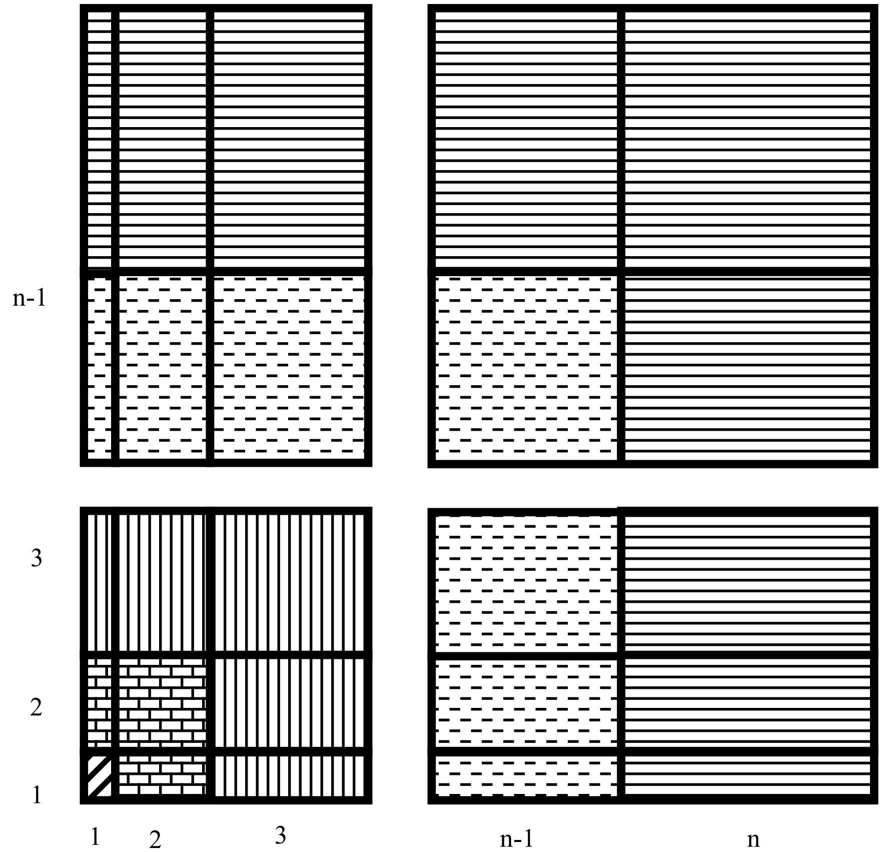

Figure 2

Sum of the Diagonal Region

$1 = 1^3$

Sum of the Bricked Regions

$1 \times 2 + 2 \times 2 + 1 \times 2 =$

$2 \times (1 + 2) + 1 \times 2 =$

$2 \times \left(\frac{2 \times 3}{2} \right) + 1 \times 2 =$

$2 \times 2^2 =$

2^3

Sum of the Vertical-Line Regions

$1 \times 3 + 2 \times 3 + 3 \times 3 + 3 \times 1 + 3 \times 2 =$

$3 \times (1 + 2 + 3) + 3 \times (1 + 2) =$

$3 \times \left(\frac{3 \times 4}{2} \right) + 3 \left(\frac{2 \times 3}{2} \right) =$

$3 \times \left(\frac{3 \times 4 + 2 \times 3}{2} \right) =$

$\frac{3 \times 3 \times (4 + 2)}{2} =$

$3^2 \times 3 =$

3^3

Sum of the Dotted-Line Regions

$$1 \times (n-1) + 2 \times (n-1) + 3 \times (n-1) + \cdots + (n-1)(n-1) +$$
$$1 \times (n-1) + 2 \times (n-1) + 3 \times (n-1) + \cdots + (n-2)(n-1) =$$
$$(n-1)(1+2+3+\cdots+n-1) + (n-1)(1+2+3+\cdots+n-2) =$$
$$\frac{(n-1)(n-1)(n)}{2} + \frac{(n-1)(n-2)(n-1)}{2} =$$
$$\frac{(n-1)^2(n+n-2)}{2} =$$
$$\frac{(n-1)^2(2n-2)}{2} =$$
$$\frac{(n-1)^2(2)(n-1)}{2} =$$
$$(n-1)^3$$

Sum of the Horizontal-Line Regions

$$1 \times n + 2 \times n + 3 \times n + \cdots + n \times (n-1) + n \times n +$$
$$1 \times n + 2 \times n + 3 \times n + \cdots + n \times (n-1) =$$
$$n(1+2+3+\cdots+n) + n(1+2+3+\cdots+n-1) =$$
$$n\left[\frac{n(n+1)}{2}\right] + n\left[\frac{(n-1)(n)}{2}\right] =$$
$$\frac{n^2(n+1)}{2} + \frac{n^2(n-1)}{2} =$$
$$\frac{n^2(n+1+n-1)}{2} =$$
$$\frac{n^2(2n)}{2} =$$
$$n^3$$

Now, we have as the sum of the areas of the subdivided square:

+ Sum of the Diagonal Region: 1^3
+ Sum of the Bricked Regions: 2^3
+ Sum for the Vertical-Line Regions: 3^3
+ …
+ Sum for the Dotted-Line Regions: $(n-1)^3$
+ Sum for the Horizontal-Line Regions: n^3

= Area of the square: $1^3 + 2^3 + 3^3 + \cdots + (n-1)^3 + n^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3$

We have subdivided the square, used arithmetic, and the area of the square is the sum of cubes.

Students' Explorations

Our work with students is framed by the view that posing and analyzing rich tasks for students provide windows into their thinking with ramifications for curriculum and instruction. As a result of observations of what students say and write, and how they represent mathematical situations, researchers make decisions about appropriate ongoing investigations to clarify or validate early assertions. In this case, we sought to probe the perspectives of pre-service high school mathematics teachers for this combination of the arithmetica and geometria.

The students in our senior-level university mathematical problem-solving course were visual and non-visual (as determined by the instructors' analyses of students' portfolios) using Presmeg's (1986) theoretical framework. During the 2003 Fall semester, all 25 aspiring teachers in the class had been studying a variety of these *geo-arithmetic* problems (as we termed them).

Our strategy was to present arithmetic representations, which in this case were series problems, or a picture of a geometric object that represented a series problem. After several weeks of presenting and assisting students with tasks that required a synthesis of geometry and arithmetic, we sought to explore their thinking. The final project was the problem we have presented here. The students did their work in small groups and revealed their new way of thinking as every group created representations for the geo-arithmetic. At the end of the semester, we interviewed two students. We spoke with Ryan, a student who processed mathematical information non-visually, and with Emily, who processed visually.

Ryan said he initially was frustrated by our asking him to solve the series problems geometrically. He said he had always thought "in equations." During group activity, he reported he was able to see how some students process information geometrically as he worked through the problems. What was striking was that as a result of the activities, he felt he would be a better teacher in relating to visual and non-visual learners. "They taught me how to think about a problem so that if you are trying to reach someone who does not think just in numbers, [slight pause], well, you can help the student to see the problem visually."

When asked how he might now approach these visual problems, he said he would "start with the equations, then go to the geometry and try to work my way to where I was with the equations." He justified this by saying, "I guess that's just the way [symbolically] I have always been exposed to learning. But these problems forced me to approach them visually." The visual representations for the series problems "opened my eyes to a new way of seeing things that I had never been exposed to before. I consider myself to be not just a better problem solver, but a better teacher seeing how other students are going to see things." Furthermore, he explained, "Before, I was only thinking of the equations, and I thought everyone else was too. My idea was that everyone was going to learn by my [symbolic] teaching. I wasn't open to visual teaching. Now I'm thinking differently, out of my comfort zone."

Emily, the visual thinker, did not experience Ryan's initial frustration caused by the problems presented geometrically. She stated, "I am sometimes not confident in my algebra skills. I know what I am doing but I am afraid of mistakes in my thinking. If I can do it visually, I know I am on the right track." Ryan said his first approach was to try to write an equation; but Emily's approach was much different: "My last resort is to write an equation. I look at it every way that is creative or out of the norm. It is easier for me to conceptualize it that way. I was struggling with the problems algebraically." When we asked her whether she thought these series problems were algebraic or geometric in nature, she said, "It was a blend for me. You needed to know the algebra behind it, but you had to have that geometry, spatial sense, in order to see the problem." When we asked her how she thought about the problem presented above, she responded, "With the series problems, I had to picture a physical cube, with them lined up next to each other, and figure it out from there."

Ryan, the non-visual thinker above, said that being confronted with problems presented visually had altered the way he thought about teaching. Emily, too, had reflected on her future teaching practice. We asked her, "Will this focus on pictures in the class change any thing about the way you plan to teach?" She responded:

> Before these problems, I would have had to just to go by the book. Teach by breaking the equations down into smaller parts algebraically. After these problems, I want to try to try to incorporate this (visual aspects) into my teaching, into as many lessons as possible. Because I now know I am that kind of thinker (visual), I know there are others like me. Based on this I want to try to accommodate all the different kinds of thinking. I will have

to teach it purely algebraically for those who don't think visually. I want to try to incorporate as much visual as I can, and that will help the algebra people to see it differently too. Maybe I can create a future engineer. And the people who are visual need to know the numbers, how the equations work and not have to see it visually.

Conclusion

Many of our students tended to think symbolically and not visually, and Ryan was typical of these learners. Emily was representative of those students who tended to think visually. In both cases, they felt the ways they would teach had been changed. We believe students develop mathematical power by learning to recognize an idea embedded in a variety of different representational systems and to translate the idea from one mode of representation to another. A positive result of multiple instructional representations of concepts is that students who are prospective teachers learn to construct and to present representational schemes with which they might not be comfortable.

Again, our contention is not that one student's representational scheme is superior to another, only that students often construct vastly different personal and idiosyncratic representations which lead to different understandings of a concept. Although Ryan and Emily valued two different types of representations, we believe students benefit from an ability to recognize an idea embedded in a variety of different representational schemes and to translate the idea from one mode of representation to another. A positive result of multiple instructional representations of concepts is that students learning to present ideas will become fluent with a variety of diagrams, graphs, symbols, and equations. And when students create and view mathematical objects from different perspectives, they develop power in mathematics. Otherwise, students are merely confronted with and must interpret a teacher's representational preference, and then the task for students becomes one of memorizing the presentation rather than learning to select or create representational schemes suitable to the problems they are trying to solve. Students allowed to present these schemes reveal aspects of their understanding that might not otherwise emerge. Because student-generated representations provide useful windows into students' thinking, it is essential for teachers to value these personal representations. The challenge then for us as teachers is to create learning environments that require students to become fluent with a variety of representations.

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AN EXPLORATION OF "FAIRNESS": GRADE 5/6 STUDENTS' THEORIZING ABOUT REPRESENTATIONS OF CENTRAL TENDENCY IN STUDENT-GENERATED NUMERIC DATA SETS

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Introduction

Mathematics education in the elementary grades has undergone a substantial shift in theoretical perspective, from a focus on teacher directed procedural instruction to an emphasis on student led investigative problem solving (NCTM, 2000). While "problem solving" has traditionally referred to the rote application of algorithms in response to word problems at the end of a textbook chapter, this notion has been reconceptualized significantly. It now encompasses the range of complexities inherent in the process of generating increasingly sophisticated mathematical representations or models of authentic problems that occur in the real world and for which multiple solutions are possible.

Theoretical Framework

Two independent but complementary pedagogical approaches that have emerged from this shift in perspective are: mathematical modelling (English & Doerr, 2003; Lesh & Doerr, 2003; Woodruff & Nason, 2000) and knowledge building (Scardemalia & Bereiter, in press). Mathematical modelling as a pedagogical tool offers students an innovative approach to accessing challenging mathematical concepts through meaningful authentic problem solving tasks. Mathematical models have been described as conceptual systems of relationships and operations that can be represented by such means as equations, diagrams or computer programmes (Lesh & Doerr, 2003). Further, mathematical modelling has tremendous potential to involve students in the theorizing, critiquing and higher order thinking that characterize student-led collaborative knowledge building (McNab, Nason, Moss, & Woodruff, in press; Scardemalia & Bereiter, in press). Mathematical modelling has been effectively pioneered with older students beyond the elementary grades, but its potential has not yet been fully explored with elementary students; knowledge building has been successfully incorporated into the science and language arts learning of elementary students, but has not yet been extended comfortably to mathematics. These two complementary pedagogical approaches, both representing a revisioning of elementary mathematics education theory and practice, informed the design of this inquiry.

Objectives or Purpose

This study implemented an experimental teaching intervention to investigate the potential of a knowledge building approach to mathematics learning in a Grade 5/6 classroom, supported by Knowledge Forum computer software, through collaborative mathematical modelling using authentic tasks that were relevant and meaningful to the students in this class. The study focused specifically on the Ontario Grade 5/6 mathematics curriculum topics of distribution and central tendency (mean, median and mode) in managing numerical data sets, which have been identified as persistently problematic for many students at this grade level and beyond (McGatha, Cobb, McClain, 2002).

Confusion existed amongst students around differentiating (defining and applying) the three central tendency terms—mean, median and mode—which are routinely taught together, in a procedural rather than a conceptual way offering students no means of constructing their own understanding. The deeper need for conceptual understanding implied by this difficulty suggested that these topics might be more effectively addressed through a knowledge building approach. Further, because these particular topics have clear links to real life situations, with the potential for student generated data sets and mathematical models within relevant authentic problems, they seemed to offer a rich context for student engagement, not only in the questioning, theorizing, exploration and discovery of knowledge building, but also in the generation, modification and rationalization of mathematical modelling. This study, then, set out to explore the potential of knowledge building and mathematical modelling for elementary students' more effective learning of mean, median and mode.

A key concern of this study was how transparent the mathematics curriculum content should be, given that one important aim of this research was to support the process of knowledge building where learning is moved ahead by students' own curiosity, questions, conjectures, theorizing and research, rather than by adhering to an agenda set by an instructor. The mathematical concepts were therefore presented implicitly through an exploration of the issue of "fairness" and how most fairly to represent consensus mathematically, using data sets generated by the students on topics of interest to them—specifically, in this classroom, establishing a rationale for how to generate an overall ranking of small group opinions within the class on the relative importance of competing issues in the current election for mayor of the city.

Methods or Modes of Inquiry

Participants included 22 students (11 in Grade 5; 11 in Grade 6), ages 9 through 11 years, in one Grade 5/6 classroom at the Institute of Child Study in Toronto, Ontario. All students in this study had been provided with laptop computers for use in their classrooms as part of a school wide programme, allowing wireless access to Knowledge Forum software and the database created for this project. All students were already familiar with Knowledge Forum; however, this was the classroom teacher's first use of laptops and of this software in her classroom. The teacher and a teacher intern were present for all parts of the study. The lessons were developed and taught by a doctoral student, with the support and assistance of a professor of elementary mathematics education and a research assistant.

The research design included a pre-test, teaching intervention and post-test. The pre-test was comprised of a total of 13 items, grouped into three parts: the first part (4 items) was made up of word problems requiring interpretation and understanding of mean (average) to generate values with defined relationships, without an explicit request for calculations; the second part (3 items) asked for the participant's own description of what was meant by the terms mean, median and mode; the third part (6 items) required application of these terms to unordered numerical data sets. (Prior to administration of the pre-test, this class (both Grade 5 and Grade 6) had completed work on the topic of *average* in the context of a review of long division, although this had not explicitly linked the procedural arithmetic calculations of average to the idea of the mean of a data set.)

The teaching intervention was comprised of a series of sixteen lessons, delivered twice weekly over a period of two and a half months. All lessons took place during regularly scheduled math periods. Whole class and small group discussions were video taped and written transcriptions made. Observations were recorded in field notes taken by the teacher intern and research assistant. All classroom artefacts, including entries made by students, researchers and teacher on the Knowledge Forum project database and Excel spreadsheets, as well as all lists, notes, charts and graphs produced by the students during the lessons, were retained.

The pre-test was re-administered as a post-test immediately following completion of the teaching intervention.

Results and discussion

A preliminary analysis of data indicates that the means of total scores improved significantly from pre- to post-test. Pre- and post-test scores for each of the three parts reveal that students' understanding of average (Part 1), taught previously by the classroom teacher, remained not significantly improved by the teaching intervention. (This is particularly interesting in light of investigations by Kirshner et al (2000) which suggest that initial introduction of algorithms in a procedural way actually interferes with students' subsequent ability to construct their own conceptual understanding.) However, participants' grasp of implicitly presented material (Part 2) and their ability to transfer this knowledge to new applications (Part 3) were significantly better from pre- to post-test, with all students showing significant improvement.

All but two of the students' post-test responses to Part 2 (meaning of terms, which had been explored but not explicitly taught) produced a range of individual descriptions, all accurate, but all differently expressed. One conjecture might be that these terms had acquired a conceptual meaning which the students had internalized.

While both grades showed significant overall improvement, the Grade 6s improved more than the Grade 5s.

Girls' degree of improvement was greater than that of the boys. However, the boys remained higher scoring overall on both pre- and post-tests, with the difference decreasing from pre- to post-testing.

Transcriptions and field notes indicate that the laptop computers generally, and Knowledge Forum more specifically, while used somewhat, proved not to be the main medium for sharing ideas amongst students in the class. Classroom discussions, on the other hand, were animated and complex and proved to be the primary forum for collaborative theorizing. Knowledge building took place vigorously in the classroom, and only to a lesser extent within the project database. This may have been partly because the classroom teacher was not experienced or comfortable in the use of Knowledge Forum. A second phase of this study is currently implementing the same teaching intervention in a second Grade 5/6 classroom at the same school, with a teacher who is highly experienced in the use of laptop computers and Knowledge Forum software, in an attempt to illuminate the role of computer support for collaborative knowledge building in mathematics learning.

Both phases of this larger project were designed to investigate the interplay between overarching concepts and specific mathematics applications, conducted within a framework of knowledge building and mathematical modelling. Further research is planned that will build on this complete study, to compare students' learning of central tendency in two different Grade 5/6 classrooms: in one, the math content will be transparently conveyed in an explicit traditional way; in the other, the math content will be embedded within a knowledge building context. Additionally, a subsequent study will include a second post-testing of students at the end of the school year to provide a comparison of far transfer of knowledge between the two approaches.

Further, the exploration of effective metaphors that would support students' appropriate developmental understanding of central tendency offers a rich area for investigation that would influence the direction of further research on this topic (Kirshner, 2002; Konald & Polatsek, 2002).

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STUDENTS' PROCESSES OF RECONSTRUCTING MATHEMATICAL RELATIONSHIPS THROUGH THE USE OF DYNAMIC SOFTWARE

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To what extent does the systematic use of technology favour students´ development of problem solving competences? What type of reasoning do students develop as a result of using a particular tool? This study documents features of mathematical practice that students display when they use a dynamic software in their problem solving experiences. In particular, the use of the software involves the construction of simple geometric configurations that become a platform to formulate questions that lead them to the construction or recognition of mathematical relationships. In this process, students can build their own repertoire of mathematical results and also utilize their previous knowledge to support, justify, or explain their conjectures. Here, it becomes important for students to develop methods and strategies to observe relationships, express them using specific notation, and provide arguments to demonstrate their results.

It is well recognized that students need to develop distinct strategies to identify and examine relevant information to deal with problems or situations that involve the use of mathematical resources. In this perspective, it is common that they pose questions, explore particular conjectures, use distinct representations, and develop ways to communicate their ideas or results. Here, the presence of technological tools in mathematical classrooms tends to influence not only the content and organization of curriculum to study, but also ways in which students will approach and learn it (NCTM, 2000). It is also recognized that there are multiples ways in which students can employ those technological tools and, as a consequence, there is a need to investigate what aspects of mathematical practice are actually enhanced in students' learning as result of using particular tools. This study aims at investigating ways of reasoning exhibited by high school students while using dynamic software to construct and examine a set of geometric configurations. In this context, it becomes important to characterize types of questions, conjectures, explanations, and forms of communication that students develop during their problem solving experiences.

A Problem Solving Scenario: Elements of a Conceptual Framework, Methods and Procedures

Mathematical problem solving has become a central activity in students´ learning of the discipline (NCTM, 2000). However, it is important to recognize that there are different scenarios in which students can be engaged in problem solving practices. Thus a problem solving approach may be based on creating a mathematics microcosm in the classroom where students openly discuss a set of well-selected non-routine problems (Schoenfeld, 1998). What happens when students are asked to participate in the process of formulating questions? What is the role of the use of technology in achieving this goal? To what extent do students' methods for solving problems get enhanced when students utilize systematically technological tools? A fundamental principle to frame students´ understanding of mathematical ideas is that they need to conceive of their learning as an opportunity to pose questions, dilemmas, or problems to be explored and solved. That is, students not only generate questions as a means to understand situations or problems but also some of these questions eventually become problems to be solved.

> Once you have learned how to ask questions –relevant and appropriate and substantial questions- you have learned how to learn and no one can keep you from learning whatever you want or need to know (Postman & Weingartner, 1969, p. 23).

Thus, the process of understanding particular themes, contents or situations involves the formulation and exploration of substantial questions. In this context, the use of dynamic software seems to offer students a powerful tool not only to identify potential relationships but also to explore and visualize properties of those relationships.

Eighteen high school students participated in a problem-solving course during one semester, meeting four hours a week. An important goal was to ask the participants to use dynamic software to work on a series of activities that involve:

(i) Routine problems that appear in textbooks. The idea here was to discuss the extent to which the use of the software helps transform the original nature of the task. That is, with the help of the software students were encouraged to explore connections or extensions of the initial problem.

(ii) The use of the software to construct simple configurations that were used as platform to identify and explore mathematical relationships.

Each participant had access to a computer but they were encouraged to work on pairs or small groups of three. We also suggest a particular pedagogic approach to encourage students to learn through an inquiry process. This instructional approach has emerged from systematic implementation of tasks in which it becomes relevant for students to examine mathematical themes through questions. Thus, the development of the sessions consistently showed the following structure:

1. The instructor introduces the task to the students and asks them to work on the task in groups of three students for about 20 minutes. The role of the instructor here is to monitor students' work and help them clarify (via questions) the statement of the task. Each small group hands in a written report showing the students' approach to the task.
2. The instructor asks some small groups to present their work to the whole class. During each small group presentation, the rest of the group, including the instructor, asks questions to explain what it may not be clear or need some elaboration from the small groups´ presentation.
3. The instructor identifies strengths and limitations associated to each small group's presentation and discusses within the whole class mathematical ideas, strategies, concepts and distinct representations that are relevant in students' solution to the task. In addition, the instructor may introduce a new concept or analyse extensions or possible connection of the original statement of the task or problem.
4. Students are asked to revise individually the initial task. Here each student has the opportunity to incorporate new ideas, concepts, or strategies that he/she has judged to be relevant during the development of the session.

This paper focuses on analyzing what occurred during two- two hour sessions in which one of a small group presented a simple construction that eventually led to the recognition of basic mathematical results. It is important to mention that even when a small group proposed the task and initiated the discussion, more mathematical results came out from the participation of the whole class.

Introducing the Task: The Importance of Posing Questions and Results

A small group commenced the session by drawing (with the use of dynamic software) two points on a Cartesian plane. They then asked the class "what can we do with two points? And the presenter responded, “you can draw, for example, segment AB, line AB and measure the length of segment AB” (figure 1). You can also observe that the length of the segment varies when point A or B is moved. At this stage, the presenter added new elements to this simple initial configuration and began to identify and explore particular relationships among the components of the emerging construction.

Figure1. Segment AB, line AB

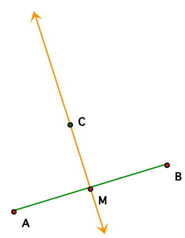

Figure 2. Perpendicular bisector of segment AB and point M

Specifically the elements added to the initial segment included midpoint M of segment AB and a perpendicular line *n* to segment AB passing by M (the perpendicular bisector of segment AB) and locate point C on that bisector (figure 2). It was observed that point C could be moved along the perpendicular bisector (figure 3).

One of the presenters mentioned that when point C is moved along the bisector, the triangle ACB is always isosceles. The immediate reaction from the class was “why”. Here the whole class began to examine and justify this statement and eventually accepted it, since it was shown that triangles AMC and BMC are congruent. Thus, the first result associate with this construction was:

> *Given a segment AB and its perpendicular bisector n, then the triangle ABC (C any point on n) will always be an isosceles triangle*

This result emerged from observing that the lengths of sides AC and CB is always the same for any position of point C. Here students were encouraged to present an argument based on congruence of triangles. That is, they eventually showed that triangles AMC and BMC are congruent (SAS postulate).

During the presentation it was also observed that when point C is located further from M the length of side AC = CB increases (figure 3). Here, another question was posed by one of the presenter: Where point C must be located to have triangle ACB equilateral? Students during this exploration assigned measures to different components of the figure and paid attention to what happened to these quantities when point C took different positions along the bisector.

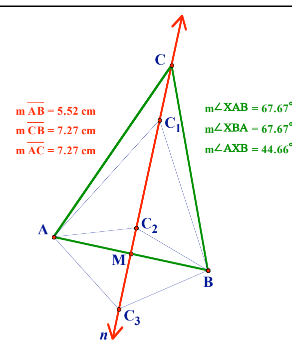

Figure 3. Three isosceles triangles

What properties do equilateral triangles have? Students drew a circle with center B and radius segment AB and locate point C' as the intersection of the circle with the perpendicular bisector. The class agreed that triangles AC'B and AC"B were equilateral (figure 4).

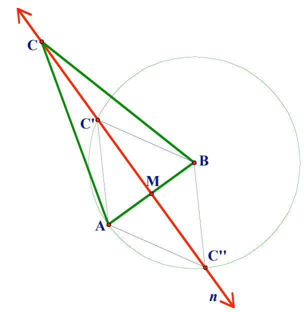

Figure 4. Constructing an equilateral triangle

Here a second result was identified:

The triangle formed by points A, B of segment AB, and a point C on its perpendicular bisector located at the intersection of that bisector and a circle with center on either point A or B and radius AB is equilateral.

Similarly, students also observed that when point C is moved along the bisector then the measure of angle ACB varies depending on how far point C is from point M. Here another question was posed: Where point C must be located to transform triangle ACB into a right triangle?

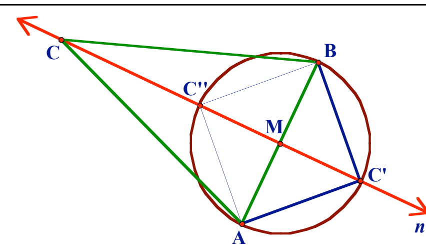

Figure 5. Constructing right triangle ACB

How to draw a right angle with vertex C? Can we move vertex C in such a way that the two sides of the angle pass by A and B respectively? If we drew a right angle ACB, then what properties this angle holds? These are examples of the type of questions that the class discussed to eventually agree to draw a circle with center point M and radius MA and locate C as the intersection point between the circle and the perpendicular bisector. Here, they stated that triangle ABC is a right triangle since side AB is the diameter of the circle (figure 5). Thus, another result was:

Triangle ABC is right triangle when point C is located at the intersection of the perpendicular bisector of segment AB and the circle with center the midpoint of AB and radius half of that segment

It is interesting to mention that the argument the students used to support this result was based on using a result that they had previously studied: If one side of a inscribed triangle is the diameter of the circle, then the triangle is a right triangle. At this point, students from the small group mentioned that they have completed their presentation; however, this task was again addressed during the following session. The idea was to add more elements to this configuration and explore the behavior of particular points as a result of moving, in this case, point C along line n.

Thus, a small group commenced drawing a perpendicular from C to the perpendicular bisector of segment AB (n) and drew the perpendicular bisector of segment CB. These two lines get intersected at P (figure 6). Again, the question to examine was: What is the locus of point P when point C is moved along the perpendicular bisector of AB?

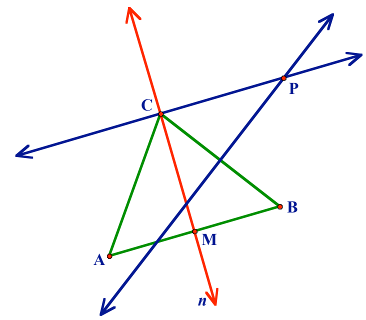

Figure 6. Perpendicular to n at C and perpendicular bisector of CB get intersected at P.

The software became a powerful tool to determine the path left by point P when point C is moved along perpendicular bisector n. Again, students rushed to mention that the locus of point P was a parabola. Here, they were asked to explain why that locus was really a parabola. What is a parabola? What properties does it hold? These types of question led students to identify point B and line n as the focus and directrix of the parabola and explained that distances PB and PC are always the same, since P is located at the perpendicular bisector of BC (figure 7).

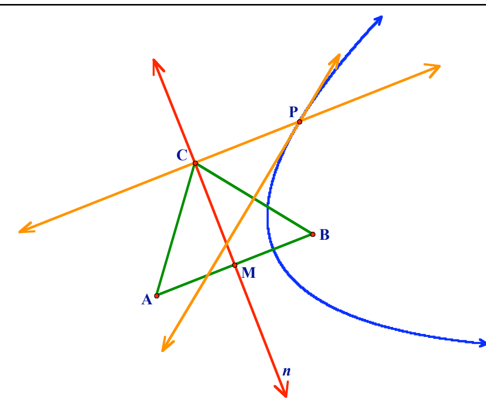

Figure 7. Perpendicular to n at C and perpendicular bisector of AC get intersected at P.

At this stage, it was evident that students recognized that the software was a powerful tool to identify and explore the behavior of particular relationships. In particular, they realized that the general properties of the parabola described before could also be verified by quantifying the distance from any point P on the locus to point B (focus) and line n (directrix) was always the same (figure 8).

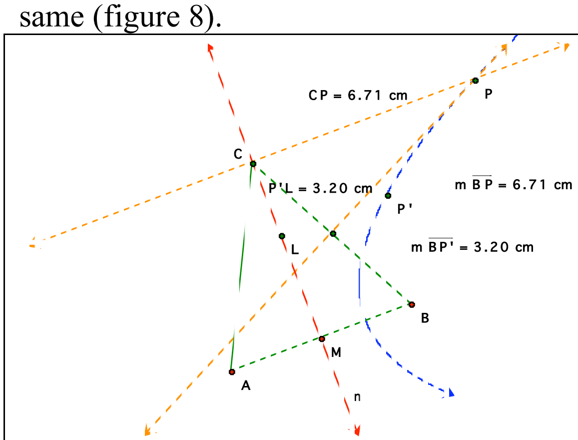

Figure 8. Verifying properties of the parabola

Students with the help of the software calculated the distance from point P on the parabola to line n (directrix) and from P to point B (focus) and observe that when point P was moved along the curve, both distance were equal. Figure 8 shows two locations of point P (P and P').

When any of the small groups identified a particular curve as a result of exploring the behavior of dynamic representations that included new elements added to the initial configuration, they were encouraged to present their construction to the whole class. Thus, the class not only examined carefully the construction, but also participated in the process of justifying properties that first appear only visually. That is, students eventually recognized that it was important to provide arguments to support their conjectures. In addition, it was observed that students developed a certain kind of sense to add other elements to the configuration in which there was a possibility to generate interesting relationships. For example, in general, they noticed that point C located on the perpendicular bisector was a key point to search for potential relationships. The fact that there was a triangle in the configuration, for example, led them to think of adding to the configuration, elements that include heights, perpendicular bisectors, and angle bisectors in that triangle and observe behavior of particular intersection points of this lines.

Students worked in small groups and there were different suggestions on what elements to add in order to identify particular relationships. A small group drew N as the midpoint of segment BC, a line MN and the perpendicular from point A to side CB (height) and located R as the intersection point between that height and line MN (figure 9). What is the locus of point R when point C is moved along the perpendicular bisector n?

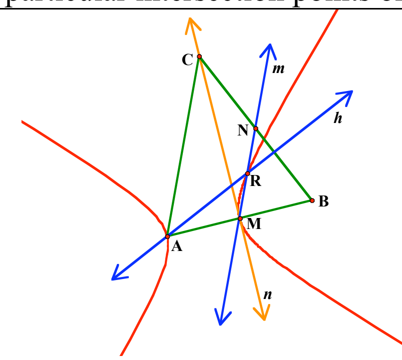

Figure 9. What is the locus of point R when point C is moved along the bisector?

Students observed that the locus seemed to be a hyperbola. Here, again the class began to examine whether the generated locus held basic properties attached to this figure; for example, they tried to identify its foci, vertices, center, etc. Here, another result emerged:

> *In an isosceles triangle ABC (C any point on bisector n), line MN (M is the midpoint of side BC) and the perpendicular line to segment BC that passes by vertex A get intersected at point R. The path (locus) left by point R when point C moves along the bisector is a hyperbola.*

Looking Back

There are aspects of mathematics practice that appeared as important during the development of the activity:

1. There is no one established or well-defined problem to be solved initially by the students. Instead, questions or problems emerge through the process of constructing a particular configuration that involves points, segments, bisectors, triangles, heights, etc. Thus, students have opportunity to observe and identify properties attached to different components of the figure in order to pose and pursue particular questions.
2. Contents or theorems that students used to relate with particular subjects (triangles, bisectors, with Euclidean geometry and conics with analytic geometry) now seem to appear connected. In fact, students not only reconstruct some particular relationships but also investigate and document new ways to generate particular figures. In addition, students are able to study properties attached to those figures. For example, by measuring particular parts of the figure, students can verify properties attached to the conic. In this case, any point located at the generated locus must satisfy the definition of hyperbola.
3. Students get involved in cycles of mathematical understanding that include the importance of posing questions or conjectures, exploring them, providing mathematical argument to support

them, and communicating their results. Hence, students realize that it is not only important to observe a particular relationship but also to provide arguments to support it. In addition, they value the need to develop a language to communicate their results.

4. In general, the process of analyzing parts of certain geometric configuration represents a challenge for students to observe and document the behavior of family of objects (segments, lines or points) within a dynamic representation. Students themselves get the opportunity to reconstruct or discover new theorems or relationships. A crucial aspect that emerged in students' problem solving instruction is that with the use of dynamic software they had the opportunity to engage in a story line of thinking that goes beyond reaching a particular solution or response to a particular problem.

Remarks

An important goal of mathematical instruction is to provide an environment for students in which they have opportunity to first exhibit their own ideas or ways to deal with mathematical problems. These initial ideas need to be challenged and expanded and the use of technological tools seems to offer an important ingredient to meet this goal. In particular, students can construct distinct types of representations with the help of technological tools that are susceptible to be studied in terms of answering or discussing questions or dilemmas posed by the students themselves.

Endnote

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WHICH WAY IS THE "BEST"? STUDENTS' CONCEPTIONS OF OPTIMAL STRATEGIES FOR SOLVING EQUATIONS

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Good problem solvers typically know many promising approaches to solving a problem and can identify some strategies that may be better than the rest. This study is an initial exploration into this capacity, with particular emphasis on the development of students' conceptions of what it means for a strategy to be the "best." 23 sixth graders were taught the basic operators of linear equation solving and then left to discover strategies on their own in three one-hour problem solving sessions. Students were regularly asked a series of questions designed to assess their developing conceptions of what it means for a strategy to be the best. Results indicate that many students developed quite sophisticated notions of best strategies, taking into account such criteria as length of solution and quickness of execution. Those with sophisticated conceptions were also stronger on measures of transfer equation solving, flexibility, and conceptual knowledge.

The ability to distinguish between effective and ineffective strategies is one of the key competencies that is integral to good problem solving in mathematics (Owen & Sweller, 1989). Good problem solvers often know many promising approaches to solving a problem and can also identify some that may be better than the rest. This study examines this capacity, with a particular focus on the development of students' conceptions of what it means for a strategy to be the "best."

Exploring what novice solvers believe it means for a solution method to be better than other methods, and how they develop this knowledge, has particular relevance in the present climate of reform. Current National Council of Teachers of Mathematics (NCTM) recommendations call for teachers to create classroom environments where students can engage in thinking deeply about mathematics (NCTM, 2000, p. 18). This form of engagement with mathematics, which requires reflection on and evaluation of students' own strategies and the strategies of peers, is believed to enhance the development of mathematical understanding (Lampert, 1986, 1992a, 1992b). It is proposed that such an approach to learning mathematics makes it more likely that students will understand *why* certain algorithms are standard or the best (Morrow & Kenney, 1998). However, evidence supporting this claim is only beginning to emerge and is limited to research in the elementary grades (e.g., Carpenter & Moser, 1984; Carroll, 2000). This paper describes an exploratory study intended to investigate this issue at the middle school level, as students initially explore the symbolic procedures of algebra.

What Makes a Strategy the "Best"? What it means for a strategy to be the "best" may seem straightforward at first glance: The best solution method is one that solves the problem in the most efficient way. Good problem solvers know multiple ways to approach problems and are also able to generate the most efficient solutions.

However, solution efficiency is a more complex construct than it appears. If the best method is the one that produces the solution most efficiently, what does it mean for a solution method to be the most efficient? A solution method may be the most efficient for a number of reasons – including that it requires the fewest steps, that it is the quickest to do, and/or that it requires the least mental effort to execute – all qualities that may or may not coincide. When these potentially defining features of efficiency suggest different choices for the optimal approach, the notion of "best" becomes much more complex. Often the most practiced or automatized solution method requires the least mental effort to execute, in that it may be the first approach a solver thinks of when looking at a problem and also is one that can be executed without a great deal of conscious effort. However, it may be the case that a solver can think of another, different strategy that results in fewer steps, if she examines the problem for a few moments before jumping in. Which solution is better – the one that comes to mind immediately and can be done automatically or the shorter one that only comes to mind after a few moments' reflection?

For example, consider the linear equation below and the two solutions provided in Table 1. Solution A follows what can be considered a standard algorithm in this domain. Many solvers who are knowledgeable in this domain can execute this standard algorithm with minimal effort and quite rapidly. Yet a solver who carefully examines this problem for a moment may be able to generate a more efficient solution, meaning one with fewer steps, as shown in Solution B. Which is a better strategy, the one with fewer steps or the one that can be generated and executed most rapidly?

Table 1: Two Strategies for Finding the Solution to a Linear Equation, $3(x + 1) = 9$

Solution strategy A:	Solution strategy B:
$3(x + 1) = 9$ $3x + 3 = 9$ $3x = 6$ $x = 2$	$3(x + 1) = 9$ $x + 1 = 3$ $x = 2$

Further complicating what "best" means is the idea that solution methods have aesthetics. The notion of a best approach may also be related to features that may or may not coincide with solution efficiency, including elegance, parsimony, symmetry, coherence, simplicity, and beauty (Silver & Metzger, 1989). Even among mathematicians, the aesthetics of solution methods are difficult to quantify or categorize (Wells, 1990), yet aesthetic judgments often become the primary means of evaluating mathematical work (Penrose, 1974). Some have argued that cognizance of the aesthetics of solution methods is a hallmark of mathematical expertise (Silver & Metzger, 1989).

The point here is that what it means for a solution method to be the best is actually quite subtle and complex. Nevertheless, this knowledge is integral to what it means to be a good problem solver. Of interest here is how this knowledge develops. As novices begin to develop knowledge of multiple solution methods in a domain, what do they think it means for a solution to be the best?

Prior Research on Conceptions of "Best" Strategies. Despite recognition of the importance of this topic (e.g., Isaacs, 1999; Schoenfeld, 1985; Taplin, 1994), the development of students' conception of best strategies has not been widely and systematically explored, particularly at the secondary school level. In elementary school mathematics, a few studies have examined students' conceptions of good and better solution strategies, with the general finding that students' conceptions of best strategies are often implicit and somewhat idiosyncratic. For example, Franke and Carey (1997) interviewed first graders about their solution strategies for solving the problem 3 + 4. Many participants indicated that their own personal strategy was the best one, without being able to articulate what made it better than the example solutions provided. Very few students recognized the role of efficiency in determining best strategies. Similar results were found in a similar study by McClain and Cobb (2001). It appears that very little is known about how and when students ultimately develop a more sophisticated ability to differentiate among different solution strategies to determine which strategies are best (and why). The present study begins such an exploration.

Method

The present study was part of a larger project that aimed to look at the development of students' strategies for solving linear algebraic equations (Star, 2004). During the summer between their 6th and 7th grade years, 23 students (12 males and 11 females) attempted a series of linear equations on videotape and answered questions about the strategies that they used. Students participated for a total of five hours over five consecutive days. On the first day, students were administered a pretest to confirm that none had prior knowledge of the four transformations used to solve linear equations (adding or subtracting to both sides, multiplying or dividing to both sides, distributing or factoring, and combining like terms).

Following this pretest, students received 20 minutes of instruction, introducing them to these four linear equation solving transformations. During instruction, students were not shown any worked-out examples of solved equations nor were they given guidance in how transformations could be chained together strategically to solve equations; rather, students were provided instruction only on how to apply each transformation individually and were left to discover how transformations could be used productively in solving equations. Following instruction, students participated in three one-hour videotaped problem solving sessions. All students solved the same problems in the same order; problems ranged in complexity, including $2(x + 1) = 12$ and $3(x + 2) + 9(x + 2) = 6(x + 2)$. At regular intervals, the interviewer asked students a pre-determined series of questions about their strategies. At the end of each day's work, the researcher asked students several concluding questions, including two questions relating to "best" solution methods: (a) *Suppose your friend told you that he/she had solved an equation in the best possible way. What do you think he/she means by the best possible way?* and (b) *How do you know when you've solved an equation in the best possible way?* Each student was asked these same questions at least twice during the study.

Students completed a posttest on the final day of the study, which was the same as the pretest. The posttest contained four types of measures: isomorphic equation solving, transfer equation solving, flexibility, and conceptual knowledge of equations. Isomorphic equations were problems that were very similar to those that were solved during the problem solving sessions. Transfer equations were novel linear equations to assess transfer of solution method; each had a feature that would have been unfamiliar to students. Flexibility, defined here as knowledge of multiple solution strategies and the ability to use a range of strategies on a given problem, was assessed using a series of questions that have been reliably used in prior work (Star, 2001, 2002, 2004). Conceptual knowledge was assessed with a series of multiple-choice questions relating to the concepts of equation, variable, and equivalency.

Analysis

Students' solutions to all posttest problems were graded on correctness of solution. Students' responses to interview questions relating to best solution strategies were transcribed and categorized (as described below) by two independent raters.

Results

Students' responses to the best strategy questions fell into seven categories, as shown in Table 2. Almost all students (20 of the 23, or 83%) mentioned more than one category in their best question responses (with a mean of about three categories mentioned per person).

Table 2: Students' Responses to Questions Relating to Best Solution Strategies

Category	% responding
Shortest way, involving fewest steps	65%
Quickest or fastest way	61%
Easiest, least complicated, or least confusing way	61%
Most accurate way, with fewest errors, and arriving at right answer	57%
Way that I'm more sure, confident, comfortable with, proud of, or happiest with	30%
Way that is most neatly written and organized	9%
It depends on various things, including the problem, how quickly a solver can execute steps, and the preferences or goals of a solver	9%

Note that students as a group captured the complexity of what it means for a solution strategy to be the best. As discussed above, determining which of several strategies is the best is quite nuanced; for experts, *all* of students' responses to this question have some merit. While it is more typically the case that the best strategies are those that are the quickest, shortest, or least complicated, it is also true that accuracy, neatness, and solvers' goals and preferences play important roles in evaluating the goodness of strategies.

For analytical purposes, students' responses were placed into two broad categories, referred to here as *naïve* and *sophisticated*, in order to identify those students whose views of best were more aligned with what would be considered typical or sophisticated in the domain. Sophisticated conceptions of best were those that included criteria of quickest, shortest, least complicated, or dependent on a variety of conditions, while naïve views included criteria having to do with confidence, neatness, and accuracy. (Other than the research cited in the opening of this paper, there are no known studies that have surveyed mathematicians' views on what criteria are used to establish the best solution method for a particular problem. However, it is assumed here that, although this issue largely depends on the problem and problem solving context, it is more generally the case that quicker, shorter, and straightforward solutions are considered to be better than longer, overly complex solutions.) Nine percent of participants held views that were exclusively naïve, 30% had exclusively sophisticated conceptions, and the remaining 61% had views that included a mix of naïve and sophisticated elements.

Sophisticated Views of Best Solution Strategies

Almost all students came to hold (at least in part) more sophisticated conceptions of best solution strategies; 87% of participants (20 of 23) mentioned at least one of the four criteria within this typical view (shortest, quickest, least complicated, and it depends). Recall that students saw no worked-out examples of equations and only minimal instruction in how to use solving transformations, so their conceptions of best strategies were developed entirely on their own, by reflecting on their own problem solving. It is quite striking that so many participants identified the same features of best strategies that experts typically use.

Over one-third (35%) of participants mentioned three of the most typical features of sophisticated views (quicker, shorter, least complicated), showing a recognition that these features are likely to be correlated. Helen (all names are pseudonyms), whose conceptions were exclusively sophisticated, was one of these students; her sense of what it means for a strategy to be the best was as follows:

> Doing it the fastest, the easiest. (*Tell me more?*) Like getting the answers as quick as you

> can, like if you need it maybe on the math quiz or something, where they are quizzing you on algebra and like stuff. They did it the best way or something, that they did it the fastest, they got the right answer. (*What does easiest mean?*) Like the least amount of steps.

However, most students only mentioned one or two of the more sophisticated components of best strategies. For example, three students felt that the best strategy was the way with the fewest steps, but did not articulate the connection between a strategy with fewer steps and one that is executed the fastest (even when pressed on this point by the interviewers). In contrast, four other students all said that the best way was the quickest or fastest way, without realizing (even when pressed) that the quickest way usually is the one with the fewest steps. Although intuitively the relationship between fewest steps and quickest method may seem obvious, most students did not make this connection.

Two participants (9%) indicated that determining the best solution strategy depended on a variety of factors, including the personal preferences of the solver, her goals, and her proficiency with equation solving. For example, Cathy (whose views were exclusively sophisticated) acknowledged the role of quick solutions with fewer steps, but also felt that personal preferences also play an important role in determining the best strategies. In particular, she felt that some students may prefer to work more slowly and carefully, while others prefer to go faster, despite an increased risk of mistakes:

> Probably the easiest way possible for her. ... So if she called me on the phone at home one night and she's like, I have found the most easiest way to do this problem. Even though it takes more time it is so easy, you could just make sure you do not miss one step. And I'll be like, well, that's great, but I don't want to be up until 11, doing my homework. So her way might be easier than my way because her whole afternoon might be blank. It doesn't really matter to her, she can just go through each problem the longest way possible, but at least she would know I've only moved one thing so you don't have to do anything else.

Naïve Views of Best Solution Strategies

Despite the prevalence of sophisticated conceptions of best strategies, 70% of participants (16 of 23) mentioned at least one of the naïve components, including accuracy of steps or solution, personal confidence in or happiness with solution method, and neatness of solution.

The most common naïve conception of best strategies focused on accuracy; 57% of participants mentioned this category at least once. Some students, such as Brad (whose responses were a mix of sophisticated and naïve conceptions), focused on the accuracy of the equation solving transformations: "Like he used the steps right. And like he added and subtracted right." For other students with this view, final solution accuracy was more central to what it meant for a strategy to be the best, as Melanie (whose views were exclusively naïve) notes: "That they used each step at the right time, and got down to the correct answer."

Personal confidence was mentioned by 30% of participants as playing a role in determining the best solution strategy. Students felt that a method in which they felt confidence in, proud of, comfortable with, and/or happy with was the best. Tracy (whose views were exclusively naïve) felt that the best way was the one that was "just the easiest way for me." And similarly, Oscar (whose responses were a mix of sophisticated and naïve conceptions) noted, "[I know my solution is the best] when I am sure, like 100%, it's the right answer and you've done your best."

In addition, two students (9%) indicated that the neatness and organization of a written solution strategy also played a role in determining whether it was the best one.

Relationship Between Conceptions of Best Strategies and Other Posttest Variables

Also of interest was whether any relationships existed between students' conceptions of best strategies and the variables assessed on the posttest – equation solving skill, flexibility, and conceptual knowledge of equations.

With respect to equation solving skill, despite the brief period of instruction and the absence of any worked-out examples to refer to, students became quite proficient at solving linear equations. Participants on average solved correctly 73% of the posttest isomorphic equations, as compared to only 18% correct on the pretest.

On isomorphic equations, there was not a relationship between students' conceptions of best strategies and their equation solving skill. In fact, the two students with exclusively naïve responses did the best on isomorphic equations, correctly solving *all* of them, as compared to those giving only sophisticated responses (75% correct) and those with mixed conceptions (68% correct).

However, on transfer equations, where students did much less well overall (posttest mean of only 7% correct), sophistication of conceptions of best strategies did appear to play a role. Neither of the two students with naïve conceptions was able to solve *any* of the transfer equations correctly, as compared to 4% correct for those with mixed conceptions and 14% correct for those with the most sophisticated conceptions of best strategies. Similar results were found in measures of flexibility and conceptual knowledge, in that those with more sophisticated conceptions of best strategies obtained higher scores on flexibility and conceptual knowledge of equation solving than those with the most naïve conceptions.

Note that the present data does not indicate the directionality of this relationship; it is not clear whether increased flexibility and conceptual knowledge were responsible for students' more sophisticated conceptions, or vice versa. But the data does suggest a relationship between improved transfer equation solving, flexibility, and conceptual knowledge and more sophisticated conceptions of best solution strategies.

Discussion

This exploratory study investigated the development of students' conceptions of best strategies for solving linear equations. There were two main results. First, many students developed reasonably sophisticated conceptions of what it meant for a strategy to be the best, including criteria for best strategies such as quickness of execution and the length and complexity of the solution method. The sophistication of students' conceptions was quite striking, particularly given the minimal amount of instruction and the discovery-oriented nature of problem solving sessions.

Second, the sophistication of students' conceptions of best strategies was not related to the ability to solve isomorphic equations. Even those students with quite naïve characterizations of best strategies became quite good at solving familiar linear equations. However, those with more sophisticated notions of best were better on transfer equations and had improved flexibility and conceptual knowledge of equation solving, suggesting a relationship between deeper conceptual and procedural knowledge in this domain and the ability to execute procedures flexibly and strategically (Star, 2000).

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WHAT DISCUSSIONS TEACH US ABOUT MATHEMATICAL UNDERSTANDING: EXPLORING AND ASSESSING STUDENTS' MATHEMATICAL WORK IN CLASSROOMS

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"What is the oldest problem of pedagogy? The appearance of learning, or 'illusory understanding', that is, the problem of people who appear to know something that they really don't know." — Lee Shulman (2000, p. 131)

The motivation for this paper comes from the great rewards we—as teachers and researchers—experience every time we listen to students' discussions about mathematics. Listening to students talk about mathematics reveals aspects of their understandings and dispositions towards mathematics that written work alone does not disclose. In particular, student discussions give us important insights into the students' relationships with mathematics. These relationships include mathematical understanding, agency, and conceptions of the nature of mathematics. Such knowledge is crucial for assessing individuals in the classroom, and can be used to help meet goals of effective, equitable teaching.

In this study we examined videotapes of 40 groups of 3-4 students working on an open-ended mathematical task for 90 minutes. Using first a broad lens of 'connoisseurship' (Eisner, 1985) we took careful note of what was revealed about students' relationships with mathematics. For each group, we compared this knowledge to what we learned from the written work students completed in response to the task. Using categories from the New Basics Project of Australia (Department of Education and the Arts, State of Queensland, 2001) as a guide, we developed a set of categories of relationships with mathematics that are particularly conspicuous in discussions. From these categories, we developed a tool to facilitate teachers in learning from student discussions. Although our tool is inspired and influenced by recently developed fine-grained research tools for analyzing student discussions (Sfard and Kieran, 2001; Barron, 2003), our tool is designed for teachers to use, in real-time, as they negotiate the busy classroom. Once our tool was refined we applied it to the videotaped discussions. We compared the results to the written work of the students, and found that the analysis of the discussions delivered a more accurate representation of students' relationships with mathematics.

Methodological Frameworks:

Recent work has demonstrated that researchers can obtain critical knowledge of student understandings and dispositions by performing fine-grained analyses of students working collaboratively in small groups. Barron's (2003) coding scheme for analyzing interactions among group members helps isolate many powerful indicators of successful student work. For example, Barron's discovery that partner responsiveness is more powerful for predicting successful student work than prior achievement or accuracy of student ideas, is a rich and valuable finding. Sfard develops a powerful analytic tool (Sfard and Kieran, 2001) for analyzing student discourse during collaborative work. Like Barron's analytic method, Sfard's tool focuses on how students respond to one another's statements. Sfard and Kieran (2001) and Kieran (2001) use this tool to carefully monitor the development of mathematical thinking in pairs of students, and to further understand how a student makes use of a partner's knowledge.

The convincing findings that materialize from both Barron's and Sfard's analytic tools suggest that teachers too could learn much from examining student discussions. Recent literature on assessment suggests that teacher must look for student understanding from multiple sources, and that student dialogue is an especially fruitful source. Wiggins & McTighe (1998) introduce the powerful concept of "backwards design", and encourage teachers to "think like an assessor", always looking at student interactions with an eye towards understanding how well students can explain, interpret, apply, critique, justify, perceive, and reveal knowledge (p.66-67). Later, Wiggins & McTighe specifically ask teachers to "use dialogue or interaction to assess" (p.85). Black and Wiliam (1998) advocate that continuous, everyday assessment of student knowledge is critical, and suggest: "Teachers can find out what they need to know in a variety of ways, including observation and discussion in the classroom and the reading of pupil's work" (p.140). These recommendations suggest that the knowledge revealed in student discussions is particularly valuable for teachers.

To produce a tool that teachers can effectively use in the busy classroom, it must be versatile and adaptable. It must be designed "through the eyes of the practitioner" (Lampert, 1985, p.180). It cannot rely on fine-grained line-by-line analysis of student discourse; rather, it must allow teachers to listen for evidence of mathematical understandings in the moment, as they travel between different student discussions. Thus, our tool is a collection of broad categories that are important for understanding students' relationships to mathematics. It is a guide to help teachers broaden the ways they look for evidence of student understanding.

Assessing Student Discussions: The Development of a Pedagogical Tool

To guide us in our development of categories of student relationships to mathematics, we studied a framework for understanding 'intellectual qualities' developed by the New Basics Project of Australia. This framework emphasizes higher-order thinking, deep knowledge, deep understanding, substantive conversation, knowledge as problematic, and 'metalanguage' as six broad categories of intellectual goals for students (Department of Education and the Arts, State of Queensland, 2001). We found these categories to be closely aligned to categories we have found in past research to be important for understanding student relationships with mathematics (Boaler, 2002a,b).

As we developed our tool, we honed in on four types of student relationships with mathematics (Gresalfi, Boaler & Cobb, 2004). The first type of relationship is students' *conceptual understandings of mathematics*. This concerns student knowledge of and proficiency with deep mathematical ideas. The second type of relationship, *mathematical agency*, concerns how students use mathematical knowledge. How able are they to draw upon diverse types of mathematical knowledge and how able are they to use mathematical knowledge to solve problems? Do they confidently see themselves as mathematicians? The third type of relationship to mathematics is students' *conceptions of the nature of mathematics*. What do students think 'doing mathematics' entails? The fourth type of relationship is *mathematical authenticity of students' work*. Are students engaging in conversations that are characteristic of mathematical work? Do students ask the kinds of questions mathematicians ask?

For each of these four types of relationships with mathematics, we developed a set of examples from classroom discussions that illuminates features of these student relationships.

From these, we developed a set of questions for teachers to ask themselves as they listen to student discussions. Some examples are: "Is the group able to abstract a problem's important elements?" or, "How precise is the students' use of mathematical language?" are questions teachers can ask to help understand students' *conceptual understandings of mathematics.* "How confident are the students at delving into the problem?" or "Do the students consider an array of different mathematical techniques as possible tools for helping solve a problem?" are questions teachers can ask to help understand students' *mathematical agency*. "Do the students see mathematical techniques as handed-down and unchangeable, or do they consider how these tools might adapt to be useful in novel situations?" and "Do the students understand that errors are an important part of mathematical progress and discovery?" are questions that will help teachers uncover students' *conceptions of the nature of mathematics*. "Do the students ask intuitive questions first, and then follow up these inquiries with rigorous work?" and "Are students comfortable engaging in long-term queries into deep problems?" are questions that help uncover the *mathematical authenticity of students' work.*

Results: What We Learned from Student Discussions

After creating our pedagogical tool, we field-tested it on our video cases of 40 groups of students working together on an open-ended task. These video cases are part of a four-year study that is monitoring the understanding of approximately 1000 students who experience different mathematics teaching approaches in three high schools. In two of the schools—Greendale and Hilltop—two different approaches to mathematics are offered, one of which is open-ended and reform-oriented, and the other of which is more traditional. At the third school, Railside, students take the traditional sequence of courses, but the mathematics is presented through longer problems that emphasize multiple connections and methods, and the students work collaboratively in groups at all times. In this larger study, we have found that students build much different relationships with mathematics depending on the teaching approaches they experience. We are hoping that our analysis of student discussions will be a useful additional data point for this analysis.

The task, called 'Rocket Boosters', encouraged students to use their collective knowledge as a resource. It was given to two classes in each of the three different schools; in the schools that offer a choice between 'open' and 'traditional' approaches we gave the task to a comparable class within each approach. The task asked students to determine the area of the region within which fuel-carrying rocket boosters would fall after being expelled from a rocket. This ultimately involved finding the maximum area of a triangle with two known sides. The task was challenging as none of the students in any of the classes had previously worked on a problem quite like it. They had sufficient background in trigonometry to solve it, but would need to draw upon multiple techniques and ideas. There was more than one way to solve the problem. Students worked on the task in groups of four and produced written answers and explanations to the questions we posed. In our analysis of their mathematical understanding we considered both their written work and their mathematical conversations.

Our analysis of the video-cases revealed valuable information about student relationships to mathematics, in each of our four categories. In particular, this information was more accurate and detailed than information obtained from the written work alone. To demonstrate, we will compare two groups whose written work looks very similar, but whose mathematical discussions reveal very different relationships with mathematics. Both of these groups performed poorly when we scored their written work alone, even with a generous scoring rubric that gave partial credit for incomplete solutions and in which the scorers scoured the work carefully for fruitful mathematical ideas. Group A, which consisted of three girls and a boy and was part of a traditional curriculum class at Hilltop, scored 0 out of 9 points on the assessment of their written work. Group B, which consisted of three boys and a girl and was part of a reform curriculum class at Greendale, scored 1 out of 9 points on their written work.

Conceptual understandings of mathematics

Despite scoring similarly on the written work, analysis of the discussions of group A and group B revealed very different understandings of mathematics. Dialogue from group A's discussion reveals deep misunderstanding of basic mathematical concepts. For example:

Girl 1: I don't think it can be a right triangle if the angle is 25. Because aren't right triangles 30-60-90?
Girl 2: 30-60-90 *(confirms)*
Girl 1: Wait, so the angle of that segment, that we're actually looking for, is actually the height?
Girl 3: No, it's A to B.
Girl 2: I believe so.

In the first two lines of dialogue, we see that Girl 1 and Girl 2 believe that all right triangles must be 30-60-90 triangles. This is a fundamental misunderstanding about the mathematics being used here. The last three lines of dialogue indicate the girls are also confused about the difference between measures of segments and measures of angles, another fundamental misconception about the basic geometry involved in this problem. On the other hand, the dialogue from group B, of which there are too many examples to include, reveals a solid understanding of the basic geometry behind the problem.

Mathematical agency

From the discussions we also learned much about students' mathematical agency. The students in group A did not deliberate over when it was appropriate to use mathematical techniques or formulations. They read formulas as a set of verbatim instructions, and did not question the meaning of the whole statement. For example, in the following statement Girl 2 tells Girl 1 how to plug numbers into a formula as Girl 1 writes exactly what she says:

Girl 2: *(reading an equation)* So you go 200 times 25, tangent equals... you have to put an equals sign... equals 93.

Girl 1 writes what Girl 2 says, and they do not engage in a conversation about whether or not it is appropriate to use this formula in this situation (it is not), and it appears they do understand what the output of this formula represents.

In contrast, group B grapples with the meaning of formulas, and questions the appropriateness of their use in each situation:

Boy 1: Can't you use 300 squared plus 200 squared equals…
Boy 2: No, 'cause it's not a right triangle.
(Boy 1 examines the formula and a few drawings)
Boy 1: Oh, I see
Boy 3: Because you have to have a hypotenuse and a right triangle to do it.

Conceptions of the nature of mathematics

In analyzing the discussions, we learn a lot about what students believe mathematics is. The students in group A make many comments that indicate they believe mathematics is a set of idealized problems passed from teacher to passive student:

Boy: God, why don't they just make this a regular triangle? (expressing frustration that the triangle is not a 30-60-90 triangle.)

Girl 1: It's not like we're supposed to get the answer, we're just supposed to try and then they take your work.

Girl 3: Yeah, I'm just going to put down any old answer. (frustrated)

In contrast, group B expects problems to be complex, like the real world, and are skeptical of problems that are too 'cleaned up', as shown by their initial reaction to the drawing that accompanies the problem:

Boy 3: That's pretty straight for the Florida coastline; I think it's a crock.

After this initial skepticism, the group later thoughtfully demonstrates that they understand this problem is, in fact, rich and complex:

Boy 3: Alright, so first we use the Pythagorean Theorem.
Boy 2: I don't think we can.
Boy 3: Oh, that's right, because it's not a right triangle.
Boy 1: No, but you can make it a right, can't you?
Boy 3: Right, but then we won't have the...
Boy 2: What do you mean we can make it a right?
Boy 1: Cut it in half!
Boy 2: It's not necessarily an equilateral, [I mean an] isosceles triangle.

Mathematical authenticity of students' work

Both group A and group B realize what is difficult about the problem—that the triangle is trickier to measure than most they have dealt with previously. Both groups develop a plan that is typical of the way mathematicians work. Specifically, they compare it to a triangle they know how to measure; group A compares it to a right triangle, and group B compares it to a isosceles triangle. However, the groups differ in how they make this intuitive connection rigorous. Group A doesn't resolve the fact that they altered the problem:

Girl 2: (reading the problem) so A and B are stationed five hundred miles…
Girl 1: Let's just pretend that it is a right triangle.
(about 30 seconds of independent work)
Girl 2: Okay, oh, wait a minute—right triangle.
Girl 3: Why would it be a right triangle?
Girl 3: 'cause I'm gonna make it one!

While these comments hint that the group has an enterprising mathematical initiative typical of authentic mathematical work, their failure to account for their adjustment indicates a defeated, incomplete relationship with math. Group B, on the other hand, carefully discusses how they can compensate for their adjustment of the triangle:

Boy 3: Yeah that's what I was thinking we could, maybe, um, make them both 300 and then when we're done with that we can take the area of the small triangle we add on, and take it out. But I think there's another way out there to do it, too.

Then, after a couple minutes of work:

Girl: But, how can you change the measurements? You can't just make it 300; how do you do that?

Boy 3: Well, because you need an isosceles triangle to be able to find it, right? So what we have right now to start with, is we have this triangle right here we have one side shorter. Well if we take it and make them both 300, then we can find that *(pointing to the larger triangle now that one of the sides is extended)*, then we can subtract this triangle that's 100 *(pointing to the small triangle added on after extending the side)*, find this base, then find the area of this triangle, right? Subtract that area from the total triangle, and then we have the area of this triangle *(pointing to the original triangle)*.

Girl: Oh yeah! Okay, okay.

Discussion: What Our Findings Tell Us About the Value of Student Collaboration

Our analysis led to two important findings. First, our analysis of the students' conversations produced a wealth of information about student thinking. Our findings were more consistent and more detailed than those obtained by our examinations of written work alone. In particular, groups with similar written work often differed greatly in the quality of their mathematical discourse and thinking. For example, we found that groups who completed the written work accurately often had discussions that were fragmented and disjoint. Some of these discussions revealed that the students harbored fundamental misconceptions of mathematical concepts (Erlwanger, 1975) and could not carry out a logical argument, despite their success on the written assessment. Similarly, some groups, such as group B above, produced sketchy or incomplete written work but had coherent discussions and harbored fewer misconceptions than even some high-scoring groups. These findings are striking and shed important light on the recent emphasis of open-ended collaborative work in mathematics classrooms. While much research has already shown that such work is valuable for improving student thinking (Boaler, 1997; 2002; Schoenfeld, 1985; Silver & Stein, 1997), our findings indicate that such discussions are also valuable for giving teachers and researchers critical insights into student understanding. Recent research on teaching suggests that teacher insight into student thinking is an important part of improving practice (Lampert, 2001). We were especially struck by the large number of student misconceptions that were apparent in the discussions, but not in the written work. An understanding of student misconceptions is highly useful for teachers in facilitating student learning (Black & Wiliam, 1998). Because a teacher does not have the time to analyze student conversations to the level of detail that Barron and Sfard and Kieran have, we have produced a framework that points to important elements of mathematical work that is useful for teachers who want to recognize and record important aspects of mathematical talk.

Our second finding allowed us to draw a number of parallels between classroom discussions of open-ended problems and high-level mathematical research. For example, many groups struggled for much of the time when working on the problem, progressing slowly, with progress coming in fits and starts. This reminded us of Koestler's (1959) analysis of the development of celestial mechanics; Koester describes long periods of no progress followed by short flurries of discovery, where great strides were made. We also saw students trying out ideas intuitively before refining them rigorously. This parallels Archimedes' (Heath, 1897) and other mathematicians' methods of mathematical discovery. It also shows a trend of intuition preceding deduction that is evident in the budding of most fields of mathematics. Striking examples are evident in the genesis of analysis (Grabiner, 1974) and group theory (Kolmogorov, 2001), two of the most important developments in mathematics. We saw students struggle and strain, exclaiming that the problem is hard—that it is like nothing they have seen before. Confrontation and strain are evident of all progress in mathematical thought—it is shown by cognitive scientists to be indicative of mathematical learning in general (Steffe, 1990), and evident historically in the discovery of irrational lengths in ancient Greece (Stillwell, 1989; Heath 1981), and in the invention of algebraic geometry (Stillwell, 1989; Hobbes, 1672) and analytic techniques (Stillwell, 1989; Hobbes, 1672) in the seventeenth century. Finally, we saw students confronted with the challenge of inventing new notations and unfamiliar pictures. This reminded us of the development of algebra and complex analysis, two fields which struggled to find a suitable notation (Stillwell, 1989; Boyer, 1968).

The striking similarities between the discussions of open-ended problems in the classroom and the development of research-level mathematics helps explain why these practices have put students at a cognitive advantage for learning mathematics. Moreover, it shows that some mathematics classrooms are succeeding in providing students with opportunities to authentically engage in the subject. This is an often forgotten goal of education: to expose students to subjects, so that they can decide if they enjoy engaging in them and direct their schooling accordingly. Boaler and Greeno (2000) and Schoenfield (1985) show that traditional classrooms often give a dishonest picture of mathematical work. Tasks that ask students to engage with each other over open-ended problems give a more authentic depiction.

Our findings are particularly important in the potential they have for strengthening connections between different groups involved in mathematics education. A recent issue of *Educational Studies in Mathematics* makes a call for research that connects the theoretical, academic work in mathematics education and the practical, in-school work of teachers (Ball & Even, 2003). Our analysis of student discussions has a two-fold utility that enables conversations between researchers and teachers. By illuminating student thinking in case study examples of classroom conversations, we present student thinking as a ubiquitous commodity in classrooms that harbor discussion—ready to be tapped by the attentive teacher or the systematic researcher, and discussed between them. In developing a method for analyzing mathematical discussions we hope to provide a useful tool for teachers, which may give them further opportunity to engage students in conversations and to formatively assess (Black & Wiliam, 1998) students' developing understandings. Another article in the same issue of *Educational Studies in Mathematics* highlights an unfortunate gap in communication between educational researchers and mathematicians (Goldin, 2003). By demonstrating parallels between classroom practices supported by educational research and practices of mathematicians, we will help show both groups that there is important common ground upon which they can work together.

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DEVELOPING UNDERSTANDING IN MATHEMATICAL PROBLEM-SOLVING: A STUDY WITH HIGH SCHOOL STUDENTS

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This study documents what high school students achieved when working with a set of tasks that involved different methods of solution in a problem-solving oriented course. During the implementation of learning activities students had the opportunity to work in small groups, present and defend their ideas to the whole class, and constantly revise their work as a result of the criticisms and opinions that appeared during the development of the sessions. The models that the students constructed in the processes of solution were documented, focusing on how they use distinct resources, representation, strategies and ways of communicating their results.

Introduction

What type of problems or tasks favors the development and understanding of students' mathematical ideas? What does it mean for students to learn mathematics? What type of instruction promotes students' learning? These are some of the questions that have been part of the research agenda in mathematics education during the past fifteen years, and were used as a guide in the development of this study. In particular, recent proposals in mathematics curriculum suggest the organization of mathematics learning around problem-solving activities (NCTM, 2000), and it has been recognized the importance for students to develop distinct resources and strategies to pose and solve different types of problems. Also, it becomes relevant to consider learning scenarios where students have the opportunity to reflect over the use of resources and processes in working with mathematics and that allow them to extend and reinforce their methods of posing and solving problems (Santos and Sepúlveda, 2003). In these scenarios the students present their ideas and listen to and examine the ideas of other students in such a way that they constantly reflect over their own ways of understanding mathematical ideas. In this study we were interested in documenting the thought processes shown by the students when they worked with a set of problems. The problems were designed with certain principles in mind: Were easy to understand and interesting for the students, they involve fundamental concepts and ideas of the curriculum and were posed in a manner that the work of the students could be analyzed and documented (Balanced Assessment Package for the Mathematics Curriculum, 1999, 2000).

Conceptual Framework

Three important themes became relevant and helped organize and structure the development of this study:

1. The idea that learning mathematics involves the development of a disposition on the part of the students to explore and investigate mathematic relationships, to use distinct forms of representation in order to analyze particular phenomena, and to use distinct types of arguments and communicate results. The NCTM suggests that it is important that the students construct their own mathematical knowledge as a result of solving distinct types of problems. Thus, relevant features that are enhanced in this process include: the motivation to express what they know; the encouragement to be open to investigate what they do not know through discussion,

experimentation and the exchange of experiences; and the recognition of the importance of the thought processes used in their attempts at problem solution.

2. The recognition that learning mathematics is a continual process that is favored in an atmosphere of problem-solving (Schoenfeld, 1998) wherein the students have the opportunity to develop ways of thinking that are consistent with the development of the discipline. In this context, the students conceptualize the discipline in terms of questions or dilemmas that must be examined, explored and solved through the use of distinct strategies and mathematic resources (Hiebert, et al., 1996).

3. The acceptance that significant problems can be incorporated in different contexts (Barrera and Santos, 2002): a context of pure mathematics where, from a simple presentation, the student must use basic concepts to analyze and solve the problem; an ordinary daily context where the student has to interpret a familiar event, use distinct mathematical resources, and establish a series of considerations to solve the problem; and an artificial context where the situation is constructed from a series of suppositions about the behavior of variables or parameters that explain the development of the situation which in the treatment of the problem the student must project the use of strategies and representations in the methods of solution.

Participants, Research Methods and Procedures

The present study is structured around six problems that were selected, and adjusted from those found the Balanced Assessment Package for the Mathematics Curriculum (1999, 2000). Twenty-four students in the eleventh grade of a public school participated in the course of 16 weeks that included the following phases of instruction:

i) Introduction to the activity. The teacher gave a brief introduction to explain the goals to the students and the importance of their participation.

ii) Discussion in small groups. Students were organized into teams of three students each, with students of distinct levels of mathematical skills. At the end of the group work the students turned in a report of their solution.

iii) Student presentations. Each group presented their solution and the other members of the group (including the teacher) had the opportunity to ask questions.

iv) Full class discussion. The teacher promoted collective discussion to analyze the different methods of solution presented by the students and when necessary, presented a summary of students work and discussed possible extensions of the task.

v) Individual work. The students had the opportunity to return to the problem and incorporate the ideas discussed during the session.

To work the task students were organized into eight small groups of three students. An attempt was made to assure that in every group the levels of mathematical skills and personalities of the students were different within each group, according to assessments made in the first sessions and the opinions of their previous teachers. The idea was that the small group organization would guarantee the participation of the students in the interaction with the other members of the group and during the presentation of their ideas to the entire class.

The sources of information that were used to analyze students' work came from:

i) the written reports from the students corresponding to the work in small groups and the individual work permitted the identification of the ideas, and application of resources used to solve the problem in two distinct moments: the first moment reflects the initial (spontaneous) way of thinking of the students about the questions that were posed when they worked in small groups; the second moment reflects the level of understanding that was acquired as a result of the interaction with the entire class and when they solved the problem individually. This permitted

the determination of whether there was an evolution in the understanding of the problem on the part of the students.
ii) Videotapes of the small group work, the students' presentations, the collective discussions, and the students' interviews. The videos permitted the analysis of the ways in which the students participated in the solution of the problem, in the presentations by the small groups, in the full class discussions and in the interviews. We could analyze with more detail the behavior of the students that helped them to solve the task. We could also identify crucial moments in which there were changes in the thinking of the students which allowed them to solve the problem.
iii) The observations of the teacher. During the development of the course, it was important to identify students' difficulties, ways of interaction in which the students used the resources and strategies that were important to analyze students' competences.

Some questions that served as a guide to analyze students' work were: Were there differences between the answers given by the students in the reports from the group work and the contents of the individual reports? How to characterize those differences? Is it possible to identify what refinement processes were presented in the initial responses of the students to the solution of the problems and those shown as a result of their participation in a learning community?

The Problem and Some Considerations

We will now present one of the problems called "Shadows" from the ordinary context which was posed along with Figure 1:

1. Alicia is 1.5m tall and is standing 3m from the base of a lamppost that is 4.5m high. How long is Alicia's shadow?
2. How does the length of Alicia's shadow vary when she gets closer or farther away from the lamppost? Draw a graph with a system of perpendicular axis. Can you find a formula for this graph?
3. Simon is 2m tall. How can the graph represent the change for his shadow? Compare this graph with the one you drew for Alicia.

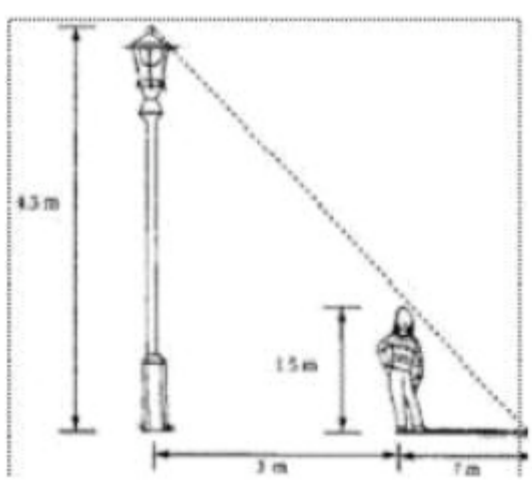

Figure 1

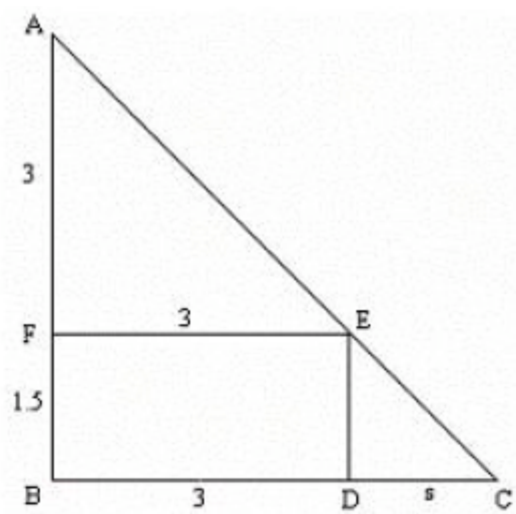

Figure 2

An important aspect for the understanding and solution of the problem is for the students to construct a representation of the situation that will help them find the relationships in the information of the problem. For example, the height of the lamppost, Alicia's height and the length of her shadow can be represented with the corresponding segments. In this manner, a figure composed of three similar rectangular triangles is obtained: one with the form of the lamppost, one with Alicia's figure, and one that describes the distance between Alicia and the lamppost when a horizontal segment is drawn over Alicia's head (Figure 2). The relationship of similarity leads to the solution to the problem, considering the segment that represents the shadow first as a fixed quantity and then as a variable. Effectively, the students drew graphs as described, with some variations; however, we observed diverse manners of employment of the mathematical relationships.

Presentation of Results

First we will present the analysis of the work with students in small groups through the written reports and applicable video segments. Then we will go to the presentations of the solutions from the different groups and collective discussions that students' participation generated. Thereafter we will present the results of individual work, making a global analysis of the interaction offered by this type of instruction. The written reports turned in by the groups demonstrated distinct approaches in the students' attempts to solve the problems. What ideas, concepts, strategies and representations were relevant in these answers? In analyzing the written reports and the video of the work in small groups, we observed that the approaches to question 1 were:

I) Those of groups D and G which assumed that there were two similar triangles (that which is formed by the lamppost and that of Alicia in the Figure 1; they remarked that the angles were equal: angles in the base of lamppost and Alicia's feet; angles in the lamp of the lamppost), without clearly justifying why this relationship existed, establishing their proportions and operations to obtain the answer.

II) Group E constructed a rectangular triangle (on Figure 1) with the vertices of Alicia's shadow, elongated the shadow, the hypotenuse, and drew it parallel to the lamppost in whose sides of the right angle they wrote as 2cm (citing the similarity of the triangle that is formed by the lamppost with that constructed; they marked the angles), calculated the acute angle as $\tan\theta = \frac{2cm}{2cm} = 1, \theta = 45°$; thereby obtaining the length of Alicia's shadow: $x = \frac{1.5}{\tan 45°} = 1.5$

III) Group C drew a horizontal line over Alicia's head, forming a triangle with the lamppost (as in the Figure 2), saying that it was the same as that formed by Alicia (there are repeated letters, they used the symbol of congruence; they marked the angles) and that the sides of the right angle were in relation to 1, and from this obtained their solution.

It is notable that without having expressly established the relation of similarity nor having precise arguments, Group D applied the proportionality between the corresponding sides of the two similar triangles, Groups E and C constructed similar triangles, one applied trigonometric resources and the other used the relation between sides of the right angle of similar isosceles triangles.

IV) Groups B and H assumed and used the expression: $\frac{4.5m}{3m} = \frac{1.5m}{x}$ without showing the elements that justified their answer. Group F came by this through the schematic application of the Rule of Three, writing in a rectangle the quantities $4.5m$, $3m$, $1.5m$, x (obtained of the Figure 1). However, this approach was later analyzed and changed during the students presentations. Based on those approaches, the students gave the following answers: D, G, E C and A coincided in that Alicia's shadow measured $1.5m$. B and F affirmed that the shadow measured $1m$, while group H reported verbally that the shadow of Alicia is the third part of the distance to the lamppost. That is to say that group H verbally agreed with groups B and F. During the presentation of Group H in the full class discussion, Andrés made the drawing on the board and verbally explained the solution that Alicia's shadow measured 1*m*; without finding arguments to answer the questions of the teacher and the other students; for example, in the following discussion, illustrates Andrés' confusion in this explanation:

Andrés: As this is 4.5 [points to the lamppost] and this is 1.5 [Alicia], then if from here to here it is 3 meters… [he stopped to think].

Teacher: Then how long is the shadow?

Andrés: One,…one meter.

Julio: How can it be one? [Julio was a member of group D].

Andrés: That's what I don't understand, but the next part I does [he explains is a low voice].

Teacher: Let's see, what do you think? Say before Andrés presents the solution to the next questions [some students say that the shadow measures 1*m* and other 1.5*m*].

Andrés: …Okay [insecure, he wants to write his answers to questions 2 and 3; he goes back to his seat].

Given that various students showed that they were not in agreement with Andrés, Victoria (of Group E) goes to the blackboard, draws the figure as stated and constructs a rectangular triangle with sides of 2cm in the vertices of Alicia's shadow on the floor, marking the right angles of the lamppost, from Alicia's feet and in the constructed triangle; the acute angle in the prolongation of the shadow she calls θ . Then she writes what the group has come up with:

$Tan\theta = \frac{2cm}{2cm} = 1, \Rightarrow \theta = 45°$; $Tan\theta = \frac{1.5}{x}$; $x = \frac{1.5}{Tan45°} = 1.5m$; then says:

Victoria: This angle θ measures 45° [points to the angle].

Teacher: Explain, why forty-five?

Victoria: Because this triangle is similar to this one [points to the constructed triangle and Alicia's form]. We measured these sides and used this function, the tangent, and it gave us 45°. With this 45° we came up with another formula [points to the triangle that Alicia forms] that is tangent. The tangent of 45° is equal to…; 1.5 divided by the unknown which gives 1.5*m*.

Still, the members of Groups F and H are not convinced. Those of Group H, unsure, accepted what was wrong because they were confused and they have not completely considered the distance from Alicia to the lamppost along with the shadow. At the invitation of the teacher, Core of Group F goes to the front, makes a drawing and says:

Core: Well, I have these triangles with this angle [marks the angle where the shadow ends], these parallel lines and these triangles [marks the right angles of the lamppost and Alicia's feet], then…4.5*m* is to 1.5*m* [speaks and writes in schematic form the rule of three] as 3*m* is to *x*; then… $x = 1m$.

Teacher: What do you think?

Class: Wrong [The voices of Andrés (Group H) and Rubí (Group D) are distinguished].

Teacher: Why? Come up [to Group D]. Look people, there are divided opinions, some say that Alicia's shadow measures 1m and other say that the shadow measures 1.5m [Rubí goes to the board]. Why is Group F's solution wrong?

Rubí: Ah, well [he explains without writing] because they said that 3m was x…but it can't be 3 meters because they lacked this piece [points to the shadow in Core's drawing].

Teacher: Then explain to us your solution here in this part of the board.

Rubí: 4.5*m* is to 1.5*m* [speaks and writes. Assumes from the beginning that the triangles are similar without mentioning this explicitly] as 3*m* plus *x* is to *x*, because you have to include all of this part because it is not here where Alicia's shadow ends but here. Then we have x, but first we do the division and it gives that 3 is equal to 3*m* plus *x* over *x*…, we have *x* equal to 3 meters divided by 2. That is 1.5*m*.

Teacher: What do you think?

Class: Very good! [Applause].

At this time Groups B and H are convinced that their solution was wrong because they had not considered the side of the triangle that was formed by the lamppost; in particular, Andrés and Karla said that Alicia's shadow is not 1*m* but measures 1.5*m*. "I said it wasn't 1" commented Julio (Group D).

When Group D presented their solution to question 2 and the class seemed to agree on what this group presented, Rubí also changed the distance from Alicia to the lamppost and noted that Alicia's shadow was half the distance to the lamppost, then drew a line that passed at the middle of the distance to the lamppost, then drew a line that passed through the points (1,0.5), (4, 2), etc., which they got making the substitution in the proportion $\frac{4.5m}{1.5m} = \frac{3m + x}{x}$; in place of 3*m* they put x ("it varies"), the shadow they now call *y* in place of *x*, to obtain $3y = x + y$; $\Rightarrow 2y = x$; finally they wrote their formula $y = \frac{x}{2}$ Joel, another member of the group, asked to go to the front and explain the solution to question 3; over the written portion on the board for question 1, he changed 1.5*m* for 2*m*, and wrote $\frac{4.5m}{2m} = 2.25 = \frac{3m + x}{x}$ and said that Simon's shadow is $x =$ 4.2 *m*. After drawing a line for Simon and writing the equation $y = \frac{x}{1.25}$ (the same procedure as in question 2, the distance from Simon to the lamppost he called *x* in place of 3*m* and the shadow he called *y*).

It seemed that the form in which Group D made their presentation, clear and orderly, contributed to unify the criteria of the other groups, as their arguments were accepted; also, from the work in small groups, the Groups C, D, E and G coincided in the answers given for the questions.

Discussion of the Results

During the presentation of the groups it was appreciated that the students used arguments of proportionality or established proportions without justifying them with precise reasoning. Although some students' approaches showed serious inconsistencies, the students had an idea of how to identify similar triangles even though they did not provide arguments as to why this relation of similarity can be established in a determined pair of triangles and from this established the proportionality between the corresponding sides. That is to say that they showed difficulties in the use of appropriate language. As a result of the interaction, some of the groups reaffirmed and defended their ideas and others modified theirs, as was the case with Groups B and H after the presentation of Group D.

With the presentations, in fact, the collective discussions began, and the students and the teacher questioned affirmations, made corrections or asked for clarification from those who were presenting. Many of these interventions came when the students explained their reasoning that brought them to consider certain relationships or the use of representations.

The class discussion was beneficial for the advances of the students, for example, Group D, after several questions, clearly posed in the proportion that solved question 1, the distance between Alicia and the lamppost is a fixed quantity (3*m*) but in questions 2 and 3 the distance varies, in the proportion they wrote *x* in place of 3*m* and they denoted the shadow *y*, whether it was Alicia or Simon. That is to say that they made the change in the designation of the variables to express the relationship proportional in the typical notation of the function. Here there was an evolution in mathematical understanding of the problem on the part of the members of this group; we think that this process could have contributed to the understanding of the students of the other groups.

Remarks

Two important aspects became relevant during the development of this work:

i) The importance of designing or reformulating activities in which the students have the opportunity to utilize previously studied mathematic resources and the process of solution demands from them the extension or consideration of new resources or concepts for the solution of problems. Here one must identify the mathematical potential of the activity before using it in the classroom. In particular, it was interesting to project the distinct methods of solution.

ii) The implementation of the activity in the instruction must consider the active participation of the students in the distinct phases of solution. In particular we recommend that initially the students work in small groups of three; afterwards each group should present their attempts at solution to the whole class. In such a way the group that is making the presentation has the opportunity to defend their methods of solution and the other students, along with the teacher can formulate questions and ask for explanations that help them understand and justify what they have presented. In particular the public presentations were a forum for discussing points related to the use of certain relationships and the necessity to justify the work in each of the groups.

In general, during the work the students on this group of problems they experienced difficulties as much in the use of the language as in the use of the resources to pose and communicate their ideas, but the form of instruction permitted a refinement of their ideas in their approximations to the problems, which permitted them to get ever closer to the solution.

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TAKING A CLOSER LOOK AT THE COLLECTIVE

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Theoretical Frame and Purpose of Research

There is a growing interest in mathematics education to establish more complex ways of understanding collectives as learning systems in the classroom (Begg, 2001; Cobb, 1999; Davis & Simmt, 2003; Kieren, 2003; Sfard & Kieran, 2001). From an ecological perspective, collectives are emergent systems that arise through the mathematical interactions of two or more individuals. It is important to understand that while the collective is distinguished as a larger cognitive system which possesses self-similar qualities to that of the individuals who come to form the pair or group, it cannot be taken to be just a 'collection' of individual agents but rather, recognized as a cohesive entity in and of itself (Bateson, 1972; Bowers, 1997; Capra, 1996; Davis & Simmt, 2003). In the same manner that an individual's understanding (Pirie & Kieren, 1989, 1994) or "structure" (Maturana and Varela, 1987) shapes his or her mathematical knowing and actions, it is the coherence within the structure of the collective that enables it too to exist and function mathematically as a unity. Moreover, given the fact that students work as individual learners *and* members of collectives in the mathematics classroom (Davis, 1996; Davis, Sumara, & Luce-Kapler, 2000), these systems are naturally interconnected and thus, interdependent. They are co-existing *and* co-evolving cognitive systems.

Research situated within an ecological perspective such as this not only necessitates the inquiry into the sense-making that takes place at an individual level but also, an examination of the mathematical ways in which learners become collectives and, how participation within these two realms affect and are affected by each other. More specifically, three key implications for how mathematical learning is conceptualized arise as a result of framing the collective within the discussed theoretical assumptions. These provided the basis for this study. They are as follows: i) identification of the mathematics that emerge from students' collective actions, ii) assessment of the mathematical structure(s) within the collective(s), and iii) the coherence and impact of the structure(s) on the students' individual *and* collective mathematical functioning.

Data Source

This research examined the collective mathematics of three fifth grade children. Allan (male, 10 years), Veronique (female, 9 years), and William (male, 10 years) all spoke English as their first language and were considered by their teachers to be meeting the provincial curriculum expectations for their grade level in mathematics (Ministry of Education, 1995). The students were given a nonroutine (Gonzales, 1994; Lesh, 1979, 1981; Papert, 1972; Saari, 1977) mathematical task involving a 3-D multilink cube pyramid. They were presented with three "cross pyramids" (see Figure 1) and asked to determine how many cubes were in the first pyramid and how many *more* cubes were required to construct the second through eighth cross pyramid. After the introduction of the first three pyramids and tasks, the students were left on their own to solve for the remaining five challenges.

The primary data for this study included the 30 minute videotape which documented the students' work on the pyramid tasks. Videotaping the session enabled the students to work on their own without the presence of the researcher. As well, the videotape provided a permanent account of the students' verbal, written, and physical mathematical (inter)actions. The data also included the 3-D cube models and all written records produced by the group.

Methods and Foci of Analysis

The first stage of analysis began by viewing the videotape several times from start to finish and relevant events in terms of the children's collective mathematics were identified. "Collective" mathematics was defined prior to analysis by the author as the mathematics brought forth through the collaboration of two or more students during the cross pyramid tasks. In contrast to what was assumed to be "individual" mathematics; that is, concepts or skills brought to the group's work by way of a student's mathematical actions that were *not* perceived by the collective unity as part of their current understanding, collective mathematics was recognized as that which unfolded *during* student interaction. Evoked by the enactions of a student or students, the mathematics then became part of the partners' or group's present and/or future mathematics. Thus, any new collective mathematics could be located in previous videotaped episodes of the students' work.

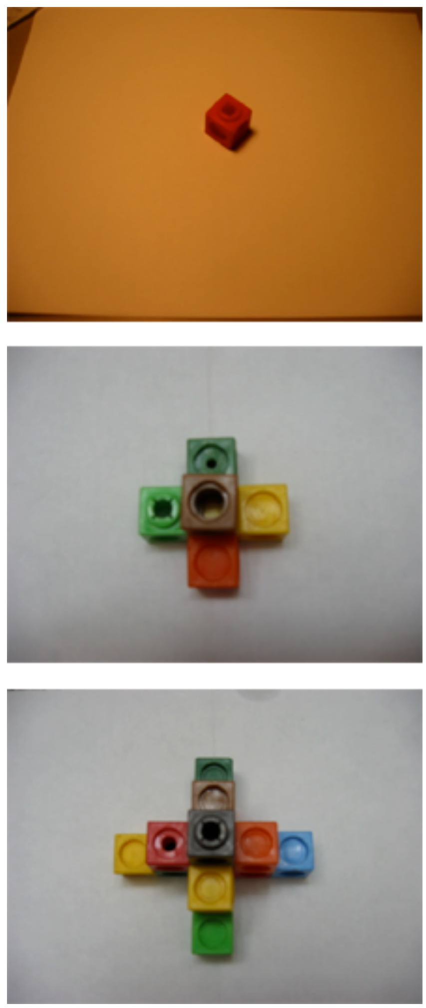

Figure 1. Digital photographs taken from a "bird's eye view" of the first, second, and third 3-D cube cross pyramids that were presented to the children in the beginning of the session.

Once the students' collective forms of mathematics were identified, a second stage of the video analysis was conducted to determine the actual "structure(s)" inherent in each of them. Given the nature of the cross pyramid tasks, this involved explicating the spatial and/or numerical underpinnings (Battista, 1994, 1999; Battista and Clements, 1996; Hirstein, 1981; Piaget & Inhelder, 1967; von Glasersfeld, 1982, 1991) that gave rise to the students' mathematics.

It was during this stage of the analysis regarding the examination of the students' conceptual structures when it became apparent that the children's collective ways of being mathematical were not uniform in nature but existed in three distinct forms. In addition to the mathematical structure(s) within the collectives, what enabled the students to function collectively was observed as being dependent on the *kind* of understanding established between the partner or group. The three forms were identified as: *adopted, intersecting*, and *integrated* understandings. Another structural layer within the collective then, these different forms of understandings evidenced in the students' building, drawing, numbering, gesturing, and verbal (inter)actions too defined how Allan, Veronique, and William came to be collaborative unities in solving for the pyramid tasks.

The third stage of analysis centred on the "impact" that each of the collective structures had on the individuals and the partner or group itself. Examination focused on explicating key aspects regarding the structure(s) within the collective that enabled or disabled the students to work as a coherent unity.

By working exclusively with the videotape for the analysis and by comparing conjectures against student artifacts and/or theoretical literature, identification of the children's collective mathematics, the mathematical and conceptual structures that occasioned the collectives to occur, and the impact that each structure had on the children's ability to function as a cohesive collective were verified (Pirie, 1996, 1997). It was only in the final stages of the analysis that transcripts of the videotape were made. These vignettes serve as illustrative examples of this study's foci. Integral to the transcripts was the recording of the students' physical actions, verbalizations, and the intonations of their voices to ensure that the episodes were portrayed as accurately as possible on paper (Clarke, 1998).]

Results

The findings of this study highlight how the collectives that were created by Allan, Veronique, and William arose as dynamic systems which co-existed and co-evolved in relation to the students' individual ways of bringing forth mathematics. The discussion is organized around the three kinds of understandings that characterized the children's collective mathematics. In each of the sections, there is a brief description of the distinguishing aspect(s) of the particular understanding and illustrative examples from the video are provided. Following this is a discussion of the mathematical structures embedded in the students' collaborative work as well as how their mathematical structure(s) together with their structures of understanding impacted the partner or group's ability to function as a cohesive collective system.

Adopted Understandings

Adopted understandings were conceptualizations brought forth by a student or students through verbal and/or physical actions that became the way of thinking for the other learner(s).

This kind of understanding was first observed when the students set to work on the third pyramid. William carried forth his previous spatial image[1] of the second pyramid's base as a *ring of four cubes attached around a middle cube* and counting method (see Figure 2) when Allan joined him and they arrived at a total of nine more cubes together (see Figure 3).

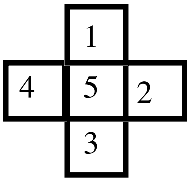

Figure 2. Diagram of the bottom of the second pyramid and William's counting strategy.

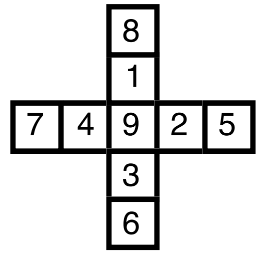

Figure 3. Diagram of the bottom of the third pyramid and William and Allan's counting strategy.

Here, William's spatializing and numbering became Allan's ways of thinking through the task and it was these coherences that enabled the two students to work together to determine the number of additional cubes required to build the third cross pyramid.

A second instance of an adopted understanding emerged during Veronique and Allan's conversation about the fourth pyramid (see also Figure 4):

Veronique: There's one here *[pointing to one of the single cubes attached to the middle cube]*
and two here. *[pointing to one of the sets of two cubes attached to the middle cube]*
So that equals three. *[cubes of four rods attached to the middle cube]*
So three times three times three times three. *(sic)*
Allan: Twelve *[takes her use of 'times' as meaning '+']*.
Veronique: Exactly! Plus one equals thirteen.

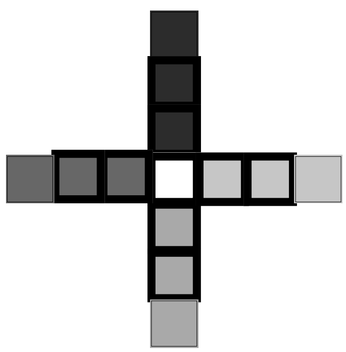

Figure 4. The visualization of the bottom of the fourth pyramid initiated by Veronique.

Veronique's perception of the pyramid's base as *four rods of three cubes each around a middle cube* and her number image of *four groups of three add one* became Allan's structure for thinking. Through repeated addition, the students determined that the fourth pyramid would have a layer of thirteen cubes. The fact that Veronique verbalized the operation of multiplication as "times" and Allan did *not* multiply the values but added the groups of three together, answered "twelve", and she exclaimed, "exactly!" highlight how the students were thinking collectively as one unity.

Intersecting Understandings

Intersecting understandings occurred when the students were solving for a task and used the same idea or mathematics but were *not* making the same individual sense. Unaware of the differences in their underpinning spatial or numerical structures, the children were able to work collaboratively for a certain period of time. The intersection of their understandings enabled them to function together but only temporarily as the collective eventually broke down due to the *in*coherence in their mathematics.

For example, Allan and Veronique took their conjecture of the fourth pyramid as requiring thirteen more cubes and proceeded to physically construct the bottom layer of cubes together. What previously existed as an adopted form of understanding; that is, Allan's adoption of Veronique's image of 'four rods of three cubes add one' had now developed into two different but intersecting understandings. This was confirmed in the following episode which captures how the two students moved on from building the fourth layer of the cross pyramid to generating a calculation for it and soon realized that they were not thinking with the same spatial and numerical structures but very different ones (see also Figures 5 and 6).

Allan: So four times four times four... *(sic)*
Veronique: *[focusing on the answer]* Is thirteen.
Allan: No, four times four times four 'cause the last one..., four times four times four...
Veronique: Is thirteen. *[louder voice]*
Allan: No, no, no, four times four times four because everything has four now.

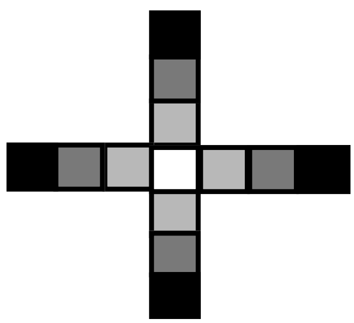

Figure 5. Diagram of the bottom of the fourth cross pyramid and Allan's visualization of three groups of four cubes each.

Veronique: This one only has *three* here. *[pointing to one of the sets of three cubes attached to the middle cube]*
So that's the one which is twelve. *[moving her finger around the around the outside of the base of the pyramid]*
Twelve, plus one *[i.e., the middle cube]* equals thirteen.

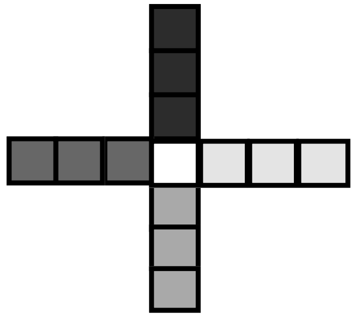

Figure 6. Diagram of the fourth cross pyramid's bottom and Veronique's visualization of the arrangement of the cubes as four groups of three cubes each.

It is clear from Allan's explanation that he saw the cubes in the pyramid's base as being organized into '*three' rings of 'four' cubes each*. For Veronique however, 'four' signified the *number of rods of cubes* while 'three' symbolized the *number of cubes in each rod*. Allan and Veronique's "taken as shared" (Cobb, Yackel, & Wood, 1992) visualization of the pyramid's growth as the addition of a horizontal bottom layer of thirteen cubes can be seen as the overlapping characteristic in their understandings that initially allowed the students to build the cube base together. And, it is their complementary images of 'three' and 'four' that do *not* allow them to generate one mathematical expression for the fourth pyramid.

It is also possible that the group's numerical solutions for the fifth and sixth pyramids could have been episodes of intersecting understandings. Although William and Allan agreed that Veronique's solutions for the fifth pyramid, "four times four times four times four plus one equals seventeen" and the sixth pyramid, "five times four plus one equals twenty-one" were correct, it would be presumptuous to conclude that the three students had structured their thinking for the two expressions around the same spatial image(s). The children did not elaborate on the significance of the values within the calculations and because of this, it is not clear whether they were in fact, making the same sense at an individual level. Even if each student was not conceptualizing the growth of the fifth cross pyramid in the same manner but in terms of either strictly numbers or spatially as rods, rings, or even outer layers[2], the calculation of "four times four times four times four plus one equals seventeen" (sic) (i.e., 4+4+4+4+1=17) fits all of these different images. It is also unclear whether Veronique was thinking of the sixth pyramid as rings of four cubes or had simply extended her last calculation of 4+4+4+4+1=13 by adding on another "4" hence, 5x4+1=21. But because of the shift in her pattern from 'four times a number' to 'a number times four', it accommodates for the ring or outer layer images with which Allan and William might have structured their thinking. Given the fact that the group later debated over the spatial significance of the values in calculation of the seventh and then the fourth through sixth cross pyramids suggests that there did exist a discrepancy not only in their present but also in their previous understandings.

<u>Integrated Understandings</u>

Unlike intersecting understandings but similar to that of adopted forms, integrated understandings were evidenced as the structural coherence established between the children's collective and individual actions. However, instead of one conceptualization becoming that of the other learner(s), this kind of understanding was distinguished as the coming together of the children's different spatial and numerical ways of making sense of the pyramid and the generation of new mathematics which extended beyond any one of the individual members (Davis and Simmt, 2003).

An episode in which integrated understandings emerged occurred immediately after Allan, Veronique, and William's unsuccessful attempts to make numerical or spatial sense out of their

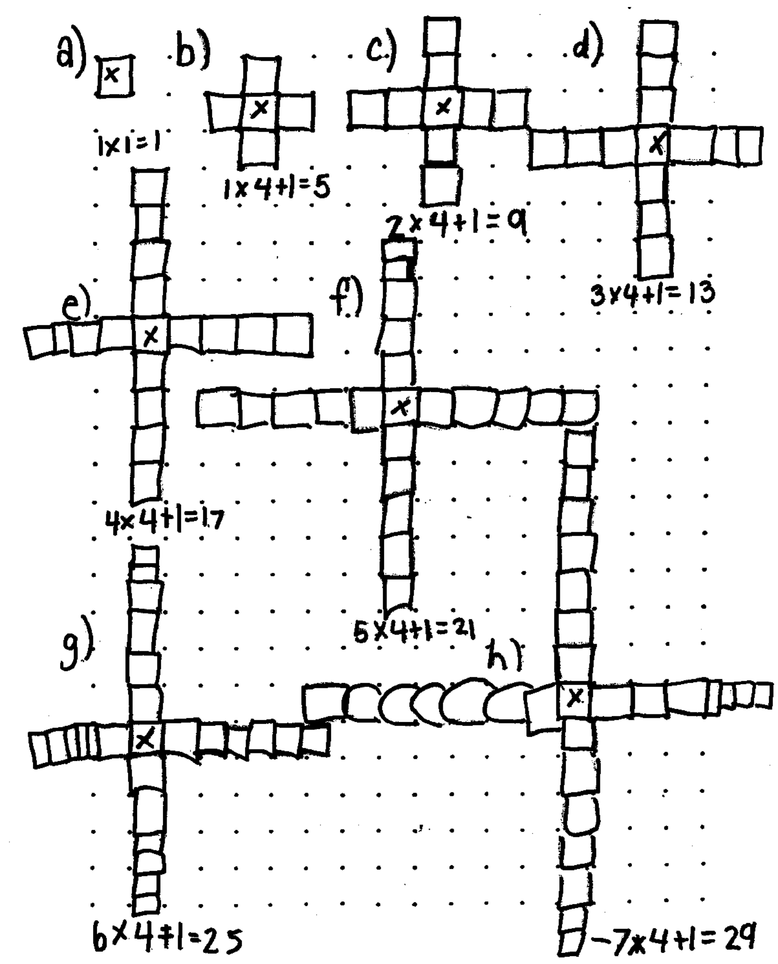

Figure 7. The group's 2-D diagrams of the bottom layers of the eight cross pyramids and the corresponding calculations.

calculations for the fourth through seventh cross pyramids. The confusion that resulted prompted the students to draw the bases of the eight pyramids on dot and chart paper (see Figure 6). The 2-D diagrams not only provided a consistent image with which the group could think but their drawings also served to occasion the students' development of a numerical pattern for *all* eight pyramids.

Veronique who visualized the pyramid as horizontal layers of *four rods of cubes around a middle rod* continued to enact this in her drawings of the pyramid bases. She did this in the same manner each time: First, by drawing one square and then around each of its four sides, she drew rods of squares. In contrast to this however, she no longer calculated the bases as 'four times a number plus one' but in a manner that expressed 'a number times four plus one'.

William who was observed enacting images of *outer layers* and *rings of cubes around a middle cube* through his methods of building and counting, now was recording more formal calculations (e.g., 2x4+1=9, 3x4+1=13, 4x4+1=17, etc.) in keeping with these. Secondly, his drawing actions for the pyramid bases were *not* that of ring-like motions but rather, rod-like ones that were identical to Veronique's.

Lastly, Allan whose thinking was seen to shift between images of rings and rods and who enumerated the fourth pyramid as 'three groups of four add one', looked at Veronique's diagram of the fourth pyramid and revealed his integrated understanding of it. Allan described the base as being "three squares, three squares, three squares, three squares" and then translated this for the group into "three times four plus one equals thirteen". One might wonder if Allan was *not* structuring his thinking around the notion of rings but with the visualization of three squares, four times. However, when Allan moved on to check his calculation by skip counting, "four, eight, twelve... thirteen", he confirmed that he was in fact, also thinking with the image of 'three groups of four'.

These coherences between the children's collective and individual actions demonstrate that their sense-making of the cross pyramid was not a piecing together of individual structures of understandings but a fluid comprehension that effectively integrated spatial and numerical images as well as that which gave rise to a generalized pattern for all eight pyramids.

Conclusion

The collective mathematics brought forth by these three children's problem solving of the cross pyramid proved to be more complex than upon first glance. Not only was there diversity in what evolved in terms of the students' spatial and numerical structuring of the cross pyramids but there were also variations in the kinds of "taken as shared" (Cobb, Yackel, & Wood, 1992) structures of understandings that existed among the members within the collectives. These characteristics inherent in the students' partner and group generated work were seen to play key roles in their ability to think, act, and exist mathematically as cohesive collective entities. By investigating the emergent internal structures that defined Allan, Veronique, and William's collaborative mathematics, this research substantiates the notion that it in order to better understand the dynamics of children's mathematical learning, it is necessary for research to be a reflexive examination of both the similarities and the distinguishing qualities of individual and collective mathematics which take place in the classroom.

Notes

1. "Image" (Pirie & Kieren, 1994) encompasses physical, verbal, and mental representations.
2. Earlier in the session, William had been building a fourth layer by attaching one cube onto each of the exposed faces of the top and sides of the third pyramid.

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SIMPLE RATE VERBAL PROBLEMS IN MATHEMATICS TEXTBOOKS FOR PRIMARY EDUCATION IN MEXICO: A THEORETICAL AND EMPIRICAL STUDY

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The theoretical constructs supporting this study were developed by Vergnaud (1991), Bednarz and Janvier (1994) and Guzmán *et al*. (1999).

We used Bednarz and Janvier's "theoretical tool", which allows for a classification of verbal problems employed in the teaching of arithmetic and algebra. Bednarz and Janvier (*ibid*) separated the mathematical-relational structure from the context elements. In other words, they identified the nature of quantities (known and unknown), the relations among them and the structure of such relations. Bednarz and Janvier (*ibid*) detected *rate* verbal problems; they classified the different classes of *rate* (Guzmán *et al.,* 1999) and found that in general the verbal problems for unknown *rate* are more difficult for the students to solve than those with known *rate* (Guzmán *et al.,* 1999).

We used Vergnaud's research study of multiplication verbal problems which are implicated by a four quantity relationship: two quantities of a certain type (for example, piles, Figure 1), and another pair of another type (for example, oranges, Figure 1). We employed his corresponding tables (Figure 1), and hierarchy of complexity of the verbal problems in terms of the operation involved. This researcher considered that the easiest for the students to solve were those verbal problems that could be approached by way of multiplication, the next being those that require division, and lastly, those that necessitate the use of the Rule of Three. Different grades of difficulty were established for the problems whose solution require division: there are simple problems, if one knows the connection of correspondence between two naturally different magnitudes, while there are difficult or complex problems whose unit value is given and it is necessary to find the number of unities of the first type that correspond to a given magnitude of the second type (problem 1).

In this study, the *simple rate* verbal problems are those that involve a relationship of comparison between two non-homogeneous quantities (Figure 2). For example:

Problem 1: Mr. Fermín filled one of the baskets with 325 oranges. How many piles of 5 oranges did he put into the basket? (Avila *et al.,* 1995, p. 28)

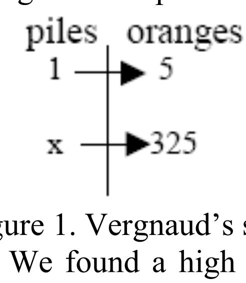

Figure 1. Vergnaud's scheme

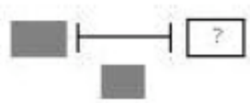

Figure 2. Bednarz and Janvier's scheme

We found a high percentage of *simple rate* verbal problems in mathematics textbooks for primary education in Mexico, they were characterized by a known *rate* (Vargas and Guzmán, 2000). We designed a questionnaire with six of these verbal problems taken from different levels. We took care that the statements contained only the necessary information sufficient for solving the problems, and we also took into account the known or unknown of the *rate*.

The participants in the exploratory test were a full class of fifth grade student in a public primary school. We gave them the written test, and thereafter selected students to interview.

We established conjectures related to the difficulty of the *simple rate* verbal problems in the questionnaire and classified these from Vergnaud's perspective: according to the operation involved and whether the useful value was given or not. Also, we classified them in terms of known or unknown for the *rate* involved (Guzmán *et al.,* 1999).

The number of the students' correct responses in the questionnaire reveled that the percentage of success obtained in solving the verbal problems coincided with the proposal of Guzmán *et al*. (*ibid*). However, our findings did not agree with Vergnaud's studies (1991).

Analysis of the students' responses showed various procedures and difficulties. There were those students who employed pictorial representations, others made mental calculations (did not register their operations) when the verbal problem contained small quantities. Various students presented as a solution a part of the information given for the verbal problems; others did not check their results, and those that did try to see if the selected operation was correct worked with trial and error. Some students met with difficulty in understanding their own numerical graphs. Occasionally their processes of multiplication and division were incorrect, wherein they used quantities that contained a unit or there were various zeros, demonstrating deficiencies in the adequate use of the algorithm of division. One notable difficulty was the fact the students did not always understand the statement of these verbal problems.

In conclusion, given that the *simple rate* verbal problems are a representative class of verbal problems identified in the analyzed textbooks, we wished to find more familiarity for the students towards them, from the method of approaching the verbal problems to having more success in the process of solution.

If it is true that the type of operations involved in the *simple rate* verbal problems has an influence on the difficulty of their solution for the students, it is not easily determined from only this criteria. To attempt to understand this we must consider the known or unknown of the *rate*, along with the quantities involved.

As to the theories employed as the theoretical framework of this study, we can say that these were important tools, as they permitted us to classify *a priori* the *simple rate* verbal problems and helped us to clarify the difficulties that the students had in the solution of these verbal problems.

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PROBLEM POSING AS A PEDAGOGICAL TOOL: A TEACHER'S PERSPECTIVE

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This exploratory study examined the use of student problem posing to support the learning of mathematics content within a college level mathematics course (precalculus) with a widely-accepted curriculum and a traditional text. A broad definition for problem posing was utilized, and the focus was the teacher's perspective concerning student problem posing as a pedagogical strategy. To facilitate an intimate knowledge of the teacher's thinking, the researcher was the teacher in the study. The presentation will provide an overview of the study including: motivation, theoretical framework, methodology, some results, and avenues for future research.

Mathematics education literature advocates student problem posing, yet it is rarely used in the classroom. Few research studies have investigated mathematical problem posing within instructional settings. At the university level, mathematics problem-posing pedagogy literature has almost exclusively been set in nontraditional course contexts, and it has focused on problem-solving and problem-posing skills rather than on the acquisition of content knowledge. Thus, this study examined student problem posing in a traditional content-focused course.

The theoretical framework for the study consisted of a "quasi"-grounded theory approach informed by Stoyanova's (1998) problem-posing framework, with the study situated in a modified version of Simon's (1995) constructivist teaching cycle. A Vygotskian socio-cultural perspective and the emergent perspective were employed to describe the influence of the traditional text and the teacher-researcher's views of learning respectively. Focused on the instructor's perspective, research questions investigated the following: (a) means of utilizing student problem posing to support the learning of mathematics, (b) roles and purposes for student problem posing in mathematics instruction, (c) the interplay between instructional situations and the nature of problem-posing tasks, and (d) aspects of instructor decision making in the use of student problem posing. Data sources included the instructor's reflective journal, instructor planning notes, problem-posing tasks, samples of student work illustrating instructor thinking, and transcripts of classroom video footage.

In data analysis, purpose, context, task, and teacher considerations were dealt with in an integrated manner. Problem-posing implementation and purposes differed from those described in practitioner literature. Purposes often were content focused, but also included providing students with "true" problem-solving experiences rather than exercises, giving students voices in the direction of instruction, and providing opportunities for students to reflect on their own thinking and understanding.

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THE RELATIONSHIP BETWEEN BILINGUAL STUDENTS' LANGUAGE SWITCHING AND THEIR GROWTH OF MATHEMATICAL UNDERSTANDING

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Rationale and Objective of Study

The notions of "language switching" and "growth of mathematical understanding" are of particular interest in our interdisciplinary research practices. Both have currently received much attention in our educational community (Qi, 1998; Martin *et al*., 2000). This paper will examine their relationship in the context of middle-school students in Tonga. The students' first language is "Tongan", and their second language, and the language of instruction at the middle and high school levels in Tonga, is English. Based on the work of Fasi (1999), there are two major reasons why this investigation is significant. Firstly, there is a rarity of any studies carried out in Tonga. Secondly, the middle school level in the Tongan school system is a critical period as their bilingual students make the transition from instruction mostly in their native language at the elementary school level, to mostly English at the middle and high school level.

In 1993, NCTM addressed and recognised the roles of language and culture of the minority people by publishing a series of manuscripts to help "all readers develop a deeper understanding of, become more sensitive to, and stimulate a desire to learn more about Asian and Pacific Island students and their unique characteristics" (Edwards, 1999, p. vi). In addition, NCTM in the *Curriculum and Evaluation Standards for School Mathematics* (1989) addressed the same interest on problems and the issues concerning such bilingual students' learning of mathematics by announcing that "students whose primary language is not the language of instruction have *unique* needs" (p.142). This paper is therefore set on the premises that we can transform our knowledge of *how* the bilingual students learn and understand mathematics (*through* both languages) to how teachers, educators and researchers might shape the art of teaching and instructing of mathematics.

Theoretical Framework

The theoretical framework for this study is the Pirie-Kieren Dynamical Theory for the Growth of Mathematical Understanding. This theory offers a *language* and way of observing the growth of mathematical understanding of a specific topic, by a specific person, over time. Pirie and Kieren (1991) explicitly state that "growth of understanding" occurs through a continuing movement back and forth between different layers or modes of understanding to re-member and to reconstruct new understanding, based on the learner's current and previous knowledge. It is the dynamical back-and-forth flow between the layers which characterises the uniqueness of the theory and defines what is meant by growth of understanding (Pirie & Kieren, 1994). The categorisation of the bilingual students' language switching within the data follows Glaser and Strauss' (1967) "constant comparative method". The purpose here was to allow categories of "sources", "forms" and "outcomes" of language switching to emerge from the analysis and so be grounded in the data.

Method and Tasks

In investigating the research question, a set of related tasks were developed to explore the growth of understanding of 'patterns and relations'. Each task consisted of either a pictorial sequence involving diagrams made up of square blocks or a sequence made up of cubes, both of which were accompanied by a set of related questions. Between September 2001 and October 2002, two separate but similar studies were conducted using these tasks in four different high schools in Tonga (Manu, 2002). A total of 30 middle school bilingual students (equivalent of grades 7-9) were asked to solve the mathematical tasks in pairs or groups of three. As relatively little was known about the way language switching affected or enhanced bilingual students' growth of mathematical understanding, the methodological approach was interpretive and based on in-depth qualitative case studies and video-recordings. The videotapes form the main data of the study, although they are supplemented with audiotapes, field notes, written work of the students, together with "video-stimulated recall" – a follow-up session with the students to comment on their mathematical working and thinking – to clarify and to elaborate what was observed.

Forms of Language Switching

Verbal mixing – the *integration* of words (or phrases) between the two languages – is categorised into *verbal substitution* and *verbal borrowing*. In 'verbal substitution', the substituted or 'embedded' words have been *built-in* and *stored* in the bilingual students' primitive knowing. The kinds of words involved in this 'replacement' process are further classified into two types, *equivalent* and *Tonganising words*. Tonganising words are particularly special in substitution, not because they are as interchangeable in nature as the equivalent words, but because they were borrowed through 'transliteration'. Because both of these kinds of substituted words are *rooted* in the bilingual students' primitive knowing, they become part of the bilingual individuals' existing knowledge in both language and mathematics and therefore are easily accessible when involved in group discussion or mathematical discourse.

Unlike substitution, 'verbal borrowing' denotes the process in which words, say, from the second language, act as 'stand-in' words within the discourse of the bilingual students in their first language. The kinds of words involved in this 'loaning' process are classified according to various degrees defined by the 'sources' into three types: non-equivalent words, keywords from the task or borrowed words from a peer. The non-equivalent words, mostly from the second language (English), do not have direct translation equivalent words in the bilingual students' first language. Keywording demonstrates the ability of the bilingual students to *attend* to specific characteristic of the language or particular image(s).

Verbal grouping – the alternation between clauses or groups of words – is classified into two main forms: "verbal translation" and "verbal shifting". Verbal translation is characterised by a change in the mode of external re-presentation of the given information. The crucial characteristic of verbal translation is that it usually involves *no additional new information*. A "direct" translation is seen as a *one-to-one* correspondence between two languages. An "indirect" translation on the other hand can be seen as a *many-to-one* correspondence between two languages. However, the addition of new information is often characterised when a bilingual student is said to be "shifting" in his or her language or verbal mode of representation. 'Verbal shifting' often reflects the competency of the bilingual individuals both languages, and like verbal translation, it implies a different cognitive function in both the internalisation and externalisation processing.

Findings and Discussion

The outcomes of language switching in the growth of mathematical understanding is determined by the nature of three interrelated components: (i) The actual 'form' of the verbal switch a bilingual student is engaged in; these forms were identified as *verbal substitution, verbal borrowing, verbal translation* and *verbal shifting*; (ii) The 'mathematical content' which includes the mathematical properties, images, meanings and concepts that are situated in each form; and (iii) The 'usefulness' of the content in how 'applicable' or 'appropriate' it is to the task at hand. In other words, the mathematical content of the switch must also be *purposeful* to the task. The effectiveness of language switching in the growth of mathematical understanding relies on the individual's internal *awareness*. This internal awareness reflects the bilingual individual's attentiveness to his or her *available resources*, which include his or her primitive knowing and existing knowledge, the *limitations* and *obstacles* of his or her present mathematical understanding and of what kind of verbal and nonverbal actions must be undertaken to use or extend them, and a *purposeful actions* in solving the task for such acts to be useful.

Verbal translation functions not only in decoding the given information into the preferred language but also in determining what information – that is, meaning, concepts and images – are *re-presented* internally and externally in a particular language. The bilingual individual's growth of mathematical understanding is therefore observed to be *dependent on* and *determined by* his or her own verbal translation or re-presentation of the given information. *Verbal shifting* not only *confirms* the existing understandings but also *informs* the mathematical activity in a particular re-presentation (verbal or nonverbal). The effect of any 'substituted' or 'borrowed' word lies in how it is *conceptualised* by the bilingual individual. This conceptualisation is either 'validated' by the use of particular words to *confirm, inform* and *elaborate* on one's existing and current understanding, or 'evoked' by *activating* specific images within a particular layer of understanding including one's primitive knowing. There exist a 'gap' between spontaneous recognition in borrowing and conceptual understanding in substitution. My proposed model of "Shared Underlying Conceptualisation" attempts to bridge this gap.

Conclusion

Halliday (1978) suggests that since languages "differ in their meanings, and in their structure and vocabulary, they may also differ in their paths towards mathematics, and in the ways in which mathematical concepts can most effectively be taught" (p. 204). As a result, bilingual students appears to pay attention to different characteristic of a particular language and further research could be done to explore if ESL students create new pathways towards understanding mathematics. Thus, the knowledge of these interdisciplinary findings will further enforced "a deeper and better understanding of the psychological aspects of teaching and learning mathematics" of bilingual students' ways of understanding mathematics as imposed by the goals set-out by PME-NA.

INTERACTING LEVELS OF DESIGN: STUDENTS AND TEACHERS REFLECTING ABOUT MATHEMATICAL MODELING

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This poster presents results from an ongoing multi-tier design experiment (Lesh, 2002) that investigates the parallel development of:

1. Middle-school teachers' understandings of what constitutes "better" teaching of mathematical modeling, and
2. Middle-school students' understandings of what constitutes "better" mathematical models and modeling

In this case, the primary focus is on the interaction that occurred between the two tiers of development during one particularly salient design iteration.

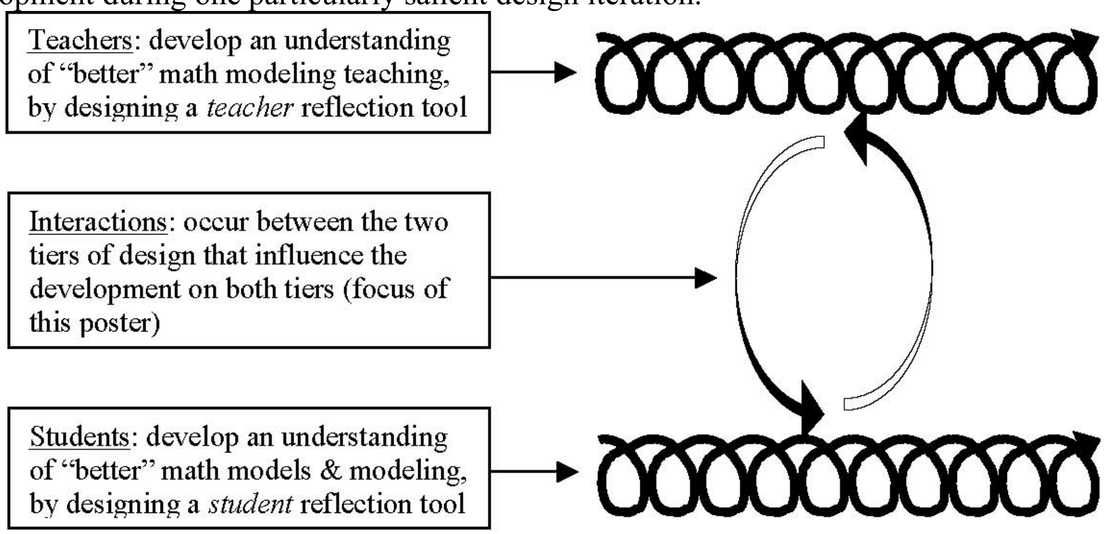

Figure 1: Interactions between the teacher and student design tiers

The interaction occurred when the teachers tested out their first conceptions of what constitutes good teaching of mathematical modeling, while their students worked in groups on Model Eliciting Activities – complex mathematical problem-solving activities that are simulations of real-life modeling situations (Lesh et al, 2000). This involved advising students *when* to use certain problem solving heuristics and metacognitive strategies, as the teachers saw fit. This poster will discuss how the students' reactions to their teachers' efforts influenced the teachers' subsequent design revisions. And conversely, it will show how the teachers' suggestions influenced students' revisions of what good mathematical models and modeling involves.

