

IMPROVISATIONAL COACTIONS AND THE GROWTH OF COLLECTIVE MATHEMATICAL UNDERSTANDING

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In this paper we characterize and describe the growth of collective mathematical understanding as a process of improvisational coaction. Drawing on the theoretical work of Sawyer (2003) in improvisational jazz and theatre, we explain how mathematical understanding can be observed to emerge from the complex, improvisational ways that a group of learners work together mathematically. We also distinguish coaction from interaction to highlight the importance in improvisational flow of acting with the contributions of other group members, rather than merely acting on these, suggesting that the growth of collective mathematical understanding is a truly shared process.

Mathematical Understanding as a Dynamical Process

In discussing mathematical understanding, we are influenced by, and employ in our analysis of data, elements of the Pirie-Kieren Theory for the Dynamical Growth of Mathematical Understanding. This theory is well established and has been extensively presented at previous PME meetings and elsewhere (e.g., Pirie & Kieren, 1994, Martin, Towers, & Pirie, 2000). The theory characterises mathematical understanding as an on-going process in which a learner responds to the problem of reorganising his or her knowledge structures by continually revisiting existing understandings to generate ‘thicker’ understandings. Pirie and Kieren have termed this process ‘folding back’. The theory considers understanding in terms of a set of embedded levels or modes of knowledge building activity. (Towers & Davis, 2002, p.318)

It is important to note that although the levels or layers of the model grow outwards from the local to the general, this does not imply that understanding grows in this way. Instead, growing mathematical understanding occurs through a complex movement backwards and forwards through the layers of understanding. Thus, mathematical understanding is not a static outcome or product, but instead is seen to emerge through the actions of the learners, moment by moment, as a process of mathematical engagement.

The theory also draws significantly on the notion of “images”, meaning any ideas the learner may have about the topic, any “mental” representations, not just visual or pictorial ones. When Image Making, learners are engaging in specific activities aimed at helping them to develop particular ideas about a concept or topic. Image Making often involves the drawing of diagrams, working through specific examples or playing with numbers. By the Image Having stage the learners are no longer tied to actual activities, they are now able to carry with them a general mental plan for these specific activities and use it accordingly. This frees the mathematical activity of the learner from the need for particular actions or examples.

Collective Mathematical Understanding

Collaborative working, whether in small groups or as a whole class, and the associated practices of interaction and discussion, continue to be strongly advocated organisational strategies in the mathematics classroom. However, whilst it is now widely recognised that

discussion is an essential part of developing mathematical understanding, and collective mathematical action is an area in which there has been significant research interest in recent years, we still know relatively little about how collective actions contribute to students' growth of understanding. Drawing on socio-cultural and distributed views of learning, a range of theories for characterising mathematical thinking as "an inherently social process" (Bowers & Nickerson, 2001, p.2) have emerged, and continue to be developed. (See, e.g., Cobb, 1999; Crawford, 1999; Lave, 1997; Saxe, 2002; Sfard & Kieran, 2001). Whilst important, and widely applicable, much of this existing work is not explicitly concerned with the growth of collective mathematical understanding as such, nor does it offer to the researcher or teacher a means to document this at a detailed level of emergence – something that the Pirie-Kieren Theory offers for individual learners. Thus, to more fully make sense of the observed growth of understanding, it becomes necessary to move beyond merely focusing on the individual learners, and to also look at what we term the "coactions" of the group, and the ways in which mathematical understanding emerges from these. Our research (Martin, Towers, & Pirie, 2006; Martin & Towers, 2003) therefore attempts to contribute to the field by systematically exploring the interplay between the individual and collective in the mathematics classroom through the development of a new theoretical perspective and through the analysis of the implications of the application of that perspective to classroom practice. In this paper we will focus on one element of our developing framework, that of coactions, and illustrate how this, together with elements from the Pirie-Kieren Theory, can be used to describe the growth of collective mathematical understanding as it is seen to emerge moment by moment.

