A FOCUS ON VARIABLES AS QUANTITIES OF VARIABLE MEASURE IN COVARIATIONAL REASONING

Jason Silverman
Saint Joseph's University
jason.silverman@sju.edu

Recent emphasis in mathematics education has been placed a covariational conception of function, but focusing on "quantities varying," makes evident a subtlety in covariational reasoning. This paper will analyze the potential difficulties one encounters when one's understanding of involves a graph constraining the ways in which variables vary, particularly an inability to reason about why fairly complex functions behave as they do.

Introduction and Purpose.

The predominant conception of function in mathematics today can be described as functions as correspondence, or "a rule that assigns each element x in a set A exactly one element, y, called f(x), in a set B." This current definition of function persists despite the fact that many mathematicians and mathematics educators (Eisenberg, 1991, Thompson, 1994, Wilder, 1967) criticize this conception on pedagogical grounds. In response to the criticisms of the correspondence conception of a function, a number of researchers (Carlson, 1998, Carlson, Jacobs, Coe, Larsen, & Hsu, 2002, Confrey & Smith, 1995, Thompson, 1994, Thompson & Thompson, 1994) have proposed a covariational conception of a function. The covariational conception of a function is based on Euler's notion of function: "[when] some quantities depend on others in such a way that if the latter are changed the former undergoes changes themselves, then the former quantities are called functions of the latter quantities" (Kleiner, 1989).

It is true that this notion of function is consistent with "reform" mathematics, which calls for a shift in attention in the mathematics curriculum from functions as rules and formulas to functional relationships in both pure and applied settings, however there are subtleties to the current conceptualization and frameworks for understanding and classifying students' understanding of function as covariation of quantities and covariational reasoning. Two strands of research which address these issues are the work of Carlson, Jacobs, Coe, Larsen & Hughes (Carlson et al., 2002) and APOS (Action-Process-Object-Schema) Theory (Dubinsky & Harel, 1992).

Mental Action Description of Mental Action

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MA1	Coordinating the value of one variable with changes in the other
MA2	Coordinating the direction of change of one variable with changes in the other variable
MA3	Coordinating the amount of change of one variable with changes in the other variable
MA4	Coordinating the average rate-of-change with uniform increments of change in the input variable
MA5	Coordinating the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the function.

Figure 1: Carlson, et al's (2002) Covariation Framework

Carlson, et al (2002) present a covariation framework which describes five "mental actions" and five coordinated levels of covariational reasoning ability. The covariation framework

Alatorre, S., Cortina, J.L., Sáiz, M., and Méndez, A.(Eds) (2006). Proceedings of the 28th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Mérida, México: Universidad Pedagógica Nacional.

"contains five distinct developmental levels. … [One's] covariational reasoning ability has reached a given level of development when it supports the mental actions associated with that level and the actions associated with all lower levels" (p. 357). The mental action form of their covariation framework is shown in Figure 1.

Covariational reasoning has also been related to the work in APOS theory on functions. Thompson (1994) notes that many mathematics students tend to see a function as a "command to calculate" and that early algebra students are no more likely to see the expression x (12 (x – 5)) as representing a number as elementary students are to see that the expression 4 (12 (4 – 5)) represents anything other than something to do. Researchers (Asiala, Brown, DeVries, Dubinsky, Mathews, & Thompson, 1996, Dubinsky & Harel, 1992) have labeled this such a conception of a mathematical concept as an action conception. A process conception of a function involves the learner automating lengthy sequences of operations into an expression that, in his or her image of it, "evaluates itself" (Thompson, 1994). When a student possesses a process conception of function, he or she can imagine the function as something that performs the sequences of operations but no longer needs to actually think about the chain of operations when envisioning the result of the evaluation. The development of covariational reasoning is related to the progression from an action to a process conception: once a person conceives of a function as the covariation of quantities, they "can begin to imagine 'running through' a continuum of numbers, letting an expression evaluate itself (very rapidly!) at each number" (Thompson, 1994, p. 26) and can therefore conceptualize the way in which the quantities covary.

It is this notion of "quantities varying," though, that makes evident a subtlety in covariational reasoning. Though the varying quantities are a conceptual precursor to a fully-developed conception of a function, these variable quantities are often not the focus of instruction or analysis of students understanding of functions. Both conceptualizations of understanding the concept of function discussed above rely on the underlying notion of an independent variable varying and a dependent variable varying accordingly. The way these two variables vary is quite different: the independent variable is free to vary, but the dependent variable is "constrained," in that as the independent variable varies, the dependent variable must vary in a particular way. As an example, consider a function representing the height of a roller coaster car at any given time during the ride. In this example, that constraint on the dependent variable makes sense – why would one be concerned with the height of the car in a vacuum? The variation of that quantity would give us no useful information about the velocity, acceleration, energy, etc. of the situation.

In this paper, we will examine a teaching experiment that placed the variability of quantities in the foreground and was designed to better understand how students develop covariational reasoning abilities that position them to analyze functional relationships. The study was grounded in Saldanha & Thompson's (1998) conceptualization of covariation as (1) an understanding of a variable is a measurable quantity (it has a magnitude) whose measure can vary; (2) the coordination of two variables, each of which can be envisioned as varying independently; (3) envisioning a graph as a collection of points; (4) envisioning the collection of points as being generated by keeping track, simultaneously, of two quantities whose values vary; and (5) envisioning that every point in a graph represents, at once, simultaneous values of two quantities.

Methods and Data Sources.

The data for this study was gathered as part of a constructivist teaching experiment (Steffe & Thompson, 2000). The analytic tools employed fall under the heading of "grounded theory" (Glaser & Strauss, 1967). Analysis involved the development and refinement of hypotheses through a process of continual review, constant comparison, and revision. The analysis consisted of multiple iterations of the generation and refinement of hypotheses, first from a global perspective (reviewing the entire data corpus to identify segments of theoretical importance) and second from a local perspective (line-by-line coding of the segments identified and continual development and refinement of hypotheses).

Participants for this study were a group of 11 undergraduate mathematics and secondary education dual majors at a large, private university in the Southern United States. For this study, we focus primarily on four class sessions that took place in the first few weeks of the Fall semester. Each class session was videotaped and immediately following the class, was transcribed and annotated. All class artifacts are also analyzed as part of this study.

Results and Conclusion.

The results to be described in this paper focus on students initial inability to reason covariationally about the straightforward, but quite complex question: Explain why the graph of f(x) = sin(nx) behaves as it does. Though their explanations would qualify as covariational reasoning (and at a minimum) MA1, analysis of students' initial explanations of the behavior of these functions indicate their reliance on graphs as the primary image underlying their explanations. Further analysis indicates that their understandings and explanations were grounded a more advanced conception of function, but something falling shy of covariation: their focus was not on how the quantities covaried but how the graph behaved. For one to explain the behavior of these functions, they must keep track of (at least) three quantities: x, nx, and sin(x), and only two of these quantities appear in the graph. A conceptual explanation of this problem involves an understanding of how the quantities x, nx, and sin(nx) each vary independently, an understanding of how x and nx covary, how nx and sin(nx) covary, and finally how x and sin(nx) covary. An explanation of this sort enables one to understand why the family of functions behaves as it does. The proposed paper will discuss the students' developing understandings as they took part in a course that focused on appropriate images of variables and functions that would support covariational reasoning. The affordances of this developing understanding, both in the students' mathematical development and in their pedagogical conceptualizations (all were future teachers) will be discussed.

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