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DESIGNING MATHEMATICAL MODELING ACTIVITIES FOR FIRST-YEAR ENGINEERING

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Modeling activities were designed for a first-year engineering course in problem solving and mathematical tools (Excel and MATLAB). The activities were designed via collaboration between engineering and education faculty and graduate students (Capobianco, Zawojewski, & Diefes-Dux, 2004). Through this collaboration, tasks were designed as simulations of real engineering contexts which asked teams of four students to report to a client (often an executive at a company) with a procedure or method for solving an engineering problem. The generation of a procedure meant that students had to explain their mathematical model clearly for someone to follow and the procedure had to apply to similar problem situations. For example, in a task designed for fall 2003, teams of students had to design a procedure for arranging lights around a building such that sidewalks would be illuminated sufficiently and the cost of the lights would be economically feasible. The procedure needed to be reused for other buildings around the campus. As a result of explaining the procedure, the students should articulate the assumptions they make about the problem context as well as revealing the mathematical procedures they employed. In this sense, the activities are thought-revealing (Lesh, Hoover, Hole, Kelly, & Post, 2000) by requiring the externalization of procedures, algorithms, and mathematical thinking within the engineering context.

As task designers, we adopted the principles for designing model-eliciting activities for middle school students (Lesh et al., 2000). However, we had to re-interpret those principles for an engineering context and a first-year context. The driving question then became, how can we design tasks within an authentic engineering context? As related questions, we also had to consider that many students would not have experience with engineering problem solving and content. In addition, there was a limited amount of time within the course to introduce students to a task context. For instance, in the lighting task, the students need to be introduced to basic principles of lighting and the problems that would be inherent in illuminating sidewalks (e.g., obstacles, costs, formulas for calculating illumination by a light at a particular height).

As we have developed a process for task design and analyzed student responses to the lighting task and the other tasks, we have begun to classify the types of methods students employ to solve the tasks. The analysis of responses serves three complementary purposes. First, the analysis improves the task design by revealing whether or not the tasks are eliciting student thinking and providing students a meaningful context. Second, the analysis improves the instruction by providing feedback about instructional methods for the course. Finally, the analysis allows the research community to understand more fully the mathematical thinking of engineering students as well as beginning to understand the problem solving experiences required for high-quality engineering education.

The poster will include examples of the tasks we have designed as well as samples of student responses.

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Rational Numbers

"A FOURTH IS A HALF OF A HALF": CHILDREN'S USE OF RELATIONSHIPS TO COMPARE FRACTIONS

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This study examined the interaction between the use of physical models and children's understanding of fractions as demonstrated through their ability to compare and order fractions. Clinical interviews and in-class observations and were conducted over a three month period with thirteen students from a third, fourth, and fifth grade class. The analysis of the data identified the relationships students attended to when comparing and ordering fractions. Extending Smith's (1995) work, these relationships were grouped into eight perspectives (Limited, Pieces, Part-Whole, Unit Fraction, Within-Fraction, Between-Fraction, Equivalence, and Transform).

The *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics, 2000) states that, "Representing numbers with various physical materials should be a major part of mathematics instruction in the elementary school grades" (p. 33). Yet research has demonstrated that children do not automatically understand the relationship between a concrete model and the underlying mathematical concept (Gravemeijer, 1997; Thompson & Lambdin, 1994), even though these relationships are readily apparent to adults who understand the concept (Behr, Lesh, Thomas R. Post, & Silver, 1983; Gravemeijer, 1997; 2000). Ball(1993) asserted, "We need more theoretical and empirical research on representations in teaching particular mathematical content… We need to map out conceptually and study empirically what students might learn from their interactions with [representations]" (p. 190). Even though there is an emphasis on using models to teach students about fractions, there is limited research that connects the understanding that students develop about fractions with the use of physical models. For example, the Rational Number Project focused on children developing conceptual understanding (Behr, Wachsmuth, Post, & Lesh, 1984; Cramer, Post, & delMas, 2002; Post, Behr, & Lesh, 1986), but did not connect their understanding to specific physical models. We need to understand more about the connections between children's thinking about fractions and their use of physical models before we can determine effective ways to use physical models in the elementary mathematics class. This study examined the relationships students attended to when they were comparing and ordering fractions while using physical models.

Conceptual Framework

The framework for examining children's thinking about fractions when they compare and order fractions was based on Smith's (1995) research. Researchers identified a variety of strategies students use for solving order and equivalence problems (Armstrong & Larson, 1995; Behr et al., 1984; Cramer et al., 2002; D'Ambrosio & Mewborn, 1994; Post et al., 1986), but Smith's list of strategies are the most detailed. Based on these strategies, Smith labeled four perspectives students used for solving comparing and ordering fractions: Parts, Components, Reference Point, and Transform. He explained each perspective was "a distinct and relatively general way to conceptualize and reason with fractions and rational numbers" (Smith, 1995, p. 15). These four perspectives provided an overarching framework for describing how students solved order and equivalence problems and were based both on how students described their solution strategies and their actions in solving problems.

Data Collection and Analysis

To examine how children's understanding of fractions and their use of physical models impacted their ordering and comparing of fractions, I conducted this study in a class where the teacher used Cognitively Guided Instruction (CGI) (Carpenter, Fennema, Franke, Levi, & Empson, 1999) for teaching mathematics. This class provided a rich learning environment for students because the teacher expected them to solve problems using their own strategies, required them to justify their thinking, and encouraged them to construct a relational understanding of mathematics. The teacher facilitated this development by the problems presented and questions posed during class discussions.

Data collection included clinical interviews, classroom observations, and documentation of student work. Thirteen students, consisting of four third-graders, five fourth-graders, and four fifth-graders from the same multi-grade class participated in videotaped clinical interviews prior to and at the conclusion of the unit. Eight students also participated in interviews approximately mid-way through the unit. The interviews included fraction problems with equal sharing, ratios, computation, paper folding, and order and equivalence. I used between-method triangulation by observing students, talking to them about their strategies for solving problems and keeping copies of students' work. On the clinical interview protocols, I included the same problem types with different numbers to allow for within-method triangulation.

The primary focus of the analysis was order and equivalence questions presented symbolically and through story problems during the clinical interviews. I coded the interview data based on how students were focusing on the fractions in the problems by recursively identifying, describing, reviewing, and revising the emerging categories multiple times using the constant comparative method described by Glaser and Strauss (cited in (Erlandson, Harris, Skipper, & Allen, 1993). I named the categories, described them, and provided examples of student work on tables to further clarify my understanding of each category. After I finished coding the data, a mathematics education colleague and I established inter-rater reliability through a process of coding, discussing and recoding.

Results

In my study, the relationships that students attended to in the physical models shaped their developing understanding of fraction concepts. Relationships developed from using physical models, and symbolic representations of fractions were important to students in their process of understanding fractions and learning to solve fraction problems efficiently. In examining the relationships that students used, I found eight perspectives that both overlapped with and diverged from Smith's descriptions. These perspectives, which are summarized in Table 1, are organized from least sophisticated to most sophisticated to some degree. Important relationships that describe the ways that students thought about the fractions as they made judgments about relative size are identified for each perspective. Within each perspective, there were different levels of complexity in students' approaches to solving problems. This provides some information about the variability within a single perspective.

Table 1: Summary of Perspectives for Comparing and Ordering Fractions

Perspectives	Description of the perspective	Key Relationships	Levels of complexity
Limited Perspective	Student does not have an understanding of fractions that allows him/her to answer questions about comparing and ordering fractions.		

Perspectives	Description of the perspective	Key Relationships	Levels of complexity
Pieces Perspective	Focus on fractions as pieces independent of the whole	Size of fraction is seen as an absolute amount Relationship of the piece to the whole is not mentioned May include equivalent relationships between pieces	1) Pieces of a certain size or shape represent specific fractions 2) Recreate fraction by drawing similar to the manipulative 3) Maintain relationships between different-sized pieces
Part-Whole Perspective	Focus on fractions as parts of a whole	Relationship to the whole is always apparent Initial relationships are derived by repeated halving Use recall facts to make equal-sized pieces that maintain relationships	1) Divide whole into the correct number of pieces 2) Try to make equal-sized pieces 3) Use facts to create equal-sized pieces
Unit Fraction Perspective	Focus on unit fractions	Unit fraction is based on how many pieces in whole Iterate the unit fraction to make a composite fraction	1) Only compare unit fractions 2) Extend to non-unit fractions 3) Recognize when insufficient data to make comparisons
Within-Fraction Perspective	Focus on the relationship between the numerator and denominator $\frac{N}{D}$	Numerator is half of the denominator, equals ½ AND Numerator and denominator are equal, equals 1 whole Approximate relationship to compare to 0, 1/2, 1	1) Exact relationship between numerator and denominator 2) Approximate relationship between numerator and denominator
Between-Fraction Perspective	Focus on the relationship between numerators and/or denominators $\frac{N \rightarrow N}{D \rightarrow D}$	Identification of relationship between like terms (i.e. numerators) Relationships maybe additive or multiplicative	1) Double/halve numerator and denominator 2) Maintain ratio relationship

Perspectives	Description of the perspective	Key Relationships	Levels of complexity
Equivalence Perspective	Focus on the relationship between equivalent fractions	Relationships between fractions in the physical materials Relationships derived by extending beyond the physical materials	1) "Recall facts" based on relationships from physical models 2) Relationship between unit fraction and dividing it in two parts 3) Extend relationships based on recall or generated facts
Transform Perspective	Focus on using rules to compare and order fractions	Use of other perspectives to explain why transformation rules	1) Multiply/divide by n/n to compare or generate equivalent fractions 2) Convert to common denominator to compare fractions

Comparison to Smith's (1995) Perspectives

Since Smith's (1995) perspectives provided the framework for this study, I compared the related perspectives that emerged from my data to this framework and previous research. The primary overlap was in the *transform perspective*, although Smith (1995) observed it more frequently. This was probably due to the fact that his students were in fifth grade or higher, in a traditional mathematics class, and his analysis of the textbook confirmed that transform strategies were explicitly taught.

My results indicated two approaches for solving order and equivalence problems embedded in Smith's (1995) parts perspective. The less sophisticated approach that emerged from my data was the *pieces perspective*. The primary relationships were based on the fraction as pieces, and students did not make connections between the pieces and the wholes. The pieces perspective was manifested in both students' use of the physical materials and their drawings. For example, one student drew a large rectangle and divided it into half and called it two-thirds. He was more concerned with making the two pieces, and he labeled the pieces thirds. Armstrong and Larson (1995) found similar results where students used direct comparison of the parts. The more sophisticated approach was the *part-whole perspective* aligned with Smith's description of the parts perspective. Although Smith required equal-sized parts, many students in my study did not use equal-sized parts and were successful at creating equal-sized parts only when they drew upon other mathematics facts to help them such as using the multiplication fact 2 x 3 = 6 to make sixths by first making halves and then dividing each half into three additional sections.

Smith (1995) defined the components perspective based on students' use of natural number relationships either within or across numerators and denominators whereas I divided these into two perspectives since students focused on different relationships with each of these approaches. Students who used a *within-fraction perspective* recognized that the relationship between the numerator and denominator was important. I included benchmark strategies in the within-fraction perspective because students considered the whole number relationships between the numerator and denominator before they could make a comparison. The students in my study divided the denominator in half and often referenced the half-way point. For example, half of five is two and a half, so one half equals two and a half fifths. Using reference points or benchmarks strategies is well documented in the literature (Behr et al., 1984; Post et al., 1986), but Smith include this strategy in the separate reference point perspective.

Students with a *between-fraction perspective* used relationships across numerators and across denominators to make comparisons. Research has shown that student initially use doubling or halving strategies for generating equivalent fractions (Brinker, 1997; Smith, 1995; Streefland, 1993). Similar to previous findings (Behr et al., 1984; D'Ambrosio & Mewborn, 1994; Post et al., 1986; Wearne-Hiebert & Hiebert, 1983), I found that students used both additive and multiplicative reasoning for within-fraction and between-fraction relationships.

Extension of Smith's (1995) Perspectives

I identified three additional perspectives that were not included in Smith's (1995) framework: limited, unit fraction, and equivalence. The *limited perspective* demonstrated students did not have valid relationships to help them solve order and equivalence problems, especially on the pre-interview. Some of these students did not seem to have a physical or mental representation that they could connect with the fraction terms. Other students relied on whole number relationships, similar to results from previous studies (Ball, 1993; Behr et al., 1984; Streefland, 1993; Vance, 1986). Students also used additive relationships instead of multiplicative relationships (Behr et al., 1984; Post et al., 1986; Smith, 1995; Wearne-Hiebert & Hiebert, 1983). For example, Christina decided that she could find other equivalent fractions by adding 1 to both the numerator and denominator so 8/12 was equal to 9/13 and 2/3 was equal to 3/4. Smith identified strategies that were invalid or limited, but he categorized them into his parts and components perspectives. I chose to keep them as a separate perspective because students demonstrated they needed to learn new relationships to help them compare fractions.

Smith (1995) described several strategies in both the parts and components perspectives that I believe belong in their own category, which I called the *unit fraction perspective*. Students used the inverse relationship between the number in the denominator and the size of the fraction to order problems with the same numerator (Behr et al., 1984; Smith, 1995). This understanding was based on the part-whole relationship: each piece is smaller as there are more partitions, which students expanded to compare fractions with different numerators. During a class discussion, Mark explained that 4/7 was larger than 3/8 because 1/7 was larger than 1/8 and there were more sevenths. When these conditions were not met, some students used qualitative reasoning to decide whether the size of the piece or the number of pieces had a greater effect (Behr, Harel, Post, & Lesh, 1992; Smith, 1995). For example, Bobby compared 6/10 and 7/15 by explaining, "This is 6/10 and that's almost the same as 7. And so, 10 is way smaller than 15. Not way, but smaller. So I know that fraction [pointing at 6/10] is bigger than that fraction [pointing at 7/15]." Since the numerators were only one apart, Bobby reasoned that the larger size of the tenths made up for having one less piece. Even though this strategy worked for some problems, it was insufficient for comparing other fractions. Bobby decided 3/5 and 4/6 were equal because, "Six is one more than 5 so that would be a smaller fraction, but it has 4 and that only has a 3," but Mark realized that he did not have enough information to compare these fractions. As students considered both the numerator and denominator, they treated composite fractions as iterated unit fractions. Behr et al. (1983) claims this understanding helps "children develop a stronger quantitative notion of rational numbers" (p. 123). The Fraction Project hypothesized that students needed to develop this understanding of fractions as an iterable unit fraction so they planned instruction that focused on this concept (D'Ambrosio & Mewborn, 1994; Tzur, 1999). By teaching fractions using the Stick microworld, Tzur (1999) found that students understood fractions "as a single quantity," which he expressed demonstrated a stronger understanding than the "parts of a whole" meaning of fractions. My study demonstrated that students can construct their understanding of composite fractions based on iterating unit fractions.

Smith's (1995) study did not address the *equivalence perspective* in his framework. He mentioned the recalled fact strategy in his dissertation, but he observed it so infrequently that he did not include it in any of the perspectives (Smith, 1990). Students in my study remembered facts, often due to their work with physical models. Students tended to use the relationship that resulted from splitting a unit fraction into two equivalent pieces, such as 1/5 equaled 2/10. Students built on these relationships by incrementing, so if 1/5 equaled 2/10 then 2/5 equaled 4/10 and 3/5 equaled 6/10. On the mid-interview, I asked students to compare 2/8 and 3/12 knowing that they only had a physical model for only one of these fractions. After talking quietly to herself and moving around manipulatives, Christina said that 1/4 was equal to 3/12. She explained, "I know 1/2 is 6/12. And I know that a half of a half is 3/12. So I knew that a fourth is a half of half, so I put a fourth on here," as she pointed at the 1/4 piece on the half. Since Christina did not have the fraction pieces for twelfths available in her kit, she figured out equivalent relationships beyond the physical materials. Even though using equivalent relationships was a powerful approach for the students in my study, I have not found similar strategies documented in the literature.

Conclusion

This study demonstrated both the similarities and difference between the perspectives that Smith (1995) and I identified. Some of the differences are due to how we categorized students' general approaches. Many of the variations between the results of our studies were due to the differences in the settings and participants. First, the students in my study were younger, so they were still developing their understanding of fraction concepts. Secondly, since the students in my study were in a constructivist class, they were learning mathematics in a very different manner from students in a traditional class. Lastly, through the interaction of grade levels and the learning environment, students used physical models including manipulatives and drawings as an integral part of how they were learning about fractions. The students in my study who moved between different perspectives and chose efficient strategies were more successful in solving problems and in explaining their answers, thereby demonstrating the importance of using number relationships to understand fraction concepts.

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EXTENDING AND REFINING MODELS FOR THINKING ABOUT DIVISION OF FRACTIONS

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Many students experience great difficulty when studying topics related to fractions, especially division of fractions. One explanation for this may be that learning how to divide fractions is often taught devoid of meaning. The lack of sense making in carrying out algorithms without making connections to concrete or other types of representations contributes to the inability of students to use previously taught algorithms to solve new problems, especially after long periods of time have elapsed. In this paper we explore the flexibility and durability of knowledge that students acquire when they study this topic in a way that encourages understanding.

Introduction

The difficulties that many students have experienced while solving problems involving fractions have been well documented (cf. Tzur, 1999; Davis, Hunting, and, Pearn, 1993; Davis, Alston, and Maher, 1991; Steffe, von Glasersfeld, Richards and Cobb, 1983; Steffe, Cobb and von Glasersfeld, 1988). It is therefore particularly important to find ways to help students overcome these difficulties. Fortunately, many researchers have also documented instances in which students have successfully been able to build ideas relating to fractions (c.f. Steencken, 2001; Steencken and Maher, 2002; Ma, 1999; Cobb, Boufi, McClain and Whitenack, 1997, Kamii and Dominick, 1997). In particular, Bulgar (2002; 2003a; 2003b; Bulgar, Schorr & Maher, 2002) [1] reports on the conceptual development of ideas relating to division of fractions amongst fourth grade students participating in a teaching experiment. Further, Bulgar reports that when this teaching experiment was replicated as part of the regular teaching practice in another classroom (her own), similar outcomes were achieved.

In this paper we report on the latter group of students, with a particular focus on how they extended, modified, revised and ultimately generalized their ideas relating to division of fractions during the following school year. This is done with a focus on mathematical flexibility, and the nature of the models that were used, and how they evolved during the following school year. In particular, we focus on how students initially used continuous linear models, how these models evolved into discrete area models and how these students moved easily back to linear models when they found them to be more appropriate.

Theoretical Framework

Our framework for analysis is based primarily upon a models and modeling perspective with a specific focus on the durability and flexibility of the models that are built over time. Briefly stated (see Schorr & Koellner-Clark, 2003, for a more complete description) a model can be considered to be a way to describe, explain, construct or manipulate an experience, or a coordinated variety of experiences. A person interprets a situation by mapping it into his or her own internal model, which helps him or her to make sense of the situation. Once the situation has been interpreted into the internal model, transformations, modifications, extensions, or revisions within the model can occur, which in turn provide the means by which the person can make predictions, descriptions, or explanations, for use in the situation at hand. Models help us to organize relevant information and consider meaningful patterns that can be used to interpret or reinterpret hypotheses about given situations or events, generate explanations of how information is related, and make decisions about how and when to use selected cues and information.

We wish to distinguish between the conceptual "models" that are embodied in the representational media that students use, and the "mental" models that reside inside the minds of learners (Lesh and Doerr, 2003) to which we refer above. In this work, we will be attending to both, with an emphasis on the nature of the mental models that are born out in the representational media that the students use, especially as evidenced in their mathematical flexibility. We document the nature of the models that the students have built in terms their "mathematical flexibility" not just during or shortly after the instruction took place, but rather over a more extended period of time. We address flexible thought in the context that follows, and because it is relevant to this study.

Carey (1991) describes flexibility by saying "… children become more flexible in their choice of solution strategy as a result of changes in their conceptual knowledge, so that they can solve problems using a variety of strategies that do not model directly the action in the problem" (p. 267). Heirdsfield (2002) notes that flexibility is the capacity of students to exhibit various invented strategies or a large repertoire of problem-solving strategies over time. She referred to the use of a single strategy consistently as inflexibility. Further, Gray and Tall (1994) describe flexible thinking in terms of an ability to move between interpreting notation as an instruction to do something (procedural use of notation) and as an object to think with and about (conceptual use of notation). Flexibility, as denoted in the work by Spiro & Jehng (1990) entails the ability to spontaneously restructure one's knowledge, in adaptive response to changing situational demands. Krutetskii (1969) characterizes flexible thinking as reversibility of thought—another much needed characteristic for students as they consider ideas related to fractions over time. Other researchers including Warner, Alcock, Coppolo & Davis (2003) and Warner and Schorr (in progress) emphasize that a critical aspect of mathematical flexibility is the ability of students to use multiple representations for the same idea, and to link, extend, and modify those representations to a broader range of situations, involving a broader range of models. Since the goal of our instruction was not simply to have students retrieve facts or procedures, or to display understanding only for very specific situations and for limited time periods, we believe that mathematical flexibility is particularly relevant, as defined by all of the researchers above.

Mathematical flexibility is particularly important if students are to use knowledge across a wide spectrum of ideas. Fosnot and Dolk (2001) note, "The generalizing across problems, across models, and across operations is at the heart of models that are tools for thinking." (p.81). They report on a class in New York City wherein a third grade teacher provided students with three different contexts that lent themselves to different models but produced the same answer. In each case the children produced different models that were closely linked to the context. Fosnot and Dolk go on to state that it is easy for students to notice that the answers are the same but that the important issue is for them to see the connections among the models to develop a generalized framework for the operations. In the work that follows, we focus on the nature and type of representations that students build, retrieve, and use over time, and how this relates to their mathematical flexibility.

The students in these studies had, essentially, used three main strategies to solve a particular series of problems (Bulgar, 2002; 2003a; 2003b; Bulgar, Schorr & Maher, 2002). There were no strategies other than these three observed in either the classroom-based studies or the teaching experiment.

These strategies consisted of the following:

1. Reasoning involving natural numbers
2. Reasoning involving measurement
3. Reasoning involving fraction knowledge

The predominant solution method observed in the fourth grade study of the same students (see Bulgar, 2002, 2003a, 2003b) consisted of reasoning involving natural numbers. Essentially, these students built models that converted the meters to centimeters, thereby substituting the fraction division with natural number division, a topic generally prominent in fourth grade mathematics curricula in New Jersey (NJ Mathematics Coalition and NJ State Department of Education, 2002). However this solution method was seen in the work of only one fifth grader in the replicated study, and even when it did appear, there was a claim by the student that it was developed after the problem was solved using reasoning involving fraction knowledge (Bulgar, 2003a, b). All of those students in the fifth grade who drew representations, created linear models to represent the division of a piece of ribbon into various-sized bows.

Methods and Procedures

Background, Setting and Subjects

The study currently addressed took place during the 2001-2002 school year, when the subjects, thirteen girls, were in sixth grade. Twelve of these students had been taught mathematics by the same teacher, the first author of this paper, during fifth grade. The students attended a small parochial school in New Jersey, which attracts children from several surrounding communities. A fundamental premise of the instructional environment was that in order to build mathematical ideas, students needed to be engaged in mathematical activities that promote understanding (Davis & Maher, 1997; Maher, 1998; Cobb, Wood, Yackel & McNeal, 1993; NCTM, 2000; Klein and Tirosh, 2000; Schorr, 2000; Schorr and Lesh, 2003). Therefore, conditions established during the fifth grade, were set up to create a classroom community in which student inquiry and discovery were of paramount importance. The classroom environment was one in which students' ideas were always respected. Students were questioned and encouraged to explain their solutions, developing their own sense of accuracy. Alternate strategies were encouraged, shared and discussed, as students were invited to discuss their thinking and to submit ideas in writing. Students were not taught algorithms. When they recognized patterns and could justify that these patterns were valid, they created generalizations, which they could apply to future problems. Questions were used to elicit explanations, to guide students towards persuasive justifications of their solutions and to redirect them when they were engaged in faulty reasoning. Justification of solutions became a part of the classroom culture.

Because essentially the same group of students who were taught by the first author in fifth grade were grouped together again in sixth grade, (Twelve of the students were from the original group and there was an addition of one new student.) and taught mathematics by the same teacher, an opportunity was presented to closely examine longitudinal development of mathematical ideas within the framework of regular teaching practice. The Tuna Sandwiches task, the problem that is the subject of this paper, was the first one assigned as these students began sixth grade.

Data

The data examined here consist of artifacts of actual student work, which were collected over the course of approximately six weeks. Written notes from the teacher were attached to some of the work, usually in the form of questions and answers to these questions also appear in the students' writing.

Tasks

The primary task studied here, "Tuna Sandwiches", was created by the first author to be isomorphic with the problem done during the previous year called "Holiday Bows"2, which introduces division of a natural number by a common fraction. The Tuna Sandwiches problem follows.

> Mr. Tastee's restaurant serves four different kinds of sandwiches. A *junior sandwich* contains 1/4 lb of tuna; a *regular sandwich* contains 1/3 lb of tuna; a *large sandwich* contains 1/2 lb of tuna and a *hero sandwich* contains 2/3 lb of tuna. Tuna comes in cans that are 1lb, 2 lb, 3lb and 5 lb. How many of each type of sandwich can you make from each size can? Find a clear way to record your information. You will need to write a letter to the restaurant owner, Mr. Tastee, and give him your findings.

One of the goals in creating the "Tuna Sandwiches" problem was for it to lend itself to be represented by an area model rather than a linear model, as was the case with "Holiday Bows". Fosnot and Dolk (2001) state that just because we create a problem with certain models in mind, we cannot be assured that this model will be used by students. By creating a problem that was essentially isomorphic to the "Holiday Bows" problem, (the one that was completed by both the fourth graders in the teaching experiment and the fifth graders in the regular classroom of the first author), yet embodied in a different type of representation, an area model, the notion of flexibility could be explored as well as an examination of the durability of the knowledge the students had demonstrated during the previous year.

Results and Discussion

All of the sixth grade students solved the problem using the approach of reasoning involving fraction knowledge. That is, they reasoned that if a sandwich requires 1/4 of a pound of tuna, four such sandwiches could be made from every pound of tuna, so what was necessary in order to find the solution was to multiply the number of pounds of tuna in a can by four. In both the fourth grade and the fifth grade studies, dividing by the non-unit fraction, 2/3, had proven to be more problematic. One might conjecture that the linear model used by students would be more conducive to solving problems such as 2 ÷ 2/3, because it is a continuous model. Yet, several fourth and fifth grade students who had used reasoning involving fraction knowledge had difficulty with this because it was arduous to give meaning to the piece that was "left over"; it was not clear how many two-thirds there were in one. One student in the fourth grade group stated the following when explaining how many bows, each 2/3 meter in length could be made from a piece of ribbon that is 2 meters long.

> Alex: There's three thirds [in one meter] so there's two-thirds and one-third and one-third that's two-thirds and you still have one two thirds left over…[*while drawing picture*] ... so then... so you only have one third so then you have to get the other third. This is two thirds so then you have two more [one] thirds left over.
>
> Jon: [*pointing to Alex's drawing*] And there are six ones [1/3] is in each, and it would be two-thirds is one [bow], two-thirds is again [a bow] and two [one] thirds left.
>
> Alex: I think it's 4 [bows].

Alex looked at the two one-meter parts of his two-meter ribbon as two discrete entities. He did not seem to realize that the two one-third meter pieces that remained at the end of each meter could be used to make another bow. Although all of the fifth grade students eventually were able to find out how many bows, each 2/3-meter in length could be made from the various lengths of ribbon, they also had greater difficulty with this set of problems than they had when dividing by the unit fractions.

None of the thirteen sixth graders used a linear model to solve the Tuna Sandwich Problem. Ten of the thirteen students actually drew area models to represent their solutions and three merely explained their thinking without referring to a model. It is interesting to note that each of these area models included discrete drawings for each pound of tuna. One would think that the problems involving the hero sandwiches, those which each required 2/3 lb. of tuna, would be more difficult to solve when using discrete area models. Yet, there was no mention of greater difficulty. In fact, several students stated that each one-pound of tuna would yield one and one-half hero sandwiches. It appeared that the shift in unit was made seamlessly. One-third pound of tuna was recognized to be one half the quantity needed to make a hero sandwich, which required two-thirds pound of tuna.

Though they were not asked to do so, most of the sixth graders spontaneously formed some kind of graphic organizer to structure their results. Seven of the thirteen students formed a matrix indicating the amount of tuna required (for each sandwich) as one dimension and the different-sized cans of tuna as the other dimension. Four of the students indicated their solutions in an organized listing. One of these students had both an organized listing and a matrix.

Since the students specified their solutions using reasoning involving fraction knowledge by looking first at how many sandwiches of each type could be made from a one pound can of tuna, it is interesting to note that very few used proportional reasoning, using multiplicative structures to arrive at solutions involving multiple-pound cans of tuna. Most used additive structures. Stephanie begins by alluding to proportional reasoning when she writes the following as she explains her solutions for finding out how many regular sandwiches, those requiring 1/3 pound of tuna, could be made from each of the various sized cans.

> Stephanie: [sic] You can only make 3 sand. With one lb of tuna because 3 thirds make 1. (3/3=1) With one more lb of tuna (2lb) you can make twice as many sand. So you have 6 sand. With 3 lb of tuna you can make 3 more sand. (9 altogether) because you have one more lb of tuna which make 3 sand. Because 3 thirds (3/3) =1. Now with 5 lb. you add not 3 sand. But 6 because it is not 4 lb, but 5 lb of tuna.

Stephanie seems to be going back and forth between multiplicative ideas and additive ones, adding on multiples of three sandwiches. When Stephanie explains her solution to the hero sandwich problem, the one involving division by a non-unit fraction, she states the following.

> Stephanie: [sic] So with a 1 lb can you can make 1 sand. and a 1/2 of another because it is /3 of a lb of tuna [required for each hero sandwich] so you have 2/3 left which is 1/3 left which is 1/2 of 2/3. A 2 lb can of tuna you can make 3 sand. easily and the excess is 1/3 from both so that makes 3… Now for a 5 lb. can you can make 6 ½ sand. Because you can make 5 easily and 2 1/2 more with the extra of each lb.

Though Stephanie's solution of 6 1/2 sandwiches is not consistent with her explanation, she has demonstrated an understanding that 1/3 of a pound of tuna represents 1/2 of a hero sandwich, an idea that students had more difficulty understanding the previous year when they worked with the linear model suggested by the Holiday Bows problem. It would appear that she is first counting the complete sandwiches that can be made from each pound, the ones she refers to as being made "easily", and then is gathering up the remaining 1/3 pounds from each can to combine them in order to make additional sandwiches. This kind of thinking was also observed in the representations of other students, such as Gabriella, Lynn, Amy, Sarah and Bea, who drew connecting lines to the "leftover" one-third pound of tuna in each representation of a one-pound can.

After completing a lengthy explanation of her solutions, Eve wrote the following reflection on her work.

Eve: P.S. When I was figuring this out for you I noticed something interesting. I noticed [sic] That by the junior sandwich (1/4 lb.) you added 4 by every can of tuna. This is because every time the can get bigger by 1 lb (from which you can make 4 sandwiches) so you just add another 4 and the 5 lb., it is 2 more lbs. So you add 8 instead of 4.

Though Eve used reasoning involving fractional knowledge, she applied additive reasoning to get the solutions.

Sarah used multiplicative reasoning in finding the solutions. She wrote the following.

Sarah: [sic] Out of 3 pound you can make 12 junior. There is 4 in each and 4 x 3 = 12.

Sarah included a diagram of 3 circles divided into four sections or fourths. She numbered the sections from one to twelve. She used this structure for all of her solutions.

Gabriella also used multiplicative reasoning. She drew five circles, divided them in half vertically and stated the following.

Gabriella: [sic] How much large sandwiches can you make from 5 pounds. Let's try those imaginary pounds [her drawings]. Well 2 in each of the 5 pounds 5x2 = 10!

In the summative class discussion of the Tuna Sandwiches Problem, students talked about the problem and how it was just like the problem they had done the previous year called "Holiday Bows". Those who did not recognize it at first agreed when their peers noted the isomorphism. They recognized that the problem required division of fractions and easily explained their solutions using symbolic notation. For example, when summarizing that three hero sandwiches could be made from two pounds of tuna, they were able to create the number sentence, 2÷2/3 =3. Some of the number sentences that the students provided were recorded on an overhead projector transparency. These number sentences are seen as solutions representing conceptual understanding derived from the use of student-generated models, rather than as algorithmic answers. Once the students agreed that they had solved these problems involving division of fractions, they were assigned numerical problems, one at a time. The first problem was 2 ÷ 3/4. They were told to build a model to solve the problem and to explain how the model can be used to find the solution. Some (Michelle, Amy and Rose for example) wrote the problem as "How many 3/4's are in 2?" This would indicate an understanding of the meaning of division. Subsequent to providing solutions for this problem, students worked on the problem, 5/8 ÷ 2 1/2. What is interesting to note here is that when drawing models to solve these problems, students invariably went back to linear representations. Many referred specifically to Cuisenaire Rods® when they discussed their linear models. They had worked with these materials early in fifth grade to build basic concepts about fractions. The activities in which they were engaged using these materials were modeled after those used and documented in another study (Steencken, 2001).

Conclusions

Students in the sixth grade were able to retrieve ideas they had built about division of fractions during the previous school year, and these ideas were used and extended appropriately. Many students demonstrated flexible thought in the way they indicated their grasp of division of fractions and extended their understanding to more complex division of fractions problems.

When first confronted with a task involving division of a natural number by a fraction in fifth grade, they made use of linear continuous models. Our results show that when a similar problem was given to the same students a year later, one that lent itself to an area model, students demonstrated flexible thinking in their ability to seamlessly move to a discrete area model and durability of the ideas they built the previous year in their ability to effortlessly move from linear models to area models and back to linear models as needed. Many recognized and verbalized that the Tuna Sandwiches Problem was "the same" as the Holiday Bows Problem. They revealed their flexible thinking in their ability to use a variety of representations for the same idea, division of fractions, and to link, extend and modify those representations to a variety of situations (Warner, Alcock, Coppolo, 2003; Warner & Schorr, in progress). They moved easily back and forth between area models and linear models as they worked on contextual tasks and used models to solve numerical problems. This is significant because as Fosnot and Dolk (1991) indicate, models represent strategies used to solve problems and thereby develop into mathematical tools. Generalization is characteristic of this development.

Endnotes

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2. For a full description of this task and results see Bulgar, 2002; Bulgar, 2003a; Bulgar, 2003b; Bulgar, Schorr & Maher, 2002.

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LEARNING TO USE FRACTIONS: EXAMINING MIDDLE SCHOOL STUDENTS' EMERGING FRACTION PRACTICES

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There is a large body of literature, both empirical and theoretical, that focuses on what is involved in learning fractions when fractions are the focus of instruction. However, there is little research that explores how students learn to use what they have learned about fractions outside instruction on fractions. In response, this research explores how middle school students learn to use fraction knowledge, the fraction concepts and skills studied in formal curriculum units, in mathematical instructional settings where fractions are not the main focus of study, but rather support the development of other mathematical content. The purpose of this paper is to describe the practices a class of sixth-grade middle school students engaged in when using knowledge learned about fractions in two contexts: (1) area and perimeter and (2) decimal operations.

Theoretical Framework

I draw upon Scribner and Cole's (1981) practice account of literacy where literacy is best understood as a set of social practices people draw upon and use in certain situations. Rather than focus on the separate skills that underlie reading and writing, studying literacy involves studying the social practices associated with a particular symbol system. In this study, fraction literacy can be thought of as a shift from studying the separate skills that underlie fractions (i.e., fraction addition or equivalent fractions) toward understanding how students make use of fractions.

Data Sources and Analysis

The participants were a class of 23 sixth-grade students and their teacher. The middle school, located in a small middle-class mid-western community, used the Connected Mathematics Project curriculum. Data collection across nine lessons included field notes, copies of students' written work, video-recordings of whole class conversations and small-group interaction of one group of four focus students. Two audio-recorded interviews were done with each focus student.

When studying literacy, literacy events and literacy practices are the basic units of analysis (Barton, 1994). I define fraction literacy events as situations where students have to use their fraction knowledge and fraction literacy practices as stable identifiable patterns of behavior students make use of during these events. The focus of data analysis was to identify the fraction literacy practices students engaged in as part of learning to use fraction knowledge.

Results

Across all nine lessons the teacher and students engaged in the practice of determining appropriateness. The conversations indicated that students were trying to understand how fractions and the context in which they were being used interacted. Students were not asking how to add or how to find equivalent fractions. Rather, the conversations indicated that students were trying to determine what was an appropriate way to use fractions in the problem situation. These questions capture what students thought about and made sense of when trying to use fractions.

- Do concepts used in whole number settings apply when fractions are used in that setting?
- Can fractions be used to make sense of mathematical ideas when not explicit in the problem?
- Is it appropriate to use the standard multiplication algorithm in this situation?

- Is it appropriate to use equivalent forms (mixed or improper) in this situation?
- What is the appropriate way to represent a fraction with a repeating decimal, or one that involves using a decimal approximation when operating?
- Is it appropriate to use a decimal representation in this situation?

The last two questions are reflective of a whole-class discussion about a problem that involved finding the width of a rectangular storm shelter with a floor area of 24 square meters and a length of 5 1/3 meters. A lengthy conversation took place where various decimals were offered to represent 5 1/3 when operating to find the missing width. As possible solutions and solution paths were offered and rejected I argue that students were trying to determine the appropriate way to represent and use 5 1/3 in this situation.

Cathy: I did 5.3 × 4.5 and got 24.3.

Bryan: I did 5.3 × 4.5 and got an even 24.

Trevor: Well 5 1/3 is not equal to 5.3. It is 5.3 with a line over it. So let's say that times 4.5 which equals 23.99999. It is pretty close to 24.

Here Trevor rejects his own approach because it does not produce exactly 24. The conversation shifts to looking at division as an approach. A student, Corey, offers that he divided 24 by 5.333 and got 4.528. Next, a student showed that if you use fractions that 5 1/3 × 4 1/2 is exactly 24.

Teacher: So that works. 5 1/3 rows of 4 1/2. But how did you get that?

Corey: Maybe like I did. [Recall that he offered 24 ÷ 5.333]

Teacher: Okay, but you got 4.528. What would explain that?

Corey: I rounded off.

Teacher: That's the problem with sometimes switching to a decimal. If I don't I can get the exact answer.

In the end approaches were rejected because using approximations for 5 1/3 did not lead to an exact area of 24. Although there are many situations where using decimal form rather than fraction form is appropriate, in this situation where you want an exact measure switching 1/3 to decimal form to operate is not appropriate.

Conclusions

These findings support a practice account of literacy (Scribner & Cole, 1981), in this case fraction literacy, where literate use of fractions develops out of understanding situations where fractions are used. These situations extend beyond ones typically developed when learning about fractions. When students learn about fractions, conversations often center around how to do something (i.e.: add fractions or make equivalent fractions) as well as why ideas make sense. One might ask "Are these two forms equivalent?" rather than "Are these two forms equivalent here?" It is through making sense of situations where fractions are used and coming to realize the potential for using fractions that students learn to use them as a tool.

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A STUDY OF FOURTH-GRADE STUDENTS' EXPLORATIONS INTO COMPARING FRACTIONS

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The purpose of this paper is to describe the growth of mathematical understanding in a class of twenty-five fourth-grade students in a suburban New Jersey school. These students were invited to investigate the comparison of pairs of fractions as part of a yearlong teaching experiment. Despite never having been taught the formal, traditional algorithms for fractions, students were able find the differences and to express their answers with equivalent fractions by constructing multiple models of different lengths with the Cuisenaire Rods™. This research builds upon the work of Steencken (2003) and Bulgar (2003), who examined components of this teaching experiment.

Led by Carolyn A. Maher and assisted by Amy Marino from Rutgers University in New Brunswick, NJ, this classroom experiment was designed under the premise that with the appropriate conditions, students could develop a deep understanding of mathematical ideas as they actively participate in their learning. These children were challenged with mathematical explorations and given both the opportunity to construct concrete models and the time and conditions to think deeply about their ideas. This study focuses on the class as it determines which, if either, of a pair of fractions is bigger and by how much.

Five sessions (approximately 60 - 90 minutes each) from October will be described and analyzed. In addition to multiple videotapes of each session, other data include students' drawings of the models that they built, students' written explanations and researcher field notes. The videotapes were analyzed following the model proposed by Powell, Francisco and Maher (2003).

Students worked in pairs and small groups to construct models using the Cuisenaire Rods™ to find the difference between the fractions. Using number knowledge and their emerging fraction schemes (Steffe, 2002), students then built other constructions for the same pair of fractions and were encouraged to look for clues that would help them build future models. Through these hands-on activities and the ensuing mathematical discourse, students often made conjectures about the relationship between the fractions and then tested their ideas through model-building. For example, when comparing two-thirds and three-fourths a student was able to argue convincingly that the shortest model that they could build would have a length equivalent to twelve white rods. By analyzing their constructions, students discovered a generalized solution that enabled them to build and to envision other representations. They showed evidence of developing ideas of ratio and proportion as they realized that by doubling the length of a model they could generate additional models. This insight allowed them to determine that the difference between two-thirds and three-fourths was one-twelfth, two twenty-fourths or four forty-eighths. They realized that if they could build "trains" (putting multiple rods end-to-end) for their "one" unit, then they could also use "trains" to represent unit fractions. This discovery enabled them to build models that were four times as long as their original.

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MAKING MEANING OF PROPORTION: A STUDY OF GIRLS

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The purpose of this study is to test and validate four levels of proportional reasoning, (e.g. the Hypothetical Learning Trajectory (HLT)) proposed by Carpenter et al. (1999). The research questions were:

1. How well does the HLT describe the pathway of a population of Icelandic girls before, during, and after they have engaged in a unit focused on proportional reasoning?
2. What evidence is there for the existence of Level 2, Level 3, and Level 4 ways of reasoning in students' verbal protocols?

Many studies on children's proportional reasoning provide evidence of various influences on students' thinking about proportion. Among these is the numerical structure, which refers to the multiplicative relationship within and between ratios in a proportional setting. A "within" relationship is the multiplicative relationship between elements in the same ratio,[1] the "between" relationship is the multiplicative relationship between the corresponding parts of the two ratios.

Researchers have hypothesized that students' learning of proportional reasoning can be described as a learning trajectory (Carpenter et al., 1999; Inhelder & Piaget, 1958; Karplus et al., 1983). By learning trajectory, I am referring to the path that student reasoning travels as students' understanding of proportion develops. As students reasoning develops, so too does student ability to solve increasingly complex problems and their strategies get more complex and mathematically sophisticated.

The literature on proportional reasoning reveals a broad consensus that proportional reasoning develops from qualitative thinking to multiplicative reasoning (Behr, Harel, Post, & Lesh, 1992; Inhelder, & Piaget, 1958; Kaput & West, 1994; Karplus et al., 1983; Kieren, 1993; Resnick & Singer, 1993; Thompson, 1994). While earlier research on students' reasoning relied on within-ratio and between-ratios strategies to analyze students' thinking (Abramowitz, 1975; Karplus et al., 1983; Vergnaud, 1983) Lamon (1993a; 1993b; 1994; 1995) offered a different lens through which to understand students' development of proportional reasoning. Lamon proposed two processes, unitizing and norming, as central to the development of proportional reasoning. Unitizing involves the construction of a reference unit from a given ratio relationship. Norming refers to the reinterpretation of another ratio in terms of that reference unit (Lamon, 1994; 1995).

Lamon's (1993a; 1993b; 1994; 1995) ideas provided a basis on which to create a more complete picture of students' developmental pathway, known as the Hypothetical Learning Trajectory (HLT). Using Lamon's (1994, 1995) operation of unitizing and norming, Carpenter et al. (1999) identified four levels of students' proportional reasoning.

Level 1, students showed limited ratio knowledge. Level 2 is characterized by the perception of the ratio as an indivisible unit. Students at this level are able to combine the ratio units together by repeated addition of the same ratio to itself or by multiplying that ratio by a whole number, but they cannot solve proportion problems in which the given ratio has to be partitioned such as, problems in which the target ratio is a noninteger multiple of the given ratio (e.g., $\frac{8}{12} = \frac{42}{x} or \frac{8}{3} = \frac{2}{x}$).

Level 3, the given ratio is thought of as a reducible unit. Therefore, students at Level 3 can scale the ratio by nonintegers. An example of a Level 3 strategy combines the reduction of the given ratio with a buildup strategy by using either addition or multiplication. Students at Level 4 think of ratios as mere than just as unit quantities. They understand the proportion in terms of multiple relations. They recognize the relation within the terms of each ratio and between the corresponding terms of the ratios. They are able to look for the integral relationships that will make the computation easiest.

The subjects of this study are the 26 fifth-grade[2] girls in two classrooms at one of Reykjavik's public schools. I observed every math class throughout the course of the study, taking on the role of "participant observer". During data collection, students worked on 24 problems that were created during instruction. Each set of problems was composed of three problems with the same contextual structure but with different multiplicative relationships in the proportion. The numbers were chosen to further students' understanding of proportion and to aid their recognition of the multiplicative relationships in the two ratios in the proportion. By varying the multiplicative relationship in the problems, sets of problems were created to distinguish between Level 2 and Level 3 students and between Level 3 and Level 4 students

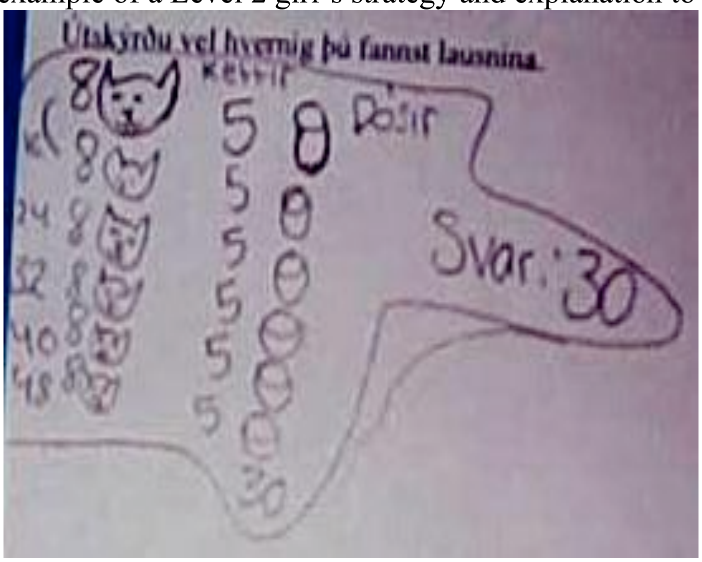
The pretest and the posttest were created using the same criteria as the instructional problems in regard to the number structure of the problems. The pretest comprised of 18 problems in three sets with different multiplicative relationships. The posttest comprised 12 problems. Students' problems solutions strategies were collected. I collected all their written work from both tests.
During instructions students worked both individually and in groups on their problems. All the written work the students produced and artifacts from their work were collected. Also all whole-classroom discussions were videotaped and transcribed. Approximately 40 percent of students participating in a group work at any given time were videotaped or audiotape and transcribed.
The four-level model of proportional reasoning proposed by Carpenter et al. (1999), proved to be a beneficial tool to analyze their work. Analyzing the pretest the classification of students' solutions resulted in the creation of a transitional level "emerging Level 3". On both pre- and posttest the results show a perfect fit; students on Level 2 were not able to solve any of more complex problems that emerging Level 3 students were able to solve successfully, nor were the emerging Level 3 students able to solve any of the most complex problems that Level 3 students were able to solve with success.

The problems were structured to discriminate between students at different levels of reasoning. Problems that could be solved by students reasoning on Level 2 had an integer relationship between the ratios and involved enlarging (e.g., $\frac{2}{8}=\frac{x}{24}$). Students reasoning on Level 3 could solve problems that were previously mentioned as well as problems that have a noninteger relationship between the ratios ($\frac{5}{6}=\frac{x}{21},\frac{15}{10}=\frac{6}{x}$). Problems that proved to be transition problems from Level 2 to Level 3 were the problems that had a scale-down number structure such as $\frac{8}{24}=\frac{2}{x}$. The difference between Level 2 and Level 3 reasoning is the need to scale down the given ratio. During the emerging Level 3 stage, students are able to scale down by whole numbers but they cannot use their knowledge of scaling down within other number structures. Strategies that students used to solve the problem distinguished between Level 3 and Lev el 4 reasoning.

On the pretest, 35 percent of the girls displayed Level 1 reasoning. Around 40 percent exhibited Level 2 reasoning. Twenty-three percent of the girls were emerging Level 3. One girl showed Level 3 reasoning on her pretest.

On the posttest only 3 girls reached Level 4 thinking, whereas more than 80 percent reached Level 3 thinking. Therefore, it is evident that reaching Level 4 thinking involves a very complex thinking that most of the girls had not yet adopted.

Throughout the course of the study, girls were thinking about the given unit as a single entity that they then operated on to reach their target number. The buildup strategy, the most common strategy, provides clear evidence of the ways in which students understand the given ratio as a single unit that they can then build up or build down. Common explanations from the girls were related to the idea that everything they did had to apply to both terms of the ratio. Following is an example of a Level 2 girl's strategy and explanation to support that argument.



$$\frac{5}{8} = \frac{x}{48}$$

It is lunch hour at the humane society. The staff members have found out that 8 cats need 5 large cans of cat food. How many large cans of cat food would they have to have if they were to feed 48 cats?

Student:	I did it—like, here is 8 and then 5 cans of food, and then again—then there is 8 and 16 cans of food until I…reached 48 cats, and then the answer is 30 cans of cat food.
Teacher:	How did you know that you should have 8 groups?
Student:	Well, I did not know that because I did 8:5 and 8:5 and 8:5 and added the 8s together until I had 48.

She explained her strategy in terms of the unit as an entity. She operated on the unit of 8:5 until she reached her target number of 48. She did not think in advance about the number of groups she had to use; rather, as she is building her units, she is adding on until she know where to stop.

Endnotes

1. Here I define ratio as the relationship between two quantities that have two different measure units.
2. Fifth grade in Iceland refers to the same age group as in the U.S.

Reasoning and Proof

REASONING STRATEGIES IN THE SOLUTION OF GEOMETRIC PROBLEMS THROUGH CABRI GÉOMÈTRE

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The goal of this paper is to show some of the reasoning strategies that students use when solving geometric problems with Cabri Géomètre (Cabri). The experimental phase was performed with high-school students who did not have any previous experience with the Cabri method. This phase was developed in two stages: familiarizing them with Cabri and the experimentation itself. Each work session the students were put to work in teams, and afterwards there was a plenary session.

Introduction

One of the topics of interest among Mathematics Educators over the past years is related to the proving processes in school context (Arzarello et al., 1998; De Villiers, 1999; Balacheff, 1999; Hanna, 2001; Furinghethi et al., 2001, 2002; Olivero & Robutti, 2001, 2002).

Nowadays, problems are put forward that have always been focused on by geometric teaching, for example, the confusion between the drawings and the geometric objects. The corresponding difficulties to the problem in this study are considered as related with the interaction between the ideas of drawing and constructing, and the different ways of interpreting a geometric representation. A possible cause of the confusion between drawing and the geometric object may be the forms of representation used for these geometric ideas. Therefore, it would be necessary to describe and analyze what happens when these new technologies of representation are used, such as the Cabri to show these ideas. This unblocking allows the teacher to observe and evaluate how far mathematics comprehension goes on behalf of the student. Therefore, our report aims at investigating the question: How does Cabri modify the reasoning strategies of the students?

Theoretic Orientations

The new technologies provide dynamic representations of mathematical objects. Particularly, Dynamic Geometry provides an exploration field that is not feasible through representations with pencil and paper. The representation that is generated is *dynamic*, and remains unchanged when the objects are deformed by *dragging*.

To analyze the reasoning strategies used by students as a result of the interaction with Cabri, it is necessary to understand the kind of manipulation that students have performed on the Cabri objects, as well as how tools used in each work session function, to solve the proposed activities. For that reason, the modalities will be analyzed, as well the dragging (Arzarello, et al., 1998) as the measurement (Olivero & Robutti, 2001).

Methodology

Fourteen students participated in the experimental phase with elementary knowledge of Geometry and without any previous experience with the Cabri tool. The participants were selected based on the former questionnaire. The experiment was performed at a high school in México City for three months, after classes. This participation was voluntary and not subject to any grading. This phase was developed in two stages. The first one, the familiarization with

Cabri and the second, the experimentation itself. Each work session was divided in the following two parts: 1) Development of the activity in teams of maximum three students, and 2) a plenary discussion. Each session, a different team is video taped and another one is audio taped.

Description of results: example of an open problem

Stating of the problem: A is the center of a circle and AB its radius. Draw the perpendicular bisector of radius AB. (The perpendicular bisector of a segment is the perpendicular straight line that divides it in two equal parts.) In one of the points where the perpendicular bisector crosses the circumference, place point C. Join points A, B and C.

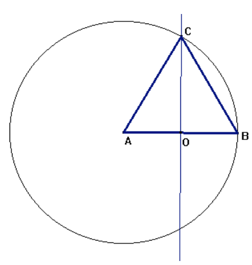
What figure do you obtain when joining points A, B and C? Explain.

This problem can be solved in different ways. This problem was exposed in a questionnaire in such way that it could be solved with pencil and paper. The main difficulty that the students faced was to understand the statement of the problem; this was evidenced in their arguments and in their figures. Moreover, the drawing converted itself into an obstacle for the majority of them. This ratifies what Sandoval (2001) found.

Reasoning strategies used by Cabri: in the first part of this activity, the students organized into 7 teams. Once finished with the construction, the first strategy to establish the type of figure formed was to use measurement. This means, the triangle was equilateral because its sides are equal and each of its internal angles is 60°.

Plenary activity: second part. We will present some examples to show the reasoning strategies developed by students.

First strategy. Nancy and Enrique were able to *see* the geometrical configuration containing the perpendicular bisector, the radio AB and two right triangles. They illustrate their ideas using a Cabri-construction built by Gustavo. It must be emphasized that the measurement command was not used and, besides, that the reasoning used the static figure. Nevertheless, we remark that dragging and measurement were two very important tools in the exploration stage and for conjectures formulation.



(a) *(b)* *(c)*

Figure 1. (a) Reproduction of Nancy-Enrique's construction; (b) and (c) the students show how the two right triangles are determined by the perpendicular bisector.

Nancy: "I thought it was due to the equal measures [she indicates the segments AO and OB; O is the mean point of AB]. The altitude… [she refers to OC]. When joining them…the measure at both sides will be the same."

Researcher: "What measure?"

Nancy: "This one [indicating AC] and that one [indicating BC]."

We can infer, from this description that Nancy perceives the geometrical configuration formed with the perpendicular bisector that divides the triangle ABC in two right triangles AOC and BOC. Students have not studied yet, congruence criteria for triangles. Nevertheless, this is what they seem to be trying to express as resulting from their exploration.

This dialogue shows a reasoning strategy close to geometric theory. They use two basic geometric elements: the radii of a circumference and the definition of perpendicular bisector to answer the researcher questions about why a triangle is equilateral. Their first answer is supported in the definition of the perpendicular bisector: Enrique specifies that the perpendicular bisector divides to radius AB in two equal parts and allows forming two right triangles. Then, he asserts the segments AB and AC are radius of the same circumference. In his intervention, Enrique showed partially, a conceptual control. This student did not use the measurement in his construction and its reasoning was on the static figure.

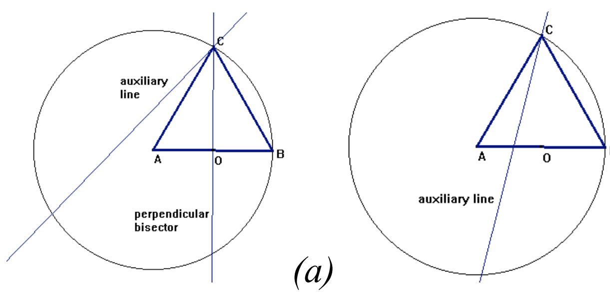
Strategy 2. Jorge presents a complete solution even if he did not participate in the team work. His reasoning is similar to Enrique's but much more systematic.

Jorge: "We have already drawn the segment from this (indicating A, and AB) The radius is always the same (moving his hand along the circle)…All right, we have already drawn the perpendicular bisector…that passes through the mean point of AB. That is the altitude. Now, we take the intersection point formed by the perpendicular bisector and the circle. The radius is always the same, so this length (AB) is always equal to this one (AC). Then, this (AC) is equal to this (BC)."

There are no measures at all on the screen.

Researcher: "Why?"

At the beginning, Jorge is unable to support his assertions. A bit later, he produces the following dynamic description: (Figure 2).



(a) *(b)*

(c) *(d)* *(e)*

Figure 2. Sequence of the Jorge's idea. (a) and (b) Reproduction of the Jorge's construction; (c), (d) and (e) Jorge's exploration used dragging.

Jorge: "I built everything…then I hid the perpendicular bisector and draw a line through the intersection point of the circumference and AC. Afterwards, I began dragging the line and saw that this line [he refers to the line that is superposed on the segment AC] has the same length from point C to the point A and its length is equal to that of segment AC. Then I drag the line up to point B, and I discovered that they have the same length. Then I measured the distance from point B to point C and it results the same. But I need to create a point in the line so that I can explain that well to you.

The researcher suggests the use of *Compass* command. Then, Jorge continues.

Jorge: "Now, I can explain. Now, I can drag it. This point hits (he refers to the point P created by *compass* on the line) with A and with B. Then, both segments are equal."

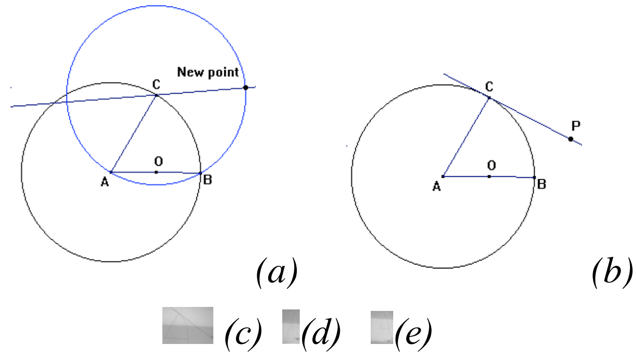


Figure 3. Sequence of the strategy of the Jorge's reasoning. (a) and (b) Reproduction of the Jorge's process. (c), (d) and (e) Jorge's explanation supported by dragging.

The following lines show how Jorge tries to convince his fellows about the appropriative of his reasoning. Afterwards, Jorge uses measures of segments to produce more evidence supporting his assertions.

Jorge's argumentation shows another type of strategy which exemplifies a dynamic reasoning. This strategy illustrated a conceptual control on what he saw in the screen (visualization process), being the dragging a fundamental tool in this process. The measurement, in this case, was used in the transition of the theoretical level to the perceptual level (Olivero & Robutti, 2001a). The reasoning outlined by this student was based on the relation between the radii of the same circumference and the theoretical status of the compass.

The foregoing examples illustrate that, as a result of the interaction with Cabri, all teams partially established the explanation why triangle ABC was equilateral. This (incomplete) justification was based on the relation between the radius of the circumference itself. The former shows a possible change of the status of the representation, that is, of an interpretation of the drawing towards one closer to the geometric object. It may be affirmed that these tools (dragging and measurement), pertaining to Cabri, allowed the students to confront their perception with the internal theory of the machine as a control mechanism.

All the groups conjectured the triangle ABC was equilateral, and they justified (initially) it with the measurement and the dragging, in *test* modality. The dragging was used in different modalities (Arzarello *et al*., 1998). The modality of *wandering dragging* was used, by the students, when they did not have explanation to his conjectures or they did not know what to do, "exploration phase". In some occasions the dragging was used in addition to the measures along. The *test dragging* was used to validate the construction. At the beginning, it went only at perceptual level (appearance), later it included the measurement to validate this perception, which complemented with a numerical verification. An example of *lieu muet dragging* was when Jorge explains why the three segments are equal by means of the superposition method.

We emphasize that the Jorge's activity illustrates how the tools of dragging and measurement were used in the transition to the perceptual level from the theoretical level and vice versa.

The reasoning strategies of the students clearly show how they changed the relationships among the geometrical objects pertaining to the situation under study: beginning at the *intuitive level*, based on the perception of the form (appearance); then going through the *empirical* stage based on numerical results of several representations, until reasoning with certain elements from *mathematical formalism*. The relationships between the perceptual and theoretical acts, was awaken in the students thanks to the use of the internal theory of Cabri (which controls the electronic drawing).

Final comments

The Cabri drawings provided the students with a major evidence level than the drawings with pencil and paper. These drawings are a mirror of the internal geometrical universe of the machine and, finally, it is closer to the geometric object. Given these characteristics, it seems possible a bridge exists between the geometric controlled evidence of Cabri and the geometric argumentation developed in the classroom.

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STUDENT UNDERSTANDING OF GENERALITY

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This paper presents results from a multi-year research project[1] *exploring the development of middle school students' competencies in justifying and proving. In particular, we present and discuss results from a written assessment completed by 394 sixth through eighth grade students. The assessment questions that are the focus of this paper targeted the idea of generality—both the idea that a general argument (i.e., proof) offers an absolute guarantee regarding the truth of a statement or result and the idea that empirical evidence does not suffice as proof.*

The nature and role of proof in school mathematics has been receiving increased attention in the mathematics education community with many advocating that proof should be a central part of the mathematics education of students at all grade levels (Ball, Hoyles, Jahnke, & Movshovitz-Hadar, 2002; Knuth, 2002; Schoenfeld, 1994; Sowder & Harel, 1998). Such attention is also reflected in current mathematics education reform initiatives. In contrast to the status of proof in previous national standards documents, its position has been significantly elevated in the most recent document (National Council of Teachers of Mathematics [NCTM], 2000). In particular, the *Principles and Standards for School Mathematics* (NCTM) recommends that the mathematics education of pre-kindergarten through grade 12 students enable all students "to recognize reasoning and proof as fundamental aspects of mathematics, make and investigate mathematical conjectures, develop and evaluate mathematical arguments and proofs, and select and use various types of reasoning and methods of proof" (p. 56). These recommendations, however, pose serious challenges for school mathematics students given that many students have found the study of proof difficult. In fact, research has painted a relatively bleak picture of students' understandings of proof (e.g., Balacheff, 1988; Bell, 1976; Healy & Hoyles, 2000; Porteous, 1990; Senk, 1985, Sowder & Harel).

Although students' difficulties with proving have been attributed to a variety of factors, one factor, an understanding of generality, is critical to developing an understanding of the concept of proof. One aspect of generality concerns the idea that a proof offers an absolute guarantee regarding the truth of a statement or result. A number of researchers have investigated student understanding with respect to this particular aspect; such studies have found that many students do not seem to have an understanding of this aspect of generality. For example, Chazan (1993), studying high school geometry students, found that some students viewed deductive proofs as verifications of single cases that were subject to possible counterexamples. Porteous (1990) examined the type of evidence students found to be convincing. His results indicated that when presented with a particular case, over half the students empirically checked it rather than appealing to the proof of the general case that they had previously been shown and had presumably accepted. Similarly, Fischbein and Kedem (1982) found that most of the students in their study opted for supplementary checks of an already proven statement, one with which they had previously expressed their full agreement. A second (related) aspect of generality concerns the idea that empirical evidence does not suffice as proof. Again, for many students this aspect of generality appears to be one that they do not adequately understand—a finding that predominates the results of many studies is students' reliance on the use of examples to prove the truth of a statement or result (e.g., Balacheff, 1988; Healy & Hoyles, 2000). For example, Healy and

Hoyles, in their study of high attaining 14- and 15-year old students, found that empirically-based arguments dominated the nature of justifications students provided in response to the researchers' assessment questions. Chazan also reported a similar finding: students believed that empirical evidence allows one to make general claims about the truth of a proposition.

Although the aforementioned studies suggest that students do not have a very robust understanding of (either aspect of) generality, little research has specifically explored the nature of students' understandings of generality itself. The purpose of this paper is to provide insight regarding students' understandings of (the limitations of) empirically-based arguments as well as their understandings that a general argument (i.e., a proof) treats the general case. In particular, we address the following two questions: To what extent do students think that examples suffice as proof? and To what extent do students recognize that a proof treats the general case?

Methods

Data were collected from 394 middle school students (grades 6-8); the students all attended the same middle school. The middle school recently adopted the reform-based curriculum *Connected Mathematics Program*; the adoption of this particular curricular program is noteworthy given the program's emphasis on mathematical reasoning (Lappan, Fey, Fitzgerald, Friel, & Phillips, 2002). The primary source of data was student responses to written assessment items. The two particular items that are the focus of this paper targeted the idea of generality. In the first item, students were given two arguments—one examples-based and one general (i.e., proof)—justifying the truth of a statement. Students were then asked to decide (and explain their decision) which argument demonstrated that the statement was always true. In the second item, students were given a statement and asked if the statement was true for a small set of numbers. In this latter item, students could either attempt to construct an argument demonstrating the statement's truth in general, or they could use the method of proof-by-exhaustion to demonstrate its truth for the specified set of numbers. A follow-up question then asked students if their justification also demonstrated that the statement was true for any number (not just those numbers included in the initial set). The students' responses to the assessment items were analyzed in terms of the two aspects of generality described previously. In particular, students' responses were coded using the following general coding descriptions (cf. Knuth, in progress; Waring, 2000): students consider checking a few cases as sufficient; students are aware that checking a few cases is not sufficient, but do not seem aware of the need for a general argument; students are aware of the need for a general argument, but perceive general arguments as limited (e.g., examples still need to be verified); students are aware that a general argument treats the general case.

Results & Discussion

Due to the page length limitations of the conference proceedings, results from the two focus items are briefly presented and discussed here (more detail as well as additional results will be presented during the conference session).

Assessment Item 1

Prior to presenting the results for this item, it is worth noting a relevant finding from the previous year's assessment: consistent with findings from previous research, students demonstrated an overwhelming reliance on the use examples as a means of demonstrating and/or verifying the truth of a statement. For example, the majority of students at all three grade levels "proved" that the sum of any two consecutive numbers is always an odd number by providing several examples demonstrating that the statement was indeed true; very few students attempted to provide a general argument. Students' reliance on the use of examples led us to wonder whether students actually believe that examples suffice as proof or whether it may be the case that they are aware of the limitation of empirical evidence, yet, are unable to produce a general argument themselves and thus examples-based arguments are their only recourse in attempting to justify (Healy & Hoyles, 2000, makes a similar observation). To address this issue we presented students with the following statement: *When you add any two consecutive numbers, the answer is always odd*. Students were then given two arguments justifying the statement (the arguments were attributed to fictitious students). The first argument, Samari's, shows three examples of consecutive numbers adding up to an odd sum, and a concluding statement that the given statement is true for all consecutive numbers because it is true for the three examples. The second argument, Ellen's, uses a deductive chain that begins by stating that with two consecutive numbers you always get one odd and one even number, and since an odd number and an even number sum to an odd number, any two consecutive numbers will always sum to an odd number. Following presentation of the two arguments, students were asked the following question: *Whose response tells us that if we were to add any two consecutive numbers we would get an answer that is an odd number? Explain your answer.*

Given that the majority of students produced examples-based justifications when asked to prove the statement on a previous assessment, one might conjecture that such a justification would also be the most popular choice among the students. To some extent this was indeed the case; across all three grades, approximately 40% of the students selected Samari's argument. Typical student explanations included:

> *Samari's response because it gives examples of 2 very different numbers, and it explains very well* (7th grade student).
>
> *Samari's. Because she explains it and she also gives examples to prove it* (7th grade student).
>
> *Samari's response because she actually has an answer to give that proves this. So you add two consecutive numbers together you will get an odd number* (8th grade student).
>
> *Samari's is correct because she can give proof and Ellen's just tells* (6th grade student).

A significant proportion of students (~30%), however, selected Ellen's argument as the argument that proves the statement. The following responses are representative:

> *Ellen's response makes more sense because Samari's response worked for those two numbers but it doesn't prove it always would* (6th grade student).
>
> *Ellen's because she tells why it will always be an odd number and Samari's show some examples that show some consecutive numbers and their answers* (6th grade student).
>
> *Ellen's because she tells us numbers go even, odd, even, odd, etc., and that when you add an even number with an odd number, the answer is always odd which Samari doesn't tell us, she just gives examples* (7th grade student).
>
> *Ellen's. Samari's response proves it's true for 2 pairs of numbers only. Ellen's proves it's true in all cases* (8th grade student).

The results suggest that when given the choice between a general argument and an examples-based argument, a significant proportion of students selected the general argument as the one that demonstrates the truth of the given statement for all cases. Thus, it may be that although many students are unable to produce general arguments themselves, they do seem to recognize the difference between a general argument and an examples-based argument and, moreover, they may view the general argument as a proof.

Assessment Item 2

In a written assessment presented to students during the previous school year, they responded to the following item (cf. Porteous, 1990): *Sarah discovers a cool number trick. She thinks of a number between 1 and 10, she adds 3 to the number, doubles the result, and then she writes this answer down. She goes back to the number she first thought of, she doubles it, she adds 6 to the result, and then she writes this answer down. [The preceding text was also accompanied by a worked out example using the number 7.] Will Sarah's two answers always be equal to each other for any number between 1 and 10?* Although the majority of students "proved" that Sarah's two answers would always be equal to each other by using examples, a significant proportion (~20%) used the method of proof-by-exhaustion to justify that the two answers would always be equal to each other. The students' use of this method prompted us to question whether students were knowingly (in a mathematical sense) exhausting the set of possibilities or whether they were simply testing examples (albeit the complete set). In other words, did these students perceive a difference between checking some cases and checking all cases in justifying the truth of a proposition? To address this question, we presented students with the same item the next year and, in addition, included the follow-up question: *Does your explanation show that the two answers will always be equal to each other for any number (not just numbers between 1 and 10)?*

Students' responses to the first part of the question were similar to the results from the previous year; however, their responses to the second part were perhaps the most interesting. Not surprisingly, the majority of students used examples as their method of justification for both parts. The following are representative of the responses (for both parts) from such students:

> *Yes because I tried some of the other numbers and for all of them I got the same answers. It applies for all numbers because I tried it with different examples [student shows two examples greater than 10]* (8th grade student).
>
> *Yes, the answers will always be equal because I tried her method using 5 and the results came out equal. I also tried 8 and the answers came out equal. Yes, it is true with every number because I tried that method with 11 and the answers came out equal* (6th grade student).

In contrast, some students seemed to recognize the limitation of examples as a means of proof:

> *[Student correctly works out an example using 8.] No, just because you do two examples [i.e., the given example and the student's worked out example] doesn't mean that if you do another two that they'll be the same. No, because if you do 3 more problems like these it doesn't mean that they will be equal to one another* (6th grade student).

Somewhat similar in nature were responses from students who seemed to recognize that they could use proof-by-exhaustion to justify that the two answers would always be equal when the choice of numbers was limited to those between 1 and 10, and that this method of justification would not suffice for justifying that the two answers would be equal for numbers outside that range. The following responses are representative:

> *Yes, all results will be equal between 1 and 10 [student shows examples for all numbers 1-10, except 7 which was worked out as a part of the item]. No, my explanation shows the two answers will be equal just for numbers 1-10* (8th grade student).
>
> *Yes [student shows examples for numbers 1-10]. No, I only gave explanations for 1-10* (7th grade student).

These latter responses seem to suggest that for some students examples do not suffice as proof, and that they do in fact recognize the limitation of empirical evidence as a means of justification.

Interestingly, a number of students, who also used the method of proof-by-exhaustion for the first part, concluded (sans a general argument) that the two answers would be equal for all numbers. In some cases, students simply stated that because they tested the numbers 1-10 and the two answers were always equal, then the number trick would also work for numbers outside that range. For example, one student responded:

> *Yes [student shows examples for all numbers 1-10]. Yes, because I did it for each number between 1 and 10* (8th grade student).

In other cases, students based their justification for the second part on further examples:

> *Yes, Sarah's two answers will always be equal to each other for any numbers between 1 and 10 because I tried every number between 1 and 10 and it does work. My explanation shows that the two answers will always be equal to each other for any number not just numbers between 1 and 10 because if you tried 56+3=59; 59x2=118. 56x2=112; 112+6=118* (7th grade student).
>
> *Yes, Sarah's two answers will always be equal for 1-10. I know that because I tried each number 1-10. Yes, the two answers will always work for any number. I checked by trying different numbers, both large and small, odd and even* (6th grade student).
>
> *Yes, because I did examples for 1-10 numbers [student shows examples for numbers 1-10]. Yes, it does because if you do 12 or even 15, it will equal the same number* (8th grade student).

These latter two sets of responses suggest that these students may not perceive a difference between using proof-by-exhaustion as a method of *proof* and simply using examples as a method of justification—these students may have simply tested 10 examples and then tested additional examples to "widen" their examples-based justification.

Lastly, the data also include responses from students who demonstrated the ability to produce a general argument as their means of justification. In such cases, these students recognized the underlying mathematical relationship, although their articulation of the relationship varied in terms of clarity; nevertheless, it is clear that these students were attempting to treat the general case. Sample responses include:

> *Yes they will be equal because when you add 3 and multiply by 2 it's the same as multiplying by 2 and adding 6. Yes it does because: (a+3)2 = 2a+6. They are the same thing and ANY number will work* (8th grade student).
>
> *Yes, Sarah's answers will always equal to each other for any number between 1 and 10 because both number tricks are doing the same thing to the number. Both number tricks double the number. Since Sarah adds three before she doubles the number, she has to add six to the other trick because she doubles before she adds. [Student shows examples using 23 and 83 as the starting numbers.] Yes, this number trick will work for any number because each of the tricks is doing the same thing to the number, just written differently* (8th grade student).

There were also some interesting responses in which students presented a general argument for the first part, but then seemed to feel that the general argument was not "general." For example, one student felt that his general argument only applied for the numbers 1-10:

> *Yes, because if she always adds 3, then doubles your answer and gets that answer. And then goes back and does it in a different sequence she'll always get the same answer. No,*

because she only did numbers between 1 and 10 (6th grade student).

Another student responded similarly initially, but then had a change of mind—she seemed to realize that the general argument she presented was indeed general:

> *Yes, I think it would work because each time you get the answer you will get the same answer as the one before. I think this because the 1st time through, you add 3 and then times by 2. But the 2nd time through, you times by 2 and then add 6. The 2nd time through since you don't add 3 when you add 6, that is what doubles the 3 like the 1st time through, so both ways you add 6. No, it does not show that numbers higher than 10 would work. [The student then seems to change her mind.] As I showed above, it [the justification] is the same thing that would be used here* (6th grade student).

Finally, a number of students who presented a general argument felt the need to "prove" that their general argument was general by demonstrating with examples that it worked. The following student's response is representative:

> *The two answers will be equal because you're doing it [the steps] in reverse order and the reason there is a 6 instead of a 3 is because the 3 gets doubled in the second way. Yes, it does and to prove it I will show you [students shows that the number trick works using the number 17]* (6th grade student).

Concluding Remarks

In closing, the results suggest that many middle school students lack an understanding of generality. Yet, the results also suggest that some students do possess an understanding of generality: Students produced and selected general arguments, recognized the limitation of examples as proof, and correctly used proof-by-exhaustion. If more students are to develop their understanding of generality—and of proving more specifically—then they must be given opportunities to engage in activities which highlight important ideas about proving.

Endnote

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CONCEPTIONS OF PROOF AMONG PRESERVICE HIGH SCHOOL MATHEMATICS TEACHERS

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The goals of the present study are to take an in depth look at preservice high school mathematics teachers' conceptions of proof and to identify the main differences between the students of the four years of university training. We began by conducting individual interviews with 12 preservice teachers and then we distributed a written questionnaire to 309 students. The questions were designed to identify the students' conceptions regarding different mathematic and pedagogic aspects of proof. An implicit teaching of proof in combination with the lack of the teaching of certain elements is, in our opinion, the cause of certain weaknesses in the conceptions of proof that we observed among our participants. In this article we will present the main results of our study and offer certain recommendations in order to improve the teaching of proof in university training.

Problematique and general theoretical framework

For the past two decades, we have witnessed a world wide trend toward a gradual return to the teaching of proof in high school program of studies. The current situation in Québec is in line with this world wide trend. In the curriculum of the 90's there was a reinstatement of the teaching of proof and the next one, scheduled for 2005, promises to make even more room for proof. Before re-integrating the teaching of proof in our high school program of studies, there seems to be an important question that should be answered: do teachers have the necessary competencies to adequately teach proof it at the high school level? Some studies have highlighted certain problems regarding the conceptions of proof among mathematics students and in the university training of preservice mathematics teachers. (Alibert et al, 1987; Almeida, 1995 ; Anderson, 1999 ; Gardiner and Moreira, 1999 ; Harel & Sowder, 1998 ; Jones, 2000; Mingus & Grassl, 1999; Moore, 1994; Tall, 1991). However, only very few studies have been conducted regarding this subject and they only present an image of the participants' conceptions at a fixed moment in time, instead of trying to show the evolution of their conceptions over a certain period of time. Such concern seems essential as we do not only want to show our participants' conceptions of proof at a specific moment in time during their university training, but rather examine the role of university training on the development of the conceptions of proof among our preservice high school mathematics teachers.

Subsequent to Knuth's (1999) study concerning the conceptions of proof among high school mathematics teachers, he proposed the following recommendation: "to trace the development of teachers' conceptions of proof during the process of learning to teach secondary school mathematics". In his opinion: "Studies of this nature would provide a more complete picture of the factors that influence teachers' conception of proof" (Knuth, 1999, p. 169-170). He adds that this type of study could also help better target the elements within the process of the education system of preservice teachers, enabling the possibility of change regarding the conceptions of proof among teachers. With this in mind, a general research question arises: What conceptions of proof do preservice teachers develop during their university training?

In order to identify the conceptions, we felt that it was imperative to carry out an exhaustive review of the literature dealing with the different factors that make up the conceptions of proof.

Knuth's (1999) model of the conceptions of proof, based on the work of Harel and Sowder (1998), strongly influenced our work regarding the identification of these different factors. The general structure of Knuth's model is comprised of three main elements: the nature of proof, the role of proof, and the schemes of proof. These three elements are dealt with according to two different categories: proof in mathematics and proof in an education context.

The first element, the nature of proof, refers to a certain type of mathematics culture of proof shared by the participants. It can be associated to a global, an occasionally philosophical, or a more pedagogic view of the process of validation. According to Knuth (1999), the conceptions associated to the process of validation influence the way teachers incorporate mathematics reasoning and proof in the classroom and the teaching methods they use to convince their students of these concepts. The second element, the role of proof, refers to more pragmatic views of this notion. Knuth (1999) concedes that teachers' conceptions in relation to the role of proof affect their beliefs about what constitutes proof and justification in class. Lastly, Knuth (1999) defines the schemes of proof as being the processes that an individual adopts or accepts in order to eliminate his or others' doubts concerning the validity of a statement.

Methodology

Our sample was comprised of 321 preservice teachers enrolled in one of the four years of a university mathematics high school teaching program. The participants were selected from four different universities in the province of Québec.

We conducted individual interviews with a first group of 12 preservice teachers (6 from the first year and 6 from the fourth year). The questions used during the interviews were developed with the intention of targeting the different elements in our model of the conceptions of proof. Each interview was tape recorded and transcribed. We then meticulously analyzed their content, which enabled us to describe the conceptions of each of the participants in relation to the model of conceptions of proof. Based on these results, we developed a written questionnaire comprised of multiple choice questions. We then distributed it to 309 preservice teachers enrolled in any of the four years of the program. The construction of the questionnaire, like the interviews, was based on the elements found in the model of the conceptions of proof.

The questionnaire included two main types of questions. The first type of question, inspired by the work of Almeida (1995), was comprised of different affirmations related to proof. The participants had to rate their level of agreement of the affirmations on a scale ranging from -2 to 2. The affirmations incorporated mathematical and philosophical aspects of proof, for example:

1- A mathematician's regular practice is principally dominated by the writing of perfectly formal and rigorous proofs.	______
10- For professional mathematicians, a mathematic proof is only valid if it is rigorously and formally written, with the use of the usual symbolism.	_____

Other affirmations incorporated elements that were more pedagogic in nature:

21- What is most important for high school students is not to learn how to write proofs, but rather to be able to understand them.	______
24- The best way to develop students' ability to write proofs is by giving them many examples and then by making them practice as much as possible.	______

The second type of question, based on the work of Healy & Hoyles (1998), consisted of different proofs of the same statement. The participants had to judge certain aspects of proof including its validity, its explicit and convincing nature, and its pedagogic properties.

In order to analyze the results obtained by our questionnaire, we opted for a statistic analysis of the participants' answers. We began by conducting a descriptive analysis of each of the questions. Average, standard deviation, frequency, or relative frequency, were used in the analysis, depending on the question. This first analysis gave us a general picture of our participants' conceptions of proof. Next, we conducted comparative analyses between the participants enrolled in each of the 4 years of the program, in order to try and identify if there were indeed differences in the participants' conceptions between these sub-groups. In order to do this, we used multiple analyses of variance. Lastly, we conducted linear regressions between the sub-groups' averages for the same question. Our intent behind this procedure was to verify if the differences observed between the averages varied consistently from one sub-group to another for the same question.

Results

Among the majority of the participants, the analysis of the collected data highlighted the presence of a moderate formalist view of proof and the process of validation. This majority sees proof as being a mandatory ritual and the mathematician's main duty. Also, when validating proofs, they feel that rigor and the use of the usual formalism constitutes a *sine qua non* condition. Thus, formalism and rigor are, in their opinion, strongly linked to the writing of proofs and to the guaranty of their validity. For the most part, our results show that our participants consider proof as a mechanical construction that follows precise rules and uses a formal and high level of rigor, rather than a human construction in which validity depends on a social consensus of a group of experts.

However, some of the results observed with the help of our questionnaire tend to contradict this formalist view, thus lessening this weakness. A majority of the students acknowledged that the discovery process in mathematics is characterized by intuition and a recursive trial and error process. In addition they recognized the need of a more informal, less technical language to explain proofs.

When comparing the mathematical and pedagogical points of view, we observed that the pedagogical point of view put less emphasis on the importance of rigidity, while on the other hand it put more emphasis on the explicative properties inherent to proofs. Our results show, however, the presence of relatively simplistic conceptions about the different pedagogical proof teaching methods that exist. Our participants preferred traditional low student interaction teaching methods combined with repetitive drilling. In their opinion, the best way to develop students' proof writing abilities is by providing them with many different models and by giving them as many opportunities as possible to reproduce these models in other contexts.

Our different results also demonstrate that our participants have an exaggerated preoccupation regarding the form of proof in high school. In fact, they heavily insisted on the importance of respecting a strict proof writing format, represented mainly by the two column model; affirmation-justification. For many of them (mainly the participants in the beginning of their training), familiarizing students to this type of writing format should be the principal element of proof teaching. We also noticed that this exaggerated interest for this model of proof could influence their judgment when they were required to evaluate the validity of a proof. In fact, the majority of the participants chose proofs of this type as being those that they would have done in some of their university classes even though they didn't think that they were the most convincing. In addition, it seems that this two column model influenced many participants (more than one third of them) to identify a proof as being valid even when it wasn't. According to Schoenfeld (1989), the rigid format of this two column model of proof writing imposed on students, forces them into a position of passivity or powerlessness regarding the lack of freedom they have when writing proofs, thus leaving the proof writing exercise void of any pragmatic meaning. This exaggerated insistence for this model of proof seems to us as being an important weakness in their conceptions.

We have also observed some underdeveloped conceptions regarding the diverse possibilities and limitations of using the Cabri-géomètre software to teach proof. The students at the end of their training were, however, those who displayed the best knowledge about the different possibilities the software offers for the classroom. Contrary to the students in their first year of training, they recognized the pertinence of letting students experiment with the software in order to discover the geometric properties of figures. The majority of the participants in the beginning stage of their training felt that Cabri could only be beneficial to the teacher when helping reinforce visual representations written on the board. However, none of the participants mentioned the time and effort saving function that Cabri offers allowing the in-class reproduction of the mathematical empirical process of discovery and validation, nor that manipulations leading to the discovery of an invariant or of a proposition could facilitate the subsequent production of proofs by students, as pointed out by Olivero (2000) and Mariotti (2000).

Regarding the different roles of proof, our participants showed very little interest in recognizing the role proof plays when convincing students of the exactitude of theorems taught in class. However, this role is identified by certain authors, Arsac et *al* (1992), as being for students, the role that best justifies the teaching of proof. Instead of attributing the role of proof in class to functions associated to a process of validation, the participants attributed it to more educative functions. Firstly, they mentioned that proof allowed for the development of logical and deductive reasoning. Secondly, they associated an explicative role to proof by recognizing its ability to enhance comprehension.

The main differences of conception between preservice teachers from the four different years of training

In general, we have noticed little difference between the conceptions of the participants that are in different years of training, even between those in the beginning of their training to those at the end of their training.

We first noticed that, for the most part, our participants rarely recognized the social element of proof in mathematics, those just beginning their university training demonstrated the most ignorance regarding this subject. These same beginning students also demonstrated the most fragile and underdeveloped conceptions about the global functioning of the mathematical processes of discovery and creation. We found that the participants at the end of their university training had a better mathematics culture as they demonstrated a deeper knowledge about the general functioning of mathematics and its developmental history, as well as demonstrating more solidly rooted and less fragile conceptions about the nature of proof.

We also observed, among our participants beginning their university training, more weaknesses when judging the validity of proofs. They had more difficulties than the other groups when identifying non valid proofs as valid. However, the proofs that they judged were from the grade five level of high school and thus accessible to all of the participants in our study. This result, somewhat surprising, has also been found by Recio and Godino (2001). It brings us to believe that deductive reasoning and abilities to write simple proofs continue to develop throughout university training, even if such proofs are not necessarily taught in mathematics classes.

Discussion

The different weaknesses observed among our participants do not seem as if they can be attributed to an explicit teaching of proof at the university level. As many authors point out, (Alibert et al, 1987; Gardiner and Moreira, 1999; Hersh, 1997), we tend to believe that these problems stem from an implicit teaching of proof in combination with an absence of the teaching of certain elements. Looking at proof and its usage and place in the mathematics world with a more philosophical nature is, in our opinion, probably ignored or at best done in a superficial way, leaving the door open for the writing of proofs. Faced with this type of situation, preservice teachers find themselves having to construct their own conceptions from interpretations based on insufficient knowledge and on an experience that will unfortunately be rather short.

Different observations lead us towards this hypothesis. Firstly, the presence of formalist views observed among our participants. In fact, according to Hersh (1997), the popular beliefs that can inspire preservice teachers as they develop their conceptions are fed by the fondationnistes philosophical perspectives in which one finds an idealist and mythical view of proof. The hypothesis of implicit teaching is also supported by the large amount of variation in the responses of our participants to the questions dealing with the nature and the role of proof in mathematics and in school. In fact, we think that the differences of opinion between the students from the same year show that they received little to no explicit teaching about the nature and role of proof. Also, the small amount of differences observed between the students beginning their training and those at the end, lead us to believe that the students have, for the most part, the same conceptions from one year to the next. It seems to us that a very small amount of changes take place between the beginning and the end of the students' university training regarding their conceptions about the nature and the role of proof in mathematics and in school. On the academic level, the rather traditional and simplistic views about the way proof is taught, as well as the limited knowledge about the diverse possibilities that Cabri offers, lead us to believe that the preservice teachers' conceptions are based on their former experiences as students in secondary school.

Recommendations

The weaknesses observed in our participants' conceptions in combination with the small amount of evolution of their conceptions throughout their university training, leads us to make certain recommendations about university training of preservice teachers.

It first seems imperative that preservice teachers should have a rather precise idea about the role and place of proof in mathematics as well as on the academic level. With this in mind, it seems important that a mathematics history course should explain the evolution of the process of validation by pinpointing the methods, the ways to conduct validations, and the place occupied by this process in the mathematician's work throughout history. With this type of course, it could also be possible to give an overview of the philosophical perspectives that have most influenced the activity of validation in mathematics.

During the preservice teachers' training, more emphasis should be put on the studying of proof as a teaching tool. It is not easy for teachers in training to understand the ''didactical transposition'' that takes place when proof changes from a method of validation in mathematics to that of an object of teaching. In fact, as Arsac (1987) believes, proof or its demonstration undergoes a particularly important change (didactical transposition) that can not be ignored if one wants to teach in a way that is adapted and meaningful to students.

It is also our opinion that it is essential for students to receive an explicit teaching of proof's different properties and of the mathematical process of validation. Such properties are not necessarily discovered only by the practice of writing proofs, but by an explicit description of this process. Also, we think that the teaching of the two column model of proof writing in high school must be reconsidered. Our intention is not to judge this model, however, we would like to point out that many authors (Alibert and Thomas, 1991; Leron, 1983; Schoenfeld, 1989) have criticized the abusive usage of this type of model of proof. Thus, it seems necessary to expose preservice teachers to different models of proof accessible to high school students, while insisting on their pedagogic characteristics.

In closing, we think that software such as Cabri-géomètre deserve a more in depth study, going further than just looking at the simple development of technical competencies in figure construction or activity creation. The ways of integrating this tool into the classroom as well as its advantages and inconveniences on the development of proof writing abilities, must also be presented and discussed.

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FROM CONCRETE REPRESENTATIONS TO ABSTRACT SYMBOLS

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We report on the transition from personal representation to formal notation for a group of five students from a community of students engaged in doing mathematics for several years. The group was introduced to standard combinatorial notation after they had already investigated concepts in counting using personal representations that they built over several years. We describe the strategies used by the students to make sense of their ideas and report on how they came to represent their ideas using standard notation as they worked together to share and connect representations.

Theoretical Framework

It is an expectation that students learning mathematics eventually become proficient in the use of conventional mathematical notation. Standard notation offers a common language for communicating mathematically; appropriate notation can be helpful in recording the important features of a mathematical problem. Davis and Maher (1997) observe that students who are provided with varied mathematical experiences build repertoires of representations. These representations are used for building new mathematical ideas. Given rich and challenging investigations and ample time to explore and revisit ideas, students have an opportunity to construct new representations and connect these representations to other knowledge.

According to Muter and Maher (1999), in the process of revisiting earlier ideas, learners extend and refine their representational strategies, moving from objects to symbols. In studying the problem-solving behavior of a group of five students who apply their earlier representations and ideas to make sense of a general solution to a problem using standard combinatorial notation, we explore the following question: How do students use personal representations in developing an understanding of standard notation?

Method of Inquiry and Data Sources

This research uses archived data from a longitudinal study (Maher, 2002) that has followed the mathematical thinking of a group of public school students (Ankur, Brian, Jeff, Michael, and Romina) from first grade through high school (1988-2000) and new data following the same students through university (2002-2003). All sessions were videotaped, most with two cameras, one following the movements of students and the other following their written work. Videotapes, student work, and researcher notes provide the data for the analysis. Summaries were made of all sessions, and they were coded for critical events (events related to students' representations and use of standard notation). All critical events were transcribed and reviewed for accuracy.

Students worked on the following three problems (with variations and extensions) over several years. Limitations in space prohibit a report on students' initial work on these tasks. For detailed reports of their initial experience with these investigations, see Maher and Martino (1996), Maher and Kiczek (2000), Kiczek, Maher, & Speiser (2001), and Powell (2003). This paper focuses on how these students recalled and continued to extend their earlier work for some time after the after-school sessions were concluded.

1. **The Pizza Problem**: Students were asked to find how many pizzas it is possible to make

when there are various numbers of toppings to choose from. C*(n,r)* (the r^{th} entry in the n^{th} row of Pascal's Triangle) gives the number of possible pizzas with exactly *r* toppings when there are *n* toppings to choose from.

2. **The Towers Problem**: Students were asked how many towers of various heights they could build from Unifix cubes when there are two colors to choose from. *C(n,r)* gives the number of towers *n* cubes tall containing exactly *r* cubes of one color.
3. **The Taxicab Problem**: Students were asked to find the number of shortest paths from the origin (a point in the top left corner of a rectangular grid) to various points on the grid, when the only allowed moves are to the right and down. *C(n,r)* gives the number of shortest paths from the origin to a point *n* segments away, containing exactly *r* moves to the right.

The students first worked on versions of the towers and pizza problems during elementary school. For towers problems, they first built towers with Unifix cubes and for pizza problems, they first drew pictures (Maher and Martino, 1996; Maher and Kiczek, 2000). They went on to use tree diagrams, letter codes, and lists with varying degrees of organization. On revisiting these problems in later years, they developed more formal representations, first using tables and numerical codes, and then working with a binary notation (1 to represent a topping on the pizza and 0 for a topping not on the pizza) developed by Michael in 1997. Here, we attend to their later work in connecting meaning to symbols as they explored Pascal's Triangle and Pascal's Identity (the addition rule for Pascal's Triangle).

Results

In the May 1999 session, Ankur, Jeff, Michael, and Romina were asked to write the numbers in Pascal's Triangle in a standard combinatorial notation. They went on to generate Pascal's Identity and to explain it to Brian, who arrived after the session had begun.

In the May 2000 session, Brian, Jeff, Michael, and Romina investigated the taxicab problem and referred to Pascal's Triangle in their investigation.

In individual interviews (2002-2003), Ankur, Michael, and Romina built on and extended their earlier ideas.

Episode 1: Writing Pascal's Identity

In the May 1999 session, Ankur, Jeff, Michael, and Romina were asked to write a general row of Pascal's Triangle and to write and explain the general addition rule. They wrote row N of Pascal's Triangle as shown in Figure 1. In response to the researcher's request, the students generated Pascal's Identity shown in Figure 2.

$$\binom{N}{0}\cdot\cdot\binom{N}{X-2}\binom{N}{X-1}\binom{N}{X}\binom{N}{X+1}\binom{N}{X+2}\cdot\cdot\binom{N}{N}$$

Figure 1: Row N of Pascal's Triangle

$$\binom{N}{X}+\binom{N}{X+1}=\binom{N+1}{X+1}$$

Figure 2: Pascal's Identity (1999)

The researcher asked the group to explain the meaning to Brian (B). In response, Jeff (J) referred to the pizza problem. Romina (R) and Michael (M) contributed.

J: If you added another topping onto your whole- Say we're doing pizzas. If you add another topping onto it-

R: You know how we get the triangle and how we go one two one and add those two together?

B: Yeah.

R: That's what we're doing right there.

M: You know why we add, though?

J: We were explaining why you add.

B: All right, keep going.

J: If it gets a topping, that's why it goes up to the X plus 1. [Jeff points to the right side of the equation.] And since it doesn't get anything, it'll stay the same. And in this one it's staying the same, right? [Jeff points to the second term of the left side of the equation.]

M: Yeah.

J: Make sense?

B: Yes. It actually does.

J: So that would be the general addition rule in this case.

Episode 2: The Taxicab Problem (2000)

In 2000, Brian (B), Jeff (J), Michael (M), and Romina (R) worked on three instances of the taxicab problem (finding shortest paths to three points at different distances from the origin). Their notations included color-coded paths drawn on grids representing streets (representing possible taxicab paths) and lists of paths organized according to complexity as measured in number of turns. Refer to Powell (2003) for further details. During this investigation, they moved from the three specific problems to an investigation of a general solution. In so doing, they noticed that the numbers in Pascal's Triangle appeared in the solutions list of the general taxicab problem.

Michael and Romina made explicit the link between taxicabs and towers in two specific cases. First Michael linked the 3-tall tower containing exactly two Unifix cubes of one color to a 3-move path with exactly two moves in one direction.

M: You have a tower of three and you have, you know, two colors. So one, it's either, you know, color *x* and two of color *y*. Well, this is direction *x* and two, two directions of *y*.

A short time later, Romina described a similar link between specific 4-tall towers and specific taxicab paths of length four:

R: The four is still three and one but then it's three across and one down so it means it's three of one color and one of the other color.

Finally, at the end of the session, Romina (R) and Brian (B) explicitly made the general connection. They had been using *x* and *y* to discuss general movement on Pascal's Triangle. The researcher (R2) asked for clarification:

R2: And the x's and y's- What does x correspond to again?

R: x is across.

B: Going across. And y is down.

R: Or a topping or a color. All the same thing. And all our y's are down, toppings, color.

Episode 3: Michael's Interview (2002)

Michael's interview took place in April 2002, when he was in the second year of college. The researcher (R1) referred to the 1999 problem-solving session and asked Michael to talk about Pascal's Triangle.

R1: It was looking at how the triangle grew. And, um, that was the question. How can you talk about how that triangle grows? And you had used the example of pizzas to think about the movement from one row of the triangle to the other.

Michael (M) responded with a description of row 2 of Pascal's Triangle and then a generalization to following rows.

M: OK. If you had no toppings, that would be one pizza.

R1: OK. So, where is that on the triangle?

M: Well, I'm going to just draw it. And then we'll find it. You know. If you're having only, using just one topping, you can make two possible pizzas with that. And then if you have all, all the toppings, that's one. Right. And then automatically you, I see that, that relates to this row. [Michael points to row 2.] And I'm pretty sure it would go down, this is like a third topping, and a fourth topping. [Michael indicates rows 3 and 4.] Now I think the way I, um, thought about it is, like, the row on the outside would be your plain pizza. [Michael refers here to the 1s down the left side of Pascal's Triangle.] And there's only one way to make a plain pizza. And the next one over would be how many pizzas you could make, um, using only one topping, and then so on until you get to the last row which is, um, all your toppings. [Michael refers here to the last number in the row.] And, once again, you can only make one pizza out of that.

The researcher next asked Michael to write Pascal's Identity:

R1: And, um, at that session, what the students did, I asked them to write an equation to show, for instance, how that might happen from one row to the next. Um, so can you just do that, write.

M: Like a general equation?

R1: Well, um, that was what I was going for ultimately.

M: To, uh, give an amount for any spot. All right, so I guess we'll give, uh, these, you know, the row a name. Um, call that *r*. And, um, I guess the spot in the row, like, you know, zero topping, one topping. Call that, *n* sounds fine. Just, just to like pick, you know, one spot and then see what-OK? Um. [There is a pause as Michael writes.] I'm just going to like work this out in my head and see if it actually works. [After a brief pause, Michael writes the equation show in Figure 3.]

$$\begin{pmatrix} r \\ n \end{pmatrix} + \begin{pmatrix} r \\ n+1 \end{pmatrix} = \begin{pmatrix} r+1 \\ n+1 \end{pmatrix}$$

Figure 3: Pascal's Identity (2002)

It is interesting to note that Michael's notation was internally consistent (and correct) but represented differently from both what the group had done in 1999 and from the standard textbook notation.

Later in that interview, Michael was asked to talk about his method for answering mathematical questions, in particular how his group approached the taxicab problem in 2000.

M: I don't remember that specifically, but I know, I haven't taken math courses in a while, but usually I, the way I think, uh, make everything into a problem first, OK?

Episode 4: Romina's Interview

When Romina (R) was interviewed in July 2002 (just before her third year in college), she recalled how the towers problem could be used to explain the numbers in Pascal's Triangle. The researcher is identified as R3.

R3: Michael would explain things by talking about, you're going from a number of pizza toppings to a different number of pizza toppings, and how the addition rule works there. And you seemed to think of towers as the primary way to do Pascal's Triangle. So can you tell me anything? I sort of refreshed your memory a little bit. Do you remember anything from how you guys worked on this or how the addition rule would apply here? Or maybe you can just tell me in this one how you did the addition rule.

R: For this one? I think this is how many toppings. Like the top number, like the one choose one or one choose zero would be how many toppings. Or I mean if we were talking about towers. [Romina points to row 2 of Pascal's Triangle.] This would be with two high with zero reds, one red, two reds. And it just keeps going like three high, zero reds, one red, two, then three reds. So it would be like three high and like out of those, you choose how many blocks of each color.

Later, Romina made the connections among pizzas, towers and the binary notation that the group had originally used in 1997 to enumerate all possible pizzas:

R3: Can you look at this in terms of pizzas too?

R: You have two toppings to pick from. And then, what he [Michael] did with this one [Romina indicates row 2 of Pascal's Triangle.] is either you could- now, you could add a third toppings to your pizza. [Romina indicates row 3.] Like you have three options, you could either not add anything to the pizza. Or you could just add one more topping.

R3: All right. So when you said, "add one more topping," or "not add one more topping," can you relate that to red and blue?

R: You either, you add one more red block, or you just keep it consistent and add, just add another blue. So blue would be like nothing, like not an ingredient, and red would be an ingredient. Like, his binary [Michael's binary notation], it does the same thing. A zero would be blue or no topping. And a red one, which would be a one, would be a topping.

Episode 5: Ankur's Interview

Ankur was interviewed in July 2002, just before his third year of college. He watched a videotape of the May 1999 session and wrote Pascal's Identity as shown in Figure 2 as he viewed the discussion. At first, he noticed that the bottom number in the result always came from the rightmost number on the left side of the equation. The researcher (R3) asked for an explanation in terms of towers. Ankur (A) explain a specific case in terms of towers – one builds the 4-tall towers with two red cubes by adding a red cube to the 3-tall towers that have one red cube and by adding a blue cube to the 3-tall towers that have two red cubes:

R3: This is a three-tall tower with one red. [Refer to Figure 4; R3 indicates the first term.] It's three tall, so it's got one red and two blues. And then you changed that into a 4-tall tower with two reds. So how are you going to do that?

A: With two reds? You just add a red.

R3: You have to put a red on it. OK. But this one's a three-tall tower that already has two reds. [R3 indicates the second term in Figure 4.] And it goes to a 4-tall tower that has two reds.

A: So you got to add a blue.

R3: So you're going to add a blue to that one.

A: Uh-huh.

R3: OK. So this one combines with this one in the sense that-

A: You add all blues to this one [second term] and all reds to that one [first term]. A red to all three of those and a blue to all three of those and that's how you get- that's why the bottom number's X plus 1.

The researcher asked Ankur to follow up this reference to the general case:

R3: So tell me again with the general one. All right, here's an N-tall with X reds. [R3 indicates the first term in Figure 2.] And how are you going to get down there? [R3 indicates the last term in Figure 2.]

A: You're going add a red.

R3: And you go from there [the second term in Figure 2] to there [last term]?

A: By adding the other color.

$$\binom{3}{1}+\binom{3}{2}=\binom{4}{2}$$

Figure 4: Generating 4-Tall Towers With Two Red Cubes

Conclusions

Davis and Maher (1997) suggest that students can learn new mathematics by building on powerful representations (mental or written) with which they are already familiar. Many of the abstract ideas with which mathematics is involved have concrete early origins, such as building towers, making pizzas, and finding taxicab routes. Over the years students had opportunity to build and extend their early ideas and to extend and refine them. The earlier ideas became the building blocks for the more abstract and sophisticated concepts about counting illustrated in this report. Michael's use of notation across interviews over the years suggests that he was not just recalling a memorized formula but that he was using notation that made sense to him. Michael talked about "making things into problems." In doing mathematics, Michael connected symbols to problem situations with which he was already familiar. He represented these situations with symbolic notations that had meaning for him. His use of formal notation expressed Michael's generalizations of his ideas.

The students located familiar numbers from their work with building towers and making pizzas in Pascal's Triangle. They investigated Pascal's Triangle to explain whether those problems were related. They explained Pascal's Identity in terms of the rules for generating successive answers to the towers and pizza problems. When a standard notation was introduced and its relationship to Pascal's Triangle was observed, they expressed a connection between their personal representations and the standard notation using Pascal's Triangle. The meaning derived from their earlier work seemed to guide their exploration into Pascal's Identity and the generalization of their earlier ideas seemed to facilitate their use of standard notation. A year later, when they encountered the Taxicab problem, their understanding of Pascal's Triangle became an important representation to detect the structural similarity in spite of the surface differences. Finally, some years after their last investigation into Pascal's Triangle, they continued to maintain an impressive ability to explain and generate its numbers.

Our research is abundant with examples showing that, over the years, this community of students built ideas that came from extensive personal experience. The personal experience was accurate, relevant, and important in doing real mathematics in the sense discussed by Davis and Maher. The students expressed the way they thought about ideas that were new to them by referring to their own personal representations. In so doing, they recognized the structural equivalence among three problems that, on the surface, did not appear to be the same. Moreover, this understanding was durable over time.

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IDEAS, SENSE MAKING, AND THE EARLY DEVELOPMENT OF REASONING IN AN INFORMAL MATHEMATICS SETTING[1]

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We report on initial results of a three-year longitudinal basic investigation into the development of mathematical ideas and forms of reasoning that students in middle school build in an informal after-school environment. Thirty sixth-graders from an economically depressed, urban school district of 98% African American and Latino students, investigate well-defined, open-ended tasks in eight, one and a half hour sessions during the fall of 2004. From students' observations of and actions with physical objects (Cuisenaire rods) while solving problems, we report on their initial explorations and give examples of their reasoning. Through their actions, observations, and reasoning, we observe students building a foundational understanding of ideas about and operations with fractions.

Recently, educators and educational policy makers have identified the critical need for opportunities for academic and social development based on student initiative in contexts outside of traditional school hours (Urban Seminar, 2001; National Research Council, 2002). This is a particularly pressing need within minority communities in urban school districts. Martin (2000) reports on studies that show the expression of positive attitudes toward mathematics by African American children. Yet, when researchers examine their course-taking and persistence patterns, eighty percent take no more mathematics than what is minimally required to graduate. He further indicates the scarcity of research on studies focusing on academic success among African-American students. Even fewer studies address those students who are successful, raising issues of individual agency, success, and persistence, which, according to Martin, remain largely underconceptualized. We agree with Martin that individual agency is pivotal in the involvement of African American and other minority students with mathematics and with overcoming school-engendered failure in the discipline. To this end, we are investigating individual agency by looking for evidence of initiative and ownership of ideas through the analysis of the student-to-student discursive practices as individual students in collaboration with peers build mathematical ideas and forms of reasoning.

The theoretical framework that guides our analysis comes from extensive research on the development of representations (Davis & Maher, 1997; Kiczek, Maher & Speiser, 2001); work that traces the intricate and complex pathways for the learning of individuals within a larger community (Maher & Davis, 1995; Maher & Martino, 1996a, 1996b); the Pirie-Kieren Dynamical Theory for the Growth of Mathematical Understanding (Pirie & Kieren, 1994); and recent research on the discourse and inscriptions of learners as windows into learners' development of mathematical ideas, heuristics, and reasoning.

Objective

The current research is centered on African American and Latino students from a low-income, urban community, investigating how they build mathematical ideas and forms of reasoning in an after school, informal setting. The investigation is designed to document student discourse and to promote the exercise of agency by inviting students to engage in meaningful mathematical tasks and to study over time how they change their participation role in mathematics from what Larson (2002) describes as, "overhearers" to "authors" of mathematical ideas and texts. We are studying this through the analysis of the student-to-student discursive practices as individual students collaborate with peers to extend and refine mathematical ideas and build forms of reasoning to justify their solution to problems. A question that guides our work is: What mathematical ideas and forms of reasoning do students from disadvantaged, urban middle schools develop and use as they investigate well-defined, open-ended tasks?

Methods

Twenty-four of the thirty sixth graders in the after-school mathematics program are participants in our study, which focuses on two sessions that took place early in the first school year of the project. The main sources of data are as follows: (1) discourse patterns and other activity of students as they work on mathematical investigations recorded on videotape; (2) students' inscriptions; (3) researcher and observer notes, and (4) research team's planning and debriefing session scripts; session notes, and reflective diaries.

In all eight research sessions, one and a half hours in length, there are four cameras, two technicians for each camera, one for video and the other for sound. Our focus is on the students doing mathematics. Two cameras follow the introduction of the task as well as the facilitation of the whole-group session as the work and ideas of the students are made public. Teachers from the school district who are interns in the study as well as graduate students were trained as ethnographers and assigned to observe and record the actions of students at particular tables.

The research employs a framework for analysis that developed from our earlier work (Powell, Francisco, & Maher, 2003). We begin by observing each videotape and then describing each one in five-minute intervals, identifying *events*, *critical events*, *traces,* and collection of critical events that determine a *pivotal mathematical strand*.

Our study concerns the development of reasoning in students. To facilitate this investigation, in conjunction with the research participants and teacher interns, we establish an environment negotiated around specific research norms (Maher, 1998). They include researchers posing problematic situations that are thematically related within a strand of mathematics and that progress from complex to simple ideas. Researchers monitor participants' activity and thinking, organize and reorganize participants into work groups, facilitate participants to share their ideas and to listen to each other, encourage participants to justify and develop convincing arguments, and pose extensions and new problems. The problematic situations or tasks are thought provoking and encourage sense making. Participants build representations and heuristics some of which are manifest in the models they build, the gestures they make, and the inscriptions they write. The questions researchers ask participants inquire into the sense they make of their activity. Participant responses provide a window into schemes of reasoning that they are developing.

Theoretically, we posit that under certain conditions ideas dawn and mature over time. In our approach to interacting with participants in research sessions, we invite them to engage in mathematical tasks, to make public their ideas, and to share, discuss, and revisit them. As they respond to our invitations, they shape the way they work with other and with us. The participants take risks in making their ideas public since their ideas could be ignored, rejected, or criticized. The invitation to be heard, listened to and have their ideas and their responses considered seriously in time becomes norms of their evolving mathematical microculture. Eventually, participants' ideas naturally become reflected on deeply, presented publicly, submitted to challenge, available for negotiation, and subject to modification, and reconsidered and refined. That is, the essence of developing and understanding mathematical ideas is often a protracted, iterative, and recursive phenomenon (Pirie & Kieren, 1994), occurring over more time than is usually appreciated or acknowledged in practice in classrooms and in reports in the literature (Seeger, 2002). Our research norms include providing participants with time to work individually and collaboratively as well as opportunities to revisit earlier ideas and build on them.

Results

We report on Sessions 3 and 4 consisting of 12 hours of videotape. The students had available Cuisenaire rods, paper and pencil, as well as colored pens and overhead transparencies. The two principal investigators (indicated below as R1 and R2) assumed the roles of teacher-researchers.

Examining One's Ideas for Reasonability (Session 3: 11/19/03)

R1 posed the following problem: Someone told me that they think 4 white rods together are half as long as the blue rod. Do you or don't you agree? How many of you think that this is true? (No one responds.) I want you to talk together about this at your tables. If you believe that it is not true, how can you convince us?

In response, Ian at his table built a train of nine white rods alongside a blue rod, and said: "If you put four whites along the blue rod, there are one, two, three, four, five left over." R1 invited him to the overhead projector to build his model and explain his reasoning. Using translucent, overhead-projector rods, Ian placed a train of five white rods side by side and above a blue rod and then a train of four white rods side by side and below the blue rod and said: "As you can see," while counting the number of white rods below and then the number of white ones above the blue rod.

Also using rods a the overhead projector, Kori claimed, "Blue doesn't have a half because it's like an odd number." Another student added: "It's nine. Nine is an odd number, because if you add four and five it equals nine. And, if you put four white rods and then a yellow, it will equal a blue rod." In response to a request by R1 to say what she was thinking, Kori replied that she and her partner placed four white rods and a yellow rod along a blue one. She said, "The yellow equals 5 and that shows that the blue is not an even number." She explained that they tried to use each of the different colored rods to see if any "equaled up to the blue, but none of them did." She indicated that four reds would be less than the blue, two yellows would be higher, two blacks would be higher, two purple would be too low. She added, "We did find one that equaled blue, but it took three light greens."

Nia joined Kori at the overhead projector to share ideas inscribed on a transparency that they made in the previous session. Then they placed the transparency on the overhead projector. Kori explained, "Blue doesn't have a half because it's like an odd number." Another student offered, "It's nine. Nine is an odd number. Because if you add 4 and 5 it equals 9. And also if you put four white rods and then a yellow it will equal a blue rod."

During the remainder of the session, students explored relationships of the light green rods to the blue rod, reaching a final agreement that if blue were given the number name one, light green would have the number name one-third. In the examples above, to explain their reasoning, students referred to the length of the rods either by their color names or by numbers, odd or even, assigned to their lengths.

Using One's Ideas to Convince Others (Session 4: 11/20/03)

The students came to the session agreeing that if the blue rod were given the number name 1 then the number name for light green is one-third and for white, one-ninth. R1 posed to the students a problem based on the idea that the number name for blue was one: *What name should we give to the red rod given that two white rods are the same length as a red?*

In response to the question students developed a heuristic to determine the number name for the red rod and then use it to explore the number names of the remaining rods in the set. The students began to use models to work on a solution. For example, Chanel staircase, that is, an arrangement of the rods according to length, with each level a train of the rod plus one white rod (one-ninth) to illustrate the idea of growing ninths and counted allowed:

Chanel: One-ninth, two-ninths, three-ninths, four-ninths, five-ninths, six- ninths, seven-ninths, eight-ninths, nine-ninths, ten-ninths. Oh! I have to think about that one, nine-tenths?

The indication of surprise and her expression that she needs to think further indicated some disequilibrium. R2, who was observing Chanel's group, noticed Chanel's conflict and suggested to the group that they talk about it.

R2: Chanel has an interesting problem that she wants you hear about. Can you tell him [Dante] what you have here [pointing to her arrangement of rods]?

Chanel immediately responded to the invitation:

Chanel: See this is one-ninth, two-ninths, three-ninths, four-ninths, five-ninths, six-ninths, seven-ninths, eight-ninths, nine-ninths. What would this one be [pointing to an orange rod]?

Dante: That will be ten-ninths. I mean - actually that should be one. That should start a new one -This [blue rod] would be the old one, and this [orange rod] should start the new – or – it should be one-tenth.

Michael: The orange should be called a whole.

Chanel: Yea, one-tenth then get something bigger. It would be one-half, one-third.

Michael: No - You mean two-tenths.

R2: What are you saying the orange should be? What should the orange rod be called?

Michael: The orange should be called a whole.

Dante and Chanel both reply that the orange rod should "start the new one, one- tenth."

Chanel: This [the orange rod] should start the new one, one-tenth.

Michael: You lost me. I would call orange a whole.

R2: I'll be back. You think about the problem.

R2 left the group, first requesting that the students continue to think about the problem. At this point, it appeared that the students were switching which rod represented the unit. Another source of confusion for the students might have come about from their use of the familiar language "whole" to represent the unit.

After R2 left, the cameras captured the students working from their models, recording their observations, and talking softly with each other. They drew each rod, beginning with a white rod, lining up white rods alongside each different colored rod in a staircase arrangement, and writing the number name in ninths beside each, leaving the problem of the orange rod unresolved until they finally got to that rod in their recording.

In referring to the length of the orange rod, Dante labeled it ten-ninths and asserted, "I'm for calling it ten-ninths."

After recording for about four minutes, Michael and Dante remained with different number names for the orange rod. They also exchanged ideas about whether the numerator of a fraction can be greater than its denominator.

Eventually, R1 came by, asking the students, "What do we have here?" Chanel responded, pointing to her staircase that began with two white rods alongside a red rod, then with a white rod on top of one of each the larger sized rods, and explained:

Chanel: We have one-ninth, two-ninths, three-ninths, four-ninths, five-ninths, six-ninths, seven-ninths, eight-ninths, nine-ninths and [when she points to the orange] one whole and that starts over so it's going to be one-tenth.

R1: I see up to nine-ninths.

Chanel: The blue ends it, so the orange starts ……

R1: If we agree that we have to keep the white one-ninth, what is the length of the orange?

Dante: Ten-ninths.

R1: Persuade Chanel.

Chanel: I don't believe it

Michael: I thought it was a whole.

R1: They are all "wholes" - each of the rods is a whole rod - we need to find the number name for each.

What surfaced was the issue about whether the numerator can be larger than the denominator.

Dante: But how can a numerator be bigger than a denominator?

R1: It can - it is! This is an example

Chanel: But the numerator can't be bigger than the denominator!

Michael: That's the law of math!

R1: Who told you?

Chanel: My teacher!

The point is not whether her teacher told her that a fraction's numerator cannot be greater than its denominator but rather that beliefs can collide with the logic of learners' emerging reasoning even with based on their interactions with manipulative materials. Indeed, Chanel's reasoning triggers a cognitive conflict in her. She says something that conflicts with what she remembers a teacher telling her and what she arrives at trying to make sense (ten ninths). Her cognitive dissonance is signaled when she states that she has to think about the situation and switches to calling the orange rod nine-tenths. The researcher tries to facilitate the building of community encouraging an exchange between two individuals who have thought about same situation.

The students continued their recording, drawing each rod and giving it a fraction name as the appropriate number of ninths, concluding with the orange rod as ten-ninths.

Dante: [Finishing his recording for all the rods.] Ten-ninths. That's what she said.

Michael: The denominator can't be higher than the numerator

Dante: Yeah, that's what our teachers told us!

Space limitation prohibits the presentation of the reasoning from other groups, their struggle with improper fractions, and the variety of reasoning that eventually led them to name the orange rod ten ninths**.**

Discussion

Students, through their actions, observations, and reasoning, progressed in building a foundational understanding of ideas about and operations with fractions. In particular, when invited to share and support their ideas with others, their arguments became more detailed and refined. They referred to the models they built when conflicts arose, either with the ideas of their others in their group or with earlier beliefs, and relied on whether their ideas made sense in reaching a decision about the correctness of their solution. With the rods in front of them, they continued to check out their ideas and the ideas of others. The physical models of the rods were replaced by drawings and then by number names. Other mathematical ideas that arose and were discussed included the following: the meaning of one-half in relation to a unit; the possibility of a non-unit fraction as a name for one of the rods; adding fractions with like or unlike denominators, using the rods to test their reasoning; comparing fractions using the length of the rods as tools for estimating; and equivalent names for fractions. The multiple representations served as a rich source of images to reason with and examine earlier held ideas about fractions.

Endnote

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THE IMPACT OF TEACHER ACTIONS ON STUDENT PROOF SCHEMES IN GEOMETRY

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Student difficulty with proof in geometry has been well established in the literature, yet research connecting teachers' actions to students' understanding of proof is limited. In a classroom-based interpretive study, we explored the relationship between teaching and understanding proof in a geometry class. More specifically, we tried to determine how one teacher's actions influenced his students' developing ideas about what constitutes a proof. An episode was chosen to illustrate some of the teacher's typical actions such as encouraging student conjectures, encouraging others to evaluate and justify conjectures, coaching students to modify arguments, and modeling deductive reasoning. In this episode, we see at least one student attempt to make sense of the teacher's suggestions and move from an empirically based proof scheme to an analytical proof scheme. In this teacher's class, students appeared ready to make the transition to an axiomatic proof scheme. The teacher's actions appeared to provide an effective means for helping students learn to develop arguments within the axiomatic system.

Objectives

In order to make sense of mathematics and to communicate mathematical ideas, it is essential to be able to assess and produce mathematical arguments, including formal proofs. Although student difficulty with proof has been well established in the literature (Chazan, 1993; Hart, 1994; Martin & Harel, 1989; Senk, 1985), research connecting teachers' actions in the classroom to students' understanding of proof is limited (Herbst, 2002). We focus on reasoning and formal proofs in Euclidean geometry because, in the U.S., students typically are first required to write formal proofs in the context of Euclidean geometry in the secondary school. In fact, the geometry content is often taught by building a formal system of postulates, definitions, and theorems.

In this paper, we describe a classroom-based interpretive study in which we explore the relationship between teaching and understanding of proof in a proof-based geometry class. In particular, we address the following question:

> What relationship exists between teachers' actions and students' actions within the context of the mathematics classroom in terms of the development of students' proof schemes?

This question is addressed by exploring teacher and student interactions that occurred over a four-month period in a high school geometry class. Through field notes, interviews, and classroom videos, we captured a 'record of practice' (Ball & Cohen, 1999) from which we drew conclusions about the relationships between teacher and student actions within the social context.

Theoretical Perspectives

The theoretical lens through which we view classroom interactions is the emergent perspective as described by Cobb and Yackel (1996). This framework is useful because it attempts to describe individual and collective learning in the social context of the classroom. In this study, we use this lens to focus on classroom activity related to student understanding of proof. More specifically, we identified classroom norms and mathematical practices established through joint negotiation between the teacher and the students as they participated in discussing concepts and tasks related to mathematical proofs. We then identified ways in which these norms influenced or were influenced by students individual perspectives.

Our interpretations of student understanding of proof have been influenced by researchers such as Balacheff (1991), Harel and Sowder (1998), and Senk (1985). These researchers have characterized students' understanding of proof in terms of beliefs about what constitutes a proof, reasoning ability and formal proof-construction ability. For example, Balacheff suggests that students' reasoning ability progresses through stages, whereas, Harel and Sowder provide a framework for classifying students' proof schemes, or what, for students, constitutes a convincing argument. Harel and Sowder's three main classifications of proof schemes are *external conviction proof schemes* (in which students appeal to an external authority to determine mathematical validity), *empirical proof schemes* (in which students appeal to specific examples or perceived patterns for validation), and *analytic proof schemes* (in which students use logical deductions to validate conjectures). Senk, on the other hand, describes students' strengths and weaknesses in terms of formal proof writing at varying levels of difficulty. Although each of the three models influenced the data collection and analysis phases of our research, we focus on Harel and Sowder's proof schemes model for interpreting student actions in the episodes contained in this paper.

