Procedural and Conceptual Knowledge in Mathematics

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A Teaching-Research report on the Procedural-Conceptual divide
1) Written Conceptual Thought in Remedial Mathematics: The case of decimals
2) Written Thought that reveals Pitfalls in understanding the Limit Concept in Calculus

We present the results of a teaching-research projects that continue the study of the relationship between procedural and conceptual knowledge in the mathematical classroom began in (Rittle et.al.,2002) in both projects the students were adults at a community college in New York City. Both experiments involved conceptual orientated writing exercises that were developed and integrated with instruction of procedural knowledge. In the first experiment the relationship between these thought processes was assessed in the first context of the students' ability to assimilate procedural knowledge and solve word problems. In the second experiment writing was used to shed light on student's difficulties translating definitions(phrases) from a textbook into an understanding of the limit concept.

In many remedial mathematics courses, a large extent of the textbook and classroom presentation begins with a brief review of definitions and then focuses on computational modeling of procedural knowledge. In this 'traditional' curriculum, concept development is viewed as arising from computational proficiency with relevant procedures. "Knowledge of structures is regarded as meta-knowledge, growing out of arithmetical procedural proficiency, and has been developed as an extension of arithmetical knowledge." (Morris,1999) In like manner, instruction in calculus often emphasizes procedural knowledge grounded in algebra. Aspinwell and Miller express this view when they state: "students regard computation as the essential outcome of calculus and thus end their study of calculus with little conceptual understanding." (Aspinwell et al., 1997)

The perception that, use of this traditional approach encourages rote learning of procedural knowledge and leads to weak understanding of the relationships between such knowledge and the objects they act upon has lead to reform efforts that focus on underlying conceptual knowledge. Reform efforts that introduce conceptual instruction in mathematics have been extensively studied with mixed results. Nesher (1986) reviewing such studies in arithmetic concludes there is no evidence of a, "relationship between success in algorithmic performance vs. success in 2 understanding." Such reform efforts have frequently been founded on the hypothesis that, student's conceptual knowledge will necessarily increase their procedural proficiency, the so called "simultaneous action view." (Haapasalo, and Kadijevich , 2000) It is perhaps not surprising that writing which for Vygotsky (1986) epitomized conceptual thought has likewise been extensively studied with mixed results. This is evidenced by Powell and Lopez (1989) who issue a challenge, to reform efforts that use writing, to establish evidence on the role of writing in promoting, "conceptual development or increased mathematical maturity."

Theoretical considerations

One developmental model that can be used to justify the “traditional” approach to learning mathematics with its emphasis on procedural knowledge has been labeled the “dynamic action view.” (Byrnes et. al., 1991), (Haapasalo et. al., 2000) In this model an individual learns by applying procedural knowledge to an existing conceptual foundation, then increasing proficiency with procedural knowledge opens the door to expanding upon one’s conceptual knowledge. The notion that there are stages of development in mathematics and learners typically go through a procedural orientated phase before they can effectively use their conceptual knowledge is studied in (Davis, et. al.,2000) and is based upon Piaget’s understanding of how procedural knowledge can be integrated or “assimilated” into one’s conceptual schema, “the heart of the process involves assimilating the new material into appropriate knowledge networks or structures.” (Heibert and Lefevre, 1987)

As mathematics instructors, using the tradition curriculum with its focus on computational modeling of procedural knowledge we had noticed that, although many of the students do well enough to pass during the semester they perform poorly on the final exam. (equally true in prealgebra or calculus) Acting on the Vygotskian hypothesis that, the conscious thought engendered in the writing process will assist the individual in assimilating their procedural knowledge into their schema and thus lead to retention, writing was introduced into the classroom pedagogy.

As researchers, we had two goals the first was, to assess the benefits of written mathematical thought during the semester in the assimilation (retention) of procedural knowledge and problemsolving skills on the final exam. The second goal was to extend the results of our work in (Rittle- Johnson et. al., 2002) in which we hypothesized that, “throughout development, conceptual and procedural knowledge influence one another in mutually supportive and integrated ways.” In 3 particular we reflect upon whether the relationship between procedural and conceptual knowledge is marked by iterations of first one and then the other type of knowledge or whether procedural and conceptual knowledge influence one another continually throughout the developmental process.

Project - Writing in Pre-Algebra

While the classroom instruction involved extensive use of computational modeling, serious attention was given to the underlying conceptual knowledge that the students would need to solve word problems and answer writing questions. Data was gleamed from initial diagnostic exams, partial exams (with written as well as computational mathematics) and a departmental final exam.

The results of this study confirm that initial conceptual knowledge does indeed tend to dominate initial procedural knowledge in determining student’s proficiency on subsequent partial exams. Furthermore, procedural knowledge on the partial exam is more important that written conceptual knowledge on the partial exam in determining students proficiency with procedural knowledge on the final exam (retention of procedural knowledge). Thus in this case the iterative or “dynamic view” can effectively be used to uphold the traditional method of classroom instruction. However, we note that, as the number of student included in this study grow each semester the role that written conceptual thought had in predicting the student's retention of procedural knowledge grew and became statistically significant (well beyond the 0.05 level) in relationship to the role of computational proficiency.

In retention of problem-solving skills, our results indicate that, the traditional method does not prepare students nearly as well as it did with retention of procedural knowledge and can clearly be helped by written conceptual thought. In this case the optimal environment for learning clearly involved coordination of procedural and conceptual knowledge during the semester. We close in noting that these results do not imply the simultaneous view that, a focus on conceptual knowledge will result in procedural proficiency, but they do provide evidence against the converse or dynamic action view.

Project on Conceptual knowledge in Calculus

In this experiment written conceptual thought was integrated or coordinated with computational instruction to better understand student's difficulties understanding calculus 4 concepts. The difficulties with coordinating terminology and instruction into conceptual understanding are presented. In particular, we note student's written responses that highlight how textbook phrases which intend to bring the limit concept of calculus within student's reach can instead create more confusion.

Several more progressive textbooks had used an interesting phrase describing the limit of a sequence: The number L is the limit of a sequence a_n if we can find terms of the sequence as close to L as we wish for n sufficiently large." The phrase is most probably thought as best conveying the idea that "L is the limit if for any epsilon larger than zero there exist N such that for any n>N, we have $| a_n – L|<e$ " without the formal meaning of the concept. However, students' essays indicate that the phrase causes probably exactly the same amount of confusion as it wants to avoid by that "conceptualizing".

> For n sufficiently large? If one would be speaking to an individual without much math experience, that individual would still be lost…It is hard to explain what for n sufficiently large is…a better explanation in English could have been, the larger n becomes, the closer to zero the value becomes.

A analysis of that phrase reveals high level of ambiguity which, unfortunately, allows to subvert the original structure of the definition into incorrect verbal interpretation. We will submit students written responses for the discussion as to what is the proper manner of conceptualizing calculus.

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