Collective Mathematical Understanding as an Improvisational Process

In a recent paper (Martin, Towers, and Pirie, 2006) we offered the beginnings of a theoretical framework that proposed a way to look at, and account for, the growth of mathematical understanding at a collective level. We defined collective mathematical understanding as the kinds of mathematical actions and learning we may see occurring when a group of learners, of any size, work together on a piece of mathematics. More specifically, we suggested that by using the lens of improvisational theory, it was possible to observe acts of mathematical understanding that could not simply be located in the minds or actions of any one individual, but instead emerged from and existed in the interplay of the ideas of individuals, as these became woven together in shared action, as in an improvisational performance. We thus suggested that, as when listening to or watching an improvisational performance, in considering the growth of collective mathematical understanding the observer's attention should focus on the group as a whole, and not simply on what each of the participating individuals is contributing.

Coactions and the Growth of Mathematical Understanding

In elaborating the notion of creative, improvisational group performances, Sawyer (2003) talks of improvisational activity as being conceived of "as a jointly accomplished co-actional process" (p. 38). For us the use of the term coaction rather than interaction emphasises the notion of acting with the ideas and actions of others in a mutual, joint way. More precisely we use the term coaction as a means to describe a particular kind of mathematical action, one that whilst obviously in execution is still being carried out by an individual, is also dependent and contingent upon the actions of the others in the group. Thus, a coaction is a mathematical action that can only be meaningfully interpreted in light of, and with careful reference to, the interdependent actions of the others in the group. (Martin, Towers & Pirie, 2006, p.156)

Sawyer (2003), in talking about improvisational performance, suggests that when an idea is offered to the group, they can respond in a variety of ways. In particular, they collectively have the option of accepting the innovation (by working with it, building on it, making it “their own”), rejecting the innovation (by continuing the performance as if it had never occurred), or partially accepting the innovation (by selecting one aspect of it to build on, and ignoring the rest). This evaluative decision is a group effort, and cannot be identified clearly with any single individual. (Sawyer, 2003, p.92)

This notion places a responsibility on those who are positioned to respond to an offered action or innovative idea, as much as on the originator, and it is this “social process of evaluation” (p.92) wherein the group collectively determines whether, and how, the idea will be accepted into the emerging performance, that we suggest is the key to the emergence of collective mathematical understanding. We therefore see coaction as being a specific kind of interaction, but whereas interaction allows for reciprocal, complementary collaboration, without the requirement to be mutually building on the just offered action, coaction goes beyond this and requires mutual joint action. This is an important distinction in our work, as it must also be recognised that collective mathematical understanding is not an automatic or simple occurrence whenever two or more people are collaborating or working together. In such cases, what is observed may instead be a set of individual understandings occurring simultaneously, even though there is a high level of interaction. It is the nature and form of the collaboration which may (or may not) give rise to the growth of mathematical understanding at the collective level, and it is on this process that we focus in this paper, by offering some necessarily brief examples of coactions and explaining how collective mathematical understanding emerges from these.

Coactions in action: The emergence of collective mathematical understanding

We now turn to some extracts of data to illustrate both the improvisational character of the growth of collective mathematical understanding and how this can be seen to emerge through coactions. The examples we offer are drawn from a set of video data, collected with the aim of seeing cases of collective mathematical understanding, and with the purpose of facilitating this. We worked with a number of students preparing to be elementary school teachers. The students were invited, over a series of one-hour lunchtimes, to come and work on some mathematical problems, which, we hoped, as well as serving our research purpose, would also help them with their own mathematical and pedagogical knowledge. They were allowed to form their own groups of three or four, and to choose tasks from a booklet supplied, which contained nine tasks covering different areas of mathematics. In this paper we shall discuss one of the videotaped sessions, with three students known here as Mary, Shauna and Hilary. Our discussion will focus on the observed growth of mathematical understanding of the students, and will employ elements of the Pirie-Kieren Theory, with a specific focus on the ways in which it is coaction that leads us to characterise the growth as collective in nature. The group has chosen a task that requires them to label sixteen different triangles as equilateral, isosceles or scalene. The question asks them to do this by considering “side length properties”. Each triangle is drawn on a 3 x 3 grid of dots.