From a social perspective, elements of the classroom microculture as described by Cobb (2000) are important to consider when examining teacher-student interactions. In particular, social norms, sociomathematical norms, and classroom mathematical practices develop concurrently with students' individual understanding of proofs. Using the emergent perspective, we also identify teachers' actions, recognizing the significant impact these actions may have on the evolving microculture of the classroom as well as on students' understanding of what constitutes a proof (Martin & McCrone, 2003). Because classroom events transpire in a social environment, teacher decisions not only influence social aspects of the classroom, such as evolving norms, but these decisions are influenced by the developing social fabric of the classroom as well. Thus, teacher actions may result from carefully considered pedagogical choices or from spontaneous reactions to classroom events. Examples of teacher decisions include choice of mathematical tasks, methods for modeling particular mathematical processes or constructs (in this case, proof), instructional strategies (such as questioning, direction instruction, cooperative learning), and teacher's expectations for student performance.

Methods and Data Sources

We investigated the nature of tasks, the discourse, and patterns of interaction in a high school classroom as one way to begin to understand the complex processes of teaching and learning proof in geometry. We collected data in Mr. Drummond's honors geometry class in a large school in the Midwestern United States. (This name and all others that follow are pseudonyms.) Mr. Drummond followed a textbook that developed Euclidean Geometry as an axiomatic system and required students to construct formal written proofs on a regular basis. Researchers observed, recorded field notes, and videotaped the class almost daily for the four months in which proof was a major focus of the curriculum. Transcripts of classroom episodes as well as field notes and student work were the sources of data upon which we based our analysis. Our analysis of the data was based on the "three-part-analysis" proposed by Miles and Huberman (1994). The components of Miles and Huberman's analysis process include data reduction, data displays, and conclusion drawing. These refer to the processes of simplifying and transforming data, organizing data into compressed form, and identifying clear patterns or emergent trends in the data. The results discussed below highlight the patterns found in daily classroom excerpts that supported the development of students' understanding of proof in the class.

Results

As part of the data reduction and display process, we annotated and coded classroom transcripts. When focusing on the teacher's actions we observed some recurring themes related to the category "Teacher's expectations." By analyzing interactions from a social perspective, we identified several social norms related to the teacher's expectations. For example, Mr. Drummond frequently reminded the students that they were honors students and, as a result, he had high expectations for their performance. One of these reminders occurred when a student complained that a postulate that Mr. Drummond told the class to write down, was already in their notes. Mr. Drummond replied to the student, "Just go ahead and write it down. You can determine what you need to write down. You guys are honors students. If you don't want to write anything down, don't, but I wouldn't suggest it." In an interview, Mr. Drummond echoed his special expectations for honors students. When asked if he gave students points for participating in discussions, Mr. Drummond responded, "No, not in honors geometry. The reason I do that is because I'm not going to reward them for something I expect. There should never be a question whether or not honors students participate…. With honors students I think they understand, many of them understand, the need for participation and attention and usually are good enough at it without needing me to reward them for it." In addition, Mr. Drummond expected his students to analyze each other's arguments or formal proofs and to assess the validity of the arguments presented. He demonstrated this expectation by asking the class questions such as "what do you think?" when a student suggested a reason for her or his conjecture or "what's wrong with this reason?" when discussing a formal proof that was written on the board. (The question did not necessarily imply that there was something wrong with the proof, just that students were to analyze the indicated portion of the proof to determine if it was correct.)

Two other collections of codes that emerged from focusing on the teacher were categorized under "Instructional strategies" and "Proof modeling." Codes in the "Instructional strategies" collection included questioning, direction instruction, and cooperative-learning situations, the three most commonly used strategies in Mr. Drummond's classroom. Although "Proof modeling" may be thought of as an instructional strategy, we identified it as a distinct subcategory of teacher's actions because the process of proving was a major focus of the research. Some of Mr. Drummond's proof-modeling actions that we identified from coded transcripts included developing an outline, connecting the big picture to the details, and focusing on appropriate terminology and format. This modeling occurred as new proofs were developed as part of the teacher's presentation of new material or as finished proofs were critiqued in teacher-led whole class discussions.

Proof modeling was an important vehicle for introducing both structural and logic requirements for a proof. For example, during a discussion of a group-presented proof a student from a different group asked if a segment (such as segment AC in Fig. 1) had to be marked congruent to itself in a diagram in which a common segment served as a side in two distinct triangles, Mr. Drummond replied, "Yes. And you have to say it in a proof. You can't assume it. Because what happens then, is … you say by SAS, those two triangles are congruent (referring to triangles such as triangle ABC and triangle ADC). But you've only told me about one pair of sides and one pair of angles. You need to be specific about what other side you used. So that's why we have to mark it reflexive and say in your proof this is congruent to itself. So, when I see

SAS …as a reason in your proof, I better be able to look up in your proof and see a pair of sides, a pair of angles, and another pair of sides. I should be able to see the S, the A and the S all stated in your proof.”

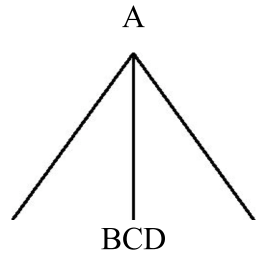

Figure 1. Adjacent Triangles

Several other groups of codes emerged from analyzing student actions within the student-teacher exchanges in the classroom. Included in these code groups were: ‘Beliefs about proof’ and ‘Reasoning strategies.’ Within the ‘Beliefs’ classification emerged several categories that were isomorphic to Harel and Sowder’s (1998) proof schemes. For example, when a student asked whether two angles in corresponding triangles were congruent, the teacher responded “yes” and then asked the student, “Why are they congruent?” The student’s response, “because you’re the teacher,” hints at the possibility that the student held an *authoritarian proof scheme*, believing that reliance on an authority is one way to determine the validity of a statement.

By displaying several sequences of codes in the order in which they appeared in the transcripts, we were able to conclude that patterns were evident in the interactions between the teacher and the students. The classroom excerpts below illustrate typical teacher-student interactions in Mr. Drummond’s classroom, and show how student and teacher actions influenced one another in the classroom. These excerpts also illustrate the negotiation of mathematical meaning and how the teacher’s actions may have helped students see the flaws in their beliefs and move toward more sophisticated proof schemes.

Episode 1

This first episode is part of a teacher-led discussion on characteristics of congruent figures. Mr. Drummond had posed an open-ended task, asking students to record everything they knew about a pair of congruent pentagons, as shown in Figure 2. (Note that the dashed segments in each pentagon were not in the original sketch made by Mr. Drummond.) In the exchange that follows, one student, Nigel, asked if the distances between corresponding non-adjacent vertices were equal. (See dashed segments AC and QS in Fig. 2.) Mr. Drummond encouraged all students to explore this conjecture.

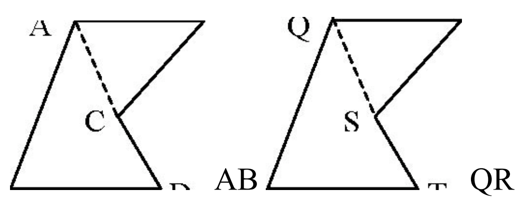

Figure 2. Congruent Pentagons

Nigel: I have a question.

Mr. D: Yeah.

Nigel: On that one [referring to pentagon ABCDE and pentagon QRSTU] …, could you say that the measure between A and C is congruent or equal to the measure between Q and S?

Mr. D: Yeah. [pause] That's good enough for you? Yeah? Don't you want to know why?

Nigel: Why?

Mr. D: [Addressing the entire class] Nigel asked, is the distance from A to C, for example, the same as the distance from Q to S even though there's no segment drawn there? I'm going to tell you yes. Anybody want to venture a guess why? [Sam] Jones?

Sam: Because they're congruent.

Mr. D: The sides [of the pentagons] are congruent, but how do you know that these segments that aren't drawn in there are congruent? [Sam] Jones, keep trying.

Sam: Well, the segments are arranged in the exact same way.

Mr. Drummond did not comment on Sam's last remark, but started to lead the class in a paper-folding activity that would help them further investigate the conjecture.

An analysis of Episode 1 highlights some of Mr. Drummond's expectations of students as well as his mode of questioning during class discussions. We also see interactions with the students that demonstrate how he encouraged all students to investigate the justification process leading to proof. Sam's responses provide a glimpse of his understanding of what constitutes a valid justification. First, because Nigel's conjecture followed a series of conjectures made by other students in the class, it is clear that the classroom atmosphere is one in which students are expected to make conjectures and feel comfortable doing so. Mr. Drummond responded to Nigel's conjecture by listening to what he said and redirecting the conjecture back to the class. In particular, we note that Mr. Drummond followed up on Nigel's conjecture even though it led the class away from the intended lesson plan (to define and share examples of congruent figures). Mr. Drummond's request for an explanation to support Nigel's conjecture also may have contributed to the development of a sociomathematical norm, namely that conjectures should be justified or refuted.

A second important aspect of this episode is the way in which Mr. Drummond encouraged students to justify the conjecture. He initially coaxed the students into "venturing a guess" or trying to develop a reasoned argument of their own. Sam responded to the prompt by providing some information to support the Nigel's claim ("they're congruent"). Mr. Drummond than took Sam's "guess" and demonstrated that more needed to be done to provide a strong argument. He posed another prompting question and encouraged Sam to try again. Sam again responded, this time providing a warrant to support his original attempt. Here, the term *warrant* refers to a statement used to demonstrate the legitimacy of prior information in justifying a claim (Krummheuer, 1995). We call this encouragement and coaxing from Mr. Drummond *coaching* because it involves supportive and directive behaviors that are typical of an athletic coach.

Lastly, we note that Sam's responses give insight into his conception of a valid justification. His response, which is interpreted by Mr. Drummond and then confirmed by Sam to be a statement about the congruent pairs of corresponding sides, suggests a naïve understanding of proof. Sam is only able to refer to the original diagram and given information. He does not use this information to provide further warrants for the validity of Nigel's conjecture. Mr. Drummond's lack of response to Sam's final suggestion indicates that he was not satisfied with the justification attempt. Rather he devised a brief paper-folding activity to help Sam and other students move beyond the basic information highlighted by Sam (congruent corresponding sides).

The series of tasks and interactions between teacher and student as in Episode 1 illustrates a common pattern in Mr. Drummond's lessons. His choices and expectations often led to discussion of student reasoning, some proof modeling, and refinement of an argument. In the next episode, the discussion resulting from Nigel's conjecture continues as Cathy interrupts Mr. Drummond during his paper-folding directions and attempts to justify the conjecture (refer to Figure 2 and Episode 1 above).

Episode 2

In the following episode, Cathy attempted to provide data and warrants for the truth of the conjecture originally posed by Nigel. Mr. Drummond and Cathy negotiate this new contribution before other students are called on to contribute.

Cathy: If you were just, like, to put a dotted line to connect them [*point A to point C and point Q to point S*], you know that they have to be equal, because that's making a triangle and the two, like $\overline{AB}$ is congruent to $\overline{QR}$ and …[$\overline{BC}$] is congruent to $\overline{SR}$. So the other one has to be congruent too, because they're congruent. There's no other way you can make a triangle that was going to be congruent without three sides matching.

Mr. D: Now you said two very important things, but you said them in opposite ways. You said first that if we looked at this triangle, QRS and ABC [*The teacher has drawn in red dotted lines from A to C and Q to S*] since these two [triangles] are the same size, then these two [$\overline{AC}$ *and* $\overline{QS}$] have to be the same size. That's what you said first. And then by the time you finished you said if we have a triangle with all three sides [*points to triangle QRS*] and all three sides [*points to triangle ABC*] then they have to be congruent or they have to be the same size triangle, right?

Cathy: … I'm just saying that if $\overline{AB}$ is congruent to $\overline{QR}$ and $\overline{SR}$ to $\overline{CB}$ then $\overline{QS}$ has to be congruent to $\overline{AC}$ in order to form a triangle that would be congruent to the other one.

Mr. D: Is that true? If these two sides [*points to* $\overline{QR}$ *and* $\overline{SR}$] in this triangle are congruent to these two sides [*points to* $\overline{AB}$ *and* $\overline{CB}$] in this triangle, they don't have to be congruent to each other, but if this pair [$\overline{AB}$ and $\overline{QR}$] and this pair [$\overline{SR}$ and $\overline{CB}$] are congruent, do those red ones have to be congruent?

After receiving conflicting opinions from students in response to his last question, Mr. Drummond proceeded by providing a counterexample to Cathy's argument, that if you know two pairs of corresponding sides are congruent, it is not necessarily true that the third pair of sides will be congruent. Eventually, other students offered new arguments that led to the conclusion that in order to be sure the triangles were congruent (Cathy's assumption), the included angles between the two pairs of corresponding congruent sides would have to be identified and shown to be congruent (commonly referred to as the Side-Angle-Side Postulate).

Cathy's actions in this episode illustrate her struggle to provide adequate warrants organized into a logical chain of reasoning to bridge from Sam's contribution ("the segments are arranged in the exact same way") to Nigel's claim (segment AC is congruent to segment QS). Cathy's approach initially rested on her ability to use the diagram to identify relevant relationships. In particular, Cathy referred to auxiliary lines in the original diagrams that allowed her to discuss a pair of corresponding triangles embedded within the pentagons. Cathy then assumed congruence of the corresponding triangles to claim congruence of a pair of corresponding sides in the triangles. Cathy's actions may be characterized as indicative of a *transformational proof scheme*, within the larger class of *analytic proof schemes* (Harel & Sowder, 1998). That is, her reasoning is based on general aspects of the situation, perceiving underlying structure behind a pattern.

Although this tendency to think about relationships in general terms is more sophisticated than many other proof schemes (such as those that rely on examples or authority to determine validity), Cathy's warrants are not adequate to serve as a formal proof in the context of this classroom. A formal proof would require a chain of logical reasoning that uses already established definitions, axioms, and theorems to serve as warrants for claims in the argument.

The teacher's actions complemented Cathy's attempt to justify the conjecture. He listened carefully to Cathy's reasoning and encouraged her to try again, similar to the *coaching* techniques noted in Episode 1. He supported Cathy's effort to refine her warrants by providing an analysis that allowed Cathy to restate and clarify her argument. The teacher then rephrased Cathy's argument as a question and asked others to assess its validity and continue refining the argument. In this way, the teacher's actions guided students toward an *axiomatic proof scheme*, another scheme within the *analytic proof scheme* classification, (Harel & Sowder, 1998). Someone who possesses an axiomatic proof scheme believes that adequate proofs are constructed by working within an axiomatic structure. By exposing flaws in Cathy's argument and encouraging Cathy and others to provide warrants from within the axiomatic system, the teacher drew students into the process of working within an axiomatic system. Thus, the teacher coached the students as they learned the rules of the game (how to construct formal proofs) by actually playing the game.

Conclusions

Episodes 1 and 2 demonstrate a few of the patterns of teacher-student interactions in the social context of Mr. Drummond's classroom. Table 1 summarizes the student actions, teacher actions, and social factors evident in these and other episodes that were coded and analyzed.

The left column of Table I lists students' actions or contributions during class discussions that relate to their understanding of proof. The middle column includes actions taken by Mr. Drummond. The right column contains aspects of the social environment or classroom microculture that became taken-as-shared in Mr. Drummond's class. For example, in Episodes 1 and 2, the teacher-student exchange followed an identifiable pattern of actions that included many of those actions described in Table I. First, Mr. Drummond posed an open-ended question. Nigel made a conjecture in response to the question. Mr. Drummond encouraged and listened to student responses, then rephrased the statements as questions and posed them to particular students or to the entire class, which, in turn, placed the responsibility for justifying or refuting student conjectures in the hands of the students. As Sam and then Cathy provided warrants for their arguments, Mr. Drummond coached them by offering praise, encouragement, or critique of their claims. When necessary, Mr. Drummond offered direction to the students in the form of prompting questions or by providing an example or counterexample. Although not demonstrated in the chosen episodes, Mr. Drummond also modeled proof-writing skills such as the development of chains of reasoning and helped students revise and improve their own arguments. He also followed students' suggestions, even when they were incorrect or led the discussion in a direction he had not planned. As various students participated in the exchange, the class began to establish shared understandings about standards for developing valid arguments.

Table I Student and Teacher Actions that Contribute to the Learning of Proof

Students' Actions	**Teacher's Actions**	**Classroom Microculture**
Make Conjectures Make claims or ask questions about relationships	Question Use rebound, clarifying, or prompting questions	Justification All conjectures must be justified or refuted
Provide Warrants Justify claims or supply reasons for others' claims • Use geometric property or relationship • Appeal to logical structure	Listen and Redirect Listen to and follow up on student ideas • Acknowledge ideas • Pursue suggestions • Repeat or rephrase in order to highlight key ideas or expose flaws in reasoning	Role of Proof Proof is used to: • Establish validity of statements • Explain why conjectures are true
Build Chain of Reasoning Create argument consisting of several connected statements	Analyze Comment on student reasoning, including dissecting arguments	Standards for reasoning Reasons must meet certain standards in order to be valid
Use Diagram Use diagram to identify relationships and illustrate reasoning	Coach Offer encouragement or praise	
	Model Proof-Related Skills Demonstrate specific proof-writing techniques such as: • Build chain of logical reasoning • Use diagram • Provide a counterexample	

Even though Mr. Drummond made pedagogical choices that necessarily involved the students, his choices may also be seen as limiting, in that he did not allow time for *all* students to think and independently investigate conjectures before calling for a justification. The quick pace of a classroom discussion that focused on one student's conjecture and an argument developed by only a few students left other students to be passive observers in the proof development process. Although some students who were not involved in the verbal interaction in the class still may have been engaged intellectually in the activities, these students did not benefit from the class' and teacher's responses to their ideas. Only the verbally active students participated in the class-level negotiation of what counts as a valid argument and other taken-as-shared sociomathematical norms and classroom practices. If all students were expected to provide warrants and build chains of reasoning before any of these justifications were shared among the group, there may have been an opportunity to examine a variety of reasoning strategies. Such a discussion could have engaged more students in a richer dialogue and allowed them to make their own decisions about the validity of arguments. Even so, these and other episodes provided rich data on a few individual students' evolving sense of the process of proving.

Through our analysis of classroom data we have described a cycle of teacher and student interactions that promotes student involvement and allows students to test the waters of proof and reasoning. Using the lens of the emergent perspective (Cobb, 1999; Cobb & Yackel, 1996) to analyze classroom interactions, we found that viewing events from a psychological perspective drew our attention to the actions of the students. Similarly analyzing the interactions from a social perspective drew our attention to how students responded to and attempted to live up to the teacher's expectations. However, we also found it necessary to focus explicitly on the teacher's pedagogical choices, as manifested by his actions, in order to capture a critical component of the cycle that draws students into the social fabric of the classroom.

In proof-based mathematics courses, teachers often model proof-construction that presupposes an axiomatic proof scheme (Harel & Sowder, 1998). It is often assumed that students believe valid proofs may only be constructed by using a chain of deductive reasoning within a particular axiomatic system. Research (Harel & Sowder, Balacheff, 1991) has shown that this assumption is not necessarily true. In this teacher's class, students appeared ready to make the transition to an axiomatic proof scheme. His modeling of deductive reasoning, along with coaching students as they attempted to construct proofs, was an effective means for helping students learn to develop arguments within the axiomatic system. However, the same type of pedagogical choices may not be as effective with students who possess more naïve proof schemes. For example, students who believe that examples constitute a proof may not respond in the same way as those who understand the generality requirements in a deductive proof. Further research is required to establish a more robust connection between pedagogical choices and individual understanding of proof.

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THE EVOLUTION OF FORMAL MATHEMATICAL REASONING: A CASE STUDY OF THE UNIQUENESS THEOREM IN DIFFERENTIAL EQUATIONS

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The purpose of this research report is to describe key processes by which students can come to understand and apply the uniqueness theorem for first order differential equations. More generally, the analysis is framed as a paradigm case for the production of formal mathematical reasoning. Analysis of classroom data revealed four critical components that characterize the evolution of formal reasoning with the uniqueness theorem: the negotiation of what counts as an acceptable explanation, the engagement of an intuitive theory, a cognitive reorganization about a central idea, and the owning of a formal statement.

Advances in technology and an increased interest in dynamical systems are prompting new directions in many first courses in differential equations. Simultaneously, research is beginning to illuminate student thinking about central ideas and methods of analysis associated with these new directions. A recent review of the literature (see Rasmussen & Whitehead, 2003) highlights the primary findings to date, including delineation of students' strategies, understandings, and difficulties with (a) coordinating algebraic, graphical, and numerical representations, (b) creating and interpreting various representations including phase portraits and bifurcation diagrams, and (c) making warranted predictions about the long-term behavior of solution functions. A notable omission to this small but growing body of research is students' thinking about and application of the uniqueness theorem. The purpose of the proposed research report is to contribute a new component to the literature by describing key processes by which students come to understand and apply the uniqueness theorem. More generally, the analysis is cast as a paradigm case for the emergence of formal mathematical reasoning.

Originating in the seventeenth century as a technique for solving geometrical and mechanical problems, the study of differential equations initially centered on attempts to find analytic solution techniques. As practitioners of mathematics moved toward increasingly analytically intractable differential equations extracted from physical and graphical situations, they became motivated to ask the questions of first existence and then uniqueness of solutions. The first to draw attention to the unspoken assumption that there exists a solution to a given differential equation was Cauchy, who in the 1820s gave a rigorous proof for the existence of a solution. Several decades later, the Lipschitz condition provided the first guarantee of unique solutions to first order ordinary differential equations.

Theoretical Background

Analysis of students' reasoning about uniqueness of solutions draws on social constructivist theories of learning in which students' mathematical reasoning is both constrained and enabled by their current understandings. Piaget (1970) emphasized that learning is a process involving a constant interaction between the learner and her environment. This process of equilibration involves the integration of things to be known with existing cognitive structures, as well as reorganization of cognitive structures as students participate in the evolving norms and practices of the classroom community. As Siegler (1996) points out, Piaget was interested in variability of thinking during transitional periods and viewed this variability as critical to cognitive change. In keeping with this emphasis on documenting production and evolution of thinking (rather than say, delineating stages), this report describes the process of change as students progress in their understanding and application of the uniqueness theorem.

Method

The analysis is based on data collected during a 15-week classroom teaching experiment (Cobb, 2000) conducted at a mid-sized university during the Fall 2002 semester. Classroom videorecordings from two cameras were the primary source of data, triangulated with copies of students' written class work, copies of exams and homework, and videorecordings of individual student interviews. Typical class sessions consisted of cycles of small group work followed by whole class discussion. The project team consisted of the teacher, who was an experienced mathematician, and two researchers who attended each class session. Informed by the theory of Realistic Mathematics Education (Freudenthal, 1991; Gravemeijer, 1999), the course materials, largely developed in previous teaching experiments, were modified as needed by the project team. An essential characteristic of the instructional design was the creation of a learning environment in which students could reinvent important mathematical ideas and methods as they engage in a series of challenging tasks.

Analysis of the classroom videorecordings proceeded through cycles of examining and interpreting the data, which involved transcribing and writing interpretive notes. Through an iterative process, specific classroom episodes were selected and four overarching components regarding the evolution of students' reasoning about uniqueness emerged. Sample episodes from this iterative process are used in this report to illustrate and clarify the main ideas about the evolution of formal mathematical reasoning with the uniqueness theorem. To sharpen the discussion I focus primarily on the reasoning of three students, Bill, Adam, and Joe.

The instructional design had two goals related to the uniqueness of solutions. First, students would come to view the issue of uniqueness as personally relevant (historically this took a very long time). Second, the Lipschitz condition for uniqueness of solutions would be a formal description of students' observations and explanations for why graphs of solutions do or do not touch. One of our goals in the teaching experiment was to explore an alternative to instructional approaches that tend to superficially treat the uniqueness theorem. Detailing the critical components by which formal mathematics can actually grow out of students' informal mathematical work is needed and this report makes a contribution in this direction.

Discussion

Evolution of students' formal mathematical reasoning surrounding the uniqueness theorem is traced in terms of four critical components: the negotiation of what counts as an acceptable explanation, the engagement of an intuitive theory, a cognitive reorganization about a central idea, and the owning of a formal statement. I illustrate and clarify the first three of these four components and highlight the fourth. To the extent possible within the page limitations, I point to connections between these components and the role of the teacher in the evolution of the four components.

Component 1 – Negotiation of Acceptable Justifications

Analysis of the classroom videotape data points to an important interplay between empirically based justifications and justifications based on mathematical relationships. Justifications were deemed empirical if they were (1) based on observed or imagined graphs or

(2) based on an imagined, real-world phenomenon. By design, instructional sequences drew heavily on geometric approaches and the framing of problem situations in terms of real world phenomena. Thus, it is perhaps not surprising that students' justifications were, at least initially, grounded in observed or imagined graphs or imagined real world events. What is significant is

that there was a shift in the nature of students' justifications over the course of the semester – from those with an empirical basis to those with a basis in mathematical relationships.

For example, prior to work on the sequence of tasks dealing explicitly with the uniqueness theorem the class discussed whether or not an imagined solution graph, one that was initially increasing, would ever actually reach or touch the equilibrium value of 12.5. Under scrutiny was the differential equation dP/dt = 0.3P(1-P/12.5), which was intended to model population growth. Tangent vector fields (without curve sketching capabilities) and analysis of the differential equation were the primary mathematical ideas and tools available to students. The following whole class excerpt succinctly captures the interplay, and tension, between empirically based and mathematically based justifications.

Bill: What Jeff and I were thining was that eventually, ideally, this would seek and equilibrium, fluctuating up and down around 12.5. But obviously, you can't have half a deer running around, so you know, it's gonna at some point go above 12.5, then it goes in negative, in the uh slope, so it'll drop below 12.5. Then, then you're back positive, and it'll, so it'll be rising and falling up and down around, around 12.5.

Joe: How do you rise up and down when you have a zero tangent?

Adam: Maybe theoretically, but that's not what our equation's saying. Our equation's saying that uh 12.5 is gonna be the limit. It's gonna go up, it's gonna, it's gonna be what's it called, asymptotic to 12.5 or, I think that's -

Joe: Well it's not actually [inaudible] It's when P to the 12.5 is one.

Adam: Yeah. Yeah.

Prof: Okay. What's so important for this 12.5? It seems that some of you think it's positive if P is less than 12.5?

Stds: Yes.

Joe: It's positive if it's less than 12 and a half. It's negative if it's greater than 12 and a half, and it's zero at 12 and a half. And so I have a problem with it being able to fluctuate around 12.5 because if you have a zero. If you had zero change,

Bill: Okay well my thinking was -

Joe: I mean, it doesn't change over time no matter [inaudible].

Bill: Well my thing was that you talk about a population, you're talking about a population, you have to have whole numbers.

Bill explained that his group's initial idea was that the graph would oscillate around 12.5 with decreasing amplitude. No justification for why such asymptotic behavior might be the case was offered. Bill said that he and Jeff then rejected this conclusion that the population would settle down to 12.5 because "you can't have half a deer." Joe and Adam immediately rejected any kind of oscillation based on the mathematical relationship between the slope of a graph as dictated by the differential equation (in particular there should be a tangent with zero slope at 12.5) and the shape of graph. This clarification was, in part, solicited from the teacher when he asked "What's so important about 12.5?" In response to Adam and Joe's point, Bill then clearly stated that his reasoning was based on the need to have "whole numbers" due to the population setting. Bill's justification falls within the realm of empirically based justifications while Joe and Adam's justification falls with the realm of justifications based on mathematical relationships.

As the discussion continued, Joe, Adam, and a third student, Jake, argued further against Bill's conclusion.

Joe: What you think a population would be doesn't mean that that's what that equation is going to do.

Adam: We're talking about a model here.

Jake: Yeah, that's just a representation, I mean like that's like a thousand times 12.5, or three thousand times 12.5.

Prof: So it's like a very large number and uh,

Adam: So the fluctuation wouldn't really, you wouldn't see it.

Joe: Yeah. Adam: It's just a model!

Jake: It would be at equilibrium at 12.5. It would level off as it approaches 12.5, the rate of change.

Prof: Yes, certainly we cannot have half fish, or half deer. But you're saying that if we have a huge number for population then, then this, although it's not really a smooth curve, but for the model, we have a, we have smooth curve. Is that what you're saying?

Jake: Yes.

Prof: Yes.

In addition to Adam's argument that the differential equation is something other than an exact fit to the population setting ("it's just a model"), Jake argued that 12.5 could very well be 12,500, for example. The teacher clarified for the class that in terms of actual population values, which would be discrete, the solution graphs of interest to the class are continuous. Although the teacher's voice is not prominent in these excerpts, he plays an essential and proactive role in shaping the classroom discussion and norms for justification. Among other functions, he is the one who selected and made possible the conversation about whether or not a solution graph would touch 12.5, he is the one who worked to set up a classroom environment in which students felt safe to voice their ideas, even if they turn out to be rejected, and he is the one who at once honored Bill's conclusion ("Yes, certainly we cannot have half a fish, or half a deer") while implicitly reinforcing the need for conclusions to be based on mathematical relationships.

As the semester progressed justifications based on mathematical relationships become more and more routine (cf., Yackel & Cobb, 1996), even though the problems posed to students continued to be framed in terms of imagined real world settings in which prediction of future quantities was important. The significance of this component in the evolution of formal mathematical reasoning in relation to the other three components is that it (a) opens a space for the teacher to become aware of students' intuitive theories (Component 2), and (b) makes explicit discussion about central mathematical ideas more viable, which in turn opens spaces for students to refine and reorganize their conceptions of these ideas (Components 3 and 4).

Component 2 – Engaging Intuitive or Informal Theories

Students exhibited an intuitive theory that non-equilibrium solution functions will approach equilibrium solution functions asymptotically (Rasmussen, 2001). Students' intuitive theory about asymptotic behavior in this classroom took on one of two forms, either oscillations with decreasing amplitude toward a fixed value or strictly increasing/decreasing behavior toward a fixed value. Both of these intuitive theories were evident in the excerpts provided in the previous section. Recall the following statement made by Bill: "What Jeff and I were thinking was that eventually, ideally, this would seek an equilibrium, fluctuating up and down around 12.5." Adam, on the other hand, assumed that the graph of the solution would approach 12.5 asymptotically in a strictly increasing manner. Recall Adam's statement that the graph is "gonna go up, it's gonna, it's gonna be what's it called, asymptotic to 12.5." Later in this same excerpt Jake also stated that the graph "would level off as it approaches 12.5."

Which of the two forms of asymptotic intuition was engaged appeared to depend on the imagined real-world setting. For settings in which oscillation of quantities was reasonable, we saw both types of asymptotic intuition. In other settings, like the one discussed in the next section, only strictly decreasing asymptotic intuition was engaged.

Although asymptotic behavior is true in many cases, it is not always the case (e.g., consider solutions to $y' = -y^{(1/2)}$). From a student's perspective, such intuitive theories make sense because they originate in extensive mathematical experiences. In his seminal work on intuition, Fischbein (1987) characterized intuitions as self-evident statements that exceed the observable facts. Being self-evident, justifications often do not accompany statements that engage intuitive theories, as was the case in the previous excerpts. When pushed for justification, students tended to provide circular arguments. For example, as the conversation about solutions to $dP/dt = 0.3P(1-P/12.5)$ continued, the teacher asked students what should be the graph of the solution if "we base it just on the differential equation model?" To which Bill responded,

Bill: Then I agree that P approaches 12.5, and as it gets closer and closer to 12.5 the rate of change will get smaller and smaller and yeah, I don't think it would ever reach 12 and a half. I would just keep getting closer and closer, but never quite make it.

Prof: Why do you think that?

Bill: Because, because, the closer it gets, the rate of change keeps decreasing. You know, never going to zero. But it keeps, it keeps holding it back. The rate of change does not let it get to 12 and a half.

From an instructional design and teaching perspective, awareness of students' intuitive theories is important because it informs subsequent work with students in efforts to promote cognitive refinements and reorganizations (Component 3). Students' intuitive theories, although often not generalizable to all situations, can serve an important function in the learning process. In particular, when students encounter instances that conflict with their intuitive theories, that is when they encounter disequilibrium, they are more likely to become explicitly aware of and search for refined and reorganized conceptions of a central mathematical idea. The point is not to replace the intuitive theory, but rather to use students' intuitive theories as opportunities for refining and reorganizing a related central mathematical idea, such as rate of change. Consistent with Brousseau's (1997) theory of cognitive obstacles, engaging intuitive or informal theories points to how such theories can function as not only as a constraint, but also as a resource.

Component 3 – Reorganizing a Central Mathematical Idea

Students' thinking about rate of change, as expressed in a differential equation, was initially either an "adjective" to describe the slope or steepness of an observed or imagined solution curve or a "predictor" of the direction a solution graph should take. Bill's previous excerpt speaks to both of these ways of thinking about rate. Specifically, Bill had a solution graph in mind (one that is increasing toward 12.5) and he used rate of change as an adjective to describe the graph. "… as it [the graph] gets closer and closer to 12.5 the rate of change will get smaller and smaller …". Here rate of change is a descriptor for an already imagined graph. Rate of change is used to describe qualities of the graph. Bill also used rate of change as a mechanism or predictor for how the graph should proceed. For example, in this same excerpt, Bill stated, "But it [the rate of change] keeps, it keeps holding it [the graph] back. The rate of change does not let it [the graph] get to 12 and a half." In this last quote, rate of change acts as control mechanism for how the graph will unfold, rather than as an adjective for an already unfolded graph.

In subsequent problems students compared graphs of solutions to two different differential equations and reasoned about the rate of change of the rate of change to account for why one set of solutions touched an equilibrium solution and the other set did not. In accomplishing this goal, rate became an object with its own adjectival properties.

To clarify, consider the analogy of a red ball. Initially the ball is the primary object of interest. Red is an adjective that describes the ball. This is analogous to an imagined solution function and its slope or rate. Now imagine there is another ball. The new one is a deep, intense red and the old is a pale, pinkish red. Through comparison we develop a need to describe redness

itself. So we might say, "That is a deep red" or an intense red. Now deep is an adjective modifying red. The focus of our attention has shifted to red. The ball is in our subsidiary awareness while our focal awareness is on the nature of redness. This is analogous to shifting one's attention from an imagined graph where rate or slope is an adjective of that graph or predictor of a graph to analysis of rate of change with own properties that need to be described. For example, as one student commented, "The rate of change of the rate of change increases as y approaches zero." In this quote, rate is not a property of a solution graph, but rather an object analyzed for its own properties.

The differential equations under scrutiny were dh/dt = -h and dh/dt = $h^{1/3}$, both of which were offered as models for the height of a descending airplane and both of which have constant solutions h(t) = 0. Using dynamic tangent vector fields (without curve sketching capabilities) students engaged in figuring out if one, both, or neither differential equation predict a landing for the plane. The issue of uniqueness therefore is relevant since landing would mean that two solution graphs touch (in particular there would be two different solutions that meet at h = 0). As illustrated in the following quote, Adam used rate as adjective and rate as predictor to argue that graphs for dh/dt = -h would not touch zero.

Adam: If you plot your vector field or whatever, your slope is gonna gradually taper off. As your number gets smaller and smaller, your slope's gonna get smaller and smaller [rate as adjective]. So there's no way you're ever gonna be able to get to zero on you're height because your slope is gonna slow it down [rate as predictor or control mechanism].

Adam also stated that the other differential equation is "kind of the same as the first one." It is likely that his conclusions for both differential equations engaged asymptotic intuition. Other students claimed that dh/dt = -$h^{1/3}$ predicts that the plane would touch ground. The justification for this claim was the observed vector field, in which students experienced and observed that vectors "snap" to zero. This is a form of empirical justification. Accounting for the snap then became a topic of conversation and further analysis. Part of this analysis involved finding analytic solutions, which yielded conclusive evidence that solutions to dh/dt = -h do not touch zero while solutions to dh/dt = $-h^{1/3}$ do touch zero. The analytic solutions did not, however, offer students insight into why solutions were or were not unique. This insight was gleaned by reexamining the "snapping" of tangent vectors near zero (or not snapping) in light of the how the rate of change changes. That is, framing the snapping in terms of the rate of change of the rate of change.

Adam: O.k., um on our slope field it looks like the rate of change of our, the differential equation, is going down kind of slow for the $-h^{1/3}$. Then after it passes, what is it, one, it snaps. It starts snapping to zero [Lynn: Why?] Why? When you re-write the derivative of your differential equation. So it's negative one third, uh, 'h'. Well, next to the three, pull out your, yeah, there you go. Yeah, when you look at that [d($-h^{1/3}$/dh] of we can see that, um, as h gets smaller, it just blows up. But that means that the, when you're at zero it's undefined. But as you go to zero it's getting bigger and bigger and bigger. As you get close to zero, you get a really big number. So as you go to zero it's not defined.

Chris: What's the 'it' that is undefined?

Joe: The rate of change of the rate of change becomes infinitely large. As you approach zero the rate of change of the rate of change goes to infinity.

This cognitive reorganization in rate (from rate as adjective or predictor to rate as object with its own adjective properties) opened a space for the teacher to then formally state the conditions for uniqueness in conventional terminology and symbolic expressions. From students' perspective, the "formalization" of the theorem was a recognizable recasting of their analysis of the rate of change of the rate of change.

Component 4 – Owning a Formal Statement

Expressing students' thinking about the rate of change of the rate of change in terms of the Lipschitz condition was critical to formally stating the uniqueness theorem. The process of coming to own the theorem, however, continued. The coming to own the theorem process involved application of the theorem in other settings, a fine-tuning of how to interpret ideas such as unbounded and partial derivatives, and what one can logically infer (or not) when the conditions of the theorem are not met.

Concluding Remarks

The four components surrounding the uniqueness theorem (the negotiation of what counts as an acceptable explanation, the engagement of an intuitive theory, a cognitive reorganization about a central idea, and the owning of a formal statement) address the production and evolution of formal reasoning in classroom settings. Rather than setting out distinctions between experts and novices, these four components offer a way to conceptualize the process of developing formal mathematical reasoning in a way that sees value in normative forms of argumentation and practice (Components 1 & 4) and cognitive variability (Components 2 & 3).

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REASONING AND PROVING IN SCHOOL MATHEMATICS CURRICULA: AN ANALYTIC FRAMEWORK FOR INVESTIGATING THE OPPORTUNITIES OFFERED TO STUDENTS

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There is widespread agreement that reasoning and proof should be a central feature of all students' mathematical experiences. Yet, research shows that students often have serious difficulties acquiring competency in this domain. Students are unlikely to develop proficiency in reasoning and proving on a large scale unless attention to this mathematical practice is woven into curriculum materials. Too little is known about the opportunities mathematics curricula offer students for reasoning and proving; a curriculum analysis is needed to illuminate the nature of the opportunities provided so they can be further developed. This study seeks to provide needed knowledge by: (a) conceptualizing reasoning and proving *in a way that is sensitive to the mathematical discipline and promising to push forward the conceptual work on the nature of this activity in school mathematics; and (b) using this conceptualization to develop and validate an* analytic framework *that provides a reliable and comprehensive way to analyze the opportunities mathematics curricula provide students to engage in reasoning and proving.*

Proof and mathematical reasoning signify mathematical activity, but their role in school mathematics has been unclear. In recent years, however, there has been a growing appreciation of the importance of *reasoning and proof* in school mathematics, primarily because of its central place to both mathematics as a discipline and learning mathematics with understanding—a prominent educational goal nowadays. The increased emphasis on reasoning and proof is reflected in both researcher and curriculum framework calls about this activity to pervade students' work throughout their schooling (e.g., NCTM, 2000; Yackel & Hanna, 2003).

Although numerous studies show that students of all grade levels have serious difficulties in reasoning and proof (e.g., Healy & Hoyles, 2000; Knuth et al., 2002), the findings of developmental (e.g., Foltz et al., 1995; Galotti et al., 1997; Klaczynski & Narasimham, 1998) and cognitive psychology (e.g., Girotto et al. 1989; Light et al., 1989) provide compelling support to the claim that even young children can engage successfully in problems involving deductive reasoning and proof. Furthermore, existence evidence from mathematics classrooms demonstrates that exceptional teaching can make reasoning and proof accessible to children in their everyday experiences as early as the elementary grades (e.g., Ball & Bass, 2000; Lampert, 1990; Zack, 1997). This kind of teaching, however, deviates from the norm in U.S. mathematics classrooms, as indicated by both the TIMSS 1995 and the 1999 Video Studies (see Hiebert et al., 2003; Manaster, 1998).

Students are unlikely to develop proficiency in reasoning and proof on a large scale unless attention to this activity is woven into curriculum materials. Research shows that curriculum significantly influences the selection and sequencing of the topics taught (e.g., Romberg, 1992); it influences teachers' beliefs about teaching and learning (e.g., Clarke, 1997; Wood et al., 1990); and its basic features—content, organization, and sequencing—impact students' conceptions of proof (e.g., Chazan, 1993; Healy & Hoyles, 2000; Hoyles, 1997). It is reasonable to assume that curriculum materials that claim to embody both the *Curriculum and Evaluation* (NCTM, 1989) and the *Principles and Standards* (NCTM, 2000) recommendations (i.e., standards-based curricula) would offer students reasoning and proof opportunities. Do they? If so, how and with what characteristics? How do these characteristics compare with those offered by curriculum materials that do not identify themselves with the curriculum-reform proposals of the *Standards* (i.e., non-standards-based curricula)? The unavailability of contemporary curriculum analyses that focus on reasoning or proof suggests a need for further investigation; a curriculum analysis is needed to illuminate the nature of the opportunities provided so they can be further developed.

The first and most important step in conducting such a curriculum analysis is to produce an analytic framework that can be used as a tool to investigate the opportunities mathematics curricula provide students for reasoning and proving. This endeavor is quite challenging because the notion of reasoning and proving has been unclear in the context of school mathematics (Reid, 2002; Steen, 1999) and does not yet cohere in an integrated conception of this practice in the teaching and learning of mathematics, rooted in the nature of reasoning and proving in the mathematical discipline and framed for use in curriculum analyses. This paper contributes to this research domain in two interrelated ways. First, by conceptualizing *reasoning and proving* in a way that is sensitive to mathematics as a discipline, considerate of the complexity of this mathematical practice, operationalizable in a curriculum analysis, applicable in different mathematical domains, and promising to push forward the conceptual work on the nature of this activity in school mathematics. Second, by using this conceptualization to develop and validate an *analytic framework* that provides a reliable and comprehensive way to investigate the opportunities mathematics curricula provide students to engage in reasoning and proving.

Method

The process of producing the analytic framework was comprised of five stages. In the first stage, we used the related literature and how reasoning and proving plays out in the mathematical discipline to develop a preliminary framework. In the second stage, we made an initial test and validation of the framework. We analyzed tasks from both the *Connected Mathematics Project* (CMP) (Lappan et al., 1998a/2002a) and the *Mathematics Applications and Connections* (MAC) (Collins et al., 1993)—the most popular standards and non-standards-based middle school mathematics curricula, respectively (U.S. Department of Education, 2000)—to ensure that the framework is considerate of, and applicable to analyzing, both kinds of curriculum materials. At this stage, we used the framework as a tool to categorize the tasks of ten CMP investigations and four MAC half-chapters in different content areas and grade levels. In the third stage, experts validated the framework. Two research mathematicians evaluated both the framework's potential to capture the opportunities curricula provide students for reasoning and proving and the validity of its definitions. After interviewing the two mathematicians and getting their feedback, we revised the framework accordingly. In the fourth stage, we tested the applicability of the new version of the framework by analyzing more curricular tasks (four CMP investigations and three MAC half-chapters). This analysis led to slight refinements of the definitions of some of the framework categories and subcategories. In the fifth stage, we tested the inter-rater agreement of the coding system; an acceptable level of inter-rater agreement around 90% was obtained.

Analytic Framework

In this section, we present the analytic framework and connect it with relevant research. The limited attention of curriculum analyses to reasoning and proof turned us to other genres of scholarship to develop the framework, especially those that examine reasoning and proof from a disciplinary perspective.

The thinking processes that characterize research mathematicians' proving activity involve various stages with *proof* usually being the last. Earliest stages involve what Polya (1954) calls 'mathematics in the making' and frequently consist of the identification and arrangement of significant facts into meaningful *patterns*; the extension of the patterns to formulate *conjectures* and the testing of the conjectures against new experimental facts; and the effort to understand and provide *arguments* about why things work the way they do. Following the premise that school curriculum should represent the structure of the discipline and that students should encounter 'rudimentary versions' of the subject matter, progressively refined through their schooling (Bruner, 1960; Schwab, 1978), we define *reasoning and proving* to encompass the breadth of the activity associated with *identifying patterns*, *making conjectures*, *providing proofs*, and *providing non-proof arguments*. The first two activities are captured under the more general notion of *making mathematical generalizations*—the passing from the consideration of a set of given objects to one for which the original set is a proper subset—and the last two under *providing support to mathematical claims*.

The definition of *reasoning and proving*—to be elaborated further below—and the way its various components relate to one another in the mathematical discipline were used in the development of the *analytic framework* presented in Figure 1. The analytic framework is comprised of two strands. The first strand—components and subcomponents of reasoning and proving—includes the four components that encompass reasoning and proving together with a further breakdown of these components into subcomponents. The second strand—purposes of pattern, conjecture, and proof—concerns the purposes (functions) these three mathematical elements may serve in the curriculum. This strand is intended to provide a more comprehensive description of students' opportunities for reasoning and proving and to facilitate a closer examination of the nature of this engagement. For example, the previous discussion suggests that, in the mathematical discipline, an important connection between patterns, conjectures, and proofs is that patterns can generate conjectures which, in turn, can give rise to proofs. Analysis of the purposes patterns and conjectures serve in the curriculum will reveal whether there are opportunities available to students to understand this connection.

In the rest of this section, we first describe how the analytic framework can be used to categorize curricular tasks. Next, we present the two framework strands in more detail. We conclude with examples that illustrate the framework.

How the Analytic Framework can be Used to Categorize Curricular Tasks

Because in a curriculum analysis we do not deal with actual student work, to categorize curricular tasks we need to find a reasonable way to make plausible inferences about the *expected formulation* of the tasks—that is, the path students are anticipated to follow to solve each curricular task. In the analytic framework, we consider that the expected formulation of the tasks depends on what a particular community assumes as shared at a given time—the *base of public knowledge* in Ball and Bass's (2000) terms. We take the *community* to be the hypothetical classroom that implements serially all the parts of a curriculum program. The expected formulation of the tasks is obtained by working the tasks out and considering together the following three dimensions: (1) The approach suggested by the student's textbook; (2) The approach suggested in the teacher's edition; and (3) The student's expected level of knowledge and understanding when encountering a certain task; this is established with reference to the base of public knowledge at a given time. The base of public knowledge is in turn determined by the notions, axioms, theorems, definitions, mathematical conventions, and methods agreed upon, presented, or discovered prior to a specific task, together with the content covered up to that point; all these are identified by considering what preceded the given task in both the student's textbook and the teacher's edition.

	Reasoning and Proving			
	Making Mathematical Generalizations		Providing Support to Mathematical Claims	
	Identifying a Pattern	Making a Conjecture	Providing a Proof	Providing a Non-proof Argument
Components and Subcomponents of Reasoning and Proving	• Definite Pattern (Transparent, Non-transparent) • Plausible Pattern (Transparent, Non-transparent)		• Generic Proof • Deductive Proof	• Empirical Argument • Rationale
Purposes of Pattern, Conjecture, and Proof	• Conjecture Precursor • Non-conjecture Precursor	• Proof Precursor • Non-proof Precursor	• Explanation • Verification • Falsification • Generation of New Knowledge	

Figure 1. Outline of the analytic framework.

Components and Subcomponents of Reasoning and Proving

Our definition of reasoning and proving suggests that a comprehensive description of the opportunities curriculum materials provide students to develop proficiency in this mathematical practice should describe the instances that students are offered to *identify patterns*, *make conjectures*, *provide proofs*, and *provide non-proof arguments*. Below we elaborate separately each of these four reasoning and proving categories.

Identifying a Pattern

Pattern is a general relation that fits a given set of data. We distinguish between *definite* and *plausible patterns* according to whether or not it is possible mathematically to provide conclusive evidence for their selection over other patterns that could also fit the given data set, respectively. Within each of these subcategories we identify two kinds depending on the extent to which the *nature of the pattern*—whether or not one can provide conclusive evidence for its selection—becomes visible to the students (either in the expected formulation of the task that yields the pattern or in the expected formulations of subsequent tasks). We call the patterns whose nature becomes visible to the students *transparent*, and the ones whose nature remains invisible to the students *non-transparent.*

The distinction between *transparent* and *non-transparent patterns* is meaningful because engagement with patterns entails the danger for the students to develop the expectation that patterns always generalize in ways predicated on the basis of the regularities found in the first few terms (NCTM, 2000). To avoid this incorrect conception, it is necessary to offer students opportunities to understand that not only it is important to find a pattern, but also to see *why* the generalization holds (Harel & Sowder, 1998). In case the students identify a plausible pattern without realizing that there are other patterns that also fit the given set of data (non-transparent plausible pattern), it is likely that the incorrect expectation described earlier will be reinforced. Similarly, if students generalize a definite pattern without connecting the pattern identification to the reasons that make the pattern definite (non-transparent definite pattern), students lose an opportunity to understand that, in some cases, we *can* be sure that patterns will generalize in ways expected on the basis of the regularities found in the first few terms.

Making a Conjecture

Conjecture is a reasoned hypothesis based on incomplete evidence. The term 'reasoned' in the definition is intended to exclude hypotheses that are primarily based on guesses or hunches.

This category and the one presented previously share some common characteristics, the most evident and important being their generalizing aspect. The two categories differ in an important way, though. In *conjecturing*, the solver puts forth a hypothesis that is not considered to be true or false and is subject to testing. Therefore, further examination of the conjecture always seems to be meaningful. In *pattern identification*, however, the solver is looking for a relation that fits a given set of data and, once this relation is found, the solver often no longer feels the need for further investigation and validation.

Providing a proof

Proof is an argument from accepted truths for or against a mathematical statement. An *argument* is considered to be a logically-connected sequence of assertions. *Accepted truths* are defined broadly to include the notions, axioms, theorems, mathematical conventions, methods (Leddy, 2001), and definitions that a particular community may take as shared at a given time.

A proof can either be *generic* or *deductive*. *Generic proof* is a general argument illustrated in a particular case seen as representative of a whole class (in parallel to Balacheff's [1988] *generic example* and Movshovitz-Hadar's [1988] *transparent pseudo-proof*). *Deductive proof* is a logically necessary argument based on properties or structural relationships (e.g., definitions) that connects data with conclusions without relying on specific examples. This kind of proof also includes situations where the solver provides a counterexample or reduces a task into a finite number of cases, record those cases, and check them in a systematic and exhaustive fashion.

The above suggest that for an argument to qualify as proof it needs to treat somehow the general case. Therefore, the definition excludes empirical arguments from being counted as proofs. This way of defining proof in school mathematics is not only faithful to the way the same concept is generally conceptualized in the mathematical discipline, but is also considerate of the large bodies of research in developmental psychology, cognitive psychology, and mathematics education that demonstrate children's ability, as early as the elementary grades, to reason mathematically and to engage successfully in tasks that involve deductive reasoning and proof.

Providing a Non-Proof Argument

Non-proof argument is an argument in support of a mathematical claim that does not qualify, according to the definition of proof, to be considered as proof. It can take one of two forms: *empirical argument* or *rationale*.

Empirical argument is an argument that provides inductive, non-conclusive evidence that a mathematical statement is true. In particular, the solver checks a proper subset of all the possible cases and uses those to support the validity of a mathematical statement. In this type of argument, the use of examples is the main (if not the only) element of conviction (Marrades & Gutiérrez, 2000). *Rationale* is some kind of a non-proof, non-empirical argument to a mathematical claim (e.g., 'the relation is linear because the points fit on a straight line').

Purposes of Pattern, Conjecture, and Proof

The tasks coded in the identifying a pattern, making a conjecture, and providing a proof categories are further analyzed according to the *purposes* patterns, conjectures, and proofs, respectively, serve in these tasks. We elaborate the purposes of each of these mathematical elements separately.

Purposes of Pattern

As we noted earlier, a major role patterns serve in mathematics is to give rise to conjectures. Specifically, a relation that is found to apply to a given set of data (a pattern) is often extended and used to make a hypothesis about a set of objects relative to which the original one is a proper subset (a conjecture). Therefore, a pattern can either forerun a conjecture opportunity associated with it in the curriculum (*conjecture precursor*) or not (*non-conjecture precursor*).

Purposes of Conjecture

Conjectures can be to proofs what patterns are to conjectures. Specifically, a conjecture can either forerun a proof opportunity associated with it in the curriculum (*proof precursor*) or not (*non-proof precursor*).

Purposes of proof

Proof can serve several purposes in the curriculum, similar to its purposes in the mathematical discipline: *explanation*, when it provides insight into why a statement is true (de Villiers, 1999) or false; *verification*, when it establishes the truth of a given statement (Bell, 1976); *falsification*, when it establishes the falseness of a given statement; and *generation of new knowledge*, when it contributes to the development of new results (Kitcher, 1984).

Examples

Next, we present two examples from CMP that help illustrate most parts of the framework.

Example 1

Problem: Make a conjecture about whether the sum of two even numbers will be even or odd. Then use the models or some other method to justify your answer. (slightly modified from Lappan et al., 1998c/2002c, p. 29) Discussion: This task would be triple-coded as *identifying a pattern*, *making a conjecture*, and *providing a proof*. The students are expected (a) to examine a few cases and notice the *pattern* that the sum they get in all cases is an even number, (b) use that pattern to formulate the *conjecture* that the sum of any two evens will be even, and (c) use their representation of even numbers as rectangles with a height of two tiles to provide the following *deductive proof* for their conjecture: The sum of two even numbers is even because we can combine two rectangles with height 2 to get another rectangle with height 2. The pattern is *transparent definite* because the students, through the proof they provide, are expected to realize that the pattern they came up with is the only acceptable pattern that fits the given set of data. With regard to purposes, the pattern is a *conjecture precursor* because it gives rise to a conjecture, and the conjecture is a *proof precursor* as it foreruns a proof. Finally, the proof serves the purposes of *explanation*, *verification*, and *generation of new knowledge*.

Example 2

Problem: A. Cut a sheet of paper as Alejandro did, and count the ballots after each cut. Make a table to show the number of ballots after 1 cut, 2 cuts, 3 cuts, and so on. [Alejandro starts by cutting a sheet of paper in half. He then stacks the two pieces and cuts them in half, and so on.]

B. Look for a pattern in the way the number of ballots changes with each cut. Use your observations to extend your table to show the number of ballots for up to 10 cuts.

C. If Alejandro made 20 cuts, how many ballots would he have? How many ballots would he have if he made 30 cuts? (Lappan et al., 1998b/2002b, p. 6) Discussion: In this problem, parts A and C would be coded as non- reasoning and proving, while part B as *identifying a pattern*. The pattern is *definite* because the context defines uniquely the pattern that needs to be chosen. In other words, the context does not allow for equations that fit the data other than the following to be considered as valid: Number of ballots after n cuts = 2^n. The definite character of the pattern, however, does not become evident to the students because they are expected to generalize the pattern from the values in the table rather than from the context of the task. Therefore, the pattern is *non-transparent*. Also, the pattern does not give rise to a conjecture and, therefore, the purpose it serves is *non-conjecture precursor*.

Conclusion

As Yackel and Hanna (2003) note, if we are to improve all students' understanding of, and ability for, reasoning and proving, we need to provide teachers with the means to develop "forms of classroom mathematics practice that foster mathematics as reasoning and that can be carried out successfully on a large scale" (p. 234). Given the central role of curriculum in practice, it appears that one of the most promising ways to support teachers' efforts in this domain is to equip them with a curriculum that is considerate of current research and curriculum framework recommendations about reasoning and proof, and has careful organization and sequencing. A first step toward this direction is to describe the different kinds of opportunities mathematics curricula offer students for reasoning and proving. A major contribution of this study is that it has produced an *analytic framework* that can be used in future curriculum analyses with a focus on the opportunities available to students of different grade levels and curriculum programs to develop proficiency in reasoning and proving. The results of these analyses can in turn be used to guide curriculum development and revisions.

The analytic framework developed can also contribute to several other domains of research and practice. It can be used to: (a) Push forward both the conceptual work on the nature of reasoning and proving in school mathematics and the methodology for conducting curriculum analyses on specific mathematical practices; (b) Study how teachers treat textbook content with respect to reasoning and proving; (c) Guide professional development and teacher preparation programs regarding different possible learning opportunities for reasoning and proving teachers can offer; (d) Examine whether students' difficulties in reasoning and proving relate to the kinds of opportunities emphasized or neglected in the curriculum materials they are using; (e) Explore the ways in which inductive and deductive reasoning are being promoted in the curriculum through the relative emphasis various framework categories and subcategories receive (empirical arguments, conjectures, and non-transparent patterns promote more inductive than deductive reasoning; deductive proofs mainly require deductive reasoning; transparent patterns require a more of a balance of inductive and deductive reasoning activity); (f) Investigate whether the opportunities made available to students by the curriculum materials they are using to engage in reasoning and proving facilitate or impede their transition from empirical to more deductive forms of argument.

Endnote

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HOW DO MATHEMATICIANS VALIDATE PROOFS?

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Abstract. This paper reports on an exploratory study addressing how mathematicians determine whether an argument is a valid proof. Eight mathematicians were presented with six to eight arguments from number theory and were asked to think aloud while determining whether each argument constituted a valid proof. The analysis in this study focuses on the reasoning the mathematicians used to determine that an assertion in a proof was valid. Our main findings were that: a) when mathematicians doubted an assertion, it was unusual for them to construct a full sub-proof to establish that assertion, b) the mathematicians sometimes used inductive (i.e., example-based) reasoning to validate an assertion, and c) the inductive reasoning strategies employed often relied on a sophisticated understanding of number theory. Pedagogical and epistemological implications of this data are discussed.

1. Introduction

Many mathematicians, philosophers, and mathematics educators have examined the nature of, and processes used in, advanced mathematical reasoning. Work in this direction includes, but is not limited to, mathematicians' introspections on their own mathematical thinking (e.g, Hadamard, 1945; Thurston, 1994) as well as conclusions drawn by philosophers and educators based upon theoretical, philosophical, and historical arguments (e.g., Lakatos, 1976; Davis and Hersh, 1981; Ernest, 1991).

Research in this direction has generally focused on learning and understanding mathematical concepts and the construction of proofs. There has been comparatively little research on how one determines whether an argument constitutes a valid proof—a process that Selden and Selden (2003) refer to as *validation*. There appears to be a relative consensus, at least among mathematics educators, that proof validation is a social process that is not solely comprised of examining the logical structure of mathematical arguments (cf., Lakatos, 1976; Hanna, 1991; Thurston, 1994). Manin (1977), for instance, argues that acceptance of a proof is aimed more at weighing the plausibility of the argument being presented than at verifying the deductive process. Hanna (1991) suggests that even non-mathematical factors, such as the reputation of the proof's author, may influence one's acceptance of a proof more than the logic in the proof itself.

The research outlined above describes general factors that influence a community's acceptance of a proof. However, it often does not explicate *specific* processes that *individual* mathematicians use in deciding whether an argument constitutes a proof. Further, the arguments put forth by these researchers are usually based on the introspection of mathematicians or philosophical/historical arguments and not on systematic empirical studies. In this paper, we address this gap by reporting on an exploratory study in which eight mathematicians were observed while determining whether a set of arguments constituted valid mathematical proofs.

We believe this research may have important consequences for the pedagogy used in proof-oriented mathematics courses. The ability to validate proofs is a critical ability for students of mathematics to possess. In order for a student to be convinced of a theorem by reading a proof of the theorem, that student would need reliable methods for determining whether a proof is valid.

If a student cannot determine whether or not an argument establishes a theorem, then that argument cannot legitimately convince him or her that the theorem is true. The educational literature, while limited, suggests that students often lack the ability to validate proofs. For instance, Selden and Selden (2003) presented four arguments to eight undergraduates in an introductory proof course and asked them to determine whether each of these arguments would prove a given statement. They found that students' performance on this task was essentially at chance level.

One reason that proof validation is difficult to teach is that currently the specific processes used in the act of proof validation are not fully understood. Dreyfus (1991) argues that research into the processes used in advanced mathematical reasoning are valuable since students will often not learn or use these processes on their own if the instructor does not make the students aware of them. An important goal of the research reported here is to highlight some of the knowledge and thought processes that are used in proof validation and to argue that these processes should be introduced in the classroom. We also hope that the data presented in this paper can contribute to the ongoing debate about the epistemological status of proof and proof validation. In particular, our data may be germane to the following questions: What constitutes a proof? By what processes does an argument become a proof?

2. Research context

2. 1. Data collection

Participants. Eight mathematics professors at a mathematics department at a regional university in the southern United States participated in this study. *Materials*. Eight purported proofs of number theoretic statements were used in this study. The first four of these arguments were identical to those used in Selden and Selden's (2003) study on proof validation. These four arguments purported to prove that "If n^2 is divisible by 3, then n is divisible by 3". One of these arguments was a valid proof, while the other three arguments contained blatant logical errors. The other four proofs were more complex and at the level of sophistication that one might expect from a proof in an undergraduate textbook on number theory or an expository mathematics journal. Three of these four proofs were valid, while one invalid argument was included as a foil. For instance, one of these arguments was Holdener's (2002) proof that "if n is an odd perfect number, then $n\equiv 1 \pmod{12}$ or $n\equiv 9 \pmod{12}$" as it appeared in the *American Mathematical Monthly*. *Procedure*. Each participant met individually with the first author of this paper. Each participant was asked to "think aloud" while attempting to validate the arguments described above. This stage of the experiment continued until the participant had validated all eight of the arguments or until 45 minutes had elapsed. Participants were then asked a series of questions concerning the processes that they used to determine whether an argument constituted a proof. Such questions included: How do you determine whether or not an argument is a proof? Does intuition ever play a role when you are determining if an argument is a valid proof? Finally, after analyzing the data, some of the participants were asked further questions about specific comments that they made during their proof validations.

2. 2. Data analysis

In this paper, we will focus on the reasoning that mathematicians used in their line-by-line validation of a proof— i.e., the reasoning used to determine that a particular assertion in a proof is valid. We argue elsewhere that to accept a particular assertion in a proof as legitimate, one must not only believe that the assertion is true, but also that the assertion is warranted- that is, there is a legitimate mathematical justification for why that assertion follows from previous statements in the proof (cf., Weber and Alcock, submitted). This paper will examine how mathematicians construct these justifications and what types of justifications they accept as legitimate. Other important aspects of proof validation (e.g., how one examines the structure of the proof) will only be discussed briefly, but will be the subject of future reports.

3. Results

The participants' validations could be divided into two distinct stages. In the first stage, participants examined the assumptions and conclusions employed in each of the proof's sub-arguments to determine the structure of the proof. If the proof methods employed were judged to be reasonable, the participants would proceed to the second stage of line-by-line verification. If the proof methods were not reasonable, (e.g., if the proof began by assuming what it was claiming to prove), the participants immediately rejected the proof as invalid.

In total, the mathematicians collectively read 225 assertions whose validity could reasonably be judged. (Assumptions and statements that introduced variables were not included among these assertions). Of these 225 assertions, 122 were accepted immediately by the participants without a justification. Twenty assertions were rejected as invalid. There were also six cases where a participant could not decide whether an assertion was valid. Our analysis concentrates on the reasoning used by the mathematicians to understand why the remaining 77 assertions were valid. We divided these assertions based on the nature of the justifications that mathematicians formed to convince themselves that the assertions were valid. We define *property-based justifications* to be justifications of an assertion based on properties of the objects and structures to which the assertion pertains. We define *inductive justifications* to be justifications of a general assertion based solely on the examination of specific examples. Ten assertions could not be coded into either class because no justifications were offered (e.g., the participant misunderstood the assertion, re-read it, and then decided it was valid) or the justifications were idiosyncratic.

3. 1. Property-based justification

There were 49 instances in which mathematicians accepted an assertion on the basis of a property-based justification. We further categorized these justifications into proof-like justifications and sketches of proofs. Each of these terms is defined below and then illustrated with an example. For each example, we list the assertion that was validated, relevant prior assertions that appeared earlier in the proof, and a transcription of the mathematicians' utterances while validating the assertion in question. *Proof-like justifications* (N = 15). A justification was considered to be a proof-like justification if the mathematician explicitly employed definitions, theorems, or other established facts to show that the assertion in question was a logical consequence of previous work. The professor's utterances or written work, with minor modifications, would constitute a valid sub-proof that established the assertion. An example is provided below.

Prior assertion:	$n\equiv 3 \pmod 4$.
Assertion:	Note that n is not a perfect square.
Prof C:	So if you take an odd number and square it, $2k + 1$, and I assume that when you square it out, that would be, yeah, that would be 1 mod 4. OK, note that n is not a perfect square. OK, I think I'm OK with that.

Sketch of a proof (N = 33). A justification was considered to be a sketch of a proof if the following two conditions held: a) The justification was based on properties of the objects and structures to which the proof pertained. These properties may have been theorems or previously established assertions that appeared earlier in the proof, or they may simply be properties that the mathematicians believe are true. b) Considerable work would be required to transform the justification into a proof. Such work might include establishing that objects did in fact have the properties that they were conjectured to have. An example is provided below.

Prior assertions: $ab = n$. $n\equiv 3 \pmod 4$.

Assertion: Either $a\equiv 3 \pmod 4$ and $b\equiv 1 \pmod 4$ or $a\equiv 1 \pmod 4$ and $b\equiv 3 \pmod 4$.

Math C: I guess that's a fact. I mean, how do you take two numbers in $\boldsymbol{Z}_4$ and multiply them together to get three?

While this mathematician's reasoning expresses the essence of why the assertion is true and suggests that a proof could be constructed by examining cases, his comments alone would hardly constitute a proof.

3. 2. Inductive justification

There were 19 instances in which the mathematicians accepted a general assertion in a proof solely by inductive reasoning—i.e., the examination of examples. The mathematicians used examples in different ways, some of which are described below. *Identification of a systematic pattern.* There were four instances in which mathematicians accepted a proof by examining systematically chosen examples, noticing a pattern in these examples, and then conjecturing a general statement that would hold for a wider class of objects based on this pattern. Consider the example below:

Previous assertions: $n\equiv 3 \pmod 4$.

Assertion: Note that n is not a perfect square.

Prof A: SO: I'm using examples to see what, where the proof is coming from. So 5^2 is 25 and that's 1 mod 4. 36 is 0 mod 4. 49 is 1 mod 4. 64 is 0 mod 4. I'm thinking that, ah! So it is… 24 times 24, that's 0 mod 4. So a perfect square has to be 1 mod 4, doesn't it? n^2 equals 1 mod 4 or 0 mod 4. Alright.

From his inspection of the integers 5, 6, 7, and 8, Professor A conjectured the property that perfect squares were only congruent to 0 or 1 mod 4, and then tested his conjecture by seeing if this held true for 24. He then concluded that n could not be a perfect square since it was congruent to 3 mod 4. In the interview following the validations, Professor A indicated that he would not ordinarily used examples in his validations, but found such reasoning to be appropriate in this particular number-theoretic context.

I: I noticed that at times you used examples to help you validate the proofs.

Prof A: Yes… I think with the proofs with number theory, they [examples] are a little easier because what you do is you try to show it's true for some and then there should be easy induction arguments, hopefully, show that it's true for all of them. Topology [Professor A's area of research] you don't quite have that.

Inspecting a generic example to see why a general assertion is true. There were seven instances in which the professor first verified that the general assertion held for a specific example, form an explanation of why the example satisfied the assertion, established that the reason also applied to a wider class of examples, and conclude that the assertion was valid.

Previous assertions: $n\equiv 3 \pmod 4$. Note that n is not a perfect square.

Assertion: $\sum_{d|n} n = \sum_{d<\sqrt{n},\, d|n} \left(d + \frac{n}{d} \right)$

Prof F: I don't understand this statement so let me look at an example. Let me look at 8. Eight has four factors and they add up to… 15. Only two are less than $\sqrt{8}$. OK so we add 1 and 8 and 2 and 4 and that's 9 plus 6 and, well what do you know? It worked! It worked for this example… Oh I see, each of these numbers multiplied to get 8, one will be less than the square root of 8 and one will be greater… yeah, OK, I see.

Professor F initially examined the specific instance where n was equal to eight to try to gain an understanding of the statement. He then saw that the statement was true for eight and formed an explanation for why this was true that was not based on this particular instance, which in turn lead him to accept the general assertion as true. It is interesting to note that professor F examined this general assertion with an instance that did not satisfy assumptions that appeared earlier in the proof. It was assumed that n was congruent to 3 (mod 4), yet the professor set n equal to 8. *Failure to find a counter-example.* There were six instances in which mathematicians sought to contradict a general assertion by constructing a counter-example. When the search was unsuccessful, they accepted the assertion as valid. It should be noted that the search to find a counter-example was not random, and seemed to rely on their conceptual understanding of number theory. Consider the example below.

Previous assertions: n is a natural number.

Assertion: There exists an odd integer m and a non-negative integer l such that $n=2^l m$.

Prof E: Hmm… can we express every integer in that way? Well, 1 is $l = 0$, $m = 1$. 2, 4, and 8 are powers, but can we express every integer in that way? What about 3? Um, let $m = 3$ and $l = 0$. And 5, let $m = 5$ and $l= 0$. What about 6? 6 is 2^3 times 3 [sic]. OK I guess this sounds reasonable.

In his interview after his validations, Professor E indicated that his search for counterexamples was partially dictated by his understanding of the integers.

Prof E: In the case that we were looking at, with $n = 2^l m$… yes, since it concerned numbers being written in a way involving powers of 2, the first thing I did was check powers of 2. Then I checked numbers other than the powers of 2, but realized that they were all odd numbers. So I tried even numbers that were not powers of 2 and saw that they worked too.

Validation from a single example. There were two cases in which mathematicians accepted a general assertion as valid by verifying that it held for a single example. It seems likely that the mathematicians may have initially been more or less sure that the assertion was true and checked a single instance to confirm their intuition.

4. Discussion

The purpose of this exploratory study was to investigate the processes and reasoning used by mathematicians in proof validation. It is certainly possible that there were important processes used for validation that we did not observe. For example, some of the reasoning used by the mathematicians may have been sub-conscious and not observable by verbal protocol analysis. Likewise, if mathematicians were given more time to validate these proofs or if non-number theoretic proofs were given, different types of reasoning may have been exhibited. Nonetheless, it still seems reasonable to say that this study can be used to identify *some* of the processes that mathematicians use to validate proofs.

The analysis in this paper focused upon how mathematicians determined that questionable assertions in a proof were valid. In our study, it appears that formal logic was not often invoked in this regard. While mathematicians sometimes constructed sub-proofs to establish these assertions, they did not do so frequently. More often, mathematicians constructed only partial proofs, sometimes based on properties that were stated intuitively or were not formally established. Other times, their reasoning was based on inductive reasoning. It should be noted that the informal or inductive reasoning that mathematicians used was not applied haphazardly. From their comments in the interview following their validations, the mathematicians indicated that their conceptual understanding of number theory helped them decide when their inductive strategies were appropriate and how they should be applied.

There seems to be a contrast between the standards that are used for presenting a formal proof and the processes used to validate it. An argument that relied on assertions purported to be true because they were difficult to contradict, the inspection of well-chosen examples, and intuitive explanations would not be considered a proof. Indeed, such arguments are often described as the types of invalid proofs that naïve students sometimes produce. However, it appears that some mathematicians use these arguments in the validation of proofs. In this sense, proof validation appears to be a *less* formal process than proof construction. In constructing a proof of a statement, one must strive to convince an enemy, or at least a skeptical mathematician who understands the subject, that the statement is true (cf., Mason, Burton, and Stacey, 1982; see also Davis and Hersh, 1981). This requires forming an argument that meets the mathematical community's relatively rigorous standards of what constitutes a formal proof. However, in validating a particular assertion that appears in a proof, it seems that the mathematicians only wanted to convince themselves that the statement was true, and that their personal standards for obtaining conviction were sometimes less stringent than the community's proof standards.

Many of the less formal reasoning processes described above currently receive little to no attention in proof-oriented university mathematics courses. In fact, we argue that such strategies may be implicitly discouraged. For example, in such courses, students are often told that one can never determine that a general assertion is true just by looking at examples. This may lead students to believe that such reasoning is not only inappropriate for proof presentation, but also are not applicable for proof validation (and proof construction). As students have difficulty validating proofs and do not appear to invoke these processes, giving explicit attention to these strategies in a thoughtful manner may help improve students' ability to validate proofs.

We suspect that one reason these strategies are not taught is that students, who lack the mathematicians' knowledge base, may not be able to apply these strategies effectively. Further, a full description of some of these processes may be difficult for the professors to explicate and hence might be difficult for students to understand. We also speculate that one way in which these strategies are learned and refined is by adjusting them when their application is unsuccessful. For instance, if one erroneously concludes an assertion is valid based on the inspection of several examples, he or she might learn that checking examples is inappropriate for validating this type of assertion, or alternatively, that a different class of examples also needed to be considered. We would thus argue while students may initially err when validating proofs using non-formal reasoning, these failures can serve as important opportunities for students to learn to use this type of reasoning effectively. If our argument is correct, discouraging students from using non-formal reasoning in proof validation may be denying them the very experience they need to validate proofs effectively.

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IMPLEMENTING THE NCTM'S REASONING AND PROOF STANDARD WITH UNDERGRADUATES: WHY MIGHT THIS BE DIFFICULT?

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Using a synthesis of the theories of Harel (1998) and Brousseau (1997), data from a series of experiments with undergraduate mathematics and science majors was analyzed to determine how students' understandings of mathematical induction evolve. The data is used to illustrate the tensions and constraints K-16 teachers may face as they endeavor to implement the Reasoning and Proof Standard in classrooms.

Throughout the *Principles and Standards for School Mathematics* (NCTM, 2000), teachers are encouraged to create communities of inquiry and to facilitate the development of mathematical reasoning as a "habit of mind." Within the standards, proof is not viewed as a concept to be learned but rather as a theme of mathematics. "Reasoning and proof should be a consistent part of students' mathematical experience in prekindergarten through grade 12" (NCTM, 2000, p. 56). Within this discussion of expectations of K-12 students it is acknowledged that proof is a very difficult area for undergraduate mathematics students (see *e.g.*, Alibert & Thomas, 1991; Harel & Sowder, 1998). It is then argued that post-secondary students' difficulties may be due to a lack of experience writing proofs and a "limited perspective (of proof)" (NCTM, 2000, p. 56).

Though it may be the case that post-secondary students lack experience, there is also an alternative explanation for their difficulties with mathematical proof: students at the post-secondary level lack experiences that facilitate the development of an intellectual need, on the students' behalf, for proof. In other words, it is not that students should simply write more proofs but rather that students should encounter more situations that foster, in the eyes of the student, the need for proof. The position taken in this paper is that attempts to facilitate the development of the student understandings described within the Reasoning and Proof Standard, which do not take into consideration the development of an intellectual need on the students' behalf, may simply result in unproductive shifts in the didactical contract.

Theoretical Perspective

This research was informed by a theoretical perspective that is a synthesis of two independent theories: the Theory of Didactical Situations (Brousseau, 1997) and The Necessity Principal, Harel's (1998) theory of intellectual need[i]. Brousseau's Theory of Didactical Situations is a theory that aims to account for the communication and circulation of mathematical knowledge (Brousseau, 1997; Herbst and Kilpatrick, 1999; Sierpinksa, 2000). It is a theory that acknowledges the constraints imposed by didactic institutions and how these can transform the meanings available to the learner. It aims to provide the means to understand and ultimately control these transformations (Brousseau, 1997).

Within the theory, the interactions that occur in didactical settings are said to be constrained by a set of tacit expectations: a didactical contract. "It is the contract that specifies the reciprocal positions of the participants on the subject of the task, and that specifies the deep meaning of the action under way, of the formulations or the explanations furnished" (Brousseau & Warfield, 1999, p. 47). Thus, the didactical contract determines how one's actions and questions are to be interpreted. As such, it constrains the meanings available to the learner. In her discussion of the didactical contract, for instance, Sierpinska argues that if the teacher both presents and solves the problems and the students are simply required to reproduce solutions then "it is the social behavior, not the mathematical knowledge that the students will learn" (Sierpinska, 2000, p. 6).

The notion of a didactical contract is used within the theory not only when exploring the meanings available to the learner, but also when explain responses that might otherwise seem rather disturbing. For example, Herbst and Kilpatrick (1999) use the notion of a didactical contract to provide a sense of rationality to students' response of "36 years" when asked, "On a boat, there are 26 sheep and 10 goats. What is the age of the captain?" They argue that such responses can be accounted for in terms of students' expectations of such problems: "students are supposed to use key words and the relative nature of the numbers given as a heuristic to separate relevant information -*i.e.*, to find the operation 'hidden' in the problem" (Herbst & Kilpatrick, 1999, p. 4).

The complementary nature of the theories of Brousseau and Harel becomes evident if one considers the hypotheses upon which the Theory of Didactical Situations is based: the constructivist and the epistemological hypotheses. The constructivist hypothesis is simply that students construct their own meaning as a result of resolving situations that, in part, appear problematic to the student. The epistemological hypothesis is that "problems are the source of meaning of mathematical knowledge" (Balacheff, 1990, p. 259). In other words, meanings are developed within a milieu when such meanings become necessary for resolving the problematic situations that arise, or for functioning, within that milieu. What is problematic in the eyes of the student, however, is not always the same as what is viewed as problematic by the teacher. Thus, the teacher must be concerned with the creation of situations that foster the development of an intellectual need on the students' behalf, *i.e.*, that appear problematic to the student. Harel's discussion of various approaches to linear independence exemplifies how one might begin to consider the issue of intellectual need. He asks:

> But, is our student likely to view this as a problem? Can he or she in this stage of the course understand its importance? Can he or she see how "independence" contributes to its solutions? In other words, what is our student's intellectual need –as opposed to social or economic need – in learning the concept of "independence" (Harel, 1998, p. 501)?

Harel posits that when "a situation that is incompatible with, or presents, a problem that is unsolvable by our existing knowledge" (Harel, 1998, p. 501) an intellectual need arises and the student "sees" the necessity of what we intend to teach them. This is Harel's Necessity Principle. The primary implication of this principle is that we must attend to the students' ways of thinking, *i.e.*, their apparatuses for filtering what we intend to teach them (Harel, 1998), if we are to foster the development of situations that appear problematic to the students.

Data Collection and Analysis

As part of a series of studies focused on undergraduates' understanding of mathematical proof, I conducted three teaching experiments with small cohorts of Calculus II students (n = 3) at large, urban state universities.[ii] The purpose of the experiments was to document how students' understandings evolved as mathematical induction arose as the means to solve a class of problems. The first course, TE 1, was a three-week teaching experiment. Students met with the investigator for a total of six 75-minute sessions. The second and third teaching experiment, TE 2A and TE 2B, respectively, occurred simultaneously and ran for a period of eight weeks, for a total of twenty 65-minute sessions. Data was obtained in the form of investigator field notes, student work, and videotapes and transcripts of each classroom session and student interview. The teaching experiment methodology employed was that described by Steffe and Cobb (1983).

Initially the analysis of the data relied heavily on Harel and Sowder's notion of a *proof scheme*: "the student's apparatuses for removing doubts during the processes of ascertaining and persuading" (Harel & Sowder, 1998). To investigate students' proof schemes Harel and Sowder's descriptions and extensive categorizations of undergraduates proof schemes were used as a framework to classify students' ways of ascertaining and persuading as indicated by students' written work and remarks during student-to-student and student-teacher interactions. The method of analysis was then revised and events were accounted for in terms of (a) the evolving local didactical contract and (b) the specific needs the students' actions might satisfy.

Results

The data indicate that, with an alternative curricular approach, the students' understandings of mathematical induction progressed through three stages: pre-transformational, restrictive transformational, and transformational. During the *pre-transformational stage* the students employed empirical-inductive proof schemes in situations of validation, *i.e.*, the students conjectures were validated by appeals to physical facts or sensory experiences (Harel & Sowder, 1998). During the *restrictive transformational stage* the students' empirical-inductive proof schemes were subsumed by restrictive transformational proof schemes, in particular, the generic and the constructive proof scheme. It was during this stage that mathematical induction emerged as a means to construct an infinite sequence of items from a collection of "actual" objects. During the *transformational stage* mathematical induction arose as the means to establish the existence, as opposed to the construction, of an infinite sequence of items.

Key to the progression from one stage to the next was a classroom renegotiation of what constituted a general solution. These negotiations initially revealed the students' robust empirical-inductive proof schemes. For example, in each teaching experiment the students were asked to solve the following, a modification of the *Towers of Hanoi* problem (Figure 1).

> *Three pegs are stuck in a board. On one of these pegs is a stack of 1275 disks graduated in size, the smallest being on top. How many moves will it take to transfer the stack to one of the other two pegs by moving the disk one at a time in such a way that a disk is never placed on top of a smaller disk?*

Figure 1. Modified *Towers of Hanoi* Task

The students' responded by to this task by computing the total for a sequence of cases and then identifying a pattern so as to arrive at a formula. Once a formula was identified the investigator attempted to facilitate the development of a situation of justification by asking questions such as "Is there a way of verifying that that would actually be the case?" To this question Boris responded, "Well, by having done three, I guess I'd be … that's kind of like your basis for saying that." Further probes into the students' ways of thinking indicated that the students viewed finding a formula that matched the data as an act of justification. Rather than perturb the students' sense of conviction, investigator-posed questions of validity simply clued the students into a shift in the local didactical contract and the expectation that something else be included in their solution. For example, consider Calvin and Jill's response:

Calvin: I don't understand what ... what you really want to know?

Jill: She wants us to prove it mathematically.

As the instructor, I was faced with a dilemma: I could either continue to pose questions about the validity of the proposed formula and ignore that these questions were alien to the students or

I could attempt to facilitate the development of a need for non-empirical reasoning. The following examples illustrate the issues that arose as a consequence of choosing the latter.

To facilitate the development of a need for non-empirical reasoning the students were asked to solve the modified *Chords of a Circle* problem (Figure 2.) Each cohort of students responded to this task by (a) calculating the number of regions created for the cases n = 1,2,3,4,5,6 and (b) dismissing as errors the results for n = 6.

> *Suppose you have a circle with n points marked on the circumference. By connecting each pair of points with straight-line segments the circle can be partitioned into a number of regions. Is there a function for calculating the number of regions?*

Figure 2. Modified *Chords of a Circle* Task

For example, in TE 1, Boris asked "How many did you get, Jill" and the following exchange occurred.

Jill: Thirty-one but it has to be wrong ... because it has to be even.

Calvin: I got thirty-one too ... but it's wrong.

After repeated attempts to produce the desired number of regions (32), the students began to investigate why their formula (2^{n-1}) failed for the case n = 6. Ultimately, this investigation resulted in the development of a rationale, even though no such rationale was solicited in the problem statement. What is particularly interesting about this response is not that the students investigated an event that contradicted their expectation that a formula existed but rather the impact this experience had on the students' ways of thinking about empirical-inductive arguments. For example, in a subsequent episode the investigator asked the students to revisit their solution to the *Towers of Hanoi* task. During this discussion both Boris and Jill expressed doubts about their solution's validity. For example, Jill remarked "I don't know why it works ... since I don't know why it works there're some doubts." These remarks stand in contrast to those made earlier in the experiment when a collection of instances was offered as an explicit rationale for a statement's validity. Subsequent responses further indicated that the students' experience with the *Chords of a Circle* task had created a need for non-empirical justification. For example, Jill remarked to Calvin, "I understand what you're saying here, if it works for this one it's going to work for that one but it ... what if at one point it doesn't? Like the circle thing?"

In TE 2B, the students responded similarly to the *Towers of Hanoi* task. For instance, after identifying a formula that generated the numbers of moves for the cases n = 1,2,3,and 4, the students "tested" their formula against the case n = 5. The students' subsequent responses indicated that this collection of cases was sufficient evidence of the formula's validity. For example, Johan provided the following rationale "We have five numbers that follow a pattern." As in TE 1, when the students were asked to solve the *Chords of a Circle* task the students generated a table of data and then dismissed the totals for n = 6 as errors. However, unlike the students in TE 1, the students in TE 2B proceeded by dismissing the task as a "trick question" rather than by investigating what might cause the formula 2^{n-1} to fail at n = 6.

Paula: Maybe it's a trick question. There is no answer.

Susan: (overlapping with Paula) them all like that.

SB: (responding to Paula) There are no trick questions in here.

Susan: Sure there are… because there's two different ways to make it and it doesn't give you a way to make it … it makes two different answers. [Johan: Uh-huh] How are you going to get a formula … if once you get to six you start getting possible different … I mean I don't know five might end up with possible answers too.

Such an interpretation of the *Chords of a Circle* task is the result of a specific expectation of teacher-posed questions of existence: questions of existence imply existence. This way of interpreting the task played an unproductive role in the students' mathematical activities, for it enabled the students to dismiss data that did not conform to their expectation that a function existed. It also enabled the students to preserve their ways of reasoning about patterns in numeric data. Thus, to foster shifts in the students' ways of thinking about empirical justifications subsequent tasks needed to be developed that could not be dismissed by the students as "tricks" and that would foster a reconsideration of their ways of reasoning about patterns.

To accomplish this goal, the students were asked to consider a second solution to the *Chords of a Circle* task. After indicating their surprise over the existence of a quartic polynomial which "matched" the function 2^{n-1} when n =1,2,3,4,5, the students noted that the formula 2^{n-1}"broke down" for n > 6, and then explored why multiple totals might be produced. Thus, the students began to recognize the complexities of the mathematical setting rather than simply dismiss the task as a "trick." That these activities resulted in a shift in the students' ways of reasoning about patterns was indicated in the next episode when, after having verified the claim "The sum of n positive integers is n^2" for a large collection of cases, the students attempted to work toward a non-empirical rationale. For example, Susan argued "It works, that's up to ninety-nine. Now, let's do something better."

Thus, it was critical to each cohort of students' progress that shifts occur in the students' ways of reasoning about patterns. My initial attempts to facilitate these shifts were of the form of instructor-posed questions of validity. The students' responses indicated these questions were alien to the students, for they saw no basis for the questions I posed. Consequently, these questions resulted in unproductive shifts in the didactical contract as the students attempted to preserve their ways of reasoning while also attempting to meet my expectation of a "different" justification. It was not until the students encountered situations that supported a need to move beyond empirical justifications that the desired shifts occurred. These shifts in the students' ways of reasoning were the first in a series of shifts that eventually resulted in the development of deductive ways of reasoning.

Conclusions

The primary claim of this paper is that we must attend to students' ways of thinking if we are to create authentic situations of justification rather than facilitate unproductive shifts in the didactical contract. The examples provided illustrate the constraints and tensions we, as teachers, may need to manage as we attempt to create authentic situations of validation. They also demonstrate why we must be careful to not underestimate the complexities of the steps involved with meeting the expectations outlined in the Reasoning and Proof Standard (NCTM, 2000), in particular, as we attempt to foster the following understandings:

- In grades 3-5, "students should learn that several examples are not sufficient to establish the truth of a conjecture" (p.188);
- In grades 6-8, students should learn to "be cautious when generalizing inductively from a small number of cases" (p. 267);
- In grades 9-12, "students should understand that having many examples consistent with a conjecture may suggest that a conjecture is true but does not prove it" (p. 345).

Endnotes

i. Harel has since advanced this theory and has written extensively about the DNR framework.

ii. This work is part of a doctoral dissertation completed by the author at San Diego State University and the University of California at San Diego, under the supervision of Larry Sowder.

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KEY IDEAS: THE LINK BETWEEN PRIVATE AND PUBLIC ASPECTS OF PROOF

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In this paper, a model is proposed to characterize some of the ways mathematical proofs are generated and understood. The model uses the notion of "Key Idea", which is defined and illustrated in three different mathematical contexts.

In many discussions about the nature of mathematical proof, researchers have used dichotomies to characterize different types of proof. There are informal and formal proofs (Schoenfeld, 1991), syntactic and semantic proofs (Weber and Alcock, in press), proofs that explain and proofs that demonstrate (Steiner, 1978; Hanna, 1989). However, as pointed out by Schoenfeld (1991), the classification of proofs in terms of dichotomies masks an important, if not essential, part of mathematics—the connection between the two dichotomous poles. This paper discusses this connection and the role it plays in generating and understanding proofs. A model of proof processes is developed, building off the notion of "key idea" proposed by Raman (2003). This model is applied to a proof from a pre-service course for secondary teachers to show how one can find deeper meaning behind what might be considered a routine task.

Heuristic ideas, Procedural ideas, and Key ideas

One type of idea used in proof production is called a heuristic idea. This is an idea based on informal understandings, e.g. grounded in empirical data or represented by a picture, which maybe suggestive but does not necessarily lead to a formal proof. A heuristic idea provides a sense that something *ought* to be true, but by itself does not constitute a formal proof. Heuristic ideas are often used behind the scenes—for instance, as a mathematician tries to develop an intuition for why a claim is true. With few exceptions (such as Polya (1968)) these types of ideas rarely make their way into the final exposition of what most mathematicians (and the textbooks they write and use) would call a proof. Because of this, we say that heuristic ideas constitute a private aspect of proof.

Another type of idea used in proof production is called a procedural idea. This is an idea based on logic and formal manipulations that lead to a formal proof. A procedural idea demonstrates *that* something is true, but is not necessarily explanatory or personally meaningful. One might be able to follow, or even produce, all the steps of a formal proof without being able to understand it. (For example, after producing a formal proof of a difficult theorem, the Field's Medalist Deligne said, "I would be grateful if anyone who has understood this demonstration would explain it to me." (Alibert and Thomas, 1991)

Procedural ideas, in contrast to heuristic ideas, generate precisely the type of arguments found in most mathematical textbooks, journal, and the like, though the level of rigor that one uses to express a procedural idea may vary from context to context (e.g. one would need more rigor for a journal article than in a discussion with a colleague).

Because procedural ideas lead to proofs that are publicly acceptable, we say that they constitute a public aspect of proof.

So far we have done little more than to introduce more dichotomies in characterizing aspects of proof. Proofs have either a public or private aspect; they are generated by either heuristic or procedural ideas. However, the characterization of proof so far misses what we consider to be the crucial aspect of proof—the key idea. A key idea is a mapping between heuristic idea(s) and procedural idea(s). It links together the public and private domains, and in doing so provides a sense of understanding and conviction. The key idea is the essence of the proof, providing both a sense of why a claim is true *and* the basis for a formal rigorous argument.

The notion of key idea, while not discussed as such, has precedents in the literature. For example mathematician Bill Thurston, another Field's Medalist, explains how he reads a mathematical paper in a field in which he is conversant:

> I might look over several paragraphs or strings of equations and think to myself, "Oh yeah, they're putting in enough rigamarole to carry such-and-such idea." When the idea is clear, the formal setup is usually unnecessary and redundant—I often feel that I could write it out myself more easily than figuring out what the authors actually wrote. (Thurston, 1994)

To the mathematician, what is important about a proof is the idea it expresses. The symbols and formalism used to express that idea are just 'rigamarole' for carrying that idea.

The problem, from a pedagogical standpoint, is that students do not view proof this way. For them, the public and private aspects of proof are disconnected (Balacheff, 1988, Raman 2003, Schoenfeld, 1985). This disconnect not only reflects an immature view of proof, but also stands in their way of generating and understanding one. Below a model is proposed to show the role the key idea can play in generating and understanding a mathematical proof.

Key ideas in action

There are at least three ways key ideas play a role in generating and/or understanding a formal proof. The model here only purports to show how key ideas *can* be involved in these processes. One can of course generate a proof without having a key idea, as illustrated by the Deligne quote above, and those proofs can be in some way meaningful.

In the first case, one begins with heuristic ideas, perhaps generated by looking at examples or more general exploration (see figure 1a below). Looking across these patterns, one identifies the key idea that convinces oneself personally that the claim is true. One then tries to rigorize this key idea into a publicly acceptable proof. As an example, consider how you would prove the claim: the derivative of an even function is odd.

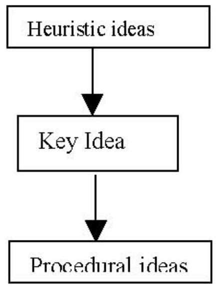
In a study reported in Raman 2002, one mathematician began to prove the claim by sketching a generic even function and thinking about the relationship between the tangent lines on either sides of the y axis. He reasons:

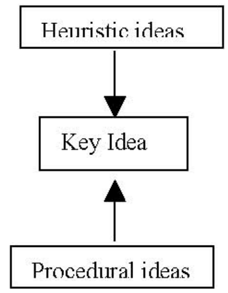
> Prof A: If it is even then it has to be the same to the left and to the right. If it is the same to the left and to the right then clearly, if you draw the tangent line, that is reversed when you flip across the y axis. The function is preserved because it is even, but the slope is reversed.

The fact that the slope is reversed is what we call the key idea in this case. Prof A goes on to write his idea out algebraically and gets what one could consider a formal proof.

> Prof A: Then you could say, well, suppose you want to write it out in formulas. You could say that means, (writes) f of x is equal to f at minus x. And then you could say, if it's differentiable, you could differentiate both sides. And then you'd say, well ok that means f prime at x is equal to negative f prime at negative x by the chain rule. This is odd, the f prime function.

By writing the key idea "in formulas", Prof A is translating the key idea into procedural ones that generate the proof. In a proof one often starts with definitions of what is given and works, through the key idea, to get to the definition of what you want. Since the key idea is evident to Prof A, he writes the proof with little difficulty.





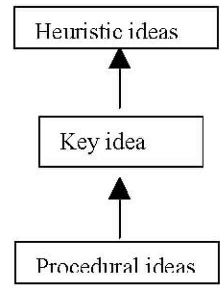


Figure 1a Figure 1b Figure 1c

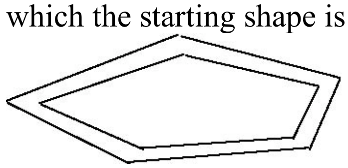
In the second case (Figure 1b), one might start with either heuristic ideas or procedural ideas. It is not immediately clear how any of the heuristic ideas lead to proof, nor how to proceed with the formal proof. One goes back and forth between the informal and formal approaches, and in doing so, one finds the key idea that connects them. This allows one to complete the formal proof as well as provide one with a deeper sense of understanding.

Consider the following high school geometry task. Suppose ABC is a triangle. Let D be the midpoint of AB, E be the midpoint of BC, and F be the midpoint of AC. What is the relationship between the area of ΔABC and ΔDEF? In a high school geometry class one might want students to be able to see that ΔDEF has 1/4 the area of ΔABC and to be able to prove. One strategy (described in Raman and Weber, under review) would be to let students begin with some sort of geometer sketchpad activity to develop some intuition about why the claim is true, and then work with the geometrical properties of the figure to try to develop a rigorous proof. As one goes back and forth between the intuitive ideas and rigorous proof, one gets closer to what turns out to be the key idea, which is that the proof rests on all four triangles being congruent (similarity between ΔABC and each of the smaller triangles gives the needed insight to establish the congruence.)

The third case (Figure 1c), is the case where one begins by producing a formal proof, and then trying to get a deeper sense of the underlying ideas. The example we use is a generalization of a standard calculus problem called the box problem. One is given a piece of cardboard of particular dimensions and asked to cut squares out of the corners and fold up the sides to make a box. The question is what the cutout size should be to maximize the volume of the box.

The typical approach to this problem is to assign a variable to the cutout size and express the volume of the box in terms of this variable. One can take the derivative of this cubic function, set it equal to zero, and find the two roots. One of them is impossible, and the other gives the desired solution.

In contrast to this analytic solution, one can also think about the problem geometrically (see Usiskin et al (2003)). This proof actually looks at a generalized version of the box problem in which the starting shape is any convex polygon.



The idea of this proof is to assume one has the maximum volume and then look at what happens when one changes the cutout size slightly in either direction. This is essentially the same idea behind the analytic solution presented above. This idea of looking at small deviations, appears to give some sense of why a cutout size of a particular size gives a maximum volume.

However, if one examines the proof more closely, one finds another, deeper, result. The key algebraic move that leads to the desired conclusion expresses a key feature of the box problem that provides a satisfying characterization of the geometric properties of the box of maximum volume. It turns out that the area of the sides must equal the area of the base. This is the key idea.

Once one has the key idea, one can verify it empirically. It turns out to hold not only for all convex polygons, but also for circles. Performing these calculations strengthens one's sense of understanding, and perhaps even believability, of the claim.

Conclusion

Looking at three different proof contexts gives us insight into three different roles key ideas have in the act of generating a mathematical proof. The key idea can serve as a bridge from more intuitive, heuristic ideas to the more formalized, procedural ones (as we saw in the case of the professor proving that the derivative of an even function is odd). The key idea can also work in the other direction, beginning as an algebraic or analytic piece in a rigorous proof, and then providing meaning for the problem situation (as we saw with the box problem). And finally, in what is perhaps the most common role, the key idea is the result of ping-ponging between the intuitive/heuristic realm and the formal/procedural realm. This ping-ponging may occur in fact with any of the proofs above, but we illustrated it only with the high school geometry proof. In that case it was not clear apriori what the key idea was, but once it was identified, it served as a bridge between the informal geometer sketchpad activity and the act of writing a formal proof.

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WHAT COUNTS AS PROOF? INVESTIGATION OF PRE-SERVICE ELEMENTARY TEACHERS' EVALUATION OF PRESENTED 'PROOFS'

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This study examines the perceptions of preservice elementary school teachers as to what constitutes a valid proof. The participants (n=75) were presented with several statements together with arguments meant to validate these statements, some acceptable as proofs and some unacceptable. They were asked to consider whether the given argument could be considered as a proof. In the case where the argument was not acceptable, participants were invited to edit, augment, or change the argument to create what they perceived as a proof. The theoretical framework for analyzing short proofs of numerical statements – referred to as 'one line proofs', previously developed by the authors, was used to design the tasks. This framework describes five competencies that are necessary in developing short proofs. The results confirm findings of previous research regarding participants' tendencies towards empirical verifications. Further, the results provide an insight into the nature of empirical verification preferred by the participants.

What counts as *proof*? This is a simple question that everyone who is involved in doing serious mathematics has asked himself or herself at least once in their lifetime. As simple as the question is, however, the answer to it is rather complicated. The main reason for this is that, although there is an expectation that every individual mathematician should have an operational understanding of what a *proof* is, there seems to exist no succinct definition of *proof*. The main reason for this is that "there is no consensus today among mathematicians as to what constitutes an acceptable proof and there never has been" (Hanna, 1983, p. 29).

The varying interpretations of what constitutes a proof revolve largely around the notion of rigor. In particular, arguments vary with respect to the degree of rigor that is required to preserve the privileged position that mathematics occupies among the sciences as the discipline that is most precise. The mathematical proof provides the certainty that is demanded in a field where precision and exactness is the currency of practice. According to Rav (1999, cited in Hanna, 2000) proofs are the "mathematician's way to display the mathematical machinery for solving problems and to justify that a proposed solution to a problem is indeed a solution". As such, proof is not only an important part of mathematical practice, but also of mathematical teaching and learning (Hanna, 1989). Unfortunately, proof is also one of the most misunderstood notions in mathematical teaching and learning (Schoenfeld, 1994) and is, therefore, one of the greatest challenges that is faced by researchers and teachers alike.

In recent decades much research has been done with regards to the diagnosis of students' difficulties with generating proof (Dreyfus, 1999; Harel & Sowder, 1998; Moore, 1994). However, only a few studies (e.g. Martin & Harel, 1989; Raman, 2002; Selden & Selden, 2003) have investigated students' ability to evaluate the correctness of a given 'proof', that is, the ability to judge whether a given argument, or sequence of arguments, proves a given statement. We see this ability as an important precursor to the ability to generate correct proofs. Furthermore, we believe that this ability is of extreme importance for teachers, who are the facilitators of mathematical understanding of their students. Teachers must be able to see the students' ideas in a broader mathematical perspective and be able to structure their lessons in such a way as to encourage and support mathematical discussion, for it is through discussions that students will begin to value the contribution that proofs can make to their arguments (Richards, 1996).

As such, we have chosen to focus our investigation on preservice elementary school teachers' ability to judge the validity of presented arguments as proofs. More specifically, we focus on the tendencies and trends in the accepting or rejecting of arguments as proofs and examine the role that numerical examples play in participants' reasoning.

Theoretical Framework

In this study we use the framework developed by Gholamazad, Liljedahl, and Zazkis (2003) for analyzing the complex coordination of competencies that are required for composing short algebraic proofs, referred to as "one line proofs". It was developed through the consideration of two sources: (1) a fine grain analysis of an 'exemplary', that is, correct and complete, short proofs related to number properties and (2) the analysis of students' efforts and errors in composing these proofs. Consider for example the following statement and its exemplary proof:

Statement: The set of odd numbers is closed under multiplication.

Proof: Let (2m+1) and (2n+1) be two odd numbers.

Then (2m+1)(2n+1) = 4mn+2m+2n+1 = 2(2mn+m+n)+1, which is itself odd.

The generation of such seemingly simple and short proof is deceivingly intricate, requiring an appreciation of the need for, and the coordination of many competencies.

The framework describes five competencies necessary for the generation of a complete and correct proof.

Recognition that a proof is required for the purposes of establishing the truth of a statement. From a mathematical perspective, such a requirement is obvious. The establishment of the validity of a statement requires the treatment of the statement in general. Students that have this competency recognize that a confirming example, or several confirming examples, do not constitute a proof, that examples are acceptable only when they exhaust all the possible cases, and that one counterexample is an acceptable means of refuting a statement.

Recognition that the treatment of a general case requires the selection of some form of representation. The treatment of the statement in general requires some form of representation. Representations have many roles in mathematics; in particular they serve as tools for symbolic manipulation and communication. We focus here on algebraic representation, acknowledging that other representations, such as numerical or pictorial, are also possible.

Selection of a representation that is correct and useful for the given case. Awareness that some kind of representation is needed to provide a general argument is not sufficient. The chosen representation must fit the requirements of the problem. For example, representing two odd numbers as X and Y is, in itself, not wrong, but simply useless in order to consider the parity of their sum or product.

Ability to manipulate the representation correctly. The skill of manipulating the given representation is often taken for granted in the discussion of proofs. However, to complete the proof students must be able to perform correctly any manipulations necessary to transform the expression into the form that clearly represents the nature of the number. In the example above this involves adeptness with algebraic manipulation in order to mould the expression into one that clearly expresses its inherent 'oddness'.

Ability to interpret the manipulation correctly. The competency of interpretation either overlooked or is simply included in the skill of manipulation. However, the manipulation encompasses both ability and intent; that is, the understanding of how the result should be represented to draw the required conclusion and recognition of this representation.

Itemized competencies of this framework not only detail what is needed in generating short proofs of number properties, but also provide a tool for the diagnosis of possible obstacles in generating such proofs. As such, we used this framework as a guiding tool in the design of the instrument for this study.

Method

The seventy-five participants in this study were preservice elementary school teachers enrolled in the course "Principles of Mathematics for Teachers", which is a core course in a teacher education program. At the time of the study this course was taught by one of the authors. The course had been designed with the intention of providing its enrollees a foundational understanding of elementary school mathematics. There is a focus on conceptual understanding of specific strands of mathematics such as geometry and number theory. There is also an attempt to integrate an underlying appreciation for mathematical thinking and reasoning across all strands of the course.

The participants responded to a written questionnaire in which they were asked to consider the validity of arguments purporting to 'prove' five different statements related to set closure. They were asked to examine the arguments and decide, in each case, whether the argument was acceptable as a proof for the given statement or not. In the case that an argument was not acceptable, the participants were asked to provide an acceptable proof either by editing or by augmenting the presented argument as necessary. In particular, they were invited to delete parts of the presented arguments that they perceived as unnecessary. The topic of set closure was explored in class prior to the administering of the questionnaire. The time for completing this activity was not limited.

As mentioned, these 'proofs' were constructed from plausible errors as indicated by the framework. We examined the participants' awareness of when a proof can rely on exhaustive consideration of all possible cases (1) and where one example is sufficient (2). Furthermore, we examined the participants' awareness of the need for representation when a multitude of examples does not constitute a proof and (3) the existence of valid argument not involving algebraic symbolism (4). The final item (5) addressed the participants' attentiveness to the correctness of symbolic manipulation. In what follows we present the statements along with their 'proofs' that were used in our study.

(1) Statement: The finite set B = {0,1} is closed under multiplication.

Proof: $0 \times 1 = 0$

$0 \times 0 = 0$

$1 \times 1 = 1$

$1 \times 0 = 0$

$\therefore$ the set B is closed under multiplication

(2) Statement: The set of prime numbers is closed under addition.

Proof: The set of prime numbers = {2,3,5,7,11,13,17,19,23,29,31,37,…}

$2 + 3 = 5$ is a prime number

$2 + 5 = 7$ is a prime number

$17 + 2 = 19$ is a prime number

but $3 + 5 = 8$ is not a prime number

and $19 + 13 = 32$ is not a prime number

So, the set of prime numbers is not closed under addition.

(3) Statement: The set of multiples of thirteen is closed under addition.

Proof: The set of multiples of thirteen =

{ 0, 13, 26, 39, 52, 65, 78, 91,104, 117, 130, 143, 156, 169, …}

13 +26 = 39 is a multiple of thirteen

39 + 52 = 91 is a multiple of thirteen

65 + 78 = 143 is a multiple of thirteen

91 + 104 = 195, 195 = 15 × 13 so it is a multiple of 13

117 + 156 = 273, 273 = 21 × 13

130 + 169 = 299, 299 = 23 × 13

195 + 143 = 338, 338 = 26 × 13

1300 + 2613 = 3913, 3913 = 301 × 13

We have seen that the sum of two multiples of thirteen is another multiple of thirteen so we can say this set is closed under addition.

(4) Statement: The set of multiples of five is closed under addition.

Proof: True, because for a multiple of five the last digit is 0 or 5.

When we add up two numbers, which are multiples of five, then the last digit could be 0+0, 0+5, 5+5, which would be again a number with the last digit 0 or 5. Therefore, the set of multiples of five is closed under addition

(5) Statement: The set of odd numbers is closed under multiplication.

Proof: $O = \{1,3,5,7,9,11,13,15,\ldots\}$ = set of odd numbers

For any $n,m \in W$, $(2n + 1) \in O$ and $(2m + 1) \in O$

[2n+1 and 2m+1 are two odd numbers]

$(2n + 1)(2m + 1) = 4nm + 1 = 2(2nm) + 1 = 2k + 1 \in O$

[2nm is a whole number like k]

So, the set of odd numbers is closed under multiplication

Results and Analysis

The arguments presented as 'proofs' for the first three items relied on numerical examples. We wanted to see whether participants could distinguish between the cases for which numerical examples are sufficient as a proof and the cases where they are not. The results indicate that:

The majority of participants accepted, as valid, proofs that consider all possible cases in a finite set. The main purpose for including the first statement in this study was to call the students' attention to the difference of working on finite and infinite sets. While for an infinite set the examination of the closure property through the consideration of every case is not possible, it is sometimes possible for a finite set. As such, in the case of the first statement, an exhaustive consideration of all numerical examples could serve as a proof. Students' feedback on the given proof for the first statement showed their high tendency towards this approach. Among the 75 participants in the study, 58 students accepted the given argument as a valid proof for the first statement. The rest of the students did not show any objection toward the given proof but they tried to improve it by adding or eliminating some part of it. In this group, seven students wrote, $0 \times 1 = 0$ is redundant when $1 \times 0 = 0$ has already been mentioned. Furthermore, ten students showed their dissatisfaction with having just numerical examples and tried to complete the given argument with more explanation. Most of the explanations referred to the definition of the closure property of a set. For example one of the students wrote:

This one line explanation should be added –
"All possibilities exhausted and answers are all within the set."

Other explanations focused on number properties in order to make the argument more clear:

Because the number that is started with are 0 & 1 the number that you get when you multiplying are still 0 & 1.

The majority of participants were not satisfied with the use of a single counterexample to disprove a claim. To establish that the second statement is false, one must realize that the given statement is universal and that the negation of a universal statement is existential. In other words, since the claim is made about all prime numbers, a single counterexample will serve to show that it is false. The students' feedback, to the given 'proof' for the second statement, showed that all but three of them knew that the rejection of the claim will need a counterexample. However, only 19 students crossed out all but one of the counterexamples. The rest of them showed their tendency towards having more than one counterexample.

There were also students who had a tendency to complete the given argument by adding explanations to it. Again, these explanations focused on the properties of numbers. For example, one of student added:

The set of prime no's is not closed – primes are odd numbers, and odd+odd=even, which will always be even, or composite.

The majority of participants accepted confirming examples as a valid method of proof. Proving a property for a set requires a supporting argument that shows the considered property works for all the elements of the set. In the case of proving a statement on a finite set, checking all the possibilities would constitute a proof. However, as mentioned above, the same approach is impossible when working with an infinite set. In such cases the use of theoretical tools, such as algebraic notation, or explanation of a general case is necessary to construct a proof. Findings from the students' reflections on the third question indicate that the majority of students did not recognize this necessity in working with the infinite set of multiples of thirteen.

Fifty-one of the student (68%) indicated that the 'proof', as given for the third statement, was acceptable. Twenty of them even crossed out some of the examples, and made statements such as "two or three examples are enough to make sure the given statement is correct" or "there is no need for so many". Some of the students also augmented the given 'proof' by adding some explanation about the closure property of the set of multiples of thirteen under addition. For instance, one of the students added:

The prime factorization of any multiple of 13 will have a 13 in it (because a multiple of 13 has to be multiplied by 13) adding 2 multiples of 13 will still have 13 in the prime factorization ∴ it is closed under addition.

One of the students, after crossing out all the examples except for two, wrote:

Multiplication is simply repeated addition, so if you take any 2 multiples of a given number and add them the result will be a multiple of the original. Ax+bx=cx, x is the multiple.

Surprisingly, only 14 students (19%) presented a valid proof by using a correct algebraic representation for the multiples of thirteen. An additional ten students recognized that the treatment of a general case requires the selection of some form of representation, but could not overcome the obstacles presented in the chosen representation. Some of them did not choose an appropriate representation for multiples of thirteen, and the others, who chose an appropriate one, could not manipulate it correctly. Three examples of this are presented below:

These examples didn't exhaust the set.
Rewrite proof: Number1 = 13m, Number2 = 26n

$13m + 26n = 39mn = 13(3mn)$
Whenever 2 multiples of 13 are added their product will always be a multiple of 13 because we are always multiplying by 13.

or

$(n + 13) + (m + 13)$
$13(m + n)$

or

$13n + 13m = 26mn \rightarrow 13(2mn)$
$\therefore$ the set of multiples of 13 is closed because 13 times any # plus 13 times any other # is divisible by 13.

The conventional form of presenting the proof seemed to play no role in the decision of a proof's validity

In general, the construction of a proof in number theory requires some kind of representation. However, in some particular cases (often dependent on number properties), the use of a reasonable explanation can be used as a proof. As such, the argument presented for the forth statement was specifically constructed to avoid the use of either algebraic representation or numerical example in order to see if students could validate a correct explanation in the form of a text. Results show that for more than half of the participants (39 out of 75) the presented argument for the fourth statement was acceptable. However, 11 students accepted the proof as valid only after adding some numerical examples to the given argument, such as "5 + 10 = 15" or "35 + 45 = 80".

There were also several students who rejected the given explanation in favour of their own. In these cases, the students displayed a preference for certain language. For example, one of the students stated:

When you add 2 multiples of 5, the result is still a multiple of 5 since you can pull the 5's out of it.
So the set is closed under addition.

Seventeen students also attempted to replace the given 'proof' with one using the algebraic representation of multiple of five. Surprisingly, some of the students who preferred the algebraic proof for the fourth statement were also those who approved of the empirical reasoning for the third statement as being a valid proof. Among these 17 students there were also a few (5 students) who could not manipulate their chosen algebraic notations correctly. For example:

$5m + 5n = (5 . m) + (5 . n) = 25 + 5m + 5n + mn$
all multiples of 5.

The majority of participants did not detect the error in algebraic manipulation for the last item. As already mentioned, and demonstrated, the inability to manipulate algebraic notations is a serious obstacle in generating a valid proof. The results of students' feedback to the presented argument for the fifth statement confirmed this concern. Only ten students could detect the error in algebraic manipulation for this item. Twenty participants considered the given 'proof' to be perfect, while for some others (7 participants) the given 'proof' seemed too complicated to understand. For example, one of these students crossed out the given 'proof' and wrote:

This is ridiculously complicated; they could just give a couple of quick examples and then you can skip all of the complicated crud.

This again shows the deeply rooted belief that the existence of some confirming examples is enough to validate a claim. This belief is further evidenced in a number of instances in which students did not alter the presented proof, but augmented it with series of numerical examples. This is exemplified in the following:

Don't understand all of this but it looks good

$3\times\cdot 5 = 15$ $5\times\cdot 7 = 35$ $3\times 11 = 33$

Exhausted all / most possibilities, answers always odd #.

ODD $\times$ ODD = ODD.

The rest of the students (34 out of 75) tried to validate the given 'proof' by editing it, albeit, none of them were successful. For example, after crossing out the last three lines of the given 'proof', Jane wrote:

I would leave it at $(2n + 1)(2m + 1) = 4mn + 1$

Even number + 1 will always be odd.

Conclusion and Discussion

Students live in a world in which the term *proof* may mean many different things in many different contexts. As such, they often believe that non-deductive arguments constitute a proof. This claim agrees with Schoenfeld's (1985) observations of the empirical nature of students' belief about mathematics and their failure to use deductive reasoning as a mathematical tool. The results of Dreyfus' (1999) research on students' conception of proof show that most high school and college students either do not know what a proof is, or do not know what it is supposed to achieve. According to Harel (1998), a primary reason that students dislike advanced mathematics is that they feel no intellectual need to establish the truth of the seemingly obvious statements that are proven in their course. The results of this study were no different.

The participants' feedback demonstrates that, although the concept of closure was generally well grasped, the concept of proof was not. For the majority of participants it seemed so clear that the sum of two multiples of five would be a multiple of five, or the product of two odd numbers would be an odd number, that they were unable to see the need for anything more than a few confirming examples as support. In the cases where the truth value of the statements was not as 'obvious', the results still showed the tendency of the preservice elementary school teachers to acknowledge empirical verification as an acceptable proof. Together, these results confirm the findings of prior research (Harel & Sowder, 1998; Martin & Harel, 1989; Fischbein & Kedem, 1982) that suggests a strong reliance on empirical proof schemes. However, an interesting contribution of our study deals with a question of how many examples constitute a 'proof', as perceived by our participants. For the majority, two or three examples seemed to be sufficient, as evidenced by the way in which the participants deleted or added numerical examples to the provided arguments.

Although, some research (Vinner, 1983; Selden & Selden, 2003) suggests that students tend to judge a mathematical argument on its appearance, we did not find high reliance on this 'ritualistic' aspect of proof. Instead, the arguments in items four and five were either augmented or verified with numerical examples before they were accepted as 'proofs'. This finding, however, can be explained by the participants' mathematical background. Preservice elementary school teachers experience only minor exposure to the rituals of the proof, as compared to the participants (mathematics majors) in the aforementioned studies. As such, mathematical proof does not become a part of the preservice teachers' mathematical culture and beliefs in the same way that it does for mathematics majors.

Considering the important role of elementary school teachers in establishing the foundation of the mathematical knowledge of the next generation of students, we believe that their ability to establish the validity of provided argument is essential. It appears that this skill is not acquired from exposure to and writing of proofs. Therefore special training in evaluating arguments could be beneficial in teacher education.

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STUDENTS' DEVELOPMENT OF MEANINGFUL MATHEMATICAL PROOFS FOR THEIR IDEAS

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This study documents the mathematical development of a group of eleventh-grade students who solved challenging combinatorics tasks and then developed convincing arguments to justify their ideas to themselves and to others. In doing so, they extended their mathematical reasoning and developed meaningful mathematical proofs, including proof by cases, proof by induction, and proof by contradiction. In addition, the students were able to apply their proofs and justifications to another isomorphic problem. The results of this qualitative study suggest that in an appropriately supportive environment, students are capable of constructing and justifying mathematical ideas.

Introduction and Theoretical Framework

The National Council of Teachers of Mathematics (NCTM) Standards state that the mathematics curriculum should include numerous and varied experiences that provide opportunities for the development of mathematical reasoning and proof making by students (NCTM, 2000). However, Hanna (1995) asserts that changes during the last thirty years have resulted in less emphasis on proof in the curriculum. Why does this inconsistency exist? One explanation offered is that traditional demonstrations of axiomatic proofs that attempt to teach students proof making have been generally unsuccessful (c.f. Anderson, 1995; Senk, 1985; Speiser, Walter & Maher, 2003). The NCTM (2000) recommends that reasoning and proof should be developed through consistent use in many contexts from prekindergarten through twelfth grade.

Prior research (Healy and Hoyles, 2000) documented that a group of high-achieving fourteen and fifteen year-old students held two different conceptions of proof: arguments that would receive high grades versus arguments that they would adopt for themselves. The latter, generally speaking, were arguments that they could evaluate for themselves, would be convincing and explanatory, and were presented in their own language. Sconyers (1995) states that proofs need not be rigid in order to be rigorous. He maintains that the essence of mathematical proof is not in the format, but rather in the concepts of necessary inference and logically compelling arguments. Hanna and Jahnke (1993) state that understanding is primary for a learner to accept that a new theorem has been proved, with rigor only secondary. They argue that students are likely to gain a greater understanding of proof when emphasis is on the communication of meaning, rather than on the formal derivation. The NCTM's assertion that "The particular format of a mathematical justification or proof, be it narrative argument, 'two-column proof,' or a visual argument, is less important than a clear and correct communication of mathematical ideas" (NCTM, 2000, p. 58) provides further support for these criteria for students' meaningful mathematical proofs.

The purpose of this study is to document, within the context of problem-solving situations that involve combinatorics, how a group of students built solutions and then developed convincing arguments to justify their ideas to themselves and to others. In doing so, they extended their mathematical reasoning and developed meaningful mathematical proofs, including proof by cases, proof by induction, and proof by contradiction.

Methods and Procedures

Background and Setting

As part of an ongoing longitudinal study involving the development of children's mathematical ideas, initiated in 1989, students have been engaged in problem-solving explorations that involve strands of mathematics including algebra, probability, combinatorics, and pre-calculus.[1] Combinatorics activities were presented beginning in grade two, before formal class instruction of algorithms. During these sessions, students worked together to find solutions to problems and to build justifications for their ideas.

At the time of this component of the study, nine students investigated combinatorics tasks in eleventh grade after-school sessions in a middle-class suburban school district in New Jersey. The students worked together in pairs or in small groups, and each session lasted approximately one and three fourths hours. Students were invited to explore ideas, develop representations, invent notations, make conjectures, devise strategies, test their methods, discuss their ideas with their peers, and to justify their solutions. The teacher's role was to step back to give students freedom to pursue their ideas, to observe the students' work, and to listen carefully in order to decide when an appropriate intervention was necessary. An appropriate intervention on the part of the teacher might be to ask a question or to pose a modification of the task to encourage students to explain their ideas and reasoning. Problems, or similar ones, would later be revisited, so students would have the opportunity to think about their ideas over time.

Subjects

Nine eleventh grade students, sixteen and seventeen years old, volunteered to participate in this research project. The students, Amy-Lynn, Angela, Ali, Magda, Michelle, Robert, Shelley, Sherly, and Stephanie, rearranged their part-time work schedules and other extracurricular activities in order to participate in the program. Five of these students, Amy-Lynn, Michelle, Robert, Shelly, and Stephanie, were a subset of the original group that had been involved in the longitudinal study in grades one through eight.[2] In grades three through five, they explored the same tasks that are the subject of this study, the Tower Problem and the Pizza Problem.

Tasks

The tasks that provide the basis for this study are the Tower Problem and the Pizza Problem. The Tower Problem presents the following:

How many different towers exactly four cubes tall can be built from unifix cubes when there are two colors available for use?

How do you know you have them all?

Can you convince us that you have all possibilities, that there are no more or no fewer?

The Pizza Problem states:

A local pizza shop has asked us to help design a form to keep track of certain pizza choices. They offer a plain pizza, that is, "cheese with tomato sauce." A customer can then select from the following toppings: peppers, sausage, mushrooms and pepperoni.

1. How many different choices for pizza does a customer have?
2. List all the possible choices.
3. Find a way to convince each other that you have accounted for all possible choices.
4. Suppose a fifth topping, anchovies, were available. How many different choices for pizza does a customer now have? Why?

The Tower and Pizza Problems have isomorphic mathematical structures, and their solution can be represented by a generalization that may be justified by either a proof by cases or a proof by induction.

Data

At least two cameras were used to videotape each session, supervised by a videographer and a sound technician. One camera focused on the actions of the students; the other camera focused on the students' written work. All of the students' written work was collected, both scratch work and more carefully written solutions and explanations. In addition, mathematics education graduate students were present at a distance to record field notes of their observations. The videotapes were digitized and converted to MPEG format to be used with vPrism software. The videotapes were transcribed, verified independently, and analyzed. The videotapes, students' written work, field notes, transcripts, and analyses for each session provide the data for this research.

A qualitative methodology for data analysis was employed. To manage the large amount of data that were analyzed, a visual representation of the flow of students' ideas and justifications was developed. Combinatorics ideas under consideration, contributions made by each student or partnership, and teacher/researcher interventions were charted with corresponding timecodes. Students' representations, strategies, justifications, connections, and interactions, as well as the role of the teacher/researcher were coded, and the codes were used to identify and trace the students' development mathematical reasoning and proof.

Results

The Tower Problem: November 13, 1989

Six students were present at the first eleventh grade after-school problem-solving session. Seated side-by-side from left to right, were: Sherly, Ali, Magda, Angela, Michelle, and Robert.

Angela and Magda were paired as partners. Magda began by building towers randomly, and Angela began by using a local organization, termed an elevator pattern because a blue cube is moved down or up one level in each tower, to find all of the towers with one blue cube. They worked together cooperatively to check for duplicates and then focused their constructions on the number of blue cubes in each of the towers. They developed a global organization by grouping their towers by cases, which they referred to as one blue, two blues, three blues, and four yellows.

To justify the completeness of the case with one blue cube, Angela and Magda used a proof by contradiction. They argued that they moved the blue cube down, into every possible position, in each of the four towers. To place the blue cube in another position would require the tower to be taller and would, therefore, contradict the given parameter of the problem, that a tower is four cubes tall. They used the same argument to justify the completeness of the case with the three blue cubes; they moved the one yellow cube down into each possible position in each of the four towers.

Sherly and Ali worked together as partners. For the case of towers four high with two blue cubes, Sherly and Ali originally built and organized their towers in pairs of opposites, but they stated that they were not sure that the six towers that they had built were all of the towers for this case. Approximately one hour later, for towers five high with two blue cubes, they organized the towers into subcases: towers with two blue cubes together; towers with two blue cubes separated by one yellow cube; towers with two blue cubes separated by two yellow cubes; and towers with two blue cubes separated by three yellow cubes. Sherly used a proof by contradiction to justify having all of the towers five high with two blue cubes; she explained that you could not have towers with two blue cubes separated by four yellow cubes because there would then be six cubes in the tower.

Robert and Michelle were paired as partners. Although Michelle initially built towers with their opposites, Robert began by building and organizing the towers by cases, focusing on the blue cubes. He used a global organization that accounted for all possible cases to build the towers and used a proof by cases as a justification. He explained that he focused on the blue cube: zero blue, one blue, two blue, three blue, and four blue. The teacher/researcher asked Robert if he thought that he had all of the towers in the case with two blue and two yellow cubes, and if so, how did he decide that he had all of those towers. Robert justified his argument by demonstrating that he repeatedly controlled one variable.

Robert used a proof by induction as a convincing argument for having all of the towers three tall. He explained that his "bottom" towers, those that were built upon, included all possible towers two tall. Then "you just take this [bottom] and add a blue and a yellow to the top and that is all the combinations for this." Each two tall tower generated two new towers three tall, one with a blue cube added atop and one with a yellow cube added atop. He continued to explain that these were the only possibilities for each new tower, and this process could be continued "all the way through" for each bottom tower to produce all possible towers one cube greater in height. Robert explained that the process could be continuously repeated, thus the doubling of the towers.

01:30:19	Teacher/Researcher	Are you sure you have got all of them by doing that?
01:30:23	Robert	Yeah, cause if you are sure you have all of this, and there is only like two ways you can change this, and that is by putting one on top of each, and there are only two ways you can change this, by putting the yellow and the blue on top.
01:30:38	Teacher/Researcher	Couldn't I put one onto the bottom?
01:30:40	Robert	But then it would be the same thing as something over here. So, this is just, always add onto the top and keep going. And this is all that is possible for two, and if you just add one to the top of them for each of them, for different colors, I guess you would have three. And do that for the three until you get all for four, etcetera.

Robert and Michelle extended their inductive reasoning to justify their generalization that the total number of towers for any given height is x to the h, where x is the number of color choices and h is the height of the tower.

During a single session, lasting approximately one hour and forty-five minutes, the students solved the Tower Problem and justified their solution using three forms of proof: proof by contradiction, proof by cases, and proof by induction. The students then used inductive reasoning to extend their ideas and generalized the number of towers that could be found for any height with any number of colors available to choose from.

The Pizza Problem: March 1, 1999

Approximately four months after the Tower Problem session, students met again after school and were presented with the Pizza Problem. Eight students were present at this session. They were organized into two groups with four students in each group, and each group worked across the room independently of the other. The four students at Table A, Robert, Stephanie, Shelly, and Amy- Lynn, had participated in the earlier investigations in grades three through five. Of the students at Table B, Angela, Magda, Michelle, and Sherly, only Michelle had explored pizza problems in the early grades.

When presented with the problem, Shelly, Stephanie, and Amy-Lynn discussed the fact that they "just did this in school, combinatorics stuff," and after using their calculators to try to solve the problem, they said that it was pathetic that they did not remember what they had been taught. Stephanie suggested that they employ a strategy that they had used with the Shirts and Pants Problem [which they had investigated in the longitudinal study in grades two and three] or the Tower Problem, and Shelly recalled that they had used a tree diagram.

The students organized their pizzas by cases according to the number of toppings and used a proof by cases to justify their solution. They found that for pizzas with four toppings available, there were 1 4 6 4 1 topping combinations, for a total of sixteen pizzas, and recognized these numbers as a row in Pascal's Triangle. They used the Triangle to determine the number of possible pizzas with five available toppings, the next row 1 5 10 10 5 1, for a total of thirty-two pizzas.

The students connected the numbers on Pascal's Triangle to the corresponding topping combinations and used pizzas to explain the addition rule for generating rows on the Triangle. Stephanie explained the addition of the three [pizzas with one topping] and the three [pizzas with two toppings] on the fourth row to produce the six [pizzas with two toppings] on the fifth row.

00:52:58	Stephanie	So then here, um, you have six pizzas with two toppings. Now you already have three pizzas with two toppings. So these three pizzas with one topping get an extra topping added on.
00:53:09	Teacher/Researcher	Okay. [Teacher/Researcher nods.]
00:53:10	Stephanie	So these become three pizzas with two toppings. And then three pizzas with two toppings plus three pizzas with two toppings equal six pizzas.

Robert generalized the solution for the total number of pizzas as 2_n, for n available toppings, based on a pattern that he observed. When Stephanie explained how pizzas could be moved to two different places on the Triangle, in one move they remain the same and in the other move they get an extra topping added to them, Amy-Lynn connected this two with Robert's 2_n, to provide a justification for his generalization.

The students at Table A also connected the numbers on Pascal's Triangle to towers and explained the addition on the Triangle using towers. Furthermore, Robert explained the isomorphism between the tower and pizza problems. He stated that the number of toppings [available] corresponds to the height of a tower; a pizza with four toppings [available] would be a tower four high. He added that the two colors would indicate with or without toppings.

Conclusions and Implications

Students approached the Tower Problem using random methods to create combinations and check for duplicates. To help simplify the task, they developed local organizations, such as opposites. When these local organizations proved inadequate to justify having all of the combinations, the students focused on cases, which supported the movement to a more global organization. This new organization became the framework for the students' development of a proof by cases. A proof by contradiction was also created to justify the completeness of an individual case. When the students solved the problem for towers of different heights, they observed a doubling pattern, which lead to a generalization and the development of a proof by induction.

The students solved the Pizza Problem and justified their solution using a proof by cases. They connected their topping combinations to the numbers on Pascal's Triangle and explained the addition on Pascal's Triangle using pizzas. They also noted the doubling pattern as the number of available toppings increased. In addition, the students at Table A, who had participated in the longitudinal study and explored the tower and pizza problems in grades three through five, explained their reasoning for the doubling rule using both pizzas and towers. Furthermore, they explained the addition on Pascal's Triangle using towers as well as pizzas. Finally, they constructed a three-way isomorphism between the Tower Problem, the Pizza Problem, and the numbers on Pascal's Triangle.

The students were presented with challenging problems and given the responsibility to solve them. They did not work in isolation; rather, they were active participants in a learning environment where ideas were shared and discussed. In the course of making sense of their observations and of what their peers were saying and doing, they built an understanding of important mathematical ideas and developed justifications for those ideas. Furthermore, since the students created personal, meaningful proofs, rather than passively receiving instruction from a teacher, they extended their reasoning and made connections to other ideas in combinatorics.

This research indicates that when given challenging problems in an appropriately supportive environment, these students can, and did, construct "wonderful ideas" (Duckworth, 1996), including sophisticated mathematical proofs and convincing arguments generated to justify their ideas to themselves and to others. This has important implications for teachers and researchers, who, as the NCTM standards suggest, wish to incorporate the idea of mathematical reasoning and proof making into the curriculum in a meaningful manner.

Endnotes

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2. For further discussion of the earlier work of these students, see Tarlow, 2004.

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IMPROVING THE FORMULATION COMPONENT OF PROOFS USING A NETWORK CHAT ENVIRONMENT WITHIN DYNAMIC GEOMETRY ACTIVITIES

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In this paper we present results from a case study derived from a teaching experiment using dynamic geometry activities in a distance communication setting (network chat) for the construction of geometrical proofs. We use as theoretical framework, the work of Balacheff who defined three components to analyze the transit from pragmatic proofs *to* intellectual proofs *culminating in a mathematical proof: the* knowledge component*, the* language *or* formulation component*, and the* validation component. *As a result of the chat activities, whose aim was to help develop a functional language, students did show an improvement on the level of the formulation component. In contrast, in the short time available, the students weren't able to construct intellectual proofs using the acquired language: the attention on the formulation component seems to lessen the one placed on the validation component.*

Introduction and theoretical framework

For several years we have been investigating the influence of technology on the teaching and learning of mathematical proof. The use of technological tools brings the possibility for different types conceptualizations of mathematical objects, which may help or hinder the processes involved in the development of proofs. Most research regarding the use and influence of technological tools in proving has been done with regards to dynamic geometry environments (Hoyles & Jones, 1998; Balacheff, 1999; de Villiers, 1998, 2002; Jones, Gutierrez & Mariotti, 2000). In this paper, we will present data from a teaching experiment involving the use of dynamic geometry activities for the construction of geometrical proofs with the use of a distance communication setting. One of our interests has been to promote and observe how students develop the *functional language* that is necessary for the construction of *intellectual proofs*. Thus, we tried to complement the dynamic geometry activities with a network chat setting in order to develop the language component.

Some researchers promote the use of proofs as a means to create meaningful experiences (Hanna, 1998). Constructing proofs using technological tools such as dynamic geometry environments can provide an opportunity for exploration, discovery, conjecturing, refuting, reformulating and explaining (de Villiers, 2002). Computer tools can be used to gain conviction through visualization or empirical verification, but as de Villiers (ibid) points out, proofs have multiple functions that go beyond mere verification and that can also be developed in computer environments: such as *explanation*, *discovery*, *communication, intellectual challenge*, *systematization*. From these, we will focus here on some aspects of the communication function.

Balacheff (1987, 1999), in particular, has introduced certain ideas that we consider important for our analysis. In accord with his theory, a proof is conceived as an explanation regarding the truth of a proposition that is accepted by a community (as explanation we understand a discourse that tries to make sense of procedures, results or mathematical propositions). The community could be a school community of a particular level, or it could be the mathematical community; in this last case, proof means a *mathematical proof.* Balacheff (ibid) has also proposed a distinction between *pragmatic proofs* and *intellectual proofs*; emphasizing the role of language in the passage from the former to the latter. Pragmatic proofs are those based on effective action carried out on the representations of mathematical objects. They lead to practical knowledge that the subject can use to establish the validity of a proposition. Intellectual proofs demand that such knowledge is reflected upon, and their production necessarily requires the use of language that expresses (detached from the actions) the objects, their properties and their relationships. The transition from pragmatic proofs to intellectual proofs culminating in mathematical proof is looked upon within the more general context of mathematical activity, which can be divided into three main components: the *action* or *knowledge component*, the *language* or *formulation component*, and the *validation component*.

As summarized by us in Figure 1, the first component is constituted by the knowledge that is involved in decision-making and it evolves from procedural knowledge to theoretical knowledge (it includes: the nature of knowledge: knowledge in terms of practices — "*savoir-faire";* knowledge as object; and theoretical knowledge). The second component relates to the language through which knowledge is transmitted and it evolves from ostentation language to formal language (it includes: ostentation, familiar language, functional language, formal language). The third component is formed by the procedures used when validating the truth of knowledge, that is, the types of rationale underlying the produced "proofs": from pragmatic, to intellectual, to mathematical proofs; these start with naïve proofs (the most basic type of pragmatic proof) and evolve to mathematical and even formal proofs (the latter is not included in the diagram). Each level of a component corresponds to levels of each of the two other components (the diagram in Figure 1 is not all-inclusive but shows the easily identifiable levels).

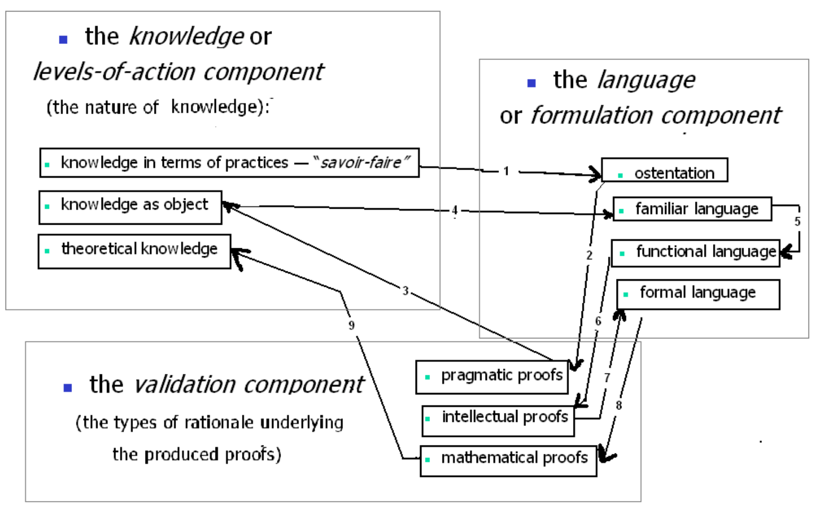

Figure 1: The three components involved in the development of proofs

We are concerned with the problem of stepping from pragmatic proofs (in dynamic geometry environments pragmatic proofs can be very powerful) to higher levels of proofs. In previous research (e.g. see Sánchez & Mercado, 2002; Sánchez & Sacristán, 2003; Mercado, 2004) we have observed that through Cabri-Géomètre-based explorations, students can discover and consider as plausible, geometrical propositions. In fact, students are able to construct "dynamic proofs", that is, pragmatic proofs based on dragging. In contrast, they aren't as able to develop a functional language that is characterized by the use of symbols and the construction of precise geometrical instructions. If students are not able to develop a functional language, they will be unable to construct intellectual proofs.

The functional language, in particular, is a language for talking about mathematical objects and for communicating ideas related to them, independently of the situation, school context, or of the persons with whom the communication takes place (e.g. the teacher). Thus, the development of a functional language involves processes of "de-temporalization" and "de-contextualization".

Additionally, we can consider that the process of language-acquisition takes place through the completion of stages (not necessarily sequential) that correspond to the discursive functions of a language, as described by Duval's (1995) semiotic theory: *designation*, the *construction of complete statements*, *discursive expansion* and *reflexivity* (the use of language in the study of language itself). We consider that there are analogous functions in the acquisition process of the geometrical language. The function of *designation* involves, in a geometric problem, the construction of the definition and the assigning of symbols to the geometrical objects. The function of *construction of complete statements* involves putting together several language elements (definitions, symbols, natural language) in order to produce a higher order statement with a geometrical meaning; it implies being able to synthesize. The function of *discursive expansion* involves integrating the statements that result in the text that describes a construction or a proof.

Based on the above ideas, we devised a situation which we hoped would help students develop their functional language in geometry, by stimulating them to carry out processes of de-temporalization and de-contextualization: the situation, part of a teaching experiment, involved, among other things, having students and communicate problems and results via a computer network chat, as described below.

A teaching experiment for developing functional language

Our original aim was to investigate the use of dynamic geometry activities for improving high-school students' abilities in proving geometrical propositions, and the conditions necessary for promoting the transition from pragmatic proofs to intellectual proofs. As mentioned above, in the course of our previous research, the lack of development of a functional language emerged as a fundamental aspect in the learning of mathematical proof by students.

For this we designed a teaching experiment involving 8 Mexican high-school students (ages 16 to 17). We should point out that in all of the middle school and high-school Mexican curriculum topics of mathematical proofs are not included. It is also worth mentioning that in most Mexican schools there is little emphasis put on writing and, even less, on mathematical writing. This is a local cultural problem, which forces us to put more interest on the development of a functional language. (This problem may also explain the poor results that we reported in Sánchez & Mercado, 2002.)

The teaching experiment consisted of two phases. In a first phase, the students were taught a Geometry-with-Cabri course that lasted some 20 hours, placing emphasis on the writing of propositions and proofs. In a second phase we had special sessions where students had to communicate a problem to a partner via a network chat, and they had to solve it together and construct a proof. (Data of students' performances was collected during both phases.)

During the first phase geometry course, students were taught the use Cabri, and they reviewed basic geometry concepts and the characteristics, purpose and construction of proofs.

The second phase added a communication setting where students could put into practice their writing abilities in a natural way. The premise was that a distance communication setting could influence the process of de-contextualization in two ways: a) If the two participants do not have a common geometrical figure, they need to use language to communicate how to reconstruct the figure. b) In this situation the student who is given the problem, is aware that his partner doesn't know the proposition they will work on which forces him to explain as best he can the problem and his solutions.

We teamed students in pairs. The two students of each pair were separated, working in different rooms on network-linked computers, so that they had to communicate by "chatting" (in written form, without the possibility of sending graphics) through the network. The teacher presented and explained a geometrical problem to one of the students; that student had to communicate the problem to his/her partner through the network chat; the pair had to work collaboratively to solve the problem, by communicating to each other their ideas and progress. Students were provided with the Cabri software as a tool to solve some problems presented to them (see further below). Students worked in this way for six 3-hour sessions. In between these working sessions, there were teaching sessions where the teacher commented on what the students did in their past sessions. During these interventions, the teacher made suggestions on the use of symbols (the designation function), the construction of complete statements, and the discursive expansion function.

Here we present excerpts from the case study of one pair of students, Pedro and Israel, during the second phase. The data was analyzed in terms of the evolution of all the three components involved in the development of proofs: the thread knowledge-formulation-validation. In particular, the data produced during the chat sessions was analyzed from the point of view of how students evolved in terms of the formulation (language) component, and this is what we shall present here.

Case Study of Pedro and Israel

First session. In the first session (of the second phase), Pedro and Israel were each given, verbally on a blackboard but without symbols, a separate problem. They each had to communicate their problem to his partner via network chat and they jointly had to find a solution. Pedro was given the following problem. (Note: All problems and transcripts are translated from the original Spanish.)

Problem 1: In a triangle, draw two perpendicular segments to a median through two vertexes. What is the relationship between these segments? Write a corresponding proposition and a proof.

Below is the way Pedro tried to explain the problem on the network chat to his partner. He uses natural language without symbols, and the definition of the problem is vague.

Pedro: Israel, the problem is about a triangle on which a median is drawn, that is, a line that divides the triangle into equal parts, when a perpendicular is drawn to the right and another to the left, the segments measure the same. Why? That's the problem. Did you understand it or should I explain it differently?

Israel was unable to understand Pedro's problem so he focused on solving the problem that was given to him and there was little communication between the two students. In subsequent sessions, only one student was given a problem at a time, in order to push the students to communicate and jointly find a solution.

Second Session. In the second session, Israel was given, orally, the following problem, which he had to communicate to Pedro. As in all the sessions, there was no use of symbols in the presentation to the student so as to not suggest a particular use of symbols.

Problem 3. Given a triangle and an orientation of its perimeter, successively find the symmetrical point to each vertex in relation to the following vertex. The three resulting points form a triangle. What is the relationship between the area of this triangle and that of the original triangle?

We erroneously thought that the necessity to communicate the proposition in a written form during the chat, would promote a spontaneous use of symbols by the students. As shown in the transcript below, Israel did use symbols to label the vertexes of the triangle but neither he nor Pedro ever used them again.

Israel: We have a triangle A, B, C, we have to find the symmetrical of each point, after joining the resulting points we get a major triangle. What is the relationship between the areas of the triangles or rather how many times does the small triangle fit into the big triangle?…

Third Session. Before the third session, the teacher suggested to the students that they use symbols; the teacher showed them how in the previous sessions they had problems in their statements because of the lack of use of symbols. Pedro, influenced by the teacher's suggestion, made an effort in the third session to use symbols for denoting the vertexes and segments. We should note, however, the non-conventional type of notation that he used for the segments:

Pedro: Israel the problem is as follows, if you have a triangle A, B, C, and you find the mid-point of each side of the triangle, call the mid-point of A to C, A1, the mid-point of B to C, B1 and the mid-point of B to A, C1 then join each of these points, you will see that a triangle is formed, What is the relationship between the formed triangles? Do I make myself clear?

An immediate consequence of the use of symbols by Pedro, was that Israel was able to understand the problem much more easily than in the previous session and construct the figure in Cabri. Just the fact of using some symbols, even if they were used in a rudimentary way, allowed the students to make some progress in the de-contextualization process. In this session, the students were able to share the object of which they were talking. On the other hand, the use of symbols was abandoned as they progressed in finding a solution. The solution Pedro presented to the problem contains the main elements for a proof, but he no longer used any symbols, and again reverted to the use of colloquial terms. Israel, however, was able to understand it because, in contrast to what happened in the first two sessions, the two students now had constructed a common object to which they could refer.

Pedro: look, I have the following: the triangles which have same areas because the central triangle forms a parallelogram with each of the remaining triangles and each parallelogram that is formed is divided by a diagonal, which divides the parallelogram in two triangles of equal area and that shows why they have 4 equal areas

Fourth Session. Before the fourth session, the teacher again made an intervention, in order to go a step further in the development of a functional language: he emphasized the need to write complete statements for the construction of geometrical objects, showing the students examples and discussing the deficiencies in the transcripts from the previous sessions. The problem, given to Pedro, was as follows:

Problem 5: Given any parallelogram, select an orientation for its perimeter. Draw the symmetrical point to each vertex with respect to the following vertex. Join the resulting vertexes in the obtained order to form a quadrilateral. What is the relationship between the area of the original parallelogram and the area of the resulting quadrilateral? Would the obtained relationship be applicable to other quadrilaterals?

Pedro began trying to state the problem very clearly; he was careful to use symbols and to give as complete statements as possible. Israel understood the problem but he realized that he didn't know how to construct a parallelogram. Pedro reverted to the use of familiar language in his instruction on how to construct it.

Pedro: Look, first build a segment, then mark a point outside the segment and then draw a parallel line to the segment that you have with respect to the drawn point, then draw a segment that begins in one end of the original lone to any point in the parallel line, finally draw a perpendicular to the second segment

However, he soon realized the need for symbols and a more functional language and reformulated his statement.

Pedro: another way is: build a segment call it A, then mark a point outside the segment call it G, then draw a parallel line to segment A that passes through point G then mark a segment B that begins in one end of line A to any point on the parallel line, finally draw a perpendicular to segment B with respect to…

From the point of view of the validation component, during the first two sessions, Pedro was constantly aware of the need of giving a proof (give an explanation), and produced "proofs" or explanations although using familiar language. In the case of Problem 3 (second session) his explanation was close, in familiar language, to a correct mathematical proof:

Pedro: An answer would be to find the symmetry for each of the segments of any triangle and the resulting segments are joined in the middle by other segments three diferent triangles are produced, now those triangles you can get the median that converts the figure into 7 triangles of the same area, this happens because each triangle shares or has in common the same base and the same height with respect to the original triangle which is at the center and shares its sides with each of the 6 leftover triangles

In contrast, by the fourth session, the emphasis was placed on the use of symbols and a functional language, and the students were unable to build a proof (even though Problem 5 was a simple variation of Problem 3 and we had assumed it would be easy for them to produce a proof.) In this session we observed a change in Pedro's attitude: he seemed reluctant to discuss the generalization of the result and did not even attempt to construct a proof: perhaps he was exhausted from his effort to state his ideas using a functional language. The validation component seems to take a step backwards when more emphasis is placed on the formulation. Although the fact that they did not produce a proof in this last session may be due to time constraints (although the session was 3 hours long for only one problem), another explanation is that the transition from familiar language to symbolic language implies a cognitive effort that diverts the attention from the aspects related to the validation component. We liken this to the difficulties one experiences when trying to think with a newly learned foreign language.

With regard to the knowledge component, we observed again that students "understood" the propositions and were able to detect geometrical relationships. It is worth noting, that Pedro, who had shown a better performance on the level of the knowledge component, and also on the level of the validation before his attempts to use a more functional language, had a greater ability for developing the latter component. This leads us to hypothesize that a higher development on the level of the knowledge component provides a good foundation on which to develop functional language.

Conclusions

As reported elsewhere (Sánchez & Sacristán, 2003), we found that the use of the dynamic geometry software, helped students make progress on the level of the *knowledge component.* Students seemed to grasp the meaning of some of the theorems (and explain the fundamental idea behind it) thanks to the phenomenological approach that the use of Cabri makes possible which allow students to discern the necessary elements needed for the proof.

On the other hand, during the first phase, their statements and results were expressed in a vague and loose language that made its comprehension difficult for anyone unfamiliar with the context of the problem; this indicates a deficient process of de-contextualization. The purpose of the second phase (the chat sessions) was precisely to help overcome this deficiency by placing emphasis on the level of the *formulation component***.**

During the second phase students did improve considerably on this level, as illustrated above in the case study of Pedro and Israel. While in the first chat sessions, the geometrical objects were described using a familiar language (no use of symbols and vague definitions), by the fourth session, the value of a functional language was realized: symbols and complete statements were used and recognized as necessary for efficient communication.

However, the attention placed on the formulation led students to neglect the *validation component.* While in the first sessions there was a constant awareness of the need to explain and produce a proof, by the fourth session the time spent on efficiently communicating the problem to each other prevented them from having time to find a proof.

In summary, the Cabri-Géomètre activities seem to allow progress on the level of the knowledge/action component; pragmatic proofs are enhanced. The communication (chat) setting and the teaching influence positively the use and acquisition of a functional language. We observed that the use of symbols and the effort to construct complete statements improved greatly the communication between the students. But the validation component seems to take a step backwards when more emphasis is placed on the formulation. The development of a functional language demands a great deal of attention on the part of the subject that in a way inhibits (maybe temporarily) the use of other abilities belonging to the knowledge and validation components. However, as already stated, the chat setting did help in the development of the formulation component and we are researching its potential so that with more practice students' use of a functional language becomes more established leading perhaps to an improvement on the level of validation.

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MATHEMATICAL IMAGES FOR WORKPLACE TRAINING: THE CASE OF JOHN, A PLUMBING APPRENTICE

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This paper presents some initial findings from a multi-year project that is exploring numeracy and the growth of mathematical understanding in a variety of construction trades training programs1. In this paper, we focus on John, an entry level plumbing trainee as he attempts to solve a pipefitting problem. We explore the ways in which he tries to decide which calculation to perform when faced with a multiplication sum required by the task. We suggest that while he may have an appropriate image for the act of multiplying, he does not access it in this task, and that he needs to either make or remake an image that will help his understanding grow in this context. We contend that it can not be assumed that the images held by adult learners for basic mathematical concepts are necessarily appropriate or accessible, particularly when being used in new, specific workplace contexts.

Mathematics and Workplace Training

Using mathematics is a fundamental part of workplace practice in most credentialed trades. To function effectively and efficiently, tradespeople are expected to use and apply a range of mathematical skills and understandings in a wide range of situations. (e.g. Nicholson, 1998; Folinsbee, 1995). Most construction trades require workers to be credentialed, and apprenticeship training courses and the associated examinations generally have considerable mathematical content. Although recent years have seen an increase in the attention paid by researchers to mathematics in the workplace, there is still only a limited body of work, which considers cognition and understanding in a vocational setting (E.g. Noss, Hoyles & Pozzi, 2000). Our research focuses on mathematical understanding in workplace training, and explores the ways in which understandings are used, modified and learned in specific workplace training contexts.

Theoretical Approach

The study is framed by the Pirie-Kieren theory for the dynamical growth of mathematical understanding (Pirie & Kieren, 1994). This theory provides a way to look at, describe and account for developing mathematical understanding as it is observed to occur in action. Adults in workplace training are often re-learning mathematics for which they have existing images and understandings. As they engage in mathematical activity during training they need to re-visit these existing understandings and images, make sense of them again in specific trades contexts, and modify and extend them and if necessary construct new understandings. We use elements of the Pirie-Kieren theory, specifically the notion of “images” to describe the way that John uses the mathematical concepts of multiplication and fractions within a pipe-fitting task. We highlight the potential limitations of his existing images for these concepts, at least in a form that is helpful in this context.

Methods and Data Sources

The larger study, currently underway, is made up of a series of case studies of apprentices training towards qualification in various construction trades in British Columbia, Canada. The trainees and their instructor were observed and video-recorded over a number of sessions. The episode on which this paper focuses involved a small group of students in the shop working to calculate the length of a pipe component required for a threaded pipe and fitting assembly to be built to given specifications. This activity followed a formal lesson on this procedure in the classroom. The second author acted as a participant observer in this session and engaged with individual trainees as they worked on the task. The video recording of this episode was analysed using the Pirie-Kieren theory with a particular focus on identifying the mathematical images held, accessed, made, modified and worked with by John as he completed the task. It should be noted that the transcript offered below represents a very short extract from a number of hours of taping, and some of the comments and conclusions we offer draw on data beyond that presented here.

John and the pipefitting task

A drawing of the pipe assembly to be constructed is shown in figure one. The students were assigned the task of constructing this assembly with a *centre-to-centre measure* (C-C) of ten inches. The values for the fitting allowance (A) and thread makeup (E) were provided elsewhere, and the length of cut pipe (P) had to calculated.

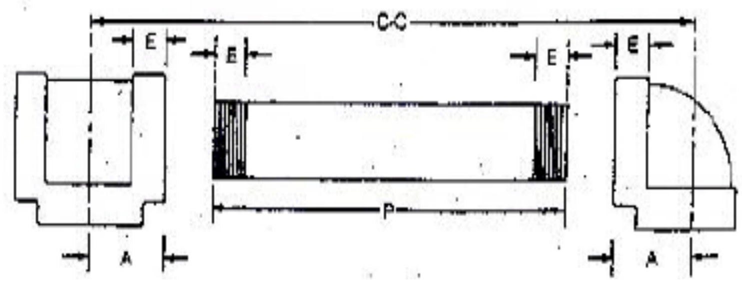

Figure One: Pipe assembly to be constructed by John using standard pipe fittings and a cut piece of pipe.

The following episode begins at a workbench in the shop as John works to make sense of the calculations needed for this task.

John: Ok. So what I do now is, I know that it's ten, what I've got to have total.

Researcher: Yeh

J: I got to multiply the take-off twice, because on each end, right? (Here John uses the term 'take-off' incorrectly to refer to fitting allowance (A)).

R: Yeh.

J: So what's got me.

R: So you multiply one and three quarters twice?

J: Yeh. One and three quarters gives me three and one sixteenths. Right?

In this first episode John has recognised the need to obtain the total *take-off* amount to accommodate fittings to be attached at each end of a pipe to make a 10 inch centre to centre pipe assembly. He has obtained the correct measurement for the *fitting allowance* (1 3/4/") from an industry standard reference table for his calculation and written down:

1 3/4 x 2 =

on his sheet of paper. Throughout this process John refers to his own notes of a similar example discussed in class. He has an understanding of what is involved in solving the problem, and knows that he now needs to translate this understanding into a numerical calculation that can be carried out on a calculator to provide an answer he can use in the actual measuring and cutting of the pipe.

Once John has translated his understanding of the fitting allowance for both ends into an appropriate mathematical calculation, specifically 1 3/4 x 2 =, he uses his calculator to find this product. He enters:

1 3/4 x 1 3/4 =

on his calculator a number of times while he works on the problem, getting an answer of 3 1/16 each time. While we cannot say with certainty why he chose to perform the calculator operations that he did, we would suggest that one factor is the mathematically ambiguous way (from our perspective) that he frames the required operation for himself, and the image, or lack of image, that underlies this.

John says, "I got to multiply the take-off twice, because on each end, right?" Here he is shifting from his appropriate (non-mathematical) pictorial and physical image for the problem – of an amount to be taken off each end – to one that is exclusively symbolic/numeric in form. However, in using language to re-formulate his understanding symbolically, he states that you have to "multiply the take-off twice" which easily lends itself to a symbolic representation of 1 3/4 x 1 3/4. It would seem that even although he wrote 1 3/4 x 2, he may have been reading this as "one and three quarters multiplied by itself", seeing this as a representation of the visual problem rather than a calculation.

We contend that John's difficulty lies in the images that he has for the mathematical concepts being used here, especially that of multiplication, and that his idea of "multiply the take-off twice" translates into an incorrect calculation. In trying to solve the problem, we see John having a viable visual and physical image of what is required for the task, and then needing to find or construct an appropriate mathematical model. To do this, he needs to draw upon his understanding of numbers, and his images for addition and multiplication--specifically that "when you multiply something by a number, it is the same as adding it to itself that many times." Accessing such an image would perhaps have allowed him to return to the problem with an appropriate piece of mathematics (e.g. 1 3/4 + 1 3/4, leading to 1 3/4 x 2) to then apply to the question at hand. While he may have an understanding of multiplication as "repeated addition" he does not access this here, nor does he generate it from his pictorial image for the problem, nor is it implicit in his choice of operation.

John does not see the problem in terms of "putting" the two take-off lengths together to produce a single piece to be cut from the pipe, something that would lend itself to thinking in terms of an addition sum rather than a multiplication. Having an image of multiplication as a short-cut for repeated addition would perhaps have helped John to produce a correct calculation to carry out. There would seem to be a need for John to fold back to his understandings of addition and multiplication, find and then translate the appropriate mathematical understanding (and procedure) within the new context. Of course, should this understanding not exist he would need to spend some time working on the actual mathematical concept, before being able to apply it to the problem. For example, in the present case, that might include working with similar examples using whole numbers exclusively, prior to considering fractions.

What is interesting here is that John is not convinced that his answer is correct, and he expresses this concern to the researcher, who then works with John on the problem:

Researcher: …Let's pull out a ruler. Here. Show me an inch and three quarters. *(John pulls out his tape measure.)*

John: Ok.

R: Just show me with your finger.

J: That's an inch. That's my inch and a half. That's my inch and three quarter right there. Am I not correct?

R: Just put your finger there so I can see it. Ok. So there's your inch and three quarters right there. Add an inch and three quarters to that. And I'd go one step at a time. Like add an inch, and then add another three quarters.

J: Ok, so, I go, I've got an inch and three quarters right here. Which is right here. (pointing to 1 3/4 point on tape measure.) So to add another an inch, and another three-quarters to it?

R: Yeh.

J: (pause) Ok, hold on. Right here (pointing at tape measure with pencil).

R: Yeh

J: *(long pause)* That would be three right here, right? No. That's one.

R: Right. That's one and three quarters, clearly. That's an inch and three quarters. Now if you add another inch and three quarters to that.

J: Ok. I'm stumped

R: Ok.

J: I'm stumped. It's simple math here, needed. That's all. I'm not doing it.

R: Give me your right hand. Replace the finger on the tape measure. If that's an inch and three quarters (pointing to this point on the tape *measure)* then another inch would be to,

J: Add?

R: Another inch would be to where?

J: Well wouldn't it be to here?

R: To there.

J: Right.

R: And now add three quarters. (Indicating intervals on the tape measure) One quarter, two quarters, three quarters. How much?

J: It would be three and a half.

R: Yeh. (Then a long pause, with no response from John.)

Here we see the researcher taking John out of the context of the problem, and working with him to think about the addition of two lengths of 1 3/4 inches. In doing this the aim is to both explore the images that John has for addition (especially for adding fractions) and also of course to help him possibly make a new, more appropriate one that will help him with the pipe-fitting problem. It should be noted that at this point in the conversation the researcher was not aware of the incorrect operation performed by John, but does know that his answer is wrong. The researcher engages John in an image making activity, involving a tape measure – a familiar and readily accessible workplace tool - hoping that John will make an image for the addition of fractions as counting-on on a number line. This is of course a useful mathematical image for the physical actions involved in cutting the required pipe length. Although John engages with the activity he struggles with it, and clearly is not sure how to count-on using the scale on the rule.

More significantly, we also suggest that John does not know why he is being asked to do this. As noted earlier he gives no indication of seeing the required calculation as one involving addition. John is certain that he needs to multiply, but is simply not sure that he has chosen the correct procedure. Whilst he is happy to work on this addition problem with the researcher, he does not indicate that he relates this in any way to the pipe-fitting task. For the researcher, and perhaps the reader, with powerful and versatile images for the concepts of multiplication and addition, the link is an obvious one, but we should not expect that John will automatically make this connection. John and the researcher are working with two different images for calculating the total take-off length, and as such even when the correct answer is achieved, John is not sure what he should now do with it.

In the next few minutes John explains how he obtained 3 1/16, and the researcher becomes aware of the error John has made. John quickly abandons his original procedure and now multiplies by two, carrying out the correct calculation. However, we suggest that John still does not connect the procedure with a mathematical image. He knew that either he had to multiply the fraction by itself or by two, but with no understanding of *why* one is correct. For him, it was a choice between the two procedures, and, as he now knows one was incorrect he performs the other. The researcher probes further:

Researcher:	So, *(pointing to crossed out 3 1/16 on paper)* did one and three quarters flag that for you, or no matter what you would have got you still would have been thinking about it?
John:	I still would have been thinking about it, because I would have known that, I still know in the back of my head, either you times it by, like I'm thinking to myself, times it by itself or you times it by two.
R:	Ok
J:	That's what I'm thinking, all the time. So, and I'm looking yesterday's, yesterday's theory, I had no problem doing that. *(points to written calculations from previous day.)*
R:	Yeh
J:	I made that in as dummy's terms as I can get. Right. So all I did was change the number here (points to written computation in notes from the *previous day).* The formula still stays the same. And, that's my problem. I didn't want to look at that. I go on memory.
R:	Yeh,
J:	and what I want to do now is get a fresh sheet of paper and start over again before I cut this pipe. Ok. So that's what I'm doing. I don't want to go any step further, even though I know the answer. That's not going to help me when I do my test.

John shares his uncertainty about which operation was the correct one for the problem. He notes that on the previous day he was able to follow an example done in class by his instructor, but also that he does not want to simply mimic what he had written there. He accepts that being able to remember what to do would be acceptable. He is not really concerned at any point with knowing *why* multiplying by two is mathematically correct, but multiplying the fraction by itself is not. John is content to now know which operation he should use, and as he comments to "go on memory." His mathematical images for multiplication as repeated addition are unchanged, and we would suggest that although he has now successfully completed the calculation his understanding of *how* and *why* the mathematical operation is the correct interpretation of his visual image is unchanged and limited. Naturally, we question the reliance of a learner on procedural memory and return to this in our conclusions.

A few minutes later in the same session, with this now correct answer for the length of pipe calculated, John went on to measure out the length of pipe that he needed using an imperial units tape measure. Although there is not space here to present this extract in detail, again, John's limited images for fractions were problematic. He did not have an image for a fraction as a location on his measuring tape i.e. as a point on a number line, and did not see the relationship of the numerator and denominator of a written fraction to the part-whole of an actual inch. This meant that he could not easily, nor reliably, locate a given measurement on his tape. The difficulty was compounded by the complex configuration of markings of imperial tape measures, which unlike their metric equivalent, involve the super-imposition of many different units on the same number line (i.e. halves, quarters, eighths, sixteenths).

Conclusions

There is not space here to comment in any depth on the complex role that mathematical understandings and images can play in trades training, but we suggest that trades educators should expect that their trainees may not come with a useful and easily applied repertoire of images for the mathematical concepts used in their training. It is not clear whether John had existing and appropriate images for the mathematical concepts required to successfully complete the pipefitting task, but it is clear that he did not access these and work with them in the creation of an appropriate mathematical model for the problem. For example, John would likely have benefited from an opportunity to fold back and re-make (or make) an image for multiplication as repeated addition that he could see as being appropriate to the task. Working with whole numbers initially may have helped him to then be able to use fractional amounts. Clearly, the measuring tape is a fundamental part of working in the construction trade, and the ability to use this, and to understand the mathematics that is captured by this tool is essential for a worker. Whilst we acknowledge that such understandings are not likely to be made explicit during every

task, the possession of a powerful and flexible set of mathematical images related to this offers something to fold back to, should memory fail, or the need arise to work in a new context. Certainly for John, being able to connect multiplication by two with the image of placing the two equal lengths of pipe together, and of then understanding how this can be represented on a measuring tape could have been a valuable experience, as would be the exploration of fractional units on the tape.

We contend that in the apprenticeship training classroom there is a need to re-visit concepts such as addition, multiplication, fractions etc. and to go beyond learning merely how to operate on and with numbers. In particular, there would seem to be a need to explore the existing understandings that trainees bring with them, to consider the appropriateness of these images for vocational related tasks, and to occasion the construction of new images as needed, drawing upon the use of common workplace tools and resources as appropriate.

Endnotes

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USING A WARRANTED CONCEPTION OF IMPLICATION TO VALIDATE PROOFS

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Introduction and theoretical framework

Implication is an essential structure in proof-oriented mathematics, but is also a topic that causes students serious difficulties (e.g., Deloustal-Jourrand, 2002; Durrand-Guerrier, 2003). In this presentation we will focus on the way in which one needs to interpret implications if one is to reliably validate proofs. We will present a theoretical analysis and data drawn from students' responses to an invalid 'proof' in real analysis.

To frame this discussion we make use of Toulmin's (1969) model of argumentation, in which a presenter puts forward *data* in support of a *conclusion*; their explanation for why the data necessitate the conclusion is referred to as a *warrant*. At this point, the audience may accept the data but question the validity of the warrant. Using this model, we will contrast material implication as it is commonly taught in introductory proof courses with a *warranted* conception that we claim is needed in order to validate proofs.

Under a material conception, an implication is said to be true if and only if the antecedent is false or the consequent is true. Hence, in Toulmin's model we may say that the conclusion is the statement "if p, then q" itself, the data to support this conclusion can either be "p is false" or "q is true", and the warrant is the logical equivalence between the statements "if p, then q" and "not-p or q". To illustrate the inadequacy of this interpretation for validating proofs, consider the following 'proof' that 1007 is prime.

Proof. 7 is prime.
If 7 is prime, then 1007 is prime.
So 1007 is prime.

The statement "If 7 is prime, then 1007 is prime" is true since 1007 is prime. However, establishing the truth of this statement does not establish the validity of the 'proof'. Instead we claim that one needs to use a *warranted* conception of implication. By this we mean that one needs to interpret the consequent of the statement (1007 is prime) as the conclusion and the antecedent (7 is prime) as the data. The warrant, in the form of a true general principle, must often be inferred. In the above case, perhaps the most obvious such warrant is a statement of the form "for every prime number n, 1000+n is prime", which is readily identified as invalid (e.g. 5 is prime but 1005 is not).

We suggest that when a validator reads an assertion in a proof, (s)he needs to interpret this as a conclusion and seek to identify the corresponding data and warrant, inferring them if necessary. If the inferred warrant is acceptable in the current context, the new assertion should be accepted as valid. If the inferred warrant is false, this assertion and the entire proof should be declared invalid. If the warrant is plausible, but not acceptable as sufficient in the current context, the proof is usually said to have a "gap".

Students' validation of an argument

We next present data to examine the degree to which students' evaluations of a proof adhere process described above in the context of a first course in real analysis. Students were asked to check a proof that the sequence ($\sqrt{n}$) $\infty \rightarrow$, which ended with the lines: $n < n+1$ so $\sqrt{n} >$

$\sqrt{n+1}$ for all n. So ($\sqrt{n}$) ∞ → as $n \to \infty$ as required. The author appears to be implicitly using the statement that if ($\sqrt{n}$) is an increasing sequence, then it diverges to infinity. This is invalid if interpreted using a warranted conception of implication, since it is not the case that all increasing sequences diverge. Hence, the argument should be rejected. We present data on the following responses:

Group 1: Three students inferred the warrant as asserting that increasing sequences diverge to infinity and were able to produce counterexamples to this assertion. In our view, these students successfully used a warranted conception of implication.

Group 2: Three students criticized the argument since it did not invoke the definition of divergence, but they did not cite a problem with the last line of the proof. In our terms, they considered the data inadequate, and did not infer a warrant for the final conclusion.

Group 3: Five students initially accepted the argument as a valid proof. From their comments, it appears that they focused exclusively on the data and conclusions and did not consider what warrant was used.

Overall, seven incorrectly accepted the proof as valid, and only three spontaneously considered the warrant used to justify the fourth line of the proof. On a more positive note, the interviewer subsequently asked whether the second of these lines followed from the first, at which point five of these inferred a warrant and rejected the proof.

Implications for practice

Our findings are consistent with a study on proof validation conducted by Selden and Selden (2003), who found that undergraduates in an introductory proof course performed at chance level when they were asked to determine whether arguments constituted proofs,

but that questions by the interviewer could improve students' performance at this task. This, together with the theory presented above, suggests that students can and should learn to use a warranted conception of implication when reading proofs. That is, that they should be taught to identify data and conclusions and infer and evaluate warrants.

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MOTIVATIONAL BELIEFS AND GOALS OF MIDDLE SCHOOL STUDENTS IN DISCUSSION-ORIENTED MATHEMATICS CLASSROOMS

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Purpose

Students' participation in mathematics classroom discussions may lead to active sense making and the development of mathematical communication and reasoning skills, but helping all students reach these goals remains challenging. Adolescents may face unique obstacles to participation, as they can be "reluctant to stand out in any way during group interactions" (NCTM, 2000, p. 61). Characterizing students' motivations for participating in classroom discussions can provide insights for building upon students' experiences in order to support their learning of mathematics.

Background

Discussion-oriented mathematics classrooms are typically studied for the purpose of understanding how teachers can orchestrate discussions more effectively. For example, upper-elementary mathematics classroom teachers who scaffolded instruction through techniques such as transferring responsibility were more likely to foster high-involvement classrooms (Turner et al., 1998).

An alternative to examining teachers' roles in classroom discussions is an examination of how students interpret their experiences in discussion-oriented classrooms, beyond traditional motivational constructs such as learning goals (Ames, 1992). One approach to conceptualizing students' motivations addresses expectancies and values (Eccles et al., 1993). Students' expectations include their beliefs and assumptions about learning and doing mathematics. Students' values include what they hope to obtain from their actions, or their goals.

Method

This study is an analysis of interviews with 15 seventh-grade students about their experiences in two mathematics classrooms in a rural middle school in mid-Michigan whose teachers incorporated whole-class discussion when implementing the Connected Mathematics Project textbook series (Lappan, Fey, Fitzgerald, Friel, & Phillips, 1997).[1] Participants were purposely sampled (Patton, 1990) to capture diversity in gender, achievement, and frequency of classroom participation. Interviews were analyzed using constant comparative methods (Strauss & Corbin, 1994) and an analytic framework developed by the author that included an examination of linguistic cues (Bills, 1999), repetition (Tannen, 1989), and affect (Hannula, 2002) for a qualitative assessment of beliefs and goals in students' talk.

Results

Students' epistemological beliefs interacted with their perceptions of the level of risk associated with classroom participation. Students (*n*=8) similar to Allen (below) who spoke of learning as a process of receiving knowledge, or obtaining knowledge from an authority, also tended to speak of the act of sharing their thinking publicly as carrying a high degree of risk.

> ...you gotta pay attention so you know what's going on, and if you don't, then you're pretty much lost, and you won't be able to really catch up real fast, it might take you a while to catch up, so you gotta really pay attention, and you gotta listen a lot. ...when I'm put on the spot, I kind of go off track. I don't know how. Every time I'm put on the spot in front of

an audience, I just panic and I can't really think straight.

Alternatively, other students (n=7), such as Molly, expressed beliefs about negotiated knowing, or learning mathematics as an exchange of ideas, and expressed a lower level of risk associated with classroom participation.

> …if you talk through it, and you, like, talk about it, you might realize something you did wrong, if you talk about it, you might say, oops, I timesed when I was supposed to divide or something. ...if I get something wrong, then I can see what I did wrong, and they'll, like, they'll help me and show me how to do it.

Beliefs about negotiated knowing interacted with a lower perception of social risk. The benefits of publicly exchanging ideas appeared to outweigh the risks for these students.

Additionally, students were motivated by social goals, such as helping classmates exercising appropriate behavior, appearing competent, and gaining status, and the academic goal of completing their tasks. Some of these goals were more community-focused, such as helping and behaving, while the others were more self-focused.

Discussion

Results suggest that the conceptualization of adolescents' "productive" disposition toward learning mathematics in reform settings should attend to students' social concerns as well as academic concerns. Students' lived experience of whole-class discussions about mathematics includes a strong focus on relatedness and social goals. While engagement in learning mathematics goes beyond classroom participation, adolescents' goals and beliefs related to their participation can be built upon to create higher involvement in whole-class discussions about mathematics.

Endnote

1. The Connected Mathematics Project texts consist of contextualized problems and lack worked examples.

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EXAMPLES AS TOOLS FOR UNDERSTANDING PROOF IN GEOMETRY

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The purposes of this study are to describe the role examples play as tools in the process of proving for university geometry students and to extend the research done by Balacheff on example usage. Sfard (2002) describes one type of symbolic tool as *inscriptions,* "graphical displays created and used for the sake of communication." When the inscription is a member of the class of objects being discussed, it is also considered an example.

In several proof scheme frameworks, example usage is placed in the empirical proof scheme and seen as a less sophisticated proof strategy, since the proof relies heavily on one or more examples (Hart, 1994 & Harel and Sowder, 1998). Balacheff (1987) characterizes four hierarchical levels of proof based on example usage: naïve empiricism, crucial example, generic example, and thought experiment. When students use examples in the form of *naïve empiricism,* their examples are created to help understand the issues in the conjecture. The examples may not even meet the premise of the statement. The *crucial experiment* example is more purposeful. It is used to push the boundaries of the statement. If the statement is true for n > 5, there is a crucial experiment is at n = 5. The *generic example* represents a class of objects. Finally, the *thought experiment* takes place when students think past the example and begin the deductive process. These levels of understanding directly address students' uses of examples in the process of proving a claim, where the example is part or all of the justification. These frameworks do not account for other uses of examples by the students.

Methods

The data for this study was collected as part of a semester long teaching experiment in an upper division geometry course. The curriculum consisted of a series of activities in which students would need to define, conjecture, or prove results in geometry on the plane or sphere. Students were routinely encouraged to use a clear plastic sphere as they worked. The classroom was videotaped using two cameras. Data came from the transcripts, written work and videotapes over the course of the semester. Each transcript was coded for the example usage based on Balacheff's framework. Other uses of examples, not in the framework, were also noted.

Results and Discussion

The students tended to use examples they had drawn or created as an inscription or tool with which to construct suitable justification for their ideas. Although the example was not the sole reasoning of the argument, the examples constructed still mirrored the types of example usage given by Balacheff. When students used examples to help flesh out a conjecture, their example usage often took the form of naïve empiricism or crucial examples.

In one activity the students were asked to determine the sum of the interior angles of a triangle on a sphere. They began by guessing it must be larger than 180^{o} and less than 360^{o} then began drawing on the Lenart sphere. Alexis said "We think it also can be greater than 270 because I was able to draw a triangle with three obtuse angles and obtuse angles are greater than 90 so it is greater than 270, but as far as in between. I don't know. So we have 90^{o} to play." The triangle Alexis drew was an inscription which allowed them to determine the sum of the interior angles of a triangle on a sphere could be greater than 270^{o}. Since the students began drawing triangles on the sphere with little direction, we would call the use of this example triangle naïve empiricism. This discussion led to a different inscription: one focused on testing the 360^{o} boundary, thus it is a crucial experiment.

S: What if you had a 120 for each of the angles? That would be 360.

E: But you could, could you?

S: I don't know. You would have to draw it. How do you draw it on a sphere?

Sue's question to the group allowed the group to begin exploring the boundaries of their conjecture further. As the group discussed their ideas Sue interjected, "I think the angles have to be less than 360 each," as she points to a drawing on her paper, "This is that whole funky one where this is inside a sphere." Her sketch was an inscription, referring to an example on the sphere. It was a small triangle and the "whole funky" triangle was the exterior of the small triangle, which was also a triangle on the sphere. Although the angles in her example were each about 300^{o}; while examining the inscription she was able to think past the example, consider the possible angle measurements, and predict a boundary for each angle. So this example was used as a thought experiment. Using this reasoning, the group to proceeded to find a correct answer.

In another activity the students were asked to determine, "If given a triangle on a sphere with two of its sides congruent, then are the two angles opposite those sides congruent?" During the whole class discussion a proof was presented which reflected the triangle and then used ridged motion to align to two triangles and prove they were congruent. The inscription presented by the group was a small triangle. The group asserted that their proof worked for large triangles. Alice stated, "And the picture we have up on top is just for one case because we have said perpendicular so we went through point B actually. And it works even for big triangles, not big triangles, it works if instead of connecting it the short way from A to B you connect it the long way around." The group's example was representative of a class of triangles. It was a generic example. The class was not completely convinced by Alice's example. In fact at the end of class Dawn asked the instructor to show the proof for a specific example of a large triangle, "that weird picture that you have on the board." Their discussion focused on this triangle as a representative of the class of large triangles and as such it was also a generic example.

The nature of geometry affords students the ability to use examples in a variety of ways. In both activities the students exhibit example usage within Balacheff's framework, where the example is an inscription used to further the group's understanding of the idea at hand. Our study explores how students use examples as inscriptions which aid in the process of developing a proof, not as the sole reasoning for the proof, as in an empirical proof scheme.

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OVERCOMING DIFFICULTIES WITH PROOF AS A GROUP

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The purpose of this paper is to describe how a group of students in a university geometry course is successful in developing a proof due to a series of requests and responses, which become a mechanism to overcome the mathematical difficulty they encounter. Typically undergraduate students struggle with proof writing, particularly with applying theorems, using symbols, unpacking logical statements and choosing a suitable approach (Selden & Selden, 1995; Weber, 2001). Knuth (2002) suggests that proving is a public activity, hence students benefit from writing proofs in a social setting. Goos et al. (2002) found small groups are more successful at problem solving when members evaluate each other's ideas.

Methods

The data for this study was collected as part of a semester long teaching experiment in an upper division geometry course. Data consisted of videotape recordings of each class session, and copies of students' written work. The curriculum consisted of a series of activities in which students would need to define, conjecture and/or prove results in geometry on the plane and/or sphere. The videotapes and written work were analyzed using open and axial coding.

This study focuses on one particular day late in the semester in which students were asked to prove one direction of the bidirectional statement Euclid's Fifth Postulate (EFP) implies Playfair's Parallel Postulate (PPP). EFP states, "If a straight line crossing two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if extended indefinitely, meet on that side on which are the angles less than the two right angles." PPP states, "For every line and every point not on the line there is a unique line through the point that does not intersect the original line." The instructor told the students that the two postulates are equivalent and gave them the option to "*use*" EFP in order to prove PPP or vice versa. Our discussion will focus on the journey of one group of four students who did successfully prove EFP □ PPP by the end of the class discussion. This particular day shows the entire process of students completing a proof and presenting it in a whole class discussion.

Results and Discussion

The interactions of the group can be characterized by several factors. Even though students think individually during some moments of silence and make individual contributions to the group, the students consistently use the word "we" to describe what they are assuming, what they have proven and what they are trying to prove. Students request input from the group regarding both (1) resources (facts, theorems or strategies that would allow conclusions to be made) and (2) (following appropriate mathematical logic, only using statements that we have already proven, using accurate resources). In the following transcript excerpts we illustrate both types of requests and discuss how they and students' responses to them help students overcome the primary mathematical difficulty that they encounter in writing this proof.

Since EFP and PPP are both conditional statements, "EFP $\Rightarrow$ PPP" has the overall structure of a conditional implies a conditional, i.e. $(p \rightarrow q) \Rightarrow (r \rightarrow s)$. This structure is complex and was difficult for many students in the class. In particular, this structure was the primary mathematical difficulty for the students in this group. In clarifying or negotiating what they are trying to prove Stacy states, "But if we are assuming the whole thing though, we are assuming they are less than π." Several of her statements seem to suggest that she is assuming that we are given *both* the premise and the conclusion of EFP, as evidenced by her statement, "Cause it's all assumed. This whole thing is. We are assuming $\alpha + \beta < \pi$. We don't care if it's equal to π." The group does not notice the error in their assumptions.

Through a series of requests for resources and evaluation, the group realizes their difficulty and is able to overcome it. In the following excerpt Nate requests resources from the group.

N: … Can we use our parallel transport proof to show that the boundary condition when they are equal to 180, that this angle is congruent to this angle and therefore they are parallel and therefore they don't intersect?

In response to Nate's request Paul suggests theorem 8.2. The group then questions the use and correct statement of theorem 8.2. This is a request for a local evaluation as to whether or not the group is following the rules of the mathematical domain.

P: Did we prove that?

N: Yeah, we did, but we started with a parallel transport.

A: We proved that two parallel transported lines are parallel in the sense that they never intersect.

After resolving the resource had been proven and clarifying the exact statement, Nate then made a second request for evaluation, this is a global request for evaluation of the entire proof.

N: Alright, does anyone find any flaws in that?

The group thinks silently for a minute and a half.

P: So, we don't even need to necessarily have the three cases do we? Just we need to prove that one case, uniqueness on that one. Because we are assuming they meet on this side...

In Paul's response to Nate's request for evaluation we see the mathematical difficulty resurface. Based on Nate's requests and subsequent argument Paul is able to articulate to the group the problem with assuming $\alpha + \beta = \pi$, if they have already assumed it is less than π. In further discussion Stacey and Paul's mistake in assumptions is finally illuminated.

P: Yeah, I think we were starting with this drawing [EFP] instead of starting with that drawing [PPP] in our head. That's what I was doing anyways.

At this point the group has developed a proof which is convincing to the entire group. In this case we can see both how the requests and the responses served as a catalyst in illuminating the mathematical difficulty and how they were useful for the development of the proof. As a result the group overcame their difficulty with the structure of the proof.

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THE INSTRUCTOR'S ROLE IN THE DEVELOPMENT OF A CLASSROOM COMMUNITY IN AN UNDERGRADUATE NUMBER THEORY COURSE: A PRELIMINARY REPORT

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The development of an understanding of mathematical proof is regarded as one of the benchmarks of a major in mathematics (Tall, 1992). In particular, it is vital for future teachers to be able to construct, understand, and validate formal mathematical arguments (CBMS, 2001; Selden & Selden, 2003), yet research shows that many students ultimately do not succeed in developing an appreciation for mathematical proof by the end of their undergraduate programs (Harel & Sowder, 1998; Knuth, 2002).

In this poster we will describe the preliminary results from an exploratory study of the impact of the Modified Moore Method in a "transition" course at a large southern university in the United States. In particular, we will discuss the role of the instructor during the first few weeks of the semester and how his actions facilitated the development of a classroom community in which discussion and argumentation formed a basis for learning to construct proofs.

The Modified Moore Method (MMM) is sometimes called an "inquiry-based" method of teaching (Renz, 1999). In a typical MMM course, students present proofs of theorems they construct on their own, and class sessions are centered around discussions of these proofs. In this way, the MMM is similar to teaching strategies such as Cognitively Guided Instruction, though its proponents do not adhere to or base their teaching upon a particular theory of learning.

As part of an intensive case study of an exceptional MMM undergraduate course, we videotaped an undergraduate number theory course during the fall 2003 semester. The course was taught using the MMM by a respected and experienced instructor. Preliminary findings demonstrate that the instructor played an active role in the discussions at the beginning of the semester, and that his actions facilitated the development of a classroom culture that may have enabled the students to develop a conception of proof that is quite mature for undergraduates. He made careful choices in directing class discussions, often using his physical position to direct attention to and away from a student presenter. Through his instruction he encouraged a view of mathematics as a human, social activity. The students in the course appeared to change their conceptions of the nature of mathematics and of proof as a result.

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LINKING GEOMETRIC ALGEBRAIC AND COMBINATORIAL THINKING

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Over the past two decades, researchers have focused on the ways that students think about functions (Dubinsky & Harel, 1992; Leinhardt, Zaslavsky, & Stein, 1990; Romberg, Fennema, & Carpenter, 1993). This research has shown that many students conceptualize functions as mathematical objects that (a) can be represented by a formula, (b) are differentiable or smooth, and (c) are continuous. These conceptions can cause difficulty as students enter calculus and are asked to begin considering functions that may not fit any of the above three criteria. This poster will be a presentation of a portion of a study designed to investigate students' conceptual models of continuity.

Participants in this study were 32 high school Advanced Placement (AP) calculus students. They were drawn from two schools adjacent to a midsize city in the northeastern United States. The first school, Monroe School (all names are pseudonyms), is an independent, private school serving students in grades pre-K through 12. Twelve of the student participants were from this school. The students were drawn from two classes, with six out of 17 from each section participating. At Monroe school, at the time the questionnaire was given, the seniors had already finished for the year, so all responses are from students in the 10^{th} and 11^{th} grades. The second school, Washington High School, is a public school serving grades 9 through 12, with approximately 700 students. Twenty of the student participants were drawn from this school. These students were also drawn from two classes, with 16 out of 20 participating from the first section and four out of 12 participating from the second section.

In analyzing the questionnaires, I began by recording coded answers to each of the problems from the questionnaire. For example, for the first problem, I recorded whether the student had said that the function was continuous or discontinuous, and also a code for their reasoning. Once I had this data compiled, I looked at the frequencies of different responses among the entire sample, and the subsets of the sample corresponding to the individual classes, and to the schools.

Findings confirmed that students have difficulty distinguishing continuity from the existence of the function and the function being differentiable. Results elaborated the contextual dependence of reasoning about continuity that suggests a link between the functional representation used and students' determination of whether a function is continuous. The poster will contain the complete questionnaire, as it was given to students. Additionally, data in tabular form from the analysis of the questionnaire will be provided. The poster will be entirely in paper format.

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Research Methods

USING STUDENT VOICE TO DECONSTRUCT TRADITIONAL STRUCTURES OF COOPERATIVE MATHEMATICAL PROBLEM SOLVING

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With all of its history and traditions, cooperative learning is a firmly established pedagogy present in both classrooms and curricula. A significant body of research and literature claims the implementation of cooperative learning in mathematics classrooms results in increased achievement, motivation, and social skills among students. Influenced by this tradition and research, I incorporated cooperative learning into my secondary mathematics classrooms for a decade. Closely following the methods proposed by Johnson & Johnson (e.g. 1990), I observed the positive results among my students reported in research studies and discussed in literature. For my students, the most surprising result was the increased enthusiasm they had for learning mathematics with their peers. They often and excitedly spoke of this phenomenon to each other and to me.

It was not until I led workshops for fellow teachers that I began to question some assumptions I had made about cooperative learning. As teachers implemented procedures discussed in our workshops into their own classrooms, they reported a variety of reactions and results. Frustrated with their perceived lack of success with cooperative learning, these teachers began asking difficult questions—What exactly is cooperative learning? Why does it work? How does it work? Why doesn't it work for me? Disconcerted by my lack of answers to questions so fundamental to cooperative learning, I began my search for a different perspective and new understanding of the pedagogy I so zealously advocated.

Wrestling to find answers, I went back to the research studies that touted the benefits of cooperative learning. In general, the research on cooperative learning in mathematics classrooms fell into one of three categories: outcome based research on the effects of cooperative learning on students, interviews and surveys on students' opinions of cooperative learning experiences, or observational studies about cognitive processes of students during cooperative learning activities. A poststructural critique framed my review of this familiar literature with questions that challenged my beliefs and assumptions about the results, methodology, and implications of previous cooperative learning research. As I read, new questions developed that propelled the direction of this study. Was I so attached to positive results of research that I failed to notice the absence of a common definition for cooperative learning, and thus making it difficult to

synthesize results? Was I so sure that my students' enthusiastic emotional responses to cooperative learning reflected increased mathematical thinking and learning? Was there an "unwillingness [on my part] to read and think about the theories that … critiqued [my] fondest attachments and… the effects on real people of whatever system of meaning [my] attachments produce"(St. Pierre, 2000, p. 500)?

As a result of this poststructural critique, I became aware that while researchers were careful to define cooperative learning by a set of procedures and to provide detailed descriptions of observed cognitive processes, students themselves were rarely asked to define or explain the cooperative learning experience in more than affective terms. Students have been tested, observed, videotaped, and analyzed, yet their voices seem somehow missing in the literature. It is important to understand students' social, emotional and mathematical experiences of cooperative problem solving team from students' perspectives. With the goal of giving students a voice in educational literature and research, this study was designed as research with (not on) mathematics students in order to better understand how the experience of mathematical cooperative problem solving affects the mathematical activity of individuals within a group. Believing that providing students an opportunity to engage in their own poststructural critique of cooperative learning promotes a synergistic view of their experiences, the following research questions framed this inquiry:

1. How do students engage in and experience cooperative mathematical problem solving?
2. What binary tensions are present or emerge within cooperative mathematical problem solving?
3. How are these tensions related to students' individual mathematical activity?

The research design of this qualitative study with three female college students reflected an interpretive, constructivist paradigm. Data collection centered around three videotaped problem solving sessions. As a group, the participants met once a week in order to investigate a mathematical problem. Each problem solving session was immediately followed by a group interview in which the participants discussed their mathematics, the roles and effects of group members, ways in which each participant felt helped or hindered in her mathematical thinking, group problem solving strategies, etc. The following day each student participated in an individual, ninety-minute interview in which she and I viewed the videotape together pausing often to discuss specific instances pointed to by both each participant and by me. Because the research design was emergent and analysis was ongoing, I transcribed and initially analyzed data between sessions. The participants also helped to analyze data as they discussed previous interviews both with me and with each other. Our combined observations and perspectives directed discussions and influenced interview protocols.

The next phase of data analysis occurred as I addressed my first research question. I used the data from the interview and videotape transcripts to represent the experiences of three PSSs in detail from the perspective of both the participants and myself. Reflexive analysis continued as I used the next two research questions and theoretical frameworks as tools to explore further both the raw data and the newly written data stories. Using Earley's (1997) theory of Levels of Consciousness, I first identified binary tensions (a perceived choice between the good of the group of the good of the self) felt by the participants during mathematical cooperative problem solving experiences. Examples of factors that contributed to these tensions occurred within three general components of cooperative learning: the environment, the group members, and the individual. The size, number and access to manipulatives along with roles that group members took on were the aspects of the environment promoting or creating self/other tensions. Issues of being polite (or being selfish), sharing (or not sharing) ideas, resisting (or submitting to) perceived power were shared by the participants as potentially problematic aspects of working with group members and were often difficult to reconcile. As the students reflected on and discussed the notion of individuality (the place of the self) within a cooperative group, they agreed that both a desire for individual ownership of the mathematics and a subsequent search for autonomy led to yet other binary tensions.

Once binary tensions were identified, I investigated ways in which these tensions affected the individual mathematical activity of the participants. As the students articulated how specific tensions were affecting their mathematics, they individually and collectively found ways to change their cooperative learning practices to protect or enhance their individual mathematical thinking. By deconstructing (Derrida, 1997) the individual experience of cooperative learning, the participants began to rethink self/other binaries tensions at a level of conscious participation. As they became aware of both levels of participation (the individual and group activity), the participants expressed and demonstrated a sense of freedom to move between these while simultaneously being aware of the other. It was this recognition and freedom that provided a stronger sense of self along with a stronger sense of community as the participants investigated mathematics together.

Traditional cooperative learning literature and suggested pedagogical strategies therein contribute to, and in some cases create, self/other binary tensions that inhibit individual mathematical activity. Thus, the need to continually question the traditions and structures of cooperative learning is vital. Deconstruction, however, is "not about tearing down but about rebuilding; it is not about pointing out an error but about looking at how a structure has been constructed… and what it produces" (St. Pierre, 2000, p. 480). In order to illustrate both the limits and the possibilities cooperative learning creates, I will present the conclusions of this study within the framework Johnson and Johnson's (1990) proposed structures. The five previously noted conditions they claim must be present for effective cooperative learning in the mathematics classroom were:

4. Teachers must clearly structure positive interdependence within each student learning group.
5. Students must engage in promotive (face-to-face) interaction while completing math assignments.
6. Teachers must ensure that all students are individually accountable to complete math assignments and promote the learning of their groupmates.
7. Students must learn and frequently use required interpersonal and small-group skills.
8. Teachers must ensure that the learning groups engage in periodic group processing. (pp. 105-106)

The paper presentation will be based on one of the central principles of Derrida's deconstructive methodology. He "examines a hierarchical binary opposition… in which one term is privileged over the other… and reverses the binary opposition by reprivileging the other" (Graves, 1998, 2nd para). As illustrated in this study, I believe the group has become the privileged term within cooperative learning traditions. Thus, I will discuss each of Johnson and Johnson's above conditions with respect to its tendencies to promote self/other binary tensions and to the possibilities of re-privileging the individual within a group in ways that enhance individual mathematical activity.

WORKING WITH LEARNER CONTRIBUTIONS: CODING TEACHER RESPONSES

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This paper comes from a larger project in which I am trying to understand different ways in which mathematics teachers work with learners thinking. A broader goal of the project is to find ways to describe pedagogy. Drawing on work on classroom discourse, I develop a coding scheme for analyzing teacher moves, particularly teacher moves that follow up on learner contributions. An analysis of video data of four South African teachers shows that the codes do distinguish between the teachers, in ways that go beyond superficial distinctions such as 'reform' and 'traditional' pedagogy.

Introduction

This paper comes from a larger project in which I am trying to understand how mathematics teachers work with learner thinking. A key purpose of this work is to find appropriate ways to describe pedagogy in mathematics classrooms in a context of curriculum change. Such descriptions are important if we are to understand the influence of pedagogy on learning outcomes. The lens of teacher responsiveness to learner contributions is useful for a number of reasons. First, understanding teacher-learner interaction is important for understanding how teaching and learning happen in classrooms. Second, working with learner thinking is key to many reform visions in different countries. Third, many teachers who might not be considered "reform" teachers, nevertheless do work with learner thinking. So teacher responsiveness is a lens that cuts across narrow definitions of pedagogy such as "traditional" and "reform", but at the same time captures some of the key visions for curriculum change in many countries. In what follows, I will describe the context of the study, the theoretical and methodological tools on which the study draws, and a coding system for teacher moves developed by the study.

Context of the study

The subjects in this study are five secondary school mathematics teachers and their learners (one grade 10 or 11 class for each teacher), in five differently resourced schools in Johannesburg, South Africa. For the purposes of this paper, data from four of the teachers was analyzed. Two of these are in schools that are in poor socio-economic areas, are under-resourced, and serve exclusively black learners (Mr. Nkomo and Mr. Peters). One is in a school in a lower-middle class area, with some resources and with a racially diverse learner profile (Mr. Daniels). The fourth is in a private school serving very wealthy learners, who are predominantly white (Ms. King). Each of the four teachers has between 7 and 15 years of mathematics teaching experience in secondary schools. They were selected from a larger group who were enrolled in an in-service degree program at Wits University in Johannesburg. The original five teachers had expressed particular interest in working in a study that addressed the teaching and learning of mathematical reasoning in their classrooms.

South Africa has a national curriculum, and in the years after 1994 much work went into developing a post-apartheid curriculum which would signal a clean break with the past (Jansen, 1999a). This curriculum embodies similar ideas to the "reform" visions in other countries, in particular responsive and relevant pedagogy. Given the positioning of the new curriculum as the post-apartheid, liberatory curriculum, many teachers are aware of and committed to the ideals and ideas in the curriculum. However, since much teacher development around the new curriculum is at a generic pedagogical level rather than subject related, many teachers' understandings of the pedagogical implications of the ideas are relatively superficial (Chisholm et al., 2000; Taylor & Vinjevold, 1999). I therefore thought it important to work with teachers who were reasonably well informed mathematically. While my sample of teachers is "special" in this way, my initial classroom observations and interviews with the teachers suggested a range of teaching styles as well as a range of mathematical and pedagogical content knowledge among them.

Describing Pedagogy

Pedagogical practice is extremely complex and not easily described. An important methodological issue is the grain size of the analytic tools (Boaler & Brodie, 2004), i.e. how broadly or how closely do we describe and analyze practice. Studies that compare effects of teaching practice on student outcomes tend to use very broad descriptors of practice. A key set of descriptors, used particularly in the United States, distinguishes between 'traditional' and 'reform' practice (Boaler, 2002; Hickey, Moore, & Pelligrino, 2001; Hiebert & Wearne, 1993). However, what counts as 'reform' practice and how it is recognized differ among studies. In her study in England, Boaler (2002) spent more than a year in each of two schools, provided ethnographic descriptions of traditional and reform pedagogies and related them to student outcomes. Hickey et al (2001) had district support workers decide which teachers used NCTM standards-aligned pedagogy. Hiebert and Wearne used the curriculum as an initial indicator, and then went on to compare the kinds of tasks and discourse in the classrooms. Boaler et al's work distinguishes between traditional and reform curricula, and shows that the same curriculum can give rise to very different teaching approaches (Boaler & Brodie, 2004). In South Africa, 'traditional' and 'reform' have even less meaning, since all teachers are expected to work with a reform curriculum, and at the same time, very few do (Brodie, 1999; Brodie, Lelliott, & Davis, 2002; Chisholm et al., 2000; Jansen, 1999b; Taylor & Vinjevold, 1999). Moving away from the rhetoric and the dichotomy of 'traditional' and 'reform', Askew et al (1997) use three terms to describe orientations towards teaching mathematics: connectionist, transmission and discovery. They describe these orientations primarily in terms of teachers' knowledge and beliefs and they suggest that knowledge and beliefs are in fact most important in understanding teaching. However, if these are crucial explanatory factors, we should be able to see how they play out in practice, otherwise we have no way of understanding how they impact on learning.

In this paper, and the larger study from which it is drawn, I am looking for ways to describe pedagogical practice that can describe a range of teaching approaches; that can take into account aspects of both traditional and reform pedagogies that teachers might be using; that work at the level of classroom practice; and that allow for comparisons across classrooms. I found the literature on classroom discourse most helpful for my analysis.

Classroom discourse

Almost 30 years ago Sinclair and Coulthard (1975) and Mehan (1979) identified a key structure of classroom discourse, the Initiation-Response-Feedback/Evaluation (IRF/E) exchange structure. The teacher makes an *initiation* move, a learner *responds*, the teacher provides *feedback* or *evaluates* the learner response and then moves on to a new *initiation*. Mehan calls this basic structure a *sequence*. Often, the feedback/evaluation and subsequent initiation moves are combined into one turn, and sometimes the feedback/evaluation is absent or implicit. This gives rise to an *extended sequence* of initiation-response pairs, where the repeated initiation works to achieve the response the teacher is looking for. When this response is achieved, the teacher positively evaluates the response and the extended sequence ends. A number of sequences and extended sequences are organized together at the level of content to form *topically related sets*. Neither Sinclair and Coulthard nor Mehan evaluate the consequences of the IRF/E structure. Other researchers (Edwards & Mercer, 1987; Nystrand, Gamoran, Kachur, & Prendergast, 1997; Wells, 1999) have argued that it may have both positive and negative consequences for learning, depending on the nature of the elicitation and evaluation moves. Edwards and Mercer (1987) identify two "ground-rules" of classroom discourse: teachers ask questions to which they already know the answers; and repeated questions imply wrong answers (also identified by Mehan). Edwards and Mercer argue that these discourse patterns function to mark important knowledge and ideas for learners, and serve to build a joint understanding and context for classroom knowledge. At the same time, because the teacher is looking for particular answers, they serve to limit what can be produced in classrooms, leading to ritual (procedural) rather than principled understandings.

Initiation moves are often in the form of questions, and a number of researchers have focused on teacher questions. Nystrand et al (1997) distinguish between "test" questions and "authentic" questions. "Test" questions aim to find out what students know, and how closely their responses correspond to what the teacher requires. "Authentic" questions on the other hand are questions which do not have pre-specified answers, which convey the teacher's interest in what students think, and which serve to validate student ideas and bring them into the lesson. They distinguish high-level evaluations from the more conventional evaluations of the IRE structure. High-level evaluations ratify the importance rather than the correctness of a student's response, and allow the contribution to modify or affect the course of the discussion in some way (Nystrand & Gamoran, 1991). They also develop the notion of uptake*,* which they describe as follows: many of the teacher's questions are partly shaped by what immediately precedes them; the teacher takes the students' ideas seriously, and encourages and builds on them in subsequent discussion; the teacher's next question is contingent on the student's idea, rather than predetermined; the teacher picks up on student ideas "weaving them into the fabric of an unfolding exchange"; and the student's ideas can change the course of the discussion. Nystrand et al (1997) found a positive relationship between authentic questions, high-level evaluations and uptake, and student learning in Grade 8 literature classrooms.

Researchers in mathematics classrooms have identified a broader range of questions. Hiebert and Wearne (1993) have four categories: recall; describe strategy; generate problem; and explain underlying features. Boaler and Brodie (2004) have nine categories which include: getting information; probing; exploring concepts and relationships and generating discussions. Both these studies show that while traditional and reform teachers both tend to ask a significant number of questions that ask for information and recall, traditional teachers ask only these kinds of questions, while reform teachers ask a broader range of questions, some of which enable conceptual engagement with mathematics. While some of these questions may or may not be authentic, questions that require learners to explore meaning and relationships do distinguish between different kinds of teaching and have positive influences on learning (Boaler & Brodie, 2004; Hiebert & Wearne, 1993). At the level of the feedback or evaluation move, researchers have shown that teachers often begin with more exploratory, higher-order questions and tasks, but teacher and students often work together to reduce the demands of the task, asking narrower questions (Stein, Grover, & Henningsen, 1996) and funneling towards answers (Bauersfeld, 1980). Some reform proponents suggest that a complete shift of the IRE structure is necessary to achieve the goals of student engagement and inquiry. Classroom discussions become more like conversations, with the teacher being a participant in similar ways to the students. However, aside from the enormous challenges involved in creating such conversations, such suggestions ignore the particular roles that the teacher must play in classroom discussion, which entail evaluating students. Some argue that the IRE is a particular form of classroom discourse that can be used for a range of functions (Wells, 1999). O'Connor and Michaels (1996) describe the "revoicing" move, where the teacher repeats or rephrases a student's comment. Revoicing amplifies the student's contribution, and sometimes reformulates it in more precise language, while still maintaining the student as owner of the contribution. In this way revoicing positions the student's contribution in relation to the discipline while simultaneously affirming it. A second function of revoicing is to make a student's idea a focus of the discussion, which facilitates other students' responses to it.

Analyzing Teacher Moves

My coding scheme was developed from my data, informed by the above literature. Since all of the above research was done in non-South African contexts, I looked for ways in which South African classroom discourse is both similar to and different from other, predominantly "first-world" contexts. I saw that my data could be described in terms of Mehan's structure, with one key adaptation for the South African context, which will be discussed below. It was particularly striking how few sequences and how many extended sequences there were, suggesting that the evaluation move and the subsequent initiation move were in fact fused much of the time. Because of the structure of my data, and because of my research interest in how teachers work with learner ideas, I chose to focus on the fused evaluation/initiation move as one move, which became my coding unit. Usually a move coincided with a turn of teacher talk, although there could be more than one move per turn. I chose not to focus on teacher questions as a unit because I wanted to account for all the moves that the teachers made, and not all of the data could be accounted for as questions. Nevertheless, many of the teacher moves are questions, and the research quoted above helped to develop and situate my codes.

My coding system was developed to distinguish between when a teacher follows up on a learner response or does not, and how s/he follows up. The coding system has two levels: level 1 codes are indicated in table 1; level 2 codes are indicated in table 2 and are subcategories of the code *follow up* in level 1.

Table 1: Level 1 codes

Affirm	*Affirms* learner contribution as being right or good. Can be indicated by "yes", "good". A repeat of a previous idea can be *affirm* if it restates a correct contribution for the class in a way that indicates an affirmation of the contribution. *Affirm* often achieves closure of an extended sequence, and is followed by a move to another idea.
Direct	Asks someone to do something or calls on learner. Classroom management is often *direct*, although sometimes it is *inform*, *or follow up* if following up on a breach of a new norm. *Direct* can be more or less directive, and does not actually need to be complied with.
Initiate	Tries to get a mathematical idea from learners but does not directly follow up or respond to previous idea.
Inform	Gives information or explains idea, either mathematical or non-mathematical. If this is in response to a previous contribution, then coded as *follow-up* > *insert*. Meta-statements about what is happening are usually *inform*.

Follow up	Picks up on a contribution made by a learner, either immediately preceding or some time earlier. Usually there is explicit reference to the idea, but there does not have to be. Usually the idea is in the public space but does not have to be, for example if a teacher asks a learner to share an idea that she has seen previously in the learner's work. The learner's idea continues to be part of the conversation. Repeating a contribution counts as *Follow Up* if the discussion continues on that point. If a teacher repeats a contribution to *affirm* it and the discussion ends, then the move is coded *affirm,* not *follow up. Follow Up* can relate to social norms in the classroom, if the teacher follows up on a particular social act and uses it to explicitly teach a norm.

Table 2: Level 2 Codes -Subcategories of *Follow up*

Confirm	Checks whether has heard the learner correctly. There should be some evidence that the teacher is not sure what s/he has heard from the learner, otherwise it is *press*.
Maintain	Maintains the contribution in the public realm for further consideration. Repeats the idea, ask others for comment, or indicates that the learner should continue talking.
Press	Pushes or probes the learner for more on their idea, to clarify, justify or explain more clearly. Does this by asking the learner to explain more, by asking why the learner thinks s/he is correct, or by asking a specific question that relates to the learner's idea and pushes for something more.
Elicit	While following up on a contribution, the teacher tries to get something from the learner. She *elicits* something else to work on learner's idea. *Elicit* should be clearly linked to a previous learner idea, otherwise it is *Initiate.*
Insert	Adds something in response to the learner's contribution. Elaborates on it, corrects it, answers a question, suggests something or makes a link.

Some of these codes are informed by various parts of the literature discussed above. *Elicit* is probably closest to Mehan's (1979) elicitation moves, Edwards and Mercer's (1987) "repeated questions imply wrong answers" or Bauersfeld's (1980) "funneling". I chose to include them under *follow up* and not as a distinct category, because they do represent instances where the teachers follow up on learner ideas in a particular way. It might be that teachers use *Elicit* to follow up particular kinds of contribution, and I need to be able to discern these. It also might be more helpful for teachers and teacher educators to distinguish between different kinds of *follow up*, rather than to exclude a range of moves intended as *follow up* from this category. *Press* comes from the mathematics reform literature that focuses on learners' reasoning and justification. It is similar to Boaler and Brodie's (2004) third question category "probing" and includes Nystrand et al's (1997) "authentic" questions, but is not limited to them. A teacher might *press* when she wants the learner to articulate her thinking more clearly or more deeply, for the learner's own benefit or for other learners. *Maintain* is similar to revoicing, and often involves a repetition or rephrase of the learner's contribution. However in the classes in my study, it did not function in the same way as O'Connor and Michael's (1996) revoicing move, so I did not call it that.

The categories *confirm, elicit, insert* and *press* all function to maintain a learner contribution. The main difference between *maintain* and *confirm* and the other three codes is that *elicit, insert* and *press* transform the contribution in some way while maintaining it as the focus of for the class. *Maintain* maintains the contribution in a way that is very similar to how the learner said it. Even when the teacher rephrases slightly, she does not push for more, insert or try to elicit something more. *Confirm* merely checks the accuracy of the contribution with the learner. *Press* is also different from *elicit* and *insert* in that it stays with the learner's contribution, asking for more, rather than trying to elicit something related or inserting a relevant point. So I can arrange these moves on a continuum as follows: *confirm* is where the teacher makes very little intervention, she merely tries to establish what the learner said; *maintain* is where the teacher makes very little intervention, rather she repeats the contribution, in order to keep it going, either for later intervention or transformation, or for other learners to do something with the contribution; *Press* tries to get the learner to transform her own contribution and can be done with more or less mathematical intervention by the teacher; *elicit* tries to get learners to transform a contribution by contributing something else; and *insert* is where the teacher transforms the contribution by making her own mathematical contribution. Each of these moves serves a range of functions in the classroom and takes different forms. They can be discussed in significantly more detail, but that is beyond the scope of this paper. Examples of each of the moves will be given in the presentation.

An important feature of the discourse patterns in many South African classrooms is "chorusing" (Adler et al., 1999). Chorusing occurs when the teacher pauses at the end of a sentence, indicating that the learners should join in on the final words. If the interaction is going well, a significant number of learners will chorus with the teacher. Often the teacher repeats the chorus afterwards as well. Arguments about the consequences of this move are similar to those discussed above in relation to other ground rules of questioning and evaluation. On the one hand, chorusing serves to mark what is important in the current discourse. It also attempts to keep learners' attention, because they have to be alert as to both when and what to chorus. On the other hand, chorused answers are usually short and very often obvious, so they often only require learners to participate on a superficial level. There was chorusing in two of the four classrooms in my study (Mr. Peters and Mr. Nkomo). I found that the codes I developed could adequately take account of chorusing, because the fact that the learners complete the teacher's move does not detract from the function of the move.

Distributions of Teacher Moves

For each teacher, I coded all the whole-class teaching in one week of videotaped lessons. All of the lessons were transcribed and coding was done on the transcript while watching the video. Codes were checked with two different research groups, one in South Africa and one in the United States. Discussions with these groups led to refinement of the coding scheme. Table 3 gives the distribution of level 1 codes across teachers in percentages and actual numbers. A chi-square test gives $p<0.01$.

Table 3: Level 1 distributions

Teacher	Grade	Affirm	Direct	Follow up	Inform	Initiate	Other	Total
Mr. Daniels	11	4% (11)	22% (63)	61% (174)	5% (13)	2% (6)	6% (16)	283
Mr. Nkomo	11	8% (26)	4% (13)	70% (238)	6% (20)	8% (26)	5% (17)	340
Mr. Peters	10	10% (62)	12% (78)	68% (432)	2% (11)	5% (34)	4% (23)	640
Ms.	10	19%	8% (32)	52% (209)	7%	9% (38)	5% (20)	404

King		(76)			(29)			

Table 3 indicates differences in the total number of teacher moves. In the case of Mr. Nkomo and Mr. Peters, these can be accounted for by the fact that Mr. Nkomo had fewer whole-class sessions than the others, and Mr. Peters had more. Mr. Daniels and Ms. King had the same number of coded minutes, more than Mr. Nkomo and less than Mr. Peters. So Mr. Daniels' smaller number of moves is interesting, and reflects the fact that significant parts of his lessons were devoted to learner-learner conversation, so there were a number of turns of interaction where he did not make alternate moves. This did not happen at all in Ms. King's or Mr. Nkomo's lessons and it happened only occasionally in Mr. Peters'.

The percentages in Table 3 indicate that the teachers look more similar than different at Level 1. The most striking result is the predominance of *follow up* moves in all the classrooms. This suggests that all of these teachers do take account of learners' contributions in some way as they proceed with their lessons. Minor differences across the teachers at this level are the higher percentages of *affirm* for Ms. King and *direct* for Mr. Daniels. More *affirms* show that Ms. King has fewer extended sequences and is more likely to end a sequence with an affirmation. The predominance of *directs* in Mr. Daniels' can be attributed to an extended discussion learner-learner discussion that required more explicit management moves by the teacher. Mr. Peters' combined score on *affirm* and *direct* suggest similar patterns to Ms. King and Mr. Daniels, to a lesser extent on each of the single dimensions than Ms. King (*affirm*) and Mr. Peters (*direct*), but to a similar extent when taken together.

Given the similarities among the teachers at Level 1, and the surprising finding of substantial *follow up*, Level 2 codes which describe the kinds of follow up moves are possibly more important than the Level 1 codes. Tables 4 give these distributions. A chi-square test gives significance $p<0.01$.

Table 4: Level 2 distributions

Teacher	Insert	Elicit	Press	Maintain	Confirm	Total
Mr. Daniels	24% (41)	5% (9)	20% (35)	42% (73)	9% (16)	174
Mr. Nkomo	18% (44)	10% (24)	13% (30)	50% (119)	9% (21)	238
Mr. Peters	24% (103)	23% (99)	20% (85)	30% (128)	4% (17)	432
Ms. King	31% (65)	21% (43)	7% (14)	39% (82)	2% (5)	209

Table 4 shows interesting differences among the teachers. The first point of note is that half of Mr. Nkomo's *follow up* moves are *maintain*, which means that he repeats the contribution, asks others for comment or asks the learner to continue. The other three teachers also do substantial *maintaining*, however, they also do more of *elicit*, *insert* and *press*, which means that as they follow up learner ideas they work to transform them in some way. Ms. King and Mr. Peters *elicit* more than the other two teachers. Mr. Daniels and Mr. Peters both *press* more than the other two teachers. Also noteworthy is how seldom Mr. Daniels *elicits* and how seldom Ms. King *presses*. Finally, all of Mr. Daniels, Ms. King and Mr. Peters do a reasonable amount of *inserting*.

Conclusions and Implications

The above distributions provide a first level description of pedagogy. Further work will look at sequences of moves in relation to particular learner ideas. This will help to provide more connected descriptions. However, even this broad description gives some useful information. The predominance of *maintain* moves in Mr. Nkomo's lessons suggests that he takes a relatively neutral stance. Mr. Nkomo looks different from the other teachers and his pedagogy seems to fit a profile of a less interventionist teacher. How and when he *maintains* contributions in relation to when he *presses*, *elicits* and *inserts* may illuminate the extent of his neutrality. More *affirms* and fewer *follow ups* in Ms. Kings lessons might suggest a more 'traditional' teacher. However, she still does a significant amount of *follow ups*. Her predominant *follow up* moves are *maintain* and *insert*, suggesting a possibly interesting mix of some neutrality and some explicit intervention. The predominance of the *maintain* move for all the teachers suggests that it is worth following up the extent to which it functions similarly and differently across the classrooms. This will be reported in subsequent papers.

Mr. Peters and Mr. Daniels both *press* for 20% of the time. Pressing learner thinking is a move often associated with reform orientations. It is likely that teachers press in different ways, and how and when they press will be useful to explore. Some initial work suggests a continuum from more neutral presses: "can you say more" to more specific presses: for example when a learner suggests that another learner made a mistake, the teacher presses with "what was their mistake". Other *presses* position learners in relation to each other, for example: "is what you're saying the same or different from her".

An important issue in relation to all of the teacher moves is the extent to which these constrain and are constrained by particular learner contributions. So another piece of the analysis will be to categorize learner contributions and look at the teacher moves in relation to these. This will show whether particular concentrations or sequences of moves enable or are enabled by particular kinds of learner contributions.

As part of the broader project to describe pedagogy, these codes can serve a number of purposes. First, together with the codes of learner contributions, they may enable relatively 'global' descriptions of practice, which account for a range of teaching practices without dichotomizing into 'traditional' and 'reform'. These global descriptions could then be linked to learning outcomes. Second, as part of a quantitative analysis, distributions of the individual codes could be linked with learning outcomes. In order to achieve the above, the codes would need to be applied to a broader range of teachers, both in South Africa and elsewhere, in order to improve their robustness. However, based on the coding of even this small sample, these codes provide a way into describing teacher-learner interaction that could be useful for teacher-education programs. Thinking about a range of follow up moves may enable teachers to build on the follow up work they are already doing and work with new possibilities.

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LIFE STORIES AND CONCEPT MAPS: A MEANS FOR UNDERSTANDING MATHEMATICS BELIEFS AND PRACTICES

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Over the past decade, narrative inquiry and life history research have become a widely utilized methodology in psychology and personality research (McAdams, 1993), as well as in studies of teaching and teacher development (Beattie, 1995; Carter, 1993; Clandinin & Connelly, 1994, 2000; Goodson & Spikes, 2001; Goodson, 1992; Hatch, 1995). More recently this methodology has emerged in the field of mathematics teacher development and change (Drake, 2000, 2002; Drake, Spillane & Hufferd-Ackles, 2001; Drake & Hufferd-Ackles, 1999; Polettini, 2000; Reeder, 2002). Specifically, studies have utilized mathematics life histories as a viable means for understanding the impact of past mathematics experiences on current beliefs and teaching practices (Drake 1999 – 2002, Reeder, 2002). Drake (1999) asserts that mathematics life histories "may be a particularly useful method for understanding teachers' belief systems, one which reduces the frequently cited disparity between teachers' professed and attributed beliefs" (p.714). In a related body of research, concept mapping techniques have been used as an alternative means for teachers to express and pictorially represent their understanding of the interrelationship between beliefs and mathematics practices (LeSage,1999; Roulet,1998; Raymond, 1994). The rationale for employing either methodological tool is to develop a thorough understanding of the individual; their circumstances and beliefs, as well as factors which influence their development.

Given the methodological promise of each technique used independently, it seemed feasible to marry mathematics life histories with concept mapping to develop a more thorough understanding of teachers' beliefs, practices and factors influencing both. Each methodology provides unique, yet related data for analysis. Concept maps provides teachers with the opportunity to develop a schematic representation of past and present influences on their current practices and beliefs. By contrast, mathematics histories provide a means for understanding the origin and duration of teachers' current beliefs, not only their currently held beliefs and assumptions.

The Methodological Tools

Math history interviews are designed to elicit teachers' understandings of their experiences learning and teaching mathematics and to highlight significant events which have influenced the narratives they re-tell. The interview protocols used in this study are an abbreviated version of those used by Drake (2002), which were based on McAdams' (1993) original works.

The concept mapping activity was modified from earlier studies by Raymond (1997) and LeSage (1999). Participants were provided with the key components of the conceptual framework guiding the study, including: Mathematics Beliefs, Mathematics Teaching Practice, Prior School Experiences, Immediate Classroom Situation, School Factors, Teacher Confidence, Creativity and Personality, then asked to create a schema to best represent the influences on their mathematics beliefs and practice. The participants were encouraged to add, remove or refine categories and components as they wished. Through observing the participants create their models, listening to their meta-cognitions, and asking additional and/or clarifying questions, significant data and insights are gleaned into the teachers' views of the inter-relationship between beliefs, practice, past experiences, confidence, creativity and personality.

Discussion

Through combining mathematics histories and concept mapping activities it is possible to develop a more comprehensive understanding of teacher's past experiences and the impact of those experiences on the development of espoused beliefs. Through the process of retelling their stories and completing the mapping activity, the participants developed a more thorough understanding of themselves as evolving teachers. For example, through verbalizing their thought process while creating the concept map, participants revealed beliefs and experiences with mathematics that were not expressed overtly during the math history interviews. Thus, participants were provided with alternative means of expressing their ideas – verbally and pictorially. The concept mapping activity also provides "access" to the participants meta-cognitions. As the participants placed each component within their developing framework, they expressed their thoughts aloud. Thus allowing access to their thinking – this opportunity provided additional information that was yet untold concerning their mathematics narratives.

As a more thorough understanding of teachers' past and current experiences with mathematics is developed, a more explicit awareness of their teaching practice emerges. It becomes more apparent why specific pedagogies dominate; why various instructional decisions are made; or what impacts teachers' willingness to change their practices. Thus, through listening to teachers' stories and meta-cognitive processes, researchers are provided with a potential means to understand the psychological components of teaching, including personal and professional beliefs about self, and self-perceptions of efficacy, confidence and personality: all of which are key contributors to teachers' ability and willingness to change their teaching practice.

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Socio-cultural Issues

DISPARITIES IN NUMERACY LEARNING FOR FIVE- TO ELEVEN-YEAR-OLDS IN NEW ZEALAND

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The purpose of this study was to explore disparities in numeracy learning as a function of ethnicity, gender and socioeconomic status, and was based on a secondary analysis of data collected initially to evaluate the effectiveness of a major government initiative in mathematics education in NZ. The Numeracy Projects offered whole-school professional development for teachers together with a research-based number framework and individual diagnostic interviews to assess students' mathematical thinking. The study found that European and Asian students started higher on the framework and made greater progress than Maori or Pasifika students. The findings show that "achievement gaps" between Maori/Pasifika students and others increased slightly, rather than narrowed as intended. The importance of culturally responsive pedagogy is discussed.

New Zealand, like many other Western countries, responded to its poor results on interntional comparisons of mathematics achievement (eg, SIMS, TIMSS) by focusing attention on mathematics learning and teaching in schools, putting a particular emphasis on numeracy and the usefulness of mathematics for everyday life. Professional development (PD) programs were created for teachers working with students at various levels of the education system (Early Numeracy Project [ENP]: Yrs 1-3; Advanced Numeracy Project [ANP]: Yrs 4-6; Intermediate Numeracy [INP]: Yrs 7-8; Secondary Numeracy Project [SNP]: Yrs 9-10), beginning in 2000. In 2002, a program in te reo Maori (the Maori language) was developed for use in Maori medium settings (Te Poutama Tau; Christensen, 2003). The PD programs used a research-based number framework to describe progressions in mathematics learning, and individual task-based interviews to assess children's mathematical thinking (see New Zealand Numeracy Project Material, 2004). The Numeracy Projects sit within a wider Literacy and Numeracy Strategy, which has several major goals (Ministry of Education, 2001). These include raising expectations for students' progress and achievement, lifting professional capability throughout the education system, and developing community capability by encouraging and supporting family and others to help students learn.

The initial analyses looked at single variables, such as ethnicity, gender, and socio-economic status, to ascertain whether or not particular student subgroups benefited from the programme (Higgins, 2002, 2003; Irwin, 2003; Thomas & Ward, 2002; Thomas, Tagg & Ward, 2003). Comparisons were made of the gains in framework stages from the beginning to the end of the project. It soon became apparent that the framework stages did not constitute a linear scale, and it was easier to progress through several stages for those whose starting point on the framework was low. Indigenous and minority students (ie, Maori & Pasifika) students made gains of about the same magnitude as those of European or Asian students. However, Maori and Pasifika students tended to begin lower on the framework than European or Asian students, and hence should have made larger average gains, if they had been benefiting to the same extent. Using gain scores as a measure of improvement disguised the fact that Maori and Pasifika students made less progress than European and Asian students who had started at the same point on the framework (see Young-Loveridge, 2004). Differences in the distributions of students at the various stages on the framework as a function of ethnicity, gender, and socio-economic status could provide a more telling picture. The initial evaluation analyses did not consider the possibility of several variables having a cumulative impact on students' progress. The analysis reported in this paper looked at the impact of multiple variables on students' numeracy learning.

Method

Participants

Data on more than 100,000 New Zealand students in years 1 to 6 (ENP & ANP projects) was forwarded to the Ministry of Education in 2003. The composition of the 2003 cohort was approximately 60% European (Pakeha), 20% Maori (the indigenous people of NZ), 10% Pasifika (of Pacific Islands descent), and 5% Asian. The remaining 5% were from "other" ethnic groups.

Procedure

Data on students' number knowledge and strategies was gathered by their teachers using individual diagnostic (task-based) interviews (the Numeracy Project Assessment: NumPA) and forwarded to a secure web-site for later analysis by project evaluators. The interview had been developed alongside a number framework, outlining progressions in the number knowledge and mental strategies used by students to solve problems (see Appendix A). Students were assessed at the beginning of the professional development program, and then again at the end (Note: Data for stages 0 to 3 was aggregated into a single level). The analysis explored patterns of performance at the beginning and end of the project, and progress (gains relative to initial framework stage) as a function of ethnicity.

Results

The research on which this report is based set out to explore the impact of the numeracy project on students' numeracy learning by looking at both *absolute* (initial and final framework stage) (see Figure 1), and *relative* (final framework stage as a function of initial framework stage) performance on the strategy component of the number framework (see Figure 2). The analysis explored these two outcome measures as a function of ethnicity, gender and socio-economic status.

Figures 1 shows that of the four main ethnic sub-groups, Asian students started the project with the greatest proportion at stage 6, the top stage of the framework for Addition/Subtraction (11.5%), followed by European (6.4%), then Maori students (2.7%), and Pasifika with the least (1.8%). It is also evident from Figure 1 that although all four groups improved from the beginning of the project to the end, the relative differences between groups remained the same, or further increased; that is, Asian students showed the greatest improvement and Pasifika students the least. A similar pattern was found for Multiplication/Division and for Fractions/Ratios, with more Asian and European students at the upper framework stages and a greater increase in the percentages who had reached one of the upper stages by the end of the project. There were more Maori and Pasifika students at the lower stages on the framework and a smaller increase in the percentage of Maori/Pasifika students who had reached one of the upper stages by the end of the project.

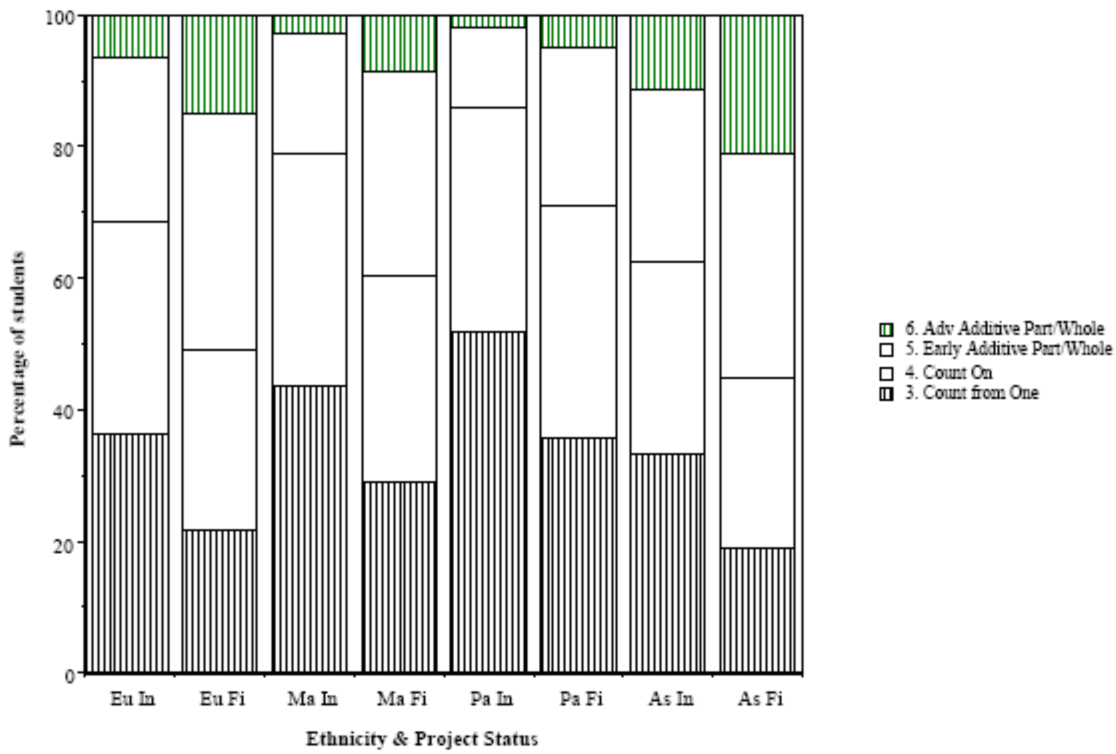

Figure 1. Percentages of Year 1-6 students at each framework stage on Addition/Subtraction at the beginning (In) and end (Fi) of the project as a function of Ethnicity (Eu: European, Ma: Maori, Pa: Pasifika, As: Asian)

Table 1 shows the increase in the percentage of students at particular framework stages from the beginning of the project to the end. It is clear from Table 1 that the increase in proportion of Asian students at stage 6 (Advanced Additive Part/Whole) was much greater (9.6%) than the corresponding increases for the other ethnic groups. The smallest increase was for Pasifika students (3.1%). The increases at stage 6 were consistently greater for boys than girls for all four sub-groups.

Table 1
Increase in the percentage of Yr 1-6 students at framework stage 6 on Addition/Subtraction from the beginning to the end of the project as a function of ethnicity, gender, and socio-economic status (SES)

Stage	European *(n=66702)*	Maori *(n=25770)*	Pasifika *(n=10842)*	Asian *(n=5454)*	Overall *(n=113,573)*
Boys	10.1	6.5	3.3	10.2	8.5
Girls	7.4	5.0	2.9	9.0	6.5
Low SES	7.5	5.2	2.6	6.7	5.2
Medium SES	8.6	6.4	4.5	10.9	8.0
High SES	9.9	8.6	6.5	11.0	9.8
Overall	8.7	5.8	3.1	9.6	7.6

Figure 2 presents the percentage of students who progressed to a higher framework stage for Addition/Subtraction as a function of ethnicity and initial stage (patterns of progress). Figure 2 shows a clear advantage for Pakeha/European and Asian students over Maori and Pasifika students in terms of the proportion at a particular initial stage on the Number Framework who progressed to a higher framework stage. In most instances, Pasifika students made the least progress of the four sub-groups. The corresponding data for multiplication/division, and fractions/ratios was very similar, with Asian and European students at particular initial framework stages making the most progress and Maori and Pasifika students, the least. An analysis of the impact of gender and socio-economic status showed that being female and being from a school in a low SES area was also disadvantageous for students, and further exacerbated the negative impact of ethnicity.

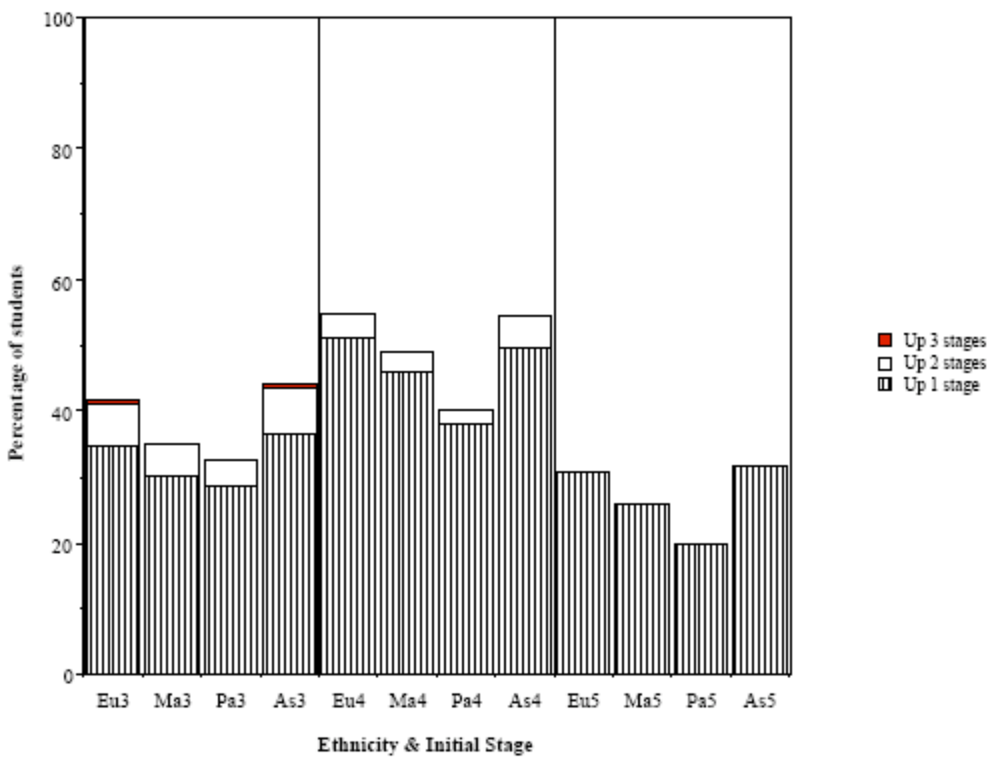

Figure 2. Percentage of Yr 1-6 students who progressed to a higher framework stage for Addition/Subtraction as a function of Initial Stage & Ethnicity

Discussion

The data analysis showed consistently that European and Asian students benefited more from their teachers' participation in a professional development program focused on numeracy learning than did indigenous and minority students. This kind of pattern, referred to as the "Matthew Effect" because "the rich get richer and the poor get poorer" (relatively speaking), was identified in the field of literacy more than a decade ago (eg, Stanovich, 1986). While there is clearly concern about disparities in mathematics achievement as a function of ethnicity, gender, and socio-economic status, little has been done to explore their impact systematically, or to help teachers better meet the learning needs of indigenous and minority students.

Equity is one of six principles proposed in the "new" Principles and Standards document of the National Council of Teachers of Mathematics in the US, and is seen as being essential to improving the mathematics education of students (NCTM, 2000). However, very little consideration has been given to the cultural appropriateness of mathematics pedagogy. Tate (1994) asserts that "connecting the pedagogy of mathematics to the lived realities of … students is essential to creating equitable conditions in mathematics education" (p. 478). Unfortunately many students get the message that school mathematics is a subject that is divorced from their everyday experiences and from their efforts to make sense of their world, the result of so-called "foreign pedagogy". According to Tate, "the curriculum and pedagogy of mathematics have been and continue to be eurocentric precepts that exclude [minority students'] experiences" (p. 479).

New Zealand academics Bishop and Glynn (1999) have written about eurocentrism, arguing that mainstream efforts to address cultural diversity in New Zealand have been "singularly inadequate" because of the way that racism is embedded in the fundamental principles of the dominant (European) culture. The NZ Curriculum Framework document states that all students will be provided with equal educational opportunities, and "all programmes will be gender – inclusive, non-racist, and non-discriminatory, to help ensure that learning opportunities are not restricted" (Ministry of Education, 1993, p. 7). The Framework document also states that "the school curriculum will be sufficiently flexible to respond to each student's learning needs [and] to a new understanding of the different ways in which people learn" (p. 6). While such rhetoric is laudable, there is substantial evidence to show that much educational practice falls short of these goals. The findings of this study raise questions about the cultural appropriateness of current teaching practices in mathematics.

Some mathematics educators such as Willis (2000) and Pinxten (1994) have questioned the appropriateness of approaches developed by the dominant (European) culture for indigenous students. Willis has written about the way that some Australian Aboriginal children can quantify collections of eight or nine objects at a glance (by subitizing), yet seem unable to count in the conventional sense. Many of the frameworks developed by education systems across the world (including those used in NZ and Australia) begin with counting-based stages, and progress to derived number facts (or part-whole thinking). Hence, some students may be disadvantaged by the assumption that counting must come before quantification. Further research is needed to explore the possibility that indigenous and minority students can develop an understanding of the number system by means other than verbal counting; for example, spatial visualization of number patterns.

Some NZ academics (eg, Clark, 1999) have suggested that teaching practices in mathematics classes need to be changed to be more inclusive of Maori students. This includes having less formality and competition by getting students to work in groups, and taking mathematics outside the classroom, using culturally appropriate and contextualized examples, resources and traditions, and helping teachers understand that mathematics is not the preserve of Western or Asian cultures – traditional Maori culture "was knowledgeable and skilled in many forms of mathematics" (Clark, 1999, p. 36).

There is considerable evidence to show that teachers' expectations of students' achievement have an impact on students' performance. Many elementary school teachers have reported having had negative experiences of mathematics learning at school themselves, and this has resulted in a lack confidence and enthusiasm for mathematics. Evaluations of the NZ Numeracy Project have shown consistently that teachers' confidence and professional capability has improved substantially from their involvement in the Numeracy Project. It may be that, before teachers can effectively tailor their teaching to the individual needs of their students, they need to have sufficient confidence and pedagogical content knowledge to teach mathematics effectively to majority group students. It is ironic that a project which was designed to narrow the gap between the most and least capable students in mathematics has, if anything, led to increased disparities. Building the professional capability of teachers in mathematics is an important first step. The challenge now is to sensitize teachers to the particular learning needs of indigenous and minority students.

Acknowledgements

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Appendix A

The New Zealand Number Framework

0. Emergent
Cannot count a collection of objects

1. One-to-one Counting
Can count a single collection, but cannot use counting to join or separate collections

2. Counting from One on Materials
Counts all objects in both collections to work out the answer to an addition or subtraction problem

3. Counting from One by Imaging
Can image visual patterns of objects (visualisation), but counts all to work out solution

4. Advanced Counting
Counts on from one collection to add the second

5. Early Additive Part-Whole
Uses knowledge of number properties to break numbers apart (partition) & recombine them to work out solution

6. Advanced Additive Part-Whole
Chooses from a range of part-whole strategies to solve addition & subtraction problems, and begins deriving multiplication from known facts

7. Advanced Multiplicative Part-Whole
Chooses from a range of part-whole strategies to solve multiplication & division problems

8. Advanced Proportional Part-Whole
Chooses from a range of part-whole strategies to solve problems involving fractions, proportions, and ratios

PATTERNS OF MOTIVATION AND BELIEFS AMONG BEFORE-PRECALCULUS COLLEGE MATHEMATICS LEARNERS[1]

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While it is generally believed that a student's motivation to learn mathematics is influenced by his or her beliefs about mathematics, few studies have investigated this connection in depth. This study concerns both the motivational characteristics of before-precalculus college mathematics learners[2] and their beliefs about the nature of mathematics, and the connections between the two. The data from this study support three student "types", which have been termed "conceptually motivated," "externally motivated," and "future value motivated." Each "type" is described in terms of psychologically central and derived beliefs that characterize those students' motivation to learn mathematics and their beliefs about the nature of mathematics.

Introduction

Students' motivation to learn has long been one of the standard constructs in educational psychology; for example, Pintrich and Schunk (2002) trace its roots back to the nineteenth century. However, much of this research has conceptualized motivation as a general construct. Fewer studies have focused on the motivational characteristics of learners in specific disciplines, such as college mathematics learners. The present study is an investigation of the motivational characteristics of before-precalculus college mathematics students.

The motivation to learn mathematics does not exist in isolation from other constructs. One variable that might be particularly important is the student's beliefs about the nature of mathematics (Kloosterman, 1996). A student who believes that mathematics is primarily the correct reproduction of procedures for solving well-defined problems in a textbook will likely be motivated to focus primarily on the problems posed for him or her at any one time. Such a student would be unlikely to focus on making connections between mathematical concepts that are not made explicitly for him or her by a teacher or a textbook. On the other hand, a student who believes that mathematics is a search for patterns and a way of making sense of the world may well be motivated to attempt to make such connections between mathematical concepts. While beliefs would seem to have a fairly strong relationship to a student's motivation to learn, little research has documented the connections between these two constructs. Because of the potential importance of beliefs in shaping a student's motivation toward learning mathematics, this study includes an investigation of students' beliefs about the nature of mathematics and the connections between motivation and beliefs.

Theoretical Framework

The present study draws strongly on theories of intrinsic motivation, particularly the self-determination theory of Deci and Ryan (1985), and on the notion of belief systems (Green, 1971; Rokeach, 1968). Deci and Ryan define self-determination as "a quality of human functioning that involves the experience of choice, in other words, the experience of an internal perceived locus of control. It is integral to intrinsically motivated behavior … self-determination is the capacity to choose and to have those choices, rather than reinforcement contingencies, drives, or any other forces or pressures, be the determinants of one's actions. … [It leads people] to engage in interesting behaviors, which typically has the benefit of developing competencies, and of working toward a flexible accommodation with the social environment" (1985, p. 38; emphasis in original). According to this theory, intrinsic motivation occurs "when a person does the activity in the absence of a reward contingency or control" (p. 34). Intrinsic motivation is "manifested as curiosity and interest, which motivate task engagement even in the absence of outside reinforcement or support" (Ryan, Connell, & Grolnick, 1992, p. 170).

In order to operationalize the construct of intrinsic motivation for the present study, other works on intrinsic motivation in education were consulted (e.g., Brophy, 1983; Corno & Rohrkemper, 1985; Harter, 1981; McCombs, 1984; Stipek, 1996). From each of these works, definitions of intrinsic motivation were extracted and then synthesized. The following definition guided the present research:

> Intrinsic motivation is present when students value the possible outcome of a task and work on it for its own sake, not for the sake of some external award. When students are intrinsically motivated to learn, they will generally exhibit the following characteristics:
> - They value learning for its own sake in the absence of extrinsic reward.
> - They exhibit interest in, and curiosity about, the task at hand.
> - They have a sense of personal control or self-determination.
> - They exhibit personal confidence and a reduction of the fear of failure.
> - They use self-directed and metacognitive learning strategies.

We can think of a *belief* as a primarily cognitive assertion held by an individual that is not necessarily consensual (as compared to a knowledge construct) and that has an affective component. Beliefs are organized in belief systems by the individual, since they "come always in sets or groups, never in complete independence of one another" (Green, 1971, pp. 41-42). Belief systems are "quasi-logical" (Green, 1971), because there is a logical structure to beliefs but that structure can differ between individuals. In addition, an individual can hold two beliefs that are logically inconsistent. Beliefs can be identified as being primary or derivative, evidential or nonevidential, central or peripheral (Green, 1971; Rokeach, 1968).

Beliefs about mathematics can be organized into a number of categories. For instance, an individual has beliefs about the nature of mathematics, the learning of mathematics, the teaching of mathematics, and the social context for teaching and learning mathematics. The present study focuses primarily on beliefs about the nature of mathematics. One component of the present study was a literature review on beliefs about the nature of mathematics; among the studies reviewed were Carter and Yackel (1989), Frank (1988), Hatfield (1991), Koch and Smith (1993), Odafe (1994), and Schoenfeld (1989). As a result of this review, six beliefs about the nature of mathematics were identified that had empirical support for being "commonly held" among individuals:

1. Mathematics is mostly computation.
2. Learning mathematics consists primarily of memorizing a set of facts, rules, and formulas. Doing mathematics consists primarily of applying these facts, rules, and formulas to clearly defined problems.[3]
3. Mathematical knowledge is certain and unfailing; it comes "from above" (from those who really understand the subject) and cannot be questioned.
4. A mathematics problem can either be done in 5-10 minutes at most, or it cannot be done at all.
5. There is usually/always only one (best) way to do a mathematics problem.
6. Some people just have a "mathematical mind"—they are naturally better at it than others.

These "commonly held" beliefs were the basis for data collection described in the next section.

Methods

The motivational profiles developed in this study were based on a careful analysis of eight students enrolled in a "Topics in Liberal Arts Math" course (taught by the researcher) at a small, private, liberal arts college in New England. This college did not offer any mathematics courses labeled as "developmental" or "remedial." Instead, the Liberal Arts Math course was the entry-level mathematics course for students in those majors that did not require a more technical mathematics course. The student body of the college is predominantly White, as were all eight students in this study. The college attracts primarily "average" students; the "middle half" of SAT scores for the cohort from which these students came was between 910 and 1070.

A number of data sources were used in this study. Each of the eight students took part in a series of five hour-long interviews during the course of a semester. Three of the interviews, conducted at the beginning, middle, and end of a semester, focused on students' motivation to learn. The other two, conducted at the beginning and end of the semester, focused on students' beliefs about mathematics. These interviews were transcribed and coded for the categories of motivation and beliefs indicated in the previous section. Additional data sources included two questionnaires, the Motivated Strategies for Learning Questionnaire (MSLQ; Pintrich, Smith, Garcia, & McKeachie, 1991) and the Mathematics Beliefs Instrument (MBI; Ibrahim,1990); videotapes from the classroom; and photocopies of students' "major" assignments (projects and exams).

Analysis of the data began with a data matrix, in which each of the eight students were rated in terms of the five characteristics of intrinsic motivation to learn and the six "commonly held" beliefs about the nature of mathematics. From this data matrix a second matrix was derived, with the number of "matches" between each pair of students on these characteristics. This analysis suggested that there were three "types" among the eight students. At this point, a preliminary description of the three "types" was developed. The descriptions identify those beliefs that are apparently central to the students, as well as other beliefs that appear to be derived from those central beliefs. The descriptions were then used to reanalyze the data to search for any evidence that would be inconsistent with the descriptions. The descriptions were then rewritten so that they were again consistent with the data. This process of revising the descriptions was continued until there was no data for any individual student placed into the "types" that was inconsistent with the descriptions. More details on methodology can be found in Johnson (2001).

Results

As mentioned in the previous section, the data suggested three "types" of students among the eight in this study. The descriptions include aspects of the students' motivation to learn, as well as their beliefs about the nature of mathematics. Based on the data and the nature of the descriptions, these three "types" are identified below as "conceptually motivated," "externally motivated," and "future value motivated." Among the eight students in the present study, two were characterized as conceptually motivated, four were externally motivated, and two were future value motivated.

Conceptually motivated

Conceptually motivated students believe that mathematics has a practical usefulness, particularly in terms of financial computations that the student believes he or she will need in the future. A second, central belief is that mathematics has a sort of "general usefulness," and that studying mathematics helps one "broaden the mind." This "broadening" can occur through studying any of the concepts of mathematics, but is especially likely to emerge through the study of logical reasoning. The purpose of studying mathematics is to make an attempt to understand its underlying concepts and to apply those concepts in other academic areas.

From these central beliefs follow a number of derivative beliefs regarding mathematics. While the conceptually motivated student may not necessarily have a high level of intrinsic motivation to learn mathematics, this student's motivational orientation is more intrinsic than extrinsic in nature. He or she wants to learn mathematics because the subject is seen as having an inherent value. At the same time, because mathematics is seen as involving reasoning and not primarily computation and memorization, the conceptually motivated student would be expected not to exhibit most of the "common" beliefs about mathematics included in this study. Moreover, because they see mathematics as reasoning and making sense of situations, such a student would be expected not to have an empirical view of mathematics.

Externally motivated

For externally motivated students, a psychologically central belief is that mathematics consists primarily of facts, algorithms, and computations. They perceive a fairly strong "split" between the mathematics they learn in school and mathematics outside of the school setting. While mathematics done outside of the school setting may be of limited use in the student's future life (such as financial calculations), school mathematics is a "meaningless game played with symbols"; it is not an integral part of the student's life. School mathematics is done in accordance with the prescriptions of "the experts," primarily teachers and textbooks, who have "figured out" mathematical truths for us. The role of the student is to imitate and reproduce this body of knowledge.

From these central beliefs follow a number of derivative beliefs regarding mathematics. School mathematics has effectively been determined for the student by "the experts," and is not necessarily related to anything important to the student outside of the classroom. Because the externally motivated student does not necessarily attribute importance to mathematical content, the student will exhibit a low level of intrinsic motivation to learn. Motivation can come only from extrinsic factors, such as grades, which have the function of "verifying" the student's abilities; mathematical knowledge is generally not verifiable by the individual learning it. In addition, because mathematics is seen as consisting primarily of facts, algorithms, and computation, the externally motivated student extends the importance of computation and "doing examples" to the very nature of mathematical truth. The "facts" of mathematics are seen as emerging from a number of examples, and it is possible that in the future some counterexample might be found that disconfirms an accepted "fact." Mathematical truth is thus seen not deductively, but empirically.

Future value motivated

For the future value motivated student, a psychologically central belief is that mathematics is important to learn for one's future. The emphasis is likely to be the mathematics needed in one's chosen career, but this is not necessarily the case. This student sees mathematics as vital for daily functioning; in fact, one can find mathematics "everywhere." The goal in mathematics classes is to learn the content that one will encounter in the future. While mathematical content is valued more for extrinsic than intrinsic purposes (exemplified by the statement "I know I need to know this in the future"), the future value motivated student still sees a genuine importance in learning mathematics.