Extract One: “None of them are equal...”

We join the group as they start the task. There is a short pause as they seem to consider where to start:

Shauna: It’s all coming back to me

Hilary: I don’t remember scalene or isosceles

S: Isosceles is this, okay? (drawing) where two are equal?

Michelle: Yeah

S: Equilateral is when they're all equal?

H: Hm hm

S: And scalene is?

M: they're all wonky?

H: This must be scalene

S: OK

H: When it has one, one sss...(pause)

M: One longer?

S: Isos, eq and scale. So the scale none of them are equal?

We see the three students begin the task at the Image Having layer. The initial statement of Shauna suggests that properties of triangles and their associated names are not new to them, and they have an existing understanding of these. However, none of the three students seems immediately able to simply recall and restate the definitions for the three different kinds of triangles, and instead what we see is individual students' definitions, posed mainly as questions, inviting the other students to accept and confirm these. For example, Shauna gives correct definitions of isosceles and equilateral, but looks for agreement from the others that her ideas are viable for the group. However, in the case of the scalene triangle, she is not sure (suggesting perhaps she either does not have or can not recall her image for this particular shape) and invites (and requires) the others to participate in what becomes a collective act of Image Having. No one student is able to simply offer a complete definition and instead the three students each offer what can be characterised as partial fragments of an image for the scalene triangle. Mary talks of it as being "wonky"; Hilary and Mary both develop the idea of "one longer", whilst Shauna extends this idea to the conception of "none of them [the sides] are equal". We suggest the students are mutually coacting on the ideas of each other, and building on what has been offered to attempt to collectively work together to have (or even re-have) a useable image. This Image Having is not located in the understandings of any one individual, but instead emerges from the way that the individual mathematical ideas are starting to intertwine, as the group collectively accepts ideas.

Extract Two: "Because of the way the pins are..."

Following this, they begin work on the task and start to label each of the triangles as equilateral, isosceles or scalene, using their definitions. However, they decide that in the case of some triangles they need to measure the sides to be able to determine their type. But, they do not have a ruler, and prior to the extract below have been trying to determine the type visually. What we now see is the emergence of a new approach for deciding what kind of triangle they are looking at. As they cannot accurately find the length of sides, they instead turn to examining the relationship between the dots (what the students call pins or dots) and the sides of the triangles:

H: I don't think there are any equilateral

M: Yeah, I was just going to say, I don't think any of them are equilateral

H: Because of the way the pins are actually...

M: ...but..like that distance should be the same as that, if they're, wouldn't these dots all be equal distance

H: Yeah, but distance (pause) yeah that's what I'm thinking, this should be the same as this

- M: So, this distance, this one would be then, yeah right, don't you think?
 H: Yeah, but they're not..see look
 S: (laughs)
 M: This one right here. No, this one down here, where this is 'cos this is one peg away and this is one peg away
 H: But that's not the same
 M: Oh but..
 H: OK, but if you have a square. See this is a square [indicating the geoboard on which triangle 2 is drawn. See Figure 1], so if you have a square. This is two and this is two and this is two and this is two. But that is not two [indicating the hypotenuse of triangle 2] and that's what the triangle is.
 M: Yeah. So none of them are equal.

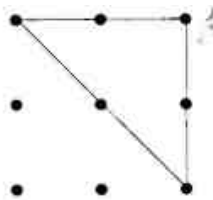


Figure 1. Triangle #2

In the above excerpt we see the students folding back to construct an image of length that is based on being able to compare the distance between different pairs of pins. Hilary begins by conjecturing that none of the triangles are equilateral “because of the way the pins are”. Mary agrees, and this prompts the group to move from thinking explicitly about sides of triangles, and instead to work with the concept of length on a pin grid. In particular, we see Hilary and Mary coacting as they offer fragments of ideas that are picked up and developed by the other. Towards the end we see Hilary drawing on, and offering to the group, an image for the properties of a square and using this to establish that two pins that are vertical or horizontal neighbours are not the same distance apart as two that are diagonal neighbours. She links this to the triangle, and at this moment the thinking of the group is returned to the task in hand, and the image they have for characteristics of triangles. However, the image they now have for an equilateral triangle is thicker than that prior to folding back. Whereas initially their image only considered the visual or measured length of sides, it now also allows for the comparison of sides in terms of pins on the grid. At the very end of the extract we see Mary using this image to state that “none of them are equal” meaning that none of the triangles in the task are equilateral.