From these central beliefs follow a number of derivative beliefs regarding mathematics. The student's focus is on learning mathematical content for his or her future. The grade one earns in a mathematics course is thus of secondary importance to the student; since mathematics has a genuine importance, the student should exhibit more of an intrinsic than extrinsic orientation toward learning mathematics. On the other hand, mathematics is valued for its extrinsic, rather than intrinsic, importance. The student's emphasis is on learning technical skills needed in the future. With this emphasis on learning technical skills and the extrinsic importance of mathematics, the future value motivated student is likely not to have questioned the "traditional" nature of the instruction they have received in mathematics in school. Thus this student would be expected to exhibit agreement with most of the "common" beliefs about mathematics considered in this study.

Discussion

Some research (e.g., Mau, 1993) suggests that many instructors of before-precalculus college mathematics courses tend to treat their students as "unmotivated" and undifferentiated with respect to motivation. This conclusion is certainly not borne out by the results of this study. The students in this study exhibited a range of motivational characteristics, as well as beliefs about the nature of mathematics. Those faculty who treat their students as "unmotivated" likely have in mind a profile close to that of the "externally motivated" student described above. It is therefore interesting to note that only half (four out of eight) of the students in this study fit this particular profile.

Clearly, one of the limitations of this study is the small sample size used. In addition, all of the students were from the same school and the same section of one course. While the author believes (based on his teaching experience) that students of each "type" presented here could probably be identified in other populations, further research would clearly be needed to determine if that is the case, and what the prevalence of each "type" might be. It would be particularly interesting to see this study replicated with students in other mathematical settings to determine the prevalence of these student "types."

The author also makes no claim that this catalog of motivational profiles is exhaustive. Further research might well uncover other motivations for students to learn mathematics, and ways in which those students' beliefs about the nature of mathematics help to inform and shape those motivations. For example, in this study there were no students who appeared to hold psychologically central beliefs about the inherent beauty of mathematics as a subject, and how such a student might be motivated to learn mathematics. This and other profiles might emerge from research on other populations of mathematics students.

Endnotes

1. This paper is based on my doctoral dissertation (Johnson, 2001). I would like to thank my dissertation advisor M. Kathleen Heid for the tremendous amount of time and effort she spent helping me make this work better than it would otherwise have been. I would also like to thank the other members of my committee, Drs. Glendon W. Blume, John W. Dawson, Jr., and Cecil Trueblood for their effort, support, and friendship.
2. The phrases "developmental" and "remedial" mathematics are frequently used with a somewhat derisive slant; I have therefore used the somewhat more cumbersome phrase "before-precalculus college mathematics" in the present study.
3. This belief is phrased in terms of learning mathematics. However, this belief concerns the "core" of mathematics; i.e., it consists of facts, rules, and formulas. In addition, one could infer that an individual who holds this belief also believes that the facts, rules, and formulas of

mathematics are unrelated and that the learner must rely on memorizing them because it is not possible to see their connections. Thus for the purposes of the present study this belief is considered to be a belief about the nature of mathematics as well as a belief about the learning of mathematics.

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THINKING, FEELING, ACTING LIKE A MATHEMATICIAN: WOMEN AND PEOPLE OF COLOR IN DOCTORAL MATHEMATICS

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Despite substantial attention paid to diversity in K-12 mathematics, little research has investigated this issue among doctoral students. I argue that doctoral students need to learn more than just the content of their disciplines, but that they also need to learn the practices of their disciplines and to develop identities as members of their professional communities. While there are some obstacles that all graduate students face, women and students of color face additional obstacles that limit their opportunities to acquire knowledge, engage in practices, and develop identities in mathematics. This model is illustrated through an in-depth interview with one female student in a mathematics doctoral program in the US. In the presentation, I will explore this framework in more depth, through interviews with 14 women enrolled in 3 mathematics departments.

While the number of women in advanced mathematics has increased over the past four decades (National Science Foundation, 2004), the proportion of women in mathematics still lags behind their male contemporaries at higher educational and professional levels. In 2001- 2002, only 32% (147 out of 465) of PhDs in mathematics in the US that were earned by US citizens and permanent residents were earned by women (Loftsgaarden, Maxwell, & Priestly, 2003). Also in 2002, women received 21% of doctoral faculty positions filled in mathematics departments in the US, and comprised 19% of the total full-time doctoral faculty in that year (Kirkman, Maxwell, & Priestly, 2003). The picture is even worse for people of color: of the US citizens and permanent residents receiving PhDs in 2001-2002, only 9 (2%; 7 male and 2 female) were Latino, 17 (4%; 10 male and 7 female) were Black, and 2 (0.4%; both male) were Native American (Loftsgaarden et al., 2003). (No data are available on participation in mathematics in higher education based on social class indicators; however, the intersections of race, gender, and class are important and worth consideration.) Admittedly, these statistics give only an approximate picture of how women fare as they progress along the educational path, as these statistics represent different cohorts of students at one fixed point in time, but the statistical picture is compelling nonetheless.

Despite the fact that both the mathematics and mathematics education communities have been aware of these statistical patterns for quite some time, little scholarship has explored women's experiences in doctoral mathematics, and even less has investigated the processes that lead women and people of color on this exodus out of mathematics. There is more to these students' experiences in graduate mathematics than these numbers show. "The question is not only one of retention in doctoral study but the more subtle one of whether women have a graduate experience that is of as high a quality as that of men." (Etzkowitz, Kemelgor, Neuschatz, & Uzzi, 1992, p.158).

A Framework for Understanding Success in Mathematics

To understand why students of some groups do not persist in mathematics at the same rate as others, we need to start with the question, *What does it take to succeed in advanced mathematics study?* Theories of situated learning posit that learning happens through participation in social practices, and that learning is inseparable from that participation (Boaler, 2000). For doctoral students, learning happens as they participate in the communities of practice found in their departments and programs. Wenger (1998) describes three dimensions that define a community of practice: a joint enterprise, mutual engagement, and a shared repertoire. In the community of practice of mathematics doctoral study, these three dimensions entail learning mathematics, developing an identity as a mathematician, and engaging in the full range of practices of graduate school and of professional work in mathematics (Boaler, 2002; Herzig, 2002). Thus, the training of mathematics graduate students requires far more than instruction in mathematics content and engaging in research for a dissertation (Bass, 2003), and success in mathematics depends on more than just mastering the content of coursework.

Bass (2003) argues that while mathematics doctoral programs in the United States provide strong disciplinary training in the core areas of mathematical scholarship, they need to do a better job of preparing students for all aspects of work within the profession of mathematics, including serious professional development for teaching, uses of technology, exposition, developing and pursuing a research program, participation in the local and broader mathematical communities, and development of a "cultural awareness in students of the significance of their discipline in the larger worlds of science and society and of the expectation that they will serve as emissaries of their discipline in the outside world" (p. 775). These categories of professional development call for students to appropriate a range of important skills for functioning as mathematicians, including acquiring mathematical knowledge, developing fluency in the practices of mathematics, and developing identities as mathematicians (Boaler, 2002). However, given that students spend the first 2 or 3 years of their graduate training isolated from the community of practice of research mathematics, the things they learn—what they acquire through their participation in their graduate program—are specific to the experiences they have. For example, they appropriate skills for taking courses and exams and, in some cases, for working as teaching assistants. The nature of the activities in which these students participate gives them only limited opportunities to develop the knowledge, practices, and identities of professional mathematicians.

Learning Mathematics

Two interleaved assumptions are often made in popular discourse about mathematics: that mathematics is a very difficult field of study, and that only some people have the talent required to be successful at mathematics (Herzig, 2002; Love, 2002; Oakes and Franke, 1999, cited in Allexsaht-Snider & Hart, 2001). Based on a review of children's motivation in classrooms, Ames (1992) found that making ability a salient feature of education interferes with students' motivation to learn, their use of effective learning strategies, and their engagement with the content of the curriculum. Not only does a belief in talent as an important predictor of success interfere with student engagement, it also removes the responsibility for instruction from the teachers. This was the perception in one mathematics doctoral program, in which faculty beliefs in the importance of talent led them to virtually ignore doctoral students in their first several years of the program and to describe the purposes of instruction as providing an opportunity for students to discover or prove if they possess that talent (Herzig, 2002). In this way, instruction moves away from fostering the development of mathematicians-in-training, to a structure for "weeding out" students who do not possess particular skills or abilities.

Faced with this type of learning environment, graduate students have cited the obstacles they faced in their efforts to learn mathematics, including limited interactions between faculty and students, absence of connections among mathematical ideas, lack of feedback mechanisms in their courses, and difficulty asking questions (Herzig, 2002; Stage & Maple, 1996).

Developing a Mathematical Identity

An important part of graduate study is adopting the identity of a mathematician, or at least that of a mathematics graduate student (Tinto, 1993). Building students' sense of *belonging* in mathematics has been proposed as a critical feature of an equitable K-12 education (Allexsaht-Snider & Hart, 2001; Ladson-Billings, 1997; National Council of Teachers of Mathematics, 2000; Tate, 1995). A sense that "I belong here" also seems to be critical in the persistence of doctoral students, with several authors arguing that students' integration into the communities of their departments is important for their persistence (Girves & Wemmerus, 1988; Herzig, 2002; Lovitts, 2001; National Research Council, 1992; National Science Foundation, 1998; Tinto, 1993). Unfortunately, women doctoral students often feel that they do not fit in the maledominated worlds of their disciplines (Etzkowitz, Kemelgor, & Uzzi, 2000; Herzig, in press-b).

Engaging in Mathematics Practices

Developing a sense of mathematical identity can also be conceptualized as a process of becoming a "full participant" (Lave & Wenger, 1991) in a community of practice. As students participate in authentic mathematical practice, their sense of mathematical identity is enhanced and they have improved opportunities to acquire mathematical knowledge. Women and students of color may face particular obstacles to participating in mathematics practice. For example, it may be difficult for people of underrepresented groups to participate in a disciplinary and departmental culture that was formed and has historically been populated by a much narrower demographic group; similarly, students with family responsibilities or other commitments outside of the program may have limited access to some of its activities (Etzkowitz et al., 2000).

Women and people of color participating in graduate mathematics

While all students face some obstacles to these dimensions of learning, women and students of color face additional obstacles that limit their opportunities to participate in the communities of practice of their programs and to acquire knowledge, engage in practices, and develop an identity in mathematics (Herzig, in press-a). For example, if a student has commitments to an ethnic, cultural, or family community (as is the case with students who are parents), it may be difficult for her to participate in the activities of the academic community. These competing communities of practice in which students participate can isolate them from the communities of their departments and programs, particularly in the case of programs that are inflexible or are built on narrow models of how students can or should be available to participate in departmental communities. Second, a student who is not accepted by the other community members or who is perceived to have a particular set of skills, abilities, and dispositions—such as is the case of women who are constructed to be lacking in confidence or autonomy—will have fewer opportunities to develop effective relationships with mentors and others. By constraining her interactions with other members of the academic community, these perceptions of her as a learner will indeed make it difficult for her to appropriate the knowledge, practices, and identity of a mathematician (Herzig, in press-a).

In this presentation, I will use this framework of for understanding the participation of women in doctoral mathematics—based on their opportunities to acquire knowledge, learn practices, and develop identities as mathematics graduate students and mathematicians—to analyze the experiences of 14 women graduate students in mathematics enrolled in three mathematics graduate programs at large, public universities in the US. In the present paper, I illustrate the framework through the experiences of one of these women.

Method

This report is based on a single open-ended interview with one female graduate student in mathematics. The interview was conducted as part of a larger study in which female and male graduate students and faculty were interviewed. Along with all the participants, Marta (a pseudonym) was recruited through a mass email sent to all enrolled graduate students in her department, inviting them to participate in an interview about their experiences in mathematics, both in and out of graduate school. She was given an outline of interview topics in advance of her interview, and was encouraged to add things she thought were relevant and delete things she did not wish to discuss (after Burton, 1999). The interviews covered her mathematical "autobiography", her reasons for attending graduate school in mathematics, her interests and goals in mathematics, and her mathematical experiences both in and out of school. She was encouraged to guide the conversation to those aspects of her experiences that she thought were most relevant. The interview was conducted in a private room on campus, outside of the Department of Mathematics. The interview was tape recorded, and the tapes were transcribed. In the text that follows, all quotes are from Marta's interview.

At the time of her interview, Marta was in her fifth year of graduate study, and was working on her dissertation research. In order to protect her anonymity, further demographic details cannot be disclosed.

Women Learning Mathematics

Acquiring knowledge

Most graduate programs in mathematics in the US are structured so that students spend their first several years taking courses and preparing for qualifying exams. The entire structure of graduate training seems to be focused around encouraging students to learn large amounts of mathematics in a short period of time. Although Marta was successful in most of her classes and passed her qualifying exams relatively quickly, she described a number of obstacles to her learning.

When she first arrived at the University, Marta found that she received very little advising about her academic program. Consequently, she ended up in a course that was over her head, and made her first year very difficult for her. Another course was taught in a way that did not inspire her interest, even though this area is ultimately the area she specialized in for her research. "It was not well taught. It was [subject] thought of from a point of view that made what I thought was an interesting subject seem very ugly somehow." The instructor of this course seemed not to care very much about his teaching; he would come to class with notes and would basically dictate his notes to the class, not allowing room for questions or discussion.

Marta had several recommendations for the teaching of graduate courses. Her own graduate coursework had been very narrow, and she took no courses at all in several classical areas of mathematics, which she described as a deficiency. She would have liked to have access to general seminars, structured to introduce younger graduate students to the big ideas in areas outside their own areas of specialty. Further, instructors should make sure that students are doing more than just coming to class and taking notes; instructors should assign homework, hold students accountable for completing assignments, give students feedback on their work, encourage students to form study groups, and make office hours a real opportunity for students to ask questions and discuss mathematics. In many of her classes, she felt these things were lacking, making it more difficult for her to learn mathematics.

Engaging in practices

Marta described a number of ways in which her graduate training did not prepare her to work as a mathematician. She was impressed with one faculty member who acknowledged that graduate students would not understand most of what went on in colloquium, and offered "debriefing sessions" for them to process the talk and "learn the general culture of math."

Working as a teaching assistant was very important to Marta, and she hoped eventually to pursue a career in college teaching in which she could share with undergraduates the joy she found in mathematics. However, her initial experiences in teaching as a teaching assistant were intimidating, as she had had not previous experience and felt overwhelmed by the responsibility and by being at the front of a classroom. She was intrigued by a graduate program at another university where students were eased into teaching over their first several years, "without diving in and hoping I could swim." In part, her interest in teaching was a sense of "justice" to make up for the lack of interest her own faculty showed in their teaching.

She complained that she had had very little opportunity to learn to give mathematical talks, which she recognized as a skill she would need as an academic mathematician. She did not only want chances to present work in her classes, but real training in how to focus a talk and how to make decisions about how to give a talk. She referred to this as "one of the . . . things that graduate students should know but no one ever teaches them."

Marta described several points in her graduate training when she struggled with her research. Particularly when her advisor was out of the country for a semester, she felt that she had no one to speak with about her research. "I didn't feel like I was on my way to my research, even though looking back at it, I had gotten some important things done but I was still new enough to the area that I didn't realize the significance of what I had done. . . . I knew what kinds of problems I wanted to solve but I didn't know how to go about it." Consequently, she felt insecure about her research. When her advisor finally returned, she received "just enough guidance from him" that she was able to solve important aspect of the problem she had been working on, and that gave her the confidence to continue.

She wanted faculty to provide more modeling about how they think about mathematics and how they solve problems. "Researchers could explain or demonstrate how their own thought processes work and how do you go from saying, 'gosh, one should be able to calculate the [mathematical quantity]' to saying 'yes well I'm going to use this technique and then I'm going to look for this kind of evidence?' " Overall, Marta felt that she was not being given an adequate apprenticeship in working as a mathematician, including all of the aspects of research, learning, speaking, writing, and other practices that are part of the professional work of a mathematician.

Developing an identity

Marta had been interested in mathematics since she was very young. While in high school, she participated in several programs for talented and gifted students, and was tutored by a professor at the local university. She entered college as a mathematics major, and attended a summer program that gave her research experience in mathematics. One of the things that appealed to her most about mathematics from a very young age was its abstraction. "Finding consistent patterns and finding abstractions that fit into consistent systems was something that I found very interesting at a younger age." This interest in patterns and abstraction was one of the primary reasons that she decided to attend graduate school. In a sense, she had developed an identity as a "mathematics person."

She did not like the "macho attitude in math" that "when you don't understand you're supposed to figure it out on your own as opposed to asking questions." She also objected to what she perceived as a tendency to present mathematics intentionally to confuse. "What's as important as explaining your work is showing people that you are capable of doing high-powered work by confusing."

Marta had been an undergraduate at a small liberal arts college, and felt very successful at mathematics in her earlier mathematical education. However, throughout her interview, she referred to her doubts about her abilities. She repeatedly described herself as "shy," and spoke of her hesitance to ask questions or to approach faculty or even other students. She had some of her most significant doubts when she was struggling to complete her research. Once her advisor returned from her sabbatical, Marta realized that she had completed a major piece of her thesis, and said, "I felt, OK, I can do this. The reasons I was having trouble were based less on my mathematical ability or lack thereof" and more on learning to manage her time. In this way, the practices in which she engaged (working on research under the direction of her advisor) affected her identity within mathematics.

Marta also described ways that being a woman in mathematics affected her. "I do sometimes walk into a room, look around, realize I'm the only woman in the room, *again*, and it has an effect." She thought she might be shy because she is a woman around not very many women.

Summary

Knowledge, practices, and identity are not independent parts of the learning process, but they are intertwined, and necessarily affect each other in complex ways. For example, Etzkowitz et al. (2000) argue that students can only act independently if they feel safe and accepted. Students who do not feel that they fit in may have more difficulty acting autonomously. In effect, autonomy and independence are double-edged swords for women in mathematics and science.

> Isolated and without interpersonal connection, a woman's ability to be playfully creative is impeded. . . . A gendered 'apartheid system' exists in which many male advisors offer support to male students, be leave women to figure things out for themselves. With no support or connection with an advisor, taking risks in the lab becomes to threatening. People only take risks when they feel safe to do so. In contrast, there is sufficient support and acceptance, by way of informal interactions with male advisors and peers, for male students to enjoy the freedom to be innovative. (Etzkowitz et al., 2000, p. 86)

That is, male students have enhanced relationships with faculty, which provide them with increased opportunities to develop identities within their disciplines, which is a pre-requisite for independent and autonomous work. Denied the same degree of relationships with faculty, female students have a more difficult time acting independently. This was certainly the case for Marta, who had only distant relationships with her advisor and other faculty members, and consequently struggled with the isolation of her work.

Within such a small space, it is impossible to do justice to even one woman's experiences in mathematics. I have tried to illustrate some aspects of Marta's experiences in graduate school that she perceived as obstacles and opportunities for her to acquire the knowledge, practices, and identity of a professional mathematician.

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PROSPECTIVE SECONDARY MATHEMATICS TEACHERS' CONCEPTIONS OF PROOF AND ITS LOGICAL UNDERPINNINGS

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Abstract: *This research study aimed to provide a snapshot of 23 prospective secondary mathematics teachers' conceptions of proof and its logical underpinnings as they were near the end of their preparation programs. When asked to complete proofs based on secondary mathematics content, only 26% of the participants were able to complete a valid proof for the direct and indirect proof items. Results suggest that some prospective teachers may have difficulty teaching proof and reasoning effectively, as outlined by the NCTM* Standards *2000 and the MAA (1998). A recycling effect seems likely as prospective teachers with an inadequate understanding of proof and reasoning return to the educational system as mathematics teachers faced with the challenge of teaching proof.*

Perspective

In the United States, current mathematics education reform efforts call for an increased emphasis in our school curricula on reasoning and proof as a stepping stone toward improving logical reasoning. The *Principles and Standards for School Mathematics* emphasized the need for opportunities in mathematical reasoning and proof for *all* students grades K – 12 and in *all* mathematics content areas (National Council of Teachers of Mathematics [NCTM], 2000). The document deemed these opportunities as essential to understanding mathematics. The goals of the NCTM were supported by a special Task Force within the Mathematical Association of America (MAA) where they recommended that students should have opportunities to learn logical reasoning, develop valid arguments or proofs, and criticize the arguments of others (Ross, 1998).

The NCTM further noted in the Teaching Principle of the *Standards* 2000 that students' understanding of mathematics, their ability to use it to solve problems, and their confidence and disposition toward mathematics are all shaped by the teaching they encounter in school. As a cornerstone to this reform vision, the NCTM (2000), the MAA (Ross, 1998) and the Conference Board of the Mathematical Sciences (CBMS, 2001) have acknowledged that the catalyst for building stronger understandings of reasoning and proof within a mathematics classroom is the teacher. This need for teachers to promote the development of their students' understanding of proof suggests that teachers themselves must have a robust understanding of proof. This robust understanding develops through opportunities to explore, conjecture, develop mathematical arguments, validate possible solutions, and recognize connections among mathematical ideas.

Knowledge of Proof

The question of what knowledge is necessary to facilitate recommended changes within school mathematics has been the focus of many in mathematics education (Ball, 1989; Epp, 2003; Galbraith, 1982; National Research Council, 2001). Ball (1989) and Galbraith (1982) voiced similar concerns about conceptions of prospective secondary mathematics teachers and what these conceptions imply about their ability to teach school mathematics. Galbraith (1982) stated, "Concern has been expressed for the *recycling effect* induced when students lacking in some essential mathematical background, return to the education system as teachers" (p. 91, emphasis added).

There is a real concern about a recycling effect when prospective teachers are faced with the challenge of teaching proof and reasoning. Proof is a very difficult area for high school students—a fact that is noted by the NCTM (2000, p. 56) and supported by many research studies that have shown that students' inadequacies and misconceptions in the area of proof and reasoning are widespread (Chazan, 1989; Healy & Hoyles, 2000; Senk, 1985, 1989; Williams, 1979). Epp (2003) stated "Unfortunately, at least in the United States, a large number of K – 12 teachers have only a weak command of the principles of logical reasoning" (p. 894). She added that it simply is not possible for such teachers to effectively teach their students' reasoning and proof as the NCTM (2000) *Standards* recommend when the teachers themselves do not have an understanding for what a valid deduction is or what it means for various statements to be true or false (Epp, 2003).

Significance of Study

Examining the skills that prospective teachers need to teach proof and reasoning in a manner that promotes critical thinking skills as outlined by the NCTM (2000) leads to the question of what conceptions they possess as they complete their preparation program. The significance of this study was three-fold. First, many in the mathematics education community have emphasized the importance of teaching students logical reasoning and formal proofs (MAA, 1998; NCTM, 2000). Secondly, the importance of preparing teachers to teach proof and reasoning, which is dependent upon their mathematical content knowledge of the nature of proof, was recognized by many organizations including the NCTM (2000), MAA (1998), and the MSEB (2001). Third, few studies have been conducted that research prospective secondary teachers' conceptions of proof after they have been "prepared to teach" proof and reasoning. The aim of this research study was to provide a snapshot of secondary mathematics teachers' conceptions of proof and its logical underpinnings as prospective teachers neared the end of their preparation programs. This research was a broad based study that replicated a research study by Galbraith (1982) that assessed prospective secondary mathematics teachers' "mathematics vitality."

The primary research question was: What are prospective secondary mathematics teachers' conceptions of proof and refutations? In the original study, two secondary research questions addressed participants' understanding of the logical underpinnings of proof and participants' abilities to complete mathematical proofs. Both secondary questions sought to answer the primary research question. For this paper, I will focus my analysis on participants' ability to complete mathematical proofs and what this implies toward the primary research question.

Theoretical Perspective of Proof

Logic is the study of methods for evaluating mathematical proof. According to Barnier and Feldman (1990) "A basic knowledge of logic is indispensable for analyzing and constructing proofs" (p. 1). They go on to suggest that to understand a proof one must know: 1) the goal of the proof; 2) the hypotheses; 3) the necessary facts and definitions of the content area; and 4) previously proved facts or laws of logic to be used in the proof.

Introductory proof courses generally provide an opportunity for undergraduates to refine their proof skills. According to Epp (2003), goals for a first course in proof and reasoning should include helping students appreciate the role of definitions in mathematical proof and reasoning and also develop an ability to evaluate the truth or falsity of mathematical statements. She further suggested that an introductory unit on the principles of logical reasoning provide a supportive framework in which students can draw from while learning various aspects of proof and disproof.

Methodology

The participants of the study were 23 prospective secondary mathematics teachers near the end of their preparation programs. Twenty of the 23 participants were seniors and 3 participants were juniors (had completed an Introduction to Abstract Algebra course). They were enrolled at three state universities in the northwestern United States. According to the state Office of Commission of Higher Education, the 2000 – 2001 enrollment of these three universities represented approximately 79% of the total enrollment at four-year schools in the state (2002). The participants represented a homogeneous sample of prospective secondary mathematics teachers completing their preparation programs. All of the participants had completed, or were near completion, of similar courses that included at least two calculus courses, an introductory proof class, and a college geometry course. Mathematical concepts in these courses included techniques and methods of proof, the logical underpinnings of proof, and methods of completing proofs.

The researcher developed a questionnaire composed of two parts in order to assess the two components of proof—1) understanding of the logical underpinnings of proof, and 2) ability to complete mathematical proofs. Items were adapted from Ball and Wilson (1990), Galbraith (1982), Knuth (1999), and Senk (1985, 1989). The content of the items centered on concepts and ideas typically found at the secondary mathematics level.

Analysis of Data

A numerical analysis was used to assess participants' responses to both parts of the questionnaire. Part I of the questionnaire addressed a secondary research question that examined the participants' understanding of the logical underpinnings of proof and refutations. It consisted of 12 multiple choice items. Participants were given a score of 0 for an incorrect answer or a non-response and a score of 1 for a correct answer.

Part II addressed a secondary research question that investigated participants' ability to complete mathematical proofs. It included three constructed response proof items that required participants to produce their own logical justifications for mathematical ideas. The responses for this part were analyzed by a numerical scoring scheme that described the approach employed by the prospective teacher in attempting the mathematical proof. The researcher categorized the responses based on guidelines adapted from Senk (1985, 1989) and the Educational Testing Services' (2002) Praxis Test on mathematical proof. The researcher and a mathematics education professor independently assessed the participants' responses.

Scores of 0 through 5 were assigned to each of the constructed response items, where a score of 4 or 5 represented a valid proof or valid disproof. These responses were classified as using what Harel and Sowder (1998) referred to as Analytic Proof Scheme—Axiomatic Proof. The response was judged as a valid justification (5 points) or valid but shows minor errors step (4 points). A score of 3 points shows some chain of reasoning by either completing half the logical steps correctly or by writing a sequence of statements that is invalid because it is based on faulty reasoning in earlier steps. Two points indicated minimal progress with at least one valid deduction; one point demonstrated a lack of understanding or an invalid proof strategy, and zero points indicated a non-response, an invalid response, or pointless deductions. The distribution of participants' responses per point value allowed the researcher to identify interesting patterns or trends in relation to the participants' understanding of direct and indirect proofs, and the refutation item.

Results and Conclusions

Data gathered from the constructed response items were analyzed and distributed in a frequency table showing the participants' score for each item, as well as their total score.

Table 1. Distribution of Data per Individual—Completion of Proof

Part II – Constructed Response Proof Items (5 points possible for each item)				
Participant	**Indirect Proof item #1**	**Refutation item #2**	**Direct Proof item #3**	**Total Points (15 points possible -%)**
1	1	1	3	**5 33.3%**
2	3	1	4	**8 53.3%**
3	1	4	3	**8 53.3%**
4	3	5	5	**13 86.7%**
5	0	1	4	**5 33.3%**
6	2	0	0	**2 13.3%**
7	2	5	4	**11 73.3%**
8	4	0	4	**8 53.3%**
9	5	5	1	**11 73.3%**
10	2	1	0	**3 20%**
11	5	3	3	**11 73.3%**
12	1	5	3	**9 60%**
13	3	0	0	**3 20%**
14	4	0	2	**6 40%**
15	5	1	4	**10 66.7%**
16	5	5	4	**14 93.3%**
17	1	0	5	**6 40%**
18	1	5	5	**11 73.3%**
19	5	0	4	**9 60%**
20	0	5	1	**6 40%**
21	5	0	4	**9 60%**
22	4	1	4	**9 60%**
23	1	5	5	**11 73.3%**
Mean	**2.74**	**2.30**	**3.13**	**8.17 54.5%**

Analysis of Proof Strategies

Item 1 asked participants to complete a proof for which the most likely strategy was the contradiction or contrapositive method of proof. Participants were asked to prove: *Let x be an integer. If* x^3 *is even, then x is even*. This item required an understanding of the definitions of even and odd integers. Table 1 shows that nine participants (39% of the sample) completed a valid proof (i.e. used an Analytic Proof Scheme). Four of these nine individuals employed the contradiction method of proof and five participants employed the contrapositive method of proof. The responses of 14 participants (61%) were scored 0 to 3 points and judged to be invalid proofs. Three of these 14 participants scored 3 points by responding with some chain of reasoning, however steps were based on faulty reasoning or had only completed approximately half of the logical steps. The responses of 11 participants (48%) lacked any chain of reasoning that could lead to a valid proof (score 0 – 2 points). Analyzing the responses of these 11 invalid proofs reveals that three participants attempted a direct proof; three participants attempted to prove the converse; three participants' responses were completely invalid or a non-response; and two participants used examples as justification, exhibiting an Empirical Proof Scheme (Harel and Sowder, 1998).

The false conjecture item (i.e. Item 2 of Part II), was adapted with slight changes from the Ball and Wilson (1990) study of novice secondary teachers. Item 2 was the following false conjecture problem: *As the perimeter of a rectangle increases, then the area of it also increases*. Below the conjecture, two rectangles were drawn—one was 3 centimeters by 3 centimeters and the other was 3 centimeters by 4 centimeters. The perimeter and area of both rectangles were given below the drawings of the rectangles. Participants were asked to reply to this conjecture. They should have questioned the truth of this conjecture and established that the conjecture was false. Table 1 show that nine participants (39% of the sample) recognized that the conjecture was false and were able to refute it with a counterexample. Eight of those nine participants completed a valid justification (score 5 points) and one participant responded with a partially correct counterexample and was given a score of 4 points. One participant scored 3 points and 13 participants or 57% responded that the conjecture was true. Six of the 13 attempted to prove it was true and seven participants responded by restating the conjecture or by an invalid response. Results show the mean for this particular proof item was 2.30 out of 5 points, the lowest mean of the 3 proofs. The responses of the participants in this study were similar to those in the Ball and Wilson (1990) study. Approximately 57% of the participants in this study responded that the conjecture was true whereas 47% responded that the conjecture was true in the Ball and Wilson study. The findings suggest that prospective secondary teachers need more exposure to recognizing false refutations

Item 3 asked participants to prove that "*If a point is on the perpendicular bisector of a segment, then it is equidistant from the endpoints of the segment*." This conditional statement is commonly referred to as the Perpendicular Bisector Theorem, a theorem that is found in traditional high school geometry texts. This item was most likely to be completed via a direct proof. The data from Table 1 show that only 13 participants (57% of the sample) constructed a valid proof. Four of these 13 participants scored 5 points and nine participants missed at most one reasoning or justification step, scoring 4 points. Four participants responded with some chain of reasoning however steps were based on faulty reasoning or had only completed approximately half of the logical steps (score 3 points). Six participants (26%) did not construct a chain of reasoning that would lead to a valid justification (scored 0 to 2 points).

Analysis of Participants' Completion of Proof

Evaluating participants' responses for all three items found that only one participant completed a valid proof (or disproof) on each of the three proofs (this participant was one of the three juniors). Four participants' responses (17%) were judged as invalid for all three items. In regards to just the direct and indirect proof items, data show that only six participants (26%) responded with a valid proof for both; seven participants (30%) did not complete a valid justification for either item. Ten participants responded with only one of the direct or indirect proof items judged as a valid justification. Of these ten participants, seven responded to the direct proof item with a valid proof. In regards to the refutation item, results show that of the six participants who responded with a valid proof for both the direct and indirect proof items, only one recognized that the false conjecture item was indeed false. Perhaps what was most disturbing was participants' poor performance in completing valid proofs, in light of the mathematical content level of the three proof items (i.e. perpendicular bisector, triangle congruency theorems, even integers, and area and perimeter of rectangles). Prospective teachers are likely to teach proofs of this difficulty level or greater.

Data Analysis for the Primary Research Question

Data from both parts of the questionnaire were used to address the Primary Research Question: What are prospective secondary mathematics teachers' conceptions of proof and refutations? There were 12 points possible for the logical underpinnings of proof items and 15 points possible for the completion of proof items. Results show 12 participants (52% of the sample), scored 60% or lower on both the multiple choice items and the constructed proof items. Eleven of these 12 participants were seniors. Comparing both parts of the questionnaire, the data presents three distinct groups within the sample. Six participants (26%) scored nine points or better on both parts of the questionnaire—demonstrating their conceptions of proof are adequate. At the other end of the spectrum, 30% of the participants (seven seniors) scored 7 points or less of the 12 multiple choice questions and 6 points or less out of 15 points on ability to complete proof. Their conceptions were judged to be inadequate. Approximately 45% of the participants' conceptions are varied and lie between these two groups.

Concluding Remarks

The purpose of the research study was to provide a snapshot of secondary mathematics teachers' conceptions of proof and its logical underpinnings as prospective teachers near the end of their preparation programs. The results from the study show some areas of concern in addressing the lofty goals of the NCTM *Standards* 2000. One concern is that the results show that only 26% of the participants were able to complete a valid proof for the direct and indirect proof items. Given the difficulty of these items, the findings are somewhat disturbing. Another area of concern is for the twelve participants, (52% of the sample), that scored 60% or lower on both the multiple choice items and the constructed proof items. This data suggests that some prospective teachers may have difficulty teaching proof and reasoning effectively, as is outlined by the NCTM *Standards* 2000 and the MAA (1998). A recycling effect seems very real as prospective teachers with an inadequate understanding of proof and reasoning return to the educational system as mathematics teachers faced with the challenge of teaching proof.

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"JUST GO": MATHEMATICS STUDENTS' CRITICAL AWARENESS OF THE DE-EMPHASIS OF ROUTINE PROCEDURES

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This is an account of part of my extended conversation with a high school mathematics class, in which I prompted the students daily to become ever more aware of their classroom language practices. Our discussion about the word just exemplified a way such "critical language awareness" conversations can draw out student perspectives on learning mathematics. The discussion also demonstrated the way language awareness can afford students new possibilities for living in their mathematics classroom discourse.

Introduction

"And you just change it to two square root five." Some time after a student said this, I asked her classmates, "What does that mean when she says *just*?" This exchange was part of a semester-long conversation in which I aimed to raise the students' awareness of their language practice in mathematics class. When the students and I disagreed over how to use the word *just*, we became aware of connections between our use of the word and the way we direct attention when communicating mathematics.

Morgan (1998), as a result of her extensive study of secondary school mathematics writing, identifies the need for students to become more aware of their language practice, but she does not say much about how teachers might help them do this. While discourse analysis has given *educators* insight into mathematics and its classroom practice (e.g. Morgan, 1998; Rowland, 2000), like Morgan, I also want mathematics *students* to benefit from increased language awareness.

Assuming that mathematics classrooms would benefit from what Fairclough (1992) calls "critical language awareness," an important question remains: how can it be brought about effectively? In Wagner (2003), I began to answer this question by analyzing transcripts of interviews in which students responded to audio-taped excerpts of themselves working on pure mathematics investigations. These interviews did not focus on language per se, but the analysis is instructive for applying critical language awareness to the mathematics classroom.

Linguists Chouliaraki and Fairclough (1999) have constructed a framework for analyzing discourse for critical purposes. With this framework, they encourage the use of discourse analysis for the identification of "the range of what people can do in given structural conditions" (p. 65). I suggest that mathematics students can benefit from exploring various ways of living within the discourse space they encounter daily. Furthermore, there is an opportunity for educators in such conversations: While conversing with students about their language practice in mathematics class, a teacher or researcher can gain insight into the unique perspective students have on their classroom discourse.

With these interests in mind – raising student awareness of language and accessing student perspectives on their discourse – I set out to answer the following question: Considering the depth and breadth of students' mathematical experience, what is the effect of their mediated use of discourse analytic tools to explore the mathematics discourse that surrounds them?

Research Method

To address the above question, I spent a nineteen-week semester with a grade 11 pure mathematics class, co-teaching the course with the regular teacher and collecting video and audio records of classroom discourse every day. By directing the students' attention to their own utterances, I tried daily to engage the students in discussion about our language practices in the class. The form of my prompts varied, as I was continually responding to the participants. In addition to our classroom interaction about language, I interviewed participant students and asked them to write accounts of their experiences with language in relation to their mathematics learning.

This research was an investigation of possibility. Skovsmose and Borba's (2000) methodology for critical mathematics education research guided me: "[I]t is by no means a simple truth that research should deal with what *is*. [...] [D]oing *critical research* means (among other things) to research *what is not there* and *what is not actual*" (p. 5, emphases theirs). I saw the original situation of the participant class as a situation that I wanted to see transformed. I imagined a situation in which students would notice aspects of their language practice and through this noticing become more aware of the nature of mathematics and of possibilities for them to relate to the mathematics. The primary data comprised transcripts of interviews and whole-class conversations about our classroom mathematics discourse. For these conversations I drew on a secondary set of data to prompt students to articulate their perspectives – transcripts and videotaped excerpts of our classroom conversation about mathematics, and excerpts of the students' mathematical writing.

My agenda was not the same as the students' agenda for this class. In fact, our agendas or imagined situations kept changing as we responded to each other. Therefore, I could not expect the classroom developments to follow my plan. Instead, I needed to expect disruption, and to welcome it. Valero and Vithal (1998) illustrate the importance of disruption in research settings and argue against typical research methodologies that assume and promote stability. Just as Valero and Vithal realized from the research they report, I am realizing that the times when I felt most resisted were frequently the most generative times, both for me and for the participant students. It was only when students actively resisted my interpretations of classroom language practices that I could be sure that they were expressing their own perspectives.

Critical Language Awareness in Action

One of these times of resistance related to the use of the word *just* in our class discourse. This stream of our ongoing conversation exemplifies a possibility opened up to mathematics teachers and learners when critical attention is directed to language practice. After I describe the relevant events from the researched classroom, I will consider a connection between mathematics practice and the use of the word *just*.

Just and Simplicity

A few months into our conversation about language practice in this mathematics classroom, the students and I were considering a transcript that related to the student-identified phenomenon of having a clear mathematical idea but no words to describe it. After we discussed the transcript, I asked students if they found anything else interesting in it. They said they did not. I then moved to my secondary mode of prompting and tried to provoke a reaction by making contentious assertions. I drew attention to a particular utterance – "Is what the, like the square root of twenty. And you just change it to two square root five, right?" Jessye, the student who had said this, was absent this day. Her absence opened up for the rest of us an opportunity to explore the *effect* of her language choices because we did not have access to her intended meaning. I circled the word *just* in the transcript that we were considering and asked, “What does that mean when she says *just*?” In the interchange that followed, Gary paraphrased his classmate’s utterance and I prompted him further (student names are pseudonyms; “DW” refers to me):

Gary: “You simply change it.”

DW: Oh. So, in other words, it’s a simple thing to do.

Gary: Well, I guess. I don’t know. Well, I guess that’s what it’s implying.

DW: It makes it sound easy. Yeah, I was just wondering. I found this interesting because we teachers sometimes say the word *just*. Do you think it kind of is insulting to students? When you say, “Well you just do this, and”?

[*many students say things all at once*]

Gary: Well, because you guys have done it for so long. You guys, like, it’s not really like, I don’t know. It’s not that big a deal, but it’s kind of implied that we, like, should get it right away, that it should make sense automatically. [*turning slightly to acknowledge Joey’s long and clear thumbs up*.]

DW: You’re agreeing, Joey? [*Joey nods yes*]

Many of Gary’s classmates agreed with him. They agreed that teachers should not use the word *just* in the sense we discussed – to suggest that a procedure is simple. Sometime in the following days’ conversations about the word *just*, Gary wrote a note to himself in his workbook (Figure 1). While he displayed significant self-confidence, both in mathematics and in discussions about mathematical language practices, Gary worried about the sense of inferiority a teacher’s use of *just* might invoke. None of the students in this class said that they themselves felt insulted. Rather, they seemed to be worried that others would feel insulted. Their concern was pedagogical.

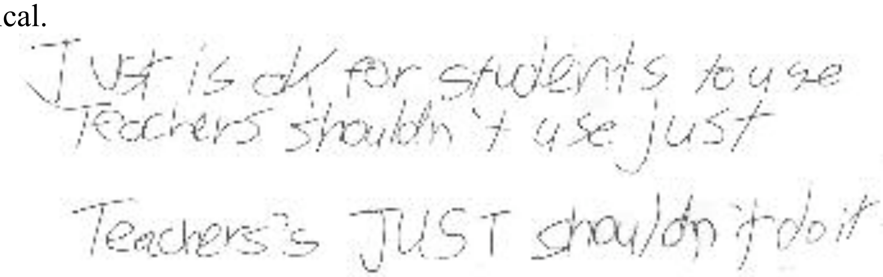

Figure 1. Gary’s note to himself about the use of the word *just*

Early in this stream of conversation, students demonstrated their sense of the importance of language in mathematics class conversations when they noted the significance of the simple word *just*. *Critical* awareness demands an exploration of a range of possibilities, but they seemed to be fixated on *one* account of the effects of teachers using the word. In addition to my interest in developing the students’ *critical* awareness of language, I also had another concern. I wanted to draw out the students’ unique perspective on their discourse. I will show how this stream of conversation illustrates the difficulty researchers face when trying to draw out student perspectives.

Hearing Student Voices

In the initial conversation, which is partly detailed in the above transcript, I initiated the vein of worry students felt about the word *just*. In my revoicing, I deliberately stretched my interpretation of Gary’s intentions, saying “Oh, so in other words it’s a simple thing to do.” This manner of responding had a number of effects, which included verifying my interpretation and provoking further discussion. O’Connor and Michaels (1996) show how teacher revoicing prompts a participant framework, in which students converse with each other. This outcome is especially promoted by revoicing that pits student ideas against each other. My revoicing cast

Gary in the role of a judge speaking against Jessye's intentions. I pushed this role further by applying his interpretation to a teacher's use of the word *just*, in effect casting him as a judge speaking against the teacher's warrant to orchestrate and control classroom relationships.

A further effect of the revoicing was significant in this research context, although it is not a concern for O'Connor and Michaels (1996), who write about *teacher* revoicing. My revoicing changed the way the students thought. When Gary said *just* was synonymous with *simply,* he did not say that Jessye's utterance implied simplicity. *I* said that. However, when I asked him if he was recognizing the implication, he seemed to agree that the implication was present in his utterance, though he was hesitant at first – "Well, I guess."

I had told the students that one of my intentions in this research would be to listen to the voices of students. I also claimed this intention in the introduction to this paper. How could I claim to be listening to student voices when I was putting words into their mouths? Though this is a significant question, another question can be used to argue against it: How can a question be asked or a response be prompted without words that could seed the answer or response? The answer to both these questions is the same – it cannot be done. It is necessary to be careful about the extent to which participants are given words to speak their ideas. And it is necessary to be aware of the influence the wording of a prompt has on the participants' understanding.

In the months preceding this stream of conversation, I had found that the students in this research tended to say either nothing or make such brief utterances that they could be interpreted in many ways. If I wanted them actually to say something, at times it seemed that they needed to be provoked into speaking. Once provoked, they might begin speaking more freely.

With my heavy-handed verbal prompting, I felt responsible for initiating the students' vein of worry about mathematics teachers suggesting simplicity by using the word *just*. Though I felt responsible for their concern, I resisted their complaint. The participant teacher and I continued to use the word *just* regularly when we taught. In order to convince the students of another more positive perspective on the use of the word, I wrote a 600-word essay for them, referring to the adverbs *just* and *simply* as "diminutives" because they suggest that the actions they describe are unimportant or trivial. This essay marked the beginning of my disagreement with the students and the emergence of their clear voice.

Diminutives like *just* can be used for pointing, I said in the essay. The de-emphasis of one procedure can emphasize another procedure or another aspect of the reasoning. With such emphasis and de-emphasis, we point attention to the important ideas we are talking about. Besides using adverbs like *just* and *simply* to de-emphasize, one can use the verbs *do* and *go* to do the same thing. When talking through my mathematics for others, I might just show my calculations and say, "and you go 'root twenty' and 'two root five'." In this case I am not saying what I am doing or how I am doing it. I am just saying to my audience, "you too can go down this path, a path which should be really obvious." When I say "just go," it is a double diminutive, suggesting that a procedure is really, really obvious, and that the procedure does not merit attention or explanation. It merely requires performance – "just do it." Presenting this reasoning in the essay, I thought I had made a clear point about a positive effect of a teacher or student using diminutives. However, the students were not convinced.

In response to my essay, the students continued to express their concern that diminutives can be insulting, that these adverbs suggest a procedure is obvious when it may not be so obvious to students. Though I considered their interest in this pedagogical issue a significant revelation, I felt frustrated that these students seemed uninterested in my suggestion that teachers and students use diminutives to point in mathematics communication. While their resistance to the alternative possibilities exposed a deficit in their critical language awareness, the resistance clarified that the concern they were expressing was important from their perspective. The students' resistance verified the role critical language awareness can play in drawing out the authentic voice of students, the articulations of their unique perspective on mathematics classroom discourse.

***Just* and Vagueness**

A few weeks after our initial discussion of *just*, on a day when Gary was absent, I resurrected the stream of conversation. I moderated a mock panel interview in which students played the part of a teacher and a student debating the merits of using the word *just* in mathematics classrooms. During this role-play, it became clear that many students no longer shared Gary's concern about the language practice in question. His classmates said that he alone had this concern, and that he still held it strongly. One of Gary's classmates, Jocelyn, expressed another concern during this discussion: "[W]hen [teachers] use *just* it's kind of an aggressive word. It's kind of like they just use *just* because they don't want to explain why it is. They just say, 'It's just that'." She resented it when her teachers gloss over any aspect of their mathematics in an explanation.

Her concern pointed at another aspect of the language practice in question. When the words *just* or *simply* are used to indicate simplicity, they actually replace a more careful explanation of the procedure indicated by the verb. For example, when a teacher says, "and we just solve that," the adverb *just* suggests that the solving is straightforward, unremarkable. As I said about emphasis and de-emphasis associated with the adverb *just*, the teacher is merely de-emphasizing the solving procedure in order to draw attention to something else. The teacher has the option to describe the solving procedure in great detail, but chooses not to do this.

Jocelyn expressed her contempt for teachers who are vague. Tharshini, another student in this class, argued against Jocelyn's concern by noting the time constraints teachers face: "Maybe they don't have time to explain." Just as Gary adamantly refused to give ground when faced with my resistance to his interpretation, Jocelyn argued with Tharshini, cutting off her utterances. This time Tharshini provided the foil to clarify Jocelyn's passionate commitment to her account of teachers using the word *just*. Again, the class' critical attention to language afforded me the opportunity to hear the students' unique perspectives on their discourse.

Vagueness is an important aspect of language, but it appears that linguists and educators have overlooked the role of adverbs like *just* in expressing vagueness. Even Channell (1994), in her extensive study of vagueness in general language practice, and Rowland (2000), in his extensive study of vagueness in mathematics learning discourse, do not consider the significance of this particular language form, though Rowland's exemplar transcripts often include the word *just*.

It should not be surprising that Jocelyn and Tharshini had very different perspectives on the vagueness-expressing *just*. Jocelyn's concern is an example of the discursive authority of Grice's maxims. Grice suggests five principles as a set of over-arching assumptions that guide the conduct of conversation. (They are only partially published by Grice, having first been delivered in a lecture, but they are widely described by others.) The maxim of Quantity states that in normal conversation people follow these rules: "[M]ake your contribution as informative as is required for the current purposes of the exchange" and "do not make your contribution more informative than is required" (Levinson, 1983, p. 101).

It appears that Jocelyn was upset with teachers who had not, in her opinion, made their oral contributions as informative as required for her purposes. Though her concern is justifiable, Tharshini and other classmates may have "required" a lesser quantity of explanation. When a teacher addresses a class of thirty, it is unlikely that all the students have the same requirements for explanation. Tharshini's responses to Jocelyn's concern correspond to the second part of the Gricean maxim – the teacher should not explain more than necessary. The disagreement between Jocelyn and Tharshini illustrates how the Gricean principles are most evident when they are perceived to be flaunted.

Directing Attention

Teachers confront this problem every day – different students want and need different degrees of explanation and vagueness. Even teachers who think they explain their mathematical examples fully cannot possibly do so. Any mathematics relies on other mathematics or on assumptions that might be questioned. It is impossible for students (or anyone) to attend to everything at once, and it is the teacher's role to direct student attention appropriately.

A particular case of this problem is particularly relevant to this research. Adler (2001) illustrates some dilemmas that are faced by all mathematics teachers, but are particularly noticeable in multilingual classrooms. She calls one the "dilemma of transparency," which recognizes that at times a degree of explicit attention to language is warranted, while at other times it is best to use language without attending to it, as though it is transparent.

Gattegno (1984) asserts that every circumstance of life involves stressing and ignoring. He adds that the process of stressing and ignoring is especially important in mathematics education because the process itself is the process of abstraction. In mathematics, the ignoring is layered with each level of abstraction: "[I]t is possible to constitute a cascade (or hierarchy) of abstractions by stressing attributes or properties and ignoring others in already-stressed items" (p. 34).

Awareness of Possibility

When Jocelyn drew attention to the effects of vagueness in mathematics communication, we had three different but interrelated accounts of the primary effect of the simplicity-implying use of the word *just* in mathematics discourse. Each of these accounts could also apply to the generic verbs, *do* and *go*, which imply simplicity or unimportance as they gloss over procedures. First, this usage suggests that a procedure is obvious. Second, it directs attention away from the procedure. And third, this diversion of attention glosses over alternative possibilities to the procedure. I feel that Jocelyn, who worried about glossed-over parts of an explanation, came closest in this discussion to meeting my hopes for student critical language awareness – an awareness that opens up alternative ways of living within the discourse.

Gary was interested in the way the word *just* suggests that procedures should be obvious. He saw the teacher's use of the word as a potential source of frustration, but what could he do differently because of this awareness? Perhaps it could mitigate his possible sense of inferiority, although he did not seem to have any sense of inferiority. One could rightly suspect that other students who become aware of linguistic forms in their mathematics classes would share Gary's strong sense of confidence. If such awareness is unnecessary for him, it might even be a distraction.

Signot, another participant student, mentioned briefly that words like *just, go* and *do* help students know what is important and what is unimportant. By this he seems to have meant that the words could help students figure out what the teacher deems important. Such an awareness of the pointing power associated with emphasis and de-emphasis might help a student or teacher direct attention effectively when communicating mathematics. With the exception of Signot's brief utterance, the students in this study seemed unmoved by the significant possibilities such awareness afforded them.

But Jocelyn, who was upset by vague language, demonstrated her awareness of a subtext in mathematics communication and opened up new possibilities for herself. She saw that alternative mathematical possibilities were being glossed over, and she could attend to these alternatives even when the speaker might deem them trivial. With an awareness of the role *just, go* and *do* can play in masking aspects of the mathematics, she could direct her awareness elsewhere. These three words could pique her attention to the ongoing stressing and ignoring that is at play in any mathematics communication. When she would hear a teacher or classmate say "[J]ust go...," she could say to herself, "Yes, there is an obvious way of doing this, but how might I go about this differently?" This kind of awareness is the goal of "critical language awareness" – to become conscious of alternative possibilities within the discourse.

Conclusion

In any discourse, it is natural to *just* fit in, to follow the language and behaviour patterns of the people around us. In mathematics class, it is understandable that students would think, "This is *just* how it is done." Alternative mathematical possibilities can become accessible to students when they come to realize that certain language patterns can actually mask these alternatives. This awareness is one possible benefit of directing students to attend critically to their language practice in mathematics class. Discussions about language can also afford teachers and researchers an insight into students' perspectives on learning mathematics.

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VOICE AND SUCCESS IN NON-ACADEMIC MATHEMATICS COURSES

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An historical overview of mathematics-for-all demonstrates uncertainty about success *for all high school mathematics students, including those who are not academically inclined. The narrative inquiry study being reported provides an effective alternative, focusing on purposes for high school mathematics that the students in the inquiry expressed and valued. The voices of four non-academic students suggest that their experiences with success in mathematics class prioritized their quest for a more positive* sense of identity *in relation to school mathematics. As we listened, their words indicated they were in the process of (re)forming their identity as mathematical thinkers, learners, individuals, and students. This paper suggests non-academic mathematics students can inform researchers in designing a legitimate curriculum where students believe their voice is valued and they experience authentic mathematical success.*

The certain conviction that all high school students should succeed in mathematics (National Council of Teachers of Mathematics [NCTM], 2000) stands in stark contrast to the uncertainty about what should be prioritized in mathematics for students who are not academically inclined. That uncertainty is reflected in the various ways that high school mathematics courses are organized for non-academic students. This paper inspects the underpinnings of one-size-fits-all high school mathematics reforms and their alternatives, giving special consideration to the needs and goals of non-academic students. To assist the mathematics education community to come to terms with those needs and goals, the paper amplifies the voices of non-academic students who, in their own terms, succeeded in high school mathematics.

High school mathematics was not originally designed for all students. Eighty years ago, when "almost one in three of the children reaching their teens in the United States enters high school" (Thorndike, 1923, p. 3), Thorndike claimed that a student of average intelligence "will be unable to understand the symbolism, generalizations, and proofs of algebra. He may pass the course, but he will not really have learned algebra" (p. 37). But that was all right, because algebra was only used "for thinking about general relations. Only a few of its abilities are used by workers … except as they become students of the sciences" (p. 47). Algebra, formal mathematics, indeed high school in general – none of the above were important to the non-academic high school student, because the non-academic student did not exist.

Things changed, of course. If we move halfway to the present time from the days of Thorndike, the ideal of all students attending high school was born. "It was not until the 1960s – just yesterday – that the nation first acknowledged an obligation to educate all students to equally high standards, both because it was fair and because our nation's health depended on it" (Meier, 1995, p. 72). Technical and comprehensive high schools offered courses and programs for non-academic students, including diluted versions of academic mathematics courses and remedial mathematics courses. However, in the eyes of many, "The non-academic, or General, route is fine – for everybody else's son or daughter. In reality it is a dumping ground, piled high with the poor, the disabled, and the newly arrived" (Dryden, 1996, p. 51).

Today, students entering high school will have logged 1500 hours or more of mathematics instruction. Even if they enter school with equivalent mathematical experiences (and they do not), it would be foolish to expect that after eight or more years of instruction, all students would have equivalent understandings of mathematics. *Streaming* – the creation of academically homogeneous class groups by academic grouping (Gamoran, 1992) – enabled teachers to offer watered-down versions of course content to low-ability groups, and go more slowly through the material. However, classroom research in streamed classes for non-academic students found that instructional practices seldom reflected the needs of non-academic students to learn in different ways (Carbonaro & Gamoran, 2002), and seldom did streamed classes provide non-academic students with opportunities to feel successful at mathematics (Schoen & Hallas, 1993; Steen, 1992). Zevenbergen (2003, p. 7), who studied the perspectives and attitudes of academic and non-academic students in Australian schools with streaming for mathematics based on students' self-reported academic ability, is representative:

> The experiences of the students fell clearly into distinct categories, whereby the students in higher groups felt that they were blessed with high-quality experiences, while the students in lower groups reported that their experiences were quite negative. … The experiences of the students could be seen to affect their relationships with mathematics profoundly and, hence, their subsequent choices as to whether or not to participate in the practices of the classroom and further study in the discipline.

Streaming of students through ability grouping seems more to alleviate the challenges of teachers facing students who vary greatly in mathematical understandings and motivations (Grossman & Stodolsky, 1995), but not the challenges faced by students of lesser abilities (Ma, 1999).

What about the possibility of teaching non-academic students a different *kind* of mathematics, and teaching them through methods *distinct* from those used with academically inclined students? This is the premise of *tracking* (Letendre, Hofer, & Shimizu, 2003). Manitoba, the authors' home province, offers an example of an extensively tracked high school mathematics program. After a one-size-fits-all mathematics curriculum for nine years, students in grade ten select among three different streams. There are two academic tracks, one more formal and symbolic and the other more technology-and application-oriented. (Many high schools differentiate even further, by offering an Advanced Placement or International Baccalaureate track for the top academic students.) There is also one non-academic track, called Consumer Mathematics, designed with the following rationale:

> In order to meet the challenges of society, high school graduates must be mathematically literate. They must understand how mathematical concepts permeate daily life, business, industry, government, and our thinking about the environment. They must be able to use mathematics not just in their work lives, but also in their personal lives as citizens and consumers. *Consumer Mathematics* has been designed to meet these challenges for those who may not use advanced abstract mathematics in their careers, but who, nevertheless, will be consumers and active citizens. They also will need to develop their cooperative, interactive, and communicative skills. (Manitoba Education, Training and Youth [METY], 2002, p. 3)

We will return shortly to look more closely at student success in Consumer Mathematics.

Our belief in the validity of designing and teaching mathematics courses that recognize the distinct needs and motivations of non-academic students is not typical within educational reform. Currently, reform efforts in high school mathematics education features detracking (Mickelson, 2001; Weiner & Mickelson, 2000), the development of a one-size-fits-all, algebra-for-all mathematics course (Schoenfeld, 2002; Strong & Cobb, 2000). Yet, this movement reflects goals of social engineering more than goals that would address the educational needs and possibilities of non-academic students (Nieto, 1994; Oakes & Wells, 1998; Slavin, 1995; Wells & Serna, 1996; Wheeler, 1992. See Linchevski & Kutscher, 1998, as an exception.)

Proponents of detracking place significant faith in teachers' abilities to differentiate within classrooms, where a one-size-fits-all curriculum "has to be much richer, more problem-oriented, and more engaging than even the curriculum of the high track. Students need a lot of opportunities to construct knowledge together as a group, to make meaning out of their experiences" (Oakes, in O'Neil, 1992, p. 21). However, researchers observed that in actual practice, "students were not up to the complex interpersonal negotiations such work entailed, with consequences for their opportunities to learn and their sense of academic identity" (Rubin, 2003, p. 563). It was and is unfair to expect teachers to spontaneously generate strategies that can simultaneously engage high-ability learners while remediating students who are weak at, or angry about, mathematics. Elmore (1995) found that results of tracking paralleled school-restructuring research, where "structural change does not necessarily lead to changes in teaching, learning, and student performance" (p. 25). A different approach to teaching non-academic mathematics students needs to prioritize opportunities for them to learn meaningfully, and repair their sense of mathematics and of themselves as mathematics learners.

Even in terms of the social engineering goals that motivate detracking, early research results are not encouraging. "The explicit goal of detracking is to contest race- and class-based inequalities in schools … Despite the best efforts of committed teachers, these inequalities were often reinforced rather than challenged" (Rubin, 2003, pp. 566-567). Programs have successfully addressed issues of social inequality and the gatekeeping functions of high school mathematics – but they encompass more than restructuring high school classrooms, and their reforms begin with students much earlier than in high school (Moses and Cobb, 2001). As well, such programs require significant levels of political will and are currently far from universal. As a consequence, educators dealing with high school mathematics today face a significant clientele of non-academic students for whom algebra-for-all is inappropriate idealism (Noddings, 2000). To remain idealistic about mathematics-for-all, we must (we believe) focus our intentions for non-academic high school students on their educational needs and capabilities (Chazan, 2000), rather than (but not to the exclusion of) social issues.

Non-academic students are not easy to teach, however. Their content knowledge is often weak. Their confidence as students is often poor. Their capabilities as learners are often under-developed. And their educational needs are not well-defined. That they are not going into engineering or the sciences at university some day tells us what mathematics they *do not* need, but it doesn't tell us what they *do* need. That they did not value or benefit from instruction in previous years based on practicing arithmetic does not tell us what they could value, appreciate, or benefit from. In general, it is not only the pedagogic pathway to success in mathematics for non-academic students that is undeveloped; the nature of success in mathematics itself for these students is as yet undefined. We now turn our attention to an empirical exploration of that question, conducted with students in a non-academic grade 10 mathematics course.

Success in Students' Words

The voices in this section are students in a semestered grade 10 Consumer Mathematics class in a large high school in Manitoba, Canada. Approximately half of the students in this study had previously attempted and failed a grade 10 mathematics course, while the remainder were in their first high school mathematics course after marginal success in grade 9 mathematics. The students' voices were amplified through a practitioner-based, narrative-inquiry-framed

(Clandinin & Connelly, 2000) research process. Data included *interactive journal writing* (Mason & McFeetors, 2002) and *portfolios* (Britton & Johannes, 2003; Morgan & Watson, 2002) generated by the students and *field notes* generated by the teacher. From this data, the teacher-researcher constructed individual *narratives of success* that provided a starting point for *informal interviews* (conversations) with each participant. There were three cycles of stories and conversations, illuminating the processes by which the teacher and students constructed success in the mathematics course and determined the nature of that success in the students' terms.

The experiences of these non-academic students suggest that succeeding in school mathematics is less a matter of learning mathematics *content* than with it is a quest for a more positive *sense of identity* in mathematics class. Students needed to see themselves as effective mathematics thinkers, as having the capabilities required for learning mathematics, as being persons whose individual qualities were suitable for learning mathematics, and as students who could succeed in mathematics as the school defined success. It seems to us that non-academic students are seldom heard when high school mathematics curriculum is being developed – in fact, we found that non-academic students needed to *develop* their voices, not necessarily so that they could be heard within the curriculum development process, but as an essential part of achieving success in mathematics. In other words, the students developing the capacity to describe and name the procedures by which they achieved success in mathematics was, in their own view and ours, an essential part of that success. Below, four students each voice a particular aspect of what we now take as success for non-academic students in high school mathematics.

Erin: Success as a Mathematical Thinker

Erin was a grade 11 student who was retaking a grade 10 mathematics credit because she had dropped out of school the previous year. Erin's summed up her entry orientation to mathematics in the opening statement in her first journal of the year: "I'm bad at math." However, as her teacher (the second author) interacted with Erin, she noticed that Erin was thinking mathematically and was learning to describe her mathematical thinking. For example, consider Erin's solution to a unit price question on the sixth test of the semester. (See Figure 1.)

Serena wants to buy some T-shirts. Her favourite department store sells T-shirts individually, or in packages of two or three. One T-shirt sells for \$2.98, a package of two T-shirts sells for \$5.49, and a package of three T-shirts sells for \$7.89.

a. Find the unit price when T-shirts are sold:

i. in a package of two. ii. in a package of three.

$$\frac{\$5.49}{2} = 2.74 \text{ per shirt} \qquad \frac{\$7.89}{3} = 2.63$$

b. Which package offers the best unit price?

the package of the 3 for \$7.89 offers you the best unit price.

c. Suppose Serena wants to buy seven T-shirts. Which combination of packages will be the least expensive? Show your calculations and the total price.

package of 3 + 2 package of 2 =
7.89 + 5.49 + 5.49 = \$18.87
2 packages of 3 + 1 package of 1 =
7.89 + 7.89 + 2.98 = 18.76 ★ *Best Buy!*
3 packages of 2 + 1 package of 1 =
5.49 + 5.49 + 5.49 + 2.98 = 19.45

Figure 1.

Figure 1.

In the last part of this question, Erin systematically created possible combinations of T-shirt packages to determine which would be the least expensive. Erin used her calculations to show her thinking and support her reason for recommending a specific combination as the best buy. NCTM (2000) supports the inclusion of this type of problem and solution as fundamental to mathematical learning: “Systematic reasoning is a defining feature of mathematics” (p. 57). They define systematic reasoning as trying all the cases for a given problem and using those cases to support an argument. As teachers we could see what Erin failed to include – those combinations that more than one T-shirt purchased at the individual-shirt price. Yet, it is reasonable to assume that in answering Part A, Erin would recognize that there was no point to considering combinations that included more than one individually packaged T-shirt. If we accept that assumption, then all reasonable cases are worked out, and they are laid out in a logical and coherent manner. Erin successfully applied systematic reasoning in her problem solving and communicated her reasoning with the combinations of packages and the matching arithmetic.

A few days later, Erin selected that question as an example of good mathematical thinking. In her test reflection she wrote, “On the T-shirt question I just got all the prices and then compared them to see which one was lowest, it was a pretty easy question.” Erin demonstrated confidence in her mathematical reasoning, even though many of her classmates found it difficult to systematically construct various combinations of packages. But her reflective statement revealed more than increasing confidence. Erin was beginning to recognize the quality of her mathematical thinking and value her success as a mathematical thinker, a type of metacognitive process Schoenfeld (1992) considers one of the “critical aspects of thinking mathematically” (p.

363). It is significant, but not enough, when a teacher sees cognitive growth in the quality of Erin's written work. What is important is that Erin herself sees and expresses certain cognitive-growth qualities evidenced in her written work. She sees and expresses the systematic structure of her thinking, rather than only describing the arithmetic she did, and she sees her approach as changing the mathematics from hard to easy. Her mathematical growth is both cognitive and metacognitive (Schoenfeld, 2002).

Erin no longer saw herself as "bad at math." Erin was (re)forming her *identity as a mathematical thinker* as she engaged in mathematical thinking, noticed her thinking, and was in discourse about the quality of her thinking. This is most apparent at the end of the course when, in her final portfolio, Erin made the statement on multiple elements, "This shows I am a good math thinker because …". Afterwards, in our third conversation, Erin defined a mathematical thinker as "just what you think yourself … when you figure out ways to do it in your own way, you're a thinker." Erin's statements demonstrate growth along Chickering and Reisser's (1993) *moving through autonomy toward interdependence* vector, where "students' overall sense of competence increases as they learn to trust their abilities, receive accurate feedback from others, and integrate their skills into a stable self-assurance" (p. 47). As Erin began to trust her ability to think mathematically, she viewed herself as a mathematical thinker and valued her evolving identity as a success more significant than the mathematical thinking itself.

Karl: Success as a Mathematics Learner

Karl, a grade 10 student, explained in his first journal that he had decided to take Consumer Mathematics because, "the other two math courses would be way to hard [sic]." In contrast with Erin, Karl believed he could succeed in the course he had chosen, and he worked diligently in class to complete assignments and learn mathematical ideas. Trigonometry, the second unit of the year, provided a context in which Karl could re-view a topic he had found too difficult in his previous mathematics course. Specific topics in this Consumer Mathematics unit included similar triangles, Pythagorean theorem, finding sides and angles in right angle triangles with trigonometric ratios, and solving word problems involving right angles (METY, 2002). In his unit portfolio, Karl's new-found proficiency at trigonometry mattered to him because he saw trigonometry as "real math," with its symbols and formulas (conversation 1). Consider several of his reflections about assignments that he selected for inclusion in his unit portfolio:

Similar Triangles Activity: This was our intro to trigonometry. It demonstrates how much I remembered from last year. This demonstrates my Reasoning ability in that I reasoned with my self as to what I should do.

Trigonometry Assignment: This Item was one of our first hand outs it shows my ability to make connections, the labels opp, hyp, adj with the correct side. It shows how much I improved over last year.

Hand-in Assignment: This was an assignment I was absent for. I did it the day I handed this in. It shows what I learned in trig. It shows my Connecting ability (what I learned and what to do).

Overview Reflection: All in all I'd say I had greatly improved in trig since last year. From mediocre marks to very nice marks in the 80's and 90's. I learnt and reviewed all of the things from the last cupple of years as well as learning new things. So in summery, I learnt many new things, reviewed older things and remembered many older things and had a blast with getting good marks.

As his reflections show, Karl used marks to define his mathematical success at the beginning of the semester, expressing pride in his "very nice marks in the 80's and 90's." However, Karl was also beginning to author a different kind of success that moved beyond the mathematics of trigonometry in his portfolio. Without offering any glimpses into what didn't work for him the year before in a similar unit, Karl was identifying and describing the behaviours that he recognized as causing his success. Although he mentioned positive student behaviours such as completing missed assignments, Karl focused more on his cognitive processes. In part, he used key words provided by the teacher for portfolio reflections, including *reasoning* and *making connections*. However, the distinctions among *remembering* and *reviewing* previously seen material and *learning new things* were Karl's own understandings expressed in Karl's own terms. *Naming* (Freire, 2000) his learning allowed Karl an opportunity to identify himself as a successful learning – an individual who is aware of how he learns, intentionally identifying and refining strategies that support his learning. Karl was beginning to see himself as a capable *learner* of mathematics, able and entitled to voice what counted as success for him.

Karl was not just succeeding at the mathematics in the trigonometry unit, he was noticing and expressing cognitive aspects of that success. More than repairing his flawed understanding of trigonometry from the previous year, Karl was repairing his *identity as a learner of mathematics*.

Nadine: Success as an Individual

Nadine began Consumer Mathematics a month late, after moving from another province. This required her to read the textbook and practice questions from the Wages and Salaries unit independently. When she struggled with percent questions regarding wage increases, she came for extra help. Although she quickly learned how to do the questions in the one-on-one session, Nadine was intent on having the solution process "make sense," a phrase she used repeatedly during the session. Nadine found ways to express in her own words the significant idea that a percentage was a comparison of a portion to a larger whole, a significant mathematical concept (Hoyles, Noss & Pozzi, 2001) that had been absent in her mathematical cognition previously. Later, when she encountered wage increase questions, she explained how her understanding of percentages could be adapted to apply to this new kind of question. During the first cycle of teacher-authored narratives and one-on-one conversations, the teacher pointed to that original extra-help session: "For me, a key moment was when you insisted that you should *say* the steps for the wage increase questions. I'm wondering if it's really *your* words that make sense to you the best." At the time, Nadine valued the idea, and searched for words to say why. "'Cause it just shows how I, like, make myself learn. Like, people learn at a different pace. I learn at a weird pace, but anyway. Like, people, like, they learn different than others."

Two months later, in the second conversation, Nadine felt the theme of voicing her mathematical thinking was central not just to her success in mathematics, but to her image of herself:

> Well, when I read [the story] yesterday, I noticed a theme about thinking in words. Like, that's the main theme of my own story. … It kind of describes how I, like, started using the thinking in words and moved on from there. … And so then, that's how I do my thinking. And I think that's generally the theme of my story. … I like my theme! 'Cause it's about me. It's how I learn. It's not how, like, Cynthia, well, I don't know. Well, Cynthia does the same thing. But just as an example, it's me. It's not Cynthia. It's not you or whoever else. It's just me. It's just about me. It's not about whoever else there is, like this. Yeah!

In her success with percentages, Nadine had been able to see a particular learning strategy as effective. On one level, then, Nadine's story is about the importance of student voice simply as a tool for mathematical learning – saying what is coming to make sense (Mason & McFeetors, 2002), negotiating the meaning of mathematics through social discourse. In this conversation, however, Nadine's comments are not just about how she learned one particular idea or topic or even all of mathematics. When Nadine reflected on this process, she constructed an image not just of a general strategy that enabled her to learn mathematics, but an image of herself as a particular individual.

Nadine was (re)forming her *identity as an individual* when interactions which helped her to learn mathematics included discourse about the learning itself. Chickering and Reisser (1993) describe the establishment of identity as forming "a solid sense of self" contributing to "a framework for purpose" (p. 181). Similar to her lived experiences in class, the theme of Nadine's narrative of success and the conversations with her teacher supported her developing sense of herself as an individual. Nadine became purposeful in her mathematical learning as she used her general learning strategy – saying things in her own words – as a strategy particular to her. In fact, she named this as a success in the course when she wrote in her final portfolio reflection, "the learning strategies that I used in Math class this semester is 'Thinking in words'." Succeeding in learning mathematics that had not yielded to independent study enabled Nadine to see and give voice to her capability as a learner and her sense of herself as an individual. A mathematics course where she could succeed was an opportunity for Nadine to (re)form her sense of who she was.

Susanne: Success as a Mathematics Student

Susanne was repeating grade 10 Consumer Mathematics in part because she had not attended classes regularly the year before. But Susanne wanted a different outcome this time, and did more than come to class regularly. As she had done with Nadine, when writing Susanne's first narrative of success the teacher pointed to a positive moment that Susanne had initiated.

> One time that sticks out in my mind was a day when you had to leave class early. The next day was going to be the first test of the year. Before you left, you asked me what you could do to review for the test. I mentioned some textbook questions. I was really impressed that you were taking lots of responsibility – and I let you know. You had a surprised reaction, and I've seen the same reaction before from other students. Why are students surprised when I'm excited about their responsible behaviour? But, the key moment for me was the strategy you explained for doing well in Consumer Math: working hard, to stay on top of things.

Susanne's score of 85% on that first test was encouraging to her. Susanne's score meant different things to her and her teacher. To the teacher, the score indicated that Susanne had learned the content: she could calculate wages, deductions, and overtime pay, and she could provide rational arguments to support hypothetical employment decisions. To Susanne, the test score suggested that she could do this mathematics, and her approach to studying was a worthwhile process.

In the conversation about the narrative, Susanne credited her mathematical success to her decision at the beginning of the semester: "'cause other years before, like, I don't like math at all and I just completely given up. But now that I have had retake it again, I thought that I should, like, try and learn that I can do it." Susanne expressed a general no-nonsense strategy in her approach to mathematics class as: "pretty much stay focused. And just, do the work." She was more explicit about her approach to studying for tests: "going through your tests from before and … I make sure from when we mark things in class that I get the right answers. It's better to study for it because you know you have the right answers."

Although the test mark was generally encouraging for Susanne, it was also validating for her in regard to her approaches to mathematics class. She recognized her general and specific strategies as effective in supporting her goal of mathematical success. For Susanne, these strategies were not aimed directly at learning, but at a related goal: to achieve success in the course as the institution defined success. In a word, these strategies can be understood as a function of Susanne's *studenting* – the way in which Susanne fulfilled her stance as a student in the classroom. In general, the stance of a student would include attendance and punctuality, engaging in classroom processes, acknowledging the authority of the teacher, completing homework and handing in assignments – generally applying oneself to achieve the marks that qualify the student for the institution's definition of student success. Susanne's methods of studying and reviewing are indicative of a thoughtful and strategic commitment of time and attention toward her goal – to succeed this year as a high school student.

We rarely find students in Consumer Mathematics who independently commit to studying for tests. Rather than depending on the teacher to review the mathematical content with her, Susanne was intentionally finding and using strategies that would support her goals as a student. Susanne progressed from asking the teacher how to prepare for tests to discussing her steadily-developing studying strategies with the teacher, but the studying remained a matter of Susanne's initiative. Susanne was doing more than passively and obediently doing the work that the teacher assigned, and her conversations with the teacher came to reflect more than recognition of the teacher's authority (Belenky, Clinchy, Goldberger, & Tarule, 1986). She was recognizing for herself what strategies made her studying effective, rather than relying on the teacher to tell her what strategies to use. Her stance as a student had approached and passed the "point at which a person sees authority as an internal agent rather than as an external agent. … It is here that one begins crossing the bridge from a submissive orientation to a position in which one's voice is a significant determiner of what one believes" (Cooney, 1994, p. 628).

Susanne was in the process of (re)forming her *identity as a competent student of mathematics* – a much different stance than she had the previous year in Consumer Mathematics. The confidence she developed as she used her strategies to complete assignments and tests correctly contributed to her authority over her studenting processes. One of the final statements Susanne made in our last conversation was, "I'm a different student because I know now that I can do it. And so I feel more confident going into my classes. And, I want to do it. I want to understand it. I want to do it better." Although Susanne experienced mathematical success, especially on the first test, the success she points to in the conversation moves beyond mathematics. The success that Susanne valued was her new-found sense of being an effective student in mathematics class. It was a success that Susanne believed would help in future mathematics courses because she would approach the course with confidence in her ability to be a student of mathematics.

Success and Non-Academic Mathematics

One of our fundamental intentions in this paper has been to illustrate the range of positive experiences for non-academic students in mathematics class when learning opportunities are compatible with their strengths and needs. Erin grew from opportunities to apply and develop her problem-solving skills. Karl benefited by successfully learning what he saw as real mathematics, mathematics that he had not been able to learn in a more academically-oriented environment. Nadine depended on opportunities to talk mathematically while she was learning. Susanne benefited from interacting with her teacher about how she personally could succeed, but then develop her own strategies as a student. As they each found different elements of the same course to be significant to them, they each achieved different kinds of personal success.

But perhaps a more basic messages can be drawn from what all four of these students experienced. All the students participated within a common course with well-scaffolded access to content that was within their competence and suited their pragmatic interests. Within a relational pedagogy (Noddings, 1984; van Manen, 1986), these students' voices were valued and encouraged. Experiencing success and giving voice to how they achieved their successes enabled all four students to engage in (re)forming their identities as mathematical thinkers, mathematical learners, individuals, and students of mathematics. In the second conversation, Susanne described the transformative nature of succeeding in mathematics.

> So, then you start, not liking it more, but liking the fact that you want to do it more. … If you get involved, you get more into it. You don't start liking it because you like math, you start liking it because you're able to understand more about it. And because you're understanding what to do and how to do that. And then you start to like it because you eventually start to understand it.

There is now a significant body of literature that documents the failures of mathematics programs for non-academic students such as streaming or tracking. Detracking appears also to be a strategy that has not considered how non-academic students' particular educational needs might be addressed. Unfortunately, programs that fail to achieve success for all students can tell us only what *not* to do in further curriculum and course development, and offers research details only about *barriers* to achieving mathematical understanding. The results of this study support the legitimacy of mathematical success for all as a goal for high school mathematics, if we provide non-academic students with access to accessible content within an appropriate pedagogy. And the details of the study suggest that non-academic students may be an under-utilized source of data for research on the range of factors that affect and comprise mathematical success. Success-for-all as a curriculum-reform ideal offers hope that we might challenge how high school mathematics currently sorts and filters students, and find ways to achieve "the desired outcome of mathematical power for all students within the chaotic reality of real students in real schools" (Steen, 1992, p. 259). More imagery about mathematical success for non-academic students is needed. We hope this study shows that both success-for-all and the nature of that success is within our collective reach as curriculum developers and researchers.

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EXPLORING AN ELUSIVE LINK BETWEEN KNOWLEDGE AND PRACTICE: STUDENTS' DISCIPLINARY ORIENTATIONS

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This paper focuses on examining the link between students' mathematical experiences and the epistemological orientations that they develop towards mathematics. Data for this paper came from a four-year longitudinal study of mathematics teaching and learning in three schools. Our analysis focused on four years of interviews from students in these schools. We found that many students developed one of three stances towards mathematics: inquiring*,* passive*, and* resistant*. These stances were made up of a range of complex beliefs about mathematics, including ideas about the* nature *and* purpose *of mathematics, the nature of* authority *and their ideas about learning. Our analysis suggests that the stances that students develop are not defined by beliefs or by learning preferences alone, but that students' experiences with curriculum, success, and outside school activities also contribute importantly to students' relationship with mathematics.*

Objectives

The purpose of this paper is to examine the link between students' mathematical experiences and the epistemological orientations and stances that they develop towards mathematics. In keeping with the theme of this conference, building connections between communities, this paper seeks to make an important connection highlighted by Schoenfeld in his AERA presidential address (1999) between research that focuses on the social perspective and research that focuses on the individual perspective. We will illustrate how it is that individuals develop stances towards mathematics through social practice and map out three different stances that students are likely to develop. Our research draws from a large data set of approximately 1000 students who experienced different mathematics approaches over four years of high school.

Perspectives

There is now widespread awareness that learning does not only involve growth in knowledge, but the development of a relationship with a particular domain, what some have defined as the development of an 'identity' as a learner (Wenger, 1998). Specifically, the kinds of opportunities that students have to interact with mathematical content have been hypothesized to have a profound impact on their sense of who they are. Although our field appears unanimous in their support of understanding students' learning identities, to this point relatively little work has explored the range and nature of identities that are available to students (except Hodge, McClain, & Cobb, 2003; Gresalfi, in progress; Nasir, 2002).

It is clear that students' beliefs about themselves in relation to a discipline are consequentially related to their performance in that discipline. For example, Martin (2000) investigated mathematically successful and failing African American students in an urban middle school. Martin found that the successful students that he studied subscribed to an achievement ideology, and had reconciled this aspect of their core identities with the normative identities as doers of mathematics. In contrast, the oppositional personal identities that the failing students were developing suggest that they experienced irreconcilable conflicts between their core identities and the normative classroom identities established in their classrooms. Like the working class boys that Willis (1977) studied in England, these students were active contributors to the processes that delimited their access to significant mathematical ideas.

D'Amato (1992) addressed similar issues in his discussion of two ways in which learning in school can have value to students. Students who attach *structural significance* to mathematics believe that achievement in school has instrumental value as a means of attaining other ends such as entry to college and high-status careers, or acceptance and approval in the household and other social networks. In contrast, students who attach *situational significance* to mathematics view their engagement in classroom activities as a means of maintaining valued relationships with peers, and of gaining access to experiences of mastery and accomplishment. This work highlights the vast differences that can emerge in students' rationale for engaging with mathematics, an idea that becomes especially intriguing when considering students who traditionally do not have access to motivators that have structural significance.

In earlier work we (Boaler & Greeno, 2000) found that students developed ideas about what it meant to learn mathematics that were linked with the kinds of opportunities that they had to engage with mathematical practices. That study drew on interviews with high school advanced placement calculus students who were in classes with very different structures: either didactic and individualized, or more inquiry-based, encouraging collaboration and discussion. Many of the students who were enrolled in the didactic classes indicated that they felt that they had to give up agency and creativity if they were to take more advanced mathematics courses. In contrast, students who were enrolled in classes that were inquiry-based developed very different ideas about mathematics, and felt that it was a discipline with which gave them space to engage with mathematics in a creative way that went beyond memorization. These positions were consistent with the epistemological positions of 'received or 'connected' knowers that were identified by Belenky, Clinchy, Goldberger & Tarule (1986).

In this paper we develop these initial analyses of epistemological positions that students take on through analysis of a longitudinal study of a cohort of students learning mathematics through four years of high school. This analysis has worked at the intersection of two approaches to studying identity: one that focuses primarily on the beliefs that students develop (Schoenfeld, 1988; McLeod, 1992), what Cobb & Hodge (in press) refer to as *core* identity, and one that focuses primarily on the establishment of particular classroom structures (Cobb, Wood, Yackel & Perlwitz, 1992), what Cobb & Hodge refer to as the intersection between a *normative* and *personal* identities This latter aspect of identity, which focuses on students' participation in a classroom system, unpacking how and when students take up opportunities that are available in their classroom settings, has received less attention than the former version of identity. Understanding how these aspects of identity get constructed in different classrooms for individual students is crucial to developing a more complete understanding of the consequences for learning that are associated with different mathematical practices.

This paper proposes that an important part of learning involves stances towards mathematics that students develop and take with them into their lives. The notion of *stance* has proved helpful to us in capturing the ways that students approach and deal with mathematics. Stance goes beyond belief and orientation to the ways in which people interact with mathematics in their lives. We believe that there is an important place for understanding stances in relation to particular *disciplines*, even though previous work has tended to focus more generally on students orientations towards knowledge, broadly conceived (c.f. Belenky et al, 1986). The teaching approaches we have studied create different positions for students in mathematics, and the act of taking up these roles and positions leads to different stances towards mathematics. These are important to understand if we are to comprehend students' mathematical capability and their life-long engagement with mathematical ways of working and thinking.

Methods

Data for this paper came from a four-year longitudinal study of mathematics teaching and learning in three schools. In two of the schools - Greendale and Hilltop - two different approaches to mathematics are offered, one of which is open-ended and reform-oriented, and another which is traditional. The traditional curriculum is made up of short, closed problems that emphasize the precise use of different procedures. The pedagogy that accompanies this curriculum is also traditional and may be characterized as demonstration and practice. At the third school, Railside, students take the traditional sequence of courses, but the mathematics is presented through longer problems that emphasize multiple connections and methods, with constant group work. In addition to large scale monitoring of student achievement and beliefs, we have chosen "focus classes" from each approach in each school each year. The data for this paper concentrates on four years of interviews from students in these focus classes. Every year a sample of students were interviewed, with the selection intended to represent different achievement levels, genders and ethnicities. Students' orientations to mathematics were probed through questionnaires and interviews given each year. Extensive reading and coding of approximately one hundred hour-long interviews produced a range of well-defined orientations towards mathematics learning. Our paper will consider these different orientations as well as the classroom participatory structures that lead to their development.

Results and Discussion

The development of students' epistemological beliefs in relation to *general* knowledge has received significant attention from many researchers (Belencky et al, 1986; Perry, 1968; Ryan, 1984; Schommer, 1990). We were interested in thinking about students' orientations more specifically, as they related to the domain of mathematics. As we studied interview data and classroom interactions we realized that students were developing very different stances towards mathematics knowledge. The three stances we most commonly observed may be described as *inquiring*, *passive*, and *resistant*. These stances were made up of a range of complex beliefs about mathematics and learning. In particular we noted that students' beliefs about the *nature* and *purpose* of mathematics, the nature of *authority* and their ideas about learning were critical in the stances they developed.

The inquiring stance that some students developed was particularly productive. Some students told us that when they saw an interesting fact or set of data, they wanted to find out more and they would use mathematics to ask questions and probe the relationships they observed. For example, the following student who studied IMP had developed such a stance:

> Like, um, I don't know. If nothing else, it's just breaking out of the pattern of just taking something that's given to you and accepting it and just, you know, going with it. Like political things that happen and, you know, media things. It's just looking at it and you try and point yourself in a different angle and look at it and reinterpret it….It's like if you have this set of data that you need to look at to find an answer to, you know, if people just go at it one way straightforward you might hit a wall. But there might be a crack somewhere else that you can fit through and get into the meaty part. (IMP4, y4)

Students who adopted a more *passive stance* tended to see mathematics as an external domain of knowledge that they did not connect with in a personal way. When they encountered an odd or intriguing mathematical situation they did not think it was appropriate to inquire further. Mathematics was a remote body of knowledge that they may have been able to use successfully, and may even have enjoyed learning but that they were not inclined to inquire about. The following student in advanced algebra captures the idea of a passive stance. When asked what she thought about mathematics she said: "Some of it just doesn't really seem like I can relate it to what I want to do or something. I don't know. Some of it just seems ridiculous to me." Later we asked what she would do if she encountered a new problem: "Oh I usually try to remember formulas that I could use for it. I'm not really good at remembering formulas, but I can look back in my notes if we have it."

The third group of students had developed a *resistant* stance with more negative ideas about mathematics. These were students who perceived themselves as unsuccessful and had withdrawn from mathematics to the extent that new mathematics problems caused them anxiety, anger or fear and they would avoid mathematics at all costs. The following student from the traditional sequence captures this stance. When asked to describe a good mathematics lesson he could not think of any. When asked to describe a bad lesson he said: "Anything… yeah stuff that is plain out math. I hate that, like what we are doing right now. Anything involved in graphing, linear equations, whatever, I hate it all."

Our analyses of the different stances revealed three dimensions that contributed towards them. Specifically, students' orientations towards *learning* mathematics (how do I like to learn?), their beliefs about the *nature* and *purpose* of mathematics (what is mathematics and why do we learn it?) and their ideas about mathematical *authority* (how do I know if something is correct?). We discovered that the stance that students developed came about through a combination of their ideas about learning and the nature and purpose of mathematics and authority, but their stance was not defined by any one dimension. For example, we found that it was possible for students to see mathematics as a set of procedures and yet be prepared to inquire actively when given the opportunity. Similarly some students liked to learn mathematics by being told methods but they saw mathematics as a set of exploratory tools and were prepared to use mathematical methods in that way. We also discovered that students' stances were strongly related to their curriculum and pedagogical experiences, but they were not determined by them. This could be heard in students' discussion of the different influences in their lives, including parental ideas and experiences and students' experience of clubs and activities in and out of schools. The following sections further unpack the dimensions of students' stances that emerged in our interviews.

How do I like to learn?

Students varied in their ideas about learning, and their preferences for different pedagogical practices. Some students wanted to be told everything, while others were concerned to take a more active role in their learning. Some students wanted to be given work that was easy and straightforward whereas others thought that it was productive to be given demanding problems.

Many students in the traditional curriculum developed a 'received' approach (Belenky et al, 1986) to learning mathematics, an approach that contributed significantly to the development of a passive stance towards mathematics. These students talked about mathematics as a discipline that was a fixed domain that they must remember and reproduce, not a set of ideas that they could relate to and understand. These ideas about learning are is reflected in the following excerpt from a student who was in his first year of the traditional program:

Int: So what do you think it takes to be successful in math?

St: A big thing for me is, like, paying attention because he'll, like, teach stuff steps at time. It'll be like here's a step, here's a step. And, like, if I doze off or, like, don't know what's going on or, like, daydreaming while he's on a step and then he, like, skips to the next step and I'm like, "Whoa. How'd he get that answer? Like,

> where am I? I'm retarded." And then he'll have to come help me. So paying attention (Algebra, y1)

In this excerpt the student communicated his belief that mathematics can be deconstructed into steps, each of which needs to be attended to and memorized. His role in this process was primarily that of attending to new information.

What is mathematics and why do we study it?

The nature and purpose of mathematics are two different concepts but they were closely related in students' ideas. Under the nature of mathematics we identified three main dimensions. The first, identified above, cast mathematics as a set of procedures. The second dimension focused on mathematics as a range of procedures connected by concepts and ideas.

> L: I think it's more about ideas, it's not "oh, I need to know how to do this exact problem" cause when it comes along in the world, it's not gonna come as that problem. It's just, the thinking, and getting the ideas, and how to come up with it.
>
> J: It's sort-of like a discipline (Railside, y4)

The third category was unexpected – students who learned mathematics at Railside, which placed significant emphasis on the communication of mathematical ideas through different representations, developed the idea that mathematics was a form of communication and its purpose was to give different insights into bigger ideas. These students saw mathematics as a communicative domain.

> I think math is kind of like a language because it's got a whole bunch of different meanings to it and no matter what the problem is there is always a solution and I think that it is communicating so I think I would call it a language. (Railside, y1)

The categories of procedural, combined (conceptual and procedural) and communicative related closely to the mathematics approaches students experienced. Only the Railside students who were consistently encouraged to communicate ideas to each other and to use different representations described mathematics as a communicative tool. The students who saw mathematics as only a set of procedures were all studying in the traditional classes. The students who saw mathematics as a combination of concepts and procedures were in all three classes. We have not got the space in this paper to consider the students' ideas about the purpose of studying mathematics, but they were closely linked to their ideas about the subject. Those who saw mathematics as a set of procedures tended to see little use for the subject other than the acquisition of grades, whereas those who saw mathematics in a broader way also thought it was a subject that would help them in life.

Authority

The other distinctly different beliefs that students developed and that influenced their stances towards knowledge could be heard in their ideas about authority in mathematics. Some students came to believe that answers were sufficient or correct only when teachers or the book said they were. Others developed the idea that the mathematical tools they learned allowed them to *reason* about situations and determine whether questions were correctly answered or not. In the traditional classes the textbook and the teacher were usually presented as the ultimate authority and students tended to look to them to know if they were moving in the right directions. In the IMP and Railside classes teachers often told students that they could determine whether they were correct or moving in the right direction by *reasoning* mathematically. In interviews the students' ideas about mathematics closely matched their classroom experiences.

The students' ideas about authority were an important contributory factor to their ultimate stances. For students to develop an inquiring stance towards mathematics they needed to believe that they had the authority to reason about ideas. It seems unlikely that students who saw mathematics as a set of rules that only the teacher or the textbook could validate would feel confident inquiring about mathematical problems.

Conclusion As has been demonstrated repeatedly in the past, students' beliefs about their ability (Ames, 1992; Eccles et al, 1993) and their ideas about particular disciplines (Boaler & Greeno, 2000) are crucial in understanding how students learn mathematics, and whether they will continue to participate in the domain. In addition, the stances that students develop towards mathematics appear to be critical to students' use of mathematics in important mathematical problems (Fiori & Boaler, 2004). Our data analysis suggests that the stances that students develop are not defined by beliefs or by learning preferences and experiences, although these play a part in their development. Rather, the stances relate to students' experiences with curriculum, success and outside school activities. The curriculum at any school did not completely define the students' experiences (Gresalfi, 2004) and some students emerged with unexpected stances. If the Railside and IMP approaches were to be evaluated based on the stances towards knowledge that students developed, both programs would be regarded as highly successful, while the traditional curriculum would not. However, our current climate values tests success more than students' stances towards mathematics, and students scored at equal levels when they learned through the traditional or IMP curriculum. We believe that it is critical to appreciate stances in evaluating mathematics approaches, partly because our data showed them to be critical in influencing students' participation and understanding (Fiori & Boaler, 2004). As the field becomes more cognizant of the importance of the identities and stances that students develop in mathematics classrooms it is our hope that policies that influence curriculum directions also pay attention to these critical dimensions.

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ENCULTURATION: THE NEGLECTED LEARNING METAPHOR IN MATHEMATICS EDUCATION

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Metaphors for learning abound in education. Sfard (1998) suggested a distinction between the *acquisition metaphor* in which skills or concepts are learned by students, and the *participation metaphor* in which Alearning a subject is now conceived of as a process of becoming a member of a certain community@ (p. 6). As she noted, there has been a shift in the pedagogical discourse in recent years from acquisition to participation metaphors. However, the National Council for Accreditation of Teacher Education (NCATE, 2002) identifies skills, knowledge, and dispositions all as important learning outcomes for educators to address, suggesting that the concerns of an era cannot be reduced to a single metaphor. Consistent with Sfard and with NCATE, in my own work I have identified three key metaphorsBlearning as *habituation*, learning as *(conceptual) construction*, and learning as *enculturation*Bthat I see as underlying current pedagogical recommendations and disputes: habituation and construction motivating traditional pedagogy; construction and enculturation motivating reform pedagogy (Kirshner, 2002).

These metaphorical notions of learning are variously addressed in learning theories. Behaviorism and some parts of cognitive science (e.g., the ACT theory of John Anderson and his colleagues) explore the conditions and processes through which skills become habituated through repetitive practice. Psychological constructivist theories stemming from Piaget=s genetic epistemology describe how conceptual structures come to be restructured and strengthened through perturbations stemming from discordant experiences. Sociocultural, situated cognition, and social constructivist theories examine how cultural dispositions are appropriated through cultural participation.

The current interest in enculturationist theory and practice is evident throughout the educational literature. The mathematics education reform documents display an especially strong interest in enculturation/participation. If we take, as do I, modes of thinking (as distinct from specific conceptual understandings) to be enculturated dispositions, the NCTM=s (1991) objectives that students come to Aexplore, conjecture, reason logically; to solve non-routine problems; to communicate about and through mathematics ... [as well as] personal self-confidence and a disposition to seek, evaluate, and use quantitative and spatial information in solving problems and in making decisions@ (p. 1) all reflect an enculturationist learning agenda.

Given the burgeoning educational interest in enculturation, and in the sociocultural, situated cognition, and social constructivist theories of learning that address it, Aneglected@ might seem to be the last adjective to apply to this learning metaphor. However, none of the theories that pursue enculturation do so unifocally. For instance, Lave (1988) Ain dialectic spirit@ describes how for situated cognition theory the Aunits of analysis, though traditionally elaborated separately [for social and individual cognitive theories], must be defined together and consistently@ (p. 146). Similarly, although Vygotsky (and the ensuing sociocultural tradition) gives clear priority to the *intermental* (social) plane (Wertsch, 1985),

> Sociocultural processes on the one hand and individual functioning on the other [exist] in a dynamic, irreducible tension rather than a static notion of social determination. A sociocultural approach ... considers these poles of sociocultural processes and individual

functioning as interacting moments in human action, rather than as static processes that exist in isolation from one another. (Penuel & Wertsch, 1995, p. 84)

This dialectic orientation for enculturation-oriented theories can be contrasted with the unifocal character of behaviorism, cognitive psychology, and (psychological) constructivism that study the individual constitution of learning. For instance, Greeno (1997) describes the *Afactoring assumption@* of cognitive science: Awe can analyze properties of cognitive processes and structures [independently] and treat the properties of other systems [e.g., social systems] as contexts in which those processes and structures function@ (p. 6)Ba characterization Anderson, Reder, & Simon (1997) readily accept. Similarly, constructivism, in its Piagetian origins and its initial radical variation in mathematics education, examined conceptual structures from a unifocal individualist perspective:

> Von Glasersfeld acknowledges a significant debt to Piaget, which may explain why he focuses on the individual knower, and pays scant attention to the social processes in knowledge construction. (Von Glasersfeld=s ... educational concerns of course lead him to address the role of the teacher. But he faces severe problems of consistency here: It is clear that in much of his writing von Glasersfeld problematizes the notion of a Areality@ external to the cognitive apparatus of the individual knower/learner. But as a result, it is difficult to see how he can consistently allow that social influences exist....) (Phillips, 1995, p. 8)

Within the rich mix of psychological theories that ground our pedagogical discourse, my concern is that the multifocal theorizations of enculturation index the second class status of this learning metaphor in teaching. Consider, for example, the behaviorist, cognitive, and situative rubrics offered by Greeno, Collins, and Resnick (1996) in their overview of learning theory and education. Whereas the first two are unifocal in their pedagogical orientation, the situative approach to education is integrative: ASequences of learning activities can be organized with attention to students= progress in a variety of practices of learning, reasoning, cooperation, and communication, as well as to the subject matter contents that should be covered@ (p. 28). Enculturating students toward modes of engagement (e.g., Apractices of learning, reasoning, cooperation, and communication@) is never addressed educationally as a bona fide pedagogical focus in its own right; only in conjunction with the (predominating) interests in developing students= skills and concepts.

This concern needs to be couched within the *crossdisciplinary perspective* (Kirshner, 2000, 2002) that frames the current analysis. Crossdisciplinarity offers a broad critique of the integrative tendency of our pedagogical discourse in which Agood teaching@ functions as a unitary construct. The basis for this concern is the simple observation that psychological theory has not yet succeeded in establishing a paradigmatic consensus about learning. Rather, in its current *preparadgimatic* state (Kuhn, 1970), multiple notions of learning compete with one another for paradigmatic hegemony. Because Agood teaching@ is teaching that supports learning, until a consensus about learning is achieved we need to be suspicious of any formulation of good teaching that claims to generality. For although integrative theorizations are offered in the situated cognition, sociocultural, and social constructivist camps, none has yet succeeded in establishing more than a toehold in the broader theoretical spectrum, and each pays a heavy price in clarity and accessibility for taking on the dialectic challenge of bridging across independently sensible metaphors for learning (e.g., Kirshner & Whitson, 1998; Lerman, 1996).

The crossdisciplinary alternative is to articulate discrete, theory based models of good teaching for the discrete learning metaphors. This process requires that each of the three metaphors be independently interrogated as to its implications for teaching, leaving to teachers the values decisions as to which notion(s) of learning to pursue with their students, as well as the tactical problems of coordination and balance in case more than a single metaphor is aspired to.

Thus from a crossdisciplinary perspective, enculturation cannot remain in the shadow of other metaphors, but must step into the limelight as a bona fide pedagogical agenda in its own right. This is the task of the present paper, a task made considerably more difficult by the fact that unifocal theorizations of enculturation processes are not available.

Enculturation as a Metaphor for Learning

I define enculturation as the process of acquiring cultural dispositions through enmeshment in a cultural community (Kirshner, 2002). I interpret dispositions broadly as inclinations to engage with people, problems, artifacts, or oneself in culturally particular ways. Thus establishing an enculturationist teaching agenda requires identifying a reference culture and target dispositions within it. In mathematics education, the reference culture usually is presumed to be mathematical culture, wherein a wide range of distinctive dispositional characteristics has been identified as instructional objectives. These include mathematical proof, the characteristic mode of argumentation by which new knowledge is established for the community through logical (rather than empirical) considerations (Lampert, 1990); a single-minded tenacity in grappling with non-routine problems (Schoenfeld, 1994), together with highly specialized heuristic approaches to solving such problems (Polya, 1957); an aesthetic appreciation of the Amathematically elegant@ solution (Yackel & Cobb, 1996); a recognition of the instrumentality of notations and the arbitrariness of definitions within axiomatic systems (Arcavi, 1994); and a propensity for posing problems, rather than just solving them (Brown & Walters, 1990). (See, also, Cuoco, Goldenberg, & Mark, 1995, for a list of Ahabits of mind@ specific to the various mathematical subbranches.)

Lacking a foundation for enculturationist learning in unifocal learning theory, I turn to social psychology for insight and inspiration to inform pedagogical methods. (Ironically, social psychology functions more as a branch of sociology than of psychology. Social psychologists tend to focus on the effects and distribution of enculturated learning, rather than the psychological processes subserving it.) A paradigm example of enculturation is explored by social psychologists under the rubric of proxemics (Hall, 1966; Li, 2001). Proxemics, or personal space, is the tendency for members of different national cultures to draw differing perimeters around their physical bodies for various social purposes. Thus, natives of France tend to prefer closer physical proximity for conversation than do Americans (Remland, Jones, & Brinkman, 1991). I count coming to participate in this cultural norm a particularly pure instance of enculturation because it is accomplished without volitional participation. Generally people within a national culture acquire proxemic dispositions through cultural enmeshment without intending it, and even without awareness of the cultural norm.

This pure form of enculturation is possible in a unitary culture in which only a single dispositional variation is present. However, one also can come to be enculturated into a subculture whose dispositional characteristics are distinctive among a range of other subcultures= (e.g., being a scientist, being a punk rocker, etc.). In such instances, inductees often seek to actively acculturate themselves to a subculture, thereby bringing volitional resources to acquiring the subculture=s dispositional characteristics. I define *acculturation* as intentionally Afitting in@ to a cultural milieu by emulating the cultural dispositions displayed therein. However, this process needs to be understood as supplementary to the more basic unconscious processes of enculturation going on around it all the time. A cultural milieu is constituted of innumerable cultural dispositions, of which only a limited number can be consciously addressed through strategies of acculturation. Note, that Vygotsky=s (1987) Zone of Proximal Development conceives of learning in acculturationist terms as an active collaboration between student and teacher: AA central feature for the psychological study of instruction is the analysis of the child=s potential to raise himself to a higher intellectual level of development through collaboration to move from what he has to what he does not have through imitation@ (p. 210).

Enculturationist and Acculturationist Pedagogies

The enculturation/acculturation distinction points to two pedagogical strategies that can be discerned in the education literature. (Here, regretfully, I make a terminological distinction between enculturation as a *learning process* that may [or may not] include an acculturationist component, and enculturation as a *pedagogical method* conceived of as distinct from acculturationist pedagogy.)

Enculturationist Pedagogy: In any teaching that aims toward students= enculturation, the teacher begins by identifying a reference culture and target disposition(s) within that culture. In enculturationist pedagogy, the instructional focus is on the classroom microculture. The enculturationist teacher works to shape the microculture so that it comes to more closely resemble the reference culture with respect to the target dispositions. Students, thus, come to acquire approximations of the target dispositions of the reference cultural through their enmeshment in the surrogate culture of the classroom. Yackel and Cobb (1996) most clearly articulate an enculturationist pedagogical agenda in their discussion of *sociomathematical norms* as the targeted dispositions of mathematical culture (e.g., the preference for mathematically elegant solutions) that come to be Ainteractively constituted by each classroom community@ (p. 475).

Enculturationist pedagogy presents the teacher with an obvious >chicken and egg= problem. Students can acquire the target dispositions only to the extent these dispositional characteristics already are constituted within the classroom microculture. However, in order for the classroom culture to embody these dispositional norms, (at least some) students must already manifest them in their interactional repertoire within the classroom. Yackel and Cobb (1996) borrow the construct of Areflexivity@ from ethnomethodology (Leiter, 1980; Mehan & Wood, 1975) to elucidate the problem:

> With regard to sociomathematical norms, what becomes mathematically normative in a classroom [i.e., the corporate dispositions of the classroom microculture] is constrained by the current goals, beliefs, suppositions, and assumptions [i.e., the individual dispositions] of the classroom participants. At the same time these goals and largely implicit understandings [the individual dispositions] are themselves influenced by what is legitimized as acceptable mathematical activity [the corporate dispositions of the classroom microculture]. It is in this sense that we say sociomathematical norms [the target dispositions of mathematical culture] and goals and beliefs about mathematical activity and learning [the currently manifest dispositions of individual students] are reflexively related. (p. 460)

(In their theoretical perspective, Cobb and Yackel, 1996, mark a terminological distinction between individual and social perspectives that I find unnecessary for a crossdisciplinary approach, hence the explanatory bracketed insertions.)

The solution to this problem constitutes the critical expertise of the enculturationist teacher. As Yackel and Cobb (1996) illustrate, through subtleties of attention and encouragement the teacher, over time, can come to exert considerable influence on the modes of engagement manifest within the classroom microculture. It is through patient and directed encouragement that targeted modes of engagement, initially manifest within the classroom microculture by happenstance, gradually come to be normative. In this way, for example, argumentation usually based on deference to authority or on empirical generalization can progress toward the norms of logicality favored by mathematical culture.

In nurturing a more sophisticated classroom microculture, the enculturationist teacher is not limited to the (relatively passive) tools of encouragement. Teachers also are members of their classroom communities, and can introduce modes of engagement through their own participation. What is crucial, however, in enculturationist pedagogy is that it is participation in the culture of the classroomBrather than emulation of the teacher as a solitary individualBthat continues to serve as the engine for students= acquisition of dispositional characteristics. To be effective, the teacher him or her self must be significantly knowledgeable about, and enculturated to, the reference culture. However, once the modes of engagement introduced or supported by the teacher come to *signify* as mathematical, this affords students who are mathematically identified the opportunity to bypass the surrogate microculture of the classroom and connect directly with the authentic culture of mathematics as manifest in their engagement with the teacher. In this case, the teaching role is significantly altered as we leave the realm of enculturationist pedagogy and verge into the acculturationist terrain with all the attendant complexities of personal identity.

Acculturationist Pedagogy: I open this section with a brief anecdote. I recently had the opportunity to co-teach a senior level university mathematics course with two mathematics colleagues. The purpose of the course was to help students understand, appreciate, and participate more fully in mathematical culture. My colleagues, both senior members of a highly ranked mathematics department, were accustomed to, and successful in, the mentoring of doctoral students. The approach they took in our course involved assigning the students problems, discussing the problems with them, and in the process modeling their own (unprescripted) solution approaches, following fascinating tangents arising from the original problem, communicating their broad perspectives on mathematics, and sharing their excitement and passion for the field. I presume these are methods they would typically employ, with good effect, with their graduate studentsBstudents already self-identified as mathematicians. However, the undergraduate students in the courseBthough seniorsBgenerally were unable to appreciate or make use of the rich cultural resources offered by the instructors.

This cautionary tale serves as an introduction to acculturationist pedagogy, a pedagogical method that builds on (or supports) students= identification with the reference culture. The acculturationist teacher is first and foremost a representative of the reference culture. The primary pedagogical activity is modeling dispositional characteristics of the culture. It is left to the students to appropriate these cultural resources and incorporate them into their evolving repertoire based on their own acculturationist goals. Or acculturationist pedagogies may seek to encourage cultural identification, as in Brown and Campione=s (1996) strategy of positioning students as experts on a particular scientific topic and involving them in email collaboration with actual scientists. The concern in the situated cognition literature for Aauthentic activity@ (Brown, Collins, & Duguid, 1989, p. 34) and Alegitimate peripheral participation@ (Lave & Wenger, 1991) are indicative of the acculturationist bent of that pedagogical movement.

In practice, the distinction between enculturation and acculturation pedagogies can be subtle. In his classic volume, mathematician George Polya (1957) described his pedagogical role in modeling the self-questioning strategies that undergird successful problem solving in mathematics. However, he was careful to emphasize the need to be unobtrusive and natural in supporting the students= own efforts with ongoing problems: AThe teacher should put himself in the student=s place, he should see the student=s case, he should try to understand what is going on in the student=s mind, and ask a question or indicate a step that *could have occurred to the student himself@* (p. 1). In this respect, Polya demonstrated an enculturationist concern for the evolving microculture of the classroom problem solving situation rather than an acculturationist appeal to the mathematical self-identity of the student.

There are some circumstances such as graduate education or after school math clubs in which acculturationist approaches seem clearly appropriate. Other circumstances, such as that described in the above anecdote, clearly are unsuitable. Those mathematics seniors needed an enculturationist pedagogical approach in which the forms of participation were interactively constituted, rather than just demonstrated or modeled. (I believe mathematics has a more pronounced problem than other subject areas in the lack of disciplinary enculturation generally achieved by undergraduates.) However, the extant pedagogical literature concerned with students= enculturation (e.g., articles cited herein) includes, without distinction, reference to both enculturationist and acculturationist techniques. This practice flirts with a variety of potential problems that will need to be addressed before enculturationist learning goals can achieve the status they deserve in education:

! Are acculturationist and enculturationist pedagogies inherently in tension with one another? Does the personal self-identification of some students with the teacher as a representative of the reference culture subvert the work of establishing a classroom microculture that serves all students; or can a skillful teacher use the acculturationist gains of the few to support and strengthen the classroom microculture for the many?
! Are there social chasms that emerge in a classroom in which the teacher reciprocally supports the identity construction of a few students? How do such chasms interact with divisions of race, class, and gender already present in the classroom? More generally, are there ethical considerations that arise in general education when a teacher places expectations of a particular cultural identification on students? If so, are such concerns outweighed by the importance for all students to have opportunities for identification with disciplinary cultures?
! Are (teacher centered) acculturationist practices in which the teacher embodies cultural dispositions used to substitute for the delicate and difficult (student centered) work of nurturing those dispositions within the evolving classroom microculture? (The analogy, here, is to lecture, understood within crossdisciplinarity as a teacher centered approach to students= conceptual development that relies on students= metacognitive sophistication to bring dissonant understandings into productive contact with one another. Otherwise, the student centered constructivist teacher must take on the responsibility for orchestrating cognitive dissonances through carefully contrived task experiences.)

The enculturationist/acculturationist distinction introduced here previously is unnoted in the literature. As a result, the possibility for a pure enculturationist pedagogy, and the potential problems of blending enculturation with acculturation pedagogies, have not been addressed. I count it a strength of the crossdisciplinary approach that unifocal attention to the learning metaphors brings forth such distinctions, with all of their attendant possibilities and problems.

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THE IMPORTANCE, NATURE AND IMPACT OF TEACHER QUESTIONS

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In this paper we explore teacher questions from a number of perspectives. We look at the broader activity contexts in which questioning takes place, we present a coding system for teacher questions and we explore qualitatively what such a coding scheme might tell us. We reflect on the grain size that is helpful in understanding differences in teaching, and we argue that teacher questions provide an important methodological lens for understanding relationships between teaching and learning. We also consider how teacher questions help shape the flow and direction of lessons.

Introduction

Teacher questioning has been identified as a critical and challenging part of teachers' work. The act of asking a good question is cognitively demanding; requires considerable pedagogical content knowledge (Shulman, 1987); and necessitates that teachers know their students well. A number of research studies have shown that teachers rarely ask 'higher order' questions, even though these have been identified as important tools in developing student understanding (Hiebert & Wearne, 1993; Klinzing, Klinzing-Eurich, & Tisher, 1985; Nystrand, Gamoran, Kachur, & Prendergast, 1997). Research on the relationships between teacher questions and student learning has produced mixed results. Some researchers argue that higher order questions do correlate with pupil achievement and higher order thinking, while others conclude that they do not (Klinzing et al., 1985). Nystrand et al (1997) show that "authentic questions," that is, questions without pre-specified answers, are asked only rarely in eighth and ninth grade English classes. At the same time authentic questions do positively influence student engagement, critical thinking and achievement in eighth grade classes. In their ninth grade classes authentic questions had positive effects in high-track classes and negative effects in low track classes. They argue that this is because the authentic questions in the low-track classrooms did not focus on the substance of the literature students were studying. Hiebert and Wearne (1993) argue that questions need to be viewed within the context of the kind of instruction that is taking place and in relation to the tasks. In a comparative study of 'traditional' and 'alternative' elementary mathematics classrooms, they showed that while teachers in 'alternative' classrooms asked a high number of questions requiring recall, they also asked a larger range of questions and asked more questions that required explanation and analysis than did teachers in traditional classrooms. Students in the alternative classrooms achieved higher gains in performance over the year.

Our study of questioning comes from a larger project in which we have worked to develop methodological lenses for the analysis of teaching and learning. Tools and methods for analyses of teaching are elusive, in part because conceptual analyses of teaching do not exist in the same ways that they do for learning (Leinhardt, 1993). Where rich accounts do exist, they are often produced by individual practitioner-scholars (Ball, 1997; Chazan, 2000; Heaton, 2000; Lampert, 2001), or by researchers employing fine-grained qualitative analyses, (McClain & Cobb, 2001; Staples, 2003). These accounts have been generative in documenting important aspects of teaching, but the methods employed are time-consuming and difficult to implement across a large number of classrooms and researchers. Larger scale, quantitative analyses, such as process-product accounts of teaching (Good & Grouws, 1977), have enabled cross-classroom comparison, but they have often missed the important subtleties of teaching that may make the difference between more and less productive learning environments.

In this paper we describe our attempts to develop and employ lenses that are of a large enough scale to be employed across a wide range of classrooms and by many different researchers, but are detailed enough to provide rich analyses. Such lenses fall between what have traditionally been conceived as quantitative and qualitative. Some progress in this direction has been made in previous studies of elementary mathematics classrooms (Hiebert & Wearne, 1993; Saxe, Gearhart, & Seltzer, 1999). A key focus of those analyses, as well as our analysis in this paper, has been teacher questioning.

Data collection Our data comes from a larger longitudinal study that follows approximately 1000 students in three schools. There are three different mathematics curricula across the three schools, two of which we characterize as 'reform' and one as 'traditional'. In two of the schools, which we call Hilltop and Greendale, students choose between a traditional curriculum (T) and a reform curriculum (R1), called the Integrated Mathematics Project (IMP). The IMP curriculum takes an open-ended, applied mathematical approach in which students work predominantly on long projects that combine and integrate across areas of mathematics. The traditional approach comprises courses of algebra, geometry and advanced algebra, taught using traditional methods of demonstration and practice. In the third school, which we call Railside, the teachers have created their own curriculum, which we consider to be a reform curriculum (R2). They draw from CPM (College Preparatory Mathematics), IMP and other sources. Their curriculum fits into the traditional algebra-geometry-advanced algebra divisions but takes a more open, exploratory and conceptual approach to mathematics within these strands. This school also employs the method of 'complex instruction' (Cohen & Lotan, 1997) where students work in groups most of the time. In addition to monitoring the students over four years, we are studying one or more focus classes from each approach in each school. In these classes we have observed and videotaped lessons and conducted in-depth interviews with the teachers and selected students.

Data Analysis and Findings

As part of our analyses of the teaching and learning environments, we conducted qualitative analyses, describing and analyzing what we saw, using 'thick description' (Geertz, 1973), drawing upon our own observations as well as reports from students. With these analyses, we were able to make some cross-classroom comparisons. However, such analyses rely heavily on individual researchers' insights. Therefore we simultaneously developed methods that would enable us to capture the important features of classroom environments through agreed upon coding schemes that different researchers could use across a wider range of classes.

Our first coding scheme was a set of mutually exclusive and comprehensive categories of classroom activities at a broad level of description. These aimed to describe how students spent their time in class. The categories were: group work, individual work, teacher questioning, teacher talking, and student focus. A team of researchers worked to agree upon what constituted each of the activities, through repeated viewing of videotapes. When over 85% agreement was reached in our coding, we coded six lessons from each of our case study classes. Every 30-second period of time was coded. When more than one activity happened during a 30-second period, we coded for the main activity. A final inter-rater reliability test again achieved 85%. Table 2 gives the results of our first, broad coding of general classroom activities.

Table 2: Percentage of time spent on each activity

Teacher	School	Approach	Group Work	Individual Work	Teacher Questioning	Teacher Talking	Student Focus	Other
Mr. Jones	Hilltop	T	15.5	38	19	22	0.5	5
Mr. Thomson	Greendale	T	6	58	12	20	0	4
Mr. Boxwell	Hilltop	R1	34	1	28	17.5	13.3	6
Mr. Morris	Greendale	R1	30	9	36	14	7.5	3.5
Ms. Nelson	Greendale	R1	0	36.5	33	13	12.5	5
Ms. Larimer	Railside	R2	72	0.5	6.5	3.5	10.5	7
Ms. March	Railside	R2	70.5	1	11.5	6	7.5	3.5

An important point about this table is that the categories were regarded as mutually exclusive. Teacher questioning, teacher talking and student focus were not coded when they occurred during group work and individual work. At Railside students worked in groups for the majority of the time and the teachers often questioned students during group work. Therefore the table above does not represent the number of questions asked at Railside (but these are captured in our later analysis). This reservation also holds for the other classrooms, but to a lesser extent. Despite this, the table does communicate some useful information. One important finding is that activities in the classes of teachers using the same curriculum are strikingly similar, even when the classes are in different schools. The two traditional teachers, for example, spent approximately 20% of their time talking to the students, explaining methods and concepts. The reform teachers spent less time talking but more time questioning. The average time spent questioning by the teachers of IMP classes was 32% whereas the teachers of the traditional curricula questioned for an average of 16% of the time. It is also interesting to note that students worked in groups for much more of the time in all of the reform classes, with the exception of one teacher. Ms. Nelson has a different profile to the other IMP teachers, as she engaged the class in more whole class discussion than others. Staples' (2003) focused analyses of this teacher showed that her whole-class teaching was extremely collaborative. Looking across all of the reform classes we see that students were the focus of attention for approximately 10% of the time. This means that they were presenting their own work or taking some responsibility for the learning of the class. This pedagogical approach was absent in the traditional classes, and is a stark difference between traditional and reform teachers. The combined time that students spent in groups or presenting, compared to individual work, contrasted vastly in the different approaches. Students in traditional classes worked individually for approximately 48% of the time, compared to 9% of the time in reform classes. Thus different curricula do give rise to different broad activity settings and these provide some explanation for differences in student learning across curricula.

While providing interesting information on similarities within the classes of teachers who use the same curriculum, this coding exercise did not capture important differences that we knew to exist between different teachers using the same curriculum. The differences were particularly evident among teachers using the reform curricula. Our detailed observations and qualitative analyses showed that the teachers generated very different classroom environments. In order to capture these differences we chose to move to a finer grain size and code the nature of teachers' questions, as our observations suggested that teacher questions were very important. We developed nine categories of teacher questions that were derived from an analysis of practice. We did not invent the categories *a priori*, rather we studied different examples of the teaching in our sample and attempted to describe and name the different types of questions we recorded. In doing this, we were informed by other analyses of questions, particularly those of Hiebert and Wearne (1993) and Driscoll (1999). Table 2 shows the categories of teacher questions we developed.

Table 2: Teacher Questions.

Question type	Description	Examples
1. Gathering information, leading students through a method	Requires immediate answer Rehearses known facts/procedures Enables students to state facts/procedures	What is the value of x in this equation? How would you plot that point?
2. Inserting terminology	Once ideas are under discussion, enables correct mathematical language to be used to talk about them	What is this called? How would we write this correctly?
3. Exploring mathematical meanings and/or relationships	Points to underlying mathematical relationships and meanings. Makes links between mathematical ideas and representations	Where is this x on the diagram? What does probability mean?
4. Probing, getting students to explain their thinking	Asks student to articulate, elaborate or clarify ideas	How did you get 10? Can you explain your idea?
5. Generating Discussion	Solicits contributions from other members of class.	Is there another opinion about this? What did you say, Justin?
6. Linking and applying	Points to relationships among mathematical ideas and mathematics and other areas of study/life	In what other situations could you apply this? Where else have we used this?
7. Extending thinking	Extends the situation under discussion to other situations where similar ideas may be used	Would this work with other numbers?
8. Orienting and focusing	Helps students to focus on key elements or aspects of the situation in order to enable problem-solving	What is the problem asking you? What is important about this?
9. Establishing context	Talks about issues outside of math in order to enable links to be made with mathematics	What is the lottery? How old do you have to be to play the lottery?

In coding teacher questions, we had to make decisions about what counts as a question. We know from work on classroom discourse (Mehan, 1979; Sinclair & Coulthard, 1975) that utterances in the form of a question often do not function as questions (for example "would you like to come and show us your idea"). Similarly, "prompts" (Ainley, 1987; Watson & Mason, 1998) which do not look like questions can function to solicit answers (for example "sixty percent of fifteen is …"). We chose to include utterances that had both the form and function of questions, and which were mathematical (i.e. we excluded questions about other aspects of students' lives, for example, "did you eat breakfast today"). We also coded repeated questions as such, and excluded them from our final counts. Two researchers worked to achieve clarity on the question types, and an inter-rater reliability exercise achieved 90% reliability. Table 3 shows our initial results from the coding of over 800 minutes of classroom lessons. The work is still ongoing and we present interim findings here.

Table 3: Percentage of different kinds of teacher questions

Teacher	School, Approach	1 (fact)	2 (term)	3 (concept)	4 (probe)	5 (disc)	6,7,8	9 (context)
Mr. Jones	H, T	97					2	
Mr. Thomson	G, T	99.5			.5			
Mr. Boxwell	H, R1	71		7	10	1.5	2	8
Mr. Morris	G, R1	69.5	1	3	8	5	5	8
Ms. Nelson	G, R1	63.5	1	8	13	8.5	2	3.5
Ms. Larimer	R, R2	61	6	21	9	2		

Our findings in this table are stark. More than 95% of the questions asked by teachers using traditional curricula were of Type 1. In the case of the reform teachers, between 60 and 75% of their questions were also of Type 1 but they asked a greater range of questions. This range varies among the reform teachers and helps us distinguish between teachers of the same curriculum, who generate different classroom environments. For example, Ms. Nelson, the teacher who engaged students in more whole class discussion, shows an interesting profile. Almost 30% of her questions were classified as probing, generating discussion or targeting concepts. This compares with at most 20% for the other teachers of the same curriculum. We show below that her questions help her to take students to quite different mathematical terrain (Lampert, 2001). We also note the high percentage of conceptual and probing questions in Ms. Larimer's classroom. Our observations suggest that the questions asked in this classroom are closely related to the teacher-developed curriculum, which is strongly grounded in exploring the conceptual links between different mathematical representations (Brodie, Shahan, & Boaler, 2004). We intend that qualitative analyses of when and how the different questions occur in the different reform classrooms will illuminate some of the intersections between curriculum and teaching approach. In what follows, we show the beginnings of such an analysis.

The following extracts come from a lesson in the IMP curriculum, which is intended to introduce students to the notion of variables. Students had been working on a unit on the pioneering families that traveled across the United States. They were told that a particular family included three generations of women and the total of the women's ages was 90. They were also told that women could not have children until they were 14. The students were asked to work out possible sets of ages for the three women. The students generated some ages and then the teacher told them that one student had represented the situation in this way: C + (C + 20) + (C + 40) = 90. The students were then asked to consider what C could mean. We join the discussion as the teacher reads from the book.

Transcript	**Question Type**
T: OK, now it says in there one student in solving this problem wrote C+ (C+20) + (C+40) = 90. So they didn't have an answer but they started writing some equation. So like the method over here was like guess and check. They were guessing some numbers and then they were checking them against the constraint,	

but somebody else decided to do their work C+ (C+20) + (C+40) is 90. **What in the world is C?** Well it has to do with this problem obviously.	3
S: It's 10.	
T: We've got C+ (C+20) + (C+40) = 90. **What are you getting with 10? What's 10?** Who is it that said 10?	4
S. I said 10.	
T. Jenny, said 10? **Could you tell me how you got 10?**	4(R)
S. Ah, 40 plus …… plus …..and then ….yeah, yeah, 'cos 40 plus 10 is 50 and then 20 plus 10 is 30 and then 15 and 30 and then 10 more is 90.	
T. **So, you, did 10 just come out of a guess?** You were trying some numbers?	4
S. I got it wrong and then….	
T. But now you got it right? **Because on your homework, what'd you do?**	4
S. I said C was something else, I said C was 20.	
T. OK. So now she's saying 10 works. **Does everybody see why 10 works?**	3
T. OK. So she said 10 plus 40 is 50, so we've got 50 here and 10 plus 20 she said was 30, and we've got 80 and we've got another 10 over here. **Now I'm curious why you put in 10 each time for C?**	3

In the extract the teacher asked for the meaning of C and received an answer (10) that was not the one she was looking for. At this point the teacher had many options, including telling the student that she was not looking for the number 10 but the meaning of C. It is typical of this teacher that she chose not to do that, but instead to probe the student to find out how she got 10. While this seemed to direct the focus of the discussion away from what the teacher originally wanted them to talk about, which was the meaning of a variable, in fact her maintaining of the discussion around the answer of 10 and why it is correct, served to create a basis from which to move. If more students were clear on what the value of C was, they might be more likely to participate in a discussion and come to understand the notion of C as a variable. Having discussed the answer of 10, the teacher reached a decision point. She knew that she needed to return to the concept of variables and she needed to decide – in that moment - how she might do that with a question. The question she chose: 'Now I'm curious why you put in 10 each time for C?' is an interesting one because it targeted the concept that the class was being introduced to at the same time as building from the student's answer of 10. This moment illustrates the complexity of teacher questioning (Boaler & Humphries, in press). The teacher had to process many forms of information in that moment. For example, she needed to consider what the student said, the mathematical terrain (Lampert, 2001), and the direction the class should move.

She then needed to connect the students' answer to the new terrain through a question that would be interesting and accessible. All of that thinking took place in an instant, in the midst of a class of students. We now rejoin the lesson.

Transcript	**Question Type**
T. **I'm curious why you put in 10 each time for C?**	3
S. Because C has to be the same.	
T. OK. That's a really important point. C has to be the same each time so we couldn't have changed it and made it a different number. **OK, what is this topic called here?** Up here. What math topic is that called?	8
S. Algebra.	
T. That's called algebra. OK. So we're doing some algebra, in fact, this unit is our algebra unit. We're gradually getting there. **But what does C in words represent?** What in the world does this algebra equation have to do with this problem we're doing?	3
S: Ummm	
T. What, what, what? **I mean how in the world did they get a bunch of these things and then they get this C junk with 30's and 40's?**	8
S: Urrrr.	
T. **Well, what could that possibly have to do with the problem?**	8
S. Uh?.	
T. Jenny says the answer's 10. **Ten what?** Ten, ten's the number that.	3(R)
S. Ten years? People?	
T. People—10 years? **How does 10 years fit in with this problem?** Ten years we were on the trail?	3

S. Ten years old, our ages. Noo, the, children.	
T. Oh, children.	
S. It's the variable.	
T. **OK, so what's the variable?**	4

The extract above illustrates the importance of the *range* of questions we observed in our coding exercise. The questions we code as '3' target the important concept of the nature of variables. But the teacher cannot only ask questions of this type, and the probing questions are critical in encouraging students to offer clarity and to justify and reason. There was an interesting moment in the lesson above that cannot be seen in the transcript, when many students who had been quiet started to become more involved. That happened when the teacher asked "Jenny says the answer's 10, ten what?" This linking of the number 10 with the meaning of C gave many students access to the ideas under discussion. In the lesson as a whole the teacher asked questions of all the types we identified, and this range seems important for the quality of the discussions that were produced. Teacher questions are traditionally linked with the demand that is placed upon students, with higher order questions leading to a greater cognitive challenge for students (Hiebert & Wearne, 1993; Klinzing et al., 1985; Nystrand et al., 1997). But teacher questions also guide students through the mathematical terrain of lessons. When we focus upon the questions teachers ask we see that they shape the mathematical landscape in significant ways.

In addition to the important role teacher questions play in shaping the nature of classroom environments and the mathematical terrain that is traversed, they also teach students to ask important questions of their own work. Our data shows that when teachers ask more conceptual questions, students start to ask conceptual questions themselves. In an analysis of the three teachers using the IMP curriculum we found that the students in Ms. Nelson's class asked significantly more conceptual questions of themselves and each other. In classes at Railside where teachers asked a significant number of probing and conceptual questions, we heard students in their groups ask questions such as, "She's going to ask us where we got the 8, where did we get it?" demonstrating the relationship between the teacher's questions and those that students learned to ask.

Conclusion

Our coding of classroom activities and different teacher questions, suggests that important differences in learning opportunities are not always captured by a broad grain size. This is not particularly surprising, most educators know that it is not the *fact* that students work in groups or listen to the teacher that is important. What is important, is *how* they work in groups, what the teacher says and how the students respond. But while this may seem obvious, most debates of teaching and learning occur at a broad level of specificity. Politicians, policy makers, parents, and others engage in fierce debates over whether students should work in groups, use calculators, or listen to lectures, for example. Our data suggests that such debates miss the essence of what constitutes good teaching and learning.

Our coding of teacher questions also illustrates the importance of the *different* questions teachers ask in shaping the nature and flow of classroom discussions and the cognitive opportunities offered to students. Our work is ongoing and we intend to develop our analyses of teachers' questions in order to capture important differences in teaching and learning environments. Our coding does not capture all that is important about teacher questions; indeed our coding of individual questions does not capture important issues of sequencing or intent. Nevertheless we regard this classification of teacher questions as important in capturing some of the important nuances of teachers' work.

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LANGUAGE AND BELIEF FACTORS IN THE LEARNING AND TEACHING OF MATHEMATICS AND PHYSICS: A STUDY OF THREE TEACHERS

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Although all forms of language are an integral part of day-to-day classroom experiences, few studies have focused on the language used in teaching mathematics concepts in both mathematics and physics classrooms. In this paper, the forms of language used by three teachers as they presented mathematical concepts associated with functions in mathematics and physics classroom are reported. Teacher beliefs influence the language and approaches used in the classroom. This study also reports the ways in which the beliefs of teachers about mathematics and physics, and about the teaching of mathematical concepts, is reflected in the language used in these classrooms.

Purpose of the Study

Students are conditioned to expect to learn mathematics in the mathematics classroom, and to learn physics in the physics classroom. But mathematical concepts are frequently encountered in the physics classroom, and physics applications are often appropriate for illustrating concepts encountered in the mathematics classroom. An understanding of any differences (and similarities) in the communication patterns that occur in mathematics and physics classrooms, as well as in teacher beliefs about mathematics teaching in these classrooms, should help to identify teaching approaches and classroom discourse practices which are most likely to support meaningful mathematics learning.

This study represents an exploration of “science talk” and “math talk” in nonintegrated secondary school mathematics and physics classrooms focusing on the mathematical concept of functions. The specific questions guiding this research are: 1) What are the main characteristics of language genres found in their classrooms? 2) How do mathematics and physics teachers’ conceptions, beliefs and knowledge, about mathematical functions, school mathematics and school physics influence their classroom practice?

Theoretical Framework

Researchers have identified unique language genres or discourse practices in classrooms (e.g., Ellerton, 1999; Lemke, 1989; Wickman & Östman, 2002). A *language genre* is an agreed-upon form and style of language which is developed in and by a particular discourse community to facilitate communication in that community (Bickmore-Brand, 1997; Ellerton & Clements, 1991; Hasan, 1996; Lemke, 1989; Wallace & Ellerton, 2004). The language genres under observation in this study were “science talk” and “math talk” as they were developed by and among the students and their classroom teacher. By observing classroom discourse the researcher can begin to identify patterns that constitute contextually-based language genres. Classroom discourse can be thought of as instances of communicating that represent dynamic actions either between others or with the self as a reflective individual (termed “self as thinking” by Sfard [2000]) that occur in a classroom setting. This communication can include verbal utterances, written texts, physical gestures, and other social contexts by teachers or students (Roth & Lawless, 2002). Furthermore, discourse is seen as both a means of communicating and a means of learning. Therefore, it is necessary for students to learn how to communicate within classroom discourses. There exist characteristic ways of communicating in science and mathematics classrooms, namely "science talk" and "math talk."

Lemke (1982; 1989) first introduced science talk; Chapman (1997) then extended the idea to the realm of mathematics education by introducing math talk. The ways in which these terms are used in the current study are consistent with the ways in which both Lemke and Chapman used "talk," and with the ways in which the linguistics literature defined and used genre. Science talk and math talk, therefore, are language genres of school classrooms.

Ways of communicating in any given situation are determined in part by the speakers and the context of the speech. Over time this way of communicating is negotiated by the participants and becomes a language genre. The negotiated components of such talk are purpose, form, compositional structure, style, and content and provide constraints for the characteristics of a genre (Hasan, 1996; Swales, 1990). Members of the community recognize and participate in the genre (Swales, 1990). However, communicating within a genre is not necessarily a conscious choice but rather a matter of communicating with other members of the community. For example, students may encounter several genres within their classroom community. Marks and Mousley's (1990) mathematics education research reported descriptions of several genres based on the function of the talk. The functions of talk discussed by Marks and Mousley were procedural, description, report, explanatory, and expository (see Marks and Mousley [1990] or Wallace [2004] for further details).

The current study focused on language factors and teacher beliefs with respect to mathematical functions. The topic of mathematical functions was chosen because of its potential overlap between the two school curricula (algebra and physics). Although functions are not explicitly taught in secondary school physics curricula, functions do play an important role in the mathematical analysis and interpretation of various physical phenomena. Furthermore, little research has been conducted on teachers' knowledge and beliefs about the teaching and learning of functions.

Studies on teacher knowledge and beliefs indicate that teachers are not well aware of student difficulties and that teachers may have a limited understanding of the concept of function (Hadjidemetriou & Williams, 2002; Norman, 1992, 1999). Norman (1992) also discussed the lack of cognition of mathematical functions by teachers and introduced the concept of functional reasoning.

Method of Inquiry

Three secondary school teachers were selected as cases for a collective case study analysis and were, therefore, examined as a group (Merriam, 1998). The teacher's conceptions (including beliefs and knowledge about mathematical functions) and classroom instructional practices provided the boundaries for each case. Semi-structured interviews with the three teachers were conducted, and the teachers' discourse patterns were examined during classroom instruction. Their classrooms were observed four to six times throughout the course of one academic year. The interviews utilized pre-determined tasks and open-ended questions organized around the concept of mathematical functions. The semi-structured nature of the interviews also allowed for flexibility in pursuing relevant ideas arising from the conversations. Interview tasks were selected and modified from current literature on functions. In addition, through an exchange of emails, further questions were pursued and clarification sought. The interviews (and email exchanges) were conducted after the completion of the classroom observations to limit their potential influence on classroom instruction. Teachers were observed and audio recorded during regular classroom instruction. Given the exploratory nature of the study, the audio recording was limited to the teacher's voice, although some student interactions were described in the researcher's field notes. Audiotapes were later transcribed to provide a more detailed account of the teachers' ways of communicating. Two of the three high school teachers who volunteered to participate in this research study were a 2nd year mathematics teacher, Mrs. Agnesi and Mr. Newton, a 10-year veteran physics teacher, both from the same Illinois High School. The third participant, Mrs. Arc was an experienced classroom teacher of 7.5 years who had taught both mathematics and science courses and was currently teaching Algebra 1 and Physics in an Indiana High School.

Results and Conclusions

To address the two research questions, interview transcriptions, email exchanges, classroom observation field notes, and classroom transcriptions were analyzed by comparing the language associated with instruction on the concept of functions in mathematics and physics classrooms. The transcriptions and other written records were analyzed using an iterative process. The data were recorded and read multiple times with successive readings being compared to initial themes in such a way to determine the common patterns or themes among the talk of the teachers during their instruction. Therefore, as themes emerged from the analysis, the themes were compared to the remaining data for confirmation and revision.

Language Genres

The use of everyday language (i.e. language used in situations familiar to students and non-technical terms) was the most common theme across all three of the cases. These teachers expressed the belief that students should be introduced to new material first through language that connects with the students, then possibly with a move towards more formal language. Therefore, the way of communicating in these classrooms relied as much if not more on everyday language as it did on technical language(s) of the discipline(s).

More specifically, everyday language was used as each teacher referred to formal mathematical language. Each teacher did so in a seemingly negative way. The teachers used these everyday expressions either in the interview, in class, or both educational situations: Mr. Newton used the word "mathwanese;" Mrs. Agnesi used the words "math garbage;" Mrs. Arc used the words "alphabet soup" to refer to mathematical language and its symbols, thus implying that mathematical language is cumbersome. Although each teacher used formal mathematical language correctly, each emphasized it in different ways, as will be discussed later.

Mrs. Agnesi alternately used everyday terms with technical terms, but did not make much use of physical situations. During her non-honors second-year algebra class, Mrs. Agnesi worked through several examples of simplifying polynomials. The teacher's role in the simplification process was to lead the students by posing questions that would lead the students through a set method, as shown in the first excerpt.

> Mrs. Agnesi: Let's do another one. $\left(x^2y^2 - xy^4 + 2x^4y + 5x^3y^3 + 2xy^4\right)$
> First they want to simplify. … So what am I looking for when I simplify? What in the world am I looking for?
> Student 1: Like terms.
> Mrs. Agnesi: Like terms, are there any like terms in this polynomial? … (Another student makes a suggestion.)
> Mrs. Agnesi: Good, this is the only one that has the same variables with the same powers, those are the only like terms here. So when I simplify I still have all this other stuff. [rewrites this on the board.] And so when I simplify I combine these two terms. What is it?
> Student 2: $-6\ xy^4$.

Mrs. Agnesi: Good, so $-6xy^4$. And the next thing and you could do this in one step if you wanted to. You are going write this in decreasing order of x and I said of x for a reason, because you are not looking for the y degrees you are looking for the x degrees so the first one would be what, student? (Classroom algebra episode)

Mrs. Agnesi did make use of everyday language and attempted to relate algebra to her students by appealing to their sense of humor. The use of the mnemonic, FOIL, in the next excerpt demonstrates the math talk in Mrs. Agnesi's classroom with respect to the use of non-mathematical situations as a means of interpreting the algebra.

Mrs. Agnesi: You are supposed to be using the laws to multiply these together. [The teacher reads the answer] OK, on this last one, (x + 2)(x - 5), is what we are going to be doing today. And that particular problem you are supposed to use "FOIL." Captain Foil is here [a large cartoon character is drawn on the board]. So "FOIL," everybody should be familiar with "FOIL," by now in your algebraic lives. "FOIL" stands for First, outer, inner, last [writes this on the board]. And what it is, is just a saying that helps you remember how you multiply polynomials. (Classroom algebra episode)

While the cartoon figure was meant to reinforce the mnemonic, the primary focus was the mathematics instruction.

Both physics teachers approached their respective courses as a laboratory, where knowledge was built inductively from the phenomenon being investigated. The physics teachers specifically referenced mathematics as a means of analyzing and interpreting phenomena. This was mentioned in their interviews and was inherent in their classroom discourse, yet physics remained the focus of instruction as illustrated by the following three transcript excerpts.

Mr. Newton: It has lots of inertia. While the earth loves this thing very much it wants to give it a huge hug. This thing is just way too cool to just go screaming to the earth because it has some serious inertia. It is just hanging tough here with some inertia. OK, mathematically when you look at this big force, big mass; small force, small mass. OK, can those two products be the same? Oh my goodness quotients, when do you get to use quotients in a sentence? Can those two quotients be the same? (Classroom physics episode)

Thus, Mr. Newton introduced technical language of physics through everyday language with simplified concepts. In the second transcript, Mrs. Arc was discussing her instructional technique with the researcher. She explained how any in-depth analysis of concepts is (and should be) preceded by informal discussion.

Mrs. Arc: In my AP course, when we introduce a new formula, we talk a lot about how the formula is set up and direct and inverse relationships and unit analysis and what cases, … would influence it. Like, for the universal gravitation what happens when r approaches infinity, or when r approaches zero. And we can do in depth analysis of that. (1st interview)

The final excerpt is drawn from an episode in Mr. Newton's class. Mr. Newton made use of a mathematical notion (reference point), but only used it insofar as it was helpful in solving a physics projectile problem. (Both Mr. Newton and Mrs. Arc solved physics problems by organizing the information given in the problem statement and that which could be derived. The information is separated into the vertical dimension and horizontal dimension.)

Mr. Newton: That is the vertical distance, put it on the vertical side of it … [Mr. Newton is filling in the horizontal and vertical "knowns" chart for the problem.] Now, just for giggles to be real specific, is that [points to a distance value written on the board] an initial distance or a final distance?

> Mr. Newton: It depends, and this is where we go back to chapter 3 and I said that you could set your reference point wherever you want. This distance, 43.9, is talking about this [points to picture height of "cliff"]. If I put my reference point here that is what this is. OK.
> Mr. Newton: The 43.9 is what this is. This would be final and my initial would be?
> [A student responds with "Zero."]
> Mr. Newton: And because this is 43.9 meters down and I call down positive that is correct. Now if I put my reference point down here, … this is what position?
> (Classroom physics episode)

Thus Mr. Newton tried to explain what the axes represented and again introduced mathematical concepts, through informal language. The numbers to which he pointed referred to values from the problem statement. Mr. Newton's physics talk, however, referenced the numerical values as more than numbers—they pointed to aspects of a physical situation and were interpreted as such. The phrase "real specific" indicates that the discussion that followed would provide such an interpretation of the numerical values.

Similarities and Contrasts Between Math Talk and Physics Talk

The physics teachers drew clear distinctions between what was mathematics and what was science. Mathematics was not simply used to obtain a numerical result, but was seen as helpful in further analysis of the physical situations; students were expected to learn the usefulness of mathematics in science. In this sense, the discourse relied heavily on the mathematical language used for developing further understanding of the science concepts and situations.

Both physics teachers introduced material by connecting science concepts to realistic situations from the students' lives. Once the informal connections had been made the material was presented or derived in more formal scientific notation. Mathematics was used to analyze and, therefore, to give more meaning to the physical situation at hand. Students were expected to interact with the teacher and each other during this presentation and when discussing new material or during laboratory discussions. The physics teachers were more likely to present new material by connecting it to previously learned material or to a realistic situation in students' lives. These teachers' knowledge of the overall science curricula helped them avoid repeating material and assisted them in making intra-science connections with old material.

The mathematics teachers were less likely to gesture and refer to diagrams and formulae in their mathematics classrooms than the physics teachers during their instruction. These physics teachers used formal terminology, but students were encouraged to conceptualize these formal ideas in more personal terms.

Teacher Beliefs

Mathematics teachers in this study perceived that physics teachers do not use mathematics in as rigorous a manner as they believed should be the case, and the physics teachers believed that mathematics teachers are overly abstract in their presentation of mathematical concepts. Thus although mathematics and physics share the language of mathematics, this language is used differently in the two contexts. Although the mathematics teachers were willing and interested in introducing mathematical concepts in non-technical language as applied to everyday situations prior to formalizing the concepts, they rarely did so in the lessons observed. When a non-mathematical situation was introduced by the mathematics teachers it was most often used to reinforce a specific set of steps or a mnemonic of a specified method.

The concept of function was approached differently in the two disciplines. There is strong evidence to suggest that the mathematics teachers were better able to formalize the mathematics and move between different representations of function. With respect to Norman's (1992) functional reasoning, the mathematics teachers were better able to provide generalizations for functions and identify patterns from algebraic representations. The mathematics teachers were also better able to communicate within functional situations.

Mathematics was used in physics as an analysis tool, often through equations and graphing of data. Mr. Newton's limited understanding of functional reasoning combined with the nature of school physics led him to use these ideas as equations and graphing tools with little exploration of the powerful mathematics involved that could extend his knowledge.

All three teachers emphasized the need to introduce material informally in the classroom through situations relevant to their students. All three teachers found formal mathematical language less than appealing and clearly set this way of speaking and thinking aside as they taught the curriculum. They implied that there is more to mathematics than symbolic language and this was reflected in their classroom instruction.

Implications

As teachers become more aware of their instructional talk they can begin to identify how their students are talking and why. Understanding the talk of students allows teachers to provide more specialized instruction. The mathematics talk by the mathematics teachers emphasized mathematics as a prescribed set of steps, whereas the mathematics talk of the physics teachers emphasized mathematics beyond a tool for carrying out calculations, and as useful for the interpretation of real-life situations.

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EMBODIED SPATIAL ARTICULATION: A GESTURE PERSPECTIVE ON STUDENT NEGOTIATION BETWEEN KINESTHETIC SCHEMAS AND EPISTEMIC FORMS IN LEARNING MATHEMATICS

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Two parallel strands in mathematics-education research—one that delineates students' embodied schemas supporting their mathematical cognition and the other that focuses on the mediation of cultural knowledge through mathematical tools—could converge through examining reciprocities between schemas and tools. Using a gesture-based methodology that attends to students' hand movements as they communicate their understanding, data examples from design research in two domains illustrate students' spontaneous spatial articulation of embodied cognition. Such embodied spatial articulation could be essential for deep understanding of content, because in performing these articulations, students may be negotiating between their dynamic image-based intuitive understanding of a concept and the static formal mathematical formats of representing the concept. Implications for mathematics education are drawn.

The growing body of literature on 'situated cognition' and 'cognition in context' (e.g., Lave & Wenger, 1991; Hutchins & Palen, 1998) is informing research in mathematics education. In particular, we are challenged to think of mathematical cognition not as "abstract" in-the-head processes devoid of concrete grounding, but as phenomenologically, intrinsically, and necessarily dwelling in student interactions with objects in their environment (Heidegger, 1962; Freudenthal, 1986; Varela, Thompson, & Rosch, 1991), such as mathematical representations. Some scholars maintain that mathematics is possible at all as a human endeavor, because the cognition of mathematics leverages and elaborates on embodied schemas that underpin thinking, such as 'containment,' 'repetition,' or 'extension' (Lakoff & Nuñez, 2000). Other scholars focus on the role that mathematical tools, such as representations and calculation devices, play in the interpersonal mediation of mathematical reasoning, such as Stigler (1984), who studied the "mental abacus." The plausibility of these parallel strands of research—the former possibly more "Piagetian" and the latter more "Vygotskiian"—invites the question of how individuals learn to use cultural tools. Specifically, if the scope of mathematical cognition is largely dictated by a repertory of embodied schema and if mathematical reasoning simulates the internalized operation of mechanical tools, how do these ends meet? Do existing schemas accommodate structures inherent in new tools? Do new tools foster the development of new schemas? Answers to these questions, and in particular a framework and terminology for describing what transpires in student–tool interactions, should be of interest to constructivist-education practitioners: Designers who strive to create learning tools supporting intuitive understanding of mathematical concepts often do not have a language to articulate what it is they are doing when they create tools that "work," and so it is difficult to evaluate and teach effective design; Teachers informed by a framework articulating types of student–tool interactions that are important for deep understanding of the concepts inherent in the tools may be encouraged to create classroom opportunities for such types of interactions. Addressing the issue of schema-driven learning versus tool-driven learning, this paper takes a position that the truth may lie somewhere in between. The objective of this paper is to spell out a theoretical position and to outline, through examples, a methodology for gathering data towards supporting this position. Future work will elaborate and expand on this theory and apply the methodology to support the position.

Learning mathematics with understanding involves students' ongoing negotiation between their embodied schemas and the cultural tools students engage with when participating in classroom activities. This theoretical position evolved through observations of students' discourse pertaining to innovative mathematical representations that were introduced into their classroom as part of design-research studies (Abrahamson, 2003, 2004a, 2004b; Fuson & Abrahamson, 2004; Abrahamson & Wilensky, 2004a, 2004b, 2004c). To support this position, it is necessary first to explain why this position has not been stated up to now. For that, we begin by focusing on the mathematical representations or, more broadly, the 'bridging tools' (Abrahamson, 2004a) that were designed for these studies. A design perspective in the study of student learning is helpful, because the agenda of designers is to create tools that "work," and this agenda informs—at least tacitly—the designers' search for mathematical representations that resonate with students' intuitions. Such resonance may be indexed by the extent of fitness between students' embodied schemas and the structures inherent in the designed tools. Following are pedagogical motivations for designing bridging tools and a discussion of gesture-based methodological lenses on student discourse. These lenses afford a distinction between evidence of students' schema-driven and tool-driven learning. Using examples from classroom interactions, we will demonstrate how embodied schemas and cultural forms are separate yet reciprocally related resources in students' learning, and how this learning can be articulated in terms of students' reconciliation between the schemas and the forms.

Bridging Tools

Abrahamson (2004a) discusses *bridging tools*, pedagogical mathematical representations that are designed to foreground and ground processes underlying a domain. The position of bridging tools between simple visual contexts and formal mathematical notation is designed to resonate with constructs, perceptual mechanisms, and schemas that are taken to be universal for the target population of students. Working with such tools, students can engage their experience and mathematical knowledge towards developing an informed fluency with more advanced concepts.

The term 'bridging tools,' although coined in the context of current design work, applies also to traditional mathematical representations that are effective in fostering learning with understanding. That is, mathematical representations that foster deep understanding are those that, either through historical "natural selection" or intentional design, are a priori tuned to accommodate learners' resources, such as their embodied cognition. Such bridging tools may paradoxically encumber the study of students' embodied cognition, because students' interactions with these tools do not easily reveal embodied cognition as a phenomenon that merits a standalone construct. All you see is "kids working with stuff"—the students attend selectively to elements within the mathematical tools, move objects from place to place, write numbers in appropriate locations, and so on. That is, the nature and order of students' attentive glances, manual operations, and solution procedures appear governed entirely by the implicit protocols afforded by the tools and by the classroom facilitator who models for students the conventional use of the tools. Thus, any putative 'embodied cognition' underpinning students' attentive operations remains a hypothetical construct of tenuous theoretical or practical standing. A theoretical decoupling of students' embodied cognition from their hands-on operating on the tools becomes more plausible upon closely examining students as they communicate what they see in the tools and what they are doing with them, such as in classroom discussions. In order to examine schemas, forms, and student negotiations between them, I use lenses from the theory and methodologies of gesture studies. These lenses purportedly reveal embodied cognition as decoupled from operations on tools yet reciprocally related to these tools.

Gesture Studies as Lenses on Embodied Cognition

The study of student gesturing, formerly a disparate intellectual pursuit associated mostly with psycholinguistics (e.g., McNeill & Duncan, 2002) and anthropology (e.g., Urton, 1997), has become a growing research effort within the community of scholars of mathematics education (e.g., Alibali, Bassok, Olseth, Syc, & Goldin-Meadow, 1999). Gesturing plays a crucial role in establishing shared meanings for new artifacts (Hutchins & Palen, 1998; Roth & Welzel, 2001). Through extended operating on new artifacts, learners develop skills of imaging these artifacts and operating on these images even in the absence of the physical embodiment of the artifacts and without any observable gesturing on these images (see Stigler, 1984, on the “mental abacus”; see Nemirovsky, Noble, Ramos–Oliveira, & DiMattia, 2003; see Urton, 1997, on tacit cultural images; see Goodwin, 1994, on how a professional culture mediates ways of seeing).

Whereas students’ internalizations of mathematical learning tools do not subsume all that students learn in mathematics classrooms, it could be that these internalized spatial–dynamic images are vehicles of mathematical reasoning upon which hinge and cohere other aspects of effective domain-specific mathematical practice, such as the modeling of situations and solution procedures (Abrahamson, 2003, 2004a; Fuson & Abrahamson, 2004). Also, there does not necessarily exist a monotonous relation between gesturing and learning, and so the extent of gesturing in communicating a mathematical idea cannot index learning in any simple way. For instance, gesturing may wax towards a moment of clarity (“aha!”) and then wane once the novelty of the new insight subsides (Goldin-Meadow, 2003) and is constituted as no longer warranting explication. Finally, bringing into classrooms innovative mathematical representations does not necessarily imply that students will not have had any prior experience with other representations that incorporate similar features. Therefore, it is probably not warranted to infer from the innovativeness of the tools that they incorporate innovative forms. On the contrary, the design principles of bridging tools together with the assumed historical reciprocity between embodied schemas and structures inherent in cultural forms means that students’ classroom negotiations are informed by prior exposures to similar forms in other contexts. That is, the embodied schemas are probably not innate and the symbolic forms are not innovative, but rather both are woven into learners’ “interconnected patterns of activity in which they [the symbols] are embedded” (Dreyfus, 1994, cited in McNeill & Duncan, 2000). This said, in the next section, we will focus on cases in which students appear to be coordinating between a way of seeing a mathematical object—a way of seeing that was not explicitly or at least not consciously conveyed by the facilitator who was the first to model the use of the object—and the format of the consensual mathematical notation associated with the concept in question.

Examples of Negotiation Between Embodied Knowledge and Mathematics Learning Tools

Students’ negotiations between their kinesthetic schemas and the forms of mathematical representations are yet uncharted territory in mathematics education. Following are two examples of this phenomenon. In both examples, students use their body so as to articulate an idea within their body space (the ‘peri-personal space’). The examples differ both in that they are taken from design studies in different mathematical domains and in that, whereas the first demonstrates a student spatially articulating a mathematical operation, the second shows a student reorganizing real objects in her immediate space towards expressing them as a mathematical relation. Such analyses and the questions that they raise inform ongoing work on a particular design, but they also can set an agenda for a research program that detects and classifies students' embodied metaphors of mathematics as observed in classrooms. A classification of students' embodied mathematics as it relates to mathematical representations may constitute a resource for design in mathematics education. The following data are presented in the *translipction* format that combines transcription, clips, and superimposed diagrams (Abrhamson, 2004; Fuson & Abrahamson, 2004).

Ratio and Proportion: Negotiating Between Schemas of Growth through Rhythmic Repetition and Constant Increments Between Products Going Down Multiplication-Table Columns

In a design for ratio and proportion (Abrahamson, 2003, 2004a; Fuson & Abrahamson, 2004), M'Buto used the multiplication table to solve a ratio-and-proportion word problem (see Figure 1, below). In the problem, an agent was advancing by increments of 5 units per some fixed time unit, and a sub goal towards the solution of the problem was for students to find out how far the agent advances in 11 increments. M'Buto added an 11th row at the bottom of his multiplication table (in front of him on his desk) that had 10 rows. In this 11th row, he wrote '55' in Column 5 (the column that has a '5' at the top; see Figure 1, on left). Ms. Winningham asks M'Buto how he knew to write '55.' In his oral response, M'Buto connects between a model of multiplication as repeated addition and the use of the multiplication table to retrieve multiplication cross products. In his gestures that complement the oral communication, M'Buto first reveals an image of multiplication as a rhythmic progression along a straight trajectory beginning at his torso and extending diagonally away, remaining in the plain of his torso (Figure 1, center). Immediately after, (Figure 1, on right) M'Buto scallops vertically down a column of a large multiplication table he is apparently imaging as if positioned directly in front of his face.

M'Buto' multiplication table with an 11th row.	"Every time you times 5 by another number	it goes up five."

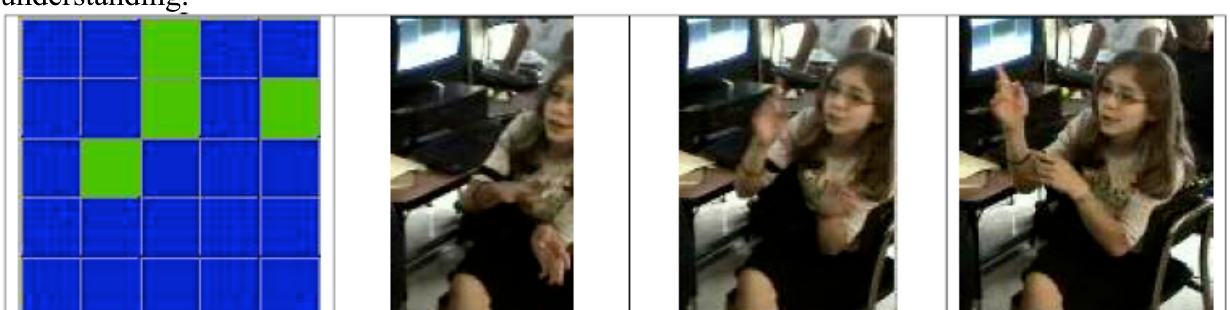
Figure 1: Negotiating between an embodied spatial–dynamic topology of multiplication (center) and an understanding of a column in the multiplication table as growing by a constant increment (on right), in explaining a strategy for determining the product of 11 and 5 (on left).

M'Buto's scalloping motion down the present–absent multiplication table that accompanies the utterance "it goes up five" (see Figure 1, above, on right) corresponds to the gesture employed by the researcher–teacher (the author), the classroom teacher, and students participating in classroom discussions on several occasions on the intervention days prior to this moment (Abrahamson, 2004a, 2004b; Fuson & Abrahamson, 2004). Also, the sweeps and scope of this gesture correspond to the physical size of the classroom multiplication table used extensively in this unit (the scope is much larger than the multiplication table that is in front of M'Buto on his desk). However, the gesture accompanying M'Buto's first words (Figure 1, above, in the center) was never employed by the teachers, at least not explicitly. Where did this gesture come from, how does it correspond to the multiplication table, and what does all this reveal about M'Buto's learning process?

There appears to be a separation, for M'Buto, at this point, between an inner sense of multiplying and how it may plot onto the mathematical representation introduced in the design. M'Buto is negotiating personal and classroom resources on several levels. He is: (a) deploying an embodied model of the multiplication operation pragmatically in problem solving and obtaining a numerical solution; (b) plotting an embodied sense of "timesing" onto the concrete multiplication-table column, perhaps mediated by the imaged multiplication table; (c) interfacing the concrete multiplication table upon which he added the 11th row with the imaged multiplication table; and (d) articulating a kinesthetic theorem-in-action within the linear constraints of the spoken communication medium. Where and how did M'Buto form or internalize the embodied spatial–dynamic image of multiplication? If this image is shared by other students, what does this mean in terms of helping students link the image with the multiplication table? What can we make of the fact that M'Buto successfully speaks of the numbers "going up" while his hand is patently going down?

Probability and Statistics: Using Embodied Hemispheres to Link to an a:b Symbolic Form

In a design for probability and statistics (Abrahamson & Wilensky, 2004a, 2004b, 2004c), Carry responds to a student–leader's prompt to explain a sample taken out of a population of thousands of green and blue squares on a computer interface (see Figure 2, below). Looking away from the computer, Carry gesturally extracts the green squares, placing them on the left "hemisphere" of her body, and then places the blue squares on her right hemisphere. Within 2.5 minutes of discussion, six students engaged the same embodied mechanism to parse and structure their seeing of the visual stimulus. This uniformity in students' gesture patterns suggests that a shared mathematical vision is being co-constructed in the classroom. This vision is design driven: The computer-based bridging tool is aimed to assist students in negotiating between, on the one hand, proportional reasoning and enumeration, and, on the other hand, the formal notation of multiplicative constructs involving proportionality, such as density (of green in the population). It could be that whether or not a gesturing person does so consciously, these gestures help other students see the tools in like ways that are conducive to similar understanding.



Fragment from the SAMPLER interface.	"Out of 25, 4 are green	and	21 are blue."

Figure 2: Negotiating between a visual metaphor of density and the formal *a:b* ratio symbol.

Embodied Spatial Articulation

Embodied spatial articulation is an individual's design-facilitated negotiation between personal and cultural resources pertaining to the visuo-spatiality of mathematical situations and representations. The personal resources are proto-mathematical action-based images and the cultural resources are the appropriate seeing-in-using of classroom spatial–numerical artifacts. I wager that embodied spatial articulation underpins human interacting with epistemic artifacts historically, developmentally, in the designer's workshop, and in classroom space–time (see Abrahamson, 2004a, 2004b, for references supporting this contention). The roles of gesturing in the teaching and learning of mathematics are in supporting the students' intra/inter-personal engagement of tacit body-based strategies for spatial modeling of mathematical concepts. This modeling serves in the co-constitution of domain-specific *epistemic forms* (Collins & Ferguson, 1993) that come from and respond to artifacts.

Educational Significance

If students achieve deep understanding of mathematical concepts by negotiating between embodied resources and cultural artifacts, then learning environments should ideally foster such negotiations—the environments should include historical and innovative representations that readily afford relevant embodied schemas as well as activities, such as individual work and group discussion, designed to create space and time for these negotiations. An embodied-cognition approach together with the gesture-based lenses on student discourse afford a methodology for obtaining nuanced descriptions of students' learning processes. This perspective responds to Confrey's (1991) call to attend to students' voice, only here we are listening to a body-based voice that has been historically neglected.

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MEANINGFUL MATHEMATICAL ACTIVITY: OPPORTUNITIES FOR LINKING IN DIVERSE MATHEMATICS CLASSROOMS

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This study researched the development and negotiation of practices of equity and mathematics reform in three high school mathematics classrooms with highly diverse populations of learners. Drawing on theoretical insights from the cultural practice and the situative perspectives, I utilized the construct of linking *to examine the operation of classroom norms and practices in affording and constraining the social and cultural practices of students. Opportunities for linking were examined in aspects of mathematical content, classroom discourse, and participation structures. Multi-level interaction analyses of videotape documentation and observation transcripts collected over the course of a school year were triangulated with interviews, surveys, and student shadowing transcripts to capture the nature of linking as it emerged in moment-to-moment classroom interaction. Results indicate that opportunities for linking were shaped by two levels of classroom norms,* framing *and* positioning *norms. These norms served to afford and constrain different forms of student participation in classroom activity, and thus, what counted as mathematics learning.*

Linking is a fundamental aspect of meaning making, both in terms of how we interpret the world around us, and how we view ourselves in it. We make sense of unfamiliar objects by noting characteristics they share with objects that are familiar. We come to understand the things that people do and the ways in which they do them by placing them within a meaningful context. We can even recognize ourselves more easily in situations when people interact with us in ways that are consistent with whom we think we are (and are becoming). We engage in linking, then, to make phenomena, situations, and people relevant to our lives.

Schools are increasingly concerned with making school-based learning (or information) relevant or authentic to students in the sense that what is being learned has meaning in students' lives. One reason for this shift is that research indicates that students have better recall of material when it is embedded in meaningful contexts (Anderson, 1995) and become more engaged in learning when they perceive the subject-matter to have intrinsic value to their lives (Deci & Ryan, 1985). A related finding is that schools organize their learning environments--the forms of knowledge they support and the norms they uphold for how students should engage with this knowledge—in ways that are more or less meaningful for different groups of students (Banks, 1993; Moll & et al., 1992). Thus, meaningfulness resides not only within the subject domain itself, but also with the ways in which different groups of individuals participate in joint activity within this domain. With respect to issues of equity and access to mathematics learning, researchers have argued that we need to consider the ways in which mathematics classrooms afford connections to purposeful and authentic participation for students of different backgrounds (Cobb & Hodge, 2002; Nasir, 2002).

There is a common misperception that the domain of mathematics is too rigid or abstract to lend itself to linking. Students do find mathematical achievement to be meaningful to them, but often in the sense that it shapes their future success in higher education and professional activities (D'Amato, 1996; Nasir & Hand, 2003). Attempts have been made to infuse relevancy into mathematical activities, for example, by linking mathematical contexts to everyday contexts with word problems, or more recently with problem-based learning. Yet all too often, students find these contexts artificial, lacking true relevance to the things that they do in their everyday lives (Boaler, 1997). In other words, to package mathematics in the trappings of the 'real world' does not guarantee increased meaningfulness.

Another way that the issue of relevancy has been addressed in mathematics education has been through a reconsideration of what it means to learn mathematics. Greeno (1997) and other researchers from situative and/or sociocultural perspectives have argued for mathematical activity to be representative of the ways that people solve problems together in the world (Lave, 1988). Thus, the development of mathematical practices, or the ways in which students and teachers engage in mathematical activity within and beyond the classroom, has become a core concern of the mathematics reform movement. One way to conceptualize the work that has been done in this area is along two broad lines of research: one line of research attempts to model classroom mathematical practices on the practices of mathematicians and other individuals who engage in real world mathematical activity, while another line of research attempts to model classroom practices on the cultural practices students engage in as members of their local and broader communities. (This is an admittedly cursory treatment of an extensive and diverse body of research in mathematics education, learning, and culture.) Taken together, what these two lines of work suggest is that it is important to consider both how the norms and practices guiding classroom mathematical activity are meaningful to real world activity, and how alignment can be promoted between the expectations held for students in their classroom participation, and the expectations students hold for themselves and others in joint social activity.

This study examined the ways that teachers and students organized their activities within mathematics classrooms towards the construction of mathematical activity that was meaningful to both parties. Mathematical activity was examined in three reform-based mathematics classrooms in a highly-diverse urban public high school that was notably successful at producing strong mathematics learners (Boaler, 2003). As an instance of meaningful mathematical activity, linking was operationalized at multiple levels—task, discourse practices, and participation structures. Multi-level interaction analyses of videotape documentation and observation transcripts collected over the course of a school year were triangulated with interviews, surveys, and student shadowing transcripts to capture the nature of linking as it emerged in moment-to-moment classroom interaction. At the first level of analysis, the moves that students and teachers made to position themselves and each other in and around mathematical activity were considered. At the second level, task affordances and constraints were considered. At the third level, the analysis concentrated longitudinally on the features of the classroom activity system: the development of classroom norms and practices, the trajectory of mathematical activity, and opportunities for links to students' social and cultural practices.

Analysis of classroom tasks that were designed to promote links to contexts familiar to students from diverse social and cultural backgrounds (*e.g.*, "youth culture") revealed that the discursive practices that emerged around these tasks were not necessarily engaging to students (and sometimes supported the development of oppositional stances). This finding supports Greeno and Hull's (2002) contention that the *contexts of information*, or settings of mathematical content, can be at least as much, if not more important than the *contexts of activity*, or settings of mathematical activity, in the design of authentic learning environments. This study found that mathematical tasks that supported linking were valuable tools in providing multiple access points into mathematical concepts; yet their location within a broader classroom activity system constrained and afforded student appropriation of these tools. Thus, it was the patterns of participation that emerged over time in each classroom, or the participation structures, that shaped opportunities for linking in significant ways.

An examination of participation structures in each of the three classrooms revealed differences among them in what was considered to be productive mathematical activity. A bi-level model of classroom norms was generated to differentiate these participation structures, and the opportunities they provided for linking. In this model, a set of *framing* norms operates at the first level to organize categories (or not) into which different forms of participation are separated (*e.g.*, mathematical or non-mathematical), while a set of *positioning* norms operates at a lower level to situate the participation that has been framed with respect to ongoing classroom activity (*e.g.*, on-or off-task) [See Diagram 1]. It is important to note that the framing norms may or may not engender distinctions between different forms of participation (e.g., mathematical and social activity) and thus support the development of categories.

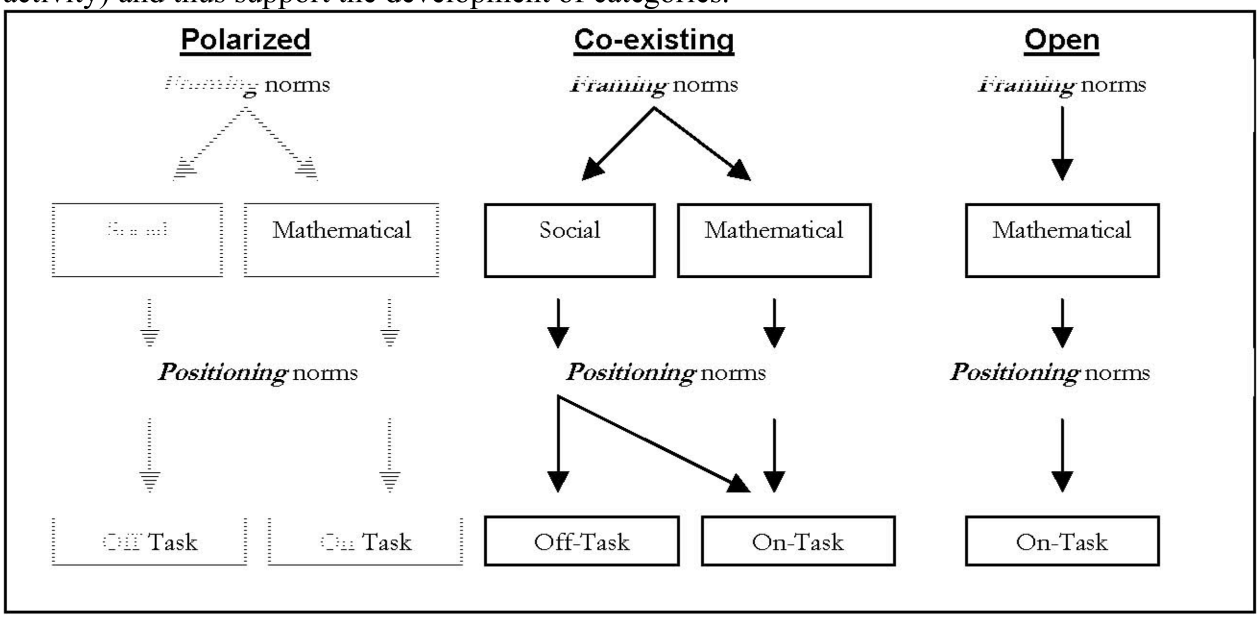

Diagram 1: A model of the operation of framing and positioning norms on classroom participation structures in mathematics classrooms.

In the first case, the polarized model, the framing norms sort participation into social and mathematical, and the positioning norms treat these two forms of participation as being in opposition to each other. In other words, there are clear distinctions made between what is and is not mathematical activity, and social activity is positioned as off-task. This model constrains linking to social and cultural practices by making *a prior* distinctions in activity.

In the second case, the co-existing participation structure, the framing norms create distinctions between forms of participation, but the positioning norms support negotiation of participation according to the features of the context that emerges in moment-to-moment classroom interaction. Thus, in one instance, a form of participation could be positioned as productive (or at least compatible) with classroom activity, while in another it could be treated as counterproductive. Co-construction of participation structures (or linking) was afforded and constrained in moment-to-moment activity.

In the third case, the open participation structure, the framing norms do not make distinctions between different forms of participation. Participation is generally positioned as being productive to ongoing mathematical activity. This model affords links to social and cultural practices to support the transformation of all participation into mathematical activity. Classroom participation structures are continuously shaped and re-shaped by the participation that students and teachers enact within them. Classrooms characterized by the open participation structure supported higher levels of engagement by all students and students themselves reported enhanced feelings of belonging and relatedness. This structure also thwarted the development of resistance and oppositional behavior among students, as different forms of student participation were often treated as a bid for students to "take up their space" and co-opted to re-engage them back to classroom mathematical activity.

These results support related findings by Gutierrez and her colleagues (Gutierrez & et al., 1995; Gutierrez, Baquedano-Lopez, & Tejada, 1999) regarding the construction of *third space*, or hybrid classroom participation structures, in classrooms. Gutierrez has studied the tension that can arise between the teacher and students when their sociocultural discourse or classroom "scripts" are polarized and reflect the reproduction of broad sociopolitical dimensions around race, culture, and ethnicity within local classroom activity. She argues that this tension can only be resolved when students and teachers attempt to move beyond given scripts towards a hybrid discourse structure or *third space*. Third spaces emerge when teachers seek out ways to enact links between student and teacher scripts.

Similarly, Lee's (Lee, 1995, 2001) concept of *culturally responsive pedagogy* supports the practice of teachers to leverage and mine students' sociocultural practices to create links to subject-matter knowledge. Lee proposes that students possess a *repertoire* (Gutierrez & Rogoff, 2003) of informal understandings in their everyday practices that can be cultivated and refined to promote domain-specific comprehension. An important assumption guiding the theories that Lee and Gutierrez propose is that the informal knowledge that students bring into the classroom is situated in ways of participating that, at first blush, may or may not be relevant to the learning goals of the classroom.

The results of this study suggest that classroom norms may afford distinctions in different forms of participation that necessarily impede opportunities for students and teachers to uncover meaningful links to mathematical activity. In contrast, mathematics classrooms that employ strategies to transform student participation into mathematical activity support the development of rich, diverse mathematical experiences that are meaningful to students. Thus, as reform-based mathematics classrooms grapple with the development of equitable mathematical practices, insights such as these may provide valuable windows into strategies for classroom design that serve diverse populations of mathematics learners.

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A MODEL FOR EXAMINING THE NATURE AND ROLE OF DISCOURSE IN MIDDLE GRADES MATHEMATICS CLASSES

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Discursive practices of middle grades mathematics teachers (grades 4 – 8) were investigated, focusing on how types of talk and verbal assessment interact to mediate mathematical meaning within whole group instruction. Grounded theory methodology, multiple-case study design, and sociolinguistic tools were applied within a social constructivist framework to analyze the discourse, especially as related to univocal (transmitting meaning) and dialogic (dialogue to generate new meaning) functions. Categories of talk (i.e., monologic, leading, exploratory and accountable) and verbal assessment (i.e., inert and generative) were identified. Relationships among the various forms of talk and verbal assessment were examined. A model representing the flow of discourse in mathematics classes was developed, suggesting potential tools for both researchers and reflective practitioners.

Introduction

Today, mathematics provides an academic passport into virtually every avenue of the labor market and higher education opportunity (Malloy, 2002). Leaders in business, government, and education recognize that effective mathematics education is the foundation for developing and maintaining the type of mathematical skills needed to become a productive citizen of the 21st century; however, many also acknowledge that mathematics education in the United States is in need of improvement (NCES, 1999, 2001; NCTM, 2000). National organizations, such as the National Council of Teachers of Mathematics [NCTM], have responded to the call for reform by developing and promoting standards that focus on content and process, such as problem solving, reasoning, and communication (NCTM, 1989, 1991, 2000). Communication, sometimes referred to as discourse, has been an integral part of the reform documents over the past two decades.

Verbal discourse, defined for the purposes of this study as, "Purposeful talk on a mathematics subject in which there are genuine contributions and interaction" (Pirie & Schwarzenberger, 1988, p. 460), can provide potentially important scaffolding to the learning-teaching process (Cobb, Yackel & Wood, 1992). Although meaningful discourse can enhance learning, the mere presence of classroom talk does not ensure that thinking and understanding follow—the quality and type of discourse are crucial to helping students think conceptually and procedurally about mathematics (Kazemi & Stipek, 2001). The predominant form of discourse in mathematics classes is univocal—that is, discourse that relates to one-way transmission of meaning (Lotman, 1988; Wertsch & Toma, 1995). In contrast, discourse that aids in the generation or construction of new meaning has been characterized as dialogic (Knuth & Peressini, 2001).

Numerous studies have focused on classroom discourse (Cazden, 2001; Sinclair & Coulthard, 1975), its functions (Brendefur & Frykholm, 2000; Nassaji & Wells, 2000), and its structures (Coulthard & Brazil, 1981; Mehan, 1985). Although discourse specifically related to mathematics instruction has been explored (Nathan & Knuth, 2003; van Oers, 2002), a coherent model of discourse related to mediating mathematical meaning has not been developed. Therefore, this study investigated interactions among categories of talk and verbal assessment in middle grades mathematics classes in order to develop a model of discourse to promote mathematical meaning.

Background of the Study

To understand the role of discourse in the mathematics classroom, it is critical to examine connections between social constructivism and theories of language. Although many linguists have viewed the primary purpose of language to be a transmitting device (Saussure, 1959), some theorists have shifted viewpoints to align with social constructivism, recognizing interrelationships of thought and speech (Vygotsky, 2002; Wertsch, 1998). This ecological approach recognizes complex links in "utterances" that are "filled with dialogic overtones" (Bakhtin, Holquist, & Emerson, 1986, p. 92). Thus, discourse can be categorized according to two main functions: univocal and dialogic (Lotman, 1988; Wertsch, 1998).

Many researchers have identified and discussed basic components of classroom discourse (Mehan, 1985; Sinclair & Coulthard, 1975). For example, Wells (1999) parsed language according to the following categories: move, exchange, sequence, and episode. The move, exemplified by a question or an answer from one speaker, is identified as the "smallest building block" (p. 236). The exchange, made up of two or more moves, occurs between speakers. Exchanges are categorized as either nuclear or bound depending upon whether they can stand alone or are dependent upon or embedded within previous exchanges. The sequence is the unit that contains a single nuclear exchange and any exchanges that are bound to it. Finally, the episode is the level above sequence and represents all the talk necessary to perform an activity.

The most common pattern of classroom discourse follows the three-part exchange of teacher initiation, student response, and teacher evaluation (IRE) or teacher follow-up (IRF) (Cazden, 2001). In the initiating move, the type of question asked guides the flow of discourse toward univocal or dialogic. Additionally, the last move in the exchange has been found to be pivotal in judging whether the discourse will tend more toward univocal or dialogic. For example, when the teacher uses the follow-up move as an evaluation tool, the intended function of the discourse is typically to convey meaning (i.e., univocal). On the other hand, if the teacher's follow-up move is related less to evaluation and more to an exploratory stance, the discourse is more likely to tend toward dialogically generating meaning. Dialogic inquiry is more probable when teachers pose questions that require exploration, negotiation, explanation, or justification, rather than a simple display of information (Wells, 1999; Wertsch, 1998).

Research has examined various categories of classroom talk such as, monologic (one person speaking with no verbal response expected), leading (where students are led to the *teacher's* understanding), exploratory (Cazden, 2001) (speaking without answers fully intact), and accountable (interactions that require accountability to accurate and appropriate knowledge, to rigorous standards of reasoning, and to the learning community) (Michaels, O'Connor, Hall, & Resnick, 2002). In any category of talk, the teacher's ongoing monitoring and verbal assessment affects the dynamics of discourse (Chazan & Ball, 1995) and its tendency toward univocal or dialogic (Wells, 1999). The flow of talk is guided by assessment: generative [GA] and inert [IA]. GA mediates discourse to promote students' active monitoring and regulation of thinking (i.e., metacognition) about the mathematics being taught, supporting tendencies toward dialogic functions. IA tends to maintain the current flow of discourse and to move discourse toward univocal functions. Exploring interactions of categories of talk and assessment has the potential to increase understanding of how discourse may be used to scaffold mathematical understanding. The purpose of this study was to uncover the types of talk and assessment that influence tendencies toward univocal or dialogic discourse, thus, providing opportunities to better understand how students make meaning in middle grades mathematics classrooms.

Research Question

This study proposed to build theory and to develop a model of classroom discourse as a tool for mediating meaning in middle grades mathematics classes. Therefore, the following research question was addressed: *What model can be constructed to explain how teachers use language (i.e., discourse) in middle grades mathematics classes to mediate mathematical meaning?*

Methodology

Participants

The participants were a purposive sample of seven middle grades mathematics teachers who were identified as having characteristics indicative of expertise (Darling-Hammond, 2000). The participants (4 females; 3 males) included three teachers with *National Board for Professional Teaching Standards* certification in early adolescent mathematics, two *Presidential Award for Mathematics and Science Teaching* awardees, and two teachers who had been recommended by university faculty for their experience in using discourse in mathematics classes. Years of teaching experience ranged from 5 to 35 years, with three participants having taught more than 20 years. The schools where the participants were employed included three urban schools, two suburban schools, and two suburban/magnet schools.

Data Collection

Data were collected via semi-structured interviews, classroom observations, field notes, audiotapes, and videotapes. Pre-observation interviews included pre-defined questions designed to uncover background traits related to effective practices; professed knowledge, goals, and beliefs associated with teaching; and the "lesson image" and "action plans" (Schoenfeld, 1998) associated with the lessons to be observed. Mathematics lessons were observed, field notes written, and classroom discourse audiotaped, and videotaped. Post-observation interviews included common themes, but were individually designed to allow participants to explain intentions and shifts in action plans, as well as to explore pivotal actions or events that occurred during the lessons. Excerpts from transcripts were read to the participants during the post observation interviews to stimulate recall of specific events or actions. Audio recordings from interviews and observations were transcribed.

Data Analysis

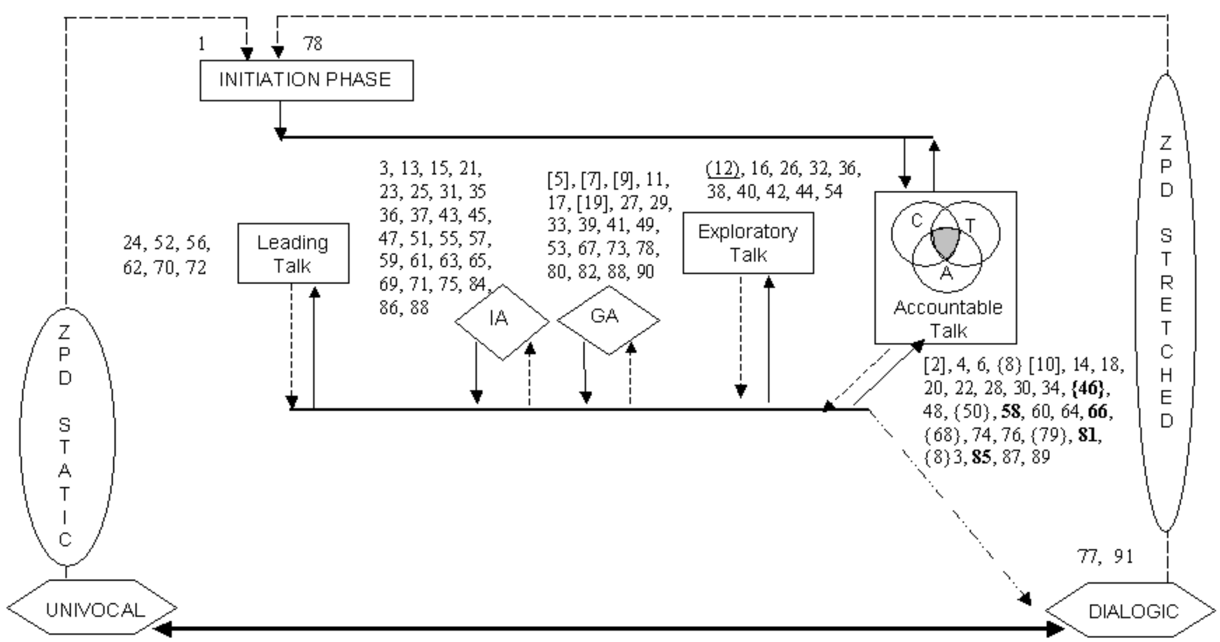
This study used grounded theory methodology to link data collection and analysis to generate a model of classroom discourse (Strauss & Corbin, 1990). Constant comparison provided evidence to inspect, test, and refine the theory and models being developed (Miles & Huberman, 1994). Using strategies adapted from Wells (1999) and Nassaji and Wells (2000), transcripts from observations were formatted in tables to facilitate coding of discursive moves.

A preliminary model of discourse was developed and used as an initial template for mapping the flow of discourse for each sequence. These diagrams, referred to as sequence maps, mapped the flow of discourse from an initiation phase, through types of talk and assessment, moving toward either univocal or dialogic functions. Numbered moves within each sequence were indicated on the sequence maps, allowing the researcher to follow the flow of the discourse. Descriptions of sequence maps were written to explain the classroom context, connecting the transcript with the diagram (see Figure 1). Constant comparison of coded data, maps, and descriptions were used to develop and refine the sequence maps and to build a graphic model for larger discursive units (i.e., episodes).

Figure 1. Example of description, map, and coded transcript from a selected sequence.

Sequence four was lengthy, complex, and dramatic, including leading, exploratory, and accountable talk and both inert and generative assessment. Students were allowed to serve in the role of *primary knower* (Berry, 1981), the teacher modeled metacognitive reasoning, and new meaning was generated. The sequence built from the previous three sequences where common language and consensus on the basic solution to the problem had been established. The teacher then used these as springboards for setting up exploration of richer concepts by conjecturing a *hypothesis* named after the student who demonstrated the initial problem's solution: "The sum of reciprocals of prime and composite factors of a number will always be one." The teacher said, "I wonder if this always works…" and proceeded to have students explore possibilities, including "corollaries" to the theorem. The teacher's knowledge of mathematics and of pedagogy was apparent. He encouraged the students to explore and to conjecture, but also supplied meaningful verbal assessments that provoked them to generalize, to question, to justify, to reformulate, and to develop new meaning, thus including tendencies toward dialogic discourse.

Sequence 4: Exploration/ Generalization - What #'s fit: Sum of Reciprocals of Prime or Composite Factors =1? — Math Procedures / Concepts/ Vocabulary/ Problem Solving



Excerpt of Coded Transcript for Sequence Four

Mp#	Ln #	Sq #	Who	Text	K1	Exch	Mv	Pros	Func	Comment
42	86	4	B8	It didn't work for 21.		Dep	R	G	Inform	Expl Tlk
43	87	4	T	Didn't work for 21! *[whistles]* Yeah?		Dep	F	A/D	Revoice/ Nom	IA
44	88	4	B4	Didn't work for 14.		Dep	R	G	Inf	Expl Tlk
45	89	4	T	Didn't work for 14. B2.		Dep	F/I	A/D	Revc/Nom	IA
{46}	90	4	B2	It didn't work for 36—which is an abundant number.		Dep	R/ I	G+	Inform	S. as K1 AT-AAK
47	91	4	T	Woe! A what? *[dramatically]*		Emb	F	A/D	Req rept	IA
48	92	4	B2	An abundant number.	K1	Emb	R	G+	Repeat	AT-AAK
49	93	4	T	An *abundant number*! What is an abundant number?		Emb	F/I	A D	Uptake Req explain	K1 status to B2

The sequence maps were inspected on multiple levels, with special emphasis on transitions from one form of talk to another, especially when tendencies toward univocal or dialogic were indicated. These pivotal points were reexamined within the coded transcripts to further explore and identify the associated moves, exchanges, and sequences in order to describe and categorize their unique properties. Along with providing evidence for understanding relationships of talk and assessment within the model, this integration of data and analysis served to refine the model itself. Both open and selective coding were applied to transcripts from interviews to establish teachers' professed expertise, goals, beliefs, knowledge, lesson images, and action plans (Schoenfeld, 1998), as related to discourse. Pre- and post-observation interview responses were compared with each other and with data from observations. Axial coding (Strauss & Corbin, 1990) was used to explore connections among the categories, focusing particularly on relationships between discursive practices and indicators of mediating mathematical learning processes.

Results and Discussion

The Model

The central result of this research is the development of a model of discourse in mathematics classes, as shown in Figure 2. This model depicts forms of talk and assessment that exist in whole group instruction in middle grades mathematics classrooms, allowing observers to map the flow of discourse and to begin to understand how discourse may be used to mediate mathematical meaning. Although other categories could be argued, these four forms of talk (i.e., monologic, leading, exploratory, and accountable) and two forms of assessment (i.e., inert and generative) have shown themselves to be adequate in mapping the discourse of the participants in this study.

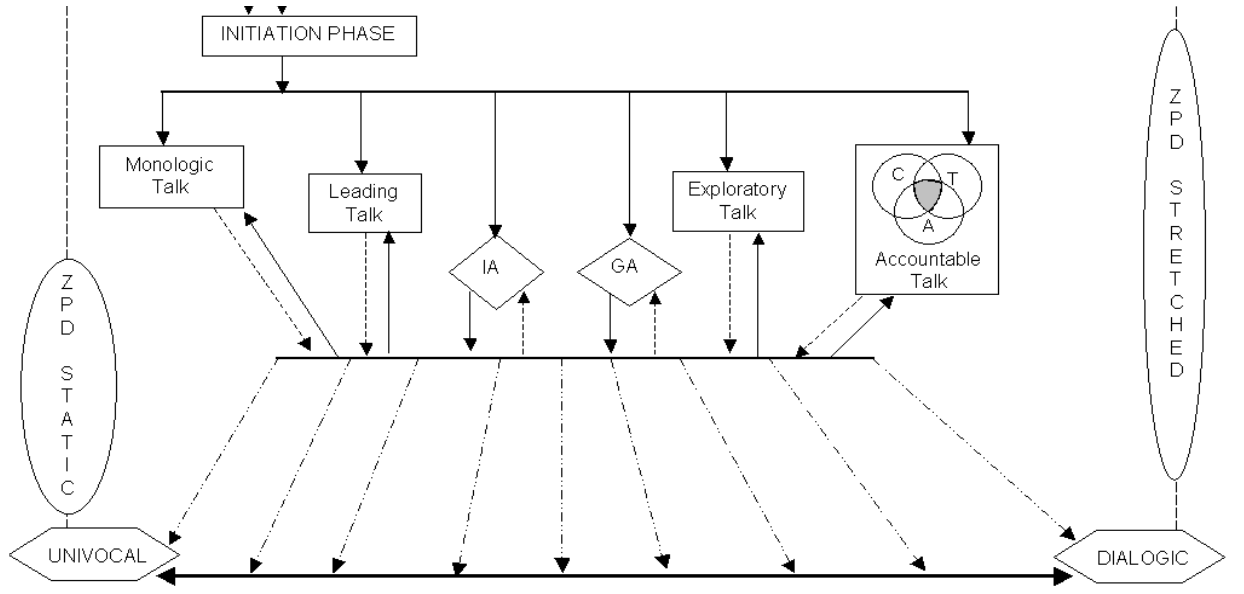


Figure 2. Model of discourse.

Description of the model

The flow of discourse as it relates to the model moves back-and-forth among the various categories of talk and verbal assessment. Within individual sequence maps the numbered moves demonstrate the paths of the talk and assessment. For example, in the sequence map shown in Figure 1, the discourse was initiated at 1, advanced to accountable talk at 2. The teacher offered an inert assessment at 3. This was followed by another instance of accountable talk at 4. A generative assessment was infused at 5. Accountable talk continued at 6, and so on. The ensuing discourse included interactions of the forms of talk and assessment that resulted in discourse that tended, in this case, toward dialogic function. It is clear that there are many paths regarding how the classroom discourse might flow.

Theoretical and practical implications of the model

The model of discourse can serve as a graphic template for creating sequence maps, suggesting uses within research and professional development activities. For instance, the model could be used for mapping teaching practices of in-service or pre-service teachers—providing evidence for research and/or self-reflection. For example, a teacher who views a sequence map that is made up exclusively of leading talk and IA may ask him/herself about the instructional intent. If the sequence was intended to perform simple classroom routines, the mapped discourse may be deemed appropriate. However, if the intent was to engage students in discussing mathematical ideas, the teacher may reflect on questioning techniques that may promote generative assessments. From a theoretical perspective, this model brings together work from mathematics education, theories of learning and socio-linguistics, and discourse analysis. Along with building on previous research, the model illustrates the relationships of the talk, assessment, and univocal or dialogic tendencies.

Mediating Meaning

In addition, the model provides an opportunity to assist in beginning to understand when students are constructing meaning. Social constructivist theory purports that dialogic speech—that occurs between and among people—can serve as a mediator of meaning to enhance speaking and thinking, and thus affect the intramental processes (Wertsch, 1998; Vygotsky, 2002). Through its focus on the interactions of talk and verbal assessment and their relationships toward univocal and dialogic tendencies, the model supplies indicators to whether meaning is being conveyed or it is being constructed. Wertsch (1998) noted that one reason for conflicting results in research related to meaning making was a "failure to appreciate the power of the mediational means involved" (p. 119). The model deals with this reported weakness by explicitly focusing on the talk and assessment as mediators of meaning.

For example, Figure 1 illustrates a sequence where talk and assessment served as mediators of meaning. First, it was very lengthy and complex, including multiple examples of both GA and accountable talk. Also, this sequence was preceded by three simpler sequences that established common language and a frame of reference from which meaning could be built; the complexity of this sequence would not have been possible without the connections to the preceding sequences. Within this sequence, leading talk was used to continue to establish and to reinforce common language (e.g., numbers 24, 52, and 56). IA was used to praise, to evaluate, and to maintain existing talk functions (e.g., numbers 43, 45, and 47). Exploratory talk (e.g., numbers) and accountable talk (e.g., numbers)were used to suggest (e.g. 16, 42, and 44) and explain (e.g., 46, 48, and 50)conjectures about a hypothesis that had been offered. GA was infused throughout the sequence (e.g., numbers 9, 11, and 17), helping to focus the discourse toward meaning making. For example, at move 77, the path advanced toward dialogic function when students voiced connections between a rich problem introduced in an earlier sequence and new knowledge established in this sequence. The flow continued, re-initiating at move 78, back to accountable talk, on to GA, and so on. Multi-level analysis, that included magnification of the sequence map and cross-referencing of transcript and interview data, resulted in a richer understanding of how the teacher mediated meaning.

Final Remarks

Providing a model of discursive practices related to promoting mathematical meaning has the potential to help teachers, mathematics educators, and researchers to make sense of mathematics instruction and, in turn, to improve it. Such a model has the potential to provide opportunities for identifying themes, examining patterns, and revealing pivotal points in classroom discourse. The model may provide possibilities for those entrusted with the charge of educating our children to begin to make sense of the nature and role of discourse within middle grades mathematics classes and, ultimately, to explore how students come to understand and make meaning of mathematics.

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COURSE-TAKING AND EQUITY: THE EFFORTS OF ONE HIGH SCHOOL MATHEMATICS DEPARTMENT

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This paper outlines the case study of one high school mathematics department that engaged in curricular redesign in order to address high failure rates in lower level mathematics courses disproportionately populated by students of color. The department implemented four major changes that were facilitated by four department characteristics. The changes, accompanied by two challenges, when considered collectively did little to increase students' likelihood of taking more higher level mathematics courses. The limited improvement of the curricular changes was influenced by the teachers' expectations of their students and their beliefs about the nature of mathematics.

Introduction

The scope of what it means to be successful in mathematics expands beyond skill-acquisition and procedural knowledge and includes sense-making and conceptual understanding. Still, mathematics achievement remains a key issue in educational research. Achievement on skills-based tests provides critical information regarding students' competencies to the educational community. These tests continue to illustrate that US students' performance is mediocre (Dossey, Mullis, Lindquist, & Chambers, 1988; McKnight, Crosswhite, Dossey, Kifer, Swafford, Travers, & Cooney, 1987; Moore & Smith, 1987; Secada, 1992; Tate, 1997) and that students' of color performance is lower than their White counterparts (Secada, 1992; Tate, 1997). Research has been conducted in various areas attempting to explain this phenomenon. Students' background factors (Moore & Smith, 1987; Rech & Stevens, 1996; Signer, Beasley, & Bauer, 1997; Tate, 1997), school level factors such as opportunity to learn (Lee, 1997; Oakes, 1990), and the relevance of content in the curriculum (Ladson-Billings, 1995; Silver, Smith, & Nelson, 1995) are some of the factors that have been examined.

In particular, research that considers students' opportunity to learn has examined course offerings and patterns in course-taking as a means to address mathematics achievement (Lee, 1997). A general argument extrapolated from this research suggests that students experience higher achievement in mathematics as they take higher level mathematics courses (Miller & Linn, 1998; Reynolds & Walberg, 1992; Tate, 1997; Winfield, 1993). Additionally, departments that have been successful at getting large numbers of students who traditionally underperform in mathematics to take more advanced level mathematics courses (Gutierrez, 1996; 2000) have been examined. Both of the aforementioned research lines have considered course offerings (or curriculum – in a narrow sense) as a means to examine successful efforts or to illustrate how advanced course offerings increases student achievement.

Objectives/Purposes

I examine the process in which one high school mathematics department engaged in order to redesign its curriculum to respond to high failure rates in lower level mathematics courses that were disproportionately populated by students of color.

Hereafter, these courses are referred to as targeted courses. The targeted courses include a geometry course, two versions of an algebra course, a pre-algebra course, Math Explorations, and Math Review. Math Explorations and Math Review were two remedial courses designed by the department and are described in greater detail later in this paper.

Previous research indicates that students benefit from taking advanced level mathematics courses. In this study, I examine how one high school mathematics department engaged in the process of redesigning its curriculum and restructuring its course offerings with the intention of improving student achievement and providing a greater number of students with access to higher level mathematics courses. An examination of this process is critical in that it provides insights to other high school mathematics departments that have identified low-achievement, disproportionate offerings of lower level mathematics courses, and course offerings that promote lateral course-taking. Among the findings, I identify aids and challenges to the department's efforts to redesign its curriculum, discuss the course offerings influence on access to advanced mathematics courses, and discuss how teachers' beliefs influenced curricular changes. This study is also significant to the field of mathematics education in that it takes an up-close look at the process to restructure course offerings while most prior research has examined course offerings that were already in place.

Methodology

Site

This case study was conducted in the mathematics department of one mid-size high school located in a small, mid-western city. Rolling Meadow High School (RMHS) (pseudonym) has a student enrollment of approximately 1500.The racial demography of the school is 65% White, 28% African American, 5% Hispanic, and 2% Asian/Pacific Islander.

Performance on state standardized exams was as follows. Thirty-nine percent of the student body met or exceeded state mathematics requirements, and statistics from various racial groups was as follows: 44% White, 19% African American, 42% Hispanic, and 71% Asian/Pacific Islander. The graduation rate was 68% for the student body, and statistics from various racial groups was as follows: 75% White, 55% African American, 41% Hispanic, and 67% Asian/Pacific Islander.

The mathematics course offerings at RMHS were broad and were categorized as a part of either the *Standard* or the *Gifted/Accelerated* curricula. Among the mathematics courses offered were pre-algebra, transitional mathematics, two versions of algebra, two versions of geometry, a second year of algebra with trigonometry, a third year of algebra with geometry, introductory statistics, pre-calculus, and two versions of calculus which included Advanced Placement Calculus. While several options in mathematics were available, research suggests that such leads to lateral movement, thereby restricting students from taking more advanced courses (Gutierrez, 1996). Moreover, Lee, Croninger, & Smith (1997) found that students learned more when they attended schools with a narrowed curriculum.

Participants

The participants included twelve of the thirteen teachers in the department, the principal, and a guidance counselor. All of the teachers were White; eight were female, and four were male. The participating teachers' average years of K-12 teaching was 10.8. Four of the teachers had less than three years of teaching experience, and three had postsecondary mathematics teaching experience. Teachers who participated in classroom observations were recommended by the school principal based on my request for teachers with varying levels of experience and who were teaching targeted courses.

Data Collection

Data collection extended over eight months and included field notes from classroom observations and observations of department meetings, a focus survey, audiotaped interviews, and school documents. Each teacher completed a focus survey that provided foundational information on their own educational background, beliefs about their students' capacity to do advanced level mathematics, and the nature of mathematics. Each teacher also participated in at least one 45-minute interview. Three of the twelve teachers were key-informants (Wolcott, 1988) participating in two additional interviews and permitting me to observe their classes. I observed five sections of mathematics classes from the list of targeted courses, twice a week for six months. The principal and guidance counselor participated in one 30-minute interview. I collected field notes from the five classes that I observed, six department meetings, and two sub-department meetings. Sub-department meetings consisted of all the teachers who taught a particular course. Among the school documents that I collected were School Report Cards, the master course schedule, the Student Handbook, course listings and prerequisites, curricula guides, and school polices.

Data Analysis

Data analysis included constant comparison analysis (Strauss, 1987), triangulation of data, and a search for disconfirming evidence. Transcripts of each interview, field notes from classroom observations and department meetings, and school documents were coded and analyzed separately and then together. While listening to the audio-taped interview, I followed the transcripts, coded the transcripts utilizing a coding scheme that was initially compiled from the research literature. I periodically revised the list of codes based on ongoing analysis. With each iteration of data analysis, I identified emergent themes and revised my list of codes accordingly. The revision of the list of codes included collapsing multiple themes, adding new codes, and eliminating codes that were deemed inappropriate. The revised list informed future data collection and my search for disconfirming evidence.

I also analyzed school documents. I examined teachers' schedules comparing their schedules with the master schedule in order to identify when and if additional courses were added to the course listing and under what circumstances. I identified which teachers typically taught the targeted courses and their level of experience. This analysis is significant since prior research has found that lower level mathematics courses are often taught by less experienced teachers (Oakes, 1990).

Findings

The RMHS mathematics department implemented four major curricular changes in their efforts to address low achievement. In this section, I describe the four curricular changes, departmental characteristics that aided and challenged the redesign process, and the resulting course offering structure and students' accessibility to advanced mathematics courses. Finally, I discuss how the changes selected by the department were influenced by the teachers' beliefs about their students' capacity to do higher level mathematics and the nature of mathematics.

Curricular Changes

The mathematics department at RMHS implemented four major curricular changes in the process of designing the mathematics curriculum in order to respond to high failure rates in lower level mathematics classes that were disproportionately populated by students of color. The first change was a switch from offering Extended Algebra to offering Modified Algebra. Extended Algebra was a two-year or four semester sequence of a high school algebra course that satisfies the basic college requirement of one year of high school algebra. Modified Algebra was a one year or two semester sequence of introductory high school algebra which also met the basic college requirement of one year of high school algebra. Algebra 1-2, a traditional one year or two semester algebra course, was also offered. Modified Algebra differed from Algebra 1-2 in that topics received less depth in coverage in the Modified Algebra course. Students received two mathematics credits towards graduation for completing Modified Algebra or Algebra 1-2. Completion of Extended Algebra satisfied the four semester mathematics graduation requirement.

The second curricular change was the addition of a Modified Geometry course. The Modified Geometry course was a two semester version of geometry that was offered in addition to a traditional, one-year geometry course, Geometry 1-2. While Geometry 12 was recommended for the *college bound* student, Modified Geometry was recommended for students who had completed Modified Algebra or students who had passed Algebra 1-2 and received a letter grade of C or below. The primary difference between Modified Geometry and Geometry 1-2 was that Modified Geometry excluded formal proof. Students received two mathematics credits towards graduation for completing either Modified Geometry or Geometry 1-2.

The third curricular change was the redesign of Mathematics Explorations. Math Explorations was a course for incoming freshmen who had scored below the range deemed appropriate for placement into Pre-Algebra. Math Explorations had previously been offered as a course for students who failed the first semester of various courses in order to fortify their skills for reenrollment into their previously failed course. The department deemed the initial design of Math Explorations as ineffective because it consisted of students from too many courses. Students received two elective credits for completing Math Explorations and were placed in Pre-Algebra the following year.

The fourth curricular change was the addition of Math Review. Math Review was a one-semester remedial course for students who had failed the first semester of Modified Algebra. Admission required a teacher's recommendation and a maximum number of missed assignments and missed days during the semester in which the student failed the first semester of Modified Algebra. Sections of Math Review were capped at 15 students, and each teacher could recommend three students for course. Establishing sections of Math Review for students who had failed the first semester of Pre-Algebra was theoretically possible. However, 15 students were not recommended even though 30% of the 150 students taking Pre-Algebra failed the first semester.

Aids and Challenges

The four major redesign efforts were significant changes for the department and required a great deal of time and effort. I identified four characteristics which aided the department in making these changes. These characteristics facilitated a relatively smooth or, at least, an essentially resistant-free implementation of the curricular changes. The first characteristic was the department's willingness to try new things. Teachers went to conferences and workshops, returned, and shared new ideas that resulted in the utilization of mathematical software and projects. Moreover, the department's implementation of the four previously described curricular changes indicated that members of the department did not shy away from implementing multiple changes simultaneously. The second characteristic was that the department had a well-organized, well-liked, and highly respected department chair. The third characteristic was the presence of enthusiastic volunteers who completed the majority of the legwork in order to implement the curricular changes. These four teachers, all of whom had taught at RMHS for less than five years, completed multiple tasks which included book selections, drafting letters to be sent to parents about new courses, and synthesizing data on students' performance in the targeted courses. The fourth characteristic was the existence of frequent department and sub-department meetings in order to engage in planning for and discussions about the curricular changes. The department had about one meeting per month, and attendance at the meetings was at least 75% with all teachers attending several of the meetings.

I identified two challenges to the curricular design efforts. The first challenge was the department's perception of the Guidance Office and the corresponding stance that the department took. Several members of the department viewed the aims of the Guidance Office as conflicting with the department's goal of students learning mathematics. Consequently, the mathematics department created policies for placement into courses that sought to limit the Guidance Office's flexibility in placing students. The second challenge was the department's rush to implement changes. The department did not engage in long-term planning when implementing curricular changes and nor did it outline possible ramifications of individual or collective changes. Moreover, the planning was quick and lacked foresight. In particular, planning for Math Review began only weeks before it was first offered, and the course was taught by the teacher who had a section of Modified Algebra dissolved as a result of having too many students to fail to sustain offering the section the second semester.

Resulting Course Structure & Accessibility to Advanced Mathematics Courses

The course structure that resulted from the four curricular redesign efforts yielded both positive and negative ramifications on students' access to advanced mathematics courses. The switch from Extended Algebra to Modified Algebra increased the likelihood that students would take a geometry course since it decreased the number of mathematics credits some students received for completing algebra. And, although the addition of Modified Geometry provided an option to students who typically did not reach a high school geometry course, the Modified Geometry course excluded formal proofs – a weakness that many members of the department acknowledged especially if students were to take mathematics beyond geometry. However, this positive change was mitigated by the addition or redesign of the two remedial courses, Math Explorations and Math Review. Overall, the four curricular changes did little to increase students' access to advanced level mathematics courses and established a *mathematics glass ceiling* with the design of Modified Geometry as a dead end course.

Influences on Curricular Redesign: Teacher Beliefs

Two factors, the teachers' expectations of their students and the teachers' beliefs about the nature of mathematics, emerged as key influences on the department's curricular redesign efforts. The teachers held low expectations of their students. Although the department had a goal of increasing the level of mathematics to which all students reached, the expectations for varying groups was different and bounded for students enrolled in the targeted courses. Teachers viewed a geometry course as a major gain for students enrolled in targeted courses, and consequently rationalized that designing the course as a dead end was not problematic since at most a few students would take a course beyond geometry. Additionally, the department's goal for students taking more mathematics courses did not equate to students taking more advanced courses or even more courses at the same level as was evidenced with the redesign of Math Explorations, a course below Pre-Algebra, and Math Review, a remedial course. Additionally, the department opted to eliminate formal proof in the design of the Modified Geometry course because it was the content that presented problems for students in the targeted courses.

The teachers held a procedural and sequential view of mathematics. An emphasis on skill proficiency was evidenced by the design of Math Explorations and Math Review as courses to fortify skills. Additionally, observations revealed that most class instruction emphasized drill and practice with little attention being paid to conceptual understanding. For example, Modified Algebra permitted more time for drill and practice and lacked depth or emphasis on sense-making. The teachers did not attempt alternative pedagogical approaches to enhance learning; rather, they designed courses that provided students with more opportunities to learn low-level mathematics and emphasized skill-proficiency.

Conclusion

The department engaged in four major efforts to address high failure rates in lower level courses. However, when taken together, these changes did little to enhance students' access to advanced mathematics courses. This study points to the significance of teachers' beliefs in making curricular decisions and the need for critical participants in the redesign process to facilitate well thought-out discussions about ramifications of curricular changes and possible unintended outcomes. These insights suggest that good intentions and a willingness to try new things are insufficient in addressing equitable mathematics education.

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CALCULUS TEXTBOOKS IN THE AMERICAN CONTINENT: A GUARANTEE FOR NOT UNDERSTANDING PHYSICS

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This report explains some relevant aspects of an investigation which shows the contrasting visions around differentials in physics and calculus courses for engineering majors. We analyze certain traits related to the differentials of discourse developed in calculus textbooks [CTB) from the US, which limit those whose knowledge of calculus depends on them. These limitations refer to many areas of physics, for example fluid mechanics, electricity, and magnetism. This criticism is based on the deeply rooted practice in physics referring to differentials that, as we show here, cannot be epistemologically reconciled when using the discourse of the CBT. The emphasis of this research is socio-epistemological in that we recognize the importance of considering the social practices influencing mathematical ideas in order to evaluate the teaching of mathematics, so that the student acquires the tools to permit progress in the study of the basic science.

Introduction

In all of Latin America and the United Status there are calculus courses for engineering majors based on US textbooks [such as Larson, 1989; Leithold, 1987; Purcell, 1993; Stewart, 2001; Thomas-Finney, 1987; Zill, 1987); these books have established the true in mathematics (philosophy and ideology), and have helped create a paradigm in teaching and learning that we still suffer from.

Criticism has been widely documented as to the emphasis on the logicodeductive character of the content in calculus textbooks (Artigue, 1995; Cantoral, 1990; Dreyfus, 1990). One indication of this emphasis is the desire to present the final state of knowledge, apparent in the index of these books: real numbers, functions, limits, continuity, derivatives, applications of derivatives, Reimann sums, etc. This form of presenting the content is associated with the students' difficulty in understanding the concepts that are intended to be communicated. The content is foreign to the student who lacks understanding of its essence. As a consequence the content will not belong to him; he can be forced to study it because of of school obligations, but will not able to learn it.

Faced with the evidence of the lack of understanding of the students, the professor tries to algorithmicize part of the content, emphasizing the learning of techniques more than the ideas. In this way, learning calculus is synonymous with solving routine exercises, and it is common that students boast that they "know calculus" because they can "derive and integrate."

The fundamental ideas of calculus and its reason for being are hard to extract from the proposals of the CTB. The recurring complaint is that students know how to "derive and integrate" but they don't know when they should use those processes. This occurs not only in countries on the American Continent but also in others like France and Spain that also suffer from a teaching of calculus that emphasizes more logicomathematical chaining of content than it encourages showing the interrelationship of relevant ideas to the student.

Purpose of the Research

This report attempts to corroborate another aspect of the content of the CTB which negatively affects learning the science, above all in engineering students. We will show the low level of articulation between what is taught in calculus courses [supported by the CTB] and the form of mathematization in other areas of the science, such as physics. Specifically, we argue the existence of a style of working in physics in which concepts are formulated and differential equations are constructed: we refer to the differential style in which the differentials (infinitely small quantities) and the setting of the differential element play a determining role; in contrast, differentials in the CTB play a confusing and contradictory role, and in fact, the notion of the infinitely small does not exist (Pulido, 1998).

This situation should be changed because the engineering students exposed to the content of the CTB will necessarily have difficulties in learning the science, in spite of their capacity or the enthusiasm of the professor, as they exposed to the crossfire of ideas coming from their calculus and physics courses.

Theoretical Framework

This research adopts a socio-epistemological perspective for the study of phenomena related to learning in the classroom, as discussed in Cantoral and Farfán (2003). This approach includes, besides cognitive, epistemological and didactic dimensions, a sociocultural component. This perspective helps center the claims we make in the analysis of certain practices around differentials that we have found in classroom physics, substantially different from those found in the CTB. In fact, it is only possible to explain this distance through the evolution of practice in key moments in the history of physics and mathematics.

Methods

By a review of the textbooks in physics we show the presence of a style of work based on the differentials, and with a historical-epistemological analysis, we show that behind the differential style, there is Leibnizian calculus in which infinitely small quantities are a fundamental part. On the other hand, an epistemological analysis of the CTB allows us to see that the discourse attempts to conform to classic real analysis, as it is called, in which real numbers are behind all of the concepts. Of these two analyses come the epistemological distancing of both presentations: the infinitely small are not real numbers; in fact, a didactic analysis of the CTB around the "differentials" shows how incoherent and confusing the discourse is as a result of adopting a definition of real number for the differential.

Conclusions

Analysis of an epistemological kind, within a socio-epistemological framework allows us to determine the incapacity of calculus courses based on the CTB to encourage students to enter successfully into the learning of physics, a serious problem for those who study engineering.

Considering the same approach, it is necessary to create new proposals of calculus that take into account the practices of the other disciplines that it attempts to support. Salinas, Alanís, Pulido et al (2002) offers one of these attempts.

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VOICING SUCCESS IN MATHEMATICS CLASS: ANDREA'S STORY OF SUCCESS

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This paper reports on the results of a practitioner-based research study conducted in a grade 10 Consumer Mathematics (entry-level) class. Using narrative inquiry, I inquired into the nature and evolution of success of students who approached mathematics with an unsuccessful stance. Because listening to student voice through story-telling is central to this study, the telling of Andrea's story illuminates the students' success: the emergent/ce (of) voice. Andrea's story of the evolution of her use of words is an exemplar that demonstrates emergent voice can be characterized as being vocal, verbal, and intentional. These characteristics address the static nature of success; however Andrea's story will also address the dynamic nature of the evolution of success – from voicelessness to emergent voice.

Success in high school mathematics has become a focus of stakeholders in education. Debates on the essentiality of students enrolling in high school mathematics has lead to students not only *studying* mathematics in high school, but also *understanding* and *valuing* the mathematics they are studying. In teaching Consumer Mathematics (Manitoba Education, Training and Youth, 2002), a course designed for students who do not intend to pursue post-secondary mathematical studies and often approach mathematics with little confidence, I had noticed that success could be fostered through relational teaching as articulated in Noddings' (1984) understanding of teaching as caring and van Manen's (1986) pedagogical relationship. However, what I did not fully understand were the ways in which these students came to be successful – leaving a stance where they perceived themselves as unsuccessful as they began to see themselves differently during the course.

My practitioner-based inquiry focused on the question: How does the nature of success of learners evolve in grade 10 Consumer Mathematics? The research methodology drew on narrative inquiry (Clandinin & Connelly, 2000), where narrative texts (data) is used to interpret the lived experiences of individuals. Eleven students participated, writing interactive journals, constructing portfolios, and participating in three conversations (informal interviews). These data pieces were used, along with daily field notes, to construct a narrative for each learner that highlighted successful moments. During each conversation, the learners and I were in discourse about their narratives of success as we came to understand the nature of their success.

In preparing for this inquiry, I was not certain about the types of success that the students would be experiencing or how I would come to understand their success. So, I prepared myself with six theoretical frameworks from educational researchers including Belenky, Clinchy, Goldberger, and Tarule (1986), Baxter Magolda (1992), Chickering and Reisser (1993), Weiner (1972), Dudley-Marling and Searle (1995), and Romagnano (1994). Throughout the inquiry, these models became secondary to the priority of the students' experiences and their descriptions of their learning and success. This paper will relate the story of one student, Andrea, making limited use of the above frameworks and will conclude by offering a generalized theme of success for Andrea that was supported by theory, but informed and shaped by the experiences of Andrea and her classmates.

Andrea's Story of Success: Using Words to Tell

Andrea approached the beginning of the school year with tentative steps. She was new to the city and to the school, so her interactions with students within the classroom were limited. She rarely asked questions in class and did not take part in whole-class interactions. Because Andrea was behind in her high school credits, she had enrolled in a variety of grade 10 and 11 courses, even though she was the same age as other grade 12 students. She wrote in her first journal that she took Consumer Mathematics to be assured of a grade 12 mathematics credit, because she believed that it was "easier than the other math courses." Her journal statement indicates her belief that she was not a successful mathematics learner in school. As well, her interactions with me at the beginning of the semester also point toward her inability to say things about herself, her thinking, her learning or her successes. Although Andrea began her journey of success saying very little, the way in which Andrea used words in class evolved over the semester. I came to understand that the theme of her story of success was "Using Words to Tell". Listen carefully to the way in which Andrea used words as I re-tell her story, in order to hear the individual steps of success Andrea took and how they contributed to her journey of success in mathematics class.

Asking a Teacher for Help

Near the beginning of the semester, Andrea's predominant interaction in class was to ask me, her teacher, questions when she did not know what to do. Andrea would use a limited number of words to indicate where she was stuck, so that I would explain clearly and slowly how to complete a question. She used this strategy to request of me to tell her exactly what to do and then she would do it. Andrea viewed me as an authority figure in terms of mathematical knowledge as well as knowledge of her as a learner. She also believed that the knowledge she needed to have could be given to her by an authority figure. At one time, Andrea had approached a resource teacher, viewed as an authority figure as well, to explain similar triangles to her. In our first conversation, Andrea related, "And finally, she told us what to do. And then, she said that what we were doing was wrong. But we were doing it the exact same way she taught us." Andrea's expectation that the teacher would listen patiently and tell her clearly how to complete the question can be seen in Andrea's frustration with the perceived "help" she had received.

Andrea's description demonstrates her stance toward knowledge and authority in the classroom. The manner in which she solicited help from authority, using words in a limited manner to point to where she was stuck while omitting her communication of what she did understand and what she had done correctly, demonstrates her devaluation of her own words and ideas. The expectation in which she waited for a teacher to tell her the mathematical steps to complete a question shows her reliance on an authority to give her knowledge. Andrea's limited use of words and her reliance on authority demonstrate an approach to knowing that Belenky et al. (1986) recognized as *silent knowing*. Silent knowers rely on authority that they perceive to be all knowing. Their lack of confidence in their ability to know and learn is exemplified in the difficulty they experience describing themselves and engaging in self-reflection. Andrea's initial stance in Consumer Mathematics is characterized by silent knowing.

Asking Peers for Help

As the semester progressed, Andrea began to ask her peers for help. When we discussed this emerging strategy in our first conversation, Andrea mentioned, "'Cause if you don't understand a question, and they do, then, if they do, then somebody you know tells you how to do it. They explain it differently." Andrea believed that her peers could explain, better than a teacher, how to do a question because they would use words that she understood. She found teacher's words hard to understand, partly because "they learned it so many times, they sometimes talk like their professors would talk." When I probed further, Andrea clarified that her peers explained the same steps to her, but just in different words. There is a shift in Andrea's actions in class when she needed help, to ask peers because their words were more understandable than a teacher's words, without a shift in her belief about the nature of knowledge.

Although Andrea still relied on others to *give her* knowledge and *tell her* how to complete questions, her prioritizing of asking peers for help demonstrated a shift of authority roles in the classroom. The shift in stance is from obtaining the teacher's knowledge from the teacher to obtaining the teacher's knowledge from peers, in peers' words. This stance demonstrates more closely the positioning of a received knower (Belenky et al., 1986), someone who still does not construct their own knowledge yet begins to use words to affect her/his identity. A received knower has emerged from a stance of silence, but still uses words in a limited way and perceives others as being authorities of the knowledge he/she views as valuable. While there was progress, Andrea was using words in this context just to complete specific questions rather than using words to *learn* mathematical ideas or skills.

Interacting with Others to Learn

As I continued to observe Andrea and talk with her in further conversations, I began to notice that asking peers for help was not the only way that she interacted with classmates. Andrea was building on that success by engaging in more complex interactions with her peers. While the previous interactions with individuals had occurred before the first conversation, I began to notice this next type of interaction between the first and second conversations. The theme of Andrea's second narrative was the way in which she used words in the classroom. In response to the narrative, Andrea told me about the unique characteristics of each of her learning partners.

When Andrea sat with Susanne, she did more than ask Susanne how to do specific questions. She stated, "And Susanne, well, we're both kind of the same, try to figure it out both … well, we had fun trying to figure out stuff for the assignments!" Rather than asking questions or telling each other how to do something, Andrea and Susanne were *figuring out* the concepts and skills that they were learning *together*. The mutuality indicated in this phrase demonstrates that Andrea and Susanne were working toward a common goal of learning. Andrea was developing a sense of authority over her actions in the classroom as she began to see the importance of *learning with* her table partner. The intentions that Andrea had for her words, to affect her learning and to affect her relationship with Susanne, was a step forward from the intention of acquiring help.

A shift in Andrea's role within the peer-to-peer relationship is evident as she moved from *listening* to *actively engaging* in the classroom discourse and her learning. Within this process, Andrea was actively involved in using words to learn with others and to learn about mathematics. Learning to use words in more complex ways, similar to constructing mathematical knowledge and understanding (Borasi, 1992; Ward, 2001), is a rich and complex task that requires the individual to be active throughout the process. By using words in more sophisticated ways, Andrea was beginning to say things about herself and her learning.

Explaining Mathematical Concepts and Skills to the Class

As Andrea built confidence in her context, both with individuals and with mathematics, the nature of her interactions continued to evolve. Just before our second conversation, I noticed Andrea was becoming more active in class, willing to provide explanations to the whole class. Because this was a new way that I noticed Andrea using words, I highlighted the moment as a part of her second narrative of success:

> You are often willing to volunteer answers in class when we are going over an assignment. Most of the time, it happens after you and I have had an opportunity to work on some

> questions together. One example is looking at food labels and working with the recommended daily intake. You explained really well to the class that you would need to eat 10 servings of cereal to receive the recommended daily intake of a vitamin because there is 10% of it in one serving. Another example was the definition of perimeter, when you told the class perimeter is just the distance around. When you answered in class, I used your words to make the notes for the rest of the class. I'm wondering if these are examples, for you, of times when you had a good math idea that you thought others should hear as well. It also makes me wonder whether putting the math ideas in your own words is what makes the ideas important to share.

Andrea took part in the classroom discourse in two different ways. First, with the daily vitamin intake question, she described to the class a solution and answer after she was sure to check with me first to confirm she was correct. She still did not view the idea as her own, but she engaged in vocalizing a mathematical idea and took ownership of giving that knowledge to her classmates. This still demonstrates a received knower's stance (Belenky et al., 1986), yet Andrea was beginning to recognize the importance of *her* words and that they had value for her and for others – a stance that demonstrated movement away from silent knowing. As I helped Andrea with the assignment and then observed her explaining it to the class afterwards, I recognized growth from her previous stance of being the individual who had to be *told*. Now, Andrea was doing some of that same *telling*.

A few weeks later in class, providing the definition and formula of perimeter was a second way in which Andrea was involved in class. Andrea recalled this example as a time when she explained something to the class without checking with me first. She felt it was a successful moment because the ideas she explained were her own and, "'Cause it's, like, it really makes you feel confident. Like, other people are using how you describe stuff." Andrea was using her words to affect her identity as a confident mathematics learner and also to affect her classmates' knowledge. As well, Andrea's belief that an idea was *her idea* demonstrates a significant shift in her use of words from our first introduction to her as a student who said very little because she believed her words and ideas held little value.

Having a Good Idea, But Remembering It?

As the semester progressed, I continued to look for moments of success where Andrea was constructing her own ideas and saying them out loud. During a perimeter/area inquiry, Andrea and Whitney were exploring rectangles with constant areas and differing perimeters. Andrea described to me how the perimeter of a "2 by …" rectangle increased by two units every time she added a column of two blocks. It was an astute pattern-recognition moment in which Andrea was generalizing a pattern she noticed and was saying the generalization to me. Andrea's mathematical success was the idea that she constructed from the inquiry activity to build a more complex understanding of area and perimeter. I was certainly encouraged by Andrea's thinking.

However, Andrea demonstrated in an extra conversation a few days later that she had no recollection of thinking the idea or saying it out loud. Andrea's success, in this example, is not contained in the event that she could not remember having a good idea. Her success is in having the good idea. Eleanor Duckworth views "the having of wonderful ideas [as] … the essence of intellectual development" (1996, p. 1). Having her own idea was enough to see this as a successful moment. What is also significant about this moment is that *I noticed* the success. I knew Andrea well enough as a student, learner, and individual, to recognize this moment of cognition as something that was significant to her lived experiences in the classroom. My noticing Andrea's success signifies that a student's shift in the use of words requires another to be inviting students to engage in the process and to be vigilant. These small shifts in stance would be easy to miss if the teacher did not live in pedagogic relationship with them and was not engaged in listening closely. As I lived with my students, I was continually looking for successful moments that I could celebrate with each learner.

Valuing Her Own Words for Understanding

Although there was fragility in Andrea's use of words as it became a more complex process, Andrea was beginning to perceive the value of her own words. In our second conversation, Andrea selected the phrase "resaying other people's words" from her second narrative to signify her success. She believed it was important because:

Andrea: Well, if they can describe it better. And you don't really have the right {inaudible words} trying, what you're trying to say. But you know the answer, but you can't write it down in your own words. Then it's easier to use somebody else's because then you can look back on it. And if there's a question on describing it, then you, later on you'll put it in your own words, instead of using other people's.

Janelle: So, for right then when you're not quite sure, you write down exactly what somebody else said.

Andrea: No, you change it a little. 'Cause they can use their words, and you might not understand it. But, you can kind of change it a little so that you will understand it. And then, later on you'll be able to put it all into your own words.

Janelle: Oh, okay. And, is it important to put it all into your own words later on?

Andrea: Yeah.

Janelle: How come?

Andrea: 'Cause then you're not relying on other people all the time.

Andrea believed that it was important to put mathematical ideas in her own words so that she could *understand* and develop an *independent* stance in her learning. Her words contained an intention to support her learning so that she could understand the problems she was solving.

Within Andrea's story of success, this interaction is significant because Andrea was coming to value the words she said as effective in learning mathematics and in identifying herself as a successful mathematics learner. Andrea had turned inward in her explanations, intending to affect her understanding and learning. Valuing her own words for understanding demonstrated a shift away from received knowing (Belenky et al., 1986) because she was engaged in constructing her own understanding with her words instead of relying on the teacher to give her the mathematical ideas *and* the words.

Andrea Telling about a Successful Moment

Andrea completed the semester approximately a week early, and in anticipation of that she was required to complete several textbook assignments independently by reading the examples in each lesson and then practicing several questions. One of the topics was capture-tag-recapture sampling, a ratio-based method used to determine wildlife population. Because Andrea's last day was not definite, she happened to be in class when the rest of her classmates learned this topic.

Although Andrea never had an opportunity to retrospectively tell me about this moment, I recorded our interaction in my field notes. This is what I wrote:

> Andrea was really happy that she had done the assignment already and knew how to do the questions. She came up to me, just before I went over the questions on the chalkboard with the class. She said (something like), "You know what I saw with these questions, Mrs.

> McFeetors? You just use the numbers backwards from the way the question is written. You start with the last number and you put it over the second last number and then that's equal to the first number over what you need to find. I noticed that [pattern] while I was working on the questions." She was really proud of herself. And, she came up to me just to tell me that.

There are several elements of Andrea's use of words in this moment that point to her success in grade 10 Consumer Mathematics. There is the metacognitive statement that Andrea made to me when she generalized the steps for the question, noticing that the steps were the same for each of the questions she had completed. There is Andrea's responsibility to complete assignments ahead of time and in a nearly self-directed manner. There is Andrea's learning how to complete capture-tag-recapture questions on her own, without the support of her peers or teacher. Rather than placing significance in Andrea's metacognition, responsibility or independence, her ultimate success was located in Andrea *noticing and expressing her success*. She used her words to say when and how she had been successful, with the intention of shaping her identity and her relationship with others. It was a pinnacle moment for Andrea's learning and self-identity because of the nature of her use of words.

Drawing Meaning from Andrea's Story of Success

The particular theme of Andrea's success was the evolution of her use of words. This theme was closely tied to Andrea's learning in Consumer Mathematics – and the particularized themes of each of the learners in the inquiry differed because of their uniqueness. As data collection and interim data analysis concluded, as an inquirer, I was left to make sense of the learners' successes. I wanted to draw a theme of success from all of the learners in order to come to understand the nature of their success and how it evolved over the semester.

Andrea was not the only student that changed the way she used words over the semester; in fact, it was a success that all the students experienced in some way. As I recognized this commonality, I considered how the six theoretical frameworks I had prepared myself with could inform this kind of success. I could not find a match between the progress that previous researchers had noticed and the way in which my students spoke in mathematics class evolved over the semester – the learners' success seemed to augment existing theory. The *epistemological frameworks* of Belenky, Clinchy, Goldberger, & Tarule (1986) and Baxter Magolda (1992) provided a starting place for data interpretation. An underlying story line that Baxter Magolda (1992) recognized in her data was the "development and emergence of voice" (p. 191). I began to consider the usefulness of this theme to illuminate the success of Andrea and her classmates because of the similarity to the way in which individuals use words to say things, especially about themselves. There was one significant difference, however, between Baxter Magolda's participants and my students – throughout her inquiry her participants were cogently self-descriptive while it is clear that Andrea could say very little at the beginning of the semester.

I noticed that the learners in my inquiry had begun the course *voiceless* (Belenky et al., 1986) in relation to mathematics and mathematics class – they could not say things about themselves, their role as students, their learning, and their success. Over the semester, their successful moments occurred as they began to say things about mathematical ideas, themselves, their learning, and their success. Through listening closely to the students, I noticed the initial utterances of internal voice that other educational researchers had not reported previously. Their *voice* was emerging. The movement away from a stance of silence embodied the *emergence of voice* for each of the learners as the *essence* of their success in Consumer Mathematics – *emergent voice*, their new voice-stance, was indicative of all the learners' success.

What I captured in the students' moments of success was the *emergence of voice*, the tentative sounds of the students' *emergent voice*. The evolution of Andrea's use of words evidence three characteristics of emergent voice drawn from the data. The students came to be *vocal* as they used words to say things they believed were worth saying. The students came to be *verbal* as they chose specific words to point toward the things they believed were worth saying. And finally, the students came to be *intentional* as they used words to affect themselves and their context. They were in the process of *becoming* – forming their identity as students, learners, and human beings.

Andrea was Vocal

The first characteristic of emergent voice is that the individual is *vocal*. Being *vocal* means that the students felt that they could *speak out*, or say things aloud (in writing or orally), and that they did *speak out*. Because an absence of words was related to an initial stance of silence, the fact that students, like Andrea, were saying things about mathematics, themselves, and their learning becomes a foundational element of emergent voice. For Andrea, her ability to speak out is apparent as she began to explain mathematical skills and concepts to the class. She vocalized a mathematical idea, with scaffolding from me, when she explained the daily vitamin intake questions. Andrea's voice was emerging as she took ownership of giving that knowledge to her classmates. She further consolidated her ability to be vocal when she explained perimeter without teacher scaffolding, demonstrating more independence. Andrea was moving in a progressive manner, even within one element of emergent voice. *Speaking out* meant Andrea had the confidence to explain mathematical ideas to the class.

As Andrea's voice emerged, she began to say more things to me about the quality of her learning. For instance, Andrea was evaluating an emerging effective strategy for learning when we talked about how she had begun to ask peers for help. She could recognize and say that asking peers was a better strategy to support her learning than asking her teacher. Expressing that one strategy was better than another demonstrated an increasing authoritative stance toward her learning. Belenky et al. (1986) noticed that individuals who are in the process of gaining a voice are also beginning to see themselves as their "own authority" (p. 54), rather than relying on external authority to tell them what to believe and to give them knowledge. Andrea was becoming an *authority* on her learning.

Freire (2000) emphasizes the significance of being *vocal* when he states, "Human existence cannot be silent . . . human beings are not built in silence, but in word" (p. 88). To begin to emerge from silence, Andrea and her classmates needed to first say something – and the content of what they said was not as significant as the fact that these students were beginning to say things to themselves and to others. As the students began to be *vocal*, they were saying with their words that the teacher was not the sole authority on their thinking and learning. They were becoming aware of their thinking and learning, and were developing a sense of *authority*.

Andrea was Verbal

Being *verbal*, the second characteristic of emergent voice, means that the individual is pointing toward specific objects through the selective use of words, rather than just putting words to thoughts and saying them aloud. Freire (2000) identifies this process as *naming*, which includes identifying significant objects in the individual's world through reflection and giving a name or label to the object. Andrea's reflective stance can be seen in our second conversation when she selected an example as a successful moment for her in Consumer Mathematics. As she identified her success by reflecting on her experiences and the narrative I authored, she used the phrase "resaying other people's words" that I had written as a label to point toward a significant moment of her success. This example illuminates the ability of Andrea to verbalize as she named her success by using a phrase to reflect on and point toward a specific moment. Andrea's use of the label "resaying other people's words" further refined the meaning and use of the name (Searle, 1983) in describing her success.

By *speaking out* and being in discourse about her successes, Andrea had already begun to develop *authority* in her emergent voice. However, with the additional element of being *verbal*, Andrea became an *author* of her own success. An *authorial* stance required Andrea to identify successful moments and label them with meaningful words that she could use to interact with the world around her. As Andrea stated a pattern for "2 by …" rectangles, she was engaged in an authorial act in naming her cognition. More than requesting others to tell her or simply noticing the pattern, Andrea spoke with an emergent voice as she authored the pattern-generalization. The *authoring* of Andrea and her classmates was at once retrospective as they talked about what they had done well, and prospective as they were beginning to say how they were succeeding in mathematics class. *Andrea was Intentional*

Intentionality is the third characteristic of emergent voice. Being *intentional* means that individuals say things to themselves and to others with specific purposes. The learners in this inquiry developed intentionality in *what* they said (chose what they wanted to say) and *how* they said their words (chose the words they wanted to say). Although some similarities to being vocal and verbal exist, learners were using their nascent abilities of speaking out and naming to say things *to* a specific audience with the intent of *affecting* the audience. The students sometimes said things to themselves, intending to affect their success, when they viewed *self as audience*. Belenky et al. (1986) observed that individuals who were gaining voice would "engage in self-expression by talking to themselves" (p. 86). Emergent voice needs to say things to self in order for the individual to internalize, author words of significance, and explore intentions imbued in statements to self and others. The students viewed *others as audience* when they said things to someone else, usually to me, about their thinking and learning, intending to affect the teacher-with-learner relationship. In this case, the students often believed they would help me understand them and notice their success in mathematics class. Emergent voice needs to say things to others in order to establish the individual's authority and to make intentions explicit to others.

As Andrea began to speak out during the semester, her intentions became more sophisticated, demonstrating complexity and confidence in speaking with her emergent voice. As well, this growth occurred in parallel with her sense of authority over her learning. Andrea exemplified a shift from asking me questions to asking peers questions in order to be given mathematical knowledge. She was *intentionally* selecting the audience with which she believed she could learn mathematics. She was using words to gather knowledge, rather than *learning* mathematics in meaningful ways. Andrea demonstrates progress from this stance as she began to interact with peers in order to learn and affect her relationship with them. Further, as she responded to questions in class and began to explain mathematical ideas to the class, she continued to select *others as audience*, but now for a different intention – to affect her *identity* as a confident mathematics learner.

Andrea's movement inward with her intentions signified that her audience for using words was evolving from others as audience to *self as audience*. As she spoke with her emergent voice, Andrea was using words to affect herself and her understanding. In our second conversation, Andrea communicated that putting ideas in her own words was important for her to understand and learn. Her intentions were directed at improving her mathematical understanding, and her emergent voice was *intentional* in supporting her success at mathematical learning. There is a complexity present in Andrea's intention with words as she comes to use her emergent voice to affect herself.

Andrea's Voice was Nascent

The three characteristics of emergent voice were indicators that the voice-stance of the students was evolving, from voicelessness to emergent voice. However, being *vocal*, *verbal*, and *intentional* are not necessarily small steps away from voicelessness, each requiring sophistication in their use. While the idea that learners spoke with an emergent voice was a significant success for them in mathematics class, their voices were just *emerging*, not reaching a fully refined state by the end of our semester. Instead, their emergent voice was *nascent*, in the act of coming into existence and in the process of being established.

Emergent voice is nascent because of its *tentative* nature, where it is prone to "fall backs" as voice emerges. Recall Andrea's expression of a pattern-generalization about "2 by …" rectangles and her inability to remember that moment days later. Within this mathematical success, there is fragility in Andrea's ability to use words to learn. Andrea's inability to remember her good idea highlights her struggle to believe in the value of her ideas and words. Rather than moving away from received knowing (Belenky et al., 1986), it would seem that Andrea was moving toward silence again. Kieren and Pirie believe that students need to "*fold back* to an inner level of activity in order to *extend* their current action capabilities and action spaces" (Kieren, Pirie, & Gordon Calvert, 1999, p. 218; italics in original). So, although at a cursory glance the *tentative* nature of emergent voice seems to denote a pause in success, it is a necessary for individuals to return to a former stance as their voice emerges. The tentative nature of emergent voice highlights that the success the students experienced was not static – rather, it was a dynamic success that captured growth as it was occurring, and that the growth itself (not the destination) was what was important in the learners' journeys of success.

Finally, Andrea's capstone moment of success, noticing and expressing her success with capture-tag-recapture, illuminates the three characteristics of emergent voice interacting as Andrea' engaged in the emergence of voice. I would encourage the reader to return to this example to notice that the emergence of Andrea's voice, from voicelessness toward emergent voice, is captivated in a moment she and I could celebrate together.

Andrea's Journey as an Exemplar

Andrea is an exemplar of the emergence of voice. An exemplar is an individual that serves as an ideal model for a group of individuals. Andrea is an exemplar because her story amplifies many of the small steps of the emergence of voice, from voicelessness. The emergence of voice occurred through her active involvement as I listened closely to her successes and as we were in conversation. The journey was not flawless, because no journey in the gaining of voice is. Rather, the return to previous stances provided opportunities for Andrea to consolidate specific characteristics of emergent voice before building on more complex successes. Andrea is not an exemplar of the emergence of voice because she experienced the largest or smallest amount of growing and stretching during our semester. Each learner's story is distinct and highlights different processes of the emergence of voice, perhaps in differing orders and also in differing intensities. What Andrea's story does exemplify, on behalf of all of her classmates, is that the emergence of voice was a tentative process and one in which I needed to listen closely to their moments of success to catch the subtle shifts in stance as they lived a process of *forming their identity*. Andrea also illustrates, on behalf of her classmates, the brilliance of each of the small steps the learners took as they moved away from a stance of silence and the fact that they were constantly in the process of building on more complex successes to the ones that they already had. The pleasure of watching and living with the learners as their voice emerged was particular to each of them, but significant for all of them.

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DRAWING ON DIVERSE SOCIAL, CULTURAL, AND ACADEMIC RESOURCES IN TECHNOLOGY-MEDIATED CLASSROOMS

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Numerous classroom technologies are being designed to support construction of mathematics and science knowledge (cf., Kaput & Hegedus, 2002; Linn & Hsi, 2000; Soloway et al., 2001; Wilensky & Stroup, 1999), but largely without regard to the social and cultural resources traditionally under-served students bring to classrooms. Carol Lee (2003) notes, "the tremendous funding being invested in the development of such computer-based tools in education may be simply reinforcing current inequities in opportunities to learn, unintentionally widening the achievement gap" (p. 58). There is little hope of attending to that gap if research and development efforts ignore issues of culture. However, with few exceptions (Lee, 2003; Pinkard, 2001), research and design in instructional technologies has not treated underserved students' social, cultural, and academic resources as central considerations.

To build understanding of the influence of culture on technology-supported classroom learning, we turn to a growing body of promising work that treats underserved students' cultural backgrounds and practices as important resources for learning (cf., C. Lee, 2001; Greenberg & Moll, 1990; Gonzalez et al., 1995). Okhee Lee (2003) argues that equity is unattainable if students are not given access to powerful discourses, but that appropriation of discourse is made more difficult if school science (and mathematics) is simply imposed on students. Appropriation is better supported by drawing on students' social and cultural practices as resources rather than as barriers to overcome (cf., Civil & Kahn, 2001; Moll & Gonzalez, 1995). For example, Haitian Creole students' story-telling and argumentation skills have been shown to support their engagement in science (Warren et al., 1992) and irony, satire, and metaphor in African American Vernacular English to scaffold students' analysis of canonical literary works (C. Lee, 2001).

We examine the unique potential of a next-generation networked classroom technology (HubNet and Participatory Simulations, Wilensky & Stroup, 1999) to draw on students' cultural, social, and academic resources to support learning in mathematics. We attend to both (1) content and representations of content; and (2) the participation/interaction of students as dual dimensions of the social space of classrooms. Attending to these dual dimensions highlights " a generative, creative tension between the structuring role of math and science and the structuring role of social activity" (Stroup, Ares, & Hurford, in press). Of particular interest are the following research questions:

(1) In what unique ways does network-mediated activity scaffold learning for underserved students?
(2) How is mathematical knowledge and practice enhanced through inclusion of underserved students' social and cultural resources in network-mediated learning?

Examining these issues will deepen our understanding of the construction of mathematical content and practice through social interaction. We hope to identify effective ways to begin to close the achievement gaps between cultural, social, and economic groups by broadening the range of tools and knowledge teachers use to allow all students to reach their highest potential.

Network Mediated Learning

HubNet and Participatory Simulations includes a wireless network of graphing calculators, being developed with funding from the National Science Foundation and Texas Instruments (Wilensky & Stroup, 1999, 2000). In Participatory Simulations, participants act as individual agents and observe how the behavior of the system as a whole emerges from their individual behaviors. The emergent results become the focus of in-class discussion and analysis. In the Elevators Participatory Simulation used in this study, the system collects students' input to individual calculators through the network (arrangement of blocks, Fig. 1, left), and displays the emergent system formed from their collective contributions in an "up front" public space (an array of all students' elevators moving together, along with position and velocity graphs; Fig. 1, right).

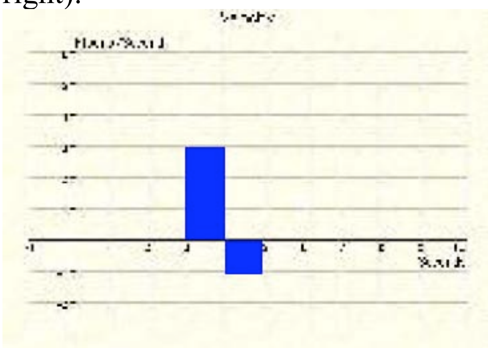

Figure 1. A sample delta blocks arrangement (left) and projected elevators screen, with position and velocity graphs (right).

Among others, qualitative understandings of the mathematics of change (Kaput, 1994; Stroup, 1996, 2002) that start from the integral and then move to the derivative are goals of this activity. Features of this system that may offer avenues for enlarging the social, cultural *and* academic practices used as resources include: 1) multiple modes of contribution (language, text, physical and electronic gestures), 2) multiple representations (texts, graphs, visual displays of emergent systems, language), and 3) inquiry-oriented discussion and analyses, "in which students are supported in making public the strategies they are employing as well as the evidence and reasoning they are using, … [and] where instructional conversations are not solely directed by teachers' intentions" (C. Lee, 2003, p. 48, 49).

Examining ways to draw on the varied, often untapped, resources available in heterogeneous classrooms can support important advances in classroom technology development by pinpointing features of their design and use that can become culturally responsive and that support the achievement of underserved students. Further, it can provide important information to teachers as they consider whether and how to incorporate networked and other classroom technologies into their teaching.

Methodology

We focus on social activity, discourse, networked technology, and learning in a sociocultural theoretical framework (Vygotsky, 1987). Learning is viewed as being mediated by social activity involving both people (teachers, peers) and tools (networked technology). Further, learning involves co-constructing mathematical knowledge and meaning through appropriation of discipline-specific content, discourses and practice. Discourse analysis (Gee, 1999) enables us to analyze students' appropriation of the content, discourses and practices of mathematics as a discipline. We situate that appropriation within the social space of classrooms that is formed by the intersection of dual dimensions of 1) social and cultural resources, and 2) content and representations. Thus, network-mediated learning in the social space of the classroom is shaped by the dynamic, mutually constitutive roles of mathematics-specific content and representations, *and* participation and cultural resources in learning (Stroup, Ares, & Hurford, 2004, in press).

Setting

Approximately 50% of the students in the high school in which this study was conducted were Hispanic, 21% African American, 25% European American, and 4% Pacific Islander, Native American, and recent immigrants from a variety of countries; those demographics were reflected in the two classes that participated. One of the two classes had 17 students; the other had 15. The teacher, Sylvia (a pseudonym, as are all names), was a European American, veteran mathematics teacher with 15 years of experience. She was interested in using networked classroom technology to support her commitment to problem-based instruction, and as complementary to her regular Interactive Mathematics Program (IMP; Fendel, Resek, Alper, & Fraser, 1997) curriculum. The two sessions taught by her but with different students were video-taped, one when the network was in use and another when it was not. Analyses centered on characterizing classroom activity both with and without the networked technology.

Quantitative analysis of classroom talk

Whole-class talk was coded by one researcher as to the type of comment or question contributed by Sylvia or students (e.g., invitation to explain, observation), drawing on the work of Brenner and Moschkovich (2002) on everyday and academic mathematics, but including codes that emerged as important as well. A second researcher used the scheme to evaluate its usefulness, add emergent codes, and delete codes. The two researchers arrived at the final coding scheme through a process of discussion, independent coding, checks of inter-rater reliability, and more discussion. The final inter-rater reliability check yielded 71% agreement. Frequency diagrams were constructed to examine how the numbers of comments or questions within categories changed over the course of each class session, to examine teacher versus student contributions, and to conduct cross-session analyses.

Qualitative analysis of classroom Discourse

Transcripts of videotapes were analyzed to identify patterns of participation, and use of social, cultural, and academic resources. Following Moschkovich (2002) and Gee (1999), we used the following questions for Discourse analysis [Discourse defined as "ways with words, feelings, values, beliefs, emotions, people, actions, things, tools, and places that allow us to display and recognize characteristic *whos* doing characteristic *whats*" (Gee, 1999, p. 19)]:

- *Discourses:* What Discourses are involved and produced in this situation? What Discourses are relevant (or irrelevant)? What systems of knowledge and ways of knowing are relevant (and irrelevant) in the situation? What Discourse practices are students participating in that are relevant in mathematically educated communities or that reflect mathematical competence?
- *Activity building:* What is the larger or main activity going on in the situation? What sub-activities compose this activity? What actions compose these sub-activities and activities?
- *Resources:* What are the multiple resources students use to communicate mathematically? What sign systems are relevant (and irrelevant) in the situation (e.g.,

speech, writing, images, and gestures)? By what means are they made relevant (and irrelevant)?