Extract Three: “An equilateral can't have a right angle in it”

Later in the session the students became confused about which distances were actually equal on the board, and were struggling to use their “pin based” image to confidently label the remaining triangles. We join them as they try to label the triangle shown in Figure 2.

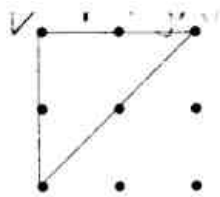


Figure 2. Triangle #9

- M: Like if that's not an equilateral triangle then what is? (She is referring to triangle #9, See figure 2)
- H: No, see to me, this one looks like its longer, shorter, shorter. To me an equilateral triangle is more like a yield sign.
- M: But that's a right angle [indicating the right angle in triangle #9]. (pause). Does that have anything to do with it?
- H: An equilateral can't have a right angle in it, I don't think.
- M: No..it can't
- S: It couldn't. Yeah, you're right. Perfect.
- M: It's like sixty, sixty, sixty.
- S: That helps.
- H: (laughing)
- S: It's all coming back to me!
- M: Slowly but surely (all laughing)

Here we see the emergence of a focus on the interior angles of the triangle and a new 'rule' – the idea that “an equilateral triangle can't have a right angle in it.” Although Mary initially suggests that the triangle is equilateral, this is an idea that is rejected by Hilary, who indicates how the triangle has two shorter sides and one that is longer. To illustrate her idea she offers a visual representation of her image – that an “equilateral triangle is more like a yield sign”. Mary picks up on this, and notes that triangle # 9 has a right angle, though she isn't sure what this means. Hilary responds and the group collectively agrees, including Shauna, that an equilateral triangle cannot have a right angle, and should have three angles of sixty degrees each. This is a collective moment of Image Having for the group, where there is a sense of confidence now that they have now recalled what they know about triangles and together re-have an image for an equilateral triangle. Again, although individuals clearly contributed to this process, the image they now have cannot be simply attributed to any one student. Their image having was collective, involving a process of offered ideas being built upon. Each student offered parts of the image, which the group collectively interweaved to have something that was acceptable to them all. Using this new image, the three students confidently and correctly identified the remaining triangles in the task.

Discussion

In all of the short extracts above we see the three students collaborating and working together. Further, we suggest that their activity can usefully be seen in terms of coactions. There is a sense of unpredictability about the pathways their collective mathematical actions, and emerging understandings, will follow. In fact, the image they make and have for an equilateral triangle in terms of the size of interior angles is not one that is hinted at by the task instructions,

which instead suggest working with the length of sides. Also, no one student is able to offer an image that is immediately viable to use in completing the task. Instead, what occurs throughout the session (and as illustrated in the extracts), is a continual process of the offering of ideas or “innovations” and the collective coacting on these. There is a sense in the extracts that no one student simply wants (or is able to) tell the others what to do, or to merely state a mathematical idea without expecting some response. Equally, those listening to the idea seem to accept their responsibility to do something with what is offered, and not merely receive it. The ways in which the group is able to interweave fragments of each individual’s knowing, to allow a shared (rather than taken-as-shared) image to emerge from their coactions, is what enables their collective mathematical understanding to grow, and is what ultimately enables them to successfully complete the task.

In theoretical terms, and to emphasise the value of researchers’ attention being oriented to coactions (rather than simply individual’s statements or even interactions), we note that the students’ actions can only be meaningfully interpreted in light of, and with careful reference to, the interdependent actions of the others in the group, and hence we propose the notion of mathematical coactions as a fruitful tool for enabling a more fine-grained analysis of the growth of collective mathematical understanding.

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