Findings

This section is organized in terms of Discourses produced, activity building, and resources, followed by examination of the dual dimensions of the social spaces that emerged in the two class sessions.

Discourses

A Discourse of school mathematics problem solving was produced in the IMP activity, while a Discourse of dynamic systems thinking was the product of the Participatory Simulation. [Dynamic systems thinking "focus[es] not on the elements of something, but on the relationships or interactions between the elements … shifting from a concern for detail complexity to a concern for dynamic complexity: concern for dynamic relationships, rather than fine distinctions" (Ramsey & Ramsey, 2002, p. 99).] The problem solving Discourse was characterized by the more mechanical aspects of mathematics (e.g., order of operations, converting fractions to decimals) being embedded in a task, in this case, determining the profit ferry owners made by transporting white settlers moving to the Western US and their wagon trains across the Kansas River. Thus, contextualizing formal mathematics and embedded arithmetic were the focus (Ares, Stroup, & Schademan, 2004). We use the term 'school math' because the ways of knowing involved were teacher- and textbook-dominated, as evidenced by the nature of students' contributions (largely procedural) and Sylvia's invitations (see Table 1), which constituted 62% of the coding.

Table 1. Nature of Students' Contributions and their Teacher's Invitations

IMP problem	Elevators Participatory Simulation
Students' Contributions	
Explanation (21)	Observe/describe (19)
Fill-in-blank answer (21)	Prediction (12)
Report result (16)	Request clarification (11)
Perform procedure (14)	Explanation (10)
Add precision (11)	Visualize (8)
	Evaluation (8)
Sylvia's Invitations	
Invite performance (18)	Invite prediction (14)
Invite explanation (16)	Invite elaboration (8)
Known-answer question (12)	Invite performance (6)
Invite elaboration (8)	Invite explanation (5)
Invite clarification (7)	Invite clarification (4)
Invite contribution (5)	Invite contribution (4)
Invite evaluation (4)	Invite play (2)

However, an important exchange in the IMP lesson drew on students' knowledge of the broader world, providing an opening for drawing on social and cultural resources. The profit formula included a wage of 40 cents per hour for the ferry operator. A student remarked, "That sucks." Sylvia asked whether there were people making that little now, and what would be a fair wage today, which precipitated a discussion of varying wages around the world, including students' own pocket money. Here, though relatively briefly, students' social resources were fodder for critical connections between mathematics in school and the world. The dynamic systems thinking Discourse involved in the Participatory Simulation was characterized by students and Sylvia exploring relationships among position and velocity graphs, and among those representations and the motion of the elevators. Here, the focus was on predicting and describing relationships, and development of increasingly sophisticated understandings of those relationships. The following exchange is representative:

Sylvia: I'd like to hear from some of you about what you think we're gonna see up front when they send all these up. .

Lydia: At the point where all of our graphs are going up and down at the same time on the…

Sylvia: …This little section? [*pointing to the worksheet plot of blocks*]

Lydia: yea, on that little section all of the ah line graphs will go up the same amount and down the same amount. [*uses hands to indicate simultaneous movement]*

Sylvia: Um. So what's it gonna look like up there? On the graph. [*pointing to the public display*]

Jose: Parallel.

Lydia: They'll be parallel [*uses arms to demonstrate parallel lines]*.

[once the simulation was run]

Sylvia: What, what's happening right here? [*points to position graph*]

Brian: Jaime.

Sylvia: What else is happening?

Jaime: No no. Everyone goes crazy again?

Sylvia: That's true. Everyone does kind of go crazy again, or do their own thing. What's happening right here? [*points to an intersection of two lines on the position graph*].

S: Someone went up and then stopped.

Sylvia: But I mean this very point, right there.

Jaime: They're crossing?

Sylvia: What does it mean?

Jaime and Lydia: They're on the same floor.

Jaime: For that second.

The ways of knowing that were relevant here and across the whole activity involved shared construction of understanding, where student's individual elevator's motions served as examples for exploration and construction of, for example, an informal metric for speed, and the graphical representations were examined for "how we did" in coordinating activity, lending a collective sensibility to this Discourse. The shared construction and collective sense invited more participation by students, evidenced by the finding that 57% of codes were attributed to Sylvia.

Activity building

In the IMP class, textbook-centered, step-by-step work through procedures for solving the profit problem was the main activity, while movement between individual creation and collective exploration of emergent mathematical objects was the main activity of the Participatory Simulation. The sub-activities that composed the IMP main activity included responding to known-answer questions with fill-in-the-blank answers, reporting results, and talking through the steps of procedures (e.g., converting fractions to decimals), balanced somewhat by explanations of procedures or why the formula "made sense" (see Table 1). The sub-activities that supported the Participatory Simulation main activity involved interpreting graphical representations, i.e., predicting and visualizing position and velocity graphs based on the arrangements of elevator floors, and observing/describing real-time development of graphs and elevators' motion (see Table 1). Important opportunities to draw on social and cultural resources of students were also in evidence, as Sylvia issued open-ended invitations: "you can do whatever you want on either side of this (required arrangement)," and, "do something interesting." Thus, students were invited to contribute social and academic resources (explorations, understandings) in creative, even playful ways to the group's efforts.

Resources

Discourses were made relevant through the use of tools or cultural artifacts, especially the textbook in the IMP class and the networked system activities and technology in the Elevators class. In addition, both activities drew on English and Spanish language, text (book for IMP, worksheet for Elevators), peers and Sylvia, calculators, and mathematical symbols (profit formula, graphs). However, the real-time public display in which students could identify their own and others' contributions, physical and electronic gestures (e.g., using arms to indicate parallel lines, individual elevator motion), and multiple representations of relationships were additional, unique resources available in the Participatory Simulation. Through the use of gesture, multiple representations, and public display, diverse ways of knowing were involved in the situation, inviting contributions of both social and academic resources, in addition to cultural resources embodied in language.

Dual Dimensions of Social Space

We examine two dimensions of the social space of the classrooms -- 1) content and representations, and 2) participation, including using of social, cultural and academic resources – to address the research questions. Comparative analyses help us pinpoint unique ways in which network-mediated activity may scaffold learning for underserved students, and how mathematical knowledge and practice may be enhanced through inclusion of underserved students' social, cultural and academic resources in network-mediated learning.

Content and representations

Not surprisingly, the content involved in each class session was markedly different (see Table 2). Both sessions engaged students in learning important content, though the variety was greater in the Elevators activity. In addition, the wider variety in the representations available for students to make sense of content in the Elevators session expanded the social space along one of the dimensions of interest, providing opportunity for students to explore multiple relationships among variables of interest. Finally, the nature of those contributions and the content addressed were such that students had much more latitude to act, given the focus in the IMP activity on mostly textbook-determined progression through a more constrained task.

Participation including use of resources

While there was considerable overlap in the types of resources available and used in both class sessions, the Elevators session did offer additional tools and avenues for participation, expanding the social space along the participation dimension of the classroom. The technological capabilities of HubNet and design of the Participatory Simulation that allow individual inputs to be collected and displayed as a real-time emergent, evolving system represented a unique activity structure in which students could identify themselves and others as individual elements, and examine the nature of a complex interaction of elements. Students were invited to both contribute and draw on diverse social, cultural, and academic resources in the situation, as they used gesture, text, language, and images as resources in their exploration.

Table 2. Content addressed in two class sessions

IMP lesson	Elevators Participatory Simulation
Order of operations; converting fraction to decimal notation, minutes to hours; substitution and evaluation of numbers for variables; rounding; profit formula; mathematizing variables and relationships among variables in profit formula; symbolizing	Change in amount; abstracted time; graphical representation of time and position; the significance of parallel position-time graphs, parallel and intersecting lines; coordination of graphs with motion; a sense of the constant of integration; visualization, prediction, explanation of graphical representations of position and velocity; extending mathematical reasoning beyond what was visible; and the initial emergence of a metric for rate

Conclusion

The results emerging from this study point to the potential for networked technologies to be culturally responsive in design and use. The Elevators class session involved increased variety of avenues for engaging content and representations, and each other, expanded social space, broadening opportunities to serve under-served students by enlarging the ways of knowing and systems of knowledge treated as important resources for learning. Further, our findings indicate that mathematical knowledge and practice may be enhanced by diversifying classrooms' academic, social and cultural resources, and drawing on those of under-served students. The point also needs to be made that connections to the world outside school in this case were stronger in IMP; the strengths of the IMP problem solving approach in inviting some diversity in experience is clear. Still, the variety of ways of acting available in the Participatory Simulation was larger than in IMP as enacted. Our findings indicate that looking closely at classroom activity can identify some existing features of networked technology design and use that have potential to better support under-served students' mathematical learning, especially in terms of academic and social resources. However, cultural practices proved to be much more difficult to pinpoint. We argue that this is due to the limits of looking only in classrooms. To move to truly culturally relevant classroom technology design and use, exploration of social, academic, and especially cultural practices of under-served students must be done across contexts, including peer and home communities, to identify those that can be important in supporting rigorous mathematical learning. This is partly or largely due to schools' historical and ongoing under-appreciation and misunderstanding of the practices of under-served students. They are often viewed as lacking, and their families and communities have been viewed through a deficit model. Until we move outside the structure of schools and traditional approaches to technology design, we will be missing important opportunities to enlarge the social space of classrooms in ways that serve all students.

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COMPARING SOLUTIONS ACROSS THE DIVIDE

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In this paper, we expand upon prior research by comparing the strategies and solutions of the students in this study, who were from a low-income, inner city high school, to students from higher SES backgrounds who had longer and more extensive exposure (at times many years) to inquiry oriented environments. Our hypothesis was that if the students in this study were provided with classroom instructional conditions that were similar to the ones present in the other studies, after a relatively short period of time, they too would exhibit the same or very similar problem-solving behaviors, solutions and justifications to the problem being explored. Indeed, we found that the students did come up with justifications that matched those of their suburban counterparts.

Introduction

"The future of American society will be determined in large measure by the quality of its urban schools. We have the responsibility and the obligation to make that future far better than the present we now know." (Noguera, 2003, p.156). Indeed, in the state of New Jersey (and elsewhere) there exist extreme, persistent, and dramatic differences in performance between students attending resource-rich schools that tend to be located in suburban neighborhoods, and students attending schools serving high proportions of economically, socially disadvantaged, or minority students that tend to be located in urban communities (Camilli & Monfils, 2002, 2004). In addition, national reports (see Education Trust, 2003a,b), document the disparities that exist in performance between different groups of students, noting that African American and Hispanic students continue to lag well behind their white counterparts.

Much has been written about the differences that exist in the instructional approaches that are used in schools serving economically disadvantaged or racially diverse students and schools serving higher socio-economic status (SES) students. For example, Anyon (1997) notes that in inner city classrooms in general, instruction is often more limited to low level, rote and unchallenging material. Regarding mathematics instruction, Ladson-Billings (1997b) reports "despite the much talked about changes in mathematics education, African American students continue to perform poorly in school mathematics" (p. 7). Knapp (1995) states that for many low-income students, the educational experience "lacks meaning and importance to the learners. Thus, students learn to work two-digit subtraction without understanding in some basic way what the two columns of figures represent or even what subtraction is, much less how it relates to their lives" (p. 1-2).

There are many reasons why mathematics instruction may be different in inner city classrooms, particularly those serving African-American populations. Some argue that state testing may be responsible for some of the differences in the ways in which teachers teach (Schorr and Firestone, 2004), in particular, a focus on testing may result in more "teaching to the test" (McNeil, 2000), thereby causing an over reliance on rote and procedural knowledge (Schorr, Firestone, and Monfils, 2003). Others, like Stiff (1998), note that classroom management may have something to do with it. He states, "Often, teachers believe that control of African American students is paramount and can best be achieved in teacher-centered classrooms." He goes on to appropriately counter this notion by stating that "control is not the goal of classroom instruction, learning is." (p. 71). Some researchers call attention to the differences in language that may exist. For example, in discussing the mathematical problem solving of African American students, Orr (1997) found that many of the difficulties that her students experienced were rooted in the ways in which they used language.

Other barriers like cultural prejudices, lack of resources, and teachers who don't really believe that all students can learn mathematics also contribute to minority students "tuning-out" of math classes (Franklin, 2003). The situation is exacerbated when the students themselves do not believe that higher education is a necessity or even an option (Franklin, 2003), and therefore mathematical study is not very important. Cultural differences also exist amongst students' home, community, and personal backgrounds, and their schools, curriculum and teachers. These differences can be quite dramatic, and manifest themselves in the ways in which students respond to problems. In a dramatic example, Ladson-Billings (1997b) calls attention to the different ways in which white suburban students and African American urban students often think about the same mathematical problem. She discusses how the white students responses "represented their very different life experiences and approaches to problem solving…they made sense of it as an abstraction. The problem had little meaning, but they knew enough arithmetic to manipulate the numbers. [While] the African American students…situated it in their own social contexts." (p. 9).

The situation gets worse as students proceed through their school careers. Koller, Baumert & Schnabel (2001), Baumert & Koller (1998) and Gottfried, Fleming & Gottfried (2001) state that, in general, interest in mathematics ebbs through high school. But for inner city African American students, the data is even more exasperating. The Educational Trust (2003a) reports that by the end of high school, "African American students have math and reading scores that are virtually the same as those of 8th grade White students" (pg. 1). Mendick (2002) makes the point that among other things, procedural work makes it difficult for students of high school age "to come to think of themselves as mathematicians and so it becomes less likely that they will study the subject further." (p. 3-329). How much more so would this be the case for African American students in a low income, inner city school? With this in mind, we report the results of a study, which compares the work of students in an inner city high school with that of their predominantly white, suburban counterparts.

Theoretical Framework

While much has been written about the differences that may exist between different groups of students, this study was designed to investigate the problem solving *similarities* that are present, particularly between African American students coming from one of the lowest performing, least advantaged high schools in the state, and their predominantly white suburban counterparts. In order to do this, we decided to focus on a task that had been used over a period of years with many different students, and in particular, many suburban students who were younger, the same age, or older than the students in this study (Tarlow, 2004; Glass, 2002; Maher, 2002; Maher and Martino, 1996) This allowed us to analyze the solution strategies of the students in the current study and compare them to those produced by the others.

The task that was chosen, unlike the one noted by Ladson-Billings (1997a,b), did not involve a "real world context" but did involve a topic with which the students had little or no prior experience. The topic involved combinatorics, and had been perceived by the students in the other studies to be interesting. In addition to being able to compare the solutions of the students, the researchers in this study felt that by using such a topic, students who had been "turned off" by traditional school mathematics might find opportunities for mathematical discovery and interesting, non-routine problem solving (Goldin, in press). Indeed, Goldin notes that when solving such problems low-achieving students can demonstrate mathematical abilities that may have gone unnoticed in the past.

While it was important to choose a task that had been used extensively, was well documented, and provided opportunities for students to experience a "non routine" type of problem, the researchers in this study also felt that an appropriate comparison could not be made unless the classroom culture and instructional environment was also similar to that of the students in the other studies (Tarlow, 2004; Glass, 2002; Maher, 2002; Maher and Martino, 1996) and consistent with the recommendations of the National Council of Teachers of Mathematics (see NCTM, 2000). In these studies, and in this study as well, students were always encouraged to formulate conjectures, test the conjectures, and defend and justify solutions in the context of an inquiry-oriented learning approach. More specifically, students were always provided with opportunities to build, modify, revise, refine, test, and extend their own ideas, discuss, question and justify solutions, make connections between different representational systems and revisit earlier ideas. Such instructional environments have been shown to be effective in longer-term interventions with inner city students (c.f. Schorr, 2000; Schorr, 2003; Campbell, 1995 as cited in NCTM, 2000; Silver and Stein, 1996; NCTM, 2000). Knapp, (1995) has noted that "the more classrooms focused on teaching for meaning—that is, geared mathematics instruction to conceptual understanding and problem solving…the more likely students were to demonstrate proficiency in problem-solving ability…all other factors being equal." (p.142). Similarly, Schorr (2000) reports that urban students who were taught in an inquiry-based manner outperformed their counterparts who were not, and Stipek, Salmon, Givvin, Kazemi, Saxe & MacGyvers (1998) found that reform-oriented instruction led to more interest and motivation to engage in mathematical activities.

This study expands upon prior research by comparing the strategies and solutions of the students in this study to students from higher SES backgrounds who had longer and more extensive exposure (at times many years) to inquiry oriented environments. Our hypothesis was that if the students in this study were provided with classroom instructional conditions that were similar to the ones present in the other studies, after a relatively short period of time, they too would exhibit the same or very similar problem solving behaviors, solutions and justifications to the problem being explored. We anticipated that this would occur despite the fact that the students in this study had not typically been exposed to inquiry-oriented instruction.[1]

In the sections that will follow, we will describe the problem, the classroom environment, and the solution strategies—along with a comparison to the strategies used by the students reported on in previous research.

Methods and Procedures

Background, Setting and Subjects

Twenty African American students from a low SES inner-city high school in New Jersey were invited to participate in an educational program at Rutgers University for 5 weeks. They were paid a stipend (as part of a grant designed to bolster the number of students who would graduate and go on to college) and participated in 11 sessions in mathematics, each spanning three hours. This paper reports on the 6th and 7th sessions.

All of the students attended a large urban public high school, where they were about to enter their senior year. The school is part of a district that is currently under state takeover. In this particular school, over the past several years, approximately 90% of the students perform below the "proficiency" level in mathematics (as noted on http://just4kids.org/).

The sessions were directed by a mathematics education professor at Rutgers University, who is the first author of this paper, with the support of two teaching assistants, one being the second author of this paper. The students worked in small groups, though some, at times, chose to work alone.

Since the type of problem solving that occurred during these sessions had not been part of their typical classroom experience, many students were unaccustomed to coming up with convincing arguments, sharing their ideas with others, or working with others to solve problems over extended periods of time. To encourage the development of thoughtful explanations, teachers constantly encouraged the students using probes such as: "I'm really interested in your thinking" or "Can you tell me how you would explain that to someone else who didn't understand?" For this particular activity, teachers also added the following probes "Can you tell me how you would explain that to someone else who found more (or fewer) towers than you did?" or "Suppose that I was sure that there were more, how could you convince me that there weren't any others?" or "How can you be sure that you don't have any duplicates?"

Data

Both authors collected detailed field notes. Careful documentation of meetings, held both for the purpose of planning and evaluation, were recorded as well. All students' original work was available for examination. Additionally, parts of the sessions were videotaped by a handheld camcorder.

Tasks

The task, which is the subject of the sessions examined here, is called The Towers Task (c.f. Maher & Martino, 1996; Maher, 2002). In this task, students are provided with unifix cubes of 2 colors and asked to stack them into what are called "towers". The task follows here.

How many towers, 4-tall, can be made when selecting from two colors? How do you know you have them all and that there are no duplicates? In an effort to push students to a deeper examination of the problem, they were subsequently given the same problem for towers 5-tall. Before beginning to work with towers 5-tall, students were asked to predict how many towers there would be. They were then encouraged to investigate their predictions in order to prove or negate them. In addition, students investigated several other extensions of the tasks.

We note that this task, in and of itself, may not be considered to be a particularly difficult one if all that is expected is a numerical solution. However, as noted above, students in this program were always challenged to defend, justify and extend their solutions.

In the sections that follow, we will highlight the mathematical problem solving of some of the students, noting in particular how they compare to those of their suburban counterparts.

Results and Discussion

In this section, we highlight the work of the entire class, and then hone in on several students, indicating their solution strategies and the evolution of their understanding of the underlying conceptualizations related to this problem. Before beginning, it is important to point out the initial reluctance exhibited by all of the students to discuss their ideas, and to delve more deeply into problems done during any of the sessions. We note that this was expected since this type of problem solving had not been part of their typical classroom experience, and they were therefore unaccustomed to sharing ideas, working with others over extended periods of time, or defending and justifying their solutions.

Their initial justifications ranged in scope from "I just can't find anymore" and "We've been looking for a long time, and there just aren't more" to partial justifications in which the seeds of a more formal proof could be found. These types of responses were completely consistent with the initial responses reported in prior research (Tarlow, 2004; Maher and Martino, 1996; Glass, 2002; Maher, 2002). As with the students whose work is documented in the above-mentioned research, in time, the students did develop two main types of proofs –a proof by cases, and a proof by induction. The students also ultimately extended their understanding to an isomorphic problem. We also note that the students, like their suburban counterparts, were able to make connections to Pascal's triangle and the Fibonacci series. We will now share some specific instances of student responses.

Overview of the Entire Class

When the problem was first distributed, one student immediately, as if intuitively, said there would be 16 towers. When he was asked why, he was unable to articulate any explanation. Rather, he wanted to build the towers and see if his intuitive response was correct. Later in the session, when he was once again questioned, he held up one tower and removed the top cube. He said that for that top you had only 2 choices of color and in every other position there were also exactly 2 choices. He was unable, at that time to offer a more complete explanation, again choosing to examine the towers more closely. We would speculate that he was beginning to formulate a proof by induction. He attempted to explain this method to his partner who was unable to follow his line of reasoning. His partner had instead chosen to record, using paper and pencil, all 16 towers without using blocks. While the partner had listed all towers, (by representing them with letters for the colors) he could not provide an explanation for his solution at the time. Ultimately, both students could justify their ideas to some extent, the first formalizing his inductive proof, and the second, though informally, using a partial proof by cases, but only after considering an extension involving the number of towers 5 high. When recording all of the towers 5-tall, the partner developed an orderly listing of the towers, indicating first the towers that would have one blue, and then the ones that would have one yellow cube. After that, the towers were listed along with their "opposites".

Some pairs of students spontaneously began to consider simpler cases in which towers two or three high could be built. They noticed that the number of towers doubled as the height of the towers increased by one, but recognized this from examining the pattern and could not give a logical reason for the apparent rule. They were content to recognize the pattern, and base their justifications on it.

Several other pairs built their towers incorporating the concept of "opposites" and several conjectured that there were 16 towers because, they explained, 4 x 4 is 16. Those who believed the latter predicted that if the number of cubes in each tower were 5, then there would be 25 towers in all. At first, some pairs did not notice the inconsistency created by this prediction, given that they believed that every tower had an opposite. When one pair built these towers, they stopped when they reached 25, believing that they were "done". They indicated that they had used their method of building opposites to build most of these. They were then asked to pair the towers with their opposites. When they attempted to do this, they found that they had a few towers, which had no opposites. When these were built, they ended up with all 32 possible towers. This pair eventually provided a proof by cases, explaining why they had all the towers with no yellow cubes, 1 yellow cube, 2 yellow cubes, 3 yellow cubes, 4 yellow cubes and 5 yellow cubes. They arranged the towers which had 2 blue (3 yellow cubes) into the following subsets.

B Y Y Y	B B B	Y Y	Y
B B Y Y	Y Y Y	B B	Y
Y B B Y	B Y Y	Y Y	B

Y Y B B Y B Y B Y Y
Y Y Y B Y Y B Y B B

They created the opposites of these subsets. In this arrangement, it appears that they began by keeping two blue cubes together and moving these two blue cubes down along the tower. Then they held one cube constant while changing the other within each subset. This, as mentioned above, led to a proof by cases.

Ultimately all students were asked to predict how many towers 5-tall would be built and then to test their hypotheses by constructing the towers. One pair of students generated an inductive solution by reasoning that since towers which were 4-tall numbered 16, there would be 2 choices for a 5th cube for each of these towers and therefore the there would be 32 towers. When they were questioned, they tried to express this idea by saying that there were 2 sets of towers, which were 4-tall, and therefore there were 32 towers 5-tall. Some students spontaneously posed the question of how many towers there would be if they were 6-tall, and then built the towers to confirm their predictions.

A Closer Look at Two Students

We will now focus in on two students, Jasmin and Lavar whose work we highlight for the purpose of documenting in greater detail, the actual proofs and justifications that they developed. These students worked together at times, but more often worked independently seeking deeper understanding of the towers task and extensions of that understanding to other related problems. They continued their work with the towers over the course of two days, not even wanting to stop for lunch. During these sessions, Lavar worked primarily in symbolic notation, not always connecting his representations to concrete towers, while Jasmin almost always built the towers prior to developing symbolic representations. Both students also decided, quite spontaneously, to explore what would happen if the towers could be built from three colors, and eventually any number of colors.

While they were seated at the same table during the lunchtime sessions, Jasmin and Lavar worked independently, at times glancing at each other's work. They repeatedly questioned each other's work, at times in very challenging ways (we speculate that some of the challenges arose out of some social dissention between them). Jasmin challenged Lavar's careless algebraic representations while Lavar alluded to an easy understanding of what Jasmin had struggled to understand, the inductive reasoning behind the doubling pattern.

Lavar tried to build a "formula" for finding the number of towers. He understood that if you take towers of any height, you could add either one of two colors to each, when building towers one taller. This would result in a "doubling rule". He wanted to represent this idea by using variables but at first, had difficulty doing so. Interestingly enough, when he was asked to explain his thinking, he would construct and then take apart (as part of his explanation) the same towers. When asked to build new towers rather than to "deconstruct" the ones he had already built, he appeared to have difficulty doing this and needed to be asked again. This seems to indicate that his understanding of the towers and how they "grew" was tightly connected to their actual constructions. Nevertheless, he demonstrated the inductive reasoning justifying why the number of towers doubled.

We found that throughout the sessions, Lavar created a model of the towers, a drawing that he kept going back to. This model looked like a stack of squares with the number one inside each square. He explained that the one represented one block tall and that for each one block tall there would be the same number of choices as there were colors. He then multiplied these numbers to get the total number of towers. When asked why he multiplied these numbers, he did not give a clear explanation. Ultimately, he assigned the variable, h, for height, to the number of cubes high that the towers were and the variable, c, to the number of colors available from which to choose. He knew that the generalization would involve exponential growth when created the formula $h=c^p$, where p represented some power. When asked how his formula could be used to represent the number of towers 7-tall, using 15 colors, he immediately said it would be 15^7. This seems to indicate that although he was unable to create a totally accurate formula using variables, which was his goal, he did understand what was happening. He was not able to map the induction to an algebraic representation even though he appeared to clearly understand it.

Jasmin spent a great deal of time handling the towers, rearranging them and thinking about what was happening. It was when she realized that the number 2 in the solution, 2^n, was not random, and was the number of colors available, that she was able to understand the inductive reasoning behind the generalization. She pursued this idea by examining what would happen if there were 3 colors available. She had a difficult time representing these towers until she stopped assuming that each tower had to have all 3 colors.

Jasmin: This [her work on towers 2-tall] is the lowest one [height of possible towers] you could start with is two 'cause it's two different colors…

Researcher1: Right…

Jasmin: and you're goin' up to three you can only start with three [height of possible towers].

Researcher2: So, can I ask you a question?

Jasmin: Sure

Researcher2: Is it not possible then to have towers 2-tall when you're using three colors?

Jasmin: No

Researcher2: How come?

Jasmin: 'Cause the point is using all three colors and having all three colors in the [unclear] positions.

Jasmin knew that there were 3^n towers possible towers based on the original problem, where n represented the height of the tower. Since in this case, n would equal 4, there would be 81 possible towers and this alerted her to the fact that she was thinking of something incorrectly, since she was unable to generate a number of towers close to 81. In spite of the disequilibrium that Jasmin experienced, she persevered and revised her hypothesis, coming up with an accurate solution.

Both Jasmin and Lavar worked on a towers extension problem which had the added stipulation that one of the two colors of the blocks, blue, was always glued to another blue block, so that when a blue block was used, it would have to be used only as a double block. This problem generates the Fibonacci sequence. In this problem, Lavar built towers before working symbolically, recording the number of towers he would get by building each successive height of tower. He noticed the pattern, and described this both in terms of the numbers and the joining of the concrete towers. It was apparent that this sequence of numbers was not familiar to him, though. Lavar recognized that his hypothesis was based on pattern recognition, but wanted to understand more. He described the recursive aspect of the pattern, and then stated "But if there's a quicker way to do that, I would definitely have to figure that out." He then conjectured that the ratio of blue blocks to yellow blocks was always 2:1.

Comparison to Students in Other Studies

The solution strategies, justifications, and explanations of the students in this study were remarkably similar to their predominantly non-minority, more affluent counterparts at the same grade level. Tarlow, (2004) studied the problem-solving behaviors of eleventh grade students who worked on the same problem. Many of these students were involved in problem-solving sessions similar in nature over the period of several years. However, all students worked side-by-side to solve the problem under discussion. Tarlow (2204) states that, "In grade eleven, Robert and his partner Michelle, two students who had been in the study for many years, used inductive reasoning to build towers and to justify having all of the towers for any given height. In addition, they used this reasoning to explain their theory that the total number of towers for height *h* and *x* available colors would be *x* to the *h*, which was an extension of their original theory for towers with two available colors and height *x,* two to the *x.* Angela and Magda, two students who had joined the study later, had also developed this idea, which they named Angela's Law of Towers" (pg. 223). She also notes that several other students "built and organized towers …into cases". We note that these types of proof almost exactly mirrored the proofs generated by the students who were the subject of our study, all of whom had no prior experience in solving this type of problem.

Conclusions

Our results support our hypothesis that the students in our study developed solution strategies and methods of proof and justification similar to the those present in the other studies, after a relatively short period of time, even though they had little if any prior exposure to problems of this type or to inquiry-based instruction. We also note the initial reluctance on the part of the students in this study. Many did not seem to know what, beyond stating the obvious, "We can't find anymore", was required to provide a convincing argument, since this was not something that they were accustomed to doing. We note, however, that after a short time, they did get the "hang of it" and were excited to be able to delve more deeply into the problem. We believe that this study documents that after a very short period of time, these students could provide convincing arguments, formulate two general types of proofs, and work over extended periods of time (often skipping lunch) on mathematical problems. We cannot say if this pattern persisted consistently in their everyday classroom environment; we can say that the level of enthusiasm was maintained over the course of our study, as we worked on subsequent problems. We can also say that some students expressed feelings of empowerment in the ability to approach and persevere in mathematical problem solving. As one student said, "If we had stuff like this in our regular school, it'd be more interesting. People wouldn't cut so much… What I see is I can be a good mathematician."

Endnotes

1. This was based upon extensive visits to the school, and to the teachers who taught the students for the past 3 years.
2. Note that the scores of the individual students in this study were not available. Rather, the reported scores reflect the proficiency levels for the entire school.

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IS THERE "CONTINUITY" BETWEEN THE MATHEMATICAL ACTIVITIES PRACTICED BY MATHEMATICIANS AND THE ONES PERFORMED IN INFORMAL CONTEXTS?

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In this paper Scientific Mathematics (ScM), School Mathematics (SM) and Informal Mathematics (IM) are characterized. According to the definitions proposed, it is argued that there is greater continuity in certain aspects between IM and ScM, than between ScM and SM.

Background

The need to provide a definition for the different types of mathematics arose during a longitudinal study which is being carried out. The purpose of the investigation is to understand the evolutionary processes of primary school children's beliefs related to the use of informal procedures to solve arithmetic problems in school as well as out of school.

In the research course it was necessary to solve two problems: a) to delimit what SM and IM are; because while the former includes a reference to school contents and practices, the latter has blurred borders; and b) to define each type of mathematics applying uniform criteria so as to make them commensurable. To solve the problem ScM was used as a parameter. The characterization became necessary because although in specialized literature a great amount of definitions can be found, they are generally based on diverse disciplinary perspectives, and consequently comparison becomes difficult.

Sense of ScM, SM and IM. Based on mathematical education research reports, Mathematics is defined as a complex system (García, 1986) constituted by:

a) a mathematical knowledge domain;
b) the cognitive activities carried out by members of a community related to mathematical knowledge; activities that take place within the framework of social interaction (Bauersfeld, 1980) from which epistemological procedures related to i) construction, communication and representation of knowledge, ii) argumentation and proof of mathematical results, iii) application of validity criteria and iv) solution of specific problems, are highlighted;
c) the axiological, attitudinal and conative aspects of the members of the aforementioned community; and
d) a culture, in the sense of a sum of customs, practices and beliefs (Wilder, 1985) which is expressed through activities and attitudes shared by the community integrants (Cobb & Yackel, 1988; Seeger, Voigt & Waschescio, 1988).

The previous generic definition provides a logical possibility of viewing each type of mathematics through the same lens. With this framework, ScM is under a permanent building up process carried out by integrants of the mathematician's community. The purpose of ScM is to solve theoretical and practical problems. The knowledge domain of ScM includes formal theories sustained by the community experts using their proof and validity criteria.

IM is used by subjects of small groups in informal contexts. The purpose of IM is to solve specific problems to make decisions (Lave, 1988; Carraher, Carraher and Schliemann, 1988). The knowledge domain of IM is formed by self-generated mathematical knowledge and/or knowledge acquired in informal and formal non-institutional ambits (Scribner & Cole, 1973) and also from compulsory schooling.

SM is performed by teachers with their groups of students in their classroom. The knowledge domain of SM is the set of mathematical contents included in the curriculum and in the textbooks teachers use; contents are linked with forms of representation and communication and with validity and proof criteria (included in pedagogical, didactical and evaluation prescriptions); all of them are related with those accepted in ScM. The purposes of SM, set up by students and teachers in a classroom community, are limited and tend to meet the requirements established by the school assessment system, even though from an institutional point of view the goal is to have the students come into contact with certain contents to acquire competencies.

In the following table, examples of epistemological activities and procedures carried out by members of the different communities are described.

	ScM	IM	SM
Self-construction processes	Always present	Very frequent	Not common: learning is limited to predetermined mathematical contents
Self-regulating processes	Always present	Very frequent	Evaluation are usually the responsibility of the teachers
Ways to validate results	The community must confirm that the requirements have been met	Based on an agreement with the rest	Usually the teacher and/or the text book are the (only) ones with authority to confirm a mathematical result
Applicability of concepts and purposes	It has a very wide range of application to actual problems	It fulfills the role of solving specific problems	It is usually applied to "school problems"
Interpretation of formal language	It has great semantic wealth	It is permanently tied to specific referents	It works preferably on written symbols (Carraher et. al, 1988) that act as blank labels (Lave, 1988)

Final comments

The three types of mathematics differ in their scopes and purposes, as well as in the culture shared by the members of the communities who performed them, which makes their practices and customs different. However, in spite of this, there are analogies among them, such as the "continuities" the authors seek to make explicit in this paper. The ideas synthesized will be used to establish an analyses framework – where ScM is placed at the core of mathematical education, whether in or out of school – for the investigation in progress.

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ISSUES TO CONSIDER IN DESIGNING DISTANCE BASED PROFESSIONAL EDUCATION PROGRAMS: A SRI LANKAN CASE STUDY

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Research shows student collaboration is important in planning and implementing distance teaching and learning of mathematics (Arnold, Shiu & Ellerton, 1996). This report considers some Sri Lankan elementary teachers' ideas for making a distance-based course more useful for their growth as learners and teachers of mathematics.

Conceptual Framework

This report is part of a study that analyzed issues in improving Sri Lankan elementary teachers' learning and teaching of mathematics in the context of CEPM, a distance-based course (de Silva, 2001). Although Sri Lanka had a large number of untrained elementary teachers, by 1998, a 3-year distance education program begun in the 1980s was being phased out, and short-term distance education was being implemented as a means of professional development. Tatto, Nielsen, Cummings, Kularatana & Dharmadasa (1991) found graduates of the 3-year program had a significant drop in their mathematical knowledge and pedagogical skills. Given the Sri Lankan authorities' plan to use distance education for ongoing professional development, it is important to implement the research findings of Arnold et al. (1996) and to include teachers' ideas in designing courses that will empower their growth beyond the duration of a course.

Methodology

My questions called for a qualitative approach where one "begins with an area of study and what is relevant to that area is allowed to emerge" (Strauss & Corbin, 1990, p. 23) as theory grounded in the respondents' experiences. In particular, I used a critical ethnographic approach (Thomas, 1993) where I raised my voice to "speak *to* an audience *on behalf* of [my] subjects as a means of empowering them by giving more authority to the subjects' voice" (p. 4, italics in original) and "became active in confronting explicit problems that affect the lives of the subjects – as defined by the subjects – rather than remain [a] passive recipient of "truth" that will be used to formulate policies by and in the interests of those external to the setting" (p. 29). Data was collected data throughout the yearlong course. Four interviews with each of 10 teacher respondents, interviews with course designers, tutors, and administrators, analysis of the 15 course modules (CEPM 101-115) and teachers' assignments, and field notes from classroom and tutorial observations allowed for triangulation of data.

Results

The CEPM course objectives were to improve elementary teachers' understanding of the mathematics they teach and the way they teach it. However, my research identified problems in the relevance of the course to the teachers and a dichotomy between what the course proposed to accomplish and what the modules, tutorials, assignments, and assessments emphasized.

For example, despite the course proposing that teachers gain a deeper understanding of elementary mathematics, the content was geared to secondary mathematics. While discussion of higher-level mathematics was considered important, respondents believed the focus on fewer topics, developed thoroughly from simple to abstract ideas (as exemplified in the first module), would be enable their ability to continue to learn beyond the course.

Further, despite the CEPM course pre-requisites asking participants to have at least five years of teaching experience, the course did little to build on teachers' this experience. In particular, discussion of issues related to teaching mathematics was rare and teachers were not encouraged to discuss their experiences or to conduct teaching experiments. Many felt it was important to continually make connections between learning and teaching mathematics: instead of having five modules on pedagogy at the end, "classroom methods and things, if we had it at the beginning, it would be good. …Then if they connected the others to them it would be more successful" (Indrani). Others suggested that all modules should involve discussion of teaching activities appropriate to their students, rather than have separate modules on pedagogy. For example, Mala proposed making a learning aid in, and a lesson plan for implementing some aspect of, each content module. Other suggestions for alternative structures to the modules, tutorials, and assignments included: Kamala's of incorporating the nature of mathematics and mathematical thinking (CEPM 111) as an ongoing discussion in tutorials; Sunila's of integrating a classroom methods (CEPM 113) discussion into each content modules' final assignments.

As continuing professional development, courses such as CEPM should reflect the needs of the times. Hence Lanka's suggestion, of analyzing the new primary mathematics curriculum to understand the way mathematical concepts build on and connect to each other, was particularly timely. (While none of the modules referred to the new curriculum, one of the eight tutors I observed did so on occasion.) Based on my respondents' concerns, it was also evident that programs for teachers must address the changing socio-economic conditions in the country. Many teachers perceived lack of nurture as an increasing problem in children's upbringing. However, as Kamala pointed out, authorities were unwilling to discuss the consequences of this lack within the context of primary teaching let alone incorporate it into the CEPM course.

Conclusion

They were going to the schools doing a project about teacher education – about different diplomas, degrees, College of Education's training, ... to ask teachers themselves how teacher education should happen. Thushara, fourth interview Given the one-way communication of instruction and policy that I had usually observed, I was delighted to hear of a systematic effort by authorities to get teachers' ideas, to hear their voices. Given my critical approach, throughout my research I had actively communicated teachers' ideas to tutors and administrators in the many discussions we had. It was heartening to know that future professional development programs might actively include teachers in course design.

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DEVELOPING CONCEPTUALLY TRANSPARENT LANGUAGE FOR TEACHING THROUGH COLLEGIAL CONVERSATIONS

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This study examines high school mathematics teachers' learning about classroom practice and subject matter through their interactions with their colleagues. In this presentation, I elaborate on how collegial conversations that (a) focus on the specificity of the classroom and (b) are purposed toward improving practice effectively supported the development of conceptually transparent language for teaching mathematics.

In particular, I analyze the conversations of teachers in an American urban high school who have worked to increase equity in their mathematics classrooms. At this school, students enroll in advanced mathematics courses at higher-than-average rates. In addition, a comparative study of student mathematical performance shows that, while these students enter high school with weaker mathematical preparation than their college preparatory peers in more affluent high schools, they actually outperformed them after two years of instruction (Boaler & Staples, 2003). These successes are particularly notable in a working-class school in which the majority of students come from traditionally underrepresented groups.

Understanding how the teachers have successfully implemented mathematics reforms would therefore greatly contribute to our larger efforts at improving practice and creating more equitable classrooms. Prior work on teachers' responses to reform emphasizes the impediments to successful implementation that arise from teachers' beliefs, existing practices, and subject matter traditions (Cohen, 1990; Stodolsky & Grossman, 1995). At the same time, research on teacher communities suggests that certain kinds of collegial environments can support innovative classroom practice (Gutiérrez, 1995; McLaughlin & Talbert, 2000; Cochran-Smith & Lytle, 1999). Yet the ways in which those collegial environments interrupt the common impediments to reform and support innovation need to be further specified.

In this study, I went inside a reform-oriented teacher community to understand some aspects of collegial environments that support innovation. During a two-year comparative ethnographic study where I taught and worked alongside the teachers in the study, I collected a variety of data targeted to capture teachers' formal and informal collegial interactions. These included audio and videotape recordings of teacher meetings, scheduled and spontaneous collegial conversations, and fieldnotes of classroom observations. Using sociolinguistic methods, I employed a sociocultural framework to specify the mechanisms by which the teachers learn from each other in informal conversations and interactions (Horn, in press). In this presentation, I focus on the ways that the teachers' conversations rendered a figurative version of the classroom, thus providing an interactional space for them to consult closely about issues of practice.

The analysis of video and audio-transcripts of the teachers' conversations showed that the classroom was often figured using two particular forms: (a) *teaching replays* — blow-by-blow accounts of classroom events that included directly quoted student voices, and (b) *teaching rehearsals* — anticipatory versions of classroom events that either reworked interactions that had occurred or defined prototypical interactions and appropriate teacher responses. The close rendering of the classroom created multiple opportunities for the teachers' collaborative pedagogical problem solving. First, by locating problems in the specific interactions of the classroom, the teachers often faced the ambiguity and complexity of their teaching choices.

Second, by sharing the normally private events of the classroom with their peers (Lortie, 1975, Little, 1990), they coordinated expectations and teaching strategies, creating a more consistent environment for their students. In addition, by taking on both the student and teacher voices in these *replay* and *rehearsal*-laden conversations, they laminated student identities onto themselves as teachers, intertwining their voices in the roles of teacher-as-teacher and teacher-as-student. By extensively taking on the student voice and perspective in their deliberations of practice, the teachers' constantly considered their students' intellectual and emotional responses to their teaching.

Part of what emerged from these classroom-specific, multiply-perspectived conversations was a conceptually transparent language for teaching mathematics. Some of this language was already in use at the time of the study. For example, in explicating the elements of linear functions (y=mx +b), the teachers introduced the idea of slope as a "grow-by number" and the y-intercept as a "start-at number." Likewise, they eschewed the conceptually opaque term "canceling out," preferring the more conceptually transparent and mathematically distinct terms "making ones" to describe the reduction of rational expressions or "making zeroes" to describe combining opposite signed terms in an expression. In this presentation, I will analyze an instance in which conceptually transparent language is shared in conversation. This study supports the notion that teachers' professional development is more effective when closely linked to classroom practice and provides a model for building such learning within a departmental community.

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AN EXAMINATION OF TEXTBOOK "VOICE": HOW MIGHT DISCURSIVE CHOICE UNDERMINE SOME GOALS OF THE REFORM?

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In this paper, I use a critical discursive framework to examine a middle school mathematics unit's "voice." Attending to the text's voice helps to illuminate the construction of the roles of the authors and readers (and the relationships between them) as well as the portrayal of mathematical knowledge. The critical discursive framework focuses one's attention on particular language forms, including personal pronouns, imperatives, nominalizations, modals, and words that develop continuity. The aim of the analysis was to examine the voice of the unit to see if the authors of the unit achieved two ideological goals put forth by the NCTM Standards (1991) document: 1) to portray mathematics as a human construction, and 2) to shift the locus of authority away from the teacher and the textbook. The findings indicate that achieving these two goals is more difficult than the authors of the Standards may have thought. I conclude by discussing the implications of these findings for curriculum developers and for future research.

Introduction

In an attempt to make the *Standards* (1991) more concrete, the National Science Foundation (NSF) announced funding for the development of curriculum materials that embodied the ideas explicated in the *Standards* documents. While the presence of some of these goals (e.g., problem solving) may be more apparent when examining curriculum materials, others require a closer look. For example, the *Discourse Standards* suggest that mathematics textbooks be positioned in a different manner than what has previously been the case: "Discourse entails fundamental issues about knowledge: What makes something true or reasonable in mathematics? How can we figure out whether or not something makes sense? That something is true because the teacher or book says so is the basis for much traditional classroom discourse. Another view, the one put forth here, centers on mathematical evidence as the basis for the discourse" (NCTM, 1991, p. 34). Rather than the textbook and teacher acting as major sources of authority, students are encouraged to rely on mathematical reasoning and evidence when discussing mathematical solutions, drawing the locus of authority away from the teacher and the textbook. Not only is the textbook to be *used* differently, but also the view of mathematics that is portrayed is intended to "represent mathematics as an ongoing human activity" (NCTM, 1991, p. 25). The question remains as to whether these deeper ideological and epistemological goals are being met.

This new positioning of the textbook and its different view of mathematics are related to "voice," i.e., "how the voice of the authors/designers is represented and how they communicate" with the reader (Remillard, 2002, p. 6). These particular goals of the *Standards* are related to the "voice" of the textbook because they are inherent in the subtle messages the textbook embodies. The discursive choices[1], which can be examined in the voice of the text, send subtle ideological messages both by constructing the roles for and relationships with the reader and by portraying mathematics as a particular type of knowledge.

The majority of analyses of mathematics textbooks have focused on mathematical ideas, their forms of representation, and their organization for student learning (e.g., Fuson, Stigler, & Bartsch, 1988; Li, 2000; Schmidt, Jakwerth, & McKnight, 1998). An additional important dimension is being ignored: the language that is being used. While you cannot read culture directly from language, language does indirectly index particular kinds of dispositions, understandings, values, and beliefs (Ochs, 1990). It is through the examination of language patterns in textbooks that ideological and epistemological issues can be scrutinized. In this paper, I argue that people who develop curriculum materials need to carefully attend to their discursive choices so that they can "hear" their voice and not undermine their own intentions.

Analytic Framework

A critical dimension of understanding how teachers use curriculum materials are the written materials themselves (Lloyd, 1999). To examine written materials, they must be viewed as an *objectively-given structure* (Otte, 1983). That is, the structure and discourse of the *written* unitnot what happens when an individual interacts with it- must be the focus of the analysis.

"Voice"

One way of attending to subtle and unintended features of a text is to examine its "presence." Love and Pimm (1996) define the "presence" of the text as encompassing the "features of the text that are usually taken for granted" (p. 379). They distinguish between the presence *of* the text and the presence *in* the text. Related to the first, they claim that the inherent authority of the text is something that teachers cannot ignore. A textbook is a codified version of what is accepted content at a given point in time; it is a message from the mathematical community outside of the school, teachers, and students. The presence *in* the text includes the author "voice", which is intimately related to discursive choices made by the authors (Love & Pimm, 1996, p. 381). The notion of *choice* is important when examining subtle and unintended meanings in text because it helps focus attention on ideological issues:

> Whenever an utterance is made, the speaker or writer makes choices (not necessarily consciously) between alternative structures and contents. Each choice affects the ways the functions are fulfilled and the meanings that listeners or readers may construct from the utterance. … The writer has a set of resources which constrain the possibilities available, arising from her current positioning within the discourse in which the text is produced (Morgan, 1996, p. 3).

The aim of this paper is to examine the voice of one mathematics unit. More specifically, I inspect the positioning of the textbook with respect to those who will be reading it and the ways in which mathematics as a body of knowledge is conveyed.

A Critical Framework for Examining the "Voice" of the Text

The discursive framework I adopted for this analysis was developed by Morgan (1995; 1996), who draws on Halliday's (1973) three metafunctions of language: the *ideational*, *interpersonal*, and *textual*. These metafunctions of language allow the analyst to examine the roles of the reader and author and the relationships between them as well as the particular view of mathematics captured by the text. I chose this framework because it offers a systematic approach to the analysis of unintended messages and thus fits the intent of this analysis.

Morgan describes the combination of these metafunctions as making up the "style of writing" and the cohesiveness of the text. The style of writing focuses the analyst's attention on the use of imperatives (or commands), personal pronouns, and modality. Imperatives implicitly address the reader and involve her in the construction of mathematics. Imperatives enable the author to speak with an authoritative voice because they are used to direct the reader's attention. First person pronouns (I & we) indicate the author's personal involvement with the activity in the text; the second person pronoun (you) directly addresses the reader. Modality focuses on the "degree of… weight or authority the speaker attaches to an utterance" (Hodge & Kress, 1993, p. 9). It appears in modal auxiliary verbs (e.g., must, could, will), adverbs (e.g., certainly, possibly), or adjectives (e.g., I am sure that…). An examination of the coherence of the text focuses one's attention on the modes of reasoning and its features that preserve continuity.

Methods

The Text

When NSF released its call to fund innovative mathematics curriculum materials, interested and qualified parties were asked to submit proposals to compete for funding. The authors of the Connected Mathematics Project (CMP) (Lappan, Fey, Fitzgerald, Friel, & Phillips, 1998a) were one such group to receive financial support.

Broadly speaking, the CMP curriculum is a middle school problem-centered curriculum where almost every problem occurs in a "real life" context. Each grade level of CMP is published as a set of separately bound units; each unit centers on a big mathematical idea and is 50 to 80 pages long. Problem sets are organized into "Investigations" which typically involve a series of related problem solving situations. At the end of each Investigation, a set of homework problems is given (called Applications-Connections-Extensions or ACE problems). Periodic reflections (i.e., "Mathematical Reflections") ask students to make mathematical connections.

The particular 64- page student unit I focus my analysis on is the *Thinking with Mathematical Models (Models)* (Lappan, Fey, Fitzgerald, Friel, & Phillips, 1998b) student edition. Because this unit focuses on mathematical modeling, it has some characteristics (e.g., experiments in which students are required to collect actual data) that are not as prominent in some of the other units. Therefore, one should not conclude that this analysis is indicative of what appears in other units[2]. Since the purpose of this analysis is to examine the unintended messages of the text, these unusual characteristics are inconsequential. I do not make claims about CMP itself; rather, I examine *Models* closely so as to raise issues about the difficulty inherent in developing curricular materials that convey unconventional messages to readers about their roles and about what it means to know and do mathematics. Even when the authors believe strongly in the goals of the reform, the analysis reveals that capturing the reform ideals in written form is a difficult task.

There are many explicit ways in which this curriculum embodies the vision of the ideas put forth in the *Standards* documents, e.g., through its goals of mathematics for all, its focus on eliciting and using student thinking, and its pervasive use of problem solving to teach big mathematical ideas in meaningful ways. However, curriculum materials also need to be examined on a more tacit level to see if deeper, ideological goals are being met.

When I examined the discursive forms of this unconventional mathematics curriculum to describe its voice, I attended to the linguistic tools described in Morgan's framework (i.e., imperatives, pronouns, modality, nominalization, features that preserve continuity, and modes of reasoning). The guiding question was: What is the nature of the voice of this text? Two related sub-questions were: What images of mathematics does it capture? How are the reader's roles and the relationship between the author and reader constructed?

Process of Analysis

In order to examine the discursive features of the unit, I used both a written copy and an electronic version of the unit and moved back and forth between them. I used the electronic version of the text primarily to do a word count for the entire text, and to search for particular words (e.g., "you") or symbols (e.g., a question mark). I followed most of the electronic searches with an examination of the written unit because I needed the text that surrounded specific word(s) to interpret their use.

I focused on the discursive elements of the text suggested in Morgan's framework. For the majority of the analysis, I examined the written copy sentence-by-sentence. To interpret these language forms, I used both the text around the forms as well as the location of the form within the larger unit. I also had to continually situate the text within the larger context within which it was being used. For example, I continually reminded myself of the presence *of* the text.

Results

Style of Writing

The style of writing focuses the analyst's attention on imperatives, personal pronouns, and modality. Each will be addressed in this section.

Imperatives

Often there were strings of imperatives listed in the book and sometimes when the authors said they were going to ask questions, they used imperatives instead. The most common imperatives include: "explain" (42) [3], "make" (35), "use" (34), "write" (29) and "describe" (33). Morgan (1996) points out that the use of conventional and specialist vocabulary in conjunction with imperatives "marks an author's claim to be a member of the mathematical community which uses such specialist language and hence enables her to speak with an authoritative voice about mathematical subject matter" (p. 6). Given the unequal relationship between the authors of textbooks and the readers in schools, it is likely the case that this combination of specialist vocabulary along with strings of imperatives is viewed as being used to inculcate students into the mathematical community.

Another characteristic that needs to be considered is whether the imperative is *inclusive* or *exclusive* (Rotman, 1988). An *inclusive* imperative (e.g., consider, define, prove and their synonyms) demands "that the speaker and hearer institute and inhabit a common world or that they share some specific argued conviction about an item in such a world" (Rotman, 1988, p. 9). In contrast, an *exclusive* imperative requires only that "certain operations… be executed" (Rotman, 1988, p. 9). Inclusive imperatives emphasize the reader's role as that of a "thinker"; exclusive imperatives construct the role of the reader as a "scribbler" who performs activities. In *Models*, 70% (221 of 315) of the imperatives were *exclusive*, including "make" (35), "use" (34), "write" (29), "draw" (17) and others (e.g., "find," "place," "copy" (106)). The choice of imperatives emphasizes the reader's role as someone who performs actions rather than contemplating and thinking about them.

Personal Pronouns

First person pronouns (I & we) were entirely absent from the unit, obscuring the presence of human beings in mathematical activity and distancing the author from the reader. The four major forms associated with the second person pronoun "you" are summarized in Table I below.

Table I

Form	Examples from the Text	**Instances**
You + verb	"You find", "you know", "you think"	**165**
You + modal verb	"You will", "you can find", "you would"	**56**
Inanimate object (as subject) + animate verb + you (as direct object)	"The graph shows you", "the equation tells you"	**37**
You + hedged verb	**"you might have found"**	**5**

Because the second and fourth forms in Table I are related to "modality," I will address those in the next section. In the first form, you + verb, the authors typically made explicit what they thought the reader was doing.

The third form offers a striking example of "nominalization"; that is, the transforming of processes into objects. In mathematics, many processes (e.g., "rotating") become objects (e.g., "rotation"). In the unit, inanimate objects (e.g., "the graph") perform activities that are typically associated with people (e.g., "tells you"). This type of nominalization depicts an absolutist image of mathematics, portraying mathematics as a system that can act independent of humans.

This is not to say that mathematical objects are the only actors in this textbook. In fact, many of the problems have human actors in them. The combination of human actors and nominalizations send mixed messages about the role of humans in the process of mathematics.

Modality

One type of modality appears in the authors' use of "hedges" which are "linguistic pointers to moments of uncertainty" (Rowland, 1995, p. 328). For example, "that is a linear model," is more direct and certain than if I use, "I *think* that *might* be a linear model." As Table I shows, there are only five instances of hedges associated with the word "you" in which the authors acknowledge that they do not know the actual readers of the text. For example, after introducing a fulcrum problem using the context of a teeter totter, the authors state, "You may have noticed…". The hedged verb "may have" indicates that this is information about the reader which the authors do not know. The other most common modal verbs are "would" (55), "can" (40), and will (40), all of which express more certainty than other verb choices like "could." This combination of certain modal verbs with few hedges indicates the authors have great conviction about their knowledge of mathematical modeling and where mathematics is used.

Coherence of the Text

Examining the coherence of the text helps inform the interpretation of the style of writing. The coherence of the text can be seen in how reasoning is constructed as well as how continuity is maintained throughout the text.

Construction of reasoning

Words like "because" and "so", which express connective reasoning, were rare in the unit. Rather, reasoning was constructed more often through a narrative telling of how someone else came to a particular conclusion. For example, the unit gave an extensive example of how a "class in Maryland" did a particular experiment and then asked the readers to repeat what the Maryland class had done. Often, after the description of the exemplar class or person, a series of imperatives were given to direct the reader's actions. According to Morgan (1996), a combination of temporal themes and imperatives constructs an algorithm to be followed, reinforcing the student's role as one of "doing".

Maintaining continuity

Continuity is maintained in the text through the author's use of recurrence: past events resurface (e.g., a class bike trip or fundraising activities), some actors appear in more than one place, mathematical terminology is introduced either in bold or italics and later is written with normal lettering. An interesting form used to maintain continuity appeared in statements like "you found." For example, the authors wrote phrases like "In the last investigation, you tested paper bridges of various thickness. You found…". In these statements, the authors give meaning to the activity, shaping, defining, and controlling the "common knowledge" (Edwards & Mercer, 1987) of the classroom. In fact, the authors need to do this because they are counting on a common readership that has done and found certain things before they can proceed to the next section of the text.

Discussion

Does this unit portray mathematics as a human activity? The analysis indicates that the textbook is devoid of first person (singular and plural) pronouns, indicating that the presence of human beings is concealed. In the second person pronouns, the authors obscure agency by having inanimate objects perform animate activities. Morgan contends that this suggests an absolutist image of mathematics as a system that can act independent of humans. In contrast, the authors have human actors throughout the textbook who both engage in and use mathematics. The type of reasoning that is used throughout is narrative in nature, valuing the human experience. Thus, the message sent to the reader is a mixed one.

How is the unit positioned with respect to authority and its epistemological stance? I have shown how almost all of the hedged forms appear where the authors draw on an experience that they cannot know if the students have been a part of or not (i.e., balancing on a teeter-totter). To soften their assertions, the authors hedge the verbs in the problem. The most prominent forms of modal verbs in the text (i.e., "would", "can", "will") indicate that the text represents a viewpoint of strong conviction.

The style of writing of the text is authoritative. The repeated imperatives in the experiments, problems and ACE construct the author of the text as having an unequal relationship with the reader; the author's role is to inculcate students into the mathematical community. Because most of these imperatives are *exclusive*, the authors emphasize the role of the student as a scribbler. Furthermore, the combination of temporal themes and imperatives support this role as students are instructed to follow particular algorithms.

Mathematics educators need to deeply consider the ways that language indexes a particular ideological stance. While there is a growing interest in how language can be used in the teaching and learning of mathematical content, less attention is given when examining written resources like textbooks and curricular materials. The findings reported here illustrate how powerful the hegemony of traditional discourse in mathematics curriculum materials can be. The curriculum developers themselves are dedicated to the vision of the NCTM *Standards*, yet they are not immune from the hegemonic discourses of traditional mathematics education.

Implications

In the following paragraphs, I address the implications of this research for both curriculum development and future research.

Implications for Curriculum Development

If we want to pursue unconventional goals in new curriculum materials, we need to expand our view of who needs to be involved in the development of those materials. Schwab's (1978) highly cited work suggests bringing together a team of people to develop curricular materials, including both subject matter specialists and experts from the social sciences. Many of these specialists are steeped in more conventional ideologies. To allow for ideological or epistemological shifts, consideration needs to be given to the choice of language in the curriculum materials. Discourse analysts could be involved in the creation and revision of new curricular materials or could be consultants to people who are trying to develop less conventional materials. Attending to discursive choices offers one way to illuminate ideological assumptions and to ensure that curriculum developers are not unintentionally undermining their own goals.

Implications for Research

The findings of this analysis raise other questions that should be pursued in educational research. A similar analysis could be done with other unconventional curricular materials to see if deeper ideological and epistemological goals are being met. Additionally, an analysis could be done to compare the more traditional textbooks to unconventional materials. A comparative analysis would be important to make claims about the prevalence of some of the discursive patterns. For example, while this text had more exclusive than inclusive imperatives (70% vs. 30%), an examination of other mathematics textbooks may uncover that this is a relatively high number of inclusive imperatives for a mathematics textbook.

Moreover, the question remains as to whether changing some of the more traditional language patterns in textbooks would make a difference in classrooms. Research on students' and teachers' beliefs in mathematics classrooms who are using less conventional curricula has shown that they have different beliefs than those using more traditional curricula (Schoen & Pritchett, 1998; Wood & Sellers, 1997). Would choosing our words more consciously so they reflect a different ideological stance change the way that students and teachers interact with the curricular materials?

Alternatively, these findings may suggest that we need to examine the ideals put forth in the NCTM *Standards* more carefully. The authors of the *Standards* suggest these shifts as if they might be unproblematic to realize. As mentioned earlier, a textbook represents a message from the larger mathematical community about what students should learn in their school mathematics experience. Some of the discontinuities between the ideas suggested by the *Standards* and the discursive choices that appear in the mathematical unit may be easier to resolve than others. Other modifications may be more difficult to make. One set of changes that would be difficult is the ones that modify the genre of a mathematics textbook beyond what an average reader may expect, given her past experience with mathematics textbooks. How might readers respond to such changes? Additionally, Apple (1986) has clearly articulated the politics involved in textbook writing and marketing, which raises a related question: would companies consider publishing such an unconventional genre of mathematics textbook? If publishers are mainly concerned with making a profit, the curriculum materials may never even go to press.

Representing mathematics as a human construction is not easily captured in written materials for at least two reasons: a) the authors have to help the readers learn particular mathematical ideas and b) the authors need to assume some particular ideas have been learned in order to write the next section of the textbook. Adopting a discovery approach to mathematics may be more straightforward for curriculum developers because they have particular mathematical ideas in mind that the students need to learn. This tension has been explored in mathematics education literature related to teachers and teaching, but has not been examined related to curriculum and curriculum developers. As a community, we need to consider more conscientiously the difficulties inherent in some of the shifts being suggested in the *Standards* documents.

Endnotes

1. Whenever I use the word "choice," I do not mean that they occur at a conscious level for the person using them.
2. Also, the curriculum is now being revised, so the findings reported might not be applicable to the next edition.
3. Numbers in parentheses indicate the number of occurrences of that word or phrase in the text.

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LEVELS OF A TEACHER'S LISTENING WHEN TEACHING OPEN PROBLEMS IN MATHEMATICS

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Here we discuss teaching/learning situations in math classrooms based on one teacher's listening and comprehension as he follows a group of four girls working on an open-ended problem. The teacher has earlier participated an in-service course about working with open problem solving. The lesson is recorded on video, which in an episode involving group work shows some of the difficulties the teacher seems to have with getting his pupils to understand the points with the tasks. We analyze the communication with a listening taxonomy together with the epistemological triangle. The result shows us that in order working with open problems will succeed, it is of paramount importance that the teacher learns to listen his pupils with understanding.

Theoretical background

The role of communication has been increasingly emphasized in mathematics education research in the last decade. Communication is central to pupils' formalization of mathematical concepts and procedures. If a teacher pays attention to this, she is compelled to listen to her pupils and to follow their thinking process before she tries to understand them, and before she tries to get them to understand her. One method of achieving this is to use discussion as an element of teaching. Very often pupils have preconceptions (or misconceptions) about the subject to be learned. The teacher may try to understand her pupils' way of thinking by listening to discussions among pupils or having a dialogue with her pupils. The comprehension she gains through such a communication can be used as a reflecting point for planning her teaching (Schoenfeld, 1987).

In his book Luhmann (1984) describes communication as composed of three components: selection of information, selection of form, and selection of understanding. The speaker understands information and selects form, while the listener has the hard task of understanding. If information and/or form is unclear, it may be difficult to listen and to understand what the speaker means. Listening and understanding will always depends on the listener's way of thinking.

In the literature, one may find many different, usually hierachic classifications for listening. For example, Covey's taxonomy (1989, 240) contains five levels of listening: 1. Ignoring the other, 2. Pretending to listen, 3. Selective listening, 4. Attentive listening, 5. Empathic listening. He states that "One should seek to understand before to be understood". Burley-Allen (1995) operates with three levels of listening – to listen only every now and then, to hear, but not really to listen, and empathic listening. These are almost the same as Coveys last three levels. In her paper, Davis (1997) considers three kinds of listening: evaluative listening, interpretative listening, and hermeneutic listening.

In comparing empathic and interpretive listening Stewart (1983) points out some theoretical and pedagogical advantages of the interpretive approach to listening. However, he considers a balanced situation, in the sense that the participants are equal. As we are focussing on a teacher-pupil discussion at least some aspects of empathic listening should be taken into account – the teacher should try to understand her pupils holistically, not other way round.

Steinbring (1999, 44) uses the theory of Luhmann and Saussure to develop an epistemological triangle that "contains very specific features with regards to the particularities of mathematical communication". The epistemological triangle consists of context of reference, sign or symbol and concept. Together with the classifications for listening we will use this idea to analyze the communication in the classroom.

Listening has been in the center of communication research more than fifty years (Stewart 1983, 379), but in mathematics education it has a shorter history. Today one may find many studies on communication (with the focus on listening), among them: Fernald (1995) describes a method to teach students in psychology to listen empathically. Pirie (1996) discussed the meaning of discussion from different perspectives - What? When? How? -and developed a model for a listening teacher. Davis (1997) reports a collaborative research project with a middle school mathematics teacher and gives some examples how the teacher listens.

Focus of the paper

Our paper concentrates on a teacher's listening to his pupils during a mathematics lesson with open problem solving. We are interested in determining, on what levels a teacher listens in such a situation. In order to look for explanations, we ask: On which levels does the teacher listen to his pupils? Why are there gaps in the teacher's listening? And we will discuss its consequences for learning.

Classroom visiting

In the spring of 2003, the researcher (the second author) visited – in order to get authentic information on classroom communication related to open problem solving – a school in Copenhagen and videotaped a mathematics lesson. The class in question was an 8th grade with 14–15 year-old pupils – 22 pupils in all. The teacher was a young man with five years of teaching experience. He had earlier participated a teacher in-service education course given by the researcher with open problem solving as one content area. Before the course the researcher had interviewed him about using open problem solving, and another interview took place after the lesson.

The teacher explained to the researcher in advance that the topic of the lesson was equations and functions. His choice of 'open approach' in the lesson resulted him to produce some open problems himself. Most of the tasks were formulated as commands, such as "Tell a story on what the arithmetic expressions could be about: 2x = , 37x = or 43,25x = ". During the active part of the lesson, the class was divided into groups of four or five. The tasks were new to the pupils and most of them did not know how to solve or give an answer to these tasks. Therefore, they discussed in their groups what to do.

It seemed to the researcher that the pupils liked their teacher, and that they wanted to find a solution or an answer to satisfy his command. At the same time they were confused about what kind of answers would be good enough, and about what kind of mathematical knowledge they should use or look for.

The following episode from the lesson is a part of the communication from a group of four girls, whom we shall call Anna, Betty, Cecilia and Doris. During the episode, the teacher visited the group every now and then. The open task in question is the one mentioned: "Tell a story […] 2x = ; 37x = or 43.25x =". The teacher's introduction to the whole class for the episode was, as follows: "As you know the letter x can be whatever you want like a dog, a ball or a number. You choose it yourselves; it is **your** story." The whole episode took about 10 minutes.

The discussion episode

ANNA: It could be a multiplication table for the number 2?]

BETTY: Or it could be two cows on the field.]

CECILIA: That is not a story.

ANNA calls the teacher; everybody is silently waiting for the teacher to come.

[4] BETTY: What do you want us to write? The multiplication task for the number 2?

[5] TEACHER: Yes, I will say it is an expression of the multiplication table for the number 2. Yes, that is perfectly correct.

[6] ANNA: Is that what we should do?

[7] TEACHER: Yes, you should find an expression and a story. When I was a little boy in first grade, this table was one of the first tables I learned. That was a good story, wasn't it?

[8] ANNA: Can I just write the multiplication task for the number 2?

[9] TEACHER: Yes, maybe someone else will come out with something else.

[10] ANNA: I still don't understand it…

[11] TEACHER: Just make a story; we will try it all together, if it is good enough.

The teacher leaves and the pupils write the multiplication table for the number 2 and after a short discussion the multiplication table for the number 37 for the story to 37x = . The next task 43.25x = gives them new troubles.

[12] BETTY: Now we have used the multiplication for the number 2 and 37, why can't we use the multiplication for the number 43.25?

Long silence while the others look confused.

[13] DORIS: Is that a table? Maybe it could be some shopping?

[14] BETTY: You don't use exactly 43.25.

[15] DORIS: More than the multiplication of the number 43.25. Maybe we buy something that costs 43.25? A bag of coffee, maybe?

[16] BETTY: Oh I don't feel like doing it. A bag of coffee doesn't cost 43.25. I don't think we should do it in this way. Shopping means to find the sum of what I buy, not to multiply. No, I give up.

[17] ANNA: Let us write that we use the money to buy something.

[18] DORIS: If we buy two bags of coffee and each cost 43.25, then we multiply?

[19] BETTY: But if we multiply by two we get the double, can't we say that we measure something?

Confusion spreads all around the table.

[20] ANNA: When do we need this?

[21] BETTY: Does it mean anything? It is unimportant? It is the point he wants.

[22] ANNA: What is the point with all this?

[23] BETTY: Let us say the multiplication table for the number 43.25.

[24] ANNA (calls the teacher saying): We are sitting here, and we are very stupid.

The teacher arrives.

[25] BETTY: Could it be the multiplication table of the number 43.25?

[26] TEACHER: It is a little unrealistic, I would say, maybe.

[27] ANNA: Maybe something with shopping?

[28] TEACHER: Yes, why are you saying shopping?

[29] CECILIA: You talk about application.

[30] TEACHER: Yes (obliging)

[31] DORIS: If we wanted to buy a football with the price 43.25 and we wanted to buy two…

[32] BETTY: No, we are not finding the sum. You can double up.

[33] TEACHER: The principle is the same. [Now follows a story about buying wood in a lumberyard]

[34] BETTY: Can't you measure something and then multiply? Like base multiplied by the height?

[35] TEACHER: (to BETTY) Maybe if you can convince your group. I think you should go on with your ideas you can do it.

[36] ANNA: If you knew how long time we already spent on this.

[37] TEACHER: Yes, it sounds reasonable too with the multiplication table of the number 43.25. I don't know what to use the multiplication table of the number 37 for either. But do you understand it?

[38] ANNA: I don't understand anything at all.

The teacher leaves and the group decide to write the multiplication table of the number 43.25.

Interpretation

In interpreting the levels of the teacher's listening, we will use Covey's five-step taxonomy (1989, 240).

The communication episode in the group can be structured in four phases, two of them without the teacher, and the other two with the teacher. In the first phase [1]-[3] the pupils suggest and reject proposals. When BETTY suggests two cows, it is in continuation of the teacher's instruction. The second phase [4]-[11] is communication between the teacher and the pupils in the group where the teacher accepts the multiplication table of number 2 as a mathematics story and gives the pupils an example of a story about this multiplication table, maybe more a 'meta' mathematics story than a normal mathematics story. The third phase is [12]-[24] discussions in the group. They cannot agree on any of the suggestions, and they seem helpless in relation to the point of the task [22]. They are going in circles when BETTY again mentions the multiplication for the number 43.25 as a solution [23]. The phase [25]-[38] is again communication with the teacher where he at last 'gives up' and accepts an 'unrealistic answer', his own words [26].

All the pupils are more or less engaged in solving the tasks, but from the beginning they are uncertain about what is acceptable as answers. The task at hand is new to them. Asking their teacher does not yield a clear answer. On one hand, he says it is perfectly correct to use the multiplication table [5], and on the other hand, he tells a story and says that someone else will surely come up with something better. Following Covey's taxonomy, the teacher ignores ANNA (level 1) who tells him that she doesn't understand [10]. In [11] he pretends to listen (level 2) using an automatic 'teacher' sentence before he leaves. The group is left to its own devices, and the pupils are unable to come to any understanding of how a story could look like. DORIS suggests several times the idea of buying coffee ([13], [15], [18], [31]), but the group did not accept her suggestion. An explanation may lie in social relations within the group – she was maybe not in a position within the group to tell the others what to do.

The leader of the group appears to be BETTY who didn't understand the idea of buying coffee. BETTY understands a shopping situation totally differently: for her, shopping means 'making sums' [16]. ANNA several times expressed that she doesn't understand what to do ([10], [24], [38]), but on the other hand, she actively tries to solve the tasks. When she calls the teacher [24], she tells him that she feels stupid, but he ignores her again (level 1). BETTY is able to get him to listen, but the answer doesn't help them solve the problem [26]. In the same utterance the teacher says that it is 'unrealistic' and 'maybe'. Now ANNA tries with shopping and the teacher listen attentively [28] (level 4). He repeats the words but when BETTY takes over once more it seems like he is only listening to her [32] – [35]. He tells another story about himself in lumberyard (difficult to hear exactly) before he encourages BETTY to go on with her ideas and to convince the group, he says to her: "You can do it." The sentence in [35] seems again to be an automatic 'teacher sentence'. ANNA tries twice to tell him that she needs help [36] and [38], but he ignores her (level 1).

CECILIA was not an active member of the group. During the whole episode, she made only two comments ([3] and [29]). The teacher listens to her once for very shortly [29] and [30] when she answers his question about shopping. A couple of times during the episode the whole group seemed to be confused [12] and [19]. But when they ask their teacher, he seemed a little uncertain himself about what answers he would accept from them [37]. It seems like he did not envisage this answer, and even though he finds it unrealistic he decides to accept it. In this sentence he gives up finding a better solution and finishes the sentence with a standard teacher sentence: "Do you understand it?" But he doesn't wait for an answer.

Discussion

The teacher in question tries to do something else than just follow the textbook. He has been in a teacher in-service course about open problem solving, and now he wants to show what he masters. Behind his resistance to change could be a wish to be seen as a respectable teacher (cf. Pehkonen 2001). Maybe he wanted to demonstrate to the researcher that he is able to use open-ended tasks in mathematics class – that he is an innovative teacher, and in that sense to be respected.

But at the same time he lacks some necessary skills in mathematics as well as in pedagogy, e.g. in asking questions and in listening. He seems to be on his very first 'steps' of using open-ended tasks (i.e. using a constructivist type of teaching, e.g. Maher 1998). This can be seen in the setting of the task "Tell a story … about 2x =…" which is a little cumbersome. Another observation is that he is not sensitive to his pupils' thinking process. He is mostly ignoring his pupils, only once he is attentively listening. Thus his listening is on lower levels of Coveys taxonomy, he does never listen empathically.

Working with open problem solving demands quick decisions from the teacher. Will he accept or reject pupils' ideas and why? How can he prepare himself? What kind of answers can he expect? How can he in the situation concentrate on his next move in the classroom situation? The teacher in question seems to be uncertain on several levels: He lacks experience with teaching open problems, and he has obvious problems with using mathematical terms for equations and functions correctly. One explanation for this uncertainty might be the fact that his teaching is videotaped.

If we follow the epistemological triangle we can first examine the teacher's intentions, and then the pupils' understanding; where is agreement and where discrepancy. The teacher wanted to develop the concept of a relation between equations/functions to applications in real life. As 'context of reference' he used 'stories as a narrative expression for the mathematical terms'. He used expressions as '2x = ; 37x = ; 48.25x = ' as 'sign or symbol'. But what happens is that none of the pupils in the group sees his point [22]. Instead of stories they see the expressions as tables for multiplication numbers, and the relation between equation and function, like seeing x as a placeholder or a variable is total overlooked. The expressions are neither equations nor functions. So the pupils are left to come up with solutions on the basis of the teacher's authority

(Steinbring, 1999) and not understanding mathematics. What the teacher thought to be a smart way to combine equations and functions confused the pupils. It was new to them and became another item in their school mathematics vocabularies without any general mathematical connection.

The teacher told the group a story about himself as a little boy and about multiplications of the number 2 [7]. It was a story *about* the story, and the pupils were left confused. Why did the teacher tell that story instead of listening to what their problems really were? An explanation could be that when the teacher speaks himself he does not need to listen. Another one is that working with new skills is demanding in the way that every thing is new and therefore it cannot be done automatically.

The problem of teachers' pretending to listen to their pupils is well documented. For example, Perkkilä (2003) described a class situation where the teacher asked her pupils questions, but she used only such answers that she could fit into her plan for the lesson. The other answers, many times good prompts, were ignored. Covey (1989) says that a consequence of non-listening may be non-understanding. During the recorded lesson, the pupils seem to be confused, and not to understand the points of the teacher's intension. And since they do not understand, they are uncertain and unable to come up with answers of any quality. Because the pupils are only searching for answers to satisfy their teacher, they only use surface strategies. In this situation the open problems are so vague that the pupils cannot see the limits for solutions. When working on a open-ended problems, a stage of confusion is very usual. But in a well-managed case, pupils develop some understanding, and based on this they can make reasonable plans for solving the problem. Such development did not happen here. The pupils accepted non-quality solutions for all parts.

Conclusion

The crucial question is how the teacher can develop his professional skills. This teacher dared to try to change his teaching in a constructivist way by working with open problem solving, but he has not listening skills required to understand his pupils. He told the researcher in an interview after the lesson, that he was very tired and aware that something was not working, but he did not recognize what was wrong or how to change it. This seems to be a good entry point for helping him to reflect on his teaching, but still there is a way to go to develop his skills. Although he seems to know what understanding is in theory, he is not able put his knowledge into practice. And in the class he acts as if his pupils are able to read his thoughts and understand what he means.

As a rule in the classroom, the teacher is not corrected when speaking or explaining, whereas the pupils are. Some pupils may ask the teacher to explain the topic in more detail, but only very seldom will the teacher stop his presentation and ask his pupils questions and listen to their problems – usually the teacher just explains the topic once more, sometimes even using the very same words, only more slowly.

A year ago the teacher in question was on a course to learn using open-ended tasks, and he was therefore eager to show how he can implement them. But the use of open-ended tasks is a very delicate process, where a teacher should have clear objectives and ideas on how to move ahead. The teacher made some of his first attempts, and was therefore primary interested in the outer form of open approach, i.e. he gave his pupils freedom to solve something that has multiple answers. It seemed that he did not have a clear mental image of what he expected his pupils to accomplish, and he was therefore not able to help them properly when they were confused and asked for his help. This lead us back to the in-service course. The teacher wants to use his new skills but it seems to be very difficult and maybe the course did not 'gave' him the skills but only appetite to do it. To change one's teaching style is a difficult process, and if it is to succeed maybe both the teacher and the researcher need to learn to understand pupils.

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TEACHING MATHEMATICS TO INUIT CHILDREN IN NUNAVIK: TAKING INTO ACCOUNT THE ENVIRONMENT AND THE CULTURE

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In the spring of 2000, the Inuit community and the Kativik School Board were pondering over the difficulties encountered by students in mathematics and the measures that could be taken to help students. One significant fact that could help explain these difficulties is that Inuit students learn Inuit mathematics (for example, a base 20 numeral system) in their own language in the first three years of their schooling and then go on to study in either French or English. It would thus seem that for these students two separate and distinct universes are cohabiting: the world of day-to-day life and the "southern" mathematical world. Faced with this dual phenomenon, the instructional situation becomes highly complex: how can these two cultures be combined and accommodated in mathematics teaching situations?

In this project we call on ethnomathematical research findings (Saxe, 1991; Bishop, 1988; Gerdes, 1985…) to help us better understand the impact of culture on the learning of mathematics and to provide methodological tools, while a collaborative approach to research guides us in our work with the teachers (Bednarz, Poirier, Desgagné and Couture, 2001; Desgagné, Bednarz, Couture, Poirier and Lebuis, 2001).

The cooperation between the researcher and teachers in creating adapted teaching situations involves a planned alternation of situation development, classroom experimentation, and feedback. We believe that a triple input is essential to the development of teaching situations, namely didactics, the teachers' experiential knowledge, and the cultural knowledge of the Inuit community. The team includes, besides the researcher, 6 inuit teachers form the Kativik School Board, 3 members of the Inuit community working as Inuit teacher educators, and curriculum development.

During this presentation, we will first talk about the environment and cultural aspects that brought the Inuit to develop their numeration system, their ways of measuring (length, distance, time…) and their great aptitudes in spatial representations. Then, we will discuss the current collaborative project that aims in the development of teaching situations adapted to Inuit classrooms.

THE BURDEN OF MATHEMATICS

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Damarin (2000) identifies the mathematically able as a socially marked category. Such groups are frequently ridiculed by society in general and function as communities in their own right. The purpose of this paper is to document, discuss, and give meaning (in light of Damarin's observation) to three coping behaviors used by mathematics department members at a teaching focused institution. These behaviors form the burden of mathematics. At once a weight and a badge of honor for department members, the burden functions to help members and the group maintain self-esteem and value mathematics. Moreover, it defines membership in the department, by drawing lines of distinction between the department and other groups on campus and by providing the measure for vetting new members. Day-to-day life in an 18 member department was observed during three extended site visits. These observation periods totaled twelve weeks (seven at the start of the fall semester, three in late November, and two at the start of the fall semester) and data collection focused on shadowing a new department member. All of his interactions with colleagues, students, and university staff were observed. Data included field notes, interviews with department members, and journaling.

Definition of the Burden of Mathematics

During analysis three types of behavior emerged as common to most collegial interactions within the department: (1) sassing students; (2) decrying the new professional evaluation system at the university; and (3) expressing exasperation at mathematical errors and innumeracy in the public. Following short descriptions below, these behaviors, collectively termed *the burden of mathematics*, are given cultural meaning by understanding their function in group definition and self-esteem protection.

Sassing students

Sassing students is joking disrespectfully about students' poor mathematics. One might laugh about students not knowing how many feet there are in a mile or trying to separate problem solving from reasoning. Sassing students is blowing off steam. It's a stress reliever. It is not part of professional teaching work and is never directed at known, specific individuals. Rather it's made up of stories about archetype students who fail in ways that seem unbelievably uninformed to anyone who values mathematics.

The faculty evaluation system

The site university was in the first year of a new system for faculty evaluation. It combined several scores in a weighted average to assign each faculty member a rating between 1 and 5, this used to compare faculty across departments and to determine all merit pay and performance recognition awards. Rolling their eyes, the department members would emphasize that "this is carried to one decimal place!" The content of this remark and of department members' feelings about the system is a belief that they, as the principal group on campus that should be professionally able to evaluate such a system, know it will not work and their objections have not been addressed. One department member said "there is a belief out there that 'because it's numbers, it's more accurate'. We've been fighting that for years, and now we've lost."

Public errors

Department members commonly regaled each other with stories about mathematical errors or innumeracy in the larger community. One popular story was about a trip the chair had made to K-mart where he found a bargain bin with prices marked "0.03¢ each". He filled an entire cart of merchandise and then made the checkout clerk and the store manager suffer as he explained the meaning of the decimal system. Department members also bemoaned a recent federal law specifying that carpet must be sold in square feet. Here the problem is the apparent need to *protect* consumers from having to convert between measurement systems.

Telling stories like these was a popular entertainment. Moreover, being able to do so and to appreciate their humor, as with participation in the sassing of students and the derision of the faculty evaluation system, was key to defining the department and its membership.

Cultural Significance

Self-esteem is that aspect of self that is concerned with protecting the identity. We may apply the concept to a social group just as to an individual. In each case self-esteem is often established through comparison with others (Osborne, 1996). In particular Gibbons and McCoy (1991) and Crocker, et. al. (1987) found that individuals will engage in downward social comparison whenever the self is under attack. The burden of mathematics is exactly a set of downward social comparisons in response to the attack represented by students who dislike mathematics and a general societal bias in favor of innumeracy. Further, participation in the burden and working to protecting group self-esteem define group membership.

The University defines department membership through job descriptions and hiring qualifications. Yet, the evidence of student clients and of the faculty evaluation system is that the university as a whole does not value the department for its skill with mathematics. Thus it falls to department members to give value to mathematics. Underlying each aspect of the burden is the common statement that participants are better than others because they understand mathematics. Students are laughed at because they have little mathematical skill. The faculty evaluation system is ridiculed as "silly" because *mathematicians* know it cannot work. The public as a whole is derided for not understanding the decimal system or needing protection from having to convert between measurement systems. It each case, an insider, a department member, is validated in his or her ability with mathematics by derogating innumeracy in others. Thus the department self-esteem is related to skill with mathematics and protected by taking up the burden of mathematics.

Beyond protecting self-esteem, the burden of mathematics is used to vet new department members on the basis of ability to take up the burden. It is the fact a new department member understands the humor and participates in each of these activities that offers real proof of membership. The Ph.D., the piece of paper, is a university requirement. Community membership is conferred to those who demonstrate, through the burden, being different from others (understanding and valuing mathematics) and who derogate innumeracy in general.

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MOVING BEYOND SERIAL PRESENTATIONS: CONDITIONS FOR INVOLVING STUDENTS IN REFLECTIVE DISCOURSE

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The extent of student participation in the conduct and direction of the verbal discourse in mathematics classrooms has been the focus of a considerable body of research since the release of the NCTM Standards. A question that has emerged from that body of literature is how to transform students' role in the discourse beyond simply *presenting* a variety of solutions. My study on mathematics classroom discourse focused on how teachers involve students in discussions in which solutions are justified, evaluated, and ultimately compared for their mathematical claims or qualities, which I classified as *reflective discourse* (Cobb et al., 1997).

In my review of the literature on discourse in mathematics classrooms (cf. Cobb, P., Boufi, A., McClain, K., & Whitenack, J., 1997;. Cobb, P., Wood, T., Yackel, E., & McNeal, B., 1992; Forman, E. A., Larreamendy-Joerns, J., Stein, M. K., & Brown, C. A., 1998; Lampert, M., 1990; O'Connor, M. C., & Michaels, S., 1996; Yackel, E., & Cobb, P., 1996), I identified three conditions for *reflective discourse* to occur. First, teachers provided students the opportunity to present and justify mathematical claims regarding solutions. Second, teachers aligned students with mathematical claims, by attributing ownership of claims to students and positioning other students with similar or competing claims. Third, teachers provided support for students to participate in the practices of evaluating and comparing claims.

The main research questions in my study were primarily to determine the extent to which teachers were able to create the conditions for *reflective discourse* and secondarily to determine the impact of *reflective discourse* on student engagement. In order to characterize the nature of student engagement, I coded each student turn as substantial or non-substantial. Examples of substantial turns included explanations, questions about the mathematics of a problem or a peer's solution or explanation, or comments on a peer's solution or explanation. I coded teachers' turns by their role in initiating or sustaining reflective discourse. The two main codes for teacher turns were *reformulation* and *seeking comments*. The *reformulation* code was used for teacher turns that made reference to a student's explanation or solution. Turns coded as *reformulation* were further classified according to whether they served to close down or extend a discussion.

I observed approximately ten lessons each of two seventh grade teachers who were implementing the *Variables and Patterns* unit of the Connected Mathematics Program. I selected up to two tasks for analysis from each lesson, for a total of 28 tasks. One teacher, SJ, had remarkably consistent discourse patterns across tasks while the other teacher, LR, had highly variable discourse patterns. I chose to do a microanalysis of nine episodes based on how representative or explanatory was the discourse pattern found in the episode. Due to the consistency of the patterns found in SJ's class, I was able to use one episode to characterize his practices in relation to reflective discourse. The variability in LR's lessons in part stemmed from four long discussions from fourteen lessons, each of which lasted over 50 turns and 5 minutes. I selected episodes from LR's class to analyze: the extent to which the three conditions for *reflective discourse* were present; whether LR used reflective discourse to close or extend a discussion; and the amount of substantial student engagement, especially in the long discussions.

SJ's turns coded as *reformulation* tended to close a discussion and, in general, SJ directed and controlled the flow and content of classroom discussions. Although SJ provided opportunities for students to present their solutions, he did not align students with mathematical claims nor provide support for students to evaluate and compare claims. Student participation in reflective discourse was limited to presenting solutions and explanations; evaluation and comparison of explanations and solutions were non-existent practices.

In the episodes in LR's class containing lengthy discussions, LR recruited explanations and comments on explanations, even when the correct answer was already given. On several occasions, LR was able to align students with clearly articulated claims. In these cases, students evaluated peers' claims and on two occasions students compared competing claims. During these episodes, there was a high number of turns rated as substantial engagement as well as considerable student-to-student interaction.

What is striking about the discourse patterns in the 28 tasks is the rarity of occasions when the classroom discourse went beyond serial explanations. In only a few cases did reflective discourse occur, and when it did the discussions were animated and lengthy. For the most part, the teachers met one of the conditions for conducting reflective discourse: providing the opportunity for students to present and justify solutions. The teachers to a far lesser extent explicitly aligned students with mathematical claims and did not create classroom norms that served to support the practices of evaluating and comparing claims. Although I was able to identify several episodes for each teacher in which opportunities for reflective discourse were not fully realized, these episodes highlighted the challenges for teachers to identify and articulate students' claims in real time.

This study shows that providing opportunities for students to present solutions is a necessary but not sufficient condition for moving beyond serial presentations. Students require more explicit articulation of claims and knowing their status relative to a claim. The challenges for teachers include being able to understand which aspects of explanations and solutions are worth pursuing as claims and then being able to align students' explanations with those claims in real time.

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ENHANCING MATHEMATICS TEACHING AND LEARNING FOR AT-RISK STUDENTS: INFLUENCES OF REFORM-BASED METHODOLOGIES AND MATERIALS

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Purpose

This study evolved out of a commitment to improve mathematics teaching and learning in an alternative high school and to begin to reverse the cycle of educational failure for students labeled "at-risk." Although research in teacher preparation has explored the ways in which preservice teachers learn to teach mathematics, few studies have focused on how teaching in an alternative high school interacts with and complicates this process. Paralleling the need to improve the preparation of preservice teachers to work with at-risk and other marginalized students is the need for more effective mathematics education programs for at-risk students. The purpose of this study was to investigate the influences of reform-based methodologies and materials on preservice teachers' instructional strategies and the mathematical development of at-risk students in an alternative high school.

Theoretical Framework

When students' home resources and experiences differ from the expectations on which school experiences are built (McCarthy & Levin, 1992), they are often at risk of not realizing their personal and academic promise. While the literature suggests that learners are at risk due to factors related to their socioeconomic status, family background, or community, it is more likely that learners are at risk because schools are not meeting their specific educational needs (Baptiste, 1992). For example, mathematics for at-risk learners is typically perceived as a hierarchy of skills that are learned in a particular sequence (Carey, Fennema, Carpenter, & Franke, 1995).

Instructional practices reflect teachers' conceptions which resonate with their own experiences and background (Thompson, 1984; Cooney, 1985; Ernest, 1991; Cabello & Burstein, 1995). Mathematics teachers' knowledge, conceptions, and attitudes about students and student learning impact the way in which these teachers interact with students in their classrooms (Calderhead, 1984). Because teachers act upon their expectations of students, negative teacher conceptions or low expectations for their students influence classroom practices and may adversely affect student performance (Brophy, 1985). Learners labeled as less capable than their peers are taught less mathematics and are presented with skill-oriented, direct instruction, and practice rather than conceptually-focused instruction promoting problem solving and understanding (Campbell & Langrall, 1993). Rote instruction often fails because it reinforces learners' negative selfperceptions and deprives them of cognitive stimulation (Silver, Smith, & Nelson, 1995). The current study draws inspiration from these earlier studies and seeks to draw together the findings from these works to inform practice.

Method and Data Sources

The methodological underpinning of this study is derived largely from orientations to research that draw attention to the importance of detailed qualitative fieldwork and the observation and analysis of participants in contexts (Goetz & LeCompte, 1984). Participants were selected using purposeful sampling strategies (Patton, 1990) and included five preservice secondary mathematics teachers and five alternative high school students. The preservice teachers were enrolled in a field-based mathematics methods course during the final semester of their teacher preparation program. The learning environment for the course is based upon the NCTM Standards (2000) and models inquiry-based, student-centered, collaborative learning. As part of the field experience component of the methods course, preservice teachers participate in eightweek cross-cultural pre-student teaching experiences in urban classrooms. The preservice teachers utilize activities from the Connected Mathematics Program and the Interactive Mathematics Program during the methods course and then incorporate these reform-oriented methodologies and materials into their mathematics lessons during their field experiences.

Consistent with the methodology of the study, qualitative data were triangulated via multiple sources of evidence, including: (1) observations, videotaping, and field notes of the preservice teachers' mathematics lessons; (2) three semi-structured, open-ended interviews with each participant; and, (3) collection of the preservice teachers' reflective journals, lesson plans, and student work. Qualitative data were analyzed utilizing a double coding procedure (Miles & Huberman, 1994) and major themes were developed using thematic analytic strategies (Spradley, 1979). Quantitative data included pre- and post-study tests, rubrics, and questionnaires designed to measure mathematics content knowledge, instructional practices, and attitudes. These quantitative data were analyzed by computing statistical means and standards deviations. Results obtained through quantitative analyses supported the findings from the qualitative data analyses.

Results and Conclusions

Four major themes emerged from the data analysis. While informative, these present a picture that is general in nature. In order to provide some insights into the totality of an individual participant's experience, a profile, in the form of an in-depth case study, was compiled for each participant. The profiles examine the themes in the context of the preservice teachers' and students' own experiences.

Instructional Practices

The findings suggest that, through their interactions with the reformoriented methodologies and materials in an alternative high school setting, the preservice teachers were prompted to implement a variety of reform-oriented instructional practices, including requiring students to think mathematically and reflectively rather than just practice. For example, Michelle, a preservice teacher who described herself as "strong in math" and attributed her success in mathematics to following a series of procedures, declared in her initial interview, "Teaching mathematics is giving students a set of skills they can apply to a variety of situations." Like the other preservice teachers, she recognized that the reform-oriented materials and methodologies were instrumental in influencing her instructional practices, "I found myself moving away from the traditional way of teaching math to encouraging student explanations and looking at many ways of solving and representing a problem."

Constructs of Student Knowledge and Student Learning

Throughout the study, the preservice teachers made frequent reference to student knowledge and student learning. As they acquired experience working with the students and the reform-oriented methodologies and materials, there were several changes to these constructs. In his initial interview, Luke, a preservice teacher asserted, "I expect that I will have to do a lot of watering down as far as math lessons are concerned." At the conclusion of the study Luke shared one of the most widely noted insights by the preservice teachers: the recognition that, "You hear the words "at-risk", "alternative high school", you think right off the bat, a bad situation, how can these kids learn, let alone do math, but they did understand and they showed us they're thinking. They had different ways to solve problems and used their life experiences. They didn't all learn in the same way."

Mathematics Content Knowledge

Pre- and post-study test results showed that the students gained in their mathematics content knowledge. These results were supported by interviews, observations, and student assignments. Because their prior mathematics education experiences emphasized a step-by-step, rote approach to learning, the students initially resisted engaging in conceptually-focused learning. As one student, Anthony, reported, "I'm used to just get a formula and just take the numbers and put them in. The teacher set it up for me." As a result of working with the reform-oriented curriculum materials, the students experienced growth in their knowledge and understanding of mathematics concepts. Anthony echoed the sentiments expressed by the students when he stated, "There's things I know what they mean now, I understand it and I can explain it and I kindof like it, you know? When you know what it means, it's kind of fun to do math."

Attitude toward Mathematics

Initially, the students did not possess positive attitudes toward mathematics. Selena, one of the students, declared, "Math's useless and boring. It's hard to figure out. It's different than the other subjects because it's a bunch of numbers and the others are words." Analyses indicated that the students exhibited changes in their attitude toward mathematics. It appears that the real-world connections, problem-solving orientation, and student-centered nature of the reform-oriented methodologies and materials increased positive attitudes toward mathematics. Selena shared the sentiments of the other students when she reflected, "I never learned math this way before. It's different, figuring things out. For the first time I actually liked going to math class.

The present study provides some insight into whether and how reform-based methodologies and materials can promote changes to preservice teachers' instructional practices and foster mathematics achievement among students who have been labeled "at-risk." An examination of what preservice mathematics teachers believe and do in response to a student population composed of at-risk students revealed that the alternative high school provided a unique environment for the preservice teachers to engage in a collaborative process of reexamining and challenging their conceptions of mathematics teaching and constructs of student knowledge and student learning. An important implication of this study is that a pre-student teaching experience in an alternative high school can foster preservice teachers' understandings of how at-risk students think mathematically and how to cultivate mathematical thinking in these students. By providing some insight into the complexities of mathematics teaching and learning in an alternative high school and revealing how reform-based methodologies and materials can begin to break the cycle of educational failure for at-risk students, this study is compatible with PMENA's goal to "further a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implications thereof."

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MATHEMATICS LEARNING IN A CULTURE OF SECRECY AND POSSESSION

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Jerry: He's copying me. I put it right there and he's like, he's suddenly like…
Randy: No I wasn't.

The ethical concerns held by these 10-11 year old children, are expressed in their words, actions and interactions. The purpose of this paper is to explore the ethical nature of *copying* as it arises in a mathematics classroom. Using the lens of hermeneutics to interpret student conversations, we investigate this phenomenon as part of the ethical dimension underlying interactions in a mathematics classroom.

Situating the Study

Over the past two decades, educational researchers in mathematics have widened their gaze from an almost exclusive focus on the individual learner to consideration of social and cultural contexts to develop understanding of learning within a classroom community (Cobb & Bauersfeld, 1995; Cobb, Yackel, & Wood, 1992; Confrey, 1999). A further expansion is needed to include the ethical dimensions of mathematics learning. As Maturana and Varela (1992) claim, every human act, "brings forth a world created with others…. Thus every human act has an ethical meaning because it is an act of constitution of the human world" (p. 247). Exploring the ethical dimensions of mathematics learning has implications for pedagogical practices, curriculum, and policy.

A consideration of the ethical only occurs if there appears to an assault upon the common code of honourable human interaction. It is primarily an emotional response, although we may rationally try to justify the inappropriateness of the behaviour (Maturana, 1988). These emotional responses occur frequently in the mathematics classroom. We investigate only one such phenomenon: the act of copying. As we listen to students interacting in a mathematical setting, how might we interpret their emotional response to an apparent breakdown in ethical behaviour? What determines whether an act of copying is viewed as unethical? Wherein lies the affront to human respect? What rational justification underlies their pointing to the unethical behaviour of copying? By focusing on the interactions between students, we broaden our understanding of the ethical implications of living in community. We wonder: In what ways are notions of ownership of mathematics significant in the development of the children's understandings of mathematics and the emergence of a complex learning system?

Ethical Acts of Copying

Students in a grade 4/5 class were part of an investigation into the nature of mathematical explanations formulated within the context of the classroom community. As we investigated ethical intonations in the classroom, we listened for emotional responses and common themes surrounding these responses. One such theme that emerged in a number of lessons was that of *copying*.

A familiar sight in the mathematics classroom is of the child hovering over his or her own page, shielding the work with the non-dominant arm. Who are they shielding the work from and for what purpose? In many cases, it is either to hide their work for fear of being wrong, but very often it is to hide it from the eyes of potential plagiarists. An incident of *copying* occurred early in the school year as the students shared their solutions to a magic square problem presented to them the week before.

Jerry: You're copying me.
Randy: No.
Jerry: Yeah, you said you were doing that and then that.
Randy: I almost…
Jerry: He's copying me. I put it right there and he's like, he's suddenly like…
Randy: No I wasn't.
Teacher: Just because he looks at it, last time I looked he had all the numbers in.
Jerry: No he didn't [group agreement]. He only had, he only had, he looked at, I was showing everybody my work and he looked at it and he put a 5 in there then.
Teacher: Well, if he knew the answer was 15, then all he had to do was figure out where the numbers went.
Ellen: Well, we said it was 15.
Jerry: Yeah, but you could put them in a different order. See, it's exactly the same order as mine.
Teacher: But I think right now that we're just sharing.
Ellen: You guys, it's okay.
Teacher: Yeah, we're sharing right now anyway. So…
Randy: So it's kind of hard for me to look at it when I'm like this.
Teacher: Okay.

It is obvious that the students are in conflict about the action that just occurred. Jerry accuses Randy of copying. What is implicit in his accusation? Perhaps Jerry's work or ideas have been stolen. Yet, he still possesses it, even if a copy or imitation of it was made. Is his work worth less because someone else has the same result? As Ellen said, they too had 15 in the middle of the magic square. Does he feel that he owns the mathematics that occurred as a result of doing the problem? Both Jerry and the teacher deny that copying took place. But these denials still uphold the unethical behaviour of copying. It is only after several exchanges that the teacher attempts to move on by claiming that if the act did occur, it occurred in the context of sharing—making it ethical. This view is upheld by Ellen. The culture of secrecy and possession remained firm in most of the students, but it was continuously challenged.

Throughout the remainder of the year, certain patterns emerged that seemed to regulate acceptable behaviour within the collective. Some instances of copying were challenged by students within the groups: "You copied me"; "It's the exact answer [as mine]"; and "Don't copy that." However, throughout the interactions, copying in the context of trading and checking answers seemed ethical. Assertions by students defending their behaviour challenge the culture of secrecy and possession: "Can I copy off you? Because you copied" and "I just need to know the answer to something. I'm not going to copy."

An analysis of students' conversations suggests that copying was acceptable when it occasioned mathematical understanding but not appropriate when the answers were stolen.

Jerry: I'm going to copy out your example because it's such a nice one.
Amil: Please don't.
Jerry: But it's so amazing I can't stand the intellectual science [laughter]. And now that I have your amazing knowledge I can complete my lovely task.

Jerry, who was so adamant that copying was wrong, now seemed to acknowledge the benefit of copying ideas of others. However, ownership of the idea is still Amil's in both his own eyes and in Amil's. What does Amil give up by allowing the example to be copied? He gives up sole ownership. He gives up possession of a precious secret.

What might be gained by challenging the culture of secrecy and possession in the mathematics classroom? We see that for Jerry, copying from Amil enlarged the space of the possible ways of acting and understanding. Viewing the phenomenon of copying through an ethical lens allows teachers and children to see themselves in relationships that build on the ideas of others and break down the stereotypes of individualistic learning in mathematics.

Endnote

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UNDERSTANDING-IN-DISCOURSE AS A TOOL FOR COORDINATING THE INDIVIDUAL AND SOCIAL ASPECTS OF LEARNING

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Mathematics education has long been about helping students understand mathematics. Thus the study of students' mental constructions has been seen as fundamental. Recently, however, the field has taken a discursive turn. Discourse is becoming increasingly more valued in the learning and teaching of mathematics. Indeed, some scholars have come to see the discursive processes, carried on in sociocultural contexts, as better describing the "understanding" of mathematics than more cognitively focused studies. It is through participation in a discourse community that the individual learns how to engage in meaningful mathematical activity (van Oers, 2001). The structured nature of practice in a discourse community points to the presence of important norms (Yackel & Cobb, 1996) or meta-discursive rules (Sfard, 2001) that guide and bound accepted practice. While such underlying norms and meta-rules can be made the subject of explicit discussion and negotiation (Cobb, Wood, & Yackel, 1993), it is often difficult for participants to engage in such a discussion due to the tacit nature of the norms and meta-rules (Sfard, 2001). Thus, these norms and meta-rules are often taught and learned unknowingly by interlocutors through participation in the discourse community. These learned norms and meta-rules are often seen as being appropriated to form at least some of the mental structures and habits of mind that we traditionally view as an understanding of mathematics.

Some researchers argue that social practice does not bring about learning; rather, participation in social practice is *constitutive* of learning (e.g., Lerman, 2001). Our view is that it makes no sense phenomenologically to ignore or devalue the existence of an inner mental life by reducing everything to practice or to participation in a discourse community. Clearly we do have a sense of "understanding" mathematical ideas, of remembering and executing mathematical procedures. Furthermore, a focus on participation in practice or discourse often is unsatisfactory as the sole lens for viewing mathematics learning, because knowing then becomes a property of the particular social context in which it took place, and thus there is no room for the notion of transfer (Sfard, 1998). This not only defies our sense that we carry something away with us from our social interactions, but also violates the underlying purpose of education, namely to enable people to participate appropriately in similar practices in a subsequent contexts that will unavoidably be different from the context in which the initial practice took place.

For the above reasons, we agree with Sfard (1998) and Cobb and Bowers (1999) that when studying the learning and teaching of mathematics, it is necessary to attend to both individuals' understanding and social contexts and practices. However, there are multiple approaches to attending to both perspectives. A common method for attending to both the individual and social is to accept the two perspectives as being complementary and useful for illuminating different aspects of learning and teaching. For example, a classroom event might be analyzed separately from the two different perspectives, which are then coordinated at the end (e.g., Cobb, 1996). In this case, the perspective not used recedes far into the background. An alternative approach might acknowledge that there is a reflexive relationship between individuals' understanding of mathematics and the norms and practices of the discourse community in which they participate (cf Cobb et al., 1993). In this case, the analysis might proceed using many shifts between “anthropological” and “cognitive” views that alternatively enrich the other. Neither view recedes far into the background, but they take turns as primary lenses. Regardless of which approach is used, the result is the alternate privileging of cognitive and discursive viewpoints. This is evident from the way that researchers switch lenses (sometimes quickly and often, sometimes more slowly), but never seem to see through both lenses simultaneously. Unavoidably, either the cognitive or the discursive is momentarily privileged.

We argue for the importance of a theoretical lens that coordinates the two perspectives and allows one to simultaneously view both the individual and social aspects of cognition. We borrow from Sfard’s (2001) notion of thinking as communicating, which implies that for one to think mathematically, one must be able to individually engage in the same kind of discursive activities that were modeled in outside discursive practices. We assert that while mathematical understanding—knowledge of important mathematical ideas and connections between those ideas—is important, that understanding must be accompanied by knowledge of the discursive practices that guide and bound mathematical activity in order for an individual to be able to use that knowledge in appropriate ways. Similarly, knowledge of the meta-discursive rules and norms for mathematical practice is insufficient to engage in doing mathematics, because mathematical understanding is also required. Thus, we argue that when utterances are analyzed for evidence of mathematical understanding, that analysis must also consider the particular discourse the participants believe they are engaged in. Similarly, when utterances are analyzed for fluency in discursive practices, that analysis must also consider individual mathematical understandings that enable participation in the discursive practices. Consequently, in analyzing any mathematical practice, understanding and discourse must be attended to simultaneously. We refer to this perspective as understanding-in-discourse.

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INVESTIGATING THE DEVELOPMENT OF A COMPUTATIONAL SCIENCE EDUCATION COMMUNITY

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A research study is looking at community impact on implementation of undergraduate computational science education initiatives. The community structure and interactions are being analyzed via social network theory. Preliminary findings indicate that effective utilization of ties and not merely the existence of ties is the keystone component to successful teaching and use of mathematics within other disciplinary contexts.

The increasingly quantitative nature of our society has remarkably influenced not only mathematics education but other disciplines as well. Computational science, the multidisciplinary overlap of computer science, mathematics tools or techniques and applications from other disciplines (Yasar & Landau, 2003), is working its way down from graduate degree programs and curriculum into undergraduate education (Swanson, 2003). This shift in educational focus has resulted in calls for more mathematical modeling, simulation, quantitative reasoning, and visualization in a wide variety of undergraduate courses and programs (MAA, 2004). Implementation is often impeded by the limited mathematics backgrounds of those instructors in client disciplines and the limited disciplinary experiences and knowledge of mathematics instructors.

The National Science Foundation is addressing the issue by educating educators through computational science education initiatives such as the National Computational Science Institute (NCSI) (http://www.computationalscience.org/). NCSI primarily targets post-secondary faculty and has as its objectives to develop a national community of undergraduate faculty interested in computational science education issues and to have participants incorporate computational tools, techniques, and technologies into their teaching. Faculty are reached via presentations at professional meetings and individual campuses, on-line materials, and weeklong workshops. Within this NCSI setting, the notion of *community* and its impact on faculty development is the focus of the authors' research. The questions that guide our study are: 1) What is the structure of the NCSI computational science education community? 2) What roles exist within the community? and 3) What effect does the community have on members' computational science education endeavors, within and outside the classroom?

The concept of *community* as "a social arena with limits defined by the capital – cultural, social, economic, and symbolic capital, that is valued and needed for individuals to legitimately participate with it," (Davis et al, 2003) has been successfully studied via *social network theory* in anthropology (Wellman, 1998). Within this framework, individuals involved with the NCSI endeavors are network *nodes*. Connections between nodes, called *ties,* can be differentiated into categories (White, et al, 2004) such as participant-to-participant collaborations and external collaborations. Wellman (1998) proposed that contemporary community ties are narrow, specialized relationships, contemporary communities are sparsely-knit, loosely bounded, frequently-changing networks, and that the nature of communities are effected by political, economic, and social milieus. We will be comparing our characterization of this computational science education community to these propositions as well as look at smaller subgroups and individuals' identities within the community.

Both qualitative and quantitative methodologies are being used in this study. Current data collection methods used by the authors, who also serve as the NCSI project evaluators, include workshop applications, surveys containing both qualitative and quantitative items, daily feedback, on-site observations, informal interviews, collection of participant artifacts, documentation of presentations given by project staff and participants, e-mail between collaborators, and case studies where the authors visit selected institutions to observe how computational science initiatives are actually being implemented in and beyond the classroom setting. These data collection tools provide both unique and overlapping opportunities to study individuals' placements within the network, as well as categorize existing ties. NetMiner™ software (http://www.netminer.com/NetMiner/home_01.jsp) will be used to generate network models to explore the patterns, structure and affiliations (ties) within the community. The NetMiner software, funded by an internal University Research Grant, also includes many statistical and social network tools to quantify tie types and strengths.

Preliminary findings show that there are many individuals connected to this community beyond NCSI staff and participants, including other faculty, administrators, students, software developers, etc. Strength of ties varies among these connections. Overall, our analysis supports Wellman's (1998) proposition that contemporary communities are loosely-bounded and sparsely-knit. However localized subcommunities, such as the NCSI staff, serve as specialized support structures within a more well-defined boundary. Members who do not have such small group ties find it more difficult to carry out their personal teaching objectives. On the other hand, merely having ties is not enough. It is the effective utilization of ties that seems to be the keystone component to successful teaching and use of mathematics within other disciplinary contexts. Effective utilization requires faculty to open themselves up to scrutiny and observation of others, to risk making mistakes or just "not knowing", and to have the drive to use their experiences as a springboard for improving their teaching. As we continue this study, we plan to further investigate these findings and incorporate not only the structural component focus but tie utilization information as well into our data collection and analysis.

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A DESIGN STUDY: THE DEVELOPMENT DIFFUSION AND APPROPRIATION OF MATHEMATICAL IDEAS IN MIDDLE SCHOOL STUDENTS

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Much of the reform in mathematics education advocates collaborative learning and approaches that require students to explain their mathematical ideas. The identification of factors that influence the development, diffusion, and exchange of new knowledge among middle school students who work collaboratively on thought revealing mathematical activities (Lesh et al, 2000) forms the basis of this presentation. The product of this classroom design study (Collins, 1992) is a set of multi-revised principles for teachers for diffusing innovative mathematical knowledge in classrooms.

Drawing on diffusion theory (Rogers, 1995) and communities of practice (Wenger, 1998), a set of implementation principles were developed for teachers to use in their work in encouraging the sharing of student-initiated ideas between students and establishing a collaborative inquiry practice. These implementation principles are intended to guide teachers in modifying the classroom environment to promote the spread and exchange of mathematical ideas, facts, concepts, problem solving strategies, tool usages.

The final set of principles has undergone three testing iterations. This poster explains the revision cycle of each testing iteration and lists the coupled draft of principles. The final set of principles are:

1. Group Interaction Principle Advancing the knowledge of the community through the diffusion of knowledge

is significantly influenced by students' communications and interactions with both members of their working group and members outside of their group.

2. Accessibility, Transferability, and Meaningfulness of Resources Principle The development, diffusion, and appropriation of knowledge is facilitated when a resource or resource-related practice is transferable to different applications, easily accessible, highly desirable from a student perspective, well promoted by the teacher or other students, and a good fit into a system of meaningful practices.

3. Shared Practice, Process, and Product Principle The key variable in developing communities that lead to the diffusion of knowledge is the concept of developing an interdependent system through giving students legitimate roles as part of a larger community. A sense of identity determines how students direct their attention. What one pays attention to is a primary aspect of sharing ideas. Identity shapes this process.

4. Metacognitive Approach Principle Participating in critiquing one's own thinking and the thinking of others promotes metacognition. This principle reflects the meteacognitive approach of advancing individual and classroom knowledge through reflecting on, testing, and, if necessary, revising one's own solution.

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PROBLEM CENTERED LEARNING AT THE COLLEGE LEVEL: STUDENTS' PERSPECTIVES

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This poster session will describe four students' interpretations of a Problem Solving university course that focused upon problem-centered learning. Students' ideas and solutions proved to be the focus of the course rather than procedures or solutions imposed by the instructor.

The purpose of this session is to present an interpretation of four students' beliefs and classroom actions as they participated in a university mathematics problem-solving course that used problem-centered learning as its model for teaching. (Wheatley 1991) Observations of this course found the classroom interactions to be significantly different from a lecture oriented mathematics course in which the teacher is the dominant figure – students were presenting their solutions and discussing their mathematical ideas while the instructor seemed to fade from the classroom.

Each of the classroom sessions was video recorded to accompany field notes. Immediately following each class session, the instructor shared his ideas, reflections, and thoughts concerning the class in video recorded sessions. Four students from the course were also interviewed throughout the semester. For analysis of the data, an interpretive approach was taken (Erickson 1986). The methodological stance was similar to that described by Voigt (1989) and Wood (1993) in which "detailed descriptions and interpretations of video recorded classes" (Voigt 1989, p. 28) were used to reconstruct patterns of interactions and routines.

The taken as shared beliefs and actions that operated in defining this course came to include collaboration, intellectual autonomy, and a focus upon heuristics and strategies rather than answers. The students discussed and elaborated upon problem solving, sharing solutions in class, the instructor's actions and expectations, and classroom control. This session will share the students' comments and elaborate upon the themes that emerged throughout the course and interviews.

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Teacher Beliefs

ONE TEACHER, TWO CURRICULA: HOW AND WHY DOES HER PEDAGOGY VARY?

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This paper discusses the case of one teacher, Jackie, whose instructional practices illuminate the importance of textbooks, time, and student/parent expectations in shaping pedagogy. Jackie teaches in the Plainview district, which offers parents and students a choice between a traditional mathematics sequence (the University of Chicago series) and an integrated sequence (Core Plus). Each day Jackie teaches two very different sections of accelerated eighth-grade mathematics using each of these curricular materials. Drawing from students' survey responses and classroom observations, we show that Jackie's pedagogy differs considerably between the two courses. An interview with Jackie shed light on the reasons underlying this variation. By examining one teacher who teaches differently in two curricular contexts, this paper highlights factors that contribute to teachers' enacted curricula—factors that have been understated in mathematics education research on teacher change.

Perspectives/Theoretical Framework

Taken together, bodies of literature about facilitating teacher development/change suggest that if we provide the appropriate policies, school-level supports, and professional development for teachers (Newmann & Associates, 1996), change teachers' beliefs (Cooney & Shealy, 1997; Stipek, Givvin, Salmon, & MacGyvers, 2001), or engage teachers in reflective activities while they teach with innovative curriculum materials (Grant & Kline, 2000) --then teachers' practice may become more reform-oriented. "Traditional" mathematics teaching is usually characterized as "provid[ing] clear, step-by-step demonstrations of each procedure, restat[ing] steps in response to student questions, provid[ing] adequate opportunities for students to practice the procedures, and offer[ing] specific corrective support when necessary" (Smith III, 1996, p. 390). In contrast, "reform-oriented" teaching (as defined by the NCTM *Standards* (1989; 1991; 2000) documents) has shifted the teacher's role to be that of a facilitator who selects tasks, models important mathematical actions, guides student thinking, and encourages classroom discourse.

At least three themes appear in the literature on teacher development that are relevant here. First, some of this literature attempts to outline developmental stages that teachers move through as they try to change their teaching practices (e.g., Franke, Fennema, & Carpenter, 1997; Goldsmith & Shifter, 1997; Spillane & Zeuli, 1999). Second, although researchers acknowledge that teachers' beliefs and practices are not always aligned with one another, the work on developmental stages has consistently focused on teachers with fairly traditional beliefs and practices moving toward more reform-oriented pedagogy. The question remains as to what happens when the reverse occurs: What happens when a teacher who prefers reform-oriented goals and practices is asked to teach "traditional" algebra?

Finally, this literature typically reports data collected in only one class period per day for each focus teacher. In the literature that examines elementary school teachers, the teachers have just one mathematics class each day. The literature on secondary mathematics teacher change is sparse and usually examines a few teachers in only one of their mathematics classes per day. This research design implies that a teacher's particular pedagogical stance is at least somewhat consistent throughout the teaching day, and can therefore be described and even categorized somewhere on a traditional-reform continuum. However, categorizing a teacher's pedagogical stance can become more complex in the current curricular context where both traditional and reform-oriented curriculum materials are available for school districts to adopt. For example, what happens when a teacher is asked to teach two very different curricula (one from a traditional sequence and one from a reform-oriented integrated curriculum)? The assumption that teachers would maintain a consistent teaching practice across these different contexts needs to be investigated empirically.

Our paper examines how one teacher, Jackie, who has many of the attributes that the literature indicates will support reform-oriented teaching, enacts the two different curricula she has been asked to teach. Additionally, we investigate why she uses different teaching strategies in each of her two teaching contexts. This case sheds light not only on the importance of reform-oriented curriculum materials in facilitating a reform-oriented pedagogy, but also on the importance of other factors that are often overlooked in teacher change literature: the expectations of both students and parents.

Context of the Study

The school & curricula

In the mid-1990's, the Plainview[1] school district changed its elementary mathematics program to include the NSF-funded *Investigations* curriculum in grades K-4. At that time, Plainview also began piloting the NSF-funded *Mathematics in Context* (*MiC*) curriculum in grades 5-8. Despite the strong support of most people involved in these adoptions, there was heated controversy in the community about the transition to *MiC*. In fall, 2000, the district introduced a four-year, integrated mathematics sequence using the *Core Plus* texts. Hoping to avoid the controversies that arose in the community upon the transition to *MiC*, district leaders decided to offer a choice between the traditional sequence (Algebra, Geometry, Algebra II, Pre-Calculus) and the *Standards*-based, *Core Plus* sequence that integrates algebra, geometry, pre-calculus, and statistics (referred to as "Integrated Mathematics"). Accelerated middle school students in 7th or 8th grade were also given the choice.

The Teacher

Jackie began teaching in the Plainview district with a propensity for teaching in reform-oriented ways. Having been a science major in her undergraduate program, she valued inquiry-based learning and, when she stayed home to raise her young children, found herself subscribing to NCTM teaching journals because she enjoyed reading about students' mathematical thinking. She received her mathematics teaching credential and came to Plainview to teach middle school mathematics at the time the district was adopting *MiC*. Jackie has been an advocate for mathematics reform in the district, even when it was unpopular. As a lead teacher of *MiC*, she participated in intensive professional development over several years.

Changes in Plainview's high school course offerings impacted the middle school, where accelerated students can now take Algebra I or Integrated. Jackie teaches three sections of *MiC* to 8th graders, as well as one Algebra I and one Integrated to accelerated students. Given Jackie's curricular situation, we were interested in the extent to which her teaching style varies by course. She is clearly an advocate for reform and she now teaches students who have intentionally chosen either traditional or reform-oriented curricular materials. In this intriguing context, we pose the following research questions: What is the nature of Jackie's instruction in each class? How is her teaching similar/different? What are some factors that influence her decisions about what and how she teaches in each class?

Data Collection and Analysis

Jackie's Algebra I and Integrated students were surveyed each fall for the past four years (n=121 in Algebra I (a sampling of over 80% of her students) and n=82 in Integrated (over 90% of her students)). Students were asked a variety of questions intended to document their experiences in their courses, as well as their reactions to the courses (see the first column in Table I for some of these survey questions). The survey questions were designed to reflect some of the primary shifts advocated by the NCTM *Standards* and the *Core Plus* texts. The question response options were based on survey items from the National Assessment of Educational Progress, which asks students to report how often various activities occur in the classroom:

"Almost every day," "once or twice a week," "once or twice a month," or "never or hardly ever."

To add to the survey data, two researchers observed Jackie's Integrated and Algebra classes for five consecutive days in March 2004. Two purposes for the classroom observations were: a) to see whether the differences reported by students were evident, and b) to capture more detailed examples of instructional similarities and differences between the two classes. One researcher took extensive field notes on the classroom interactions. The other researcher coded for various activities occurring in the classroom at 15-second intervals, following the observation system described by Foegen and Lind (2003). In coding activities for individual and small group work, a randomly selected male and female were observed, with the students reselected every 5 minutes.

The coding categories were limited to the differences that were statistically significant in the student surveys (see left column of Table 2 for the coding categories). The choice of code made by the observer was based on behaviors that were defined by the researchers prior to the observations. For example, predefined distinctions were made amongst "teacher lecture", "question and answer", and "discussion" based on the talk that occurred in the whole group activities. Talk was coded as "teacher lecture" when a teacher monolog took place. "Question and answer" took place when the talk moved back and forth between the teacher and the students, often occurring in a Initiation-Respond-Evaluate format (Mehan, 1979). The interactions were coded as "discussions" when students expressed new or novel ideas and these became the discussion topic and/or more exploratory talk was happening as students tried to make sense of ideas and the teacher tried to understand their thinking about a problem (Nystrand, 1995). Distinguishing between "question and answer" vs. "discussion" was important because literature on teacher change contends that student thinking is central to teachers who are changing their practices.

The student survey data were analyzed in SPSS. Specifically, the four ordinal response categories (daily, weekly, monthly, never) were assigned values (1-4), and then two-tailed t-tests were used to compare the means of Algebra and Integrated students in order to determine significant similarities and differences in Jackie's instruction.

To analyze the coded observational data, the percentages of time spent in various instructional activities were compared between the two courses. A 1-tailed[2] t-test was used to compare the mean percentages of time for each activity over the five days observed, using class periods as the unit of analysis. The t-test results are provided only as indicators of the strength of the Algebra-Integrated instructional differences, as opposed to suggesting that the differences would exist throughout the year, or for teachers other than Jackie. The field notes added nuance to our understanding of the similarities and differences in the two classrooms.

After analyzing the data, the results were reported to Jackie. The three researchers then interviewed her about the findings. The researchers took notes during the interview, compiled and then individually analyzed them for salient themes before coming back together to agree upon the main influential factors reported by Jackie.

Results

Results are reported for three data types: student survey, observational, and teacher interview data. The student surveys and observations shed light on the similarities in, and differences between the instruction occurring in Jackie's Algebra and Integrated classes. Jackie's interview data illuminates underlying causes of the patterns found.

Student Survey Data

There were statistically significant differences between Jackie's Algebra and Integrated students on four survey questions: frequency of group work, teacher lecture, calculator use, and students working more than ten minutes on a single problem (see Table 1). While every Integrated student surveyed indicated that *students worked in groups* almost every day, only 3% of Algebra students indicated that group work occurred daily, and almost 80% of Algebra students indicated that group work occurred only once or twice a month or never. Similarly, *teacher lecture* was reported to be much more frequent in Algebra, with over two-thirds of the students reporting that Jackie lectured at the board for the majority of the class period "almost every day," whereas only 8% of Integrated students indicated this frequency of teacher lecture. While the majority of Jackie's students reported that they *use calculators* in their math class "almost every day", this percentage was 98% in Integrated and only 82% in Algebra. Finally, twice as many Integrated (23%) as Algebra (11%) students reported that they *spend more than ten minutes on a single math problem* "almost every day", with almost two-thirds of Algebra students (compared to less than one third of Integrated students) indicating that they spend such time on a problem monthly or never.

Table 1: Student Survey Responses by Course

In your math class, how often do these things happen?	Algebra N= 121				Integrated N= 82			
	Never or hardly ever	Once or twice a month	Once or twice a week	Almost every day	Never or hardly ever	Once or twice a month	Once or twice a week	Almost every day
****Students work in groups*	39%	38%	20%	3%	0%	0%	0%	100%
****Your teacher lectures at the board or overhead for most of the class period*	4%	9%	20%	67%	18%	21%	53%	8%
***Students use calculators*	0%	4%	14%	82%	0%	0%	2%	98%
**** You spend more than 10 minutes working on a single math problem*	26%	36%	27%	11%	1%	30%	46%	23%

** Differences in means for Algebra and Integrated students are significant at p<.01 level.
*** Differences in means for Algebra and Integrated students are significant at p<.001 level.

Observational Data

We begin this section with some holistic description of the Integrated and Algebra classes we observed before presenting our analyses of the coded observational data.

When Integrated began each day, the students physically rearranged their desks so that each group of four students faced one another, rather than all of the chairs facing the front of the room as they did in Algebra. Jackie often reminded Integrated students that they were to talk about their solutions with other students. Jackie expressed frustration with two of the groups after the first observation, stating that she had tried other groupings with the hopes that some of the quieter students would interact with their peers more. In Algebra, when students worked on homework, they were asked to lower their voices and to work quietly.

During the second and third days of the week observed, Integrated students collected data for an investigation of decay and growth problems (e.g., dropping tacks on a paper plate that was divided into four equal parts). After taking roll and asking a few questions, Jackie interacted with students while they worked in small groups. On the fourth day, Jackie led a discussion about the findings of the experiments. Each group was asked to load the data they collected for one experiment into a single graphing calculator (GC) and Jackie went through a series of questions to help students recap and share their findings. Throughout this activity, more than one solution for many of the problems was discussed—sometimes Jackie requested this, other times it occurred without Jackie's elicitation. Throughout the week, students used their GCs whenever they chose to do so.

In contrast, Jackie spent the bulk of each Algebra class period at the white board in the front of the room. The lesson format was very similar to those described in traditional mathematics lessons: a) Jackie read through the solutions to the homework and answered students' questions, b) the class listened as Jackie showed them how to solve the next type of problem, and c) students worked on the assigned homework problems. Students rarely used GCs. Homework was due each day in Algebra whereas Integrated students handed in homework problems once each week.

According to our analysis of the coded observational data, there were statistically significant differences in the amount of time spent in small group work, whole-class interactions, teacher lecture, and in students' use of GCs. (See Table 2.) Specifically, whereas over one third (36%) of class time in Integrated was devoted to small group work, this occurred only 3% of the time in Algebra. Whole-class, teacher-led interactions occurred 76% of the time in the Algebra class, compared with only 37% in Integrated. During whole-class interactions there was a statistically significant difference only for teacher lecture, which occurred 39% of the time in Algebra, versus only 7% in Integrated. Finally, more student GC use occurred in Integrated than Algebra, with students using them on their own (without teacher direction) 27% of the time in Integrated, and never during the days observed in Algebra.

Despite the substantial differences, there were aspects of instruction that were similar between the two classes. In both classes, the detailed field notes suggested that Jackie stressed connections and sense making when she spoke. There were instances in both classes of Jackie taking an idea from the domain of whole numbers and connecting it to algebraic ideas. She also focused on the derivation of mathematical words in both classes.

There were no statistically significant differences in the percentages of time devoted to individual student work, teacher-facilitated questions and answers, and student/teacher discussion.[3] In fact, student/teacher discussion occurred rarely in both classes. The detailed field notes captured the nature of the whole-group interactions and during these, Jackie did the majority of the talking. While the talk moved back and forth between procedural and conceptual in both classes, rarely did Jackie ask students if they agreed or disagreed with other students' ideas or explore a unique idea in depth that was offered by a student. The difference in whole-class GC use was also not significant.

Table 2: Summary of Observational Data – Percentage of Time Spent in Various Activities

	Class	Day 1	Day 2	Day 3	Day 4	Day 5	Average
Level of Participation Structure							
Individual Work	Algebra	0%	6%	23%	0%	0%	6%
	Integrated	35%	0%	36%	0%	9%	16%
Small Group Work*	Algebra	1%	1%	13%	0%	0%	3%
	Integrated	21%	69%	38%	0%	52%	36%
Whole-Class, Teacher-Led Interactions*	Algebra	87%	91%	55%	84%	64%	76%
	Integrated	27%	23%	13%	95%	27%	37%
Nature of Whole-Class Interactions							
Teacher Lecture**	Algebra	24%	42%	43%	55%	30%	39%
	Integrated	4%	6%	11%	15%	.2%	7%
Teacher-Facilitated Questions and Answers	Algebra	61%	46%	11%	29%	33%	36%
	Integrated	12%	16%	1%	77%	24%	26%
Student/Teacher Discussion	Algebra	2%	.2%	0%	0%	0%	.4%
	Integrated	11%	1%	0%	2%	1%	3%
Graphing Calculator Use							
Using GC (Individual Use by Target Students)*	Algebra	0%	0%	0%	0%	0%	0%
	Integrated	42%	22%	11%	0%	62%	27%
Using GC (Teacher-Led, Whole-Class Use)	Algebra	9%	0%	0%	0%	0%	2%
	Integrated	1%	2%	0%	74%	0%	15%

* Differences in means between Algebra and Integrated classes are significant at $p<.05$ level, using 1-tailed test.
** Differences in means between Algebra and Integrated classes are significant at $p<.01$ level, using 1-tailed test.
Table note: The percentages of time spent in the three levels of participation do not sum to 100% because of non-instructional tasks such as taking roll.

When examining the differences in both the student survey and observational data, one is struck by the differences in key aspects of the role of students and teachers in these classrooms. In Algebra, Jackie appears to lecture the majority of the time, group work is rare, and students infrequently work for an extended period of time on any one mathematics problem. In the Integrated class, the scene is dramatically different on these dimensions.

Interview Data

When delving into why these differences exist, the most important information source is Jackie, herself. Jackie is fully aware that she used a different pedagogical style in each class and she described the students' responses as "accurate." During our interview, Jackie consistently revealed her agreement with the *Standards*, but also pointed to key barriers she faced:

> The Standards fit the way I like to do things (or vice versa), so it wasn't hard to adjust. One barrier is parent reaction – if they don't know what you are doing and why you are doing it, that can cause some problems. Also, it is difficult for me to redesign curriculum materials that are not Standards-based. Time is a huge factor in that, but so is the knowledge that many of the students who have chosen to take algebra make the choice because they want a traditional approach[4].

The reasons for Jackie's different practices appear to be related to at least four factors, some of which are highlighted in the above quote: a) the curriculum materials; b) time constraints; c) parental expectations; and d) students' reactions to the curricular materials.

Algebra and Integrated curricula provide different types of activities for the students to do and present the mathematical content in very different ways. The Algebra book is set up in a format that supports more traditional pedagogy: new information is given and homework problems are offered to practice. The connections between mathematical ideas are emphasized more in the Integrated materials, so Jackie finds it easier to talk about algebraic representations (i.e., tables, graphs, and equations) and how they are related. In Algebra, the ideas and representations are treated separately, "*so connections are not easily made*."

Jackie recognized that she could teach the Algebra course differently, but to do that, she would need to create a curriculum herself: "*If I wanted to teach Algebra as a Standards-based course, I'd have to design everything from the ground up…and I probably should do that, but then here are all these parents who didn't want that. So what is my obligation here?"* This quote highlights the complexities in teaching in the context of curricular choice: sometimes what the teacher thinks is better for students conflicts with parental expectations. This tension is not unfamiliar to Jackie, having been involved in the parental backlash related to *MiC*. Additionally, parental pressure can be felt through the administration because they "*leave the teachers alone and assume you are doing a good job unless a parent complains."*

Jackie knows that the parents make the decision to enroll their child in a particular course and have specific expectations about what will happen in each class. The students are also involved in this decision and Jackie recognizes that some students *"want to be told how to do it, see it, and then do it."* In fact, sometimes the intensive writing and talking involved in Integrated made it difficult to teach because students were not always motivated to be involved and sometimes resented having to write in their mathematics class.

Consistently these themes – curriculum materials, time, parent expectations, and student reactions, arise when Jackie discusses her choice of instructional practices. Although Jackie implements *Standards*-based instruction throughout most of the school day, these critical factors compel her to teach traditionally one period each day.

Discussion/Conclusion

Due to page constraints, we conclude with a description of our plans for further research about Jackie and her students. First, while the students are reporting different experiences in these two classrooms, we are still analyzing other data we have gathered to see how these experiences might impact student understandings and achievement. Second, while Jackie has identified these crucial components that influence what and how she teaches (i.e., curriculum materials, time, parents, and students), we still need to carefully examine how these themes impact her daily decisions. For example, a more detailed cross case study has been planned for next year to try to understand the influence of the curriculum materials on her teaching practices.

Third, part of this upcoming case study will also examine further her discourse patterns in each class. Although Jackie mentioned that she wished her students would talk more, she seemed dissatisfied that most of the interaction patterns in her classroom were of a question-answer format. This leads us to wonder if discourse patterns are one of the latter changes to be made, even when a teacher takes a strong reform-oriented stance about her teaching. As scholars seek ways to reform teachers' practices, political and social factors that influence teachers need to become more focal. These constraints need to be carefully examined so that we can better understand how to support teachers in their endeavors to teach in reform-oriented ways.

Endnotes

1. For more information about Plainview school district and its curricular history, see Lubienski (In Press). [2] A 1-tailed test was used because the purpose was to confirm the differences reported in student surveys. [3] However, it is important to note that the number of days examined was small, and if more days were examined, it is likely that some of these differences in these areas would be significant. [4] In this section, the italicized words are quotes from the interview with Jackie.

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TEACHING EFFICACY AND ATTRIBUTIONS FOR STUDENT FAILURE

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A qualitative case study was conducted to describe and understand the beliefs and experiences of Algebra I teachers who have high failure rates. This paper illuminates findings about teaching efficacy as related to teacher attributions for student failure. Self-efficacy was found to be closely related to teachers' attributions for student failure and, more importantly, to whether the teachers felt they could influence the factors to which they attributed student success or failure. As teacher educators challenge teachers' beliefs about the nature of mathematics and mathematics teaching, they must also help teachers to develop new sources of efficacy that are aligned with reformed practice.

The reasons for failure in high school algebra are undoubtedly complex, but there is no doubt about the seriousness of the problem: the National Center for Educational Statistics reports that a failure rate of 40-50% is typical (Mullis, et. al, 1991). Indeed, as minority students fail at significantly higher rates than Whites (Confrey, 1997), and with the increasing importance of algebraic reasoning in the job market, Bob Moses calls access to algebra a civil rights issue (Ladson-Billings, 1997; Williams and Molina, 1997).

An ethnographic study was conducted to describe and understand the beliefs and experiences of Algebra I teachers who have high failure rates. The purpose of this paper is to illuminate findings about teaching efficacy as related to teacher attributions for student failure.

Theoretical framework

A growing body of research provides evidence that teacher beliefs affect instructional practice (Thompson, 1992; Brown & Baird, 1993). Sigel (as cited in Pajares, 1992) defined beliefs as constructions of experience that are held to be true and that guide behavior. According to Pajares (1992), the broad construct of educational beliefs includes beliefs about the teacher's ability to affect students' learning (teaching efficacy), about confidence in oneself to perform certain tasks (self-efficacy), about the nature of mathematics (ontology) and the nature of mathematical knowledge (epistemology), and about causes of success or failure (attributions).

This study focuses on two belief constructs: attributions and teaching efficacy. Attributions, or causal perceptions about success or failure, may be classified according to locus of control (Weiner, 1983). Teaching efficacy refers to teachers' beliefs about their capacity to affect student performance. General teaching efficacy refers to the belief that good teachers can affect students regardless of their home environments. Teachers who have personal teaching efficacy believe that by trying hard they personally can reach even the most difficult students. Teaching efficacy is important because research shows that teachers with teaching efficacy expect and, in fact, elicit greater achievement from students (Ornstein, 1995).

Conceptual Framework

A conceptual framework was constructed to investigate teacher attributions for student failure. Secondary math teachers participating in graduate classes and staff development workshops were asked the reasons for the high failure rate in Algebra I. Their responses were consistent with the research on this problem and with my own experiences as an Algebra I teacher. The reasons cited by teachers were organized according to locus of control, that is, whether the factor is associated with the teacher, with the student or home environment, or with the curriculum and class structure. A concept map was developed to provide a visual display of the conceptual framework. The concept map underwent several revisions prior to the study in response to the literature, critiques and suggestions from other mathematics educators, and my reflections. Figure 1 displays the concept map at the beginning of data collection. Further modifications were made during the study, and are described in another section of the paper.

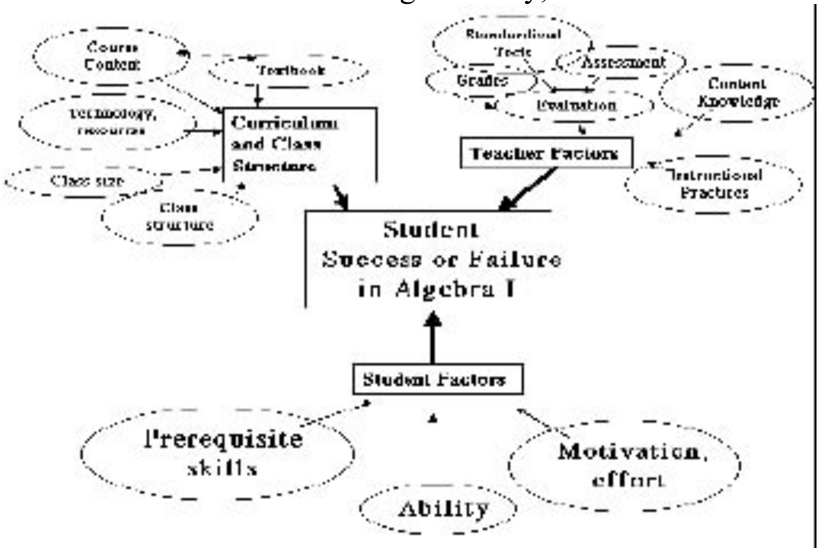

Figure 1. Factors to which teachers attribute student failure in Algebra I, organized according to locus of control.

Research Design

The research design was a qualitative case study using ethnographic techniques of (a) semistructured interviewing (b), participant observation, (c) collection of artifacts, and (d) reflective field notes. The study was conducted at a suburban high school with a diverse student population, located in a large metropolitan area in the southeast United States. The school had recently adopted an alternating-day block schedule, with four 90-minute periods daily. The high school offered four beginning algebra courses designed for students of different ability levels. It should be noted that advanced students completed Algebra I in eighth grade; students taking beginning algebra at the high school often had a history of difficulty in math classes.

Participants in the study were four mathematics teachers who taught beginning algebra courses. This paper reports results for two teachers who held very different efficacy beliefs. Beth and Chad (pseudonyms) were young and new to the field; both held masters' degrees and were familiar with the National Council of Teachers of Mathematics (NCTM) Curriculum Standards (1989).

Data were collected primarily through semistructured interviews, supported by data from classroom observations and examination of artifacts. Four interviews were conducted with each teacher; questions were planned to guide discussion, but teachers were encouraged, and in fact did, speak freely about their beliefs and practices, and their concerns about teaching and learning. All interviews were recorded and transcribed verbatim by the researcher. Data from the interviews often informed the questions planned for later interviews. Classroom observations provided data on instructional and management strategies and assessment techniques.

Text units from transcribed interviews were coded and analyzed according to a coding scheme that was initially developed from the concept map. A computer software program designed for qualitative data analysis allowed me to modify the coding scheme, assign multiple codes to a text unit, recategorize data, and search for patterns and connections. The concept map, and the coding scheme, underwent several revisions as data were collected and analyzed. In the final interview, each teacher was asked to highlight the concept map to indicate attributions for student failure, modifying the map as they saw fit. They added and deleted factors, drew lines to indicate relationships between factors, and commented on which factors a teacher could influence.

In assessing the general and personal efficacy of the teachers, I first examined the coded interview data, looking for evidence that they believed they could, or could not, impact student learning. I then examined responses to a 16-item Teacher Efficacy Scale (Gibson & Dembo, 1984), completed by each teacher during the last interview. I also analyzed the thoughts they shared during this final interview about the concept map I had developed and revised. I examined all of this data for relationships between teachers' attributions as coded in the interview data and highlighted on the concept maps, their responses to the Teacher Efficacy Scale, and the interview data coded to efficacy.

Findings

Contrary to my expectations, failure rate was not a factor in the efficacy of these teachers. In fact, none of the teachers had calculated their failure rates for the previous semester, nor had school administrators analyzed grade data. I collected from the teachers themselves the grades they had turned in to the registrar, calculated failure rates, and verified my calculations with the teachers. The different configurations of the courses and class sections made comparisons of grades by teachers or courses meaningless. Failure rates for classes ranged from 10% to 57%; Blacks failed at higher rates than Whites, and in students in lower level classes (Applied Problem Solving) were more likely to fail than students in Algebra I. Teachers were not surprised at the high failure rate in some classes, and in fact expressed greater concern that many students who received passing grades had not in fact learned the mathematics necessary to prepared them for further coursework.

During the time of this study, all four of the teachers appeared to have low general teaching efficacy, as reported on the Teacher Efficacy Scale (Gibson & Dembo, 1984); teachers were not strong in their beliefs in the ability of teachers to affect student performance. The state was in the midst of an educational reform movement, perceived by teachers to be a top-down, mandated program. Media reporting of this reasons for and potential consequences of this reform had a negative impact on teacher morale across the state.

Personal teaching efficacy, or an individual teacher's conception of his or her ability to be an effective teacher, varied for the four participants in the study. In this brief paper I focus on two of the teachers, selected because of the distinct differences in their personal teaching efficacy beliefs. A higher percentage of Beth's students than Chad's received passing grades in both Algebra I and Applied Problem Solving classes; despite this apparent success her sense of personal efficacy was much lower than Chad's.

Beth

Beth was discouraged and frequently expressed doubts about her teaching ability. She did not understand why students did not learn and retain what was taught. While recognizing that repetition was ineffective, she didn't seem to have other pedagogical strategies available to her. She felt that she ought to incorporate laboratory activities into her algebra classes, but did not know how to help students make the connections from the activities to the concepts involved. Discipline was also a factor in her decision to abandon a laboratory approach. Beth felt most successful (or least unsuccessful) with a teacher-directed lesson "when they're sitting there taking notes. Which is terrible and I hate, but…most of them listen, when you're up there talking, they sit there and pay attention and usually you have two people that are trying to nap, but…that's one of the only times I can get those 33 people in my class to be quiet and think about anything at all.” Although she felt classroom management was important to the learning process, she seemed to have no effective strategies.

Each time I visited Beth she seemed more tired and depressed by her inability to make a difference in the classroom. She even shared her feelings with students: “I'm obviously in the wrong profession since you have no idea what you're doing and we've been working on this, you know. I'm obviously not so good at my job, because nothing is getting across.”

Most of Beth’s attributions were related to student factors (motivation, immaturity) and system factors (class size, block schedule). On the concept map she highlighted almost all of the elements related to students, adding several in this area (see Figure 2). She removed classroom management, listed as a factor in the teacher’s locus of control, from the concept map, while emphasizing the student-related factors of discipline and maturity. When asked if she could do anything about any of these factors, Beth replied hesitantly that she might have some control over behavior and could try to help students with organization.

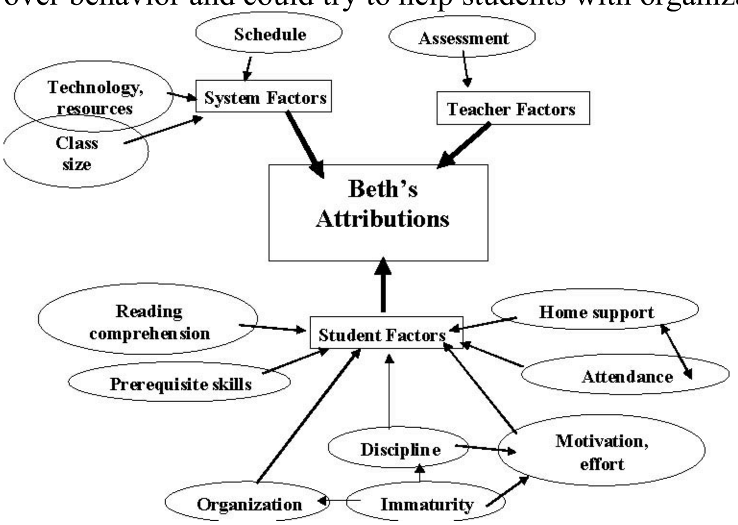

Figure 2. Beth’s attributions for student failure.

Clearly for Beth the locus of control was with the students; it was not surprising that Beth had a lack of teaching efficacy. It was unlikely that this young woman, who had only taught for two years, would remain in the profession much longer: “I know it's important. I know that there need to be good teachers, yet I don't want to do it. And I just hope that there are people that do enough to stick with it.”

Chad

Chad, an energetic young man who served as department chair, enjoyed discussing his teaching methods. Like Beth, Chad also attributed student failure predominately to factors that were related to students (see Figure 3). However, he had a great deal of personal teaching efficacy. Although Chad attributed student failure to many student-related factors, he believed a good teacher could overcome most of these. He often reflected on "what is it I can do that . . . helps them to connect" concepts, asserting, "the pedagogy really matters." For example, he added confidence to the list of student factors on the concept map, but believed that his enthusiasm and encouragement could bolster his students’ confidence in their ability. He taught organizational skills, and planned ways to help students connect new learning to existing knowledge.

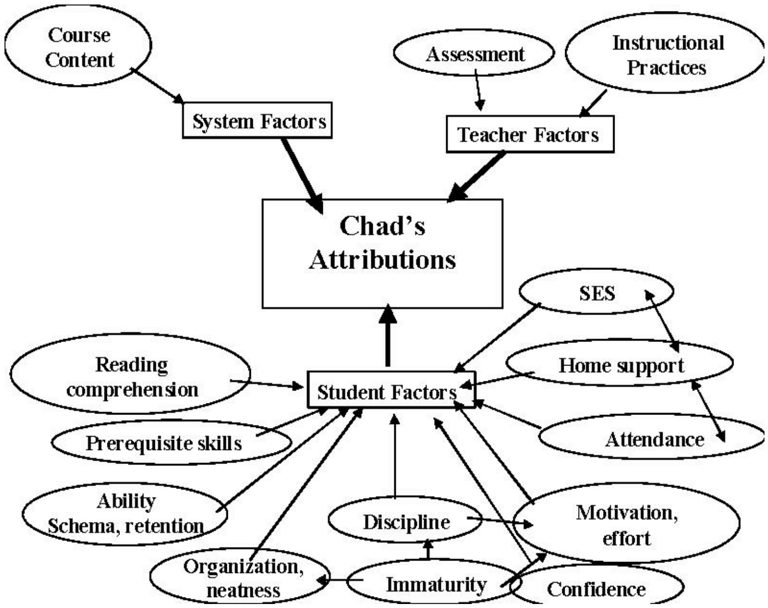

Figure 3. Chad's attributions for student failure.

This sense of control contributed to Chad's efficacy, as did the extensive time he spent creating his own instructional materials. However, the amount of time he spent at home was a major factor in his decision to leave education for computer programming at the end of the year. It was difficult to accept that someone who seemed so interested about all aspects of teaching and learning could completely leave the field of education, and I broached this subject with Chad.

Researcher: There's so much enthusiasm when you talk about teaching…Your face just lights up when you're talking about the pedagogy…

Chad: Yeah, I mean I really love it. But part of it also is just me. When I was in psychology, because I really love learning and… I knew I wasn't going to pursue that but it was just fascinating to me. I'd read on my own and I would light up talking about research or about a particular study or about cognitive development in a child . . . Then when I got into math ed at first it was just to be around kids and mentor them and, you know, be a positive influence. And then I got so into the math and the pedagogy. It's like this is just fascinating and it really is.

Researcher: So you're going to be this excited about computer programming?

Chad: Well, partly I think.

Chad was saying that he would feel efficacious in whatever he chooses to pursue. My understanding of personal efficacy was enlightened by this revelation. Was it possible that Beth, who could not commit her life to music (her undergraduate major field) and now planned to leave education for Biostatistics, would have difficulty feeling efficacious in this new venture? Or had she just not found the right match for her skills and interests? Chad seemed capable of creating, or taking control of, an environment where he felt effective. Would Beth ever find that place?

Discussion

Differences in self-efficacy were closely related to attributions for student failure and, more importantly, to whether the teachers felt they could influence the factors to which they attributed students' success or failure. Each teacher modified instructional practices in ways that enhanced personal teaching efficacy. Teaching efficacy appeared to be strongly linked to the teacher's ability to create the type of classroom environment in which he or she had experienced efficacy as a mathematics student. Beth, a disciplined student herself who found math easy to learn, envisioned a classroom where the teacher explains concepts and algorithms, and students sit quietly, listen, and follow directions. However Beth's students did not behave or learn this way, and Beth had low teaching efficacy. As a mathematics student, Chad had found it important to keep highly organized notes, to see patterns and to look for connections among concepts. His high personal teaching efficacy came from his belief that he could develop these skills in his students. That it did not work for many (as evidenced by Chad's high failure rate) did not seem to deter him. Despite the fact that both teachers were knowledgeable about the Standards, and Chad in particular spoke often of reform-based teaching, their teaching practices emphasized mastery of procedures and skills. Chad's teaching efficacy was based his ability to make choices, establish attainable goals, plan strategies for attaining them, and then achieve them. His attributions were mainly for factors he believed he could control. On the other hand, Beth's low teaching efficacy correlated with her attributions for factors outside her locus of control.

Lappan (1997) observed, "Traditional practice offers a sense of accomplishment for teachers" (p. 209). Smith (1996) discusses the strength of the traditional model of "teaching by telling" in supporting the teacher's sense of personal teaching efficacy. This model clearly defines the role of the teacher and the student, identifies a strong link between the teacher's actions and the student's learning, and provides a clear structure for daily instruction. As teacher educators challenge teachers' beliefs about the nature of mathematics and mathematics teaching, they must also help teachers to develop new sources of efficacy. According to *Principles and Standards for School Mathematics* (NCTM, 2000), effective mathematics teachers have a deep understanding of mathematics, understand their students as learners, and are able to establish a classroom environment where all students are challenged and engaged by worthwhile mathematical tasks. The authors acknowledge, "The vision for mathematics education described in *Principles and Standards for School Mathematics* is highly ambitious" (p.3). In addition to deeper content and pedagogical content knowledge, teachers must develop beliefs in teaching efficacy consistent with the vision of a mathematically rich environment where all students are engaged in problem solving, and the teacher is a facilitator of learning. In a standards-based classroom, efficacy must be derived from the teacher's ability to help all students gain mathematical power.

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THE BELIEFS AND CONCEPTIONS OF ELEMENTARY PRESERVICE TEACHERS

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Research suggests a strong relationship between what teachers believe about mathematics and teaching mathematics and the way they teach. However, we know very little about the nature of the relationship between teachers' beliefs and what they know about mathematics and how to teach it. For the past several years, we have been conducting research concerning the mathematics beliefs and conceptions of preservice elementary teachers. In this paper, we report on some of the general beliefs and conceptions of preservice teachers and those of mathematics specialists and nontraditional preservice teachers.

A substantial body of research suggests a strong relationship between teachers' beliefs and their teaching practices (Ma, 1999; Foss & Kleinsasser, 2001; Thompson, 1984, 1992; Wilson & Cooney, 2002). However, relatively little is known about the relationship between teachers' beliefs and their knowledge. Since the nature of teachers' beliefs about learning, teaching, and mathematical knowledge affects their instructional decision-making, it should therefore continue to be an integral part of research.

The overall aim of this paper is to report on research we have been conducting over the past several years concerning the mathematics beliefs and conceptions of 716 preservice elementary teachers. Specifically, in the paper we focus on three questions:

- What conceptions of mathematics and of mathematics teaching and learning do elementary preservice teachers bring to teacher education programs?
- What does taking more mathematics mean for preservice teachers' beliefs about mathematics and teaching mathematics?
- What mathematics beliefs and conceptions specifically do nontraditional students bring to teacher education programs?

The question about the amount of mathematics studied by teachers is particularly important in the preparation of elementary preservice teachers. Therefore, in addition to identifying preservice teachers' beliefs and conceptions, we have also been comparing the beliefs and conceptions of preservice teachers who had taken the minimum amount of required mathematics with those who had chosen to take an additional five or more courses of university mathematics above the minimum required for teacher certification. The study also looked at the relationship between the number of mathematics courses taken in high school and college and the preservice teachers' beliefs about mathematics and its teaching.

Identifying the beliefs and conceptions of another distinct group, nontraditional preservice teachers, is becoming increasingly important as the number of alternative certification programs, and thus nontraditional preservice teachers, is rapidly increasing. As a result of the No Child Left Behind Act's (NCLB) focus on teacher quality and mandate that all teachers in all classrooms are "highly qualified," the Department of Education has become increasingly interested at initiatives aimed at facilitating and promoting alternative routes to teacher certification and accordingly bringing a different, nontraditional group of teachers into the classrooms. Therefore, this study also has been comparing the beliefs and conceptions of traditional preservice teachers with those of nontraditional preservice teachers, students who were 23 years old or older at the time they became involved in the study.

Research Methods

The Preservice Teachers

From January 2002 through December 2003, 716 participants (about 88% female) were enrolled in one of three programs of study: (a) the regular program for prospective elementary teachers who have not chosen mathematics as their area of specialization, (b) a program for prospective elementary teachers who have chosen mathematics as their area of specialization, and (c) an alternative certification program for students who have earned a baccalaureate degree in an area other than education. A fairly small percentage of the students in the regular program had completed more mathematics than the two required college courses in mathematics for elementary teachers. The 136 mathematics specialists participating in the study had previously studied a range of mathematical topics in other courses, including calculus, linear algebra, statistics, and advanced Euclidean geometry, in addition to the two mathematics courses required of all prospective elementary teachers. The 50 alternative certification students had varied mathematics backgrounds, but about half of them had taken at least one 200 or higher level mathematics course during their baccalaureate studies.

Procedures

At the beginning of each semester, the participants completed a 75-item survey involving items related to their beliefs about mathematics teaching and learning. The survey, which used a traditional five-point Likert-type scoring format, included items from the National Assessment of Educational Progress (National Center for Education Statistics, 2001) and the *Indiana Mathematics Beliefs Scales* (IMBS) (Kloosterman & Stage, 1992). Additional items pertaining to general background information and their beliefs were also included. The items from the survey were pooled to represent four scales: (a) *Confidence in Doing Mathematics* (12 items), (b) the belief that students' *Conceptual Understanding* of mathematics is a primary goal of teaching mathematics (7 items), (c) the belief that *Effort* makes one smarter in mathematics versus the belief that one's ability is a fixed trait (7 items), and (d) the belief that mathematics is *Useful* (8 items). The reliability (Cronbach's alpha) for the four scales were as follows: .89 for *Confidence in Doing Mathematics,* .81 for *Conceptual Understanding,* .91 for *Effort,* and .82 for *Usefulness.*

Based on responses to the survey, a sample of students (over 50) with particularly "positive", particularly "negative", and relatively average beliefs about mathematics and mathematics teaching was selected to participate in in-depth interviews during the spring and fall of 2002 and spring of 2003. The interview used a semi-structured format consisting of 22 questions to look into aspects of the strengths and nature of the students' beliefs. The questions solicited information about the participants' previous educational background, feelings about mathematics, and their beliefs about the nature of mathematics and mathematics learning and teaching. During the interview, the participants also were asked to complete tasks used by Ma (1999) (specifically the tasks involved subtraction with regrouping, an error in the application of the algorithm for multiplying large whole numbers, and the relationship between area and perimeter). These tasks were used as one indicator of how the preservice teachers would deal with classroom situations involving mathematics errors and misconceptions of children. That is, the consistency of the prospective teachers' responses to the survey items were compared with their responses during the interviews in an effort to determine if what the participants claimed to be their beliefs (on the survey) were similar to the beliefs they exhibited when confronted with classroom teaching situations.

Research Results

In this section, we discuss some key results of our analyses of the survey and interview data. The discussion is organized into three sections. The first section focuses on the general results from the analysis of the survey data and addresses research question 1: *What conceptions of mathematics and of mathematics teaching and learning do elementary preservice teachers bring to teacher education programs?* The second section focuses on research question 2: *What does taking more mathematics mean for preservice teachers' beliefs about mathematics and teaching mathematics?* And, the third section focuses on the mathematics beliefs and conceptions that nontraditional students bring to teacher education programs.

General Results

In the paragraphs that follow, we report on selected results from the surveys and interviews. Note that we chose to report on those findings that we felt were often the most disconcerting, and the tone of many of the questions is negative. Primarily we report on the preservice teachers' beliefs about (a) their confidence in teaching and doing mathematics; (b) effort making one smarter in mathematics; and (c) the role of problem-based learning.

Confidence in teaching and doing mathematics

Collectively, 16% of the 716 participants did not believe that they could be effective teachers of mathematics in an elementary school, and another 23% were unsure of their ability to teach effectively. Not surprisingly, the students who were more confident in their ability to do mathematics were also more confident in their ability to teach ($r=0.68$, $p<.01$). However, many of the students reported (about 1/3 of the students interviewed) that they were only confident teaching the lower elementary grades. Perhaps most puzzling of all was the finding that about 25% of the students interviewed were not confident in their ability to do mathematics but were confident in their ability to teach.

Effort

Sixty-two percent of the students interviewed felt that "anyone can be good at it [mathematics] if they work hard," suggesting that a majority of students regard effort as the primary determinant of success in mathematics. However, 40% of the students surveyed did not believe or were unsure that "by trying, one can become smarter in math." The survey also revealed that about 34% of the preservice teachers believed that there are students who simply cannot do well in mathematics no matter how much effort they give and 26% were unsure if everyone can do well in mathematics if they try. About 22% of the students surveyed did not believe or were unsure that one's ability in mathematics can be improved by effort. Thus, it seems that these prospective teachers were divided as to the importance of effort in success in learning mathematics.

Problem-based learning

Three-fourths of the students interviewed agreed with the statement, "Elementary students remember math the best when they figure it out for themselves, and thus a good math teacher may let them struggle on a challenging problem." (Even though the interviews took place early in each semester, several of the students admitted that the methods course they were taking influenced their position on this statement.) However, on the survey, only 35% agreed that it was reasonable to expect students to "solve problems that they have not been taught solution procedures for." During the interviews, some students reported that it is okay to let elementary students struggle, but at the high school and university levels, teachers should explain more. Conversely, others felt that elementary students should not struggle, but that struggling is okay in the upper grades. For example one student reported, "Elementary teachers need to explain everything, but mathematics in college should be more challenging." So, it appears that the issue of how much students should be challenged and expected to struggle was an undecided issue for many of the participants.

Effect of Formal Mathematics Studied on Beliefs

The belief scales

The mean scores on each of the four beliefs scales (*Confidence in Doing Mathematics, Conceptual Understanding, Effort, and Usefulness*) comprising the 75-item survey were compared to determine if there were significant differences between the beliefs of those students specializing in mathematics and those not specializing in mathematics. The analyses indicated significant differences in favor of the mathematics specialists on only one of the four scales ($t=3.94$, $p<.01$), *Confidence in Doing Mathematics*, and in favor of the non-mathematics specialists on another ($t=-2.19$, $p<.05$), *Effort*. Specifically, the mathematics specialists were more confident in their ability to do mathematics, but those students who did not specialize in mathematics had stronger beliefs that effort makes one better able to learn mathematics. Note that the two groups' beliefs about the role of C*onceptual Understanding* in learning mathematics and the *Usefulness* of mathematics did not differ significantly.

Three of the four beliefs scales (*Confidence in Doing Mathematics, Conceptual Understanding, and Usefulness*) and the number of mathematics courses the students had taken in high school and college were correlated ($p<.05$) (see Table 1). The fact that the correlations between the subscales and the number of courses taken were significant ($p < .05$) indicates that the more mathematics the students had taken, the more likely they were to have positive beliefs on each of the three scales. However, despite the significant correlations, the strength of the relationship among the beliefs subscales and the number of mathematics courses taken is not nearly as strong as might be expected.

Table 1.
Correlations Among Subscales and Number of Mathematics Courses Taken in High School and College (700 □n □711)

Subscale	2	3	4	5
1. Number of Math Courses	.230**	.078*	.006	.105**
2. Confidence in Doing Math		.574**	.544**	.679**
3. Conceptual Understanding			.600**	.735**
4. Effort				.648**
5. Usefulness of math				

Note. * denotes correlations are statistically significant at $p<.05$ level.
** denotes correlations are statistically significant at $p<.01$ level.

Interesting questions

In addition to the beliefs encompassed in the four scales, we find that several of the individual questions captured interesting beliefs of these two groups of preservice teachers.

When asked about their confidence in their ability to teach, about 66% of the mathematics specialists agreed that they could be very effective teachers of mathematics at the elementary school level, and around 11% were unsure of their ability to teach. Likewise, around 61% of the non-mathematics specialists agreed that they could be effective elementary school mathematics teachers, and over 25% were unsure of their teaching ability. Note that even though the specialists appear to be more confident, the two groups' responses did not differ significantly.

Yet, when the two groups, the math specialist and non-math specialists, were asked about their beliefs about mathematics learning, they did respond significantly different ($p<.01$) to the following statement: "Memorizing steps is necessary for doing mathematics." Believing that memorizing steps is essential in doing mathematics, about 45% of the specialists and 64% of the non-specialists agreed with the statement. About 30% of the specialists and 21% of the non-specialists disagreed with the statement. Likewise, the two groups responded significantly different ($p<.01$) to the statement: "Learning mathematics is mostly memorizing facts." Again, the specialists were more likely to respond negatively to the role of memorization in learning mathematics. However, the percentage of specialists believing that memorizing plays an important role in learning mathematics was still rather high.

Beliefs of Non-Traditional Students

The belief scales

As with the mathematics specialists and the non-mathematics specialists, the mean scores on each of the four beliefs scales (*Confidence in Doing Mathematics, Conceptual Understanding, Effort, and Usefulness*) were compared to determine if there were significant differences between the beliefs of the traditional and non-traditional preservice teachers, those students (102) who were 23 years old or older at the time the survey was administered. The analyses indicated no significant differences on any of the four scales. Specifically, the two groups' confidence in their ability to do mathematics did not differ, nor did their beliefs about effort making one better able to learn mathematics, the importance of conceptual understanding in learning mathematics, and the usefulness of mathematics.

Interesting questions

Once more, we also find that several questions captured interesting beliefs of these two groups of preservice teachers.

When asked about their confidence in their ability to teach, over 61% of the traditional preservice teachers agreed that they could be very effective teachers of mathematics at the elementary school level, and about 24% were unsure of their ability to teach. Likewise, around 65% of the non-traditional students agreed that they could be effective elementary school mathematics teachers, and about 18% were unsure of their teaching ability. Again, the two groups' responses to the question did not differ significantly.

When the two groups, the traditional and nontraditional preservice teachers, were asked about their beliefs about teaching mathematics, they did respond significantly different ($p<.05$) to the following statement: "It is unreasonable to expect students to solve problems that they have not been taught solution procedures for." Over 41% of the non-traditional and 34% of the traditional students believed that students should be expected to solve problems that they have not been taught solution procedures for. And, about 30% of the non-traditional and 46% of the traditional students believed that it is unreasonable to expect students to solve problems that they have not seen before.

Discussion

Among the results we have found to date, a few are particularly noteworthy and have begun to make us rethink the mathematics content and pedagogy courses we require of prospective teachers at our university. *Studying Mathematics and the Relationship with Positive Beliefs about and Conceptions of Mathematics*

Although it is inconclusive, our data suggest that taking more mathematics seems to have some salutary effects: it enhances elementary preservice teachers' self confidence in doing mathematics, and it may promote the development of various healthy conceptions of mathematics learning and teaching, such as the belief mathematics is useful and that conceptual understanding plays an essential role in learning mathematics. These findings reaffirm Ma and Kessel's (2001) position that teachers' attitudes towards mathematics and the teaching of mathematics and their knowledge of mathematics content are interrelated. However, our results also suggest that taking more mathematics does not support the belief that success in mathematics is related more to effort than to some sort of innate ability. Rather, the results showed that those students who did not specialize in mathematics had stronger beliefs that effort makes one better able to learn mathematics. Thus, these results should be interpreted with caution for two reasons. In the first place, the fact that course taking and beliefs are related should not be interpreted as meaning that taking more mathematics alone causes healthier beliefs and conceptions. Indeed, it may be the case that these students chose to study additional mathematics because they already had positive beliefs and attitudes about mathematics. Furthermore, *what* mathematics prospective teachers study seems more important than *how many* courses they take (Ma, 1999). In our view, it would be a mistake to simply suggest that students take additional mathematics courses without also considering the nature and emphases of the courses. This position is consistent with the recommendations of a joint committee of the Mathematical Association of America and the American Mathematical Society. This committee recommends that the mathematics courses prospective elementary teachers take must be designed to give special attention to the topics of special relevance at the elementary school level and be taught in a manner consistent with an inquiry-oriented approach (Conference Board of the Mathematical Sciences [CBMS], 2001).

Teacher Education Programs Must Attend to Prospective Teachers' Beliefs and Conceptions

As we note above, an alarming number of preservice elementary teachers do not believe they can be effective teachers of mathematics. This result suggests that

preservice teachers' mathematics content and methods courses need to address more than mathematics content and pedagogy; they also need to focus on helping these prospective teachers develop healthy attitudes towards mathematics and beliefs about the nature of mathematics and how it should be taught. Even though there is growing concern in the U.S. for preservice teachers to gain adequate mathematical and pedagogical content knowledge, there should also be concern about what beliefs teacher candidates have about mathematics since these beliefs will impact not only how they will teach but also the children these prospective teachers will teach. Foss and Kleinsasser (2001) have observed: "Today, political pressure to restructure schools and concerns for quality in teaching imply that research on the culture of teacher education and the methods courses therein is as timely as research in elementary classrooms" (p. 289). As we continue with our investigation into the nature of the beliefs and conceptions about mathematics of preservice elementary teachers, we must at the same time give heed to how our programs of teacher preparation should be modified.

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PRESERVICE TEACHERS' BELIEFS ABOUT MATHEMATICAL UNDERSTANDING

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Thirteen preservice middle school mathematics teachers from a four-year teacher education program in Turkey were interviewed about their beliefs related to mathematical understanding. The analysis yielded four components of mathematical understanding with various subcomponents: Content, reasoning, applications, and procedures. The competitiveness of high school background seemed to have an effect on participants' beliefs about mathematical understanding. Participants from the most competitive high school background seemed to have richer conceptions of mathematical understanding.

Teachers' knowledge and beliefs have become a recent focus of educational research. Of particular interest is the relationship between the teachers' beliefs about the nature of and the teaching and learning of subject matter and their classroom practices (Calderhead, 1995; Koehler & Grouws, 1992; Thompson, 1992). Preservice teachers are likely to have different beliefs from inservice teachers about the nature of and teaching the subject matter since they have not met real classroom conditions yet (Haggarty, 1995). They carry their existing beliefs from precollege education to their teacher education programs (Lampert, 1990) and they learn to teach through the lenses of what they know and believe about teaching the subject matter from those years (Foss & Kleinsasser, 1996; Joram and Gabriele, 1998; Llinares, 2002). Moreover, their beliefs can affect the ways that they conduct lessons in the first few years of teaching (Feiman-Nemser, 2001) usually as a limiting factor (Cooney, 1985). Hence, preservice teachers' beliefs become an important concern in teacher education programs (Llinares, 2002). With this motivation, the present study investigates preservice middle school mathematics teachers' beliefs about mathematical understanding through asking them directly and allowing them to talk more about their own ideas.

Theoretical Background

Beliefs are considered to be a "messy construct" since many researchers have used the term to represent similar subjective constructs and provided definitions based on their own research agendas (Pajares, 1997). Thompson (1992) prefers to investigate the literature about teachers' beliefs in mathematics rather in terms of conceptions which she refers as "a more general mental structure, encompassing beliefs, meanings, concepts, propositions, rules, mental images, preferences, and the like" (p. 130). She claims that the distinction between the conceptions and beliefs might not be drastic but conceptions include many subjective terms besides beliefs. Beliefs and conceptions can be characterized as subjective in nature, likely to differ among people, free from social norms of validation and reflect differences attributable to various background and environmental factors (Thompson, 1992). Operationally, beliefs are teachers' perspectives or references that they use to analyze their teaching practice and its effect on the students (Anderson & Bird, 1995). Beliefs can have varying degrees of conviction and they are not consensual as others might have different beliefs. On the other hand, knowledge is rather objective, and has truth-value which is the same for everyone (Thompson, 1992). There are contradictory issues among the beliefs that people hold, but knowledge is true and certain.

Hence, teachers' beliefs reflect differences attributable to various background and contextual factors when compared to their knowledge.

Beliefs seem to play an important role on how teachers view and/or perceive or develop a conception of knowing, understanding and teaching the content (Calderhead, 1995). In teaching mathematics, teachers' beliefs about mathematical knowledge and understanding are found to be associated with their teaching practices (Stipek, Givvin, Salmon and MacGyvers, 2001), and with their students' beliefs about mathematical knowledge and understanding (Hiebert & Carpenter, 1992).

Teachers conceptualize teaching and learning with an eclectic collection of different beliefs that they have as a result of their experiences in classroom settings (Thompson, 1992). These experiences include their pre-college experiences and university level courses as students, as well as classroom experiences as teachers. Teachers' experiences about how to do mathematics in the school years are acquired by watching and listening to what their own teachers say and do (Lampert, 1990; Schmidt & Kennedy, 1990). Research on preservice teachers' knowledge and beliefs has concluded that the ways preservice elementary teachers have been taught mathematics, which is typically as discrete bits of procedural knowledge, affects the ways that they understand mathematics and relate the ideas to each other in their own teaching. Both preservice teachers from elementary and secondary majors generally believe that to know something in mathematics means to remember rules and to use the standard procedures without difficulty (Ball, 1990).

The nature of mathematical knowledge and understanding is generally analyzed within two main domains. The first one, conceptual knowledge, is defined to be rich in relationships and is developed through building relationships between new and existing pieces of knowledge and among the existing pieces of knowledge (Hiebert & Lefevre, 1986; Rittle-Johnson & Siegler, 1988). It seems that conceptual understanding emphasizes on awareness of one's building those relationships and how one knows what one knows as well as finding out the answers of "why" and "what" questions (Skemp, 1987). The second one, procedural knowledge, includes symbolic system, algorithms and rules such as "step-by-step instructions that prescribe how to complete tasks" (Hiebert & Lefevre, 1986, p.6), or sequences of actions for solving problems (Rittle-Johnson & Siegler, 1988). This type of knowledge and related understanding address the answer of "what" (Skemp, 1987).

Methodology

Context

The Turkish pre-college educational system that participants come from is a central system with national curriculum. The high schools in this system can be grouped into three levels of competitiveness depending on how the student population is formed in the high schools. The most competitive high schools, Exam-Track High Schools (E-HS), take their students from a competitive national exam and the teachers of these schools are also considered to be superior in terms of knowledge and skills in teaching. Most high schools in this group have one-year English preparation class. The medium competitive high schools, Foreign-Language Based High Schools (F-HS), take their students according to their middle school cumulative grade and have one year of English preparation class before the ninth grade. The least competitive high schools, Regular High Schools (R-HS), take all students regardless of their middle school cumulative grade or national exam score. Although the curriculum is the same in all high schools, there are differences in terms of the depth of the instruction and the load of class work. More competitive high schools generally have deeper instruction especially in mathematics and science compared to the less competitive high schools.

Participants

The study was conducted with the senior students of a four-year teacher education program that aims to educate mathematics teachers for the middle school grades (6^{th}, 7^{th} and 8^{th} grades). The subjects were seven female and six male fourth year students chosen among the students who volunteered for the study based on the type of the high school that they have attended. It was assumed that the competitiveness of the high school would affect participants' beliefs about the nature of mathematics and the teaching and learning of mathematics. The participants, their gender, and their background are given in Table 1.

Table 1. High School Background and Gender of Study Participants

	Exam-Track High School (E-HS)	Foreign-Language Based High School (F-HS)	Regular High School (R-HS)
Female	4	2	1
Male	2	2	2
Total	6	4	3

Participants' ages ranged from 21 to 23. These 13 students formed the 54% of the whole fourth year students. The program offered five courses on mathematics teaching, four courses on educational psychology, nine courses on mathematics, and three semesters of student teaching besides the electives. The participants had completed all requirements of the program at the time of the study.

Instrument

Semi-structured interviews including 23 main questions were conducted to investigate preservice teachers' beliefs related to mathematical understanding, teaching and learning mathematics, and high school and college settings that might have affected these beliefs. However, this report focuses on the answers to the first ten questions about the nature of mathematics and different types of mathematical understanding. The following are the sample questions from the interview considered for this study: What is mathematics for? What does it mean to know mathematics? Can you tell me what a "mathematical concept" is depending on your own ideas? How do you know that a student has understood a mathematical concept? Probing or additional questions were asked to explore the emerging issues during the interview.

Procedure

The interviews were conducted at the time and the place that subjects preferred in one-on-one settings, and took 45 minutes to one hour. The answers to the other questions were also considered during the analysis if there appeared an issue related to the focus of this study. The interviews were transcribed and similarities in the responses among all of the participants were investigated through a set of predetermined set of response types. These response types were based on an initial study with a similar focus took place in the same setting and the literature review. The results are combined under categories that include similar type of answers. All names are pseudonyms.

Results and Analysis

The answers given to the questions related to the mathematical understanding resulted in four components: Content, reasoning, application and procedures. The nature of these components and the evidence from the participants are presented below.

Content: Beyond Surface Level Knowledge

Six of the seven participants who mentioned content knowledge as being a part of what it means to know or understand mathematics emphasized that there are more things to be considered besides numbers, operations, and formulas when knowing and understanding mathematics are considered. Moreover, such knowledge can appear in rich ways, like knowing where things come from, as two of the six E-HS, three of the four F-HS, and one of the three R-HS participants mentioned. For example, Yvonne (E-HS) indicates this focus on going beyond surface level knowledge: "..knowing the meaning of the concepts, and where and what they are used for is necessary [knowledge, too]." The participants generally believed that mathematics content should not only include the surface level knowledge of rules and formulas, but also the deeper knowledge of how these formulas came out and what the concepts mean through theorems and proofs. The emphasis on how concepts are formed seems to be an evidence of relational or conceptual understanding (Skemp, 1987).

Reasoning: Logical Thinking and Multiple Views

Twelve of 13 participants expressed that understanding mathematics helps in improving the habits of thinking in general. One of these habits is thinking logically in studying mathematics or thinking about the logic that bases the concepts studied, as five of the six E-HS and one of the three R-HS participants mentioned. For example, Tom (E-HS) said that, "I think that logical thinking is fundamental to the mathematics. It involves thinking and acting logically."

Another habit of thinking is developing multiple approaches in thinking about not only mathematical tasks but also other situations. Two of the six E-HS, one of the four F-HS, and two of the three R-HS participants mentioned this view, as exemplified by Arthur (F-HS): "We probably don't realize it, but mathematics … help us in developing multiple views when we come across a situation."

The evidence that the participants sought for habits of thinking, or whether one has logical thinking and multiple approaches in studying mathematics, was one's verbal explanation of these habits of thinking and multiple approaches. Four of the six E-HS, three of the four F-HS, and one of the three R-HS participants claimed that the verbal explanation should involve one's own words and the expressions gathered in a meaningful way, as indicated by Mark (F-HS): "I would not want a student to solve me a problem, I would want him to explain me verbally [what he has done]."

The underlying issue may be that if one is logically organizing his own ideas, then he surely knows what he is doing, which seems to be related to one's awareness of what one is doing and why (Skemp, 1987). Participants considered that the specific path of reasoning that includes logical thinking and multiple views is likely to appear in verbal explanation. The verbal explanation should have a unique language, determined by one's own words and expressions, to indicate mastery in reasoning.

Application: Mathematical Dimension and Building Connections

All of the participants thought that being able to apply mathematical knowledge to other mathematical tasks or problems and to other contexts such as science and social sciences is an important task of knowing and understanding mathematics. This seems like finding the mathematics within other contexts, or building context-free relations among different types and areas of knowledge (Hiebert & Lefevre, 1986), as indicated by Gail (E-HS): "It [mathematical understanding] can be considered as adding a mathematical dimension to the concepts in other fields."

Application of mathematical ideas to other contexts also means building connections to other mathematical knowledge. This also includes the connection between the abstract and concrete knowledge and the cause-effect relationship, as two of the six E-HS and two of the four F-HS participants mentioned, as Emily (E-HS) notes: "Given a concept, if one can build connections between that concept and some other concepts, and can comment on the cause-effect relationships between them, then I can say that one has mathematical understanding."

Real-life contexts were commonly mentioned both as a special area in the application of mathematical knowledge and also in relation to the nature of mathematics, as indicated by Stacy (F-HS): "The mathematical things that we don't see make life go on. It [mathematics] is for life."

The participants believed that knowing and understanding mathematics could be seen through the skills such as applying the mathematical knowledge on other tasks and fields, building connections through various pieces of knowledge and thinking mathematics within the real-life context. The conceptions of building relationships had many dimensions and seemed to be an indication of conceptual understanding as described in the literature (Hiebert & Lefevre, 1986; Rittle-Johnson & Siegler, 1988). According to the participants, application is not simply a skill of being able to implement rules and procedures on similar situations, but being able to search for the existing mathematical ideas, rules or procedures in different contexts.

Procedures: Steps to Follow

The participants generally viewed mathematical understanding as a multi-dimensional construct. Although most of the participants could not express what conceptual understanding might be, a total of nine participants including four of the six E-HS, three of the four F-HS, and two of the three R-HS ones were able to mention a step-by-step pattern for what they believe procedural understanding might be in the context of solving a question or a problem. Mark (F-HS) was one such student; he said that understanding a procedure involving knowing, "The steps that one should follow… If we ask a question, and if the question has several steps, than one should know where to start."

If the task is a question, then having procedural knowledge allows one to decide on which steps to take to solve the problem. If it is a problem, then the participants seem to associate the step-by-step pattern with the problem solving steps, and consider the usage of such pattern a as an indication of procedural understanding. There appears to be a consistency between the participants' beliefs about procedural understanding and the way it is defined either in terms of step-by-step structure (Hiebert & Lefevre, 1986) or as problem-solving sequences (Rittle-Johnson & Siegler, 1988).

Discussion

This analysis of preservice teachers' beliefs about mathematical understanding showed that the four components include various subcomponents. In general, preservice teachers mentioned the reasoning, application, and procedural components of the mathematical understanding more often than the conceptual component. Although most of the participants could not express what conceptual understanding might be, the way they described it through student actions and task examples showed that their beliefs about conceptual understanding aligns towards how it is defined in the literature. Procedural understanding, on the other hand, was expressed through definitions and examples more fluently. Although some of the responses to the questions were clearer, some of them did not fit only one component of mathematical understanding, but had relationships to the other components or subcomponents.

High school background had some effect on the participants' views of nature of mathematics and teaching and learning mathematics. This effect seems to be a combined effect of the school type and mathematics teacher. More competitive schools have more skillful teachers and the probability of having a good mathematics teacher increases for the students of more competitive high schools. Preservice teachers from E-HS, which are the most competitive high schools, did not mention content knowledge as frequently as they mentioned the other three components (reasoning, application, and procedures). Most of F-HS participants mentioned all four components. Content knowledge was also mentioned less than the other three components by the R-HS preservice teachers, who had the least competitive high school background.

Participants generally emphasized reasoning, building connections through other mathematical ideas, and deepening the meaning of the concepts and presented richer conceptions of mathematical understanding. Preservice teachers with richer conceptions, who were mostly coming from E-HS background, did not hesitate in responding to the questions and provided deeper insights when asked to explain more. These participants seem to exhibit properties of conceptual knowledge and understanding since they mention about rich relationships between the concepts (Hiebert & Lefevre, 1986; Ma, 1999; Rittle-Johnson & Siegler, 1988) and emphasize on how the concepts are formed (Skemp, 1987).

The way preservice teachers mention real-life and other contextual applications of mathematical ideas seem to be an indication of reflective conceptions of mathematical knowledge (Hiebert & Lefevre, 1986) since they refer to the relationships among the mathematical ideas independent from the context the ideas are presented. Claims about the multiple approaches in dealing with mathematical tasks seem to provide evidence on preservice teachers' richer conceptions of mathematical understanding as Ma (1999) argues. Moreover, participants generally mentioned about the procedural knowledge and understanding in terms of step-by-step structure (Hiebert & Lefevre, 1986) and problem-solving sequences (Rittle-Johnson & Siegler, 1988).

Preservice teachers with poorer conceptions and with the least high school background, on the other hand, could not provide complete and clear expressions, especially for conceptual understanding. Hence, they were not able to provide the relationship between the conceptual and procedural understanding in terms of students' actions or hypothetical tasks. Moreover, they generally tended to give examples of measurement processes, such as pre and post testing of the students, for analyzing the students' mathematical understanding, rather than the actual student work. In general, the participants from the most competitive high schools seemed to have richer conceptions of mathematical understanding compared to the participants from the least competitive high schools.

As the teacher education research evolved through a picture of more complex relationships (see Koehler & Grouws, 1992), there appears a need to clarify the pieces in the picture. The study reported here attempts to describe preservice teachers' beliefs about subject matter, to provide preservice education and further inservice development with an insight of strength and quality of ideas that the teachers have. The results seem to confirm the previous findings about the effect of pre-college education on preservice teachers' beliefs about mathematics (Ball, 1990; Lampert, 1990; Schmidt & Kennedy, 1990) in a different context. However, more investigation with more participants from different year levels is needed to understand the differences related to the high school background and how teacher education program might have affected their beliefs. The results of this study might be considered in designing content and pedagogical content courses for preservice mathematics teachers.

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AFFECTS OF ENGAGEMENT IN REFORM-BASED PRACTICE ON A COLLEGE INSTRUCTOR'S CONCEPTIONS OF MATHEMATICS

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A case study of a college instructor's first engagement with a reform-based approach to teaching mathematics is the focus of this study. Teaching in this manner did not seem to have caused any significant, genuine changes in the participant's conceptions of mathematics. Further, because of the differences in the author's and the participant's conceptions of mathematics, there are differences in how each perceived the taught unit. The author, who has fallibilist conceptions of mathematics perceived the unit to be more successful than the participant who exhibits emerging absolutist conceptions.

Objectives

Reform practices are being advocated for by mathematics education researchers and various professional organizations involved in mathematics education, NCTM (1989; 2000) specifically. While research and literature on reform to mathematics education is prevalent, advocating of reform in traditional community colleges is less prominent. My study aims to add to the discourse surrounding the issues that relate mathematics instructors' subject conceptions and engagement with reform-based practices in mathematics education. The hope is that this research study will contribute to the goal of defining and initiating reform practices in traditional community college mathematics classrooms. Students need to engage in the *doing* of mathematics in addition to learning what has been done. Of course, students will not have the opportunity to engage in a reformed method of mathematics education if their teachers do not teach mathematics in this way. And, teachers will not teach mathematics in this way if they are not exposed to a broadened perspective of what mathematics *is* and what it means to *do* mathematics. For this reason, my research addresses the question: How are a college instructor's conceptions of mathematics and mathematics education affected by their engagement in reform-based practice in mathematics education?

Theoretical Framework

NCTM (1989; 2000) recommends fundamental changes take place in mathematics education as it is traditionally practiced. However, before substantial systemic changes occur, teachers' beliefs about the nature of mathematics need to be addressed. Legitimate reform cannot take place in mathematics education until there is fundamental change in teachers' views about the nature of mathematics. Teachers' personal philosophy of mathematics impact what teachers do in the classroom (Edward and Roberts, 1998; Ernest, 1989; Roulet, 1998; Thom, 1973; Thompson, 1992). As Thom (1973) articulates, "all mathematics pedagogy, even if scarcely coherent, rests on a philosophy of mathematics." This does not imply that changing teachers' conceptions of mathematics is sufficient to affect changes towards reform practices, but it is necessary for its commencement (Battista, 1994).

Absolutist conceptions of mathematics are characterized by accurate results and infallible procedures. Within this conception, mathematics teaching involves presentation of concepts and procedures with students acting as passive recipients of mathematical knowledge. Students are then presented with opportunities to identify concepts and perform procedures (Ernest, 1998; Thompson, 1992).

Conversely, *fallibilist* views arise from "sociological analysis of mathematical knowledge based on the ongoing practice of mathematicians" (Thompson, 1992). Unlike absolutist views, fallibilists conceive of mathematics as a mental conglomeration of ideas, through social construction, that involve conjectures, proofs, and refutations, the results of which are validated by the contemporary social and cultural milieu (Ernest, 1998; Hersh, 1997; Lakatos, 1976; Thompson, 1992). The conception of mathematics teaching that follows from this view is one that engages students in meaningful activities spurred by problem situations that require reasoning, creative thinking, discovering, inventing, communicating, collaborating, and social validating (Thompson, 1992). Knowledge of concepts and procedures then, are valued only to the extent to which they are applicable in these meaningful activities.

Recognizing that the absolutist view has been predominant in traditional practices in mathematics instruction, Thompson (1992) makes a case for the denouncement of this misrepresentation of mathematics, and a movement toward fallibilist views through a process of changing teachers' conceptual understandings of the nature of mathematics. This process is necessary to facilitate mathematics education reform (Battista, 1994). Though, analyses of research of the nature of the relationship between beliefs and practice suggest that this relationship is complex, dynamic, and dialectic rather than causal (Thompson, 1992).

Ernest (1989) identifies three philosophies of mathematics. In the *instrumentalist view*, mathematics is an accumulation of facts, rules and skills to be used in the pursuance of some external end – a set of unrelated but utilitarian rules and facts. Still within the sphere of absolutism is the *Platonist view* that mathematics is a static but unified body of certain knowledge – mathematics is discovered not created. Contrary to these two views, the *problem-solving view* recognizes mathematics as a dynamic, changing social creation, a cultural product. Mathematics in this conception is a process of inquiry; its results remain open to revision, as opposed to a finished product (Ernest, 1989; Hersh, 1997; Lakatos, 1976).

Associated to each of these views there is a generalized representation of the role of the teacher. The instrumentalist views their role as that of an instructor emphasizing mastery of skills, whereas the Platonist is an explainer disseminating knowledge for the student to receive. The constructivist (problem-solving view) however, perceives their role as that of a facilitator with students actively constructing their own knowledge. They encourage student exploration and autonomous pursuit of their own interests (Ernest, 1989).

Researchers and professional organizations, such as NCTM, advocate a reform of current practice in mathematics education from transitive methods of instruction to those of a social constructivist approach (Roulet, 1998). Various associations and organizations (NCTM, 1989; OAME/OMCA, 1993) conclude that traditional mathematical practices tends to be uniform in its approach – the major part of class time consisting of the teacher presenting new material while students listen and take notes. The change in this methodology being advocated for is not new. The groups interested in fostering this change have been campaigning a number of years. Romberg (1992) identifies a change in epistemology of mathematics in schools as the "the single most compelling issue in improving school mathematics" in closing the gap between espoused visions of mathematics and traditional school practices. Teachers' pedagogical choices then are seen as manifesting from their personal conceptions of mathematics (Roulet, 1998).

Research Method

Again, the nature of the relationship between beliefs and practice is complex, dynamic, and dialectic rather than causal. Practice here is inclusive of "everything teachers do that contributes to their teaching" (Simon and Tzur, 1999), including their beliefs and attitudes about math.

A case study approach (Merriam, 2001) was employed for the proposed research as it offers a means of investigating the complex, multi-variable nature of the relationship between engagement in reform-based practices and a teacher's conceptions of the nature of mathematics, in a real-life context. Qualitative research methods (Merriam, 2001) were utilized as a means of exploring this relationship; specifically, to understand and document the experiences of a college teacher who is engaging in a reform-based approach for the first time. I wanted to understand this phenomenon from a holistic perspective that captures as many dimensions of this experience as possible. I sought to understand this in deep ways that included the teacher's feelings, attitudes and explanations of her experience. Further, I wanted to know how the teacher's background, beliefs, and conceptions of mathematics and mathematics education caused their perceptions of this experience to be different, or similar, to mine.

Background and Population Sample

The Liberal Arts Program [LA] (pseudonym) offered at Carling College of Applied Arts and Technology (pseudonym) is an articulation program, with a rigorous course of study and high expectations. It is said to be a program that is *demanding yet forgiving*; demanding in that a high level of commitment and achievement is expected in this rigorous course of study, but forgiving in that the nature of the program takes into consideration that students entering this program, for numerous reasons, did not meet their academic potential in secondary environments. All efforts are made to actualize these students potential. Those students who meet the minimum two-year requirements of the program have the option of entering a third year at various universities with whom LA has articulation agreements. Coordinators, instructors, and support staff are strongly committed to this program and the success of its students.

Successful completion of a mathematics course is a requirement of all LA students in the first semester of their program. Prior to entrance, students write a *competency placement test*. Based on the results of this test, students are placed into one of four first semester mathematics courses ranging from basic (remedial) algebra to intermediate algebra to calculus. The level of sophistication and content of the mathematics taught within each course is left to the discretion of individual instructors, though voluntary collaboration transpires. Instructors make decisions based on what they gauge as students' prior understandings of mathematics, possible student demands and needs, and their own views of what mathematics should be taught within a minimal framework that is negotiated between the instructors and program coordinators. Thus, there is a high level of autonomy for instructors with regard to their curriculum development, teaching methodologies, textbook choices and manners of use, and assessment and evaluation methods. Kayla Adams (pseudonym), the study's participant, is a mathematics teacher within the LA program. No preordained criteria, such as specific beliefs or teaching practices were imposed other than her capacities within the program.

Data Collection

A full range of data collection techniques associated with qualitative research (Merriam, 2001) was utilized to explore the affects of Kayla's engagement in reform-based practices on her subject conceptions. Personal interviews were conducted with Kayla, structured on her personal reflections. Document notes, audio and video recordings of interviews, audio recordings of planning session discussions, and extensive notes recorded in logbooks were maintained so as to ensure the accuracy and integrity of this study. Field notes were taken during interviews, discussions, and during observations of Kayla in her classroom setting. All interviews and planning session discussions were transcribed, and forwarded to Kayla to ensure accuracy of what she had said. Further, Kayla was asked to certify that what was recorded reflected accurately the meanings she was trying to convey.

Kayla's was initially asked to independently reflect on, and complete a background questionnaire addressing questions related to her mathematics education and her images of mathematics to be used for analysis. This was followed by two one-and-a-half-hour semi-structured interviews, one prior to commencement of classes and one after completion of the unit. Also, Kayla and I engaged in five planning session discussions.

The first interview addressed issues concerning Kayla's mathematics education, her teaching experience, her confidence in mathematics, her conceptions of mathematics, and her mathematics teaching strategies.

Kayla and I then co-planned and co-taught a unit on fractions in her pre-algebra course, utilizing a reform-based approach. Kayla and I planned lessons collaboratively. The discussions during the planning sessions were recorded and transcribed for further analysis.

Upon completion of the unit, a follow-up questionnaire was given to the Kayla to independently reflect on the unit of study, and to revisit some of her initial beliefs about mathematics. This formed the foundation for the final interview that addressed her feelings and attitudes about the unit taught in a reform-based manner. As well, Kayla's attitudes and beliefs about mathematics, the mathematics teaching/learning process, and her confidence in teaching mathematics in this manner were revisited and reflected upon to gauge change, and possible reasons for this change, or lack thereof. Data was connected to Kayla's conceptions of mathematics, and examined for recurring ideas or themes.

Participant

Kayla was born and raised in a large urban city in Ontario, and completed all of her elementary and secondary schooling in that city's public board. Kayla remembers feeling "neutral" about her elementary math schooling, neither enjoying it, nor disliking it.

Kayla was surprised by her high achievement in grade nine algebra. This marked the point when Kayla's confidence in mathematics began to grow. Her continued success in grade ten mathematics prompted Kayla to continue her mathematics schooling beyond this minimum required for graduation. Kayla's success prompted her decision to do her grade thirteen maths.

In spite of alternate family expectations, and without much of their support, Kayla completed a Bachelors degree from a large, recognized university in Ontario, with a concentration in mathematics. Personal, non-academic goals, family pressure, and societal norms were factors in Kayla's decision to graduate with her three-year degree and move on to teacher training, which she completed the following summer. Kayla spent two years teaching at the secondary level before deciding to have children. Never really thinking about returning to teach at the secondary level, it was many years later that Kayla was asked to cover for an acquaintance taking leave from Carling College. Her background in mathematics helped gain Kayla a part-time teaching contract position at the College. This "evolved" into a full-time, tenured teaching position at the College, where Kayla has been teaching ever since.

Researcher's Role

As a researcher and participant in this study I moved between two roles, that of a researcher, *vis-à-vis observer as participant* and that of a *participant as observer* (Merriam, 2001). Though at times the lines between these two roles became blurred, the need to wear both a researcher and a participant's hat was essential to this study. Though this report is being written from the perspective of a researcher, I draw extensively upon my experiences as a *participant as observer*.

My role during the planning and teaching of the unit was that of a *participant as observer*. Kayla and I engaged in our planning sessions as colleagues. I acted as an "expert" in reform-based methodologies at times during our planning interactions. "Expert" here is not used in the sense that I was the bearer of professional knowledge that I transmitted to Kayla. Rather, a meta-constructivist approach was used in which I brought forth some new ideas and materials, and we collaboratively designed activities and decided how to utilize materials. Since this was Kayla's first exposure to this manner of teaching, *vis-à-vis* reform-based mathematics education, at times I acted as a mentor for Kayla and modeled ways in which this approach may be enacted upon. Further, I documented my impressions of student's reactions to the approach, and engagement with the activities. Kayla was asked to do the same. Her impressions and perceptions were analysed from her written reactions in her logbook, and her verbal reflections during subsequent discussions and the final interview.

Discussion

Kayla is a professional educator and conducts herself as such. She shows genuine concern for her students both in and out of the classroom. This includes her ongoing quest to improve her teaching. I feel that Kayla is looking for an *optimal* method for teaching mathematics, almost formulaic. This search led Kayla to a conference where she listened to "interesting speakers" that presented innovative ideas that she could add to her repertoire of teaching methods. However, Kayla states, "I don't think any of that translated into using it [the new ideas]… I would need to do something hands-on… within a classroom at the same time; so I would need to apply things." This quest for improved teaching methods is what motivated Kayla to participate in this study and engage in a reform-based approach (RBA) to teaching a unit in fractions.

Kayla notes that the differences in the RBA and her usual (traditional) way of teaching fractions were like "night and day." Particularly noticeable for Kayla was the idea of students "discovering" meanings for themselves versus placing the abstract concept of a fraction on the board and iterating, "Here's a fraction; this is what you do with a fraction." Kayla further notes, "What I normally do is very teacher-led in terms of fractions." This is not surprising as her experience with those who have taught the same, or similar courses is that "the norm is definitely teacher-led." This RBA was not only different from how Kayla has ever taught, but also from how she has ever learned mathematics.

One noticeable difference for Kayla between the RBA and her traditional approach was the amount of time needed for the unit. The RBA was not as efficient a way of teaching computation, something Kayla values as a goal of teaching fractions, as her usual method. This is discussed further below. Also, Kayla notes that her role as a teacher within the RBA was different from her usual one in which she teaches concepts in front of the class, with everyone arriving at the same answer via the same "path." In the RBA Kayla viewed her role as one of facilitating the learning. Facilitating what though is a question discussed in more detail below.

Another difference Kayla noted was that in general students enjoyed the RBA to this unit and that the RBA helped alleviate some of the "fear of fractions" that she usually notices amongst her students. She feels this is because of their more concrete understanding of what a fraction is.

A recurring area of concern for Kayla was how students are supposed to make the "leap" from concrete understandings to abstract ones. Embedded in her concern seems to be a search for how to "teach" through this leap. This indicates to me that Kayla has not fully understood the ideas of discovery learning and constructivist theories. There is reluctance on her part to render control of learning to the students. Although Kayla had been immersed in the RBA within her own classroom, this lack of understanding may be a limitation of this study. Whether this is a result of a limited engagement (one month) with the RBA, insufficient to gain a holistic perspective, or due to her conceptions of epistemological understanding of how mathematical knowledge is acquired is unclear. A longitudinal study related to this may yield greater understanding.

This issue is complicated further as Kayla and I have differing aims for teaching fractions. Kayla admittedly places much emphasis on "computation over comprehension" as her indicator of success – success here is taken as students' mastery over the material and Kayla's effectiveness as a teacher. Computational fluency is an emphasized goal of Kayla's and she questions whether the RBA will necessarily lead to this. It certainly is not the most efficient route, nor is it intended to be. For me, computational fluency as part of the automatization of operations for greater efficiency is still a goal, but a rather low priority one. Further, I feel this should only be sought after sufficient understandings of concepts have been actualized. Kayla would like to see both computational fluency and understanding occurring concurrently.

This issue seems to be a root for a further divide between Kayla and myself, that being the idea of *incorrect* answers. Kayla expresses much concern over students moving in "incorrect" directions. Kayla had to fight the urge to show students the "direct" route, i.e., the most efficient route to gaining proficiency. This reflects Kayla's lesser valuing of mistakes than myself – both as a necessary part of learning, and as part of the mathematical process that should be embraced, as opposed to stumbling blocks to be avoided. My willingness to "allow" students to continue on incorrect trajectories, producing incorrect mathematical products, stems from my perceiving of good mathematical processes of conjecturing, testing, refuting, and refining taking place in the classroom. For myself, *good* mathematical process may occur independent of *good* mathematical products. Kayla believes that as the teacher if she does not correct mistakes promptly, students will incorrectly believe something to be true. This may be due to differences in Kayla's and my conceptions of the teacher as an authoritative source of knowledge versus a facilitator of learning.

I return here to Kayla's earlier statement of the teacher's role in a RBA as a facilitator. Kayla and I have differing understandings of what this entails. My understanding of this role is that the teacher facilitates learning environments that provide students opportunities to construct knowledge. Students here build their own knowledge on previous knowledge through various means. Thus knowledge is not merely transmitted. When Kayla was asked to expand on her meaning of "facilitator," she contended that the teacher facilitates the students' getting through the curriculum. In other words, a guide as to what mathematical content students should master. Our uses of the word "facilitate" thus differ greatly.

Although by the end of the unit Kayla has begun to appreciate the use of concrete understanding more than when she began, her conceptions of mathematics as merely a final product and not necessarily the process of achieving this product remains unchanged. She also maintains utilitarian ends as her primary aims for teaching and learning mathematics. Further, she retains a dual, incoherent conception of mathematics as being both discovered and invented.

This RBA has not shifted Kayla's conceptions of what good mathematics teaching entails. Kayla views this RBA as merely pedagogically different from her traditional approach. For Kayla, good teaching includes being flexible in how you approach teaching. For her, this RBA is just part of that flexibility. Kayla doesn't seem to realize that a genuine RBA is more than simply another way of delivering curricula; that it entails a paradigm shift in terms of philosophical underpinnings, inherent aims and purposes, and broadened understandings of what mathematics is. Perhaps this is because this study did not span a long enough time to foster any significant, genuine change in her conceptions of mathematics. Or perhaps it is because these types of changes can only result from deep reflection ensuing from existing and recognized incongruence between individuals' beliefs and their personal experiences. It was hoped that Kayla's engagement with a RBA would produce this type of catalytic experience. It is conceivable that a dissonance may need time to ferment. Nonetheless, this has been a rewarding and enriching experience for myself, and cause for my own dissonance and self-reflection.

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VIEWING TEACHERS' BELIEFS AS SENSIBLE SYSTEMS

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This paper discusses theoretical assumptions either explicitly stated or implied in research on teachers' beliefs. Such research often assumes teachers can easily articulate their beliefs and that there is a one-to-one correspondence between what teachers state and what researchers think those statements mean. Research conducted under this paradigm often reports inconsistencies between teachers' beliefs and their actions. This paper explores an alternative framework that views teachers as inherently sensible rather than inconsistent beings. Through the lens of coherentism, teachers' beliefs are not seen as inconsistent; rather, researchers' interpretations of teachers' beliefs as well as teachers' abilities to articulate those beliefs are seen as problematic. Implications of such a view for research on teacher beliefs as well as for the practice of mathematics teacher education are discussed.

Numerous studies in the 1970s and 1980s focused in one way or another on describing, exploring, and explaining teachers' beliefs and possible relationships between those beliefs and the practice of teaching. In 1992, Kagan, Pajares, and Thompson each published a synthesis of research on beliefs. These three syntheses, although from slightly different vantage points, each tried to accomplish a similar goal: to portray various research agendas and the resultant research on teachers' beliefs. Kagan (1992) and Pajares (1992) discussed educational research on beliefs across disciplines; Thompson (1992) discussed primarily research in mathematics education. Pajares (1992) focused primarily on the underlying definitions of belief and belief systems necessary for quality research on teacher beliefs; Kagan (1992) focused on the variety of methodological underpinnings and implications of such research. Thompson's (1992) synthesis spanned both of these while focusing on mathematics education. All three essentially concluded that research on teacher beliefs, although fraught with pitfalls to avoid and difficulties to surmount, had great potential to inform educational research and practice and was therefore worth the effort.

Research on teachers' beliefs often takes a positivistic approach to belief structure, assuming that teachers can easily articulate their beliefs and that there is a one-to-one correspondence between what teachers state and what researchers think those statements mean. Research conducted under this paradigm often reports inconsistencies among teachers' beliefs as well as between their beliefs and their actions. This paper describes an alternative framework for conceptualizing teachers' beliefs that views teachers as inherently sensible rather than inconsistent beings. Through the lens of coherentism, teachers' beliefs are not seen as inconsistent; rather, researchers' interpretations of teachers' beliefs as well as teachers' abilities to articulate those beliefs are seen as problematic. When apparent inconsistencies arise, the framework calls for further elucidation; it calls for a deeper understanding of teachers' beliefs and a better understanding of our inferences as researchers.

Theoretical Framework: Sensible Systems of Beliefs

The word *conception* has been used by some (e.g., Lloyd & Wilson, 1998; Thompson, 1992) as a general category containing constructs such as beliefs, knowledge, understanding, preferences, meanings, and views. Educational researchers generally agree with this broad category; it is when we get down to distinguishing the members of this set that there is considerable variation (Pehkonen & Furinghetti, 2001). In particular, the relationship between belief and knowledge has been viewed in extremely different ways, although this disagreement may be more semantic that substantive (Pajares, 1992). Some choose to view knowledge as a subset of beliefs; others view beliefs as a subset of knowledge. The desire to distinguish these two constructs and yet maintain a strong relationship between them stems primarily from a desire to make this definition consistent with our everyday usage of these terms. We speak of these constructs similarly, yet differently. If there is something we claim to know for certain, such as that there are 50 states in the United States of America, it would seem odd to make the statement,

"I believe there are 50 states." Somehow, in this instance, knowing is stronger than believing (Rokeach, 1968). But that does not mean beliefs need be seen as a subset of knowledge. I have found it more useful to consider those conceptions to which we assign some truth value as beliefs, and then to refer to as knowledge a certain subset of those beliefs. How do we define that subset? Knowledge is a belief we take as fact. We may learn some conceptions as knowledge, or fact, from the beginning. Other conceptions may start out as belief and become knowledge over time. When we say we know something, we no longer state we "merely" believe it. Despite the inclusion of one within the other, it is most common in our everyday language to speak of beliefs and knowledge as separate constructs, and I will continue to do so. Although knowledge is a subset of beliefs, we tend to refer to the compliment of knowledge, rather than to the set within which it resides, as beliefs. When I use the term *belief* in this study, I am referring to the subset of beliefs we do not refer to as knowledge.

The definition of *belief* for this framework pays particular attention to the notion that what one believes influences what one does, adopting Rokeach's (1968) description: "All beliefs are predispositions to action" (p. 113). This description does not imply, however, the person holding a belief must be able to articulate the belief, nor even be consciously aware of it. It thus makes sense to discuss uncovering, discovering, and exploring one's own beliefs. In addition, a belief "speaks to an individual's judgment of the truth or falsity of a proposition" (Pajares, 1992, p. 316), but the belief may exist independently of the proposition.

In addition to inferring what teachers believe, research on beliefs often seeks to describe how those various beliefs are related to each other (often referred to as belief systems) and how these beliefs influence actions. In order to conceptualize such a system, the framework presumes individuals develop beliefs into organized systems that make sense to them. This view is informed by the philosophy of coherentism:

> Our knowledge is not like a house that sits on a foundation of bricks that have to be solid, but more like a raft that floats on the sea with all the pieces of the raft fitting together and supporting each other. A belief is justified not because it is indubitable or is derived from some other indubitable beliefs, but because it coheres with other beliefs that jointly support each other…. To justify a belief… we do not have to build up from an indubitable foundation; rather we merely have to adjust our whole set of beliefs… until we reach a coherent state. (Thagard, 2000, p. 5)

In coherentism, beliefs become viable for an individual when they make sense with respect to that individual's other beliefs. This viability via sense making implies an organization or system of beliefs, which I refer to as a sensible system. To discuss what this sensible system might look like, I turn to the works of Rokeach (1968) and Green (1971).

Green (1971) suggested three dimensions one can consider as a metaphor for visualizing a belief system. One dimension, referred to as "psychological strength" (p. 47), describes the relative importance a person might ascribe to a given belief. Both Rokeach (1968) and Green

(1971) describe this dimension as varying from central to peripheral. Assuming "the more central a belief, the more it will resist change" (Rokeach, 1968, p. 3), Rokeach introduces the idea of connectedness as a means of exploring the central or peripheral nature of a belief. Beliefs can vary with respect to the degree to which they are existential, shared, derived, or matters of taste. Existential beliefs are those we associate with our identity—with who we are and how we fit into our world. They have a high degree of connectedness and are thus more strongly held—more central. We also tend to hold more centrally those beliefs we think we share with others. If, however, a belief is derived from an association with a group, then it may be less connected and thus more peripheral in nature. Finally, "many beliefs represent more or less arbitrary matters of taste" (p. 5). These beliefs, as implied by the use of the word *arbitrary*, are less connected and thus more peripheral in nature. I find it helpful to visualize the placing of beliefs along this dimension (and each of the other dimensions as well) as a sense-making activity. Beliefs naturally go where they make the most sense to us—where they fit in.

A second dimension considers the quasi-logical relationships that may exist between an individual's beliefs (Green, 1971, p. 44). Consider the following statements:

A: Students need to learn their times tables.

B: Students should not use calculators.

Some teachers maintain there is a logical relationship between these statements. That is, for some, A implies B: IF you want students to learn their times tables THEN they should not be allowed to use calculators. And if a person believes that A implies B, and they believe that A is true, then B is seen as true because it is the logical conclusion from knowing that A is true. Green (1971) refers to the relationship as quasi-logical. Whether B does in fact follow from A is not at issue. In this person's belief system, A implies B; that is how they hold these beliefs. In this case, belief B is referred to as *derivative*, and belief A is referred to as *primary*. This quasi-logical relationship need not correlate directly with the central-peripheral dimension. That is, the same person described in the preceding example may hold belief B considerably stronger than belief A, even though belief A is a primary belief. Belief B may be much more important to the person than belief A. One of the reasons we may posit such a quasi-logical relationship is a desire to make two beliefs more coherent when considered in tandem.

A third dimension of beliefs is the extent to which beliefs are clustered in isolation from other beliefs (Green, 1971, p. 47). Beliefs seen as contradictory to an external observer are not likely to be seen as contradictory to the one holding those beliefs. In one sense, this dimension allows for the contextualization of beliefs; a person may believe one thing in one instance and the opposite in another. There are often exceptions to rules. One need not, however, be consciously aware of these beliefs. Consequently, seemingly contradictory beliefs may exist in different belief clusters with no explicit exception or delineation of context. Although not all beliefs are based on evidence (for instance, matters of taste), even those based on evidence are based on what is *seen* as evidence by the one holding the belief. In this same light, the same evidence may be used to bolster different beliefs, beliefs clustered in isolation. Thus, defining a belief to be a "conviction of the truth of some statement or the reality of some being or phenomenon especially when based on examination of evidence" (*Merriam-Webster online dictionary,* 2000) is more specific than I have chosen to be in my definition of belief. Whether a belief is "based on examination of evidence" is a question of how a belief is held; it is a question of structure.

The assumption that belief systems are sensible systems does not allow contradictions. Whenever beliefs that might be seen as contradictory come together, the person holding those beliefs finds a way to resolve the conflict within the system—to make the system sensible. As observers, we may not find the resolution sensible. It may not seem logical, rational, justifiable, or credible. But our incredulity does not diminish another's coherence. As researchers, however, it is often difficult to look beyond the beliefs we assume must have been (or should have been) the predisposition for a given action. The sensible system framework attempts to minimize these assumptions. In essence, when belief structures are viewed as sensible systems, observations of seeming contradictions are, in the language of constructivism, perturbations, and thus an opportunity to learn. Thus, teacher actions do not prove our belief inferences. When a teacher acts in a way that is consistent with the beliefs we have inferred, we have evidence that we may be on track, but we do not know what belief the teacher really was acting on at the time. When a teacher acts in a way that seems inconsistent with the beliefs we have inferred, we look deeper, for we must have either misunderstood the implications of that belief, or some other belief took precedence in that particular situation.

Evidence: Examples from the Literature

Several examples from the literature illustrate how theoretical assumptions have influenced how research on teachers' beliefs has been conducted and interpreted. In her case of Joanna, Raymond (1997) stated the following with regard to the relationship between Joanna's beliefs and her teaching practice:

> Joanna's model shows factors, such as time constraints, scarcity of resources, concerns over standardized testing, and students' behavior, as potential causes of inconsistency. These represent competing influences on practice that are likely to interrupt the relationship between beliefs and practice. (p. 567)

From the context of the article, as well as from the fact that Raymond's model only defined mathematics-related beliefs, the beliefs referred to in this last sentence are Joanna's beliefs about mathematics learning and teaching. These were defined as "personal judgments about mathematics formulated from experiences in mathematics, including beliefs about the nature of mathematics, learning mathematics, and teaching mathematics" (p. 551). The factors of time, resources, standardized testing, and students' behavior are simply described as influences; there is no mention of Joanna's beliefs with respect to these factors. Certainly Joanna has beliefs about how she should use the amount of time she is given or about what must be done in order to keep students' behavior in check. That these beliefs seemed to be more strongly held than her beliefs about learning mathematics through group work was interpreted as an inconsistency.

If instead we view Joanna's beliefs as a sensible system, the strength of Joanna's beliefs about learning mathematics through group work varies by context. In some circumstances, such as the one Joanna found herself in at the time of Raymond's research, strategies other than group work were more appropriate. This reinterpretation of the case of Joanna highlights the influence of theoretical frameworks on the analysis of research on beliefs. One need not interpret the case of Joanna as a case of beliefs being inconsistent with practice. When one defines belief systems as sensible systems, certain beliefs have more influence over certain actions in certain contexts. Joanna may have chosen to keep her students working quietly in their desks rather than working in groups because her beliefs about classroom management outweighed her beliefs about group work. If so, she was then predisposed to deal with issues of behavior management *over* issues of group work in this context. Her actions are sensible, not inconsistent, when Joanna's beliefs are viewed as a sensible system.

Raymond (1997) referred to the case of Fred (Cooney, 1985) as an example of a study that found inconsistencies between beliefs and practice. Based on coherentism, I believe there is another valid and valuable way to interpret the findings of this study. Perhaps Cooney found the meanings Fred attached to such concepts as “problem solving” and “the essence of mathematics” were different than the meanings Cooney had originally supposed. Although there is little question as to the struggle Fred had as a beginning teacher, it does not appear to be a struggle of belief. In fact, with respect to belief, the biggest struggle in this case study seemed to be similar to what others have found—the difficulty, despite an incredible amount of quality research, to get into Fred’s mind and characterize the structure of his beliefs. There is some evidence in the case of Fred to suggest that Fred’s core belief about mathematics was that mathematics is interesting in its own right. I am not sure what Fred thought “problem solving” meant, but it may have been merely a catchword he came to associate with what he enjoyed about doing mathematics. In this sense, motivating students to engage in mathematics was getting them to “problem solve”—just not in the exact same sense the researcher thought of problem solving. Thus Fred seems to have constructed a meaning for “problem solving” that differed from the intended meanings he had been taught and these two meanings differed in important ways. With this interpretation, Fred’s core beliefs are indeed manifested by his actions. Thus, the inconstancy is not between Fred’s beliefs and his practice. The inconsistency is between Fred’s practice and the beliefs Cooney thought would most likely influence that practice. As mathematics teacher educators often advocate mathematics-influenced pedagogy, it is not surprising when research presupposes that teachers’ beliefs about mathematics are the core beliefs that should influence their teaching.

This reinterpretation of the cases of Fred and Joanna is not meant to call into question the value of the research. I only mean to point out the necessity to take into account the conceptual framework for beliefs when interpreting the findings of research on beliefs. Raymond’s (1997) model only defined mathematics beliefs, defined as “personal judgments about mathematics formulated from experiences in mathematics, including beliefs about the nature of mathematics, learning mathematics, and teaching mathematics” (p. 551). Note the relationships between certain beliefs and actions implied by this definition. In addition, Raymond’s model placed Joanna in the position of being able to explicitly state her beliefs as well as the relationships between her beliefs and her teaching practice. In this sense, Raymond believed a person can not only verbally articulate their own beliefs about such complex issues as the nature of mathematics, but a person can also verbally articulate the relationships existing between their various beliefs and their teaching practices. The assumption someone can simultaneously articulate their own beliefs AND be inconsistent in their actions with respect to those beliefs is not an assumption I am willing to make. I assume, rather, when Joanna was asked to articulate her beliefs, Joanna simply took her best shot at it. I am convinced not only is it insufficient to ask someone what their beliefs are, it may be impeding. As Kagan (1992) said, “A direct question such as ‘What is your philosophy of teaching?’ is usually an ineffective or counterproductive way to elicit beliefs” (p. 66). Participants may try so hard to figure out what they are supposed to believe that their responses get in the way of sufficiently revealing what they do believe.

Skott (2001) attempted to solve the problem of viewing beliefs and practice as inconsistent by limiting the type of beliefs he studied. He did this by focusing his research on the beliefs he described as “teachers’ explicit priorities” (p. 6)—beliefs of which teachers are explicitly aware and that they can articulate. His purpose was then to study the relationships that might exist between these priorities and what takes place in the classroom. Skott focused on finding what made these explicit priorities and practices consistent rather than inconsistent. This approach is illustrated through the case of a novice teacher referred to as Christopher.

Christopher's explicit priorities with respect to teaching mathematics were that mathematics should be about experimenting and investigating, so teaching mathematics should be about inspiring students to learn independently. Much of Christopher's teaching (action) that Skott (2001) observed seemed consistent with these priorities. Christopher was seldom the center of attention at the front of the classroom and his students spent a significant amount of time working on open-ended problems in small groups. There were actions, however, that initially appeared to be inconsistent with Christopher's priorities. In particular, as Christopher moved about from group to group, Skott observed he would often use what Skott described as mathematics-depleting questioning. This kind of questioning would often replace rather than facilitate students' mathematical explorations. Rather than viewing this apparent inconsistency as something needing to be fixed, Skott tried to make sense of it. His analysis revealed there were other related yet competing priorities Christopher was attempting to manage. In particular, Christopher's priorities with respect to student learning focused on his ability to interact with as many students as possible and on each student feeling confident and successful. In light of these other priorities, Skott stated that the teaching he observed

> should not be seen as a situation that established new and contradictory priorities, but rather as one in which the energizing element of Christopher's activity was not mathematical learning. He was, so to speak, playing another game than that of teaching mathematics. (p. 24)

It turns out, as has been previously postulated, the apparent inconsistency with respect to the case of Christopher was in the researcher initially assuming Christopher's beliefs about mathematics would have the strongest influence on his pedagogical decisions. The more centrally held belief for Christopher was his belief in the importance of individuals and their need to feel successful. The importance of this belief meant mathematical beliefs sometimes took a back seat. The way Skott (2001) describes the consistency between beliefs and practice has important implications for teacher education and for future research on teachers' beliefs. It illustrates the power in searching for consistency in the participants' accounts, in viewing their beliefs as sensible systems—systems that help them to make sense of and operate in the world around them.

Conclusions & Implications

The notion of consistency is an overlooked theoretical assumption in research on teachers' beliefs. Not only is the definition of belief often glossed over, the idea of a belief system and of how this system might be related to practice is often ignored. Thus, researchers claim beliefs impact practice, then call "foul" when the beliefs they thought would most influence practice do not. Research on teachers' beliefs should focus on building coherent models of teachers' belief systems. The process of exploring and explaining apparent inconsistencies rather than merely pointing out inconsistencies facilitates a deeper understanding of the nature of beliefs and how they are held.

This understanding, in turn, has the potential to significantly influence the application of research on teachers' beliefs to the practice of teacher education. The challenge for teacher education is not merely to influence *what* teachers believe—it is to influence *how* they believe it. When it comes to making pedagogical decisions, there are certain desirable beliefs (Brouseau & Freeman, 1988) teacher educators want teachers to hold; they also want those beliefs to strongly influence practice. Coherence theory offers teacher educators a constructive approach for viewing teachers' belief systems as well as changes in those systems. By way of coherentism, teachers are seen as complex, sensible people who have reasons for the many decisions they make. When teachers' belief systems are viewed in this way, we have a basis for constructing a different type of teacher education. Teacher educators should provide teachers with opportunities to explore their beliefs about mathematics, teaching and learning. Teacher education strategies such as critiquing tradition, demonstrating by case and example, and encouraging rigorous discussion take on new meaning when beliefs are explicitly examined. In the process, teachers acquire terms and expressions requisite for ongoing, meaningful reflection on their beliefs and practice.

For example, one goal of mathematics teacher education might be to affect teachers' beliefs about mathematics such that those beliefs move high on the list of those beliefs that most influence teaching. In order to have this impact, however, teacher educators and the teachers themselves need to become aware of the beliefs that are currently filling those "most influential" roles. From this perspective, teachers' belief systems are not simply "fixed" through a process of replacing certain beliefs with more desirable beliefs. Rather, teachers' beliefs must be challenged in such a way that "desirable" beliefs are seen by teachers as the most important beliefs with which to cohere.

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THE IMPORTANCE OF BELIEFS IN DRIVING TEACHER PRACTICE

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Background

The Ontario mathematics curriculum calls for students to be actively engaged in a problems-based curriculum, while strengthening skills in arithmetic and algebra. Students are assessed in knowledge/understanding, application, communication, and thinking/inquiry/ problem solving using four levels of achievement. The reform efforts in Ontario are supported by the National Council of Teachers of Mathematics (NCTM)'s *Principles and Standards for School Mathematics* (2000), with both formative and summative assessments used. Lambdin (1998) states that there are four purposes to assessment: evaluate student achievement; monitor student progress; make instructional decisions; and evaluate programs.

Objectives

The objectives of this presentation are:

(a) discuss the factors in my study that impede or facilitate the implementation of multiple assessments when beliefs and practices are critically examined;

(b) explore, critique, and elaborate on Lambdin and Forseth's (1996) claim that "good teaching is seamless – assessment and instruction are often one and the same" (p. 298).

Perspectives/Theoretical Framework

Theories of learning and cognition view learning as a "complex process of model building" where students construct their own understanding (Lambdin, p. 98, 1998). Long and Benson (1998) argue that more emphasis needs to be placed on aligning assessment with curriculum and instruction since alignment does not occur naturally.

Raymond's "Mathematics Beliefs and Practices Model" (1997) provides a framework that illustrates the complexity of teachers' beliefs and practices. Before commencing the study, I applied Raymond's "Criteria for the Categorization of Teachers' Beliefs About the Nature of Mathematics, Learning Mathematics, Teaching Mathematics, and Mathematics Teachers' Teaching Practice" to determine where I was positioned on the continuum using her five-point scale. Edwards (2000) states that teachers who are successful in changing their practices do so through their commitment to change as well as visualizing what that change looks like.

Research Methodology/Data Sources

This qualitative study is based on action research. Berg (2001) states that action research has been used increasingly to investigate classroom teaching practices since both the researcher (educator) and subjects (students) are highly engaged in the study, and that teachers enhance their understanding between knowledge and practice that can result in making informed decisions for their students. Although various models exist, Berg states that all models view the action research process as a spiral.

This study took place in the second semester of the 2000 – 2001 school year in a grade 9 class in southeastern Ontario. A case study approach of changing my assessment practices followed a six-stage process as described by Shaw and Jakubowski (1991). Field notes were maintained to document thoughts, reactions, next steps, issues, concerns, and interpretation and analysis of the data, along with informal conversations with students.

Findings

Five factors emerged that impeded the implementation of multiple assessments:

- time (additional half-course to teach; co-curricular activities; providing feedback to journals and creating rubrics; outside responsibilities as an editor and textbook author)
- collaboration (taught in isolation)
- curriculum content (difficult to implement alternative assessments in skills-based units such as number sense and algebra; very challenging and overcrowded curriculum)
- students (lacked prerequisite content and skills; not risk-takers; weak literacy and numeracy skills; several high-needs students)
- reporting to parents (stressful to evaluate student portfolios with report card marks due within 48 hours; Parents' Night in 4th week of school did not allow for presentation of portfolios)

Three factors facilitated the implementation of multiple assessments:

- examining beliefs (use of Raymond's model before conducting the study)
- resources (ministry funding for graphing calculators, geometry software, textbooks)
- planning/organization (template to balance instruction, curriculum, and assessment; field journal for reflection)

From my study, I argue that assessment *can* affect instruction and curriculum, but it is the *curriculum content* that is the driving force.

Educational Significance

This study adds to the existing literature on classroom-based assessment and is timely to mathematics educators attempting to implement student-centred assessment. Whether teachers, as in Ontario, are mandated to align assessment with curriculum or instruction, or choose to incorporate assessments besides tests and examinations in their courses, this study will provide them with a case study of one mathematics teacher. Of particular importance is for teachers to challenge their beliefs and practices so teachers are convinced of the need to change, especially when reform uses a "top-down" approach.

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PRESERVICE TEACHERS' BELIEFS AND CONCEPTIONS ABOUT MATHEMATICS TEACHING AND LEARNING

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Teachers' beliefs and conceptions about mathematics play an important role in shaping their characteristics and efforts in the classroom (Thompson, 1992), and are, indeed, reflected in their classroom actions. Because preservice teachers are required to go through certain stages before practicing in the field, it is important to investigate their beliefs and conceptions while they are still building and forming their views, philosophies, and ways of teaching. Thus, the purpose of this study is to investigate the beliefs and conceptions of preservice teachers about mathematics and mathematics teaching and learning.

Method

This study is part of a large ongoing study of preservice teachers' beliefs, which started in the spring 2002 semester. At the beginning of each semester when the study took place, surveys were administered and interviews were conducted. The subjects, students enrolled in an elementary math methods course, completed a 75-item survey. A subsample also participated in a semi-structured interview consisting of 21 questions. During the interview, the subjects were asked to complete three scenarios used by Ma (1999). In this paper, only two students' interviews were focused on to provide in-depth information about their beliefs, and to give a clearer picture of how they perceive themselves as future teachers. The two students were selected because they were extreme cases– one was very positive about mathematics and the other very negative.

Preservice Teachers' Profiles

Tom, 23 years old, rated himself as an excellent student in mathematics and his liking of math on a scale of 1 to 10 was 9. Believing he could make a difference and teach kids some math, he felt he had "a very good grasp on concepts so that [he could] be a good teacher." Tom thought that all math teachers should have strong math skills and varieties of teaching techniques. He claimed that elementary schools should focus on problem solving rather than just computational skills. In his words, "I think problem solving is probably the way to go, if you can do the problem solving, you will be able to grasp the computational knowledge."

When asked about memorization as an aspect of mathematics, Tom argued that it is not very important, "if you figure out why you are memorizing the things that you memorize then you will be better off…[however], well memorizing speeds up the process." Further, "A peer can help and explain difficult concepts." However, in the elementary classroom, "kids might tend to just give answers instead of giving knowledge."

Emily, 23 years old, rated her math ability as below average and her liking of math on a scale of 1 to 10 was 3. Her past experience influenced her to be a teacher. She "had some really great teachers, especially in elementary school that left a really strong impression." Regarding teaching mathematics, she felt she "definitely need[s] a lot of practice because [she is] not personally good at math." Whereas she thought that it was important for teachers to have strong math skills, it was not necessary to have specific techniques because "research is so big and…you can find specific techniques that work."

Emily thought that both problem solving and computational skills are of great importance. “You have got to do both.” Moreover, memorization was not important “I do not think memorizing is a good idea.” Struggling on a concept is important. “I was not taught to think for myself and that has been a real big problem.” She continued that working with others was very important. “A student can learn more from another student. It is appropriate to work with a group when there is a new concept.”

Discussion

These in-depth interviews produced some unexpected results. Tom was an excellent student and fond of mathematics. However, despite the fact that he was proud of and pleased with his ability to understand mathematics easily, he was not able to show insightful thought while working on the tasks. In contrast, Emily was uncomfortable with mathematics and had a great fear of it. Thus, she struggled when she was asked about the tasks. She had been a weak student, striving to understand mathematics and working hard to overcome her shortcomings. She valued perseverance, yet she could barely make sense of a concept.

In spite of their individual background differences and what they think about mathematics, they have shared views on several issues, including group work, math usefulness, and the notion of problem solving. To some extent they both opposed memorization. However, they emphasized the idea that memorizing procedures may help students to some degree.

Both Tom and Emily advocated problem solving as a strategy in the teaching and learning of mathematics. However, despite the fact that Tom strongly supported teaching with conceptual understanding, he implied that the procedural method is important as well because of “we could not overlook the significance of computational procedures.” Emily thought that problem solving was the most important way of helping students to learn; however, she believed this way is not appropriate for primary grades. This idea contradicts research that suggests children should be exposed to problem solving early in order for them to be good problem solvers later on (Kloosterman & Stage, 1992).

Conclusion

It can be noted that the two preservice teachers have differing thoughts toward mathematics and the teaching and learning of mathematics. According to the results of the two interviews and scenarios, it can be said that the subjects have dissimilar attitudes toward mathematics: one was positive and the other was negative, but they have similar views on how to teach. Furthermore, the interviewees showed inconsistency between reality and how they portrayed themselves as mathematics teachers. Nevertheless, they appeared to hold narrow views and were reluctant in adopting the trends that most educators call for. Moreover, the interviewees indicated that their beliefs were influenced by a number of different factors including past experience, schooling, and teachers.

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THE BELIEFS AND EFFECTIVE PRACTICES OF COMMUNITY COLLEGE MATHEMATICS FACULTY REGARDING STUDENTS WITH LEARNING DISABILITIES

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The purpose of this qualitative research was to add to a sparsely existing body of knowledge in the area of the beliefs and practices of college mathematics faculty. This research was conducted with the intention of providing community college mathematics instructors with effective practices that promote mathematical learning in students with learning disabilities (LD) and the underlying beliefs of these instructors. Interviews with students and instructors regarding beliefs and effective practices along with classroom observations constituted the methodology of data collection. Research from the students suggested that affective social factors and particular instructional design promoted mathematical self-efficacy which affected achievement outcomes. Instructors who exhibited the most knowledge of the learner were the instructors who were rated the most effective and who had the most self-efficacy in teaching these students.

Rationale

Because mathematics is one of the most multi-modal subject areas, mathematics instructors, in particular, need adequate education in working with students who have learning disabilities. Most faculty are generally uninformed and unprepared to work with students who have disabilities (Greenbaum, Graham, & Scales, 1995). Negative attitudes of faculty are cited as a primary reason that students with disabilities fail at postsecondary institutions (Deshler, Ellis, & Lenz, 1996), whereas positive interactions with faculty are considered one of the most important elements in the students‘ college experience (Stage & Milne, 1966). Mathematics is often the gatekeeper which either allows students with learning disabilities to continue their postsecondary education or prevents their admission to four-year institutions. Faculty with faulty knowledge of learners with disabilities could result in beliefs that underlie ineffective practices in the classroom context. Thus, it is incumbent that mathematics instructors at the community college level be educated and equipped to work with this particular population of students.

Methodology

Participants in this research consisted of a purposeful sample of four community college mathematics students or alumnae with learning disabilities and their previous college mathematics instructors. Students with above average verbal IQs were chosen because these students are strongest in verbal conceptualization, reasoning in processing information, and thinking skills (Waldron, & Saphire, 1990). Since this was a qualitative research, these students could provide the thick description necessary to generate a model of what these students considered to be effective instructor practices. Based on student referrals of effective and ineffective mathematics instructors, seven of the mathematics faculty were selected and interviewed, and four of these were observed in the classroom context. Most of the students and faculty were interviewed twice, with the follow-up interview consisting of questions that arose as transcriptions and analyses were conducted. Field notes were taken during classroom observations and a researcher journal was also used in generating a model. Each interview was coded and then compared to other interviews and observations. The triangulation of the multiple data sources provided internal validity of the research project.

Results

Coding the answers of the students' interviews revealed twenty-nine practices that the students deemed as effective in the mathematics classroom. These were consolidated into the following categories and are listed by order of importance: a risk-free environment to seek help, being respected and valued as a human being, and explicit step-by-step instruction with classroom notes made available. Students often felt embarrassed and misunderstood by instructors they deemed as ineffective. The instructors interviewed that had been classified as effective also strongly emphasized the social-emotional needs of these students as well as explicit instruction. Classroom observations revealed that students were much more likely to speak out and ask and/or answer questions in the classrooms of effective instructors. These findings concur with Ryan, Gheen, and Midgley (1998) who reported that instructors who had warm, supportive relationships with their students empower even those with low efficacy to seek help. Also in the literature, Carnine (1997) recommends explicit practice as an instructional design for students with learning disabilities. A severe limitation of this research, however, is that none of the students interviewed had ever participated in a reform mathematics classroom. Therefore, generalities that all students with disabilities will benefit from explicit instruction as opposed to reform instruction are unwarranted.

Implications

Students with learning disabilities often have low mathematical self-efficacy due to past mathematical failures. Instructors who believe all students can learn mathematics and make an effort to know their students and provide a learning environment in their classrooms where all students feel the freedom to seek help, empower students with learning disabilities and motivate these students to succeed. As one student said, "Cause he made me feel, maybe I am competent in math." And her final grade in Statistics revealed that Maylynn was indeed competent in math.

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TEACHERS' BELIEFS INFLUENCING THE IMPLEMENTATION OF A PROJECT-BASED HIGH SCHOOL MATHEMATICS CURRICULUM

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The idea of the use of projects in the teaching and learning of mathematics has been around a long time, but, traditionally, has not been reflected in the mathematics classroom, particularly at the senior high school level. Like word problems, projects could "bring reality into the mathematics classroom, to create occasions for learning and practising the different aspects of applied problem solving, without the practical … inconveniences of direct contact with the real world situation" [Verschaffel, 2002, p. 65]. A project-based curriculum, then, has the potential of enhancing school mathematics. However, the curriculum by itself is unlikely to make a significant difference in the classroom. How it is realized will likely depend on the teacher. This paper reports on mathematics teachers' beliefs in implementing a new project-based high school mathematics curriculum.

In recent years, studies on mathematics teachers' beliefs have taken on significant importance as a basis for understanding mathematics education. These studies suggest that in order for us to achieve a comprehensive conception of the relationship between mathematics teachers' beliefs and practice, it is important to understand the belief content (what teachers believe), the belief structure (how the beliefs are held), and the belief function (impact on teaching). In this study the focus is on the belief content and function.

The project-based curriculum is the new Alberta [Canada] Applied Mathematics Program for students of grades 10 to 12 who are likely to not pursue an academic area at university that requires mathematics. The curriculum was implemented beginning in 2000 through to 2002. The curriculum is officially described in its introduction by the designers as follows:

> [It] focuses on the application of mathematics in problem solving. Through challenging and interesting activities and projects, students further develop their skills in mathematical operations and in understanding concepts. …The curriculum, in general, emphasizes the application and relevance of math in daily life.

The textbooks for this program consist predominantly of projects. An example of a grade 10 project topic is: In this project, you will work in a group of three to design your own line of jewelry. The members of your group will include two jewelry designers and one marketing manager. You will work with area, volume, scale factors, and metric units of measurement. An example of a grade 11 project topic is: In this project you will explore some of the mathematics of population growth. You will study different mathematical models that describe the growth of the world's population, write a report on the subject, and create a poster display.

A case study was conducted with three experienced high school mathematics teachers. They were selected because they were at the beginning of the second year of implementation of the new curriculum, they were experienced and they were willing to participate. Data collection consisted of a semi-structured interview of mainly open-ended questions. The interview focused on the teachers' thinking about the applied curriculum, their experience implementing it, their beliefs about and attitude towards the curriculum and their teaching of it. The interviews were tape recorded and transcribed. Analysis involved scrutinizing the transcripts for significant statements that conveyed the participants' beliefs and attitudes, and organizing them into themes.

The findings of the study describe the teachers' attitudes and beliefs that were related to their implementation and teaching of the new curriculum. Only the beliefs that were common to the three teachers are presented in this summary of the findings. Three themes emerged as consisting of their most dominant, common beliefs as follows:

Beliefs About Nature of Projects: The teachers' beliefs about the nature of the projects focused on the utility of mathematics and relevance to students. They viewed mathematics as meaningful to students only when it is related to applications that were relevant to them. The projects in this curriculum focused on real world applications and this made them appealing to the teachers. They believed the curriculum content was more relevant and meaningful to students because of the focus on real-world situations. This supported their positive attitude to implement the curriculum and screened out concerns about lack of depth in mathematics content that they claimed other teachers had. Thus their teaching focused on motivating students. In general, their beliefs about projects limited the scope and depth in which they implemented the curriculum. These beliefs were reinforced by the positive attitude the students displayed with these projects, which they had not encountered before.

Beliefs About Group Learning: The teachers believed that the teaching approach required for the new curriculum was the way to teach mathematics and should result in a more interesting way of learning for the students, although they did not use it in their prior teaching, in particular, the use of investigations and groups. The projects were set up as small-group investigations, which challenged the teachers' beliefs about using groups in the mathematics classroom. They considered groups as unnecessary in learning mathematics. Thus, the focus on group work was not considered an asset. For example, one teacher explained, "I don't see the point of why we have to work in groups. The projects could have been fine to do individually." Another teacher thought that this was changing a unique part of mathematics instruction.

> Mathematics classrooms had something that was a little different. In science they group up, in English they group up, in social [studies] they group up, so we [math] didn't group up, so I don't mind the grouping up, but it had strong points to not group up.

Thus their implementation of groups was very controlled. In spite of this, they recognized the importance for students' autonomy. For example:

> Anytime you can give the student the opportunity to learn first, the better it is for the student. I think that the kind of an inverse relationship, the less the teacher does, the better the teaching is. That is what this new curriculum is saying; it's great!

Beliefs About Support: The teachers believed that the best form of support was from within their schools. While they valued the workshops they attended in the school board, they considered the internal support to be more important in influencing their implementation of the new curriculum. They also believed that getting time to read the text book, to plan the lessons, and to ask questions was more important "to figure it out" than the workshops by the school system. In conclusion: The findings provide evidence of how teachers' beliefs can influence if and how a new curriculum gets implemented and an example of what teachers could believe is important to aid the implementation process. This has implication for teacher development in terms of the importance to attend to beliefs. It also draws attention to examples of beliefs that could enhance or hinder the implementation process.

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CHANGE IN PROSPECTIVE TEACHERS' PERSPECTIVES ABOUT EFFECTIVE TEACHING OF MATHEMATICS: STRUCTURAL PATTERNS AND QUALITATIVE CATEGORIES

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Educators claim that teachers' thinking and beliefs have powerful influences on their teaching (Clark & Peterson, 1986; VanLeuvan, 1997). Fennema (1990) and Li (1999) found that teachers' belief systems influence their perceptions, decisions related to classroom planning and actions, as well as students' learning, beliefs and attitudes. Suggestions for effective teaching of mathematics vary. Some educators weigh both affective and management elements equally, while others do not. Some focus on theoretical perspectives, and some specify practical strategies for effective teaching. Some consider the teacher's role as a key element of learning, while others emphasize the importance of the student's part in effective learning.

Tharp and Gallimore (1988) proposed three major mechanisms - modeling, contingency management, and feedback - that can become meta-cognitive strategies for learners to control their own learning. Bliss and her colleagues (1996) suggested that "pupils make sense of teachers' instructions in their own ways, sometimes very different from those of the teacher" (p.41). In other words, teachers can help students to organize their own experience by suggesting cognitive strategies, which can be used in developing structures for memorization or rules for storing information and experience. Wood (1991) argued that effective teaching does not always guarantee sufficient and necessary conditions for learning. He placed emphases on both the learner's need to make sense of the world and the teacher's task of initiating pupils into ways of conceptualizing and reasoning. He proposed two fundamental rules for effective teaching with contingent controls of learning: immediate increase in help or control for a child's failure; following a child's success, apply less help than was given before the success. Reynolds and Muijs (1999) reviewed the research on teacher effectiveness in teaching mathematics and summarized the elements of the active teaching models: high opportunity to learn, an academic orientation from the teacher, effective classroom management, high teacher expectations of the pupils, a high proportion of whole-class teaching, and heavily interactive teaching that involves pupils in classroom attitudes. Van de Walle's (2001) seven strategies for effective teaching are exclusively practical. According to him, teachers need to create a mathematical environment; to pose worthwhile mathematical tasks; to use cooperative learning groups; to use models and calculators as thinking tools; to encourage discourse and writing; to require justification of student responses; and to listen actively- focus more on the instructional process rather than considering all the possible factors that influence teaching. On the other hand, some studies discussed standard qualifications of effective teaching. For example, the NCTM (2000) requires the following principles for effective mathematics teaching: understanding what students know and need to learn and then challenging and supporting them to learn it well; knowing and understanding mathematics, students as learners, and pedagogical strategies; providing a challenging and supportive classroom learning environment; and continually seeking improvement. Wong & Wong (1998) suggested that a successful teacher must know and practice the three characteristics of an effective teacher. According to them, the effective teacher has positive expectations for student success, is an extremely good classroom manager, and knows how to design lessons to help students reach mastery.

The participants in this study were forty-two preservice teachers enrolled in mathematics methods courses, which is part of the University's 5-year teacher preparation program. Concept maps were used to collect data on a) students' entering and exit perspectives about effective teaching of mathematics and b) their conceptual changes throughout the teacher preparation process. The participants were asked to develop a concept map of "Effective Teaching in the Mathematics Classroom" in the beginning and the end of the program. Both qualitative and descriptive analyses were used to compare and contrast responses of the elementary education students' entering and exiting perspectives about teaching mathematics effectively.

To understand the conceptualization and structural organization of the preservice teachers' concept mapping, the total number of concept map entries, the degree of hierarchical organization, categorical centrality and specificity, and density were measured. The major finding claims that preservice teachers developed changes in conceptual and structural organization, as well as similar tendencies throughout the teacher preparation program. Overall, instructional methods, curriculum/planning, and teachers were considered as prominent factors for effective teaching. Instructional methods and classroom management showed increased centrality while curriculum/planning, social context, students, and teachers were factors that did not. In addition, the factors proportions for curriculum/planning, instructional methods and classroom management increased from premaps to postmaps, whereas social context, students, and teachers percentage decreased. This study is expected to be useful in developing a reflective teacher education program, as well as guiding subsequent program evaluation efforts.

As a whole, group mean density scores showed an increase from 2.56 on premaps to 2.78 on postmaps. There were either substantial or slight increases in the density scores for twenty-two student teachers' concept maps. With regard to the density scores by category, student teachers' attention to instructional methods increased from premaps (2.95) to postmaps (3.57). Classroom management factors also showed a slight increase, while other factors decreased from pre- to post-maps. However, as Artiles & McClafferty (1998) stated, one "cannot assume that greater density means greater understanding and vise versa" (p.198). Rather, the density scores show changes in student teachers' conceptualizations.

There was a change in the rank order of the centrality from premaps to postmaps. For the premaps, categories were centered according to centrality in the following order: teachers, curriculum/planning, instructional methods, social context, classroom management, and students. For the postmaps, the centrality rank order was: instructional methods, classroom management, teachers, curriculum/planning, social context, and students. On the other hand, a change in the rank order of the categorical specificity was insubstantial. Instruction, curriculum/planning, and teacher factors showed strong specificity for both pre- and postmaps. Even though there were some changes indicated, a similar tendency was also observed in the categorical centrality and specificity between premaps and postmaps. Instructional methods and classroom management showed increased centrality, while curriculum/planning, social context, students, and teachers showed decreased centrality from entering maps to exit maps. With regard to the categorical specificity, student teachers included the most items in curriculum/planning and instruction categories. Curriculum/planning, instructional methods and classroom management proportion increased from premaps to postmaps, while the social context, students, and teachers percentage decreased.

While this study reported the results as a descriptive study, it provides useful information for the teacher education curriculum, which will help decide course offering, topics to be emphasized in supervision, content of methods courses, and so on. The most important contribution of this study is that the results include information about the needs and desires of preservice teachers. It is also indicated that responses (items) in the concept maps and the structural organization of the mapping reflect their previous experiences as students during the methods courses, or as teachers during field experience and student teaching.

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THE TEACHING BELIEFS AND VIEWS ABOUT MATHEMATICS OF EARLY RECRUITMENT MATHEMATICS TEACHERS

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Helping teachers construct and maintain reform-based views of the nature of mathematics and beliefs about teaching of mathematics is a central goal of any teacher preparation program. These beliefs influence the type of instruction enacted in the mathematics classroom; thus giving attention to the beliefs and views of mathematics teachers is critical as teacher education programs develop new pathways for students to obtain certification. This study explores the beliefs and views of students who are participating in an early recruitment course with the hope of providing direction to those who are developing teacher preparation programs.

The students in this study participate in the UTeach program at the University of Texas, a project between faculty and staff from the Colleges of Education and Natural Sciences that prepares secondary science, mathematics and computer sciences teachers. The program coursework draws upon content and pedagogical courses, as well as field experiences in local school districts. In particular, the program has an early recruitment component, referred to as Step 1 and Step 2. These one-credit courses allow students to explore teaching as a profession prior to enrollment in the professional development program.

This poster describes the first year of a longitudinal study examining the development of mathematics teachers in the UTeach program. Data from several different sources will be examined collectively in order to determine how an early recruitment program impacts the development of mathematics teachers. Students currently enrolled in Step 1 who indicated that they were mathematics majors were contacted through e-mail about participation in the study. Those willing to participate (n = 7) were observed during the Step 1 class and interviewed. Semi-standardized interviews are used to collect background information from the participants as well as to ascertain the reason the students have elected to consider teaching as a career. Each student is interviewed regarding beliefs about teaching and learning in mathematics using the TPPI. There are eight questions, which include: How will your students learn best? How do you know when your students understand? The interviews will be assessed in order to determine if the students hold reform-based, instructional, transitional, or traditional beliefs about teaching.

Views on the Nature of Mathematics Surveys (VNOMS) – Each student also completed the VNOMS, which is a 10 question open-ended survey on their views about the discipline of mathematics and how it is best represented in the classroom. The VNOMS survey was developed from the Views About the Nature of Math (VAMS) (Carlson, Buskirk & Halloun, 2002) and it intends to identify the factors that affect teachers' understanding of mathematics and how they design and implement their instructional material. Each participant's responses will be coded as naïve, expert or a particular level of transition.

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"HOW DO I KNOW IF THEY'RE LEARNING?" AN INVESTIGATION OF A MATHEMATICIAN'S STRUGGLE TO CHANGE HER TEACHING

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The modified Moore method (MMM) has been recognized as a successful instructional strategy for teaching undergraduate and graduate mathematics (Jones, 1977; Renz, 1999). One of the salient features of the MMM is that students individually construct proofs and solve problems, and then present their solutions during class meetings. Though the MMM is not based on a particular theory of learning, it reflects current reform efforts in mathematics education emphasizing the active involvement of students in the learning process (NCTM, 2000; National Research Council, 2001). Experienced MMM instructors report that the method is the ideal way to teach undergraduate and graduate mathematics (Halmos, 1985). However, it is often difficult for beginning MMM instructors to conduct the class in such a way that improved mathematics learning is apparent. The research presented in this poster focuses on the experience of a mathematician who was implementing the MMM in an undergraduate course for the first time.

Research has demonstrated that teachers pass through several stages of concern when attempting to change from a traditional to a more student-centered teaching style: (1) concern for *self*, (2) concern for *task*, and (3) concern for *students* (Brown & Borko, 1992; Luft, 1999). Our results demonstrate that this framework may be an effective lens for examining the development of post-secondary mathematics instructors, who frequently have little or no pedagogical training. The instructor's strategies and approaches to the course changed over the semester as she moved from concern for herself as a teacher towards concern for the learning of the students. Primary teaching issues she identified during the semester included strategies for encouraging student presentations and participation in discussions, appropriateness of the use of brief lectures to introduce and explain concepts, posing questions to elicit student understanding, facilitating students'attempts to construct difficult proofs in and out of class, and strategies for encouraging the learning of the weaker students in the course. Factors that appeared to influence her progression towards a more student-centered teaching style were discussion and collaboration with both novice and expert MMM instructors, reflection on her own teaching experiences in the form of a journal, time spent talking with students outside of class, and exposure to research on the learning of mathematics, which she requested from the researchers